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Farshid Hajir, Christian Maire

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PRIME DECOMPOSITION AND THE IWASAWA MU-INVARIANT

by

Farshid Hajir & Christian Maire

Abstract. — For $\Gamma = \mathbb{Z}_p$, Iwasawa was the first to construct Γ -extensions over number fields with arbitrarily large μ -invariants. In this work, we investigate other uniform pro- p groups which are realizable as Galois groups of towers of number fields with arbitrarily large μ -invariant. For instance, we prove that this is the case if p is a regular prime and Γ is a uniform pro- p group admitting a fixed-point-free automorphism of odd order dividing $p-1$. Both in Iwasawa's work, and in the present one, the size of the μ -invariant appears to be intimately related to the existence of primes that split completely in the tower.

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Introduction

Let p be a prime number. Let K be a number field and let L/K be a uniform p -extension: L/K is a normal extension whose Galois group $\Gamma := \text{Gal}(L/K)$ is a uniform pro- p group (see section 1.1.1). We suppose moreover that *the set of places of K that are ramified in L/K is finite*.

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If F/K is a finite subextension of L/K , let us denote by $A(F)$ the p -Sylow subgroup of the class group of F and put

$$\mathcal{X} := \varprojlim_{\mathbb{F}} A(\mathbb{F}),$$

where the limit is taken over all number fields F in L/K with respect to the norm map. Then \mathcal{X} is a $\mathbb{Z}_p[[\Gamma]]$ -module and, thanks to a structure theorem (see section 1.1.2), one attaches a μ -invariant to \mathcal{X} , generalizing the well-known μ -invariant introduced by Iwasawa in the classical case $\Gamma \simeq \mathbb{Z}_p$. Iwasawa showed that the size of the μ -invariant is related to the rate of growth of p -ranks of p -class groups in the tower. For the simplest \mathbb{Z}_p -extensions, i.e. the cyclotomic ones, he conjectured that $\mu = 0$; this was verified for base fields which are abelian over \mathbb{Q} by Ferrero and Washington [9] but remains an outstanding problem for more general base fields. Iwasawa initially suspected that his μ -invariant vanishes for all \mathbb{Z}_p -extensions, but later was the first to construct \mathbb{Z}_p -extensions with non-zero (indeed arbitrarily large) μ -invariants. It is natural to ask what other p -adic groups enjoy this property. Our present work leads us to the following conjecture:

Conjecture 0.1. — *Let Γ be a uniform pro- p group having a non-trivial fixed-point-free automorphism σ of order m co-prime to p (in particular if $m = \ell$ is prime, Γ is nilpotent). Then Γ has arithmetic realizations with arbitrarily large μ -invariant, i.e. for all $n \geq 0$, there exists a number field K and an extension L/K with Galois group isomorphic to Γ such that $\mu_{L/K} \geq n$.*

Our approach for realizing Γ as a Galois group is to make use of the existence of so-called p -rational fields. See below for the definition, but for now let us just say that the critical property of p -rational fields is that in terms of certain maximal p -extensions with restricted ramification, they behave especially well, almost as well as the base field of rational numbers. As we will show, Conjecture 0.1 can be reduced to finding a p -rational field with a fixed-point-free automorphism of order m co-prime to p . These considerations lead us to formulate the following conjecture about p -rational fields.

Conjecture 0.2. — *Given a prime p and an integer $m \geq 1$ co-prime to p , there exist a totally imaginary field K_0 and a degree m cyclic extension K/K_0 such that K is p -rational.*

Although we will not need it, we believe K_0 in the conjecture may be taken to be imaginary quadratic; see Conjecture 4.16 below. Our key result is:

Theorem 0.3. — *Conjecture 0.2 for the pair (p, m) implies Conjecture 0.1 for any uniform pro- p group Γ having a fixed-point-free automorphism of order m .*

One knows that if m is an odd divisor of $p - 1$, where p is a regular prime, then for any $n \geq 1$, the cyclotomic field $\mathbb{Q}(\zeta_{p^n})$ provides a positive answer to the previous question. We therefore have

Corollary 0.4. — *Assume p is a regular prime and that the uniform group Γ has a fixed-point-free automorphism σ of odd order m dividing $p - 1$. Then Conjecture 0.1 is true for Γ .*

This circle of ideas is closely related to the recent work of Greenberg [15] in which he constructs analytic extensions of number fields having a Galois group isomorphic to an

open subgroup of $\mathrm{Gl}_k(\mathbb{Z}_p)$. The idea of studying pro- p towers equipped with a fixed-point-free action of a finite group of order prime to p occurs also in Boston's papers [2] and [3].

Our work raises the following purely group-theoretical question.

Question 0.5. — *Let Γ be a nilpotent uniform pro- p group. Does there exist a uniform nilpotent pro- p group Γ' having a fixed-point-free automorphism of prime order $\ell \neq p$ such that $\Gamma' \twoheadrightarrow \Gamma$?*

A positive answer to this question would imply that for all nilpotent uniform pro- p groups Γ , there exist arithmetic realizations with arbitrarily large μ -invariant.

In his recent work [12], Gras gave some conjectures about the p -adic regulator in a fixed number field K when p varies. In our context, one obtains:

Theorem 0.6. — *Let \mathbb{P} be an infinite set of prime numbers and let m be an integer co-prime to all $p \in \mathbb{P}$. Let $(\Gamma_p)_{p \in \mathbb{P}}$ be a family of uniform pro- p groups of fixed dimension d , all having a fixed-point-free automorphism σ of order m . Assuming the Conjecture of Gras (see Conjecture 4.13), there exists a constant p_0 , such that for all $p \geq p_0$, there exist Γ_p -extensions of number fields with arbitrarily large μ -invariants.*

In another direction, a conjecture in the spirit of the heuristics of Cohen-Lenstra concerning the p -rationality of the families \mathcal{F}_G of number fields K Galois over \mathbb{Q} , all having Galois group isomorphic to a single finite group G , seems to be reasonable (see [31]). When the prime $p \nmid |G|$, the philosophy here is that the density of p -rational number fields in \mathcal{F}_G is positive. This type of heuristic lends further evidence for conjecture 0.1.

Notations

Let \mathcal{G} be a finitely generated pro- p group. For two elements x, y of \mathcal{G} , we denote by $x^y := y^{-1}xy$ the conjugate of x by y and by $[x, y] := x^{-1}y^{-1}xy = x^{-1}x^y$ the commutator of x and y . For closed subgroups $\mathcal{H}_1, \mathcal{H}_2$ of \mathcal{G} , let $[\mathcal{H}_1, \mathcal{H}_2]$ be the closed subgroup generated by all commutators $[x_1, x_2]$ with $x_i \in \mathcal{H}_i$. Let $\mathcal{G}^{\mathrm{ab}} := \mathcal{G}/[\mathcal{G}, \mathcal{G}]$ be the maximal abelian quotient of \mathcal{G} , and let $d(\mathcal{G}) := \dim_{\mathbb{F}_p} \mathcal{G}^{\mathrm{ab}}$ be its p -rank.

Denote by (\mathcal{G}_n) the p -central descending series of \mathcal{G} :

$$\mathcal{G}_1 = \mathcal{G}, \mathcal{G}_2 = \mathcal{G}^p[\mathcal{G}, \mathcal{G}], \dots, \mathcal{G}_{n+1} = \mathcal{G}_n^p[\mathcal{G}, \mathcal{G}_n].$$

The sequence $(\mathcal{G}_n)_n$ forms a base of open neighborhoods of the unit element e of \mathcal{G} .

If K is a number field, let $A(K)$ be the p -Sylow subgroup of the class group of K . Let $S_p := \{\mathfrak{p} \subset \mathcal{O}_K : \mathfrak{p}|p\}$ be the set of primes of K of residue characteristic p . If S is any finite set of places of K , denote by K_S the maximal pro- p extension of K unramified outside S and put $\mathcal{G}_S := \mathrm{Gal}(K_S/K)$ as well as $A_S := \mathcal{G}_S^{\mathrm{ab}}$.

1. Arithmetic background

1.1. Formulas in non-commutative Iwasawa Theory. —

1.1.1. Algebraic tools. — Two standard references concerning p -adic analytic and in particular, uniform, pro- p groups are the long article of Lazard [25] and the book of Dixon, Du Sautoy, Mann and Segal [8].

Let Γ be an analytic pro- p group: we can think of Γ as a closed subgroup of $\mathrm{Gl}_m(\mathbb{Z}_p)$ for a certain integer m . The group Γ is said *powerful* if $[\Gamma, \Gamma] \subset \Gamma^p$ ($[\Gamma, \Gamma] \subset \Gamma^4$ when $p = 2$); a powerful pro- p group Γ is said *uniform* if it has no torsion.

Let us recall two important facts.

Theorem 1.1. — *Every p -adic analytic pro- p group contains an open uniform subgroup.*

Theorem 1.2. — *A powerful pro- p group Γ is uniform if and only if for $i \geq 1$, the map $x \mapsto x^p$ induces an isomorphism between Γ_i/Γ_{i+1} and $\Gamma_{i+1}/\Gamma_{i+2}$.*

Let us make some remarks.

Remark 1.3. — Let $\dim(\Gamma)$ be the dimension of Γ as analytic variety.

1) If Γ is uniform then $d_p\Gamma = \dim(\Gamma) = \mathrm{cd}_p(\Gamma)$, where $\mathrm{cd}_p(\Gamma)$ is the p -cohomological dimension of Γ .

2) [24, Corollary 1.8] Suppose Γ is a torsion-free p -adic analytic group, $p \geq \dim(\Gamma)$ and $d(\Gamma) = \dim(\Gamma)$. Then the group Γ is uniform.

3) For $p \geq 2$, the pro- p -group $\mathbb{1}_n + \mathcal{M}_n(\mathbb{Z}_p)$ is uniform, where $\mathcal{M}_n(\mathbb{Z}_p)$ is the set of $n \times n$ matrices with coefficients in \mathbb{Z}_p .

Now let us fix a uniform pro- p group Γ of dimension d . Recall that (Γ_n) is the p -descending central series of Γ . By Theorem 1.2, one has $[\Gamma : \Gamma_n] = p^{dn}$, for all n .

Let $\mathbb{Z}_p[[\Gamma]] := \varprojlim_{\mathcal{U} \triangleleft \circ \Gamma} \mathbb{Z}_p[\Gamma/\mathcal{U}]$ be the complete Iwasawa algebra, where \mathcal{U} runs along the

open normal subgroups of Γ . Put $\Omega := \mathbb{F}_p[[\Gamma]] = \mathbb{Z}_p[[\Gamma]]/p$. The rings Ω and $\mathbb{Z}_p[[\Gamma]]$ are local, noetherien and without zero divisor [8, §7.4]: each of them has a fractional skew field. Denote by $Q(\Omega)$ the fractional skew field of Ω . If \mathcal{X} is a finitely generated Ω -module, the *rank* $\mathrm{rk}_\Omega(\mathcal{X})$ of \mathcal{X} is the $Q(\Omega)$ -dimension of $\mathcal{X} \otimes_\Omega Q(\Omega)$. For more details, we refer the reader to Howson [19] and Venjakob [34].

Definition 1.4. — Let \mathcal{X} be a finitely generated $\mathbb{Z}_p[[\Gamma]]$ -module. Put

$$r(\mathcal{X}) = \mathrm{rk}_\Omega \mathcal{X}[p] \quad \text{and} \quad \mu(\mathcal{X}) = \sum_{i \geq 0} \mathrm{rk}_\Omega \mathcal{X}[p^{i+1}] / \mathcal{X}[p^i].$$

Remark 1.5. — One has $\mu(\mathcal{X}) \geq r(\mathcal{X})$ and, $r(\mathcal{X}) = 0$ if and only if $\mu(\mathcal{X}) = 0$.

There is a large and growing literature on the study of $\mathbb{Z}_p[[\Gamma]]$ -modules in the context of Iwasawa theory. We recall a result of Perbet [30], where, by making use of the work of Harris [17], Venjakob [34] and Coates-Schneider-Sujatha [5], among others, he manages to estimate the size of the coinvariants $(\mathcal{X}_{\Gamma_n})_n$ of \mathcal{X} . Recall that \mathcal{X}_{Γ_n} is the largest quotient of \mathcal{X} on which Γ_n , the n th element of the p -central series, acts trivially.

Theorem 1.6 (Perbet, [30]). — *Suppose that \mathcal{X} is a torsion $\mathbb{Z}_p[[\Gamma]]$ -module where Γ is a uniform pro- p group of dimension d . Then for $n \gg 0$:*

$$\dim_{\mathbb{F}_p} \mathcal{X}_{\Gamma_n} = r(\mathcal{X})p^{dn} + O(p^{n(d-1)}),$$

and

$$\#(\mathcal{X}_{\Gamma_n}/p^n) = p^{\mu(\mathcal{X})p^{dn} + O(np^{n(d-1)})}.$$

We now turn to applying these formulas in the arithmetic context.

1.1.2. Arithmetic. — Let L/K be a Galois extension of number fields with Galois group Γ , where Γ is a uniform pro- p group. We assume that *the set of primes of K ramified in L/K is finite*. Let $(\Gamma_n)_n$ be the p -central descending series of Γ and put $K_n := L^{\Gamma_n}$ for the corresponding tower of fixed fields.

Now let \mathcal{X} be the projective limit along L/K of the p -class group $A(K_n)$ of the fields K_n . Then, \mathcal{X} is a finitely generated torsion $\mathbb{Z}_p[[\Gamma]]$ -module. For the remainder of this work, \mathcal{X} will denote this module built up from the p -class groups of the intermediate number fields in L/K . In particular, we put $\mu_{L/K} = \mu(\mathcal{X})$ and $r_{L/K} = r(\mathcal{X})$. For this module, by classical descent, Perbet proved:

Theorem 1.7 (Perbet [30]). — *For $n \gg 0$, one has:*

$$\log |A(K_n)/p^n| = \mu_{L/K} p^{dn} \log p + O(np^{d(n-1)})$$

and

$$d_p A(K_n) = r_{L/K} p^{dn} + O(p^{n(d-1)}).$$

One then obtains immediately the following corollary:

Corollary 1.8. — *(i) Along the extension L/K , the p -rank of $A(K_n)$ grows linearly with respect to the degree $[K_n : K]$ if and only if $\mu_{L/K} \neq 0$.*

(ii) If there exists a constant α such that for all $n \gg 0$, $d_p A(K_n) \geq \alpha p^{dn}$, i.e. $d_p A(K_n) \geq \alpha [K_n : K]$, then $\mu_{L/K} \geq \alpha$.

At this point, we should recall some standard facts from commutative Iwasawa theory. First, for the cyclotomic \mathbb{Z}_p -extension, Iwasawa conjectured that the μ -invariant is 0 for all base fields, and this has been shown to be the case when the base field is abelian over \mathbb{Q} . When the base field K contains a primitive p th root of unity, the reflection principle allows one to give some estimates on the μ -invariant [30].

1.2. On the p -rational number fields. — A number field K is called *p -rational* if the Galois group \mathcal{G}_{S_p} of the maximal pro- p -extension of K unramified outside p is pro- p free. From the extensive literature on p -rational fields, we refer in particular to Jaulent-Nguyen Quang Do [22], Movahhedi-Nguyen Quang Do [27], Movahhedi [28], and Gras-Jaulent[14]. A good general reference is the book of Gras [11].

Definition 1.9. — If \mathcal{G} is a pro- p group, let us denote by $\mathcal{T}(\mathcal{G})$ the torsion of \mathcal{G}^{ab} . When $\mathcal{G} = \mathcal{G}_S$, put $\mathcal{T}_S := \mathcal{T}(\mathcal{G}_S)$; when $S = S_p$, put $\mathcal{T}_p := \mathcal{T}_{S_p}$.

A standard argument in pro- p group theory shows the following:

Proposition 1.10. — *A pro- p group \mathcal{G} is free if and only if $\mathcal{T}(\mathcal{G})$ and $H^2(\mathcal{G}, \mathbb{Q}_p/\mathbb{Z}_p)$ are trivial.*

Proof. — Indeed, the exact sequence $0 \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$ gives the sequence

$$0 \rightarrow H_2(\mathcal{G}, \mathbb{Z})/p \rightarrow H_2(\mathcal{G}, \mathbb{F}_p) \rightarrow H_1(\mathcal{G}, \mathbb{Z})[p] \rightarrow 0,$$

and to conclude, recall that $H_1(\mathcal{G}, \mathbb{Z}) \simeq \mathcal{G}^{\text{ab}}$ and $H_2(\mathcal{G}, \mathbb{Z}) \simeq H^2(\mathcal{G}, \mathbb{Q}/\mathbb{Z})^*$. \square

Remark that if \mathcal{G} is pro- p free then $\mathcal{G}^{\text{ab}} \simeq \mathbb{Z}_p^d$, where d is the p -rank of \mathcal{G} . This observation shows in particular that if the group \mathcal{G} corresponds to the Galois group of a pro- p extension of number fields, then necessarily this extension is wildly ramified.

We now explain how the Schur multiplier of \mathcal{G}_S is related to the Leopoldt Conjecture.

Proposition 1.11. — *Let S be a set of places of K , and for $v \in S$, let \mathcal{U}_v be the pro- p completion of the local units at v . If the \mathbb{Z}_p -rank of the diagonal image of \mathcal{O}_K^\times in $\prod_{v \in S} \mathcal{U}_v$ is maximal, namely $r_1 + r_2 - 1$, then $H^2(\mathcal{G}_S, \mathbb{Q}_p/\mathbb{Z}_p) = 0$. Moreover if $S_p \subseteq S$, these conditions are equivalent. In particular, assuming the Leopoldt Conjecture for K at p , \mathcal{G}_{S_p} is free if and only if \mathcal{T}_p is trivial.*

Proof. — For the case of restricted ramification, see [26]. For the general case, see [29, Corollaire 1.5]. \square

Remark 1.12. — Some examples of free quotients of \mathcal{G}_S with splitting conditions are given in §4.3 below.

Proposition 1.13 (A ‘numerical’ p -rationality criterion)

Let A_m be the p -Sylow subgroup of the ray class group of modulus $\mathfrak{m} = \prod_{\mathfrak{p}|p} \mathfrak{p}^{a_p}$ of K , where $a_p = 2e_p + 1$, and e_p is the absolute index of ramification of \mathfrak{p} (in K/\mathbb{Q}). Assume that K verifies the Leopoldt Conjecture at p . Then K is p -rational if and only if $d_p A_m = r_2 + 1$.

Proof. — First, as one assumes Leopoldt Conjecture for K at p , then the \mathbb{Z}_p -rank of $A_{S_p} = \mathcal{G}_{S_p}^{\text{ab}}$ is exactly $r_2 + 1$. Hence \mathcal{G}_{S_p} is free if and only if $\mathcal{T}_p = \{1\}$, which is equivalent to $d_p A_{S_p} = r_2 + 1$. Now, by Hensel’s lemma, every unit $\varepsilon \equiv 1 \pmod{\pi_{\mathfrak{p}}^{a_p}}$ is a p -power in $K_{\mathfrak{p}}^\times$ and so $d_p A_m = d_p A_{S_p}$. \square

Example 1.14. — Take $K = \mathbb{Q}(\zeta_7)$ and $p = 37$. Then, p is not ramified in K/\mathbb{Q} and $a_p = 3$. A simple computation gives $d_p A_{S_p} = 4 = r_2 + 1$, so K is 37-rational.

Example 1.15. — One can check easily that $\mathbb{Q}(\zeta_7)$ is not 2-rational but $\mathbb{Q}(\zeta_{13})$ is 2-rational.

Here is a very well-known case of the situation.

Proposition 1.16 (A ‘theoretical’ p -rationality criterion)

Suppose that the number field K contains a primitive p th root of unity. Then K is p -rational if and only if there exists exactly one prime of K above p and the p -class group of K (in the narrow sense if $p = 2$) is generated by the unique prime dividing p .

Proof. — See for example Theorem 3.5 of [11]. \square

Remark 1.17. — In particular, when p is regular, $\mathbb{Q}(\zeta_{p^n})$ is p -rational for all $n \geq 1$.

Now we will give some more precise statements by considering some Galois action. Let K/K_0 be a Galois extension of degree m , with $p \nmid m$. Put $\Delta = \text{Gal}(K/K_0)$. Let $r = r_1(K_0) + r_2(K_0)$ be the number of archimedean places of K_0 . Let S be a finite set of places of K_0 . The arithmetic objects that will use have a structure of $\mathbb{F}_p[\Delta]$ -modules. Then for a such module M , one notes by $\chi(M)$ its character. Let ω be the Teichmüller character, let $\mathbf{1}$ be the trivial character and let χ_{reg} be the regular character of Δ .

Proposition 1.18. — *Suppose the field K is p -rational and that the real infinite places of K_0 stay real in K (this is always the case when m is odd). Then*

$$\chi(A_{S_p}) = r_2(K_0)\chi_{\text{reg}} + \mathbf{1}.$$

Proof. — It is well-known. The character of the Δ -module $\prod_{v \in S_p} (\mathcal{U}_v / \mu_{p^\infty}(K_v))$ is equal to $[K_0 : \mathbb{Q}]\chi_{\text{reg}}$ and by Dirichlet's Unit Theorem, the character of $\mathcal{O}_K^\times / \mu(K)$ is equal to $(r_1(K_0) + r_2(K_0))\chi_{\text{reg}} - \mathbf{1}$. Then, as K is p -rational,

$$\chi(A_{S_p}) = [K_0 : \mathbb{Q}]\chi_{\text{reg}} - \left((r_1(K_0) + r_2(K_0))\chi_{\text{reg}} - \mathbf{1} \right) = r_2(K_0)\chi_{\text{reg}} + \mathbf{1}.$$

□

1.3. Genus Theory. — The literature on Genus Theory is rich. The book of Gras [11, Chapter IV, §4] is a good source for its modern aspects. All we will need in this work is the following simplified version of the main result.

Theorem 1.19 (Genus Theory). — *Let F/K be a Galois degree p extension of number fields L/K . Let S be the set of places of K ramified in F/K (including the infinite places). Then*

$$d_p A(L) \geq |S| - 1 + d_p \mathcal{O}_K^\times.$$

We will also need the following elementary fact.

Proposition 1.20. — *Let K be a number field containing a primitive p th root of unity, and let $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ be distinct prime ideals of \mathcal{O}_K . Then there exists a cyclic extension F/K of degree p in which the primes $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ all ramify.*

Proof. — Choose a prime ideal \mathfrak{p}_0 of \mathcal{O}_K in the inverse of the ideal class of the product $\mathfrak{p}_1 \cdots \mathfrak{p}_t$, so that $\mathfrak{p}_0 \mathfrak{p}_1 \cdots \mathfrak{p}_t$ is a principal ideal in \mathcal{O}_K , generated by some algebraic integer α , say. Then $F = K(\alpha^{1/p})$ is a cyclic degree p extension of K totally ramified at all the primes $\mathfrak{p}_1, \dots, \mathfrak{p}_t$. □

2. Background on automorphisms of pro- p groups

For this section, our main reference is the book of Ribes and Zalesskii [32, Chapter 2 and Chapter 4]. If Γ is a finitely generated pro- p group, denote by $\text{Aut}(\Gamma)$ the group of continuous automorphisms of Γ . Recall that the kernel of $g_\Gamma : \text{Aut}(\Gamma) \rightarrow \text{Aut}(\Gamma/\Gamma_2)$ is a pro- p group and that $\text{Aut}(\Gamma/\Gamma_2) \simeq \text{GL}_d(\mathbb{F}_p)$, where d is the p -rank of Γ .

2.1. Fixed points. —

Definition 2.1. — Let Γ be a finitely generated pro- p group and let $\sigma \in \text{Aut}(\Gamma)$. Put $\text{Fix}(\Gamma, \sigma) := \{x \in \Gamma : \sigma(x) = x\}$.

Remark 2.2. — (i) Obviously, $\text{Fix}(\Gamma, \sigma)$ is a closed subgroup of Γ .
(ii) For integers n , $\text{Fix}(\Gamma, \sigma) \subset \text{Fix}(\Gamma, \sigma^n)$ with equality when n is co-prime to the order of σ .

Definition 2.3. — For $\sigma \in \text{Aut}(\Gamma)$, $\sigma \neq e$, one says that $\Gamma^{\circ\sigma}$ is fixed-point-free if $\text{Fix}(\Gamma, \sigma) = \{e\}$. More generally, if Δ is a subgroup of $\text{Aut}(\Gamma)$, one says that the action $\Gamma^{\circ\Delta}$ of Δ on Γ is fixed-point-free if and only if

$$\bigcup_{\substack{\sigma \neq e \\ \sigma \in \Delta}} \text{Fix}(\Gamma, \sigma) = \{e\}.$$

In others words, Δ is fixed-point-free if and only if, for all non-trivial $\sigma \in \Delta$, $\Gamma^{\circ\sigma}$ is fixed-point-free.

Remark 2.4. — Clearly if $\Gamma^{\circ\langle\sigma\rangle}$ is fixed-point-free then $\Gamma^{\circ\sigma}$ is fixed-point-free; it is an equivalence when σ is of prime order ℓ .

We are interested in instances of groups with fixed-point-free action that arise in arithmetic contexts. Let us recall the Schur-Zassenhaus Theorem for a profinite group \mathcal{G} :

Theorem 2.5 (Schur-Zassenhaus). — *Let Γ be a closed normal pro- p subgroup of a profinite group \mathcal{G} . Assume that the quotient $\Delta := \mathcal{G}/\Gamma$ is of order co-prime to p . Then the profinite group \mathcal{G} contains a subgroup Δ_0 isomorphic to Δ . Two such groups are conjugated by an element of Γ and $\mathcal{G} = \Gamma \rtimes \Delta_0$.*

Proof. — See Theorem 2.3.15 of [32] or proposition 1.1 of [10]. □

As first consequence, one has the following:

Proposition 2.6. — *Let Γ be a finitely generated pro- p group and let $\Delta \subset \text{Aut}(\Gamma)$ of order m , $p \nmid m$. If $\Gamma^{\circ\Delta}$ is fixed-point-free then $(\Gamma/\Gamma_2)^{\circ\Delta}$ is fixed-point-free, where we recall that Γ_2 , the 2nd step in the p -central series of Γ , is the Frattini subgroup.*

Proof. — Let $\sigma \in \Delta$. The group $\langle\sigma\rangle$ acts on Γ , on Γ_2 and on Γ/Γ_2 . By the Schur-Zassenhaus Theorem (applied to $\Gamma_2 \rtimes \langle\sigma\rangle$), the non-abelian cohomology group $H^1(\langle\sigma\rangle, \Gamma_2)$ is trivial and then the nonabelian cohomology of the exact sequence $1 \rightarrow \Gamma \rightarrow \Gamma \rtimes \langle\sigma\rangle \rightarrow \langle\sigma\rangle \rightarrow 1$ allows us to obtain:

$$H^0(\langle\sigma\rangle, \Gamma) \twoheadrightarrow H^0(\langle\sigma\rangle, \Gamma/\Gamma_2),$$

which is exactly the assertion of the Proposition. See also [4], [15], [35]. □

A main observation for our paper is the converse of the previous proposition when Γ is uniform:

Proposition 2.7. — *Let Γ be a uniform pro- p group. Let $\sigma \in \text{Aut}(\Gamma)$ of order m co-prime to p . Then $(\Gamma/\Gamma_2)^{\circ\sigma}$ is fixed-point-free if and only if $\Gamma^{\circ\sigma}$ is fixed-point-free.*

Proof. — One direction is taken care of by Proposition 2.6. For the other direction, we first note that

$$\begin{aligned} \varphi : \Gamma_{n-1}/\Gamma_n &\rightarrow \Gamma_n/\Gamma_{n+1} \\ x &\mapsto x^p \end{aligned}$$

is a $\langle \sigma \rangle$ -isomorphism for $n \geq 2$. We thus obtain a $\langle \sigma \rangle$ -isomorphism from Γ/Γ_2 to Γ_n/Γ_{n+1} . If $(\Gamma/\Gamma_2)^{\circ\sigma}$ is fixed-point-free, then $(\Gamma_n/\Gamma_{n+1})^{\circ\sigma}$ is fixed-point-free for all $n \geq 1$. Suppose $y \in \Gamma$ satisfies $\sigma(y) = y$. As the action of Γ/Γ_2 is fixed-point-free, we have $y(\text{mod}\Gamma_2) \in \Gamma/\Gamma_2$ is trivial, so $y \in \Gamma_2$. Continuing in this way, we in fact conclude that $y \in \bigcap_n \Gamma_n = \{e\}$. \square

Corollary 2.8. — *Let Γ be a uniform pro- p group and let $\sigma \in \text{Aut}(\Gamma)$ of order m co-prime to p . Denote by χ the character of the semi-simple action of $\langle \sigma \rangle$ on Γ/Γ_2 . Then $\Gamma^{\circ\sigma}$ is fixed-point-free if and only, $\langle \chi, \mathbf{1} \rangle = 0$.*

Proof. — Indeed, one has seen (Proposition 2.7) that $\Gamma^{\circ\sigma}$ is fixed-point-free if and only $(\Gamma/\Gamma_2)^{\circ\sigma}$ is fixed-point-free, which is equivalent to $\langle \chi, \mathbf{1} \rangle = 0$. \square

Remark 2.9. — Suppose that Γ is uniform of dimension d . The restriction of $\sigma \in \text{Aut}(\Gamma)$ to $\Gamma/\Gamma_2 \simeq \mathbb{F}_p^d$ is an element of $\text{Gl}_d(\mathbb{F}_p)$. Denote by $P_\sigma \in \mathbb{F}_p[X]$ its characteristic polynomial. Then the action $\Gamma^{\circ\sigma}$ is fixed-point-free if and only if $P_\sigma(1) \neq 0$.

Remark 2.10. — When Γ is uniform and $\Delta \subset \text{Aut}(\Gamma)$ is of order m co-prime to p , testing that the action of Δ is fixed-point-free on Γ is equivalent to testing this condition on the quotient $M := \Gamma/\Gamma_2$. Let χ be the character resulting from the action of Δ on M . Then $\Gamma^{\circ\Delta}$ is fixed point-free on M if and only if, for all $e \neq \sigma \in \Delta$, $\text{Res}_{|\langle \sigma \rangle}(\chi)$ does not contain the trivial character, where here $\text{Res}_{|\langle \sigma \rangle}$ is the restriction to $\langle \sigma \rangle$. By Frobenius Reciprocity, this condition is equivalent to $\langle \chi, \text{Ind}_{\langle \sigma \rangle}^\Delta \mathbf{1} \rangle = 0$, where $\text{Ind}_{\langle \sigma \rangle}^\Delta$ is the induction from $\langle \sigma \rangle$ to Δ .

We need also of the following proposition which will be crucial for our main result.

Proposition 2.11. — *Let Γ be a finitely generated pro- p group. Let σ and τ be two elements in $\text{Aut}(\Gamma)$ of order m co-prime to p . If $\sigma = \tau$ on Γ/Γ_2 , then $\Gamma^{\circ\sigma}$ is fixed-point-free if and only if, $\Gamma^{\circ\tau}$ is fixed-point-free. More precisely, $\text{Fix}(\Gamma, \sigma) = g \cdot \text{Fix}(\Gamma, \tau)$ for a certain element $g \in \ker(\text{Aut}(\Gamma) \rightarrow \text{Aut}(\Gamma/\Gamma_2))$.*

One needs the following lemma:

Lemma 2.12. — *Let Γ be a finitely generated pro- p group. Let σ and τ be two elements of $\text{Aut}(\Gamma)$ of order m , $p \nmid m$, satisfying $\sigma = \tau$ on Γ/Γ_2 . Then there exists $g \in \text{Aut}(\Gamma)$ such that $\tau = \sigma^g$.*

Proof. — This is to be found in Lemma 3.1 of [18]. Since σ and τ coincide as elements of $\text{Aut}(\Gamma/\Gamma_2)$, there exists γ in the pro- p group $\ker(\text{Aut}(\Gamma) \rightarrow \text{Aut}(\Gamma/\Gamma_2))$ such that $\sigma = \gamma\tau$. Consider the group $\langle \tau, \gamma \rangle : \langle \gamma, \gamma^\tau, \dots, \gamma^{\tau^{m-1}} \rangle \rtimes \langle \tau \rangle$. Since $\langle \gamma, \gamma^\tau, \dots, \gamma^{\tau^{m-1}} \rangle \subset \ker(\text{Aut}(\Gamma) \rightarrow \text{Aut}(\Gamma/\Gamma_2))$, $\langle \tau, \gamma \rangle$ is a semi-direct product of a pro- p -group and a group of order m . As τ and σ are both in $\langle \tau, \gamma \rangle$, the subgroups $\langle \tau \rangle$ and $\langle \sigma \rangle$ are conjugate to each other (by the Schur-Zassenhaus Theorem 2.5): there exists $g \in \langle \gamma, \gamma^\tau, \dots, \gamma^{\tau^{m-1}} \rangle$

such that $\tau^k = \sigma^g$ for a certain integer k , $(k, m) = 1$, since σ and τ have the same order. Moreover, in Γ/Γ_2 , $\overline{\tau^k} = \overline{\sigma^g} = \overline{\sigma} = \overline{\tau}$, and so we can take $k = 1$. \square

Proof of Proposition 2.11. — By the previous lemma, $\tau = \sigma^g$, for some g in $\ker(\text{Aut}(\Gamma) \rightarrow \text{Aut}(\Gamma/\Gamma_2))$. We have that y is a fixed point of τ if and only if $g(y)$ is a fixed point of σ . \square

2.2. Lifts. — Given a uniform pro- p group Γ equipped with an automorphism σ of order m prime to p , the central question of this subsection is to realize $\Gamma \rtimes \langle \sigma \rangle$ as a Galois extension over a number field.

Proposition 2.13. — *Let F be a free pro- p -group on d generators, and let g_F be the natural map $\text{Aut}(F) \rightarrow \text{Aut}(F/F_2)$. Consider a subgroup $\Delta \subset \text{Aut}(F/F^p[F, F])$ of order m co-prime to p . Then there exists a subgroup $\Delta_0 \subset \text{Aut}(F)$ isomorphic to Δ such that $g_F(\Delta_0) = \Delta$. Moreover, any two such subgroups are conjugated by an element $g \in \ker(\text{Aut}(F) \rightarrow \text{Aut}(F/F_2))$.*

Proof. — First of all the natural map $g_F : \text{Aut}(F) \rightarrow \text{Aut}(F/F_2)$ is onto (Proposition 4.5.4 of [32]). Put $\tilde{\Delta} = g_F^{-1}(\Delta)$ and recall that $\ker(g_F)$ is a pro- p group. Then, $\Delta \simeq \tilde{\Delta}/\ker(g_F)$ which has order co-prime to p . By the Schur-Zassenhaus Theorem 2.5, there exists a subgroup $\Delta_0 \subset \text{Aut}(F)$, such that $(\Delta_0 \ker g_F)/\ker(g_F) \simeq \Delta$. Moreover, two such subgroups are conjugate to each other. \square

In fact, one needs a little bit more. The following proposition can be found in a recent paper of Greenberg [15] and partially in an unpublished paper of Wingberg [35].

Proposition 2.14 (Greenberg, [15], Proposition 2.3.1). — *Let $\mathcal{G} = F \rtimes \Delta$ be a profinite group where F is free pro- p on d' generators and where Δ is a finite group of order m co-prime to p . Let Γ be a finitely generated pro- p group on d generators, with $d' \geq d$. Suppose that there exists $\Delta' \subset \text{Aut}(\Gamma)$, with $\Delta' \simeq \Delta$, such that the module $\Delta'_{|\Gamma/\Gamma_2}$ is isomorphic to a submodule of $\Delta_{|F/F_2}$. Then there exists a normal subgroup N of F , stable under Δ , such that F/N is Δ -isomorphic to Γ and so we have a surjection*

$$\mathcal{G} \twoheadrightarrow \Gamma \rtimes \Delta \simeq \Gamma \rtimes \Delta'.$$

Since this result is essential for our construction, we include a proof.

Lemma 2.15 (Wingberg, [35], lemma 1.3). — *Let F be a free pro- p -group on d generators and let Γ be a pro- p group generated by d generators. Let φ be a morphism on pro- p groups $\varphi : F \twoheadrightarrow \Gamma$. Assume that there exists a finite group $\Delta \subset \text{Aut}(\Gamma)$ of order m co-prime to p . Then the action of Δ lift to F such that φ becomes a Δ -morphism.*

Proof following [35]. — For a finitely generated pro- p group N , denote by g_N the natural map $g_N : \text{Aut}(N) \rightarrow \text{Aut}(N/N_2)$. Recall that $\ker(g_N)$ is a pro- p group.

Let $1 \rightarrow R \rightarrow F \rightarrow \Gamma \rightarrow 1$ be a minimal presentation of Γ . Denote by $\text{Aut}_R(F) := \{\sigma \in \text{Aut}(F) : \sigma(R) \subset (R)\}$. The natural morphism $f : \text{Aut}_R(F) \rightarrow \text{Aut}(\Gamma)$ is onto (see [32, Proposition 4.5.4]). Put $\tilde{\Delta} := f^{-1}(\Delta) \subset \text{Aut}_R(F)$. Then $f(\tilde{\Delta}) = \Delta$. Now the isomorphism between F/F_2 and Γ/Γ_2 induces an isomorphism f' between $\text{Aut}(F/F_2)$ and $\text{Aut}(\Gamma/\Gamma_2)$. Hence on $\text{Aut}_R(F) \subset \text{Aut}(F)$, one has: $g_\Gamma \circ f = f' \circ g_F$. In particular $\ker(\text{Aut}_R(F) \rightarrow \text{Aut}(\Gamma/\Gamma_2))$ is a pro- p group. Now let $\tilde{f} : \tilde{\Delta} \rightarrow \Delta$. Then: (i) $\ker(\tilde{f})$

is a pro- p group and (ii) $\tilde{\Delta}/\ker(\tilde{f}) \simeq \Delta$ which has order co-prime to p . By the Schur-Zassenhaus Theorem 2.5, there exists $\Delta_0 \subset \text{Aut}_{\mathbb{R}}(\mathbb{F})$ such that $\Delta_0 \cap \ker(\tilde{f}) = \Delta$ and we are done. \square

Proof of Proposition 2.14. — As $d' \geq d$, let φ be a surjective morphism $\mathbb{F} \twoheadrightarrow \Gamma$. Put $\mathbb{R} = \ker(\varphi)$. As $p \nmid m$, the action of Δ on \mathbb{F}/\mathbb{F}_2 is semi-simple. Let us complete the $\mathbb{F}_p[\Delta]$ -module Γ/Γ_2 with a submodule M such that $\Gamma/\Gamma_2 \oplus M \simeq \mathbb{F}/\mathbb{F}_2$ as Δ -module. Let Γ' be the pro- p group $\Gamma' = \Gamma \times \Gamma_0$, where $\Gamma_0 \simeq (\mathbb{Z}_p/p\mathbb{Z}_p)^{d'-d}$ is generated by an \mathbb{F}_p -basis of M . By Lemma 2.15, there exists $\Delta_0 \subset \text{Aut}_{\mathbb{R}}(\mathbb{F})$ isomorphic to Δ , such that the morphism $\varphi : \mathbb{F} \twoheadrightarrow \Gamma'$ is a Δ_0 -morphism. By Proposition 2.13, there exists $g \in \text{Aut}(\Gamma)$ such that $\Delta_0 = \Delta^g$. We note that $\Delta_0 \subset \text{Aut}_{\mathbb{R}}(\mathbb{F})$ is equivalent to $\Delta \subset \text{Aut}_{g(\mathbb{R})}(\mathbb{F})$. Then we take $\mathbb{N} = \langle g(\mathbb{R}), g(M) \rangle$ and observe that \mathbb{F}/\mathbb{N} is Δ -isomorphic to Γ . \square

2.3. Frobenius groups. — We now review a group-theoretic notion that we need for our study of the μ -invariant.

Definition 2.16. — Let \mathcal{G} be a profinite group. One says that \mathcal{G} is a Frobenius group if $\mathcal{G} = \Gamma \rtimes \Delta$, where Γ is a finitely generated pro- p group, Δ is of order m co-prime to p , and such that the conjugation action of Δ on Γ is fixed-point-free.

The notion of a Frobenius group is a very restrictive one, as illustrated in the following Theorem:

Theorem 2.17 (Ribes-Zalesskii, [32], corollary 4.6.10). — *Let $\mathcal{G} = \Gamma \rtimes \Delta$ be a Frobenius profinite group. Then the subgroup Γ of \mathcal{G} is nilpotent. Moreover if $2 \mid |\Delta|$, Γ is abelian, if $3 \mid |\Delta|$, Γ is nilpotent of class at most 2, and more generally of class at most $\frac{(\ell - 1)^{2^{\ell-1}-1} - 1}{\ell - 2}$ if the prime number ℓ divides $|\Delta|$.*

Proposition 2.18. — *Let $\mathcal{G} = \Gamma \rtimes \Delta$, where Γ is a uniform pro- p group and such that $p \nmid |\Delta|$. Then \mathcal{G} is a Frobenius group if and only if the action of Δ is fixed-point-free on Γ/Γ_2 .*

Proof. — It is a consequence of Proposition 2.7. \square

3. Proof of the main result

Let us recall the motivating question of this paper. Given a uniform group Γ of dimension d , equipped with a fixed-point-free automorphism of finite order co-prime to p , can one realize an arithmetic context for Γ as Galois group with arbitrarily large associated μ -invariant?

3.1. The principle. — Here we develop our strategy, which is simply to emulate Iwasawa's original construction for $\Gamma = \mathbb{Z}_p$. Given a uniform pro- p group Γ , our claim is that it suffices to produce a Galois extension L of a number field K such that

- (i) $\text{Gal}(L/K)$ is isomorphic to Γ ;
- (ii) there are only finitely many primes that are ramified in L/K ;
- (iii) there exist infinitely many primes of K which split completely in L/K .

We now explain why such a construction suffices to answer our key question for Γ .

Proposition 3.1. — Suppose Γ is a uniform pro- p group and L is a Galois extension of a number field K such that:

- (i) $\text{Gal}(L/K)$ is isomorphic to Γ ;
- (ii) there are only finitely many primes that are ramified in L/K ;
- (iii) there exist infinitely many primes of K that split completely in L/K .

Then there exist Γ -extensions of number fields with arbitrarily large associated μ -invariant.

Proof. — We may assume, without loss of generality, that K contains ζ_p , a primitive p th root of unity, for if we replace K by $K(\zeta_p)$ and L by $L(\zeta_p)$, then conditions (i), (ii), (iii) still hold – we merely observe that $K(\zeta_p)/K$ has degree co-prime to p whereas L/K is a pro- p extension. Choose an integer $t \geq 1$, as well as distinct prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ of \mathcal{O}_K which split completely in L . Let K'/K a cyclic degree p extension in which each of the primes \mathfrak{p}_i ramifies for $i = 1, \dots, t$ (see Proposition 1.20). Letting $L' = K'K$, we see immediately that L'/K' is a Galois extension with Galois group isomorphic to Γ . Put $K_n = L'^{\Gamma_n}$ where Γ_n is the n th term in the p -central series of $\Gamma = \text{Gal}(L'/K')$. Let $K'_n : K'_n$. Then using the genus theory estimate Theorem 1.19 to the cyclic degree p extension K'_n/K_n , we get the following lower bound for the p -rank of the p -class group of K'_n :

$$d_p A(K'_n) \geq [K'_n : K] (t - r_2(K') - 1).$$

We conclude from Corollary 1.8 that $\mu_{L'/K'} \geq t - r_2(K') - 1$. Since we can take t to be as large as desired, and noting of course that $r_2(K')$ is fixed as $t \rightarrow \infty$, we have shown that Γ -extensions with arbitrarily large μ -invariant can be constructed over base fields which are cyclic of degree p over K (if K contains ζ_p), and over $K(\zeta_p)$ otherwise. \square

To produce a tower L/K satisfying the hypotheses of the Proposition, we realize Γ , along with its fixed-point-free automorphism, inside the maximal pro- p extension K_{S_p}/K of a p -rational field K ; in particular the condition (ii) will be automatically satisfied in our situation as the ramification in the tower will be restricted to the primes dividing p .

3.2. The case $\Gamma = \mathbb{Z}_p$. — We now review Iwasawa's strategy in [21] (see also Serre [33, §4.5]) for finding arithmetic situations with large μ -invariant.

Let K/\mathbb{Q} be an imaginary quadratic field. Let us denote by σ the generator of $\text{Gal}(K/\mathbb{Q})$. Suppose p is a rational prime which splits in \mathcal{O}_K into two distinct primes \mathfrak{p}_1 and \mathfrak{p}_2 . Let us suppose further that p does not divide the class number of K . Then for $i = 1, 2$, the maximal pro- p extension $K_{\{\mathfrak{p}_i\}}$ of K unramified outside \mathfrak{p}_i has Galois group Γ_i isomorphic to \mathbb{Z}_p . The automorphism σ permutes the fields $K_{\{\mathfrak{p}_i\}}$. Thus inside the compositum of these two \mathbb{Z}_p -extensions, if we denote by $\langle e_i \rangle$ the subgroup fixing the field $K_{\{\mathfrak{p}_i\}}$, then the subfield L fixed by $\langle e_1 + e_2 \rangle$ is Galois over \mathbb{Q} with Galois group isomorphic to \mathbb{Z}_p , and the action $\sigma(e_1) = e_2 \equiv -e_1 \pmod{\langle e_1 + e_2 \rangle}$ is dihedral.

Corollary 3.2. — Under the preceding conditions, $\text{Gal}(L/\mathbb{Q}) = \text{Gal}(L/K) \rtimes \langle \sigma \rangle \simeq \mathbb{Z}_p \rtimes \mathbb{Z}/2\mathbb{Z}$, where the action of σ satisfies $x^\sigma = x^{-1}$, x denoting a generator of $\text{Gal}(L/K)$. Thus, all primes ℓ of \mathbb{Q} which remain inert in K/\mathbb{Q} subsequently split completely in L/K .

Proof. — The first part of the corollary follows from the observations preceding it. For the second part, remark that the non-trivial cyclic subgroups of $\text{Gal}(L/\mathbb{Q})$ are of the form $\langle x^{p^k} \rangle \simeq \mathbb{Z}_p$, $k \in \mathbb{N}$, or of the form $\langle \sigma x^{p^k} \rangle \simeq \mathbb{Z}/2\mathbb{Z}$. Hence let ℓ be a prime which is

unramified in L/\mathbb{Q} . If ℓ is inert in K/\mathbb{Q} , then the Frobenius automorphism σ_ℓ of ℓ in L/\mathbb{Q} is of the form σx^{p^k} . As σ_ℓ is of order 2 (or equivalently, $\langle \sigma_\ell \rangle \cap \text{Gal}(L/K) = \{e\}$), the prime ℓ splits totally in L/K . \square

By applying Proposition 3.1 to the extension L/K , this construction allows us to produce \mathbb{Z}_p -extensions with μ -invariant as large as desired.

Remark 3.3. — We can say a bit more in the example 3.2. Let us show that a place of K above a prime ℓ splits totally in L/K if and only if ℓ is inert in K/\mathbb{Q} . For a prime $\mathfrak{q} \nmid p$, denote by $\sigma_\mathfrak{q}$ the Frobenius of \mathfrak{q} in the compositum of the \mathbb{Z}_p -extensions of K . Let ℓ a prime that splits in K/\mathbb{Q} ; let us write $\ell\mathcal{O}_K = \mathcal{L}\mathcal{L}'$. If we write $\sigma_\mathcal{L} = ae_1 + be_2$, then by conjugation, $\sigma_{\mathcal{L}'} = be_1 + ae_2$. The key point is that the maximal pro- p extension of K unramified outside p and totally split at \mathcal{L} and \mathcal{L}' is finite, see [11]. Then $a^2 \neq b^2$. Note that when ℓ is inert in K/\mathbb{Q} then $\sigma_\ell = a(e_1 + be_2)$. By reducing the Frobenius modulo $\langle e_1 + e_2 \rangle$, we note that in the case when ℓ is inert, $\sigma_\ell \equiv 0 \pmod{\langle e_1 + e_2 \rangle}$ but when ℓ splits in K/\mathbb{Q} , $\sigma_\mathcal{L} \equiv (a - b)e_1 \pmod{\langle e_1 + e_2 \rangle} \neq 0 \pmod{\langle e_1 + e_2 \rangle}$.

Remark 3.4. — One can generalize the above discussion for $\Gamma = \mathbb{Z}_p^r$, by considering a large CM-extension abelian over \mathbb{Q} . See Cuoco [7, Theorem 5.2])

Remark 3.5. — We observe that the basic principle in the constructions above is that there is a positive density of primes of a field K_0 which are inert in K , and that all of these subsequently split (thanks to the fact that the action $\Gamma^{\circ\sigma}$ is fixed-point-free) in the Γ -extension. This is the starting point for the general case.

3.3. The general case. — We will consider a uniform pro- p group Γ of dimension d having a fixed-point-free automorphism σ of order m co-prime to p . We assume that $m \geq 3$; indeed for $m = 2$, $\Gamma \simeq \mathbb{Z}_p^d$ (by Theorem 2.17 of Ribes and Zaleskii).

Proposition 3.6. — *Let Γ be a uniform pro- p group of dimension d having an automorphism τ of order m with fixed-point-free action, where $m \geq 3$ is co-prime to p . Suppose F_0 is a totally imaginary number field admitting a cyclic extension F/F_0 of degree m such that F is p -rational. Let n be an integer such that $p^n[F_0 : \mathbb{Q}] \geq 2d$, and let K_0 , respectively K be the n th layer of the cyclotomic \mathbb{Z}_p -extension of F_0 , respectively F . Then there exists an intermediate field $K \subset L \subset K_{S_p}$ such that L is Galois over K_0 with Galois group isomorphic to $\Gamma \rtimes \langle \tau \rangle$. In particular, if τ acts fixed-point-freely on Γ/Γ_2 and if $m = \ell$ is prime, then $\text{Gal}(L/K_0)$ is a Frobenius group.*

Proof. — The extension K/K_0 is cyclic of degree m and the number field K is p -rational, hence $\mathcal{G} := \text{Gal}(K_{S_p}/K)$ is a free pro- p group on $r_2(K) + 1 \geq d$ generators. Put $\text{Gal}(K/K_0) = \langle \sigma \rangle$. The extension K_{S_p}/K_0 is a Galois extension with Galois group $\mathcal{G}_0 = \text{Gal}(K_{S_p}/K_0)$ isomorphic to $\mathcal{G} \rtimes \langle \sigma \rangle$. By Proposition 1.18, the character of the action of σ on $\mathcal{G}/\mathcal{G}_2$ is $p^n([F_0 : \mathbb{Q}]/2) \cdot \chi_{\text{reg}} + 1$. Now let $\chi(\Gamma/\Gamma_2) = \sum_{\chi \in \text{Irr}(\sigma)} \lambda_\chi \chi$ be the character

of the action of τ on Γ/Γ_2 . Then $\sum_\chi \chi(1)\lambda_\chi = d$ and, for all χ , $\lambda_\chi \leq d/\chi(1) \leq d$. In particular since $[F_0 : \mathbb{Q}] \cdot p^n \geq 2d$, then necessarily, the $\langle \tau \rangle$ -module Γ/Γ_2 is isomorphic to a submodule of $\mathcal{G}/\mathcal{G}_2$. By Proposition 2.14, there exists a normal subgroup N of \mathcal{G} , stable under σ , such that \mathcal{G}/N is $\langle \sigma \rangle$ -isomorphic to Γ , and we have a surjection $\mathcal{G}_0 \twoheadrightarrow \Gamma \rtimes \langle \sigma \rangle \simeq \Gamma \rtimes \langle \tau \rangle$. \square

We now state the key arithmetic proposition we need.

Proposition 3.7. — Let L/K_0 be a Galois extension of Galois group $\Gamma \rtimes \langle \sigma \rangle$, where Γ is a uniform group of dimension d and where σ is of order m co-prime to p . Suppose that $\Gamma^{\circ\sigma}$ is fixed-point-free. Then every place \mathfrak{p} which is (totally) inert in K/K_0 and is not ramified in L/K splits completely in L/K .

Proof. — Let \mathfrak{p} a prime of K inert in K/K_0 which is not ramified in L/K . Let us fix a prime $\mathfrak{P}|\mathfrak{p}$ of L (see \mathfrak{P} as a system of coherent primes in L/K). Denote by $\widetilde{\sigma}_{\mathfrak{p}}$ be the Frobenius of \mathfrak{p} in K/K_0 and let $\sigma_{\mathfrak{P}} \in \text{Gal}(L/K_0)$ be an element of order m of the decomposition group of \mathfrak{P} in L/K_0 lifting $\widetilde{\sigma}_{\mathfrak{p}}$. Then $\sigma_{\mathfrak{P}} = \sigma^i$ in $\text{Aut}(\Gamma/\Gamma_2)$, for an integer i , $(i, m) = 1$. By Proposition 2.11, there exists $g \in \ker(\text{Aut}(\Gamma) \rightarrow \text{Aut}(\Gamma/\Gamma_2))$ such that $\text{Fix}(\Gamma, \sigma_{\mathfrak{P}}) = g \cdot \text{Fix}(\Gamma, \sigma^i) = \{e\}$, the last equality coming from Remark 2.2. Let $\widehat{\sigma}_{\mathfrak{P}} := \text{Frob}_{\mathfrak{P}}(L/K)$. As $\mathfrak{p} \in \Sigma$, the prime \mathfrak{P} is unramified in K/K_0 and then the decomposition group of \mathfrak{P} in L/K_0 is cyclic: the elements $\sigma_{\mathfrak{P}}$ and $\widehat{\sigma}_{\mathfrak{P}}$ commute or, equivalently, $\widehat{\sigma}_{\mathfrak{P}}^{\sigma_{\mathfrak{P}}} = \widehat{\sigma}_{\mathfrak{P}}$. Hence if $\widehat{\sigma}_{\mathfrak{P}} \neq e$, the element $\sigma_{\mathfrak{P}} \in \text{Aut}(\Gamma)$ has a fixed-point. Contradiction, and then $\widehat{\sigma}_{\mathfrak{P}} = e$. \square

Assembling our forces, we can now formulate our main theorem.

Theorem 3.8. — Let Γ be a uniform pro- p group having an automorphism τ of order m with fixed-point-free action, where $m \geq 3$ is co-prime to p . Suppose F_0 is a totally imaginary number field admitting a cyclic extension F/F_0 of degree m such that F is p -rational. Then there exists a finite p -extension K/F unramified outside p and a Γ -extension L/K with the following property: for any given integer μ_0 , there exists a cyclic degree p extension K' over $K(\zeta_p)$ such that $L' = LK'$ is a Γ -extension of K' whose μ -invariant satisfies $\mu_{L'/K'} \geq \mu_0$.

Proof. — We simply follow the rubric of Proposition 3.6. Choose an integer n large enough so that $p^n[F_0 : \mathbb{Q}] \geq 2 \dim(\Gamma)$, then let K_0 and K be the n th layer of the cyclotomic \mathbb{Z}_p -extensions of F_0 and F respectively. Since F is p -rational, so is K . By Proposition 3.6, we are guaranteed of the existence of a Galois extension L/K_0 with Galois group $\Gamma \rtimes \langle \tau \rangle$ and $K = L^\Gamma$. By Proposition 3.7, every inert place in K/K_0 splits completely in L/K . Now we simply apply the construction described in the proof of Proposition 3.1. \square

Remark 3.9. — Let us remark that in the construction of Theorem 3.8 (in fact of Proposition 3.6), we start with a p -rational field F and pass to another p -rational field K whose maximal p -ramified p -extension is of large enough rank to have Γ as a quotient. However, the cyclic degree p extension K' of $K(\zeta_p)$ over which we construct a tower with large μ -invariant is not itself p -rational (see [11, Chapter IV, §3]).

Remark 3.10. — In his original treatment [21], Iwasawa was able to treat the case $p = 2$ alongside odd primes p . The elements of finite order of $\text{Aut}(\mathbb{Z}_2)$ are of order 2. Then the Question 0.5 is essential for applying our previous “co-prime to p ” strategy for \mathbb{Z}_2 . Indeed, let us consider the uniform pro-2 group $\Gamma := \mathbb{Z}_2^2$ instead of \mathbb{Z}_2 by noting that $\text{Aut}(\mathbb{Z}_2^2)$ has a fixed-point-free automorphism τ of order 3. By example 1.15, one knows that the field $K = \mathbb{Q}(\zeta_{13})$ is 2-rational; the Galois group G_{S_2} is free on 7 generators. Moreover, G_{S_2} has an automorphism of order 3 coming from the unique cyclic sub-extension K/K_0 of degree 3. The character of this action contains the character of the action of τ on Γ/Γ_2 . Hence by Proposition 3.6, there exists a Galois extension L/K with $\text{Gal}(L/K) \simeq \Gamma = \mathbb{Z}_2^2$ in which every odd inert place \mathfrak{p} in K/K_0 splits completely in L/K . In particular every such place splits in every \mathbb{Z}_2 -quotient of Γ and then the Proposition 3.1 apply for \mathbb{Z}_2 .

4. Complements

4.1. A non-commutative example. — Assume $p > 3$. The nilpotent uniform groups of dimension ≤ 2 are all commutative. In dimension 3 they are parametrized, up to isomorphism, by $s \in \mathbb{N}$ and represented (see [23, §7 Theorem 7.4]) by:

$$\Gamma(s) = \langle x, y, z \mid [x, z] = [y, z] = 1, [x, y] = z^{p^s} \rangle.$$

Here the center of $\Gamma(s)$ is the procyclic group $\langle z \rangle$ and one has the sequence:

$$1 \longrightarrow \mathbb{Z}_p \longrightarrow \Gamma(s) \longrightarrow \mathbb{Z}_p^2 \longrightarrow 1.$$

Proposition 4.1. — *Let $s \in \mathbb{N}$ and let $p \equiv 1 \pmod{3}$. The group of automorphisms of $\Gamma(s)$ contains an element σ of order 3 for which the action is fixed-point-free.*

Proof. — To simplify the notation, put $\Gamma = \Gamma(s)$. First, let us remark that for $a, b \in \mathbb{N}$, one has $x^a y^b = z^{abp^s} y^b x^a$. Indeed,

$$x^a y = x^{a-1} z^{p^s} y x = z^{p^s} x^{a-1} y x = z^{ap^s} y x^a,$$

and the same holds for xy^b .

Let ζ be a primitive third root of the unity and let $\zeta^{(n)} \in \mathbb{N}$ be the truncation at level n of the p -adic expansion of ζ : $\zeta^{(n)} \equiv \zeta \pmod{p^n}$. Let us consider σ defined by:

$$\begin{aligned} \sigma : \Gamma &\rightarrow \Gamma \\ x &\mapsto x^\zeta \\ y &\mapsto y^\zeta \\ z &\mapsto z^{\zeta^2} \end{aligned}$$

Then $\sigma \in \text{Aut}(\Gamma)$. Indeed one has to show that the relations defining Γ are stable under the action of σ , which is obvious for the relations $[x, z] = 1$ and $[y, z] = 1$. Let us look at the last relation. First, as Γ is uniform, let us recall that $\Gamma_{n+1} = \Gamma^{p^n}$. We have:

$$\sigma([x, y]) = \sigma(xy x^{-1} y^{-1}) \equiv x^{\zeta^{(n)}} y^{\zeta^{(n)}} x^{-\zeta^{(n)}} y^{-\zeta^{(n)}} \pmod{\Gamma_n} \equiv z^{(\zeta^{(n)})^2 p^s} \pmod{\Gamma_n} \xrightarrow{n} z^{\zeta^2 p^s}.$$

To finish, let us show that the automorphism σ is fixed-point-free. Indeed the eigenvalues of the action of σ on the \mathbb{F}_p -vector space $\Gamma/\Gamma_2 \simeq \mathbb{F}_p^3$ are ζ (with multiplicity 2) and ζ^2 , so $\Gamma/\Gamma_2^{\circ\sigma}$ is fixed-point-free and we conclude with Proposition 2.7. \square

Remark 4.2. — If $p \not\equiv 1 \pmod{3}$, there is no element $\sigma \in \text{Aut}(\Gamma(s))$ of order 3 with fixed-point-free action. Indeed, let $\sigma \in \text{Aut}(\Gamma(s)/\Gamma(s)_2) \simeq \text{Gl}_3(\mathbb{F}_p)$, and suppose that σ is fixed-point-free. The eigenvalues of σ in $\overline{\mathbb{F}_p}$ are ζ and/or ζ^2 with multiplicity. But, as the trace of σ is in \mathbb{F}_p , then necessarily $\zeta \in \mathbb{F}_p$ and $3 \mid p - 1$.

Corollary 4.3. — *Assume $p \equiv 1 \pmod{3}$ and p regular. For each $s \in \mathbb{N}$, there exist $\Gamma(s)$ -extensions of numbers fields with arbitrarily large μ -invariant.*

Remark 4.4. — For $p = 37$, which is the smallest irregular prime, we may not resort to the construction above with $F = \mathbb{Q}(\zeta_{37})$, but we can still realize $\Gamma(s)$ -extensions with arbitrarily large μ -invariant by applying Theorem 3.8 for $F/F_0 = \mathbb{Q}(\zeta_7)/\mathbb{Q}(\sqrt{-7})$.

4.2. Counting the split primes in uniform extensions. — Let L/K be a Galois extension with p -adic analytic Galois group Γ of dimension d . Denote by Σ the set of primes of \mathcal{O}_K unramified in L/K . Assume Σ finite. For $\mathfrak{p} \in \Sigma$, let $\mathcal{C}_{\mathfrak{p}}$ be the conjugacy class of the Frobenius of \mathfrak{p} in L/K and put

$$\pi_{L/K}^{\text{split}}(x) = \left| \{ \mathfrak{p} \in \Sigma, \mathcal{C}_{\mathfrak{p}} = \{1\}, N(\mathfrak{p}) \leq x \} \right|,$$

where $N(\mathfrak{p}) := |\mathcal{O}_K/\mathfrak{p}|$.

In [33, Corollary 1 of Theorem 10], under GRH, Serre shows that for all $\varepsilon > 0$, $\pi_{L/K}^{\text{split}}(x) = O(x^{\frac{1}{2}+\varepsilon})$. Without assuming GRH, $\pi_{L/K}^{\text{split}}(x) = O(x/\log^{2-\varepsilon}(x))$.

Proposition 4.5. — *Let p be a regular prime and let Γ be a uniform pro- p group having a non-trivial fixed-point-free automorphism σ of prime order $\ell \mid p-1$. Then there exists a constant $C > 0$ and a Γ -extension L over a number field K such that*

$$\pi_{L/K}^{\text{split}}(x) \geq C \frac{x^{1/\ell}}{\log x}, \quad x \gg 0.$$

Proof. — One use the construction of section 3.3. Put $K = \mathbb{Q}(\zeta_{p^n})$ where n is the smallest integer such that $p^n(p-1) \geq 2\ell d$. Let K/K_0 be the unique cyclic extension of degree ℓ ; $\text{Gal}(K/K_0) = \langle \sigma \rangle$. Let L/K be a uniform Galois extension with group Γ constructed by the method of Proposition 3.6. Let $\mathcal{E}(L/K)$ be the set of conjugacy classes of $\text{Gal}(K/Q)$ of the elements σ^i with $(i, \ell) = 1$. By Proposition 3.7, a prime \mathfrak{q} of K for which the Frobenius conjugacy class $\sigma_{\mathfrak{q}} \in K/Q$ is in $\mathcal{E}(L/K)$, splits totally in L/K . Hence

$$\begin{aligned} \pi_{L/K}^{\text{split}}(x) &\geq \left| \{ \mathfrak{q} \in \Sigma, \sigma_{\mathfrak{q}} \in \mathcal{E}(L/K), N(\mathfrak{q}) \leq x \} \right| \\ &= \left| \{ \mathfrak{q} \in \Sigma, \exists (i, \ell) = 1, \mathcal{C}_{\mathfrak{q}}(K/Q) = \sigma^i, q^\ell \leq x \} \right| \\ &= \left| \{ \mathfrak{q} \in \Sigma, \exists (i, \ell) = 1, \mathcal{C}_{\mathfrak{q}}(K/Q) = \sigma^i, q \leq x^{1/\ell} \} \right| \end{aligned}$$

where $\mathcal{C}_{\mathfrak{q}}(K/Q)$ is the conjugacy class of the Frobenius of q in $\text{Gal}(K/Q)$. By using Chebotarev Density Theorem, one concludes that

$$\pi_{L/K}^{\text{split}}(x) \gg \frac{x^{1/\ell}}{\log x}.$$

□

Example 4.6. — For the uniform group $\Gamma(s)$ of dimension 3 discussed in §4.1, one obtains $\Gamma(s)$ -extensions L/K with $\pi_{L/K}^{\text{split}}(x) \gg \frac{x^{1/3}}{\log x}$.

4.3. On p -rational fields with splitting. — For all this section, assume that $p > 2$. By making use of fixed-point-free automorphisms, we have produced some uniform extensions with infinitely many totally split primes. Using Class Field Theory it is also possible to produce some free-pro- p extensions with some splitting phenomena, but *we do not know if it is possible to construct free pro- p extensions in which infinitely many primes of the base field split completely*. In fact this question is related to the work of Ihara [20] and the recent work of the authors [16].

Let us show how to produce some free pro- p extensions with some splitting. Let S and T be two finite sets of places of number field K such that $S \cap T = \emptyset$. Recall that $S_p = \{ \mathfrak{p} \in K : \mathfrak{p} | p \}$. Let (r_1, r_2) be the signature of K . Let K_S^T be the maximal pro- p

extension of K unramified outside S and totally decomposed at T . Put $\mathcal{G}_S^T = \text{Gal}(K_S^T/K)$; $A_S^T := \mathcal{G}_S^{T, \text{ab}}$ and $A_S^T/p := \mathcal{G}_S^{T, \text{ab}}/\mathcal{G}_S^{T, \text{ab}, p}$. Of course, one has $d_p A_S^T = d_p \mathcal{G}_S^T$ and for $S = T = \emptyset$, $A_\emptyset^0(K) = A(K)$.

We are interested in constructing an example where the group \mathcal{G}_S^T is a free pro- p group and T is not empty.

The following is a fundamental and classical result about the Euler characteristics of \mathcal{G}_S^T .

Proposition 4.7 (Shafarevich and Koch). — *Suppose that K contains the p -roots of the unity. Then,*

$$d_p H^2(\mathcal{G}_S^T, \mathbb{F}_p) \leq d_p A_T^S + |S| - 1$$

and

$$d_p \mathcal{G}_S^T = d_p A_T^S + |S| - |T| - (r_1 + r_2) + \sum_{v \in S \cap S_p} [K_v : \mathbb{Q}_p].$$

Proof. — See for example [11, Corollary 3.7.2, Appendix] for the bound for the H^2 and [11, Theorem 4.6, Chapter I, §4] for the H^1 . \square

The first inequality of the previous proposition would allow us to produce a free pro- p extension with complete splitting at the primes in T if $d_p A_T^S + |S| - 1 = 0$. On the other hand, if we apply the second line, let us read the second of Proposition 4.7 with the role of S and T reversed, we find that

$$d_p A_T^S \geq |T| - (r_1 + r_2 + |S|).$$

We conclude that, under this strategy, $|T| \leq r_1 + r_2 + 1$, so this method is rather limited in scope.

Let us take $K = \mathbb{Q}(\zeta_p)$ when p is regular and $S = S_p$. Then the group \mathcal{G}_S^T will be pro- p free when A_T^S is trivial. As the p -class group of K is trivial, one has:

$$A_T^S \simeq \left(\prod_{v \in T} \mathcal{U}_v \right) / \mathcal{O}_K^S,$$

where \mathcal{U}_v is the pro- p completion of the local units at v and where \mathcal{O}_K^S is the group of S -units of K . Now, there are several types of scenarios where the quotient $(\prod_{v \in T} \mathcal{U}_v) / \mathcal{O}_K^S$ is trivial. Let us give one. Suppose that $T = \{\ell\}$ where ℓ is inert in K/\mathbb{Q} . Then \mathcal{U}_ℓ is isomorphic to the p -part of \mathbb{F}_ℓ^\times . Hence, the global p th roots of unity will kill this part when $|\mathcal{U}_\ell| = p$ i.e. when $\ell^{p-1} - 1$ is exactly divisible by p . When this is the case, the group \mathcal{G}_S^T is free on $(p-3)/2$ generators.

4.4. Incorporating the Galois action. — We can take these ideas a bit further by studying the Galois action. Throughout this subsection, we fix the following notation and assumptions (we still assume $p > 2$). Let K_0 be a number field, K/K_0 a cyclic extension of integer m co-prime to p . Put $\Delta = \text{Gal}(K/K_0)$. Let $r = r_1(K_0) + r_2(K_0)$ be the number of archimedean places of K_0 . Let S and T be two finite sets of places of K_0 such that $S \cap T = \emptyset$ and S contains S_p . By abuse of notation, the set of places of K above S and T are again called S and T respectively. Denote by S_{split} (resp. T_{split}) the set of places of S (resp. of T) splitting in K/K_0 and by S_{inert} (resp. T_{inert}) the set of places of S (resp. of T) not splitting in K/K_0 . As in §1.2, the arithmetic objects of interest have a structure

as $\mathbb{F}_p[\Delta]$ -modules. We recall the following mirror identity from the book of Gras [13, Chapter II, §5.4.2]:

Theorem 4.8. — *Assume that $S_p \subset S$ and that K contains a primitive p th root of unity. Then:*

$$\omega\chi^{-1}(A_S^T) - \chi(A_T^S) = r\chi_{\text{reg}} + \omega - \mathbf{1} + |S_{\text{inert}}|\mathbf{1} + |S_{\text{split}}|\chi_{\text{reg}} - |T_{\text{split}}|\chi_{\text{reg}} - |T_{\text{inert}}|\omega.$$

Proposition 4.9. — *Suppose that K , which is a cyclic degree p extension of K_0 , contains a primitive p th root of unity and is p -rational. Then K_0 , as well as every number field K_n in the cyclotomic extension K_∞ of K is also p -rational.*

Proof. — It is an obvious extension of Proposition 1.16. □

Now we assume that K_0 is totally imaginary and we take $S = S_p$. By hypothesis, there is no abelian unramified p -extension of K in which p splits completely. Then for $T = \emptyset$, by Theorem 4.8, one has:

$$\omega\chi^{-1}(A_S) = r\chi_{\text{reg}} + \omega.$$

The group $A_S(K_0)$ corresponds by Class Field Theory to the Galois group of the maximal abelian pro- p extension of K_0 unramified outside S . The Galois group $\mathcal{G}_S(K_0)$ is free on $r+1$ generators. The action of Δ on A_S/p being semi-simple, the $\mathbb{F}_p[\Delta]$ -module $A_S(K_0)/p$ is isomorphic to A_S^T/p on which the Δ action is trivial.

By the Chebotarev Density Theorem, we can choose a set $T := \{\mathfrak{p}_0, \dots, \mathfrak{p}_r\}$ of places of K_0 all inert in K/K_0 , with $|T| = d_p A_S(K_0) = r+1$, such that the Frobenius symbols $\sigma_{\mathfrak{p}_i}$, $i = 0, \dots, r$, generate the p -group $A_S(K_0) \otimes \mathbb{F}_p$. By the choice of T , one has $\chi(A_S^T) = \chi(A_S) - |T|\mathbf{1}$. Thanks to Theorem 4.8 one then obtains:

$$\chi(A_T^S) = \omega\chi^{-1}(A_S^T) - \omega\chi^{-1}(A_S) - |T|\omega = 0.$$

Then the maximal pro- p -extension of K unramified outside T and splitting at S , is trivial. One then uses Proposition 4.7 to obtain:

Proposition 4.10. — *Under the conditions of Proposition 4.9, and with $S = S_p$ and T as above, the pro- p -group \mathcal{G}_S^T is free on $r(\ell-1)$ generators and*

$$\chi(A_S^T) = r(\chi_{\text{reg}} - \mathbf{1}).$$

Remark 4.11. — (i) The main point of Proposition 4.10 is that the action of σ on A_S^T is fixed-point-free, where $\Delta = \langle \sigma \rangle$.

(ii) By Proposition 4.9, the degree r can be taken arbitrarily large.

As corollary, one obtains:

Corollary 4.12. — *Assume the conditions of Proposition 4.10. Let L/K_0 be a sub-extension of K_S^T/K_0 such that $\Gamma = \text{Gal}(L/K)$ is a uniform pro- p group. Then every (totally) inert (odd) prime $\mathfrak{p} \subset \mathcal{O}_K$ in K/K_0 splits completely in L/K .*

4.5. Some heuristics on p -rationality. — Here we exploit some conjectures on p -rationality to get further heuristic evidence in direction of Conjecture 0.1. These are of two types. The first is in the spirit of the Leopoldt Conjecture – in a certain sense this is a transcendental topic; the main reference is the recent work of Gras [12]. The second one is inspired by the heuristics of Cohen-Lenstra [6].

Conjecture 4.13 (Gras, Conjecture 8.11 of [12]). — *Let K be a number field. Then for $p \gg 0$, the field K is p -rational.*

Proposition 4.14. — *Let \mathbb{P} be an infinite set of prime numbers and m an integer coprime to all $p \in \mathbb{P}$. Let $(\Gamma_p)_{p \in \mathbb{P}}$ be a family of uniform pro- p groups all of dimension d and all having a fixed-point-free automorphism of order m . If Conjecture 4.13 is true for some number field K of degree mn , then for all but finitely many primes $p \in \mathbb{P}$, the groups Γ_p have arithmetic realizations as Galois groups with arbitrarily large μ -invariant.*

Proof. — Let us take K_0 be a totally imaginary abelian number field of degree n over \mathbb{Q} and let K_1/\mathbb{Q} be a degree m cyclic extension with $K_0 \cap K_1 = \emptyset$. Let $K = K_0K_1$ be the compositum. Put $\langle \sigma \rangle = \text{Gal}(K/K_0)$. Under Conjecture 4.13, one knows that for large p depending only on K , the field K is p -rational. One then applies Theorem 3.8. \square

Example 4.15. — Let us fix $s \in \mathbb{N}$ and let \mathbb{P} be the set of primes $p \equiv 1 \pmod{3}$. Let $\Gamma(s, p)$ be the pro- p group $\Gamma(s)$ of dimension 3 introduced in §4.1. One can apply Proposition 4.14 to the family of groups $(\Gamma(s, p))_{p \in \mathbb{P}}$.

Let us consider another idea for studying p -rationality, in the spirit of the Cohen-Lenstra Heuristics [6]. For a prime number $p \geq 5$ and an integer $m > 1$ not divisible by p , let us consider the family $\mathcal{F}_m(K_0)$ of cyclic extensions of degree m over a fixed p -rational number field K_0 . Following the idea of Pitoun and Varescon [31], it seems reasonable to make the following conjecture:

Conjecture 4.16. — *Let us fix p and m as above. If K_0 is a p -rational quadratic imaginary field, the proportion of number fields K in $\mathcal{F}_m(K_0)$ which are p -rational (counted according to increasing absolute value of the discriminant) is positive.*

Obviously, Conjecture 4.16 implies Conjecture 0.1.

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FARSHID HAJIR, Department of Mathematics & Statistics, University of Massachusetts, Amherst MA 01003, USA.

CHRISTIAN MAIRE, Laboratoire de Mathématiques de Besançon (UMR 6623), Université Bourgogne Franche-Comté et CNRS, 16 route de Gray, 25030 Besançon, France