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GEVREY WKB METHOD FOR PSEUDODIFFERENTIAL OPERATORS OF REAL PRINCIPAL TYPE

RICHARD LASCAR AND IVÁN MOYANO

ABSTRACT. In this paper we investigate the conjugation of Fourier Integral Operators (FIOs) associated to Gevrey phases and symbols and the corresponding semiclassical pseudodifferential operators (pdos) in the Gevrey class. We obtain an Egorov theorem compatible with Gevrey FIOs and real principal part pdos with Gevrey symbols. As a consequence, we obtain a justification of the usual microlocal WKB expansion for Gevrey pdos which are of real principal part at a point in the phase space, with the natural Gevrey subexponential asymptotics with respect to the semiclassical parameter.

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1. INTRODUCTION

The use of Gevrey functions in the study of some linear and non-linear partial differential equations has experienced a significant development in recent years. In particular, the use of Gevrey norms have been instrumental in the context of some non-linear Cauchy problems that may be ill-posed in the usual Sobolev context [9] or the study of some singular limits difficult to describe rigorously using other topologies [21, 1].

Let us recall that, as usual, for $N \in \mathbb{N}$ and $s > 0$ a function $f \in C^\infty(\mathbb{R}^N; \mathbb{C})$ is Gevrey of order s (Gevrey- s for short, or simply \mathcal{G}^s) whenever for every compact $K \subset \mathbb{R}^N$ the following holds:

$$(1.1) \quad \exists C_K > 0, \quad \forall \alpha \in \mathbb{N}^N, \quad |\partial^\alpha f|_{L^\infty(K)} \leq C_K^{|\alpha|} \alpha!^s.$$

As can be seen from (1.1), the case $s = 1$ coincides with the usual class of analytic functions, but the cases $s > 1$ may contain smooth functions which are not analytic. Hence, the Gevrey class may be understood as a class of functions in between the analytic and the smooth context.

The use of pseudodifferential calculus associated to particular Gevrey classes, started by Boutet de Monvel and Kree in their seminal paper, [2, Sect. 1], has also been important in the past, for example in connexion with the study of propagation of Gevrey singularities of these pseudodifferential operators [14, 16] or the diffraction of waves around an obstacle [17]. More recently, the study of the FBI transform in the Gevrey context has risen some interesting questions related to the quantisation in the complex plane of symbols admitting only quasi-holomorphic extensions which in fact are not unique [11] and thus introduces new aspects with respect to the analytic framework [22, 12, 20].

In this work we are concerned with the study of Gevrey pseudodifferential operators and the resulting WKB expansion in the Gevrey class. In what follows, we introduce some definitions of these Gevrey symbols and the related pseudodifferential operators and then state our main results. We work in a semiclassical framework (cf. [23]) involving a possibly small parameter $h > 0$.

Let us note, for $n \in \mathbb{N}$, $m \in \mathbb{R}$ and $s > 0$, the set $\mathcal{S}_s^m(\mathbb{R}^n \times \mathbb{R}^n)$ of semiclassical Gevrey s symbols of order m . Following [2, Sect. 1] we write $a \in \mathcal{S}_s^m$ if and only if for some $C > 0$

$$(1.2) \quad |\partial_x^\alpha \partial_\theta^\beta a(x, \theta, h)| \leq C^{1+|\alpha|+|\beta|} \alpha!^s \beta!^s h^{-m},$$

for all $(x, \theta) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{N}^n$. Observe that this condition is more precise than the usual definition of the class $\mathcal{S}_{1,0}^m$ (cf. for example (cf. [23, Chapter 4])) because the constant C in (1.2) is uniform with respect to the multi-indices α, β . As a consequence, we shall need a specific version of symbolic calculus adapted to the class \mathcal{S}_s^m (in particular Proposition 2.3 below). Finally, recall that a pdo A is elliptic at a point $(x_0, \theta_0) \in \mathbb{R}^n \times \mathbb{R}^n$ whenever its symbol a satisfies $|a(x_0, \theta_0)| > 0$.

We consider as usual (cf. [23, Chapter 4]) semiclassical pseudo-differential operators (pdo for short) defined as suitable extensions of

$$a(x, hD, h)u(x) = \frac{1}{(2\pi h)^n} \iint a(x, \theta, h) e^{\frac{i}{h}\theta \cdot (x-y)} u(y) dy d\theta, \quad u \in \mathcal{S}(\mathbb{R}^n),$$

for a given symbol $a \in \mathcal{S}_s^m$. One has $a(x, hD, h) = \text{Op}_h(a)$ and also $\text{Op}_h(a) = \text{Op}_1(a_h)$ with $a_h(x, \theta) = a(x, h\theta)$. We shall use the notation $\text{Op}_h(a) \in \Psi_s^m$ for such a pdo.

1.1. Motivations, hypothesis and main results. The motivation of our work is twofold. On the one hand, the WKB method is known to hold true in the smooth (C^∞) and analytic categories, which raises the natural question of its validity with respect to the specific Gevrey asymptotics required in the Gevrey framework (in particular of the form (1.3)). Our Theorem 1 makes this point explicit at least microlocally. On the other hand, we obtain this microlocal WKB expansion as a consequence of Theorem 2, which is a Gevrey version of the Egorov theorem, a central result in microlocal analysis, first published in [5], having its own independent interest. We state next our main hypothesis and the context of our results.

Let $P = P(x, hD_x, h)$ a semiclassical Gevrey- s PDO of order zero in \mathbb{R}^n , with $n \geq 2$. Let $p = p(x, \xi)$ be the principal symbol of P and let H_p be the associated Hamiltonian. We assume the following hypothesis.

- (H1):** p is real,
- (H2):** $dp(x_0, \xi_0) \neq 0$,
- (H3):** and $p(x_0, \xi_0) = 0$.

If $P = P(x, hD_x, h)$ satisfies hypothesis **(H1)**, **(H2)** and **(H3)**, it is customary to say that P is of *real principal type* at the point $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n$ (cf. [7, Definition 3.1]).

The class of real principal type operators enjoys an important property allowing to reduce a general pdo to a canonical form. This is known as Egorov theorem, which in the C^∞ class states roughly the following (cf. [7, Proposition 3.1] for instance, among other references as [23, Chapter 8], [8, Section 62], [19]): If $A = A(x, D_x)$ and $B = B(x, D_x)$ are two pdo of real principal part with the same principal symbol, then there exists elliptic pdo $R = R(x, D_x)$ and $S = S(x, D_x)$ of order zero such that the conjugation $AR - SB$ is a smoothing operator. Of course, that this theorem is valid for a large class of pseudo-differential (possibly semiclassical) operators, including Gevrey symbols (see below for a precise definition). On the other hand, if we assume further regularity properties on the class of symbols at hand, for instance Gevrey regularity (see 1.1 for a definition), we may expect to get more precise information on the canonical transformations and the pseudo-differential calculus involved. In [10] gave a first Gevrey version, but the result is not adapted to the semiclassical pdos and in particular to the WKB asymptotics. In this paper we aim at proving a Gevrey version of Egorov theorem which is sensitive to a small parameter h . We do this by using the classical WKB approach.

1.1.1. Main results: Gevrey WKB expansion and Gevrey Egorov's theorem. Our main result is a microlocal semiclassical WKB expansion for real principal part operators in the Gevrey setting compatible with the usual required asymptotics in the Gevrey framework.

Theorem 1 (Gevrey WKB expansion). *Let $n \geq 2$. Let $P = P(x, hD_x, h)$ be a semiclassical \mathcal{G}^s pdo of order zero in \mathbb{R}^n of symbol $p = p(x, \xi)$ and assume that P is of real principal type at a point $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n$. Let $S \subset \mathbb{R}^n$ be a real \mathcal{G}^s -hypersurface of \mathbb{R}^n . Let $\varphi \in \mathcal{G}^s(\mathbb{R}^n)$ be given. Assume that*

$$p(x, \varphi'_x) = 0 \quad \text{and} \quad \xi_0 = \varphi'_x(x_0).$$

Under these hypothesis, one may solve the WKB problem near x_0 , i.e.: If a_0 and b are given symbols in $\mathcal{S}_s^0(\mathbb{R}^n)$, one may find some $a \in \mathcal{S}_s^0(\mathbb{R}^n)$ such that

$$\begin{cases} \frac{1}{h} e^{-i\frac{\varphi}{h}} P \left(a e^{i\frac{\varphi}{h}} \right) - b = \mathcal{O}_{\mathcal{G}^s}(h^\infty), & \text{close to } x_0, \\ a|_S = a_0. \end{cases}$$

The smallness condition in Theorem 1 is defined in (1.3). As mentioned before, we shall obtain Theorem 1 as a consequence of a particular version of Egorov's theorem in the Gevrey setting, according to our next result.

Theorem 2 (Gevrey Egorov theorem). *Let $P = P(x, hD_x, h)$ be a \mathcal{G}^s pdo with principal symbol $p = p(x, \xi)$. Assume that P is of real principal type at some point $(x_0, \xi_0) \in T^*\mathbb{R}^n$. If*

$$p(x_0, \xi_0) = 0, \quad \frac{\partial p}{\partial \xi}(x_0, \xi_0) \neq 0,$$

then P is microlocally conjugate to hD_{x_1} by \mathcal{G}^s FIOs.

We give a precise definition of microlocal conjugation in Definition 3.3, after having introduced the necessary objects for the reader's convenience.

1.2. Strategy and outline. In Section 2 we recall some non-stationary and stationary asymptotics with complex Gevrey phase and Gevrey symbols and review the symbolic calculus adapted to Gevrey pdo. We also discuss formal symbols and Carleson's theorem.

In Section 3 we introduce the class of Fourier integral operators associated to Gevrey phases and symbols that will be crucial in the proof of our main theorems.

In Section 4 we prove Theorem 2 by using a microlocal WKB expansion, involving the usual steps: reduction to an evolution equation, construction of a suitable phase as a solution to an eikonal equation and finally the construction of a suitable symbol by imposing a hierarchy of transport equations.

In Section 5 we prove Theorem 1 as a particular case of Theorem 2 proved in the previous section.

1.3. Notation. According to [11], in this work we shall say that, for $s \geq 1$ a given function g , depending on a small parameter $h > 0$, is a \mathcal{G}^s -small remainder, and we note $g = \mathcal{O}_{\mathcal{G}^s}(h^\infty)$, if there exists $C > 0$ such that for all $\beta \in \mathbb{N}^m$,

$$(1.3) \quad |\partial^\beta g| \leq C^{1+|\beta|} \beta!^s \exp\left(-\frac{h^{-\frac{1}{s}}}{C}\right).$$

As usual, we use $\mathcal{S}(\mathbb{R}^n)$ to denote the space of Schwartz functions in \mathbb{R}^n .

2. STATIONARY PHASE AND SYMBOLIC CALCULUS IN THE GEVREY SETTING

In this section we review the some results concerning the symbolic calculus in with Gevrey pseudodifferential operators, as some of them are difficult to find in the literature. We first review the non-stationary and stationary lemmas for Gevrey phases and symbols. As usual, this allows

to give a sense to the composition of pseudodifferential operators and yield suitable asymptotic expansions for the resulting symbols. Of course, all the results reviewed below are classical in the \mathcal{C}^∞ framework, according to classical references as [13, 18, 8, 23], but in the Gevrey context we need to justify that all small remainders have the particular form given in (1.3), which sometimes necessitates some modifications with respect to the smooth or analytic cases.

2.1. Non-stationary and Stationary phase method in the Gevrey setting. We consider a symbol $a = a(x, y; h)$ in $\mathcal{S}_s^{m_0}(\mathbb{R}^n \times \mathbb{R}^m)$, $m_0 \in \mathbb{R}$, for a small parameter $0 < h \leq 1$. Let $f = f(x, y)$ be a phase function of class $\mathcal{G}^s(\mathbb{R}^n \times \mathbb{R}^m)$. Consider

$$(2.1) \quad \mathcal{I}_f(h, y) = \int_{\mathbb{R}^n} e^{\frac{i}{h}f(x,y)} a(x, y; h) dx, \quad y \in \mathbb{R}^m,$$

for a compactly-supported (in x) symbol a . The goal of this section is to adapt the usual non-stationary and stationary phase asymptotics to the Gevrey setting.

2.1.1. Non-stationary phase lemma. The following result is a non-stationary phase lemma adapted to the Gevrey asymptotics in h , which has an independent interest.

Lemma 2.1 (Non-stationary phase). *Let $a = a(x, y; h)$ in $\mathcal{S}_s^{m_0}(\mathbb{R}^n \times \mathbb{R}^m)$ be compactly supported and let $\text{supp } a$ be its support. Assume $f \in \mathcal{G}^s(\mathbb{R}^n \times \mathbb{R}^m)$ is such that*

- (1) $\text{Im } f(x, y) \geq 0$,
- (2) $f'_x(x, y) \neq 0$ for every $(x, y) \in \text{supp } a$.

Then, $\mathcal{I}_f(h, \cdot)$ is a \mathcal{G}^s -small remainder, i.e., there exists $C > 0$ such that for all $\beta \in \mathbb{N}^m$,

$$|\partial_y^\beta \mathcal{I}_f(h, \cdot)| \leq C^{1+|\beta|} \beta!^s \exp\left(-\frac{h^{-\frac{1}{s}}}{C}\right).$$

Proof. As usual, as $\text{supp } a$ is compact, upon using a suitable partition of the unity, we may reduce the result to a purely local situation. Thanks to the hypothesis on f , we may assume that $\text{supp } a$ is a sufficiently small neighbourhood of a point around which it is possible to use a diffeomorphism κ straightening the phase f . Indeed, if $\text{supp } a$ is sufficiently small, let $(\tilde{x}, \tilde{y}) = \kappa(x, y)$ with $\tilde{x} = (\tilde{x}_1, \tilde{x}')$ and let κ be such that

$$\tilde{x}_1 = f(x, y), \quad \tilde{x}' = x', \quad \tilde{y} = y, \quad \text{in } \text{supp } a.$$

Observe that, as $\text{Im } f(x, y) \geq 0$ and $f_{x_1}(x, y) \neq 0$, this mapping is indeed a local diffeomorphism. Upon changing variables, we find

$$\mathcal{I}_f(h, y) = \int_{\mathbb{R}^n} e^{\frac{i}{h}f(x,y)} a(x, y; h) dx = \int_{\mathbb{R}^n} e^{\frac{i}{h}\tilde{x}_1} \tilde{a}(\tilde{x}_1, \tilde{x}', y; h) d\tilde{x}_1 d\tilde{x}'.$$

Now, taking derivatives of any order with respect to y , the usual non-stationary phase lemma gives exponential decay on h and the result follows (cf. for instance [23, Lemma 3.14]).

□

2.1.2. *Stationary phase lemma.* We turn now to the stationary phase asymptotics in the Gevrey setting. Upon using a suitable partition of unity, we may consider a purely local situation. Let us pick a point $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^m$ such that the following

$$(2.2) \quad f \text{ is real-valued,}$$

$$(2.3) \quad f'_x(x_0, y_0) = 0,$$

$$(2.4) \quad \det(f''_{xx}(x_0, y_0)) \neq 0,$$

hold. Consider (x, y) sufficiently close to (x_0, y_0) in $\mathbb{R}^n \times \mathbb{R}^m$. Then, there exists a function $y \mapsto \chi(y)$ of class \mathcal{G}^s in a neighbourhood of y_0 such that

$$(2.5) \quad f'_x(\chi(y), y) = 0, \quad \chi(y_0) = x_0.$$

We have the following stationary phase asymptotics for locally defined symbols that will be used in Proposition 2.3.

Lemma 2.2 (Stationary phase). *Assume that $\text{supp } a$ is close to x_0 so that (2.5), (2.2), (2.3) and (2.4) hold. Then, the symbol*

$$(2.6) \quad b(y, h) = e^{-if(\chi(y), y)} \int a(x, y; h) e^{-if(x, y)} dx$$

is of class \mathcal{G}^s near y_0 and enjoys the asymptotic expansion

$$(2.7) \quad b(y, h) \sim \left| \det \left(\frac{1}{2i\pi h} f''_{xx}(\chi(y), y) \right) \right|^{-\frac{1}{2}} e^{i\pi \frac{\sigma}{4}} \sum_{j \geq 0} h^j L_{f, j, y}[a](\chi(y), y),$$

where each $L_{f, j, y}[a]$ is a differential operator of degree $2j$ in x and σ is the signature of the matrix $D_x^2 f(x_0, y_0)$.

Proof. Since f is real-valued, we can use Morse lemma. By Taylor expansion of order 2 around (x_0, y_0) we find

$$f(x, y) = f(\chi(y), y) + \frac{1}{2} \langle Q(x, y)(x - \chi(y)), x - \chi(y) \rangle,$$

whenever (x, y) is close to (x_0, y_0) , where

$$Q(x, y) = 2 \int_0^1 (1-t) f''_{xx}(\chi(y) + t(x - \chi(y)), y) dt.$$

For (x, y) close to (x_0, y_0) , $Q(x, y)$ is symmetric and invertible. Setting

$$Q_0 = f''_{xx}(x_0, y_0),$$

we observe there is a map $(x, y) \mapsto R(x, y)$ such that

$$Q_0 = R(x, y)^t Q(x, y) R(x, y), \quad \text{with } R(x_0, y_0) = Id.$$

Consider the substitution

$$(x, y) \mapsto (\tilde{x}(x, y), y),$$

with

$$\tilde{x}(x, y) = R(x, y)(x - \chi(y)), \quad \text{near } (x_0, y_0).$$

Write

$$b(y, w) = e^{-\frac{i}{h}f(x(y), y)} I(y, \omega) = \int e^{\frac{i}{2h}\langle Q_0 \tilde{x}, \tilde{x} \rangle} \tilde{a}(\tilde{x}, y, \omega) d\tilde{x},$$

for a \mathcal{G}^s symbol \tilde{a} with $\tilde{x} - \text{supp}$ close to zero. Next, using Plancherel's formula

$$b(y, w) = |\det(\frac{1}{2i\pi h} Q_0)|^{-\frac{1}{2}} e^{-\frac{ih}{2}\langle Q_0^{-1} D, D \rangle} \tilde{a}(0, y).$$

Now, the asymptotic expansion (2.7) follows from [13, 7.6.7] as a consequence of the usual asymptotics for quadratic phases with nonnegative real part. \square

2.2. Symbolic calculus with Gevrey symbols. The calculus of Gevrey PDO has been established previously (where??). Let $Q = \text{Op}_h(q)$, $A = \text{Op}_h(a)$, $q \in \mathcal{S}_s^{m_0}$, $a \in \mathcal{S}_s^m$. The composition $Q \circ A = \text{Op}_h(q \circ a)$ can be written

$$(2.8) \quad (q \circ a)(x, \xi) = \frac{1}{(2\pi h)^n} \iint e^{-\frac{i}{h}(x-y) \cdot (\xi-\eta)} q(x, \eta) a(y, \xi) dy d\eta.$$

Proposition 2.3. *Let $Q = \text{Op}_h(q)$, $A = \text{Op}_h(a)$, $q \in \mathcal{S}_s^{m_0}$, $a \in \mathcal{S}_s^m$ be given. Then, the composition $Q \circ A = \text{Op}_h(q \circ a)$ satisfies, for arbitrary $N \in \mathbb{N}$,*

$$(2.9) \quad (q \circ a)(x, \xi) = \sum_{|\alpha| < N} \frac{h^{|\alpha|}}{\alpha!} D_\xi^\alpha q(x, \xi) \partial_x^\alpha a(x, \xi) + r_N(q, a)(x, \xi),$$

where $r_N(q, a)$ is a \mathcal{G}^s symbol of order $m_0 + m - N$ given by

$$r_N(q, a)(x, \xi) = \frac{h^N}{i(2\pi h)^n} \sum_{|\alpha|=N} \int_0^1 \iint_{\mathbb{R}^{2n}} e^{\frac{y \cdot \eta}{ih}} (\partial_\xi^\alpha q)(x, \xi + \eta) \partial_x^\alpha a(x + \theta y, \xi) d\theta dy d\eta.$$

We shall use (2.9) with $N = 1$ in the following sections. The proof relies on the following Lemma, which follows the lines in [18, Lemmas 4.1.2 and 4.1.5].

Lemma 2.4. *For every $t \in \mathbb{R}$, let J^t be the operator defined by*

$$J^t b(x, \xi) = |t|^{-n} \iint_{\mathbb{R}^n \times \mathbb{R}^n} b(x + z, \xi + \zeta) e^{\frac{z \cdot \zeta}{it}} dz d\zeta, \quad b \in \mathcal{S}_s^m(\mathbb{R}^n \times \mathbb{R}^n).$$

Then, J^t maps $\mathcal{S}_s^{m'}(\mathbb{R}^n \times \mathbb{R}^n)$ into itself. Moreover, for any $N \in \mathbb{N}$,

$$(J^t b)(x, \xi) = \sum_{|\alpha| < N} \frac{t^{|\alpha|}}{\alpha!} D_\xi^\alpha \partial_x^\alpha b(x, \xi) + r_N(t)(x, \xi)$$

with $r_N(t) \in \mathcal{S}_s^{m-N}$ and

$$r_N(t)(x, \xi) = t^N \int_0^1 \frac{(1-\theta)^{N-1}}{(N-1)!} J^{\theta t} ((D_\xi \cdot \partial_x)^N b)(x, \xi) d\theta.$$

Proof. Let $N \in \mathbb{N}$. By Taylor's expansion up to order N in the variable t , we have

$$\begin{aligned} J^t b(x, \xi) &= \sum_{k \leq N-1} \frac{t^k}{k!} \iint_{\mathbb{R}^n \times \mathbb{R}^n} b(x+z, \xi+\zeta) \left(\frac{z}{i} \cdot \zeta\right)^k e^{\frac{z \cdot \zeta}{it}} dz d\zeta \\ &\quad + \int_0^1 \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(1-\theta)^{N-1}}{(N-1)!} b(x+z, \xi+\zeta) \left(\frac{z}{i} \cdot \zeta\right)^N e^{\frac{z \cdot \zeta}{it}} dz d\zeta d\theta. \end{aligned}$$

As a result, we can write

$$J^t = \sum_{k < N} \frac{t^k}{k!} (D_\xi \partial_x)^k + \int_0^1 \frac{(1-\theta)^{N-1}}{(N-1)!} J^{\theta t} ((t D_\xi \partial_x)^N) d\theta.$$

Hence, J^t maps $\mathcal{S}_s^{m'}(\mathbb{R}^n \times \mathbb{R}^n)$ into itself and one has

$$(J^t b)(x, \xi) = \sum_{|\alpha| < N} \frac{t^{|\alpha|}}{\alpha!} (D_\xi^\alpha \partial_x^\alpha b)(x, \xi) + r_N(t)(x, \xi)$$

with

$$r_N(t)(x, \xi) = t^N \int_0^1 \frac{(1-\theta)^{N-1}}{(N-1)!} J^{\theta t} ((t D_\xi \partial_x)^N b)(x, \xi) d\theta.$$

□

Proof of Proposition 2.3. From (2.8), we may write

$$(q \circ a)(x, \xi) = \frac{1}{(2\pi h)^n} \iint e^{-\frac{i}{h} z \cdot \zeta} q(x, \xi + \zeta) a(x + z, \xi) dz d\zeta,$$

thanks to the changes of variables $x + z = y$ and $\xi - \eta = \zeta$. For $t \in \mathbb{R}^*$ set

$$J_0^t b = \frac{1}{(2\pi|t|)^n} \iint b(z, \zeta) e^{\frac{1}{it} z \cdot \zeta} dz d\zeta, \quad b = b(z, \zeta) \in \mathcal{S}_s^{m'}(\mathbb{R}^n \times \mathbb{R}^n).$$

Then, setting $C_{x,\xi} = q(x, \xi + \zeta) a(x, x + \zeta, \xi)$ one has

$$(q \circ a)(x, \xi) = J_0^h C_{x,\xi}.$$

So thanks to Lemma 2.4 we may write

$$(q \circ a)(x, \xi) = J^h C_{x,\xi}(0, 0) = \sum_{|\alpha| < N} \frac{t^{|\alpha|}}{\alpha!} D_\xi^\alpha q(x, \xi) \partial_x^\alpha a(x, \xi) + r_N(t)(q, a)(x, \xi)$$

where

$$r_N(t)(q, a)(x, \xi) = \int_0^1 \frac{(1-\theta)^{N-1}}{(N-1)!} J^{\theta t} ((t D_\xi \partial_x)^N C_{x,\xi}(0, 0)) d\theta.$$

Now, expanding the remainder

$$r_N(q, a)(x, \xi) = \frac{1}{(2\pi h)^n} \sum_{|\gamma|=N} \frac{h^N N!}{\gamma!} \int_0^1 \iint_{\mathbb{R}^{2n}} \frac{(1-\theta)^{N-1}}{(N-1)!} e^{\frac{z \cdot \zeta}{ih}} D_\xi^\gamma q(x, \xi + \zeta) \partial_x^\gamma a(x + \theta z, \xi) dz d\zeta d\theta.$$

Using Gevrey stationary phase lemma one may prove that $r_N(q, a)$ is a Gevrey symbol of order $m_0 + m - N$ and thus,

$$(q \circ a)(x, \xi) = \sum_{|\alpha| < N} \frac{h^{|\alpha|}}{\alpha!} D_\xi^\alpha q(x, \xi) \partial_x^\alpha a(x, \xi) + r_N(q, a)(x, \xi),$$

which ends the proof. □

2.3. Formal symbols and Borel lemma in the Gevrey setting. In this section we introduce a notion of Gevrey formal symbol of a given order and give conditions ensuring the existence of a Gevrey symbol realising a given sequence of formal symbols, based on Carleson's theorem [3].

Definition 2.5 (Formal symbols). Let $m \in \mathbb{R}$. Let $(a_j)_{j \geq 0}$ be a sequence of symbols such that

$$a_j \in \mathcal{S}_s^{-j+m}(\mathbb{R}^n \times \mathbb{R}^n), \quad \forall j \in \mathbb{N}.$$

The sequence of $(a_j)_{j \geq 0}$ is said to be a formal \mathcal{G}^s symbol of degree m if there exists $C > 0$ such that

$$(2.10) \quad |\partial_x^\alpha \partial_\theta^\beta a_j(x, \theta)| \leq C^{1+|\alpha|+|\beta|+j} j!^s \alpha!^s \beta!^s h^{j-m}, \quad \forall (x, \theta) \in \mathbb{R}^n \times \mathbb{R}^n,$$

for all $\alpha, \beta \in \mathbb{N}^n$ and $j \in \mathbb{N}$.

Carleson's theorem [3] allows us to realise Gevrey s formal symbols into symbols. This result is a Gevrey Borel theorem as follows.

Theorem 3. *Let $m \in \mathbb{R}$ and let $(a_j)_{j \geq 0} \in \mathcal{S}^{-j+m}$ be a formal \mathcal{G}^s symbol of degree m . Then, there exists $a \in \mathcal{S}_s^m(\mathbb{R}^n \times \mathbb{R}^n)$ such that for any $N > 0$, $\alpha, \beta \in \mathbb{N}^n$ one has uniformly*

$$|\partial_x^\alpha \partial_\theta^\beta (a - \sum_{j < N} a_j)| \leq C^{1+\alpha+\beta+N} N!^s \alpha!^s \beta!^s h^{N-m}.$$

Proof. Consider the sequence $(a_j(x, \theta, h) h^{-j} j!)_{j \geq 0}$. This is a $(s+1)$ -sequence in the Boutet-Kree sense, according to [2]. Using Carleson theorem we have that there exists $g(t, x, \theta, h) \in \mathcal{G}^{s+1}(\overline{R}_+, \mathcal{S}_s^m)$ such that for all $j \geq 0$

$$\partial_t^j g(0, x, \eta, h) = j! a_j(x, \theta, h) h^{-j}.$$

Setting $a(x, \theta, h) = g(t, x, \theta, h)|_{t=h>0}$ for h small, by Taylor's formula

$$|\partial_x^\alpha \partial_\theta^\beta (a - \sum_{j < N} a_j)| \leq \sup_{\theta \in [0,1]} \|\partial_x^\alpha \partial_\theta^\beta \partial_t^j g(\theta h, x, \theta, h)\| \frac{h^N}{N!}.$$

As g is a function in \mathcal{G}^{s+1} function w.r.t. t and in \mathcal{G}^s w.r.t. (x, θ) , one has for some $C' > 0$

$$|\partial_x^\alpha \partial_\theta^\beta (a - \sum_{j < N} a_j)| \leq C'^{1+|\alpha|+|\beta|+} N!^s \alpha!^s \beta!^s h^{N-m},$$

so the theorem is proven. □

2.4. Formal quasinorms in the class of Gevrey symbols. In the set $\mathcal{S}_s^m(\mathbb{R}^n \times \mathbb{R}^n)$ we may define, following [15], a family of quasinorms (cf. [2]) of the form :

$$(2.11) \quad N_m(a, T)(x, \theta) = \sum_{(\alpha, \beta) \in \mathbb{N}^d} \frac{h^m T^{|\alpha|+|\beta|}}{\alpha!^s \beta!^s} |\partial_x^\alpha \partial_\theta^\beta a(x, \theta)|,$$

where $T > 0$ is a fixed parameter. We also set

$$\overline{N}_m(a, T) := \sup_{(x, \theta)} N_m(a, T)(x, \theta).$$

By Leibniz rule, one has

$$\overline{N}_{m+m'}(aa', T) \leq \overline{N}_m(a, T) \overline{N}_{m'}(a', T),$$

for any $a \in \mathcal{S}_s^m(\mathbb{R}^n \times \mathbb{R}^n)$ and $a' \in \mathcal{S}_s^{m'}(\mathbb{R}^n \times \mathbb{R}^n)$.

These quasinorms will be important in the proof of Theorem 2.

3. FOURIER INTEGRAL OPERATORS IN THE GEVREY SETTING

In this section we present some basic results concerning the specific class of Fourier Integral Operators (FIO for short) associated to Gevrey symbols and classes and how they relate to the class of Gevrey pdo.

3.1. FIO with Gevrey symbols and phases. We work with a local class of Gevrey semiclassical Fourier integral operator defined in an analogous way to [8, 61.1] in the h -independent case and also to [23] for the h dependent case in the smooth class. These are operators of the form

$$(3.1) \quad Fu(x, h) = \frac{1}{(2\pi h)^n} \iint_{\mathbb{R}^n \times \mathbb{R}^n} e^{\frac{i}{h}(S(x, \eta) - y \cdot \eta)} a(x, \eta) u(y, h) dy d\eta,$$

where the phase function $S = S(x, \eta) \in \mathcal{G}^s(\mathbb{R}^n \times \mathbb{R}^n)$ is such that $\det S''_{x, \eta} \neq 0$ and the symbol $a = a(x, \eta)$ is in some class $a \in \mathcal{S}_s^m(\mathbb{R}^n \times \mathbb{R}^n)$. We will sometimes use the notation $F \in \mathcal{I}_s^m(\mathbb{R}^n)$ for such an operator.

For a FIO F as above, define (cf. [8, Section 61.1]) its adjoint F^* by

$$\langle Fu, v \rangle = \langle u, F^*v \rangle,$$

so that

$$F^*u(x, h) = \frac{1}{(2\pi h)^n} \iint_{\mathbb{R}^n \times \mathbb{R}^n} e^{\frac{i}{h}(S(y, \eta) - x \cdot \eta)} \overline{a(y, \eta)} u(y, h) dy d\eta.$$

Then, we have a Gevrey version of [8, Lemma 62.4].

Lemma 3.1. *Let $F \in \mathcal{I}_s^m$ and let F^* be its adjoint. Then,*

$$FF^*u(x, h) = \frac{1}{(2\pi h)^n} \iint_{\mathbb{R}^n \times \mathbb{R}^n} e^{\frac{i}{h}(S(x, \xi) - S(y, \xi))} a(x, \xi) \overline{a(y, \eta)} u(y, h) dy d\xi, \quad \forall u \in \mathcal{D}'(\mathbb{R}^n),$$

and for every $N \in \mathbb{N}$ we can write

$$FF^* = K_1 + K_2,$$

where $K_1 \in \Psi_s^{2N}$ and $K_2 = \mathcal{O}_s(h^\infty)$.

Proof. Using a Gevrey– s partition of the unity we may assume that $\text{supp}(a)$ is small enough so that the mapping

$$\Sigma : (y, \eta) \mapsto (x, \xi), \quad \xi = S'_x(x, \eta), \quad y = S'_\eta(x, \eta)$$

is a canonical transformation on $\text{supp}(a)$ of class \mathcal{G}^s . Let $\chi \in \mathcal{G}_0^s(\mathbb{R}^n)$ and $r > 0$ small enough so that $\text{supp}(\chi) \subset B(0, r)$ and $\chi = 1$ on $B(0, \frac{r}{2})$. We can write

$$S(x, \xi) - S(y, \xi) = \Sigma(x, y, \xi)(x - y).$$

Then, we have

$$FF^* = K_1 + K_2$$

with

$$(3.2) \quad K_1 u(x, h) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} \mathfrak{a}(x, y, \eta) \chi\left(\frac{x-y}{\delta}\right) e^{\frac{i}{h}(x-y)\cdot\eta} |J(x, y, \eta)| u(y, h) \frac{dy d\xi}{(2\pi h)^n},$$

$$(3.3) \quad K_2 u(x, h) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} \mathfrak{a}(x, y, \eta) \left(1 - \chi\left(\frac{x-y}{\delta}\right)\right) e^{\frac{i}{h}(x-y)\cdot\eta} |J(x, y, \eta)| u(y, h) \frac{dy d\xi}{(2\pi h)^n},$$

where

$$(3.4) \quad \mathfrak{a}(x, y, \eta) := (x, \Sigma^{-1}(x, y, \eta)) \overline{a(y, \Sigma^{-1}(x, y, \eta))}, \quad J = \partial_\eta \Sigma^{-1}.$$

Using Kuranishi's trick (cf. [23, Chapter 8]), we deduce that K_1 is a \mathcal{G}^s PDO of order $2N$.

Next, using Gevrey non stationary phase we obtain that K_2 is a Gevrey s negligible reminder, while it is easy to compute the principal symbol of K_1 , which is of order $2N$. □

3.2. Composition of FIO and PDOs in the Gevrey setting. The following result is an adaptation of [8, Lemmas 62.2 and 62.3] to the Gevrey setting, which is possible with only minor modifications thanks to Lemma 2.1.

Proposition 3.2. *Let F be a \mathcal{G}^s -FIO of the form (3.1) for some phase $S \in \mathcal{G}^s$ and symbol $a \in \mathcal{S}_s^{m_0}$. Then, for every $A = A(x, D) \in \Psi_s^m$ one has*

$$(3.5) \quad AFu(x) = \frac{1}{(2\pi h)^n} \iint_{\mathbb{R}^{3n}} c(x, \eta) e^{\frac{i}{h}S(x, \eta)} \hat{u}(\eta) d\eta, \quad u \in \mathcal{D}'(\mathbb{R}^n),$$

for some $C \in \mathcal{S}_s^{m+m_0}$ whose principal symbol satisfies

$$c_0(x, \eta) = A_0(x, S(x, \eta))a(x, \eta),$$

where A_0 is the principal symbol of A . Also,

$$(3.6) \quad FAu(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} c'(x, \eta) e^{\frac{i}{h}S(x, \eta)} \hat{u}(\eta) d\eta + R'u, \quad u \in \mathcal{D}'(\mathbb{R}^n),$$

for some $C' \in \mathcal{S}^{m+m_0}$ whose principal symbol satisfies

$$(3.7) \quad C'_0(x, \eta) = A_0(S(x, \eta), \eta)a(x, \eta),$$

and $R' \in \Psi_s^{m+m_0}$.

3.3. Gevrey wavefront and Lagrangian distributions. We end this section with some details on the wavefront set adapted to the Gevrey setting. If $u \in \mathcal{D}'(\mathbb{R}^n)$ is a distribution, the semiclassical \mathcal{G}^s -wavefront $\text{WF}_{s,h}(u)$ of u is the complementary set of the set

$$\left\{ (x, \xi) \in \mathbb{R}^{2n}; \exists U \times V \text{ nhd of } (x, \xi) \text{ such that } \widehat{\psi u} \left(\frac{\xi}{h} \right) = \mathcal{O}_s(h^\infty), \forall \psi \in \mathcal{C}_0^\infty(U), \xi \in V \right\}.$$

We recall next that two FIO $P \in \mathcal{I}_s^m$ and $Q \in \mathcal{I}_s^m$ are said to be equal \mathcal{G}^s -microlocally at a point $(x_0, \xi_0) \in T^*\mathbb{R}^n$ if there exist two open neighbourhoods $U_{x_0}, V_{\xi_0} \subset \mathbb{R}^n$ of x_0 and ξ_0 respectively such that

$$A(P - Q)B = \mathcal{O}_s(h^\infty),$$

for every A, B such that $\text{WF}_{s,h}(A) \subset U_{x_0}$ and $\text{WF}_{s,h}(B) \subset V_{\xi_0}$. This notion is analogous to the usual definition of microlocality in the smooth setting (cf. [23, Section 8.4.5]).

Definition 3.3 (Microlocal conjugation by Gevrey FIOs). Given $P, Q \in \Psi_s^m$, we say that P and Q are microlocal conjugate in the \mathcal{G}^s sense around a point $(x_0, \xi_0) \in T^*\mathbb{R}^n$ if there exists a FIO $R \in \mathcal{I}_s^m$ such that RP equals QR microlocally around (x_0, ξ_0) .

Consider a Lagrangian distribution of the form

$$(3.8) \quad u(x, h) = \int e^{\frac{h}{i}\phi(x, \theta)} a(x, \theta, h) d\theta$$

for a phase ϕ and amplitude a satisfying

$$\begin{aligned} \phi &\in \mathcal{G}^s(\mathbb{R}^n \times \mathbb{R}^N), \\ \text{Im } \phi &\geq 0, \quad d\phi \neq 0, \\ a &\in \mathcal{S}_s^{m_0}(\mathbb{R}^n \times \mathbb{R}^N). \end{aligned}$$

If $\text{WF}_h(u)$ is the semiclassical wavefront of $(u(\cdot, h))_h$, then it is standard that

$$\text{WF}_h(u) \subset \{(x, \phi'_x) \mid (x, \theta) \in F, \phi'_\theta(x, \theta) = 0\},$$

with $\text{supp}(a) \subset F$. The next result is an analogue version of this fact for the Gevrey semiclassical wavefront $\text{WF}_{s,h}(u)$.

Lemma 3.4. *Let u as in (3.8). Then, one has*

$$\text{WF}_{s,h}(u) \subset \Lambda = \{(x, \phi'_x) \mid (x, \theta) \in F, \phi'_\theta(x, \theta) = 0\},$$

Proof. This is a consequence of the results in [13, Ch. 25] with minor modifications. The Gevrey asymptotic comes from the non-stationary phase lemma in Lemma 2.1. □

4. PROOF OF THEOREM 2

The goal of this section is to prove Theorem 2. We divide the proof into several steps, according to the usual WKB method.

Step 1. Setting of the problem.

We may assume w.l.g. that $(x_0, \xi_0) = (0, 0)$. Let us choose once for all a particular real coordinate and write as usual $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $\xi = (\xi_1, \xi') \in \mathbb{R} \times \mathbb{R}^{n-1}$. As P is of real principal type by hypothesis, we can assume that the principal symbol of P writes (cf. for instance [23, Section 12.2])

$$(4.1) \quad p(x, \xi_1, \xi') = \xi_1 - \lambda(x, \xi'),$$

for some real symbol $\lambda \in \mathcal{S}_s^0(\mathbb{R}^n \times \mathbb{R}^{n-1})$ real. If $\lambda = \lambda(x, \xi')$ stands for the principal symbol of Q let us write

$$(4.2) \quad P(x, hD_{x_1}, hD_{x'}) = hD_{x_1} + Q(x, hD_{x'}).$$

Step 2. Suitable phase using the eikonal equation.

We consider now the following Cauchy problem: Find φ such that

$$(4.3) \quad \begin{cases} \frac{\partial \varphi}{\partial x_1} - \lambda(x, \varphi'_x) = 0, \\ \varphi|_{x_1=0} = x' \cdot \eta', \end{cases}$$

where $\lambda \in \mathcal{S}_s^0(\mathbb{R}^n \times \mathbb{R}^{n-1})$ given by (4.1). The Cauchy problem (4.3) has a solution φ of class \mathcal{G}^s . Moreover, since $\det \varphi''_{x', \xi'} \neq 0$ the phase function φ generates a canonical transform.

Our goal in the following sections is to find suitable h -FIOs F, G such that

$$(4.4) \quad GPF = hD_{x_1} + \mathcal{O}_s(h^N),$$

for arbitrarily large $N \in \mathbb{N}$. We shall do this treating the case $N = 1$ first and then introducing an iterative scheme.

Step 3. Reducing the order of the remainder by conjugation.

Let us introduce a h -FIO of the form

$$Fu(x, h) = \frac{1}{(2\pi h)^{n-1}} \int e^{\frac{i}{h}\varphi(x, \eta')} a(x, \eta') \hat{u}(x_1, \frac{\eta'}{h}) d\eta', \quad u \in \mathcal{S}(\mathbb{R}^n),$$

where $a \in \mathcal{S}_s^m(\mathbb{R}^n \times \mathbb{R}^{n-1})$, \hat{u} denotes the partial Fourier transform in the variables x' only and φ is given as before. Assume that a is elliptic when $x_1 = 0, x' = 0, \xi_1 = 0, \xi' = 0$.

Now, F defined above is a h -FIO of the form (3.1) associated to a \mathcal{G}^s phase and symbol. As $P \in \Psi_s^m$ by hypothesis, we can compose F and P using Proposition 3.2. Indeed, from (4.2) we get, for every $u \in \mathcal{S}(\mathbb{R}^n)$,

$$PFu = (hD_{x_1} + Q)Fu = hD_{x_1}Fu + QFu$$

Now, as the symbol a is supposed to be elliptic at the point $(x_1, x', \xi_1, \xi') = 0$, we may find a microlocal inverse of F near this point and thus write

$$PFu = F(hD_{x_1}) + F_{-1}u,$$

for some $F_{-1} \in \Psi_s$. Moreover, the symbol of F_{-1} , namely \mathfrak{f} , admits the expansion (as in [13, Theorem 7.7.7]):

$$\mathfrak{f} = \sum_{\alpha \geq 0} \frac{h^{|\alpha|}}{\alpha!} p^{(\alpha)}(x, \varphi'_x) D_y^\alpha (e^{\frac{i}{h}\rho} a)|_{y=x},$$

with

$$\rho(x, y, \eta') = \varphi(y, \eta') - \varphi(x, \eta') - (y - x)\varphi'_x(x, \eta').$$

We observe that the first term of the expansion vanishes, for

$$p(x, \varphi'_x)(e^{\frac{i}{h}\rho}a)|_{y=x} = p(x, \varphi'_x)(e^{\frac{i}{h}(\varphi(y, \eta') - \varphi(x, \eta') - (y-x))}a)|_{y=x} = hD_{x_1}\varphi - \lambda(x, \varphi'_x) = 0,$$

thanks to (4.3). Hence, $F_{-1} \in \Psi_s^{-1}$.

Let now G be the microlocal inverse of the elliptic h -FIO F near zero. Then, we can write

$$GF = I + r, \quad \text{microlocally near } (0, 0, 0, 0),$$

for $r \in \Psi_s^\infty$ and hence

$$GPFu = GF(hD_{x_1}) + GF_{-1} = hD_{x_1} + R,$$

where $R = rhD_{x_1} + GF_{-1}$. Now, as G and F_{-1} are associated to the same generating function, we have in particular that GF_{-1} is a pdo and moreover $GF_{-1} \in \Psi_s^{-1}$. Furthermore, as r is a regularising pdo, we also have $rhD_{x_1} \in \Psi_s^{-1}$, which guarantees that

$$GPFu = hD_{x_1} + R, \quad R \in \Psi_s^{-1},$$

microlocally near zero. This means that P is microlocally conjugate by \mathcal{G}^s FIOs to hD_{x_1} up to a remainder of degree -1 .

Step 4. Reducing arbitrarily the order of the remainder by conjugation.

We aim now to iterate the scheme of the previous step in order to get (4.4) for a remainder of arbitrary order. We do this thanks to the following iterative scheme, which produces a formal Gevrey symbol.

Lemma 4.1. *Let $q \in \mathcal{S}_s^{-1}(\mathbb{R}^n \times \mathbb{R}^n)$. Then, there is a sequence $(a_j)_{j \geq 0}$ of symbols such that $a_j \in \mathcal{S}_s^{-j}(\mathbb{R}^n \times \mathbb{R}^n)$ satisfying*

$$\frac{h}{i}\partial_{x_1}a_0 + qa_0 = 0, \quad \text{and} \quad a_0|_{x_1=0} \text{ is elliptic,}$$

and for all $j \geq 0$,

$$(4.5) \quad \frac{h}{i}\partial_{x_1}a_j + qa_j = r_1(q, a_{j-1}), \quad \text{and} \quad a_0|_{x_1=0} = 0.$$

Moreover, if

$$A_j = \text{Op}_h(a_j), \quad A^{(N)} = A_0 + \cdots + A_{N-1}, \quad N \geq 1,$$

one has

$$(hD_{x_1} + Q)A^{(N)} = hA^{(N)}D_{x_1} + R_N,$$

with R_N a $\mathcal{G}^s \in \Psi_s^{-(N+1)}$. Finally, if q is of order m_0 small enough the above sequence of $(a_j)_{j \geq 0}$ is a formal Gevrey s symbol in the sense of Definition 2.5.

Proof. Equations (4.5) are solved by induction setting for $j \geq 1$

$$a_j = -i\frac{a_0}{h} \int_0^{x_1} \frac{r_1}{a_0}(q, a_{j-1}) dt.$$

One has $a_j \in \mathcal{S}_s^{-j}(\mathbb{R}^n \times \mathbb{R}^n)$, in view of (2.9) and (4.5) we have R_N is of order $-N - 1$.

We now show that if q is of order m_0 small enough the above sequence of $(a_j)_{j \geq 0}$ is a formal Gevrey s symbol.

In order to prove (2.10) we use the formal quasinorms for Gevrey symbols introduced in (2.11). Upon repeating the scheme of the previous step, we may assume that $P = hD_1 + Q$, with $Q \in \Psi_s^{m_0}$ of order m_0 small, is a microlocal conjugate to hD_{x_1} by \mathcal{G}^s FIOs, modulo negligible remainders. If q is the symbol of Q , recall that

$$\text{Op}_h(q) \circ \text{Op}_h(a) = \text{Op}_h(qa + R), \quad R = R(a, q)$$

with a remainder R as in Lemma 2.3. Let us define next $\chi \in \mathcal{G}_0^s(\mathbb{R}^{2n})$ such that

$$\text{supp } \chi \subseteq \{(y, \eta); |y| + |\eta| \leq r\} \quad \text{and} \quad \chi = 1 \text{ on } \left\{ (y, \eta); |y| + |\eta| \leq \frac{r}{2} \right\}$$

and let us write

$$R(a, q) = R_\chi(a, q) + S_\chi, \quad R_\chi = \chi R(a, q), \quad S_\chi = (1 - \chi)R(a, q).$$

We deduce that S_χ is Gevrey s remainder by the non-stationary phase asymptotics in Lemma 2.1. Now if $\alpha, \beta \in \mathbb{N}^n$, if $\epsilon > 0$ small enough, we also have

$$(4.6) \quad \overline{N}_m(\partial_x^\alpha \partial_\theta^\beta a, T) \leq T^{-(|\alpha|+|\beta|)} \epsilon^{-s(|\alpha|+|\beta|)} \alpha!^s \beta!^s \overline{N}_m(a, T(1 + \epsilon)^s).$$

Now, if $q \in \mathcal{S}_s^{m_0}(\mathbb{R}^n \times \mathbb{R}^n)$ with $m_0 \leq -n$, then from the previous equations,

$$\overline{N}_{m-1}(R_\chi(a), T) \leq \tau^{2n} C_n \epsilon^{-s} M_0(q) T^{-1} \overline{N}_m(a, T(1 + \epsilon)^s).$$

If $(a_j)_{j \geq 0}$ is the sequence defined by (4.5). We claim that estimate (4.6) implies, for $j \geq 0$, $\epsilon > 0$ small that for some $M'_0 = M'_0(r, q, n) > 0$ one has

$$(4.7) \quad \overline{N}_{-j}(a_j, T) \leq M'_0 \epsilon^{-sj} j^{sj} \overline{N}_0(a_0, 2T) T^{-j}.$$

Indeed, this is true if $j = 1$ from (4.6). Assuming that (4.7) holds for $j - 1$ with $j \geq 2$, we have for $\epsilon, \tilde{\epsilon} > 0$ small

$$\overline{N}_{-j}(a_j, T) \leq \epsilon^{-s} \frac{M'_0{}^j}{T^j} \frac{(j-1)^{s(j-1)}}{(1+\epsilon)^{s(j-1)} \tilde{\epsilon}^{s(j-1)}} \overline{N}_0 \left(a_0, T(1+\epsilon)^s \left(1 + \frac{\tilde{\epsilon}}{j-1} \right)^{s(j-1)} \right).$$

We fix $\delta > 0$ small and choose $\tilde{\epsilon} = \frac{\delta}{j}$ and $\epsilon = \frac{\delta}{j}$ in the previous inequality, so that

$$(1+\epsilon)^s \left(1 + \frac{\tilde{\epsilon}}{j-1} \right)^{s(j-1)} = \left(1 + \frac{\delta}{j} \right)^{sj}.$$

It remains to check that

$$\frac{(1+\epsilon)^{s(j-1)}}{(j-1)^{s(j-1)}} \tilde{\epsilon}^{-s(j-1)} \delta^{sj} j^{sj} \leq \frac{\delta^s}{j^s},$$

which is $(1 + \frac{\delta}{j})^{-s(j-1)} \leq 1$, which is true since $j \geq 1$. So this allows to prove (4.7).

As a consequence, the sequence of symbols $(a_j)_{j \in \mathbb{N}}$ defined by (4.5) is a formal \mathcal{G}^s symbol as desired. \square

Step 5. Conclusion.

Thanks to Lemma 4.1, there exists a sequence of symbols $(a_j)_{j \in \mathbb{N}}$ defined by (4.5) of order zero. Using Theorem 3 we deduce that there exists $a \in \mathcal{S}_s^m$ realising $(a_j)_{j \in \mathbb{N}}$. As a consequence, the h -FIO associated to the symbol a and the phase φ satisfying (4.3) satisfies that $PF = F(hD_{x_1})$ microlocally near zero up to negligible remainders. This ends the proof of Theorem 1.

5. PROOF OF THEOREM 1

We start by P of the form $P = hD_{x_1} + Q(x, hD_{x'})$, where Q is a h -PDO on \mathbb{R}^{n-1} depending on x_1 of order zero. Moreover $S = \{x_1 = 0\}$.

Using FIOs of a special form we reduce P to

$$P = hD_{x_1} + Q_{-1}(x, hD_{x'}),$$

with Q_{-1} of order -1 . Then, using the Kuranishi's trick we reduce the WKB problem to the same with $x_0 = \xi_0 = 0$ and $\varphi = 0$.

Now to solve this one it is enough to solve

$$(5.1) \quad \begin{cases} \frac{h}{i} \partial_{x_1} a + e^{-i\frac{\phi_0}{h}} Q_{-1}(x, hD_{x'}) (a e^{i\frac{\phi_0}{h}}) = 0, \\ a|_{x_1=0} = a_0, \end{cases}$$

where Q_{-1} is a pdo of order -1 and ϕ_0 is the phase $x' \cdot \xi'$.

Solving (5.1) is equivalent to an Egorov's reduction

$$\begin{cases} P(x, hD_x)A = hAD_{x_1}, \\ A|_{x_1=0} \text{ is elliptic.} \end{cases}$$

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