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DE LA RECHERCHE À L'INDUSTRIE

Toward a multigrid method for the Helmholtz equation

In order to evaluate the electromagnetic behavior of 3D objects, we need to solve the **Maxwell equations**.

Numerical simulation of **Maxwell Equations** with Finite Volumic Elements



Need to solve a large and sparse linear system of equations $Ax = b$



Requires an **efficient** and **scalable** solver

A *domain decomposition* method is used in this context, but limitations appear as the computing resources are growing :

- Increase in the number of sub-domains \Rightarrow Convergence is slower
- Increase in the size of the sub-domains \Rightarrow Increase in computational complexity

We need to investigate an alternative method to domain decomposition :
multigrid methods.

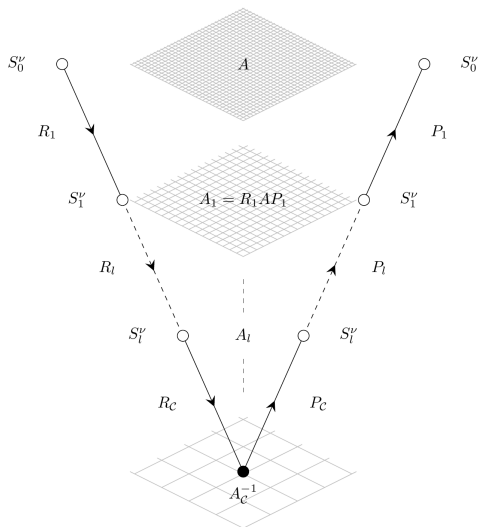


Figure 1: V-cycle of a multigrid method

In this method, the computation of the solution x is accelerated thanks to a hierarchy of coarse problems.

We introduce :

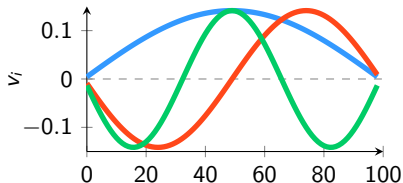
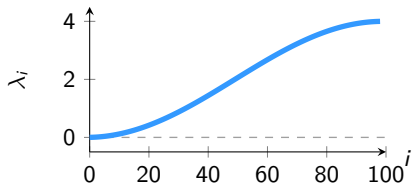
- A_l : Matrix of the level l
- S_0^ν : ν smoother iterations on the level l
- P_l : Interpolation operator of size $n_{l-1} \times n_l$
- R_l : Restriction operator of size $n_l \times n_{l-1}$

In most application

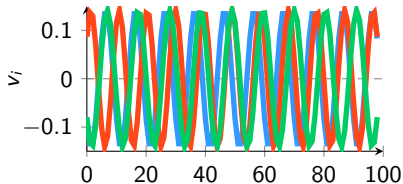
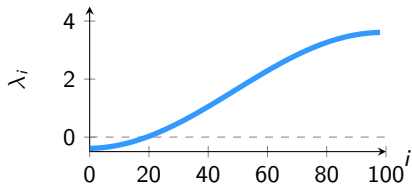
$$R = P^H. \quad (1)$$

- 1 - Multigrid applied to Laplace vs Helmholtz
- 2 - Smoother for Helmholtz
- 3 - Interpolation rules for Helmholtz
- 4 - Numerical results
- 5 - Conclusion and perspectives

(Laplace Problem) $\Leftrightarrow \begin{cases} -\Delta u = f \text{ on } \Omega = [0, 1] \\ u|_{\partial\Omega} = 0 \end{cases} \quad (2)$



(Helmholtz Problem) $\Leftrightarrow \begin{cases} -\Delta u - k^2 u = f \text{ on } \Omega = [0, 1] \\ u|_{\partial\Omega} = 0 \end{cases} \quad (3)$



Target : Find a smoother able to damp V_+ , without touching V_0 .

Problem : V_+ eigenvectors are either associated with **negative** or **positive** eigenvalues.

Alternative to usual multigrid smoothers :

- *Krylov methods* are good smoothers in the *indefinite* case but :
 - They minimize $\|r\|_2$ regardless of the eigenvalues
 - They are non-linear because of their right-hand side dependence
- *Chebyshev Polynomial Smoother* built on *normal equations* will be considered
 - Normal equations are helpful to damp both negative and positive eigenvalues
 - The Chebyshev framework is practical to find a minimum polynomial within an interval
 - This smoother has the following error propagation formula

$$q(A^2) := I - p(A^2)A^2, \quad (4)$$

giving for $Av_0 = \lambda_0 v_0 \approx 0$

$$q(A^2)v_0 = (1 - p(\lambda_0^2)\lambda_0^2)v_0 \approx v_0 \quad (5)$$

→ We seek a polynomial smoother p such that q is minimum in a given interval.

- 1 Choose an appropriate interval $\mathcal{I} = [x_{\min}, x_{\max}]$ where q must be minimum
- 2 Compute First Kind Chebyshev roots as best interpolation points within \mathcal{I}

$$x_i := \frac{x_{\max} + x_{\min}}{2} + \frac{x_{\max} - x_{\min}}{2} \cos\left(\frac{(2i-1)\pi}{2d}\right) \quad (6)$$

- 3 Construct the polynomial using the Lagrangian formula

$$q(x_i) = 0 \Leftrightarrow p(x_i) = \frac{1}{x_i}, \quad p(x) = \sum_{j=0}^d \frac{1}{x_j} \prod_{i=0, i \neq j}^d \frac{x - x_i}{x_j - x_i} \quad (7)$$

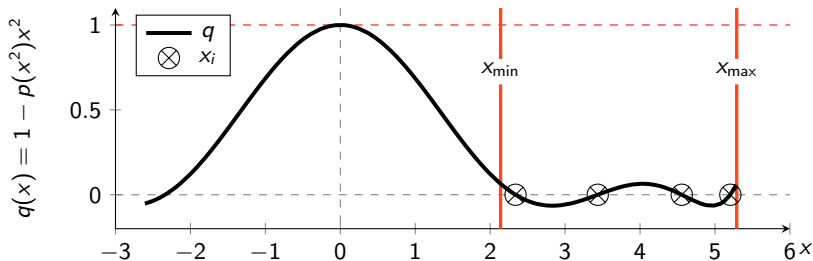


Figure 2: Spectrum of the polynomial smoother error propagation matrix

From the theory [Falgout et al. 2004], we can define, under SPD assumption of A , an ideal interpolator P_* from coarse (\mathcal{C}) and fine (\mathcal{F}) variable selection operators

$$\underset{(n_{\mathcal{C}} \times n)}{R} : \Omega \mapsto \mathcal{C} \text{ and } \underset{(n_{\mathcal{F}} \times n)}{S^H} : \Omega \mapsto \mathcal{F}, \quad (8)$$

such that $R^H \oplus S = \mathbb{R}^n$ and $RS = 0$. This operator is defined by

$$P_* = (I - S(S^H A S)^{-1} S^H A) R^H \quad (9)$$

Example : Let the coarse and fine selection operators be defined by

$$R = [0 \quad I_{n_{\mathcal{C}}}] \text{ and } S^T = [I_{n_{\mathcal{F}}} \quad 0]. \quad (10)$$

After reorganizing A by coarse/fine blocks, such that

$$A = \begin{bmatrix} S^H A S & S^H A R^H \\ R A S & R A R^H \end{bmatrix} = \begin{bmatrix} A_{ff} & A_{fc} \\ A_{cf} & A_{cc} \end{bmatrix},$$

the ideal interpolator can be written

$$P_* = \begin{bmatrix} -A_{ff}^{-1} A_{fc} \\ I_{cc} \end{bmatrix}, \quad A_C = P_*^H A P_* = A_{cc} - A_{cf} A_{ff}^{-1} A_{fc} \quad (11)$$

Remark : The ideal framework requires A to be SPD.

↪ **However the reduction viewpoint is still valid :** Removing orthogonal information that the smoother captures will correct the coarse selection operator

- ① First, we generate a set K of κ **smoothed random vectors** approximating the NKS
- ② Then we construct a better coarse selection operator \hat{R}^H by a least squares strategy minimizing the squared difference between **fine variable i values** in K and their **interpolation from coarse neighboring variables C_i**

$$\forall i \in \mathcal{F}, r_i = \arg \min_r \sum_{l=1}^{\kappa} w_l (K_{i,l} - r \cdot K_{C_i,l})^2 \quad (r_i \text{ row of } R_{\mathcal{F}}) \quad (12)$$

- ③ We plug the **new coarse and fine selection operators** defined by

$$\hat{R}^H = [R_{\mathcal{F}} \quad I_C]^T, \quad \hat{S} = [I_{\mathcal{F}} \quad -R_{\mathcal{F}}^H]^T, \quad (13)$$

- ④ And we finally **approximate the ideal interpolator** from the previous definition

$$\hat{P} \approx (I - \hat{S} X_{\mathcal{K}}^{-1} \hat{S}^H A) \hat{R}^H \quad (14)$$

where $X_{\mathcal{K}}^{-1}$ is the best polynomial approximating $(S^H A S)^{-1}$ within the sparsity-pattern constrained Krylov subspace \mathcal{K} .

2D Helmholtz problem :

$$(PC) \Leftrightarrow \begin{cases} -\Delta u - k^2 u = f \text{ on } \Omega \\ \partial_n u - iku = 0 \text{ on } \partial\Omega \end{cases}$$

- 5 pts Stencil, 10 pts per wavelength
($h = \lambda/10 \Leftrightarrow kh = 2\pi/10 \approx 0.625$)
- Multigrid parameters :
 - Normal equations polynomial smoother of degree $d = 3$
 - Krylov sub-size $m = 3$ in the construction of \hat{P}

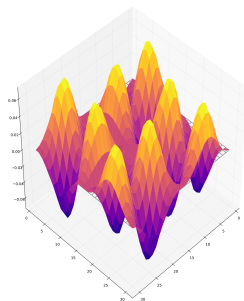


Figure 3: Solution of a Helmholtz problem

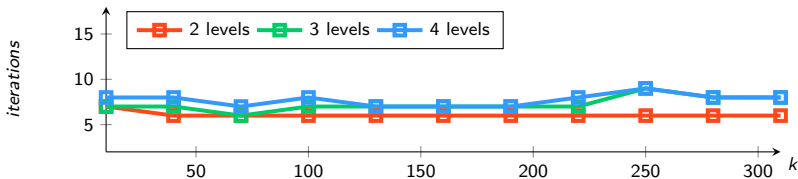


Figure 4: Number of iterations following the wavenumber k

k	70	100	130	160	190	220	250	280	310
$n (\times 10^5)$	1.28	2.59	4.37	6.60	9.30	12.46	16.08	20.16	24.7
$n_c (\times 10^5)$	0.16	0.32	0.55	0.83	1.16	1.56	2.01	2.52	3.09
$\overline{\text{nnz}}(A_0)$	5	5	5	5	5	5	5	5	5
$\overline{\text{nnz}}(A_1)$	77	78	79	79	79	80	80	80	80
$\overline{\text{nnz}}(A_2)$	299	315	316	322	332	335	337	339	341
$\overline{\text{nnz}}(A_3)$	322	342	345	359	376	380	385	386	386
$\overline{\text{nnz}}(\hat{P}_1)$	20	20	20	20	20	20	20	20	20
$\overline{\text{nnz}}(\hat{P}_2)$	63	64	65	66	66	66	67	67	67
$\overline{\text{nnz}}(\hat{P}_3)$	168	178	178	183	188	190	191	192	193

Table 1: Sparsity measurement of level matrices and interpolators following k

Helmholtz is difficult because :

- Negative components appear and require an adapted smoother
- The near-kernel space is oscillatory and requires adapted interpolation rules

⇒ We proposed an algebraic multigrid method that reaches those requirements up to a certain limit.

Further researches will focus on :

- ① Increasing the depth of the multigrid cycle
- ② Sparsity has to be improved
- ③ Construct a better coarse selection operator
- ④ Work on an adapted framework for Helmholtz