DE LA RECHERCHE À L’INDUSTRIE

Toward a multigrid method for the Helmholtz equation
In order to evaluate the electromagnetic behavior of 3D objects, we need to solve the Maxwell equations.

Numerical simulation of Maxwell Equations with Finite Volumic Elements

Need to solve a large and sparse linear system of equations $Ax = b$

Requires an efficient and scalable solver

A domain decomposition method is used in this context, but limitations appear as the computing resources are growing:

- Increase in the number of sub-domains $\Rightarrow$ Convergence is slower
- Increase in the size of the sub-domains $\Rightarrow$ Increase in computational complexity

We need to investigate an alternative method to domain decomposition: multigrid methods.
In this method, the computation of the solution $x$ is accelerated thanks to a hierarchy of coarse problems.

We introduce:

- $A_l$: Matrix of the level $l$
- $S^\nu_0$: $\nu$ smoother iterations on the level $l$
- $P_l$: Interpolation operator of size $n_{l-1} \times n_l$
- $R_l$: Restriction operator of size $n_l \times n_{l-1}$

In most application

$$R = P^H.$$  \hfill (1)
1 - Multigrid applied to Laplace vs Helmholtz

2 - Smoother for Helmholtz

3 - Interpolation rules for Helmholtz

4 - Numerical results

5 - Conclusion and perspectives
1 - Multigrid applied to Laplace vs Helmholtz

(Laplace Problem) \iff \begin{align*}
-\Delta u &= f \quad \text{on } \Omega = [0, 1] \\
\left. u \right|_{\partial \Omega} &= 0
\end{align*} \quad (2)

(Helmholtz Problem) \iff \begin{align*}
-\Delta u - k^2 u &= f \quad \text{on } \Omega = [0, 1] \\
\left. u \right|_{\partial \Omega} &= 0
\end{align*} \quad (3)
Target: Find a smoother able to damp $V_+$, without touching $V_0$.

Problem: $V_+$ eigenvectors are either associated with negative or positive eigenvalues.

Alternative to usual multigrid smoothers:

- **Krylov methods** are good smoothers in the *indefinite* case but:
  - They minimize $||r||_2$ regardless of the eigenvalues
  - They are non-linear because of their right-hand side dependence

- **Chebyshev Polynomial Smoother** built on *normal equations* will be considered
  - Normal equations are helpful to damp both negative and positive eigenvalues
  - The Chebyshev framework is practical to find a minimum polynomial within an interval
  - This smoother has the following error propagation formula

\[
q(A^2) := I - p(A^2)A^2,
\]

(4)

Giving for $Av_0 = \lambda_0 v_0 \approx 0$

\[
q(A^2)v_0 = (1 - p(\lambda_0^2)\lambda_0^2)v_0 \approx v_0
\]

(5)

→ We seek a polynomial smoother $p$ such that $q$ is minimum in a given interval.
1. Choose an appropriate interval $\mathcal{I} = [x_{\text{min}}, x_{\text{max}}]$ where $q$ must be minimum.
2. Compute First Kind Chebyshev roots as best interpolation points within $\mathcal{I}$
   \[ x_i := \frac{x_{\text{max}} + x_{\text{min}}}{2} + \frac{x_{\text{max}} - x_{\text{min}}}{2} \cos \left( \frac{(2i - 1)\pi}{2d} \right) \] 
   \hspace{1cm} (6)
3. Construct the polynomial using the Lagrangian formula
   \[ q(x_i) = 0 \iff p(x_i) = \frac{1}{x_i}, \quad p(x) = \sum_{j=0}^{d} \frac{1}{x_j} \prod_{i=0, i\neq j}^{d} \frac{x - x_i}{x_j - x_i} \] 
   \hspace{1cm} (7)

Figure 2: Spectrum of the polynomial smoother error propagation matrix.
From the theory [Falgout et al. 2004], we can define, under SPD assumption of $A$, an ideal interpolator $P_*$ from coarse ($C$) and fine ($F$) variable selection operators

$$ R_{(n_C \times n)} : \Omega \mapsto C \quad \text{and} \quad S_H^{(n_F \times n)} : \Omega \mapsto F, \quad (8) $$

such that $R^H \oplus S = \mathbb{R}^n$ and $RS = 0$. This operator is defined by

$$ P_* = (I - S(S^H AS)^{-1} S^H A)R^H \quad (9) $$

**Exemple**: Let the coarse and fine selection operators be defined by

$$ R = [ 0 \quad I_{n_C} ] \quad \text{and} \quad S^T = [ I_{n_F} \quad 0 ]. \quad (10) $$

After reorganizing $A$ by coarse/fine blocks, such that

$$ A = \begin{bmatrix} S^H AS & S^H AR^H \\ RAS & RAR^H \end{bmatrix} = \begin{bmatrix} A_{ff} & A_{fc} \\ A_{cf} & A_{cc} \end{bmatrix}, $$

the ideal interpolator can be written

$$ P_* = \begin{bmatrix} -A_{ff}^{-1} A_{fc} \\ l_{cc} \end{bmatrix}, \quad A_C = P_*^H A P_* = A_{cc} - A_{fc} A_{ff}^{-1} A_{fc} \quad (11) $$
Remark : The ideal framework requires $A$ to be SPD.

However the reduction viewpoint is still valid: Removing orthogonal information that the smoother captures will correct the coarse selection operator.

1. First, we generate a set $K$ of $\kappa$ smoothed random vectors approximating the NKS.

2. Then we construct a better coarse selection operator $\hat{R}^H$ by a least squares strategy minimizing the squared difference between fine variable $i$ values in $K$ and their interpolation from coarse neighboring variables $C_i$

$$\forall i \in \mathcal{F}, \ r_i = \arg \min_r \sum_{l=1}^{\kappa} w_l (K_{i,l} - r \cdot K_{C_i,l})^2 \ (r_i \ \text{row of } R_F) \quad (12)$$

3. We plug the new coarse and fine selection operators defined by

$$\hat{R}^H = [ \ R_F \ I_C ]^T, \ \hat{S} = [ \ I_F \ - R_F^H ]^T, \quad (13)$$

4. And we finally approximate the ideal interpolator from the previous definition

$$\hat{P} \approx (I - \hat{S} X_K^{-1} \hat{S}^H A) \hat{R}^H \quad (14)$$

where $X_K^{-1}$ is the best polynomial approximating $(S^H A S)^{-1}$ within the sparsity-pattern constrained Krylov subspace $\mathcal{K}$. 
2D Helmholtz problem:

\[(PC) \iff \begin{cases} -\Delta u - k^2 u = f \text{ on } \Omega \\ \partial_n u - iku = 0 \text{ on } \partial \Omega \end{cases}\]

- 5 pts Stencil, 10 pts per wavelength
  \((h = \lambda/10 \iff kh = \frac{2\pi}{10} \approx 0.625)\)
- Multigrid parameters:
  - Normal equations polynomial smoother of degree \(d = 3\)
  - Krylov sub-size \(m = 3\) in the construction of \(\hat{P}\)

**Figure 3:** Solution of a Helmholtz problem

**Figure 4:** Number of iterations following the wavenumber \(k\)
4 - Numerical results

<table>
<thead>
<tr>
<th>$k$</th>
<th>70</th>
<th>100</th>
<th>130</th>
<th>160</th>
<th>190</th>
<th>220</th>
<th>250</th>
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<tbody>
<tr>
<td>$n \times 10^5$</td>
<td>1.28</td>
<td>2.59</td>
<td>4.37</td>
<td>6.60</td>
<td>9.30</td>
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<td>$n_c \times 10^5$</td>
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</table>

Table 1: Sparsity measurement of level matrices and interpolators following $k$
Helmholtz is difficult because:

- Negative components appear and require an adapted smoother
- The near-kernel space is oscillatory and requires adapted interpolation rules

⇒ We proposed an algebraic multigrid method that reaches those requirements up to a certain limit.

Further researches will focus on:

1. Increasing the depth of the multigrid cycle
2. Sparsity has to be improved
3. Construct a better coarse selection operator
4. Work on an adapted framework for Helmholtz