L ∞ Stationary Solutions to Non Homogeneous Conservation Laws

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Abstract Stationary solutions, besides being relevant on their own, play a key role in a variety of analytic techniques related to conservation laws. Here, we present the construction of a (partial) *foliation* of stationary solutions to scalar conservation laws with x dependent fluxes. Differently from what happens in the x independent case, here solutions are in $\mathbf{L}^\infty$, no bound on the total variation is to be expected, and all discontinuities are entropy admissible.

1 Introduction

We construct $\mathbf{L}^\infty$ stationary entropy solutions to a scalar non homogeneous conservation law in one space dimension, i.e.

$$\begin{cases} \partial_t u + \partial_x f(x,u) = 0 & (t,x) \in \mathbb{R}^+ \times \mathbb{R} \\ u(0,x) = u_o(x) & x \in \mathbb{R}\,. \end{cases} \tag{CL}$$

We stress that these solutions are typically non smooth, may well contain entropic – though stationary – shocks and no bound on their total variation is to be expected. In this respect, the homogeneous – x independent – case is significantly simpler. There, constant solutions are sufficient to provide all necessary bounds and entropy

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conditions play a role only to select stationary shocks. In the x dependent case, entropy conditions have a central role in selecting a sufficient provision of stationary solutions.

The availability of a sufficiently rich set of stationary solutions is, in the homogeneous case, at the basis of most analytic techniques. Here, our first motivation is to produce a (non homogeneous) substitute for the well known *Maximum Principle* that applies to **(CL)** in the homogeneous case. Indeed, any $\mathbf{L}^1$–contractive semigroup is order preserving [8] and stationary solutions thus provide *a priori* $\mathbf{L}^\infty$ bounds.

Secondly, stationary solutions serve as a basis for a possible use of Crandall–Ligget [7] techniques for the construction of semigroups generated by **(CL)**. In this connection, we recall that already in [2, 3] stationary solutions are assigned a key role in selecting *good* solutions, also beyond the standard framework including, for instance, the case of fluxes with discontinuities in x.

Below, we construct a (partial) foliation of the (x, u) space, exhibiting stationary solutions above (or below) any assigned value U, (or $-U$). This completely answers to our first motivation above. We refer to [5] for results, based on this foliation, on the connection between homogeneous conservation laws and Hamilton–Jacobi equations as well as on their well posedness. Moreover, since the techniques presented below are set in a rather general framework, we expect that further uses of this construction are to be found.

The next section lists the assumptions and presents the main result. Section 3 reduces the proof to that of three lemmas. The first, not explicitly stated in [5], is considered in some details in § 4.1 while § 4.2 outlines the proofs of the remaining lemmas. For all details, we refer to [5].

2 Assumptions and Result

The framework we propose is based on these assumptions[1] on f:

Smoothness : $f \in \mathbf{C}^3(\mathbb{R}^2; \mathbb{R})$. **(C3)**

Compact NonHomogeneity : $\exists\, X > 0 \colon \forall\, (x,u) \in \mathbb{R}^2$ if $|x| > X$ then $\partial_x f(x,u) = 0$; **(CNH)**

Uniform Coercivity : $\forall\, h \in \mathbb{R} \quad \exists\, \mathcal{U}_h \in \mathbb{R} \colon \forall\, (x,u) \in \mathbb{R}^2$ if $\left|f(x,u)\right| \leq h$ then $|u| \leq \mathcal{U}_h$. **(UC)**

Weak Genuine NonLinearity : for a.e. $x \in \mathbb{R}$ the set $\left\{w \in \mathbb{R} \colon \partial^2_{ww} f(x,w) = 0\right\}$ has empty interior. **(WGNL)**

[1] In view of **(CNH)**, in **(UC)** and in **(WGNL)** it is sufficient to consider only $x \in [-X, X]$.

Likely, the smoothness assumption **(C3)** can be slightly relaxed. Condition **(CNH)**, introduced in [5], qualifies the behavior of the flux, and hence of the solutions, for $x \to \pm\infty$. Note that X plays essentially no quantitative role throughout, so that its weakening might require only technical modifications. Hypothesis **(UC)** is a restriction on the structure of the level sets of f and replaces any growth condition. Finally, **(WGNL)** has a mostly technical role, since compensated compactness is the tool used to ensure the convergence of stationary solutions to approximated problems.

Note that, under the same assumptions, the kinetic approach in [13, 17] is likely to allow for analogous results.

Definition 1 [5, Definition 2.1] A function $u \in \mathbf{L}^\infty(\mathbb{R}^+ \times \mathbb{R}; \mathbb{R})$ is an entropy solution to **(CL)** if for all non-negative test functions $\phi \in \mathbf{C}^1_c(\mathbb{R}^+ \times \mathbb{R}; \mathbb{R}^+)$ and for all $k \in \mathbb{R}$,

$$\begin{aligned} &\int_0^{+\infty}\int_{\mathbb{R}} \big|u(t,x)-k\big|\,\partial_t\phi(t,x)\,\mathrm{d}x\,\mathrm{d}t \\ &+\int_0^{+\infty}\int_{\mathbb{R}} \operatorname{sgn}\big(u(t,x)-k\big)\,\Big(f\big(x,u(t,x)\big)-f(x,k)\Big)\,\partial_x\phi(t,x)\,\mathrm{d}x\,\mathrm{d}t \\ &-\int_0^{+\infty}\int_{\mathbb{R}} \operatorname{sgn}\big(u(t,x)-k\big)\,\partial_x f(x,k)\,\phi(t,x)\,\mathrm{d}x\,\mathrm{d}t \\ &+\int_{\mathbb{R}} \big|u_o(x)-k\big|\,\phi(0,x)\,\mathrm{d}x \geq 0\,. \end{aligned} \tag{1}$$

Remark that, differently from [12, Definition 1], above we require no hypothesis on any sort of continuity in time. In fact, the results in [5] ensure that, under only **(C3)**, the above definition also guarantees uniform $\mathbf{L}^1_{\mathbf{loc}}$ continuity in time of the solution to **(CL)**.

Theorem 1 [5, Theorem 2.9] *Let f satisfy* **(C3)–(CNH)–(UC)–(WGNL)**. *Then, for all $U > 0$,* **(CL)** *admits stationary entropy solutions $u_-, u_+ \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$, i.e., solutions in the sense of Definition 1, that satisfy*

$$u_-(x) \leq -U \quad \text{and} \quad u_+(x) \geq U \quad \text{for a.e. } x \in \mathbb{R}\,.$$

3 Sketch of the Proof of Theorem 1

We list below the lemmas in [5, § 3.2] that constitute the proof of Theorem 1. We provide additional details to the proof of Lemma 1, which is proved although not explicitly stated in [5, § 3.2] and plays a key role. Indeed, this lemma is a stability result about the convergence of stationary solutions when the fluxes are locally uniformly converging. The lack of any bound on the total variation of the stationary solutions, together with the central role played by $\mathbf{L}^\infty$ bounds, suggests to rely on compensated compactness, which appears here as the natural tool.

Lemma 1 *Let f satisfy* **(C3)–(CNH)–(UC)–(WGNL)** *and fix a sequence $f_n \in \mathbf{C}^2(\mathbb{R}^2;\mathbb{R})$ that converges to f locally uniformly and each f_n satisfies* **(CNH)**. *Call*

u_n an $\mathbf{L}^\infty(\mathbb{R};\mathbb{R})$–bounded sequence of stationary solutions to **(CL)** *in the sense of Definition 1. Then, there exists a $u_* \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R})$ such that up to a subsequence, u_n converges pointwise a.e. to u_* and, therefore, u_* is a stationary solutions to* **(CL)** *in the sense of Definition 1.*

The above statement is devoted to the proof of Theorem 1. However, the procedure used in the proof of Lemma 1 is likely to yield an analogous result for non stationary solutions.

The proof of Theorem 1 proceeds with a careful construction of a particular class of fluxes whose level sets enjoy suitable geometric properties. This class is large enough to approximate any flux satisfying **(C3)–(CNH)–(UC)–(WGNL)**.

Lemma 2 [5, Lemma 3.2] *Let* **(C3)–(CNH)–(UC)** *hold. Fix $U > 0$. There exist $\Lambda \in \mathbb{R}$, $V \in \mathbb{R}$ and real monotone sequences a_n, b_n with $\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} b_n = 0$ such that if*

$$\forall\, (x,u) \in \mathbb{R}^2 \qquad f_n(x,u) \coloneqq f(x,u) - a_n u - \frac{1}{2}\, b_n u^2\,, \tag{2}$$

then:

1. *For all $n \in \mathbb{N}$, for all $(x,u) \in \mathbb{R}^2$, $f_n(x,u) = \Lambda$ implies $\nabla f_n(x,u) \neq 0$.*
2. *For all $(x,u) \in \mathbb{R}^2$, $f(x,u) = \Lambda$ implies $\nabla f(x,u) \neq 0$.*
3. *For all $n \in \mathbb{N}$, for all $(x,u) \in \mathbb{R}^2$, $|u| \leq U$ implies $\left|f_n(x,u)\right| < \Lambda$ and $\left|f(x,u)\right| < \Lambda$.*
4. *For all $n \in \mathbb{N}$, for all $(x,u) \in \mathbb{R}^2$, $u \geq V$ implies $\left|f_n(x,u)\right| > \Lambda$ and $\left|f(x,u)\right| > \Lambda$.*
5. *For all $n \in \mathbb{N}$, for all $(x,u) \in \mathbb{R}^2$, $f_n(x,u){=}\Lambda$ and $\partial_u f_n(x,u){=}0$ imply $\partial^2_{uu} f_n(x,u){\neq}0$.*

We are now ready to actually construct piecewise $\mathbf{C}^1$ stationary entropy solutions – for all fluxes in the previous generic class – by means of the Implicit Function Theorem and Sard's Lemma.

The next Lemma yields, for all $U \in \mathbb{R}$, a stationary entropy solution u_+ to **(CL)** such that $u_+ > U$. An entirely analogous result yields a stationary entropy solution u_- such that $u_- < -U$.

Lemma 3 [5, Lemma 3.3] *Let f satisfy* **(C3)–(CNH)–(UC)** *and moreover*

$$\forall\, x \in \mathbb{R} \qquad \lim_{u\to+\infty} f(x,u) = +\infty\,. \tag{3}$$

If U, V and Λ are positive real numbers such that

$$\forall\, (x,u) \in \mathbb{R}^2 \qquad u \in [0,U] \qquad \Longrightarrow \qquad f(x,u) < \Lambda\,, \tag{4}$$

$$\forall\, (x,u) \in \mathbb{R}^2 \qquad u \geq V \qquad \Longrightarrow \qquad f(x,u) > \Lambda\,, \tag{5}$$

$$\forall\, (x,u) \in \mathbb{R}^2 \qquad f(x,u) = \Lambda \qquad \Longrightarrow \qquad \nabla f(x,u) \neq 0\,, \tag{6}$$

$$\forall\, (x,u) \in \mathbb{R}^2 \qquad \left.\begin{array}{r} f(x,u) = \Lambda \\ \partial_u f(x,u) = 0 \end{array}\right\} \qquad \Longrightarrow \qquad \partial^2_{uu} f(x,u) \neq 0\,. \tag{7}$$

Then, there exists a stationary solution $u_+ \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R}^+)$, in the sense of Definition 1, to $\partial_t u + \partial_x f(x,u) = 0$ that satisfies $f\big(x,u_+(x)\big) = \Lambda$ (so that u_+ attains values in $]U,V[$).

When (3) is replaced by

$$\forall\, x \in \mathbb{R} \qquad \lim_{u\to+\infty} f(x,u) = -\infty\,, \tag{8}$$

the above procedure can be repeated with essentially only one substantial modification stated at the end of § 4.2, see [5, Lemma 3.3] for all details.

By Lemma 3, we have a sufficiently rich supply of stationary solutions, at this moment for a specific class of fluxes. Lemma 2 ensures that this class is dense in the class of those satisfying **(C3)–(CNH)–(UC)–(WGNL)**. Finally, Lemma 1 allows to pass to the limit in the flux, thus completing the proof of Theorem 1.

4 Sketch of the Proofs of the Lemmas

For all details we refer to [5, § 3.2].

We recall what we mean by entropy – entropy flux pair for **(CL)**, see [9, § 3.2].

Definition 2 [5, Definition 2.3] Let $f \in \mathbf{C}^1(\mathbb{R}^2;\mathbb{R})$. A pair of functions (E,F) with $E \in \mathbf{Lip}\,(\mathbb{R};\mathbb{R})$ and $F \in \mathbf{Lip}\,(\mathbb{R}^2;\mathbb{R})$ is an *entropy – entropy flux pair* with respect to f if for all $x \in \mathbb{R}$ and for a.e. $u \in \mathbb{R}$

$$\partial_u F(x,u) = E'(u)\;\partial_x f(x,u)\,. \tag{9}$$

4.1 Proof of Lemma 1

For any $E \in \mathbf{C}^2(\mathbb{R},\mathbb{R})$, introduce the entropy – entropy flux pair (E,F_n) with respect to f_n by

$$\begin{aligned} F_n(x,u) &\coloneqq \int_0^u E'(v)\,\partial_u f_n(x,v)\,\mathrm{d}v \\ &= E'(u)\,f_n(x,u) - E'(0)\,f_n(x,0) - \int_0^u E''(v)\,f_n(x,v)\,\mathrm{d}v \end{aligned} \tag{10}$$

for all $(x,u) \in \mathbb{R}^2$. Note that $F_n \longrightarrow F$ locally uniformly, F being a flux of the entropy E with respect to the flux f, similarly to (10).

Since (u_n) is uniformly bounded, by [11, Chapter 1, § 9, Theorem 1.46], (u_n) admits a subsequence, which we keep denoting (u_n), and, for a.e. $x \in \mathbb{R}$, a Young measure [11, Chapter 1, § 9, Definition 1.34] ν_x, which is a Borel probability measure on $[U,V]$ and such that

$$\lim_{n\to+\infty} \int_{\mathbb{R}} g\left(u_n(x)\right) \phi(x)\,\mathrm{d}x = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} g(w)\,\mathrm{d}\nu_x(w) \right) \phi(x)\,\mathrm{d}x$$

for any $g \in \mathbf{C}^0(\mathbb{R};\mathbb{R})$ and for any $\phi \in \mathbf{L}^1(\mathbb{R};\mathbb{R})$. Clearly, we also obtain that for any $\phi \in \mathbf{L}^1(\mathbb{R}^+ \times \mathbb{R};\mathbb{R})$, we have

$$\lim_{n\to+\infty} \int_0^{+\infty} \int_{\mathbb{R}} g\left(u_n(x)\right) \phi(t,x)\ \mathrm{d}t\,\mathrm{d}x = \int_0^{+\infty} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} g(w)\ \mathrm{d}\nu_x(w) \right) \phi(t,x)\,\mathrm{d}t\,\mathrm{d}x\ . \tag{11}$$

Claim 1: For any $R > 0$ and for any convex entropy $E \in \mathbf{C}^2(\mathbb{R};\mathbb{R})$, define F_n by (10). Then, $\left\{ \partial_x \left(F_n(\cdot, u_n) \right) : n \in \mathbb{N} \right\}$ is relatively compact in $\mathbf{H}^{-1}([-R,R];\mathbb{R})$.

Claim 2: For any $G \in \mathbf{C}^0(\mathbb{R}^2;\mathbb{R})$ such that $G(x,u) = G(-X,u)$ for all $x \in]-\infty,-X]$ and $G(x,u) = G(X,u)$ for all $x \in [X,+\infty[$,

$$\lim_{n\to+\infty} \int_0^{+\infty} \int_{\mathbb{R}} G\left(x, u_n(x)\right) \phi(t,x)\,\mathrm{d}t\,\mathrm{d}x = \int_0^{+\infty} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} G(x,w)\ \mathrm{d}\nu_x(w) \right) \phi(t,x)\,\mathrm{d}t\,\mathrm{d}x\ .$$

Claim 3: For any $G_n \in \mathbf{C}^0(\mathbb{R}^2;\mathbb{R})$ with $G_n(x,u) = G_n(-X,u)$ for all $x \in]-\infty,-X]$ and $G_n(x,u) = G_n(X,u)$ for all $x \in [X,+\infty[$, such that G_n converges to G uniformly on $\mathbb{R} \times [U,V]$,

$$\lim_{n\to+\infty} \int_0^{+\infty} \int_{\mathbb{R}} G_n\left(x, u_n(x)\right) \phi(t,x)\,\mathrm{d}t\,\mathrm{d}x = \int_0^{+\infty} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} G(x,w)\ \mathrm{d}\nu_x(w) \right) \phi(t,x)\,\mathrm{d}t\,\mathrm{d}x\ .$$

Claim 4: For any entropy $E \in \mathbf{C}^2(\mathbb{R};\mathbb{R})$, there exists a set $\Omega_E \subseteq \mathbb{R}$ such that $\mathbb{R} \setminus \Omega_E$ is negligible and for all $x \in \Omega_E$

$$\begin{aligned} &\int_{\mathbb{R}} \left(w\,F(x,w) - E(w)\,f(x,w) \right) \mathrm{d}\nu_x(w) \\ &= \int_{\mathbb{R}} w\,\mathrm{d}\nu_x(w) \int_{\mathbb{R}} F(x,w)\,\mathrm{d}\nu_x(w) - \int_{\mathbb{R}} E(w)\,\mathrm{d}\nu_x(w) \int_{\mathbb{R}} f(x,w)\,\mathrm{d}\nu_x(w) \end{aligned} \tag{12}$$

where F is any entropy flux corresponding to E with respect to f, according to Definition 2.

To prove this Claim, consider the vector fields

$$V_n(t,x) := \begin{bmatrix} u_n(x) \\ f_n\left(x, u_n(x)\right) \end{bmatrix} \qquad\qquad W_n(t,x) := \begin{bmatrix} F_n\left(x, u_n(x)\right) \\ -E\left(x, u_n(x)\right) \end{bmatrix}$$

and assume preliminarily that E is convex. Call F_n the flux corresponding to E with respect to f_n.

Fix an arbitrary $R > 0$. In the present stationary situation, $\operatorname{div} V_n$ vanishes. Moreover, by Claim 1, $\nabla \wedge W_n$ lies in a relatively compact subset of $\mathbf{H}^{-1}(\mathbb{R} \times [-R,R];\mathbb{R})$. By the div–curl Lemma [9, Theorem 17.2.1], we have

$$\lim_{n\to+\infty}(V_n\cdot W_n)=\left(\lim_{n\to+\infty}V_n\right)\cdot\left(\lim_{n\to+\infty}W_n\right), \tag{13}$$

and standard computations complete the proof of Claim 4.

Call $\mathcal{E}$ the countable set of all polynomials with rational coefficients and define

$$\Omega\coloneqq\bigcap_{E\in\mathcal{E}}\Omega_E\,. \tag{14}$$

Claim 5: The set Ω is such that $\mathbb{R}\setminus\Omega$ is negligible and for all $E\in\mathbf{C}^0(\mathbb{R};\mathbb{R})$ and for all $x\in\Omega$, equality (12) holds, where F^k is given, for any $k\in\mathbb{R}$, by

$$F^k(x,u)\coloneqq E(u)\;\partial_u f(x,u)-E(k)\;\partial_u f(x,k)-\int_k^u E(v)\;\partial^2_{uu}f(x,v)\,\mathrm{d}v\,. \tag{15}$$

Define for all $x\in\mathbb{R}$

$$u(x)\coloneqq\int_{\mathbb{R}} w\;\mathrm{d}\nu_x(w)\,. \tag{16}$$

Claim 6: With reference to (14) and (16), for all $x\in\Omega$,

$$\int_{\mathbb{R}} f(x,w)\;\mathrm{d}\nu_x(w)=f\left(x,u(x)\right)\,. \tag{17}$$

Claim 7: The sequence u_n converges to u, as defined in (16), a.e. in $\mathbb{R}$.

This latter claim follows, by contradiction, from the relation

$$\Big(f(x,w)-f\left(x,u(x)\right)\Big)=\big(w-u(x)\big)\;\;\partial_w f(x,w)\qquad\begin{array}{l}\text{for all } w\in\text{co spt }\nu_x\\ \text{and for a.e. } x\in\mathbb{R}\,.\end{array}$$

which violates **(WGNL)**, unless ν_x is a Dirac delta and, hence, we have the pointwise convergence of the u_n.

4.2 Key Steps in the Proofs of Lemma 2 and Lemma 3

The statements and all details of these proofs are in [5, § 3.2]. Here we point out the main ingredients.

Concerning Lemma 2, its proofs is centered on the map $G\colon\mathbb{R}^2\to\mathbb{R}^2$ defined by

$$\forall\,(x,u)\in\mathbb{R}^2\qquad G(x,u)\coloneqq\Big(\partial_u f(x,u)-u\;\partial^2_{uu}f(x,u)\,,\;\partial^2_{uu}f(x,u)\Big)$$

which, by **(C3)**, is in $\mathbf{C}^1(\mathbb{R}^2;\mathbb{R}^2)$. By Sard's Lemma [18, Chapter II, Theorem 3.1], the set of critical values of G is negligible. Let (a_n,b_n) be a sequence of regular value for G that converges to $(0,0)$. Then, each set $\big\{(x,u)\in\mathbb{R}\colon G(x,u)=(a_n,b_n)\big\}$ is discrete by the Local Inverse Function Theorem, hence it is countable. On the

other hand,

$$
\begin{aligned}
&G(x,u) = (a_n, b_n)\\
\Leftrightarrow\ &\partial_u f(x,u) - \partial^2_{uu} f(x,u)\, u = a_n \text{ and } \partial^2_{uu} f(x,u) = b_n\\
\Leftrightarrow\ &\partial_u f(x,u) - b_n\, u = a_n \text{ and } \partial^2_{uu} f(x,u) = b_n\\
\Leftrightarrow\ &\partial_u f_n(x,u) = 0 \text{ and } \partial^2_{uu} f_n(x,u) = 0 \text{ where } f_n(x,u) = f(x,u) - a_n u - \frac{1}{2}\, b_n u^2.
\end{aligned}
$$

Call $\mathcal{Q}_n := \left\{(x,u) \in \mathbb{R}^2 \colon \partial_u f_n(x,u) = 0 \text{ and } \partial^2_{uu} f_n(x,u) = 0\right\}$. Thus, each set $f_n(\mathcal{Q}_n)$ is also countable, its complement having full measure, and the same holds for the union $\bigcup_{n\in\mathbb{N}} f_n(\mathcal{Q}_n)$.

The main obstacle in the proof is now overtaken choosing Λ in $\mathbb{R}$ but neither in $\bigcup_{n\in\mathbb{N}} f_n(\mathcal{Q}_n)$ nor in the set of the critical values of f or of any of the f_n.

Further difficulties are then dealt with by careful *ad hoc* manipulations, see [5, § 3.2]. This completes the present discussion of the proof of Lemma 2.

In the proof of Lemma 3, there is a clear distinction between points where the stationary solution is smooth and those where an *entropic* shock needs to be selected, refer to Figure 1. As long as the Implicit Function Theorem can be applied,

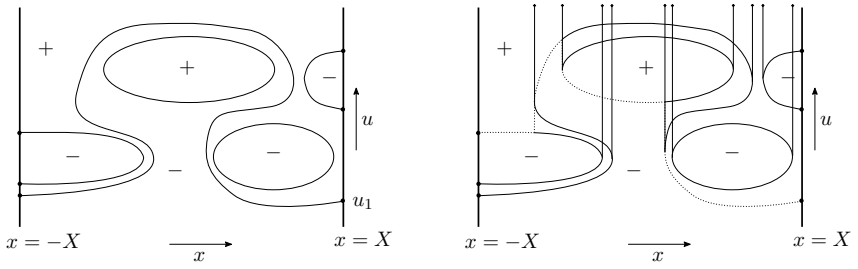

Fig. 1 Left, the level set $f(x,u) = \Lambda$, with $\pm$ denoting the regions where $f(x,u) \gtrless \Lambda$. Right, the dashed line is the graph of u_+: the diamonds indicate the positions of the points that, along the x axis, constitute the set $\mathcal{X}$ defined in (18).

a smooth stationary solution is locally constructed. Whenever the level curves of f has a *vertical tangent* in the (x,u)–plane, we have to make sure that a vertical and entropic jump can land on another connected component of the same level set. Here enters assumption (7), which in particular ensures that at the chosen level, the level curve has a non zero curvature. We thus introduce the set of possible jump points (corresponding to the diamonds in Figure 1, right)

$$
\begin{aligned}
\mathcal{X} &:= \mathbb{R} \setminus \left\{x \in \mathbb{R} \colon \text{ if } u \in \mathbb{R}_+ \text{ is such that } f(x,u) = \Lambda \text{ then } \partial_u f(x,u) \neq 0\right\} \quad (18)\\
&= \pi_x \left(\left\{(x,u) \in \mathbb{R} \times \mathbb{R}_+ \colon f(x,u) = \Lambda \text{ and } \partial_u f(x,u) = 0\right\} \right),
\end{aligned}
$$

where $\pi_x : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is the canonical projection $\pi_x(x,u) = x$. An essential remark here is that $\mathcal{X}$ turns out to be discrete, by (7). Note that the stationary solution thus constructed is actually piecewise of class $\mathbf{C}^1$.

The actual construction proceeds following pieces of level sets, as long as they are graph of a function, and jumping to another piece when necessary. To ensure that the resulting jumps are entropic, when (3) holds, this procedure has to be carried *"from right to left"*, see Figure 1. On the contrary, it is critical that the case (8) be treated *"from left to right"*, i.e., from $-X$ to X.

5 Comments and Further Questions

In the stability result of Lemma 1, entropy solutions might be replaced by quasi entropy solutions, as suggested in [15]. Coherently, the choice of admissible jumps should be adapted but the topological methods in Lemma 3 keep being effective and also Lemma 2 may remain unaltered. The need for selecting suitable stationary shocks is evident, for instance, in the selection of Riemann solvers at junctions in a variety of traffic models on networks, see for instance [4] and the references therein.

The above tools, in particular the topological methods in Lemma 3, can be reasonably expected to be generalized to the scalar multi–dimensional case, at the cost of non trivial technicalities. For instance, the present use of Lemma 2 might need to be substituted by Thom Transversality Theorem, see [10, Chapter 3].

A related situation referred to one dimensional systems of balance laws is considered in [1]. Stationary solutions, in connection with wave front tracking, are efficiently exploited to construct all general, time evolving, solutions in the case where there is no resonance between the source term and the flux. Along these lines, Lemma 3 might be extended to the case of systems. However, the need for bounds on the total variation and the need for an extension of the stability result in Lemma 1 conceal major difficulties in any extension to systems of conservation laws.

We also expect that in the many still open questions about the asymptotic behavior of solutions to **(CL)**, see [14] for a strictly related case, the stationary solutions constructed above can have a role. A further related example, pointing out also the differences between x–dependent and x–independent fluxes, is in [6, Theorem 4.1]; see also [9, § 11.11]. Moreover, as soon as these stationary solutions turn out to be unstable, a stabilization procedure can be sought, for instance along the lines of [16].

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References

1. D. Amadori, L. Gosse, and G. Guerra. Global BV entropy solutions and uniqueness for hyperbolic systems of balance laws. *Arch. Ration. Mech. Anal.*, 162(4):327–366, 2002.
2. B. Andreianov, K. H. Karlsen, and N. H. Risebro. A theory of L^1-dissipative solvers for scalar conservation laws with discontinuous flux. *Arch. Ration. Mech. Anal.*, 201(1):27–86, 2011.
3. E. Audusse and B. Perthame. Uniqueness for scalar conservation laws with discontinuous flux via adapted entropies. *Proc. Roy. Soc. Edinburgh Sect. A*, 135(2):253–265, 2005.
4. A. Bressan, S. Čanić, M. Garavello, M. Herty, and B. Piccoli. Flows on networks: recent results and perspectives. *EMS Surv. Math. Sci.*, 1(1):47–111, 2014.
5. R. M. Colombo, V. Perrollaz, and A. Sylla. Conservation Laws and Hamilton-Jacobi Equations with Space Inhomogeneity. working paper or preprint, Nov. 2022.
6. R. M. Colombo, V. Perrollaz, and A. Sylla. Initial data identification in space dependent conservation laws and Hamilton-Jacobi equations. Preprint, 2022.
7. M. G. Crandall and T. M. Liggett. Generation of semi-groups of nonlinear transformations on general Banach spaces. *Amer. J. Math.*, 93:265–298, 1971.
8. M. G. Crandall and L. Tartar. Some relations between nonexpansive and order preserving mappings. *Proc. Amer. Math. Soc.*, 78(3):385–390, 1980.
9. C. M. Dafermos. *Hyperbolic conservation laws in continuum physics*, volume 325 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, fourth edition, 2016.
10. M. Demazure. *Bifurcations and catastrophes*. Universitext. Springer-Verlag, Berlin, 2000. Geometry of solutions to nonlinear problems, Translated from the 1989 French original by David Chillingworth.
11. L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Textbooks in Mathematics. CRC Press, Boca Raton, FL, revised edition, 2015.
12. S. N. Kružkov. First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)*, 81 (123):228–255, 1970.
13. P.-L. Lions, B. Perthame, and E. Tadmor. A kinetic formulation of multidimensional scalar conservation laws and related equations. *J. Amer. Math. Soc.*, 7(1):169–191, 1994.
14. C. Mascia and A. Terracina. Large-time behavior for conservation laws with source in a bounded domain. *J. Differential Equations*, 159(2):485–514, 1999.
15. W. Neves, E. Panov, and J. Silva. Strong traces for conservation laws with general nonautonomous flux. *SIAM J. Math. Anal.*, 50(6):6049–6081, 2018.
16. V. Perrollaz. Asymptotic stabilization of entropy solutions to scalar conservation laws through a stationary feedback law. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 30(5):879–915, 2013.
17. B. Perthame and A. E. Tzavaras. Kinetic formulation for systems of two conservation laws and elastodynamics. *Arch. Ration. Mech. Anal.*, 155(1):1–48, 2000.
18. S. Sternberg. *Lectures on differential geometry*. Chelsea Publishing Co., New York, second edition, 1983. With an appendix by Sternberg and Victor W. Guillemin.