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# Conjectures analogous to the Collatz conjecture

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This paper is dedicated to the memory of Mario Bruschi

(08.08.1948-01.07.2021)

## *Abstract*

In this paper we introduce some conjectures analogous to the well-known Collatz conjecture.

## 1 Introduction

The Collatz conjecture (also called " $3n + 1$  conjecture", or with various other names) [1] [2] is, so far, neither *proven*, nor shown to be *unprovable*. It is considered "one of the most famous unsolved problems in mathematics" [1]. In this paper we introduce some analogous conjectures; and mention some related ideas and conjectures.

Mario Bruschi (MB) was a pupil (almost 13 years younger), and then friend, colleague and, over time, co-author of several scientific papers with the second author of this paper (FC); he was a very remarkable man, with many interests in life besides science; he also authored many scientific papers (as single author or with various colleagues); 2 of these papers were related to the Collatz conjecture [3, 4], and it was him who first brought the attention of FC to this topic, and who made the numerical calculations reported in a joint paper never published (the last typed copy of this paper is dated February 4th, 2019, and we now put it on arXiv [5]).

## 2 The Collatz conjecture

Consider the sequence of *positive integers*  $n_j$  ( $j = 0, 1, 2, \dots$ ) generated by the following 2 simple rules (hereafter we assume these rules to be applied *sequentially*, so that they generate a *uniquely* defined sequence of *positive integers*):  $n_{j+1} = n_j/2$  if  $n_j$  is *even*,  $n_{j+1} = 3n_j + 1$  if  $n_j$  is *odd*. The Collatz conjectures

states that, for *every* positive integer *initial* value  $n_0$ , this sequence ends up into the cycle  $\{1, 4, 2, 1\}$ . This conjecture is attributed to Lothar Collatz (1910-1990), who perhaps first thought of it in 1937 [2]; it has been checked (in 2020) for *all* positive integers numbers  $n_0$  up to  $2^{68} \approx 2.95 \cdot 10^{20}$ , and some computer experts continue to labor in order to increase this upper bound [1]; it is *obviously true* for certain *arbitrarily large* initial values (for instance for  $n_0 = 2^N$  with  $N$  any positive integer); but it has not yet been *proven* nor, of course, *disproved*, nor shown to be *unprovable*; recently it has been shown that it is *extremely probable* that this conjecture is *true*, in an appropriate mathematical framework where this statement has a *precise* meaning [6]. The reason why it might *not* be true is, of course, because there might exist an *initial* positive integer  $n_0$  such that the sequence generated by the simple rules detailed above either *diverges* or leads to a *different* cycle than the *single* one mentioned above.

Many analogous conjectures have been proposed over time (see for instance the book [7]). In this paper we mention some, possibly new, analogous *conjectures*.

### 3 The "5n + 1 Conjecture"

Consider the sequence of *positive integers*  $n_j$  ( $j = 0, 1, 2, \dots$ ) generated by the following 3 simple rules:  $n_{j+1} = n_j/2$  if  $n_j$  is *even*,  $n_{j+1} = n_j/3$  if  $n_j$  is *odd and divisible by 3*,  $n_{j+1} = 5n_j + 1$  if  $n_j$  is *odd and not divisible by 3*. The "5n + 1 Conjecture" states that, for *every* positive integer *initial* value  $n_0$ , this sequence ends up into the cycle  $\{1, 6, 3, 1\}$ . This conjecture is not new; we thought so when we wrote the previous version (<https://hal.science/hal-04046113>) of this paper; it is in fact sometimes called the *Takutani conjecture*, see for instance the Chapter entitled *Empirical verification of the 3x + 1 and related conjectures* by Tomás Oliveira e Silva (kindly made available to us by its author) in the book [7].

#### 3.1 A modified version of the "5n + 1 Conjecture"

Note that, if we eliminate the *second* rule namely we do *not* divide  $n_j$  by 3 if  $n_j$  is divisible by 3, we find several cycles: (e.g.  $\{3, 16, 8, 4, 2, 1, 6, 3\}$ ;  $\{5, 26, 13, 66, 33, 166, 83, 416, 208, 104, 52, 26\}$ ;  $\{13, 66, 33, 166, 83, 416, 208, 104, 52, 26, 13\}$ ); while starting from other numbers (e.g. 7, 11,...) the sequence *seems* to diverge.

### 4 The "7n + 1 Conjecture"

Consider the sequence of *positive integers*  $n_j$  ( $j = 0, 1, 2, \dots$ ) generated by the following 4 simple rules:  $n_{j+1} = n_j/2$  if  $n_j$  is *even*,  $n_{j+1} = n_j/3$  if  $n_j$  is *odd*

and divisible by 3,  $n_{j+1} = n_j/5$  if  $n_j$  is odd and divisible by 5,  $n_{j+1} = 7n_j + 1$  if  $n_j$  is odd and not divisible by 3 and 5. The "7n + 1 Conjecture" states that, for every positive integer initial value  $n_0$ , this sequence ends up into the cycle  $\{1, 8, 4, 2, 1\}$ . This conjecture is not new (we thought so when we wrote the previous version—see: <https://hal.science/hal-04046113>—of this paper); it is in fact sometimes also called the *Takutani conjecture*, see for instance again the Chapter by Tomás Oliveira e Silva in the book [7].

#### 4.1 A modified version of the "7n + 1 Conjecture"

Consider the sequence of positive integers  $n_j$  ( $j = 0, 1, 2, \dots$ ) generated by the following 3 simple rules:  $n_{j+1} = n_j/2$  if  $n_j$  is even,  $n_{j+1} = n_j/3$  if  $n_j$  is odd and divisible by 3,  $n_{j+1} = 7n_j + 1$  if  $n_j$  is odd and not divisible by 3 and 5. Then, for every positive integer initial value  $n_0$ , this sequence ends up into one of the two cycles  $\{1, 8, 4, 2, 1\}$  and  $\{19, 134, 67, 470, 235, 1646, 823, 5762, 2881, 20168, 10084, 5042, 2521, 17648, 8824, 4412, 2206, 1103, 7722, 3861, 1287, 429, 143, 1002, 501, 167, 1170, 585, 195, 65, 456, 228, 114, 57, 19\}$ . We checked this statement (only) for all positive integers  $n_0$  up to  $10^9$ . This is the first case we found, of a conjecture, somewhat analogous to the Collatz conjectures, which figures (only) 2 cycles.

## 5 The "11n + 1 Conjecture"

Following the same intuition that had led us to write the "5n + 1 Conjecture" and the "7n + 1 Conjecture", one might be tempted to propose many quite analogous conjectures (based on *primes*); even stating that, starting from any positive integer, one always reaches a unique cycle containing 1. However, the following observation disproves this naive idea (although some variations of this idea might merit further exploration; see below).

Indeed, consider the sequence of positive integer numbers  $n_j$  ( $j = 0, 1, 2, \dots$ ) generated by the following 5 simple rules:  $n_{j+1} = n_j/2$  if  $n_j$  is even,  $n_{j+1} = n_j/3$  if  $n_j$  is odd and divisible by 3,  $n_{j+1} = n_j/5$  if  $n_j$  is odd and divisible by 5,  $n_{j+1} = n_j/7$  if  $n_j$  is odd and divisible by 7,  $n_{j+1} = 11n_j + 1$  if  $n_j$  is odd and not divisible by 3, 5 and 7. The "the 11n + 1 Conjecture" states that, for every positive integer initial value  $n_0$ , this sequence ends up into one of the two cycles  $\{1, 12, 6, 3, 1\}$  and  $\{17, 188, 94, 47, 518, 259, 37, 408, 204, 102, 51, 17\}$ . We checked its validity (only) for all positive integers  $n_0$  up to  $10^9$ .

### 5.1 The "Modified 11n + 1 Conjecture"

Surprisingly, if we modify the "11n + 1 Conjecture" by eliminating the *fourth* rule, we seem to only get 1 cycle. Indeed, consider the sequence of positive integer numbers  $n_j$  ( $j = 0, 1, 2, \dots$ ) generated by the following 4 simple rules:

$n_{j+1} = n_j/2$  if  $n_j$  is *even*,  $n_{j+1} = n_j/3$  if  $n_j$  is *odd and divisible by 3*,  $n_{j+1} = n_j/5$  if  $n_j$  is *odd and divisible by 5*,  $n_{j+1} = 11n_j+1$  if  $n_j$  is *odd and not divisible by 3 and 5*. The "Modified  $11n+1$  Conjecture" states that, for *every* positive integer *initial* value  $n_0$ , this sequence ends up into the cycle  $\{1,12,6,3,1\}$ . We checked this statement (only) for *all positive integers* numbers  $n_0$  up to  $10^9$ .

## 6 Final remarks

Note that, for the "Modified  $7n+1$  conjecture" and the " $11n+1$  conjecture", one has 2 cycles. So—assuming these conjectures are indeed *true*—this entails the possibility to divide, in 2 different ways, *all positive integers* in 2 classes, identified by the property to generate—when used as initial value—a sequence that eventually ends up into one of these 2 cycles.

The results mentioned above clearly suggest the possibility to introduce *many*—indeed, presumably an *infinity*—of *somewhat analogous* conjectures to those described above. A very interesting *open* issue is whether the *hypothesis* that one of these conjecture be *true* (or, alternatively, *false*) might be shown to imply *that another* one of these conjectures is *as well true* (or, alternatively, *false*); and whether progress in this direction might eventually provide an avenue to finally *prove* or *disprove* the validity of *some*, or even of *all*, these conjectures.

Let us end this paper by recalling the simple idea [5], that if the Collatz conjecture is *true*, then the *3-times iterated Collatz procedure* (as described in **Section 2**) implies that *all* positive integers can be divided into 3 *distinct categories*: those such that, when taken as *initial values* of the sequence generated by that procedure, entail that it converge eventually to 1, or to 2 or to 4; hence the related conjecture —relevant of course only if the Collatz conjecture is true—that these 3 categories of integers are *equally* populated, a statement meant to be true in the same sense as the assertion that there are *as many even positive integers as odd positive integers* [5]. Clearly analogous ideas and conjectures can be associated to all the conjectures formulated above.

## References

- [1] See the item "Collatz conjecture" in Wikipedia, the References quoted there, and, if need be, the References quoted in those References (the full list would add up to *several hundred* items...).
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