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Conjectures analogous to the Collatz conjecture

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This paper is dedicated to the memory of Mario Bruschi (08/08/2048-01/07/2021).

Abstract

In this paper we introduce some conjectures analogous to the well-known Collatz conjecture, and mention some other ideas and conjectures.

1 Introduction

The Collatz conjecture (also called " $3n + 1$ " conjecture, or with various other names) [1, 2] is, so far, neither *proven*, nor shown to be *unprovable*. It is considered "one of the most famous unsolved problems in mathematics" [1]. In this paper we introduce some analogous conjectures—both some which are so far also *open*, and some which are instead easily shown to be *false*; and some related ideas and conjectures.

In **Section 2** we tersely describe the Collatz conjecture. In **Section 3** we introduce and discuss the " $5n + 1$ Conjecture". In **Section 4** we introduce and discuss the " $7n + 1$ Conjecture". In **Section 5** we introduce and discuss the " $11n + 1$ Conjecture". The title of **Section 6** is "Final remarks".

Mario Bruschi was a pupil (almost 13 years younger), and then friend, colleague and, over time, co-author of several scientific papers with the second author of this paper (FC); he was a very remarkable man, with many interests in life besides science; he also authored many scientific papers (as single author or with various colleagues); 2 of these papers were related to the Collatz conjecture [3, 4], and it was him who first brought the attention of FC to this topic, and who made the numerical calculations reported in a paper never published (the last typed copy of this paper is dated February 4th, 2019, and we now put it on arXiv [5]).

2 The Collatz conjecture

Consider the sequence of *positive integer numbers* n_j ($j = 0, 1, 2, \dots$) generated by the following 2 simple rules: $n_{j+1} = n_j/2$ if n_j is *even*, $n_{j+1} = 3n_j + 1$ if n_j is *odd*. The Collatz conjecture states that, for *every* positive integer *initial* value n_0 , this sequence ends up into the cycle $\{1, 4, 2, 1\}$. This conjecture is attributed to Lothar Collatz (1910-1990), who perhaps first thought of it in 1937 [2]. It has been checked for *all* positive integers numbers n_0 up to 10^{20} [6], and some computer experts continue to labor in order to increase this upper bound. It is *obviously true* for certain *arbitrarily large* numbers (for instance for 2^N with N *any positive integer*), but it has not yet been proven nor, of course, disproved, nor shown to be *unprovable*. Recently it has been shown that it is *extremely probable* that this conjecture is true—in an appropriate mathematical framework where this statement has a precise meaning [6]. But let us also note that, whenever the Collatz sequence of positive integer numbers characterized by the rules detailed just above is shown to end up—or viceversa *not* to end up—into the cycle $\{1, 4, 2, 1\}$ when starting from a specific positive integer number n_0 , then the same conclusion is clearly as well *true*—or as well *false*—for *every analogous* sequence starting from the *infinite* set of initial integer numbers $n_0 \cdot 2^N$ with N *any* arbitrary positive integer, and n_0 up to 10^{20} . Hence the Collatz conjecture is either *true* for *all* initial positive integers n_0 , or viceversa it is *false* for an *infinity* of initial positive integers n_0 . And analogous statements are clearly as well applicable—*mutatis mutandis*—to *all* the analogous conjectures discussed below.

Many analogous conjectures have been proposed over time, but none of them seems to us to have the same status of *neatness—simplicity* and yet *inscrutability*—as the *original* Collatz conjecture.

In this paper we mention some, presumably new, analogous *conjectures*.

3 The "5n + 1 Conjecture"

Consider the sequence of *positive integer numbers* n_j ($j = 0, 1, 2, \dots$) generated by the following 3 simple rules: $n_{j+1} = n_j/2$ if n_j is *even*, $n_{j+1} = n_j/3$ if n_j is

odd and divisible by 3, $n_{j+1} = 5n_j + 1$ if n_j is *odd and not divisible by 3* (just for definiteness, here and hereafter, let us assume that these rules are applied sequentially). The "5n + 1 Conjecture" states that, for *every* positive integer *initial* value n_0 , this sequence ends up into the cycle $\{1, 6, 3, 1\}$. We checked its validity—via a PC (using—here and hereafter—an appropriate routine of **Mathematica**)—for *all positive integers* numbers n_0 up to 10^9 . It is of course also true for *all* initial values $n_0 \cdot 2^{N_2} \cdot 3^{N_3}$ with N_2 and N_3 *arbitrary positive integers*, and n_0 up to 10^9 .

3.1 A modified version of the "5n + 1 Conjecture"

It is worth mentioning that, if we eliminate the *second* rule namely we do *not* divide n_j by 3 if n_j is divisible by 3, we have some additional cycles (e.g. $\{1, 6, 3, 16, 8, 4, 2, 1\}$; $\{13, 66, 33, 166, 83, 416, 208, 104, 52, 26, 13\}$ which is already reached starting from the initial value 5; $\{27, 136, 68, 34, 17, 86, 43, 216, 108, 54, 27\}$), while starting from other numbers (e.g. 7, 11, etc.) the sequence seems to diverge.

4 The "7n + 1 Conjecture"

Consider the sequence of *positive integer numbers* n_j ($j = 0, 1, 2, \dots$) generated by the following 4 simple rules: $n_{j+1} = n_j/2$ if n_j is *even*, $n_{j+1} = n_j/3$ if n_j is *odd and divisible by 3*, $n_{j+1} = n_j/5$ if n_j is *odd and divisible by 5*, $n_{j+1} = 7n_j + 1$ if n_j is *odd and not divisible by 3 and 5*. The "7n + 1 Conjecture" states that, for *every* positive integer *initial* value n_0 , this sequence ends up into the cycle $\{1, 8, 4, 2, 1\}$. We checked its validity for *all positive integers* numbers n_0 up to 10^9 . It is of course also true for *all* initial values $n_0 \cdot 2^{N_2} \cdot 3^{N_3} \cdot 5^{N_5}$ with N_2 , N_3 and N_5 *arbitrary positive integers*, and n_0 up to 10^9 .

4.1 A modified version of the "7n + 1 Conjecture"

It is worth mentioning that, if we eliminate the *third* rule and do not divide n_j by 5 if it is divisible by 5, we find (only!) 2 cycles. Indeed, consider the sequence of *positive integers* n_j ($j = 0, 1, 2, \dots$) generated by the following 3 simple rules: $n_{j+1} = n_j/2$ if n_j is *even*, $n_{j+1} = n_j/3$ if n_j is *odd and divisible by 3*, $n_{j+1} = 7n_j + 1$ if n_j is *odd and not divisible by 3*. Then, for *every* positive integer *initial* value n_0 , this sequence ends up into one of the 2 cycles $\{1, 8, 4, 2, 1\}$; $\{19, 134, 67, 470, 235, 1646, 823, 5762, 2881, 20168, 10084, 5042, 2521, 17648, 8824, 4412, 2206, 1103, 7722, 3861, 1287, 429, 143, 1002, 501, 167, 1170, 585, 195, 65, 456, 228, 114, 57, 19\}$. We checked this statement for *all positive integers* n_0 up to 10^9 .

5 The "11n + 1 Conjecture"

Following the same idea that led us to write the "5n + 1 Conjecture" and the "7n + 1 Conjecture", one might be tempted to propose many quite analogous conjectures; even stating that, starting from *any* positive integer, one always reaches a *unique* cycle containing 1. However, the following observation disproves this naive idea (although some of variations of this idea might merit further exploration).

Indeed, consider the sequence of *positive integer numbers* n_j ($j = 0, 1, 2, \dots$) generated by the following 5 simple rules: $n_{j+1} = n_j/2$ if n_j is *even*, $n_{j+1} = n_j/3$ if n_j is *odd and divisible by 3*, $n_{j+1} = n_j/5$ if n_j is *odd and divisible by 5*, $n_{j+1} = n_j/7$ if n_j is *odd and divisible by 7*, $n_{j+1} = 11n_j + 1$ if n_j is *odd and not divisible by 3, 5 and 7*. The "the 11n + 1 Conjecture" states that, for *every* positive integer *initial value* n_0 , this sequence ends up into one of the 2 cycles $\{1, 12, 6, 3, 1\}$ and $\{17, 188, 94, 47, 518, 259, 37, 408, 204, 102, 51, 17\}$. We checked its validity for *all positive integers* n_0 up to 10^9 .

5.1 The "Modified 11n + 1 Conjecture"

Surprisingly, if we modify the "11n + 1 Conjecture" by eliminating the *fourth* rule, we seem to only get 1 cycle. Indeed, consider the sequence of *positive integer numbers* n_j ($j = 0, 1, 2, \dots$) generated by the following 4 simple rules: $n_{j+1} = n_j/2$ if n_j is *even*, $n_{j+1} = n_j/3$ if n_j is *odd and divisible by 3*, $n_{j+1} = n_j/5$ if n_j is *odd and divisible by 5*, $n_{j+1} = 11n_j + 1$ if n_j is *odd and not divisible by 3 and 5*. The modified "Modified 11n + 1 Conjecture" states that, for *every* positive integer *initial value* n_0 , this sequence ends up into the cycle $\{1, 12, 6, 3, 1\}$. We checked this statement for *all positive integers* n_0 up to 10^9 . This conjecture is of course also *true* for *all* initial values $n_0 \cdot 2^{N_2} \cdot 3^{N_3} \cdot 5^{N_5}$ with N_2, N_3 and N_5 *arbitrary positive integers*, and n_0 up to 10^9 .

6 Final remarks

Note that, for the "Modified 7n+1 conjecture" and the "Modified 11n+1 conjecture", we found 2 cycles. So—assuming these conjectures are indeed *true*—this entails the possibility to divide, in 2 different ways, *all positive integers* in 2 classes, identified by the property to generate—when used as *initial value*—a sequence that eventually ends up into one or the other of these 2 cycles.

The numerical check of the "5n + 1 Conjecture" that we have performed is based on the following observation: suppose that it has been verified that, for *every positive integer initial value* $n_0 < N$, with N a positive integer, the sequence generated by the "5n + 1 Conjecture" ends up into the cycle $\{1, 6, 3, 1\}$. If, taking $n_0 = N$ as initial value, one finds a number $n_k < n_0$ after k iterations, the sequence generated by $n_0 = N$ ends up into the same cycle $\{1, 6, 3, 1\}$. This also implies that the number n_0 giving rise to *another* cycle would necessarily

be the *smallest number* of the *new cycle*. Of course the *minimum* number of iterations k such that $n_k < n_0$ does depend on n_0 , but its value is irrelevant in this context.

Indeed, one can check that in the " $5n + 1$ Conjecture" in **Section 3** there are no other cycles than $\{1,6,3,1\}$ for n_0 up to the integer $N > 1$ simply by verifying that, starting with *any initial value* n_0 ($n_0 = 2, 3, \dots, N$) such that $1 < n_0 \leq N$, one finds an *integer* $n_k < n_0$ ($k = 1, 2, 3, \dots$) after a sufficient number k of iterations. The computational advantage of this procedure is that one does not have to calculate all the sequence, but one stops the iteration as soon as a value $n_k < n_0$ is found. Moreover, one can exclude from the check *all* the initial values n_0 that are *multiples* of 2 or of 3, as they would immediately give an $n_1 = n_0/2 < n_0$ or $n_1 = n_0/3 < n_0$.

This method can be generalized to the cases of the " $7n + 1$ Conjecture" in **Subsection 4.1** and the " $11n + 1$ Conjecture" in **Subsection 5.1** " $11n + 1$ " which seem to feature only 2 cycles. Let us consider for instance the " $11n + 1$ ". Suppose that we have verified that any sequence starting with $n_0 \leq 17$ ends up into one of the two cycles $\{1,12,6,3,1\}$ and $\{17,188,94,47,518,259,37,408,204,102,51,17\}$. Then, if we verify that, for any initial value $17 < n_0 \leq N$ one has $n_k < n_0$ after a sufficient number of iterations k , we can conclude that there are no other cycles for any initial value $n_0 \leq N$.

Note that, in *all* the cases identified above, there obviously are *infinitely many, explicitly identifiable, positive integers* which—when chosen as *initial values* generating the relevant sequence—eventually cause that sequence to end up in the corresponding cycle, as explicitly pointed out at the end of **Sections 3** and **4**.

The results mentioned above clearly *suggest* the possibility to introduce *many*—indeed, possibly an *infinity*—of *somewhat analogous* conjectures to those described above. A very interesting *open* issue is whether the *hypothesis* that one of these conjecture be *true* (or, alternatively, *false*) might be shown to imply *that another* one of these conjectures is *as well true* (or, alternatively, *false*); and whether progress in this direction might eventually provide an avenue to finally *prove* or *disprove* the validity of *some*, or even of *all*, these conjectures.

Let us end this paper by recalling the simple idea [5], that if the Collatz conjecture is *true*, then the *3-times iterated Collatz procedure* (as described in **Section 2**) implies that *all* positive integers can be divided into 3 *distinct categories*: those such that, when taken as *initial values* of the sequence generated by that procedure, entail that it converge eventually to 1, or to 2 or to 4; hence the related conjecture—relevant of course only if the Collatz conjecture is true—that these 3 categories of integers are *equally* populated, a statement meant to be true in the same sense as the assertion that there are *as many even positive integers as odd positive integers* [5]. Clearly analogous ideas and conjectures can be associated to all the conjectures formulated above.

References

- [1] See the item "Collatz conjecture" in Wikipedia, the References quoted there, and, if need be, the References quoted in those References (the full list would add up to *several hundred* items...).
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