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# A MINIMAL MASS BLOW-UP SOLUTION ON A NONLINEAR QUANTUM STAR GRAPH

FRANÇOIS GENOUD, STEFAN LE COZ, AND JULIEN ROYER

ABSTRACT. The main contribution of this article is the construction of a finite time blow-up solution to the mass-critical focusing nonlinear Schrödinger equation set on a metric star graph  $\mathcal{G}$  with  $N$  edges, for any  $N \geq 2$ . After establishing well-posedness of the corresponding Cauchy problem in  $H^1(\mathcal{G})$ , we obtain the sharp threshold for global existence in terms of the mass of the ground state, called minimal mass. We then construct a minimal mass solution which blows up in finite time at the vertex of  $\mathcal{G}$ . The blow-up profile and blow-up speed are characterized explicitly.

## 1. INTRODUCTION

Let  $\mathcal{G}$  be a *metric star graph of size  $N$* , i.e. a vertex  $v$  to which are connected  $N$  edges  $e_1, \dots, e_N$ , each edge being a half-line. We thus identify each edge  $e$  with the interval  $I_e = [0, \infty)$ , the point 0 corresponding to the vertex. We shall use  $v$  or 0 interchangeably to denote the vertex. A schematic representation of a star graph is given in Figure 1.

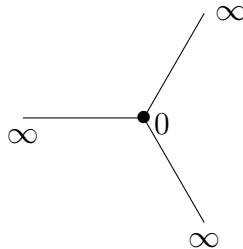


FIGURE 1. A star graph with 3 edges.

A function  $u$  on  $\mathcal{G}$  is a collection of functions  $u_j : I_{e_j} \rightarrow \mathbb{C}$ ,  $j = 1, \dots, N$ . Letting  $\mathbb{R}_+ = [0, \infty)$ , a point  $x \in \mathcal{G}$  will be identified with  $(x_1, \dots, x_N) \in \mathbb{R}_+^N$ . Thus,  $u : \mathcal{G} \rightarrow \mathbb{C}$  can be described as

$$u(x) = (u_1(x_1), \dots, u_N(x_N)), \quad x_j \in \mathbb{R}_+, \quad j = 1, \dots, N.$$

A function  $u : \mathcal{G} \rightarrow \mathbb{C}$  will be called *radial* if all its components  $u_j : \mathbb{R}_+ \rightarrow \mathbb{C}$  are identical. In this case,  $x \in \mathcal{G}$  will be identified with  $x \in \mathbb{R}_+$ , and we will simply interpret  $u$  as a complex-valued function of  $x \in \mathbb{R}_+$ .

Next, we define on  $\mathcal{G}$  the formal Hamiltonian

$$H_\gamma = -\partial_{xx} + \gamma\delta,$$

where  $\partial_{xx}$  acts as the one-dimensional Laplacian on each edge and the delta potential  $\gamma\delta$  encodes Robin boundary conditions at the vertex. The coupling constant  $\gamma$  is real, ensuring that  $H_\gamma$

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is selfadjoint; see Section 1.3 for a precise definition of  $H_\gamma$ . Our graph  $\mathcal{G}$ , equipped with the Hamiltonian  $H_\gamma$ , is a model case of a *quantum graph*; see e.g. [17] and references therein.

In this paper, we are interested in a *nonlinear quantum graph*, namely, we equip the quantum graph  $(\mathcal{G}, H_\gamma)$  with the focusing nonlinear Schrödinger equation

$$iu_t - H_\gamma u + |u|^4 u = 0, \quad (1.1)$$

where  $u = u(t, x)$ ,  $t \in \mathbb{R}$ ,  $x \in \mathcal{G}$ , is complex-valued. The *energy*

$$E(u) := q_\gamma(u) - \frac{1}{6} \|u\|_{L^6(\mathcal{G})}^6$$

and the *mass*

$$M_{\mathcal{G}}(u) := \frac{1}{2} \|u\|_{L^2(\mathcal{G})}^2$$

are the main conserved quantities associated with this nonlinear dynamical system. The usual function spaces on  $\mathcal{G}$  will be defined in Section 1.3, as well as the quadratic form  $q_\gamma$  appearing in the energy.

Our goal in this paper is to construct a finite time blow-up solution of (1.1). Let us denote by  $Q$  the positive radial ground state of the focusing mass-critical nonlinear Schrödinger equation on the line (see Section 1.2). It is well-known that the mass

$$M_{\mathbb{R}}(Q) := \frac{1}{2} \int_{\mathbb{R}} Q^2 dx$$

of  $Q$  gives the threshold between global existence and finite time blow-up for this problem. It turns out that  $M_{\mathbb{R}}(Q)$  also determines the mass threshold for global existence of solutions of (1.1) on the graph. Indeed, we shall prove in Section 3.2 that, if the initial condition  $u_0 \in H^1(\mathcal{G})$  satisfies

$$M_{\mathcal{G}}(u_0) < \min\left\{\frac{N}{2}, 1\right\} M_{\mathbb{R}}(Q),$$

then the corresponding solution of (1.1) is global. Furthermore, if we restrain ourselves to radial solutions, then the threshold for global existence becomes

$$M_{\mathcal{G}}(Q) = \frac{N}{2} M_{\mathbb{R}}(Q), \quad (1.2)$$

where, for simplicity, we have also denoted by  $Q$  the function on the star graph consisting of  $N$  half-copies of  $Q$  on each edge. In the attractive case  $\gamma < 0$ , we construct a radial minimal mass blow-up solution, that is, a solution with mass (1.2) which blows up in finite time at the vertex  $v$ .

More precisely, our main result is the following theorem.

**Theorem 1.1.** *Suppose  $\gamma < 0$ . Let  $E^* \in \mathbb{R}$ . There exist  $t_0 < 0$  and a solution  $u \in C([t_0, 0), H_{\text{rad}}^1(\mathcal{G}))$  of (1.1) such that*

$$M_{\mathcal{G}}(u) = M_{\mathcal{G}}(Q) \quad \text{and} \quad E(u) = E^*,$$

which blows up at  $t = 0$  as

$$\lim_{t \rightarrow 0^-} |t|^{2/3} \|u_x(t)\|_{L^2(\mathcal{G})} = C, \quad (1.3)$$

for a constant  $C > 0$ .

**1.1. Remarks on Theorem 1.1.** To the best of our knowledge, the present work is the first *construction* of a finite time blow-up solution for a nonlinear Schrödinger equation on a quantum graph. By contrast, for NLS equations on  $\mathbb{R}^N$  a substantial literature exists, which is partly reviewed in Section 1.4. Beside the particular blow-up solution given by Theorem 1.1, we conjecture there exist solutions which blow up in finite time outside of  $v$  as a translate of the pseudo-conformal solution  $S(t)$  defined in (1.5); such solutions should exist irrespective of the sign of  $\gamma$ .

For  $\gamma = 0$ , one may construct an explicit finite time blow-up solution on  $\mathcal{G}$  by simply putting  $N$  half-copies of  $S(t)$  on the edges. Since  $\gamma = 0$ , the function thus constructed verifies the required compatibility condition at the vertex (see (1.7)), is a solution of (1.1), and blows up at time  $T = 0$  at the vertex with pseudo-conformal speed (1.6). When  $\gamma < 0$ , one cannot construct a simple solution based on  $S(t)$ , but we believe that solutions which blow up in finite time *by concentrating their mass at  $v$*  should also be governed by the symmetric profile  $Q$  in this case. In the repulsive case  $\gamma > 0$ , there are no solutions blowing up in finite time at  $v$  with minimal mass; see Section 3.2.

Our approach to prove Theorem 1.1 will follow the strategy laid down in [37,50] with some improvements obtained by Matsui [40–46]. In particular, we shall work directly in the virial space, which allows us to avoid the localization procedure of the virial-energy functional used in [37]. We have also reformulated the blow-up profile expansion borrowed from [37], thereby making it more tractable for the proofs. In the particular case of the 2-star graph, we recover the result obtained in [52] for the mass-critical NLS on the line with an attractive delta potential.

**1.2. The mass-critical NLS on the line.** Consider the classical mass-critical nonlinear Schrödinger equation on the line

$$iu_t + u_{xx} + |u|^4u = 0. \tag{1.4}$$

Let us recall a few well-known facts about this equation. The Cauchy problem for (1.4) is well-posed in the energy space  $H^1(\mathbb{R})$ , we have conservation of energy, mass (and momentum) and the blow-up alternative holds. Of particular interest is the standing wave solution  $e^{it}Q(x)$ , where the profile  $Q$  is explicitly given by

$$Q(x) = 3^{\frac{1}{4}} \operatorname{sech}^{\frac{1}{2}}(2x)$$

and is the unique even positive solution in  $H^1(\mathbb{R})$  of the differential equation

$$-Q'' + Q - Q^5 = 0.$$

The mass of  $Q$  gives the threshold between global existence and blow-up: any solution of (1.4) with  $M_{\mathbb{R}}(u_0) < M_{\mathbb{R}}(Q) = \pi\sqrt{3}/4$  is global, whereas there exists a minimal blow-up mass solution, i.e. a solution such that  $M_{\mathbb{R}}(u_0) = M_{\mathbb{R}}(Q)$  and the associated solution of (1.4) blows up in finite time. It turns out that such a solution can be found by an explicit pseudo-conformal transform of the standing wave. Indeed, let

$$S(t, x) = \frac{1}{\sqrt{|t|}} Q\left(\frac{x}{|t|}\right) e^{-i\frac{|x|^2}{4|t|}} e^{it}, \tag{1.5}$$

then  $S$  is a solution of (1.4) and we have

$$\|S(t)\|_{L^2(\mathbb{R})} = \|Q\|_{L^2(\mathbb{R})}, \quad \|\partial_x S(t)\|_{L^2} \sim \frac{1}{|t|} \|Q'\|_{L^2}, \quad t \rightarrow 0^-. \tag{1.6}$$

In particular,  $S$  blows up at  $t = 0$  with the so-called *pseudo-conformal speed*  $|t|^{-1}$ . Furthermore, up to the symmetries of the equation,  $S$  is the unique minimal mass blow-up solution (see [47]).

1.3. **Functional setting on  $\mathcal{G}$ .** Lebesgue and Sobolev spaces on  $\mathcal{G}$  are defined by

$$L^p(\mathcal{G}) = \bigoplus_{j=1}^N L^p(I_{e_j}), \quad H^s(\mathcal{G}) = \bigoplus_{j=1}^N H^s(I_{e_j}),$$

with norms

$$\|u\|_{L^p(\mathcal{G})}^p = \sum_{j=1}^N \|u_j\|_{L^p(\mathbb{R}_+)}^p, \quad \|u\|_{H^s(\mathcal{G})}^2 = \sum_{j=1}^N \|u_j\|_{H^s(\mathbb{R}_+)}^2.$$

Observe that no compatibility condition is imposed at the vertex. We introduce below the notation  $H_D^1(\mathcal{G})$  for the space of functions of  $H^1(\mathcal{G})$  which are continuous at the vertex.

We equip  $\mathcal{G}$  with the Laplace operator with Dirac condition at the vertex, i.e. the Hamiltonian operator  $H_\gamma$  defined by

$$H_\gamma : D(H_\gamma) \subset L^2(\mathcal{G}) \rightarrow L^2(\mathcal{G}), \\ (u_1, \dots, u_N) \mapsto (-\partial_{xx}u_1, \dots, -\partial_{xx}u_N),$$

where the domain  $D(H_\gamma)$  is defined by

$$D(H_\gamma) = \left\{ u \in H^2(\mathcal{G}) : \forall j, k = 1, \dots, N, u(0) := u_j(0) = u_k(0), \sum_{j=1}^N u_j'(0) = \gamma u(0) \right\}. \quad (1.7)$$

We observe that the domain contains a continuity condition at 0 and a jump condition for the derivatives. For  $\gamma = 0$  we recover the classical Kirchhoff-Neumann conditions. For  $\gamma \neq 0$  and  $N = 2$ , we recover the case of the line with a  $\delta$  potential at 0.

The quadratic form associated with  $H_\gamma$  is

$$q_\gamma(u) := \langle H_\gamma u, u \rangle = \frac{1}{2} \sum_{j=1}^N \|u_j'\|_{L^2(\mathbb{R}_+)}^2 + \frac{\gamma}{2} |u(0)|^2,$$

defined on the domain

$$H_D^1(\mathcal{G}) = D(q_\gamma) := \{u \in H^1(\mathcal{G}) : \forall j, k = 1, \dots, N, u(0) := u_j(0) = u_k(0)\}.$$

Observe that the domain of the quadratic form retains the continuity at the vertex, but the jump condition on the derivatives now appears in the expression of the quadratic form instead of appearing in the expression of the domain. In this paper, we will mostly work on a subspace of  $H_D^1(\mathcal{G})$ , the space  $H_{\text{rad}}^1(\mathcal{G})$  of functions which are symmetric with respect to the vertex. Namely,

$$H_{\text{rad}}^1(\mathcal{G}) = \{u \in H^1(\mathcal{G}) : \forall j, k = 1, \dots, N, u_j = u_k\}.$$

We will also use the notation  $H_\gamma$  for the operator from  $H_D^1(\mathcal{G})$  to  $H^{-1}(\mathcal{G})$ . In particular, we may write

$$H_\gamma = -\partial_{xx} + \gamma\delta,$$

where it is understood that  $-\partial_{xx}$  denotes the second derivative on each edge of the graph and  $\delta$  is defined by the duality pairing  $\langle \delta u, v \rangle = u(0)v(0)$ . This allows us to split the operator  $H_\gamma$  into two parts,  $-\partial_{xx}$  and  $\gamma\delta$ , whenever needed. We emphasize that whenever  $-\partial_{xx}$  and  $\gamma\delta$  are treated separately, they are always taken in the  $H^1 - H^{-1}$  sense (the operator  $H_\gamma$  as an  $L^2 - L^2$  operator with domain cannot be split).

**1.4. History of construction of minimal mass blow-up solutions.** For the classical pure power mass-critical nonlinear Schrödinger equation on  $\mathbb{R}^N$ , a minimal blow-up mass solution is explicitly obtained from a pseudo-conformal transform of a standing wave, in any dimension, similarly to (1.5) for  $N = 1$ . In the seminal paper [47], Merle showed that it is the unique minimal blow-up mass solution up to the symmetries of the equation. Existence and uniqueness of a minimal mass blow-up solution for NLS equations which do not possess pseudo-conformal symmetry is more involved. The study was initiated by Merle himself in [48], where he established a sufficient condition for the existence of a minimal mass blow-up solution in the case of a Schrödinger equation with inhomogeneous mass-critical nonlinearity  $k(x)|u|^{\frac{4}{d}}u$ . Further contributions (see e.g. Banica, Carles, Duyckaerts [15], Bourgain and Wang [21], Krieger and Schlag [35]) treated the problem perturbatively from the homogeneous case, and required a flatness assumption on  $k$ . A nonperturbative approach was called for in order to remove the flatness assumption. The breakthrough came from the work of Raphaël and Szeftel [50], in which existence and uniqueness of a minimal mass blow-up solution for the inhomogeneous mass-critical nonlinearity was established. The approach of [50] is very robust and was used for instance by Krieger, Lenzmann and Raphaël [34] in the case of the critical half-wave equation, or by Martel and Pilod [39] for the Benjamin-Ono equation. The construction of the profile of the minimal mass blow-up solution was later refined by Le Coz, Martel, Raphaël [37] in the case of the nonlinear Schrödinger equation with a double power nonlinearity, where a minimal mass solution exhibiting a new blow-up speed was constructed. The approach of [37,50] was successfully implemented by Matsui [40–44,46] for various Schrödinger equations (e.g. with singular potentials or with a Hartree nonlinearity). Several improvements to the work [37] have been made by Matsui, in particular the observation that the blow-up profile is more naturally constructed in the virial space instead of  $H^1$ . Recently, the paper [37] was transposed by Tang and Xu [52] to the nonlinear Schrödinger equation on the line with a delta potential at the origin.

**1.5. Earlier results on star graphs.** There is now a wide literature on nonlinear quantum graphs cannot be shortly summarized. For an introduction to nonlinear Schrödinger equations on quantum graphs and their physical motivations, one may refer to the survey of Noja [49]. For star graphs in particular, one may refer to the recent monograph of Angulo Pava and Cavalcante de Melo [10]. In this introduction, we will only present the results close to our work, along with a very partial sample of the rest of the literature. Many of the works devoted to nonlinear quantum graphs focus on existence and variational characterizations of standing waves. Among the earliest studies, one finds the works by Fukuizumi in collaboration with (separately) Jeanjean, Le Coz, Ohta and Ozawa [24,25,36], which are devoted to the case of a line with a delta potential at the origin (equivalent to a 2-star graph). The first author, together with Malomed and Weisshäupl [26], studied orbital stability of standing waves for the 2-star graph with a cubic-quintic nonlinearity. The variational characterization of standing waves on star graphs was considered by Adami, Cacciapuoti, Finco and Noja [2–5]. Further developments for the study of standing waves on generic quantum graphs started with Adami, Serra and Tilli [6–8], where a topological obstruction for the existence of ground states on quantum graphs was discovered. Elements such as well-posedness of the Cauchy problem, Strichartz estimates and conservation laws on star graphs can be found in the work of Adami, Cacciapuoti, Finco and Noja [1] (along with the analysis of the collision of a fast solitary wave with the vertex, which is the main object of the paper). The 2-star graph with non-zero boundary conditions has been investigated by Ianni, Le Coz and Royer [31]. The case of a loop (which is equivalent to a segment with periodic boundary conditions) was studied by Gustafson, Le Coz and Tsai [30]. Absence of scattering of global solutions towards standing

waves was established by Aoki, Inui, Mizutani [13], while scattering on the 2-star graph was obtained by Banica and Visciglia [16]. Exponential stability in the presence of damping on one branch was obtained by Ammari, Bchatnia and Mehenaoui [9]. Existence of ground states on star graphs with finite and infinite edges was studied by Li, Li and Shi [38]. On balanced star graphs (i.e. star graphs with adjusted coefficients on the edges, see [51]), Kairzhan, Pelinovsky and Goodman [33] proved the nonlinear instability (by drift) of spectrally stable (see [33]) shifted states. Standing waves of the nonlinear Schrödinger equation with logarithmic nonlinearity was considered by Goloshchapova [27] (see also the earlier work of Ardila [14] for well-posedness and existence results). Instability of non-ground state standing waves on star graphs was obtained by Kairzhan [32] in the repulsive and attractive cases. Instability by blow-up of standing waves on star graphs for mass-supercritical nonlinearities was proved by Goloshchapova and Ohta [29]. Stability and instability results were obtained by Angulo Pava and Goloshchapova [11,12] using the extension theory of symmetric operators for star graphs with  $\delta$  or  $\delta'$  interaction at the vertex. Star graphs with  $\delta'_s$  conditions were considered by Goloshchapova in [28]. Recently, Besse, Duboscq and Le Coz [18,19] developed a Python Library [20] for the numerical simulation of Schrödinger equations on quantum graphs. A numerical approach for the calculations of ground states is studied in [18] whereas the implementation of the library and further experiments are presented in [19].

The rest of the paper is organized as follows. Section 2 provides a detailed outline of the construction of our blow-up solution. The proof of Theorem 1.1 is given there, assuming a number of propositions. Section 3 presents some basic results underlying the whole analysis: the Cauchy theory for (1.1) and some detailed properties of linearized operators. In Sections 4 to 7, the propositions used in the proof of Theorem 1.1 are proved.

**Notation.** We shall write  $f \lesssim g$  or  $g \gtrsim f$  to mean that there is a universal constant  $C > 0$  (i.e. which does not depend on the dynamical variables) such that  $f \leq Cg$ . We will write  $f \sim g$  as  $t \rightarrow 0^-$  (or  $s \rightarrow +\infty$ ) if  $f/g \rightarrow 1$  as  $t \rightarrow 0^-$  (or  $s \rightarrow +\infty$ ). When no confusion is possible, we may simply write  $L^2, H^1$ , etc. instead of  $L^2(\mathcal{G}), H^1(\mathcal{G})$ , etc.

## 2. OUTLINE OF THE PROOF

The approach we adopt is *via* a change of variables transforming a finite time blow-up solution into a solution that is global in positive time. We seek a radial solution  $u$  of (1.1) in the form of

$$u(t, x) = \frac{1}{\sqrt{\lambda(s)}} w(s, y) e^{i(\theta(s) - b(s)y^2/4)}, \quad t < 0, \quad x \in \mathbb{R}_+, \quad (2.1)$$

where the new variables  $s$  and  $y$  satisfy

$$\frac{ds}{dt} = \frac{1}{\lambda(s)^2}, \quad y = \frac{x}{\lambda(s)}. \quad (2.2)$$

We will construct  $w$  global and bounded in  $H_D^1(\mathcal{G})$ , together with *modulation parameters*  $\lambda(s), b(s)$  and  $\theta(s)$  such that  $\lambda(s) > 0$ ,

$$\lambda(s) \rightarrow 0^+, \quad b(s) \rightarrow 0^+, \quad \theta(s) \rightarrow 0, \quad s \rightarrow +\infty.$$

This type of ansatz is common in blow-up analysis (see the references in the introduction for similar constructions). The exact definition of the rescaled time  $s$  will appear in Section 6. By

straightforward calculations,  $u$  solves (1.1) if and only if  $w$  solves

$$iw_s - H_{\lambda\gamma}w - w + |w|^4w + (1 - \theta_s)w + \left(b_s - b^2 - 2b\frac{\lambda_s}{\lambda}\right)\frac{y^2}{4}w - i\left(b + \frac{\lambda_s}{\lambda}\right)\Lambda w = 0, \quad (2.3)$$

where the scaling operator  $\Lambda$  is defined for each component  $w_j$  of  $w$  by

$$\Lambda w_j(y_j) = \frac{1}{2}w_j(y_j) + y_j w_j'(y_j), \quad j = 1, \dots, N.$$

Our strategy to prove Theorem 1.1 is to seek  $w$  of the form

$$w(s, y) = P(s, y) + h(s, y),$$

for a suitable approximate solution profile  $P$  constructed using the ground state  $Q$  and the dynamical parameters  $\lambda(s)$  and  $b(s)$ . The result will then follow from (2.1) and (2.2) by proving that  $\lambda(s) \sim s^{-2}$  and  $h(s) \rightarrow 0$  in a well-chosen norm, as  $s \rightarrow +\infty$ .

The blow-up profile  $P$  is constructed as an approximate solution of the auxiliary equation

$$iP_s + P_{yy} - P - \gamma\lambda\delta P + f(P) + \alpha\frac{y^2}{4}P = 0,$$

where we have defined

$$f(z) = |z|^4 z, \quad z \in \mathbb{C}.$$

For  $\kappa \in \mathbb{N}$ , let

$$\Sigma_\kappa = \left\{ (j, k) \in \mathbb{N} \times \mathbb{N}^* : \frac{j}{2} + k < \kappa \right\}. \quad (2.4)$$

For  $u \in C^1(\mathcal{G}, \mathbb{R})$ , let

$$\|u\|_{C_{\text{exp}}^1} = \sup_{y \in \mathcal{G}} e^{\frac{|y|}{2}} (|u(y)| + |u'(y)|).$$

We say that  $u$  belongs to  $C_{\text{exp}}^1$  if  $\|u\|_{C_{\text{exp}}^1} < +\infty$ . The following proposition will be proved in Section 4.

**Proposition 2.1** (Approximate blow-up profile). *Let  $\kappa \in \mathbb{N}$ . Let  $J$  be an interval of  $\mathbb{R}$ . There exist  $C > 0$  and two families  $(a_{j,k})_{(j,k) \in \Sigma_\kappa} \subset \mathbb{R}$  and  $(P_{j,k})_{(j,k) \in \Sigma_\kappa} \subset C_{\text{exp}}^1$  with the following property. For any  $b \in C^1(J, \mathbb{R})$  and  $\lambda \in C^1(J, \mathbb{R}_+^*)$ , if we set*

$$P = P(b, \lambda) = Q + \sum_{(j,k) \in \Sigma_\kappa} (ib)^j \lambda^k P_{j,k}, \quad (2.5)$$

$$\alpha = \alpha(b, \lambda) = \sum_{\substack{(j,k) \in \Sigma_\kappa \\ j \text{ even}}} (ib)^j \lambda^k \alpha_{j,k} \quad (2.6)$$

and

$$\Psi_\kappa = \Psi_\kappa(b, \lambda) = iP_s + P_{yy} - P - \gamma\lambda\delta P + f(P) + \alpha\frac{y^2}{4}P, \quad (2.7)$$

then we have

$$\|\Psi_\kappa\|_{C_{\text{exp}}^1} \leq C(|\lambda b + \lambda_s| + \lambda|b_s + b^2 - \alpha|) + C(b^2 + \lambda)^\kappa. \quad (2.8)$$

Furthermore, letting  $\tilde{P}(b, \lambda, \theta) = \lambda^{-1/2}P(b, \lambda)e^{i(\theta - b\frac{y^2}{4})}$ , there holds

$$\left| \frac{d}{ds} E(\tilde{P}) \right| \lesssim \frac{1}{\lambda^2} \left( \left| \frac{\lambda_s}{\lambda} + b \right| + |b_s + b^2 - \alpha| + (b^2 + \lambda)^\kappa \right). \quad (2.9)$$



Finally, there exist  $(\varepsilon_{j,k})_{(j,k) \in \Sigma_\kappa} \subset \mathbb{R}$  such that

$$\left| E(\tilde{P}(b, \lambda, \theta)) - C_Q \mathcal{E}(b, \lambda) \right| \lesssim \frac{(b^2 + \lambda)^\kappa}{\lambda^2}, \quad (2.10)$$

where  $C_Q = \frac{1}{8} \|yQ\|_{L^2}^2$ ,

$$\mathcal{E}(b, \lambda) = \mathcal{E}_{\text{mo}}(b, \lambda) + \sum_{\substack{(j,k) \in \Sigma_\kappa \\ j \text{ even, } j/2+k \geq 1}} b^j \lambda^{k-1} \varepsilon_{j,k} \quad (2.11)$$

and  $\mathcal{E}_{\text{mo}}$  is the Hamiltonian of the model dynamical system, defined in (2.14).

Next, a choice of modulation parameters  $\theta(s), b(s), \lambda(s)$  can be made so that the remainder  $h$  satisfies orthogonality conditions which are useful to construct our solution. This is ensured by the following proposition, which will be proved in Section 5.

**Proposition 2.2** (Modulation parameters). *Let  $\eta > 0$ . Let  $I$  be an interval of  $\mathbb{R}$  and consider a solution  $u \in C^0(I, H^1(\mathcal{G})) \cap C^1(I, H^{-1}(\mathcal{G}))$  of (1.1). There exists  $\delta > 0$  such that, if*

$$\sup_{t \in I} \inf_{\substack{\theta \in \mathbb{R} \\ 0 < \lambda < \delta}} \left\| u(t, x) - \frac{1}{\sqrt{\lambda}} e^{i\theta} Q\left(\frac{x}{\lambda}\right) \right\|_{L^2(\mathcal{G})} \leq \delta, \quad (2.12)$$

then there exist  $\theta \in C^1(I, \mathbb{R})$ ,  $b \in C^1(I, (-\eta, \eta))$  and  $\lambda \in C^1(I, (0, \eta))$  with the following property. The function  $h \in C^0(I, L^2(\mathcal{G}))$  defined by

$$u(t, x) = \frac{1}{\sqrt{\lambda(t)}} e^{i\theta(t) - i\frac{b(t)x^2}{4\lambda(t)^2}} \left( P_{b(t), \lambda(t)}\left(\frac{x}{\lambda(t)}\right) + h\left(t, \frac{x}{\lambda(t)}\right) \right), \quad t \in I, \quad x \in \mathbb{R},$$

satisfies the orthogonality conditions

$$(h(t), i\lambda P_{b(t), \lambda(t)})_{L^2(\mathcal{G})} = (h(t), y^2 P_{b(t), \lambda(t)})_{L^2(\mathcal{G})} = (h(t), i\rho)_{L^2(\mathcal{G})} = 0, \quad t \in I.$$

We use here the notation  $P_{b, \lambda} \equiv P(b, \lambda)$ .

*Remark 2.3.* To keep a light notation in this section, we use the same letters  $b, \lambda, \theta, h$  to denote the modulation parameters and rest as functions of  $t$  or  $s$ . We will later be more specific, see (6.2).

As we shall see in Section 6.1, the modulation parameters  $b(s)$  and  $\lambda(s)$  are governed, at first order as  $s \rightarrow +\infty$ , by the nonlinear ODE system

$$b_s + b^2 - \beta\lambda = 0, \quad \frac{\lambda_s}{\lambda} + b = 0, \quad (2.13)$$

where

$$\beta := \alpha_{0,1} = -2 \frac{\gamma Q(0)^2}{\|yQ\|_{L^2}^2} > 0,$$

is the coefficient of the first term in the expansion (2.6) (see (4.2)). The system (2.13) is Hamiltonian, with conserved energy

$$\mathcal{E}_{\text{mo}}(b, \lambda) = \frac{b^2}{\lambda^2} - \frac{2\beta}{\lambda}. \quad (2.14)$$

An exact solution with energy  $\mathcal{E}_{\text{mo}} = 0$  is given by

$$b_{\text{mo}}(s) = \frac{2}{s}, \quad \lambda_{\text{mo}}(s) = \frac{2}{\beta s^2}.$$

Thus, Proposition 2.1 shows that, at leading order, the energy of the rescaled profile  $\tilde{P}$  is governed by the Hamiltonian energy (2.14). However, it should be noted that the correction appearing as a

power expansion in (2.11) does not vanish as  $s \rightarrow \infty$ . Indeed, by Proposition 2.5 below, the terms corresponding to  $(j, k) = (0, 1)$  and  $(j, k) = (2, 0)$  behave asymptotically as

$$\varepsilon_{0,1} + \varepsilon_{2,0} \frac{b^2}{\lambda} \sim \varepsilon_{0,1} + \varepsilon_{2,0} \frac{b_{\text{mo}}^2}{\lambda_{\text{mo}}} \sim \varepsilon_{0,1} + 2\beta\varepsilon_{2,0}, \quad s \rightarrow \infty.$$

Hence,

$$\mathcal{E}(b, \lambda) \sim \mathcal{E}_{\text{mo}}(b, \lambda) + \varepsilon_{0,1} + 2\beta\varepsilon_{2,0}, \quad s \rightarrow \infty.$$

*Remark 2.4.* Remarks 2.7 and 6.2 show that the energy shift  $\varepsilon_{0,1} + 2\beta\varepsilon_{2,0}$  has an influence on the asymptotic behaviour of  $b$  and  $\lambda$  as  $t \rightarrow 0^-/s \rightarrow +\infty$ .

The relation between  $E(\tilde{P})$  and  $\mathcal{E}_{\text{mo}}$  suggests that, up to a shift in energy (and a rescaling by  $C_Q$ ), one should be able to control the energy of the solution of (1.1) by the model Hamiltonian energy  $\mathcal{E}_{\text{mo}}$ . Unfortunately, as can be seen by a direct calculation, the difference

$$\mathcal{E}_{\text{mo}}(\underline{b}, \underline{\lambda}) - \mathcal{E}_{\text{mo}}(b, \lambda)$$

evaluated between the modulation parameters  $(b, \lambda)$  and a solution  $(\underline{b}, \underline{\lambda})$  of the model system (2.13) with energy  $\mathcal{E} = \mathcal{E}_{\text{mo}}(b(s_1), \lambda(s_1))$  grows logarithmically as  $s \rightarrow \infty$ . For this reason, the choice of final data  $(b_1, \lambda_1)$  is rather made using the full expansion  $\mathcal{E}(b, \lambda)$  by the following proposition, proved in Appendix 7.

**Proposition 2.5.** *Let  $\mathcal{E}^* \in \mathbb{R}$ . For any  $s_1 \gg 1$ , there exists  $b_1, \lambda_1 > 0$  satisfying*

$$\left| \frac{\lambda_1^{1/2}}{\lambda_{\text{mo}}(s_1)^{1/2}} - 1 \right| \lesssim \frac{1}{s_1}, \quad \left| \frac{b_1}{b_{\text{mo}}(s_1)} - 1 \right| \lesssim \frac{1}{s_1},$$

$$\mathcal{F}(\lambda_1) = s_1, \quad \mathcal{E}(b_1, \lambda_1) = \mathcal{E}^*,$$

where

$$\mathcal{F}(\lambda) = \int_{\lambda}^{\lambda_0} \frac{d\mu}{\mu^{3/2} \sqrt{\mathcal{E}^* \mu + 2\beta}}, \quad \lambda \in (0, \lambda_0], \quad (2.15)$$

with  $\lambda_0 > 0$  a fixed parameter such that  $\mathcal{E}^* \lambda_0 + 2\beta > 0$ .

We now define the final data which will give rise to an approximate solution of our problem by backward in time integration of (1.1). Let  $E^* \in \mathbb{R}$ . Consider  $t_1 < 0$  and close to 0. Let

$$\mathcal{E}^* = C_Q^{-1} E^*, \quad (2.16)$$

and  $(b_1, \lambda_1)$  given by Proposition 2.5. Let  $u_1$  be the radial solution of (1.1) such that

$$u_1(t_1, x) = \frac{1}{\sqrt{\lambda_1}} P_{b_1, \lambda_1} \left( \frac{x}{\lambda_1} \right) e^{-i \frac{b_1 x^2}{4\lambda_1^2}} \quad (2.17)$$

Let  $I \subset \mathbb{R}$  be the maximal interval such that  $t_1 \in I$ ,  $u_1$  exists on  $I$  and verifies (2.12). Then the asymptotics as  $t \rightarrow 0^-$  of the functions  $\theta$ ,  $b$ ,  $\lambda$  and  $h$  given by Proposition 2.2 follow from the next proposition, which will be proved in Section 6.

**Proposition 2.6** (Uniform estimates in the  $t$  variable). *There exists  $t_0 \in (-\infty, t_1)$ , independent of  $t_1$ , such that the solution  $u_1$  defined by (2.17) and its decomposition given by Proposition 2.2 satisfy, for all  $t \in [t_0, t_1]$ ,*

$$|b(t) - C_b |t|^{1/3}| \lesssim |t|, \quad |\lambda(t) - C_\lambda |t|^{2/3}| \lesssim |t|^{5/3}, \quad (2.18)$$

$$\|h(t)\|_{L^2(\mathcal{G})} \lesssim |t|^{1/3}, \quad \|h_y(t)\|_{L^2(\mathcal{G})} \lesssim |t|^{1/3}, \quad \|yh(t)\|_{L^2(\mathcal{G})} \lesssim |t|^{2/3}, \quad (2.19)$$

$$|E(\tilde{P}(b, \lambda, \theta)(t)) - E^*| \lesssim |t|^{5/3}, \quad (2.20)$$

where  $C_b = 2\left(\frac{3\beta^2}{4}\right)^{1/3}$ ,  $C_\lambda = \frac{2}{\beta}\left(\frac{3\beta^2}{4}\right)^{2/3}$  and  $\kappa \geq 7$  is the integer introduced in (2.4). Furthermore, all these estimates are independent of  $t_1$ .

*Remark 2.7.* Thanks to Remark 6.2, the first two estimates in Proposition 2.6 can be improved to

$$|b(t) - C_b|t|^{1/3}| \lesssim |t|^{5/3}, \quad |\lambda(t) - C_\lambda|t|^{2/3}| \lesssim |t|^{7/3}$$

by replacing the energy  $\mathcal{E}^*$  in (2.15) with  $\underline{\mathcal{E}}^* = \mathcal{E}^* - (\varepsilon_{0,1} + 2\beta\varepsilon_{2,0})$ .

We are now in position to prove our main result.

*Proof of Theorem 1.1, assuming Propositions 2.1, 2.2, 2.5 and 2.6.* Let  $E^* \in \mathbb{R}$  and define  $\mathcal{E}^*$  by (2.16). Choose an increasing sequence of times  $(t_n) \subset (t_0, 0)$  such that  $t_n \rightarrow 0$  as  $n \rightarrow \infty$ . For each  $n \in \mathbb{N}^*$ , let  $b_n$  and  $\lambda_n$  as given by Proposition 2.5 and  $u_n(t_n)$  defined by (2.17), with the change of notation  $t_1 \rightarrow t_n$ ,  $b_1 \rightarrow b_n$ ,  $\lambda_1 \rightarrow \lambda_n$ . For each  $n \in \mathbb{N}^*$ , the corresponding solution  $u_n$  of (1.1) satisfies Proposition 2.6, where all the estimates are independent of  $n$ . We will show that  $(u_n)$  converges to a solution  $u$  of (1.1) with the desired properties.

Let  $\chi \in C^\infty([0, \infty), [0, 1])$  be equal to 0 on  $[0, 1]$  and equal to 1 on  $[2, \infty)$ . For  $R > 0$  we define the radial function  $\chi_R$  on  $\mathcal{G}$  by  $\chi(x/R)$  on each edge. Let  $\delta > 0$ . From the formula (2.17) defining  $u_n(t_n)$  we deduce that there exists  $R > 0$  such that

$$\int_{\mathcal{G}} |u_n(t_n)|^2 \chi_R dx \leq \delta.$$

By (1.1), we have

$$\frac{d}{dt} \int_{\mathcal{G}} |u_n|^2 \chi_R dx = 2 \operatorname{Im} \int_{\mathcal{G}} \partial_x u_n \bar{u}_n \partial_x \chi_R dx.$$

Using the decomposition of the solution given by Proposition 2.2, the estimates of Proposition 2.6 for the corresponding variables  $b_n, \lambda_n, h_n$  and the exponential decay of  $P_{b_n, \lambda_n}$ , direct calculations show that

$$\left| \frac{d}{dt} \int_{\mathcal{G}} |u_n|^2 \chi_R dx \right| \lesssim \frac{1}{R\lambda_n(t)} \left( e^{-\frac{R}{2\lambda_n(t)}} + \|y h_n(t)\|_{L^2(y \geq R/\lambda_n(t))}^2 + \|h_n(t)\|_{H^1(y \geq R/\lambda_n(t))}^2 \right) \lesssim \frac{|t|^{2(\kappa-3)}}{R}.$$

Thus, integrating over  $[t_0, t_n]$ , we find a constant  $C > 0$  such that (choosing  $R$  larger if necessary)

$$\int_{\mathcal{G}} |u_n(t_0)|^2 \chi_R dx \leq C \frac{|t_0|^{2(\kappa-3)+1}}{R} + \int_{\mathcal{G}} |u_n(t_n)|^2 \chi_R dx \leq 2\delta.$$

After extracting a subsequence if necessary, we obtain that the sequence  $(u_n(t_0))$  has a limit  $u_0$  in  $L^2(\mathcal{G})$ . We denote by  $u$  the maximal solution of (1.1) with initial condition  $u(t_0) = u_0$ .

Let  $\tau \in (t_0, 0)$  and assume by contradiction that  $u$  is not defined on  $[t_0, \tau]$ . Let  $n \in \mathbb{N}^*$  such that  $t_n > \tau$ . Let  $C > 0$  be such that  $\|u_n(t)\|_{H^1(\mathcal{G})} \leq C$  for all  $n \in \mathbb{N}^*$  and  $t \in [t_0, \tau]$ . By the blow-up alternative (see Section 3.1), there exists  $\tau_1$  in  $[t_0, \tau]$  such that  $u$  is defined on  $[t_0, \tau_1]$  and  $\|u(\tau_1)\|_{H^1(\mathcal{G})} \geq 2C$ . For all  $t \in [0, \tau_1]$  the sequence  $(u_n(t))$  goes to  $u(t)$  in  $L^2(\mathcal{G})$  and has a weak limit in  $H^1(\mathcal{G})$ . Thus,  $(u_n(t))$  goes weakly to  $u(t)$  in  $H^1(\mathcal{G})$ . In particular,  $\|u(t)\|_{H^1(\mathcal{G})} \leq C$ . This gives a contradiction and proves that  $u$  is defined on  $[t_0, \tau]$ . Finally,  $u$  is well defined on  $[t_0, 0)$ .

By conservation of the mass and convergence of  $u_n(t_0)$  to  $u(t_0)$  in  $L^2(\mathcal{G})$  we have, for all  $t \in [t_0, 0)$ ,

$$\int_{\mathcal{G}} |u(t)|^2 dx = \lim_{n \rightarrow \infty} \int_{\mathcal{G}} |u_n(t_n)|^2 dx = \int_{\mathcal{G}} Q^2 dx.$$

Moreover, the fact that  $u_n(t)$  goes to  $u(t)$  weakly in  $H^1(\mathcal{G})$  implies that  $u$  satisfies the assumption of Proposition 2.2 on  $[t_0, 0)$ . Let  $b_\infty(t), \lambda_\infty(t), \theta_\infty(t), h_\infty$  denote the modulation parameters and corresponding rest given by Proposition 2.2. Then, by standard arguments,

$$\theta_n(t) \rightarrow \theta_\infty(t), \quad b_n(t) \rightarrow b_\infty(t), \quad \lambda_n(t) \rightarrow \lambda_\infty(t),$$

and, weakly in  $H^1(\mathcal{G})$ ,

$$h_n(t) \rightharpoonup h_\infty(t), \quad t \rightarrow 0^-.$$

By Proposition 2.6 we deduce that, as  $t \rightarrow 0^-$ ,

$$b_\infty(t) \sim C_b |t|^{\frac{1}{3}}, \quad \lambda_\infty(t) \sim C_\lambda |t|^{\frac{2}{3}},$$

$$\|h_\infty(t)\|_{L^2(\mathcal{G})} \lesssim |t|^{\frac{\kappa-1}{3}}, \quad \|\partial_y h_\infty(t)\|_{L^2(\mathcal{G})} \lesssim |t|^{\frac{\kappa-1}{3}}, \quad \|y h_\infty(t)\|_{L^2(\mathcal{G})} \lesssim |t|^{\frac{\kappa-2}{3}}.$$

Using  $y = x/\lambda_\infty$ , the decomposition of  $u$  given by Proposition 2.2 and the formula for  $P_{b_\infty, \lambda_\infty}$  in Proposition 2.1, it then follows by direct calculations that

$$\begin{aligned} \|u(t)\|_{L^2}^2 &= \int_{\mathcal{G}} |P_{b_\infty, \lambda_\infty}(y) + h_\infty(t, y)|^2 dy \longrightarrow \|Q\|_{L^2}^2, \quad t \rightarrow 0^-, \quad \text{and} \\ \|u_x(t)\|_{L^2}^2 &= \lambda_\infty(t)^{-1} \|u_y(t)\|_{L^2}^2 \\ &\sim \lambda_\infty(t)^{-2} \int_{\mathcal{G}} \left| -ib \frac{y}{2} (P_{b_\infty, \lambda_\infty}(y) + h_\infty(t, y)) + \partial_y (P_{b_\infty, \lambda_\infty}(y) + h_\infty(t, y)) \right|^2 dy \\ &\sim \lambda_\infty(t)^{-2} \|Q_y\|_{L^2}^2 \sim C_\lambda^{-2} \|Q_y\|_{L^2}^2 |t|^{-\frac{4}{3}}, \quad t \rightarrow 0^-. \end{aligned}$$

This proves (1.3) and, since  $\|u\|_{L^2}^2$  is constant, that  $M(u) = M(Q)$ .

To complete the proof, we now show that  $E(u) = E^*$ . By (2.10) and (2.20), there exists a function  $\varepsilon : [t_0, 0) \rightarrow \mathbb{R}_+^*$  with  $\lim_{t \rightarrow 0^-} \varepsilon(t) = 0$  and such that, for all  $n \in \mathbb{N}^*$ ,

$$|\mathcal{E}(b_n(t), \lambda_n(t)) - C_Q^{-1} E^*| \leq \varepsilon(t), \quad t \in [t_0, 0).$$

Taking the limit  $n \rightarrow \infty$  gives

$$|\mathcal{E}(b_\infty(t), \lambda_\infty(t)) - C_Q^{-1} E^*| \leq \varepsilon(t), \quad t \in [t_0, 0).$$

Hence, using again (2.10), we conclude that

$$E(P_{b_\infty(t), \lambda_\infty(t)}) \longrightarrow E^*, \quad t \rightarrow 0^-.$$

It then follows from the above information about  $b_\infty, \lambda_\infty$  and  $h_\infty$  that

$$E(u(t)) \longrightarrow E^*, \quad t \rightarrow 0^-.$$

By conservation of the energy, we deduce that  $E(u(t)) = E^*$  for all  $t \in [t_0, 0)$ .  $\square$

### 3. GENERAL CONTEXT

In this section we collect some basic definitions and results which will be used throughout the paper.

**3.1. The Cauchy Problem.** Since the operator  $H_\gamma$  is self-adjoint, it generates a strongly continuous group  $e^{-itH_\gamma}$ . Since we are working in a one-dimensional setting, the nonlinearity  $|u|^4u$  is Lipschitz continuous from bounded sets of  $H_D^1(\mathcal{G})$  to  $L^q(\mathcal{G})$ ,  $2 \leq q \leq \infty$ , and well-posedness of the Cauchy problem for (1.1) may be obtained (see e.g. [4]) following a classical line of arguments (see e.g. [22]). For any initial data  $u_0 \in H_D^1(\mathcal{G})$ , there exists a unique maximal solution

$$u \in \mathcal{C}((-T_{\min}, T_{\max}), H_D^1(\mathcal{G})) \cap \mathcal{C}^1((-T_{\min}, T_{\max}), H^{-1}(\mathcal{G}))$$

such that  $u(t=0) = u_0$ . The energy  $E$  and the mass  $M$ , defined by

$$M(u) = \frac{1}{2} \|u\|_{L^2(\mathcal{G})}^2, \quad E(u) = \frac{1}{2} \|u_x\|_{L^2(\mathcal{G})}^2 - \frac{1}{6} \|u\|_{L^6(\mathcal{G})}^6 + \frac{\gamma}{2} |u(0)|^2.$$

are preserved along the time evolution, i.e. for any  $t \in (-T_{\min}, T_{\max})$ , we have

$$E(u(t)) = E(u_0), \quad M(u(t)) = M(u_0).$$

The blow-up alternative holds, i.e. either  $T_{\max} = \infty$  (resp.  $T_{\min} = \infty$ ) or

$$\lim_{t \rightarrow T_{\max} \text{ (resp. } T_{\min})} \|u(t)\|_{H^1(\mathcal{G})} = \infty.$$

There is continuous dependence with respect to the initial data, i.e. for any  $(u_{0,n}) \subset H^1(\mathcal{G})$  such that  $u_{0,n} \rightarrow u_0$  in  $H^1(\mathcal{G})$  the associated solutions  $(u_n)$  of (1.1) verify  $u_n \rightarrow u$  in  $\mathcal{C}((-T_*, T^*), H^1(\mathcal{G}))$  for any  $0 < T_* < T_{\min}$ ,  $0 < T^* < T_{\max}$ . Finally, if in addition  $u_0 \in D(H_\gamma)$ , then  $u$  verifies

$$u \in \mathcal{C}((-T_{\min}, T_{\max}), D(H_\gamma)) \cap \mathcal{C}^1((-T_{\min}, T_{\max}), L^2(\mathcal{G})).$$

**3.2. Global existence.** We now establish some global existence results for the nonlinear Schrödinger equation (1.1).

**Lemma 3.1** (Gagliardo-Nirenberg inequalities on star-graphs). *The following inequalities hold*

$$\|u\|_{L^6(\mathcal{G})}^6 \leq \frac{3}{\|Q\|_{L^2(\mathbb{R})}^4} \|u_x\|_{L^2(\mathcal{G})}^2 \|u\|_{L^2(\mathcal{G})}^4, \quad u \in H_D^1(\mathcal{G}), \quad (3.1)$$

$$\|u\|_{L^6(\mathcal{G})}^6 \leq \frac{12}{N^2 \|Q\|_{L^2(\mathbb{R})}^4} \|u_x\|_{L^2(\mathcal{G})}^2 \|u\|_{L^2(\mathcal{G})}^4, \quad u \in H_{\text{rad}}^1(\mathcal{G}). \quad (3.2)$$

*Proof.* The inequality (3.1) is well-known to hold on the line  $\mathbb{R}$ , and the extension to star-graphs is immediate (see e.g. [4, (2.3)]). On the half-line  $\mathbb{R}^+$ , we infer from (3.1) that

$$\|u\|_{L^6(\mathbb{R}^+)}^6 \leq \frac{12}{\|Q\|_{L^2(\mathbb{R})}^4} \|u_x\|_{L^2(\mathbb{R}^+)}^2 \|u\|_{L^2(\mathbb{R}^+)}^4, \quad u \in H^1(\mathbb{R}^+).$$

Let  $u \in H_{\text{rad}}^1(\mathcal{G})$  and denote by  $u_e : \mathbb{R}^+ \rightarrow \mathbb{R}$  the function representing  $u$  on any of the branches of the graph. We have

$$\|u\|_{L^6(\mathcal{G})}^6 = N \|u_e\|_{L^6(\mathbb{R}^+)}^6 \leq N \frac{12}{\|Q\|_{L^2(\mathbb{R})}^4} \|(u_e)_x\|_{L^2(\mathbb{R}^+)}^2 \|u_e\|_{L^2(\mathbb{R}^+)}^4 = \frac{12}{N^2 \|Q\|_{L^2(\mathbb{R})}^4} \|u_x\|_{L^2(\mathcal{G})}^2 \|u\|_{L^2(\mathcal{G})}^4,$$

which establishes (3.2).  $\square$

**Proposition 3.2** (Global wellposedness). *Let  $\gamma \in \mathbb{R}$ ,  $u_0$  be an initial data and  $u$  be the corresponding solution of (1.1) such that  $u(0, \cdot) = u_0$ .*

*If  $u_0 \in H_D^1(\mathcal{G})$  satisfies  $\|u_0\|_{L^2} < \min\{1, \frac{N}{2}\} \|Q\|_{L^2}$ , then  $u$  is global in  $H^1(\mathcal{G})$ . Furthermore, if  $\gamma > 0$ , then for any solution with  $\|u_0\|_{L^2} = \min\{1, \frac{N}{2}\} \|Q\|_{L^2}$ ,  $u(t, 0)$  remains bounded on the lifespan of  $u$ .*

If  $u_0 \in H_{\text{rad}}^1(\mathcal{G})$  satisfies  $\|u_0\|_{L^2} < \frac{N}{2}\|Q\|_{L^2}$ , then  $u$  is global in  $H_{\text{rad}}^1(\mathcal{G})$ . Furthermore, if  $\gamma > 0$ , then for any solution with  $\|u_0\|_{L^2} = \frac{N}{2}\|Q\|_{L^2}$ ,  $u(t, 0)$  remains bounded on the lifespan of  $u$ .

*Proof.* The proof follows by combining (3.1) with the conservation laws of (1.1). We have

$$\begin{aligned} E(u_0) = E(u(t)) &= \frac{1}{2}\|u_x(t)\|_{L^2}^2 - \frac{1}{6}\|u(t)\|_{L^6}^6 + \frac{\gamma}{2}|u(t, 0)|^2 \\ &\geq \frac{1}{2} \left[ 1 - \left( \frac{\|u_0\|_{L^2}}{\|Q\|_{L^2}} \right)^4 \right] \|u_x(t)\|_{L^2}^2 + \frac{\gamma}{2}|u(t, 0)|^2. \end{aligned}$$

If  $\gamma > 0$ , it follows that  $\|u_x(t)\|_{L^2}$  remains bounded provided  $\|u_0\|_{L^2} < \|Q\|_{L^2}$ , and global existence in  $H^1(\mathcal{G})$  follows by the blow-up alternative. Moreover in this case, if  $\|u_0\|_{L^2} = \|Q\|_{L^2}$ , we see that  $|u(t, 0)|^2$  must remain bounded.

If  $\gamma < 0$ , the inequality  $|u(t, 0)|^2 \leq 2\|u_x(t)\|_{L^2}\|u(t)\|_{L^2}$  yields

$$|u(t, 0)|^2 \leq \epsilon\|u_x(t)\|_{L^2}^2 + \frac{4}{\epsilon}\|u(t)\|_{L^2}^2$$

for any  $\epsilon > 0$ , and it follows that

$$E(u_0) \geq \frac{1}{2} \left[ 1 - \left( \frac{\|u_0\|_{L^2}}{\|Q\|_{L^2}} \right)^4 - |\gamma|\epsilon \right] \|u_x(t)\|_{L^2}^2 + \frac{2\gamma}{\epsilon}\|u_0\|_{L^2}^2.$$

If  $\|u_0\|_{L^2} < \|Q\|_{L^2}$ , we can choose  $\epsilon > 0$  so that  $1 - \left( \frac{\|u_0\|_{L^2}}{\|Q\|_{L^2}} \right)^4 - |\gamma|\epsilon > 0$ , showing that  $\|u_x(t)\|_{L^2}$  remains bounded. This concludes the proof.  $\square$

**3.3. The linearized operators.** In this subsection, we establish some useful properties of the linearized operators

$$L_- = -\frac{d^2}{dy^2} + 1 - Q^4 \quad \text{and} \quad L_+ = -\frac{d^2}{dy^2} + 1 - 5Q^4.$$

They are seen as bounded operators from  $H_{\text{rad}}^1(\mathcal{G}, \mathbb{R})$  to its dual  $H_{\text{rad}}^{-1}(\mathcal{G}, \mathbb{R})$ . For instance, for  $\varphi, \psi \in H_{\text{rad}}^1(\mathcal{G}, \mathbb{R})$  we have

$$\langle L_- \varphi, \psi \rangle = (\varphi', \psi')_{L^2(\mathcal{G})} + (\varphi - Q^4 \varphi, \psi)_{L^2(\mathcal{G})}.$$

We can also consider the corresponding (unbounded) operators on  $L_{\text{rad}}^2(\mathcal{G}, \mathbb{R})$ . We set

$$\mathbf{L}_- = -\frac{d^2}{dy^2} + 1 - Q^4 \quad \text{and} \quad \mathbf{L}_+ = -\frac{d^2}{dy^2} + 1 - 5Q^4.$$

They are defined on the same domain  $D(\mathbf{L}_-) = D(\mathbf{L}_+) = D(H_0) \cap H_{\text{rad}}^1(\mathcal{G}, \mathbb{R})$ .

We denote by  $\sigma(A), \sigma_{\text{ess}}(A)$  the spectrum, respectively the essential spectrum, of a linear operator  $A$  on  $L_{\text{rad}}^2(\mathcal{G})$ . The following spectral properties of the operators  $\mathbf{L}_{\pm}$  are well-known in the context of radial functions on the line (see e.g. [23, 53] and references therein) and it is straightforward to transpose them to  $L_{\text{rad}}^2(\mathcal{G}, \mathbb{R})$ .

We denote by  $\Lambda$  the generator of dilations on  $\mathcal{G}$ . For  $v \in C_{\text{rad}}^1(\mathcal{G})$ ,

$$\Lambda v = \frac{v}{2} + yv' = \frac{d}{d\lambda}(\sqrt{\lambda}v(\lambda y))\Big|_{\lambda=1}.$$

**Lemma 3.3.** *The operators  $\mathbf{L}_{\pm}$  have the following properties:*

- (i)  $\mathbf{L}_{\pm}$  are selfadjoint and bounded below.
- (ii)  $\sigma_{\text{ess}}(\mathbf{L}_{\pm}) = [1, \infty)$ .

- (iii)  $-8$  is the only eigenvalue of  $L_+$ , with  $N(L_+ + 8I) = \text{span}\{Q^3\}$ .
- (iv)  $0$  is the only eigenvalue of  $L_-$ , with  $N(L_-) = \text{span}\{Q\}$ .
- (v) Setting  $\rho = L_+^{-1}(y^2Q)$ , we have the relations

$$L_-Q = 0, \quad L_+\Lambda Q = -2Q, \quad L_-y^2Q = -4\Lambda Q, \quad L_+\rho = y^2Q.$$

From these results on  $L_\pm$  we deduce similar properties for  $L_\pm$ .

**Proposition 3.4.** *The operators  $L_\pm$  have the following properties:*

- (i)  $L_+ : H_{\text{rad}}^{-1}(\mathcal{G}, \mathbb{R}) \rightarrow H_{\text{rad}}^{-1}(\mathcal{G}, \mathbb{R})$  is bijective;
- (ii)  $\ker(L_-) = \text{span}(Q)$  and  $\text{Ran}(L_-) = \{\varphi \in H_{\text{rad}}^{-1}(\mathcal{G}) : \varphi(Q) = 0\}$ .

*Proof.* We have  $\text{span}(Q) = \ker(L_-) \subset \ker(L_-)$ , and if  $\varphi \in \ker(L_-)$  we have  $\varphi \in D(L_-)$  and  $L_-\varphi = 0$ . This proves that  $\ker(L_-) = \text{span}(Q)$ .

Since  $L_- = (\text{Id}_{H_{\text{rad}}^{-1}(\mathcal{G}, \mathbb{R})} - K)(-\partial_x^2 + 1)$  with  $K = Q^4(-\partial_x^2 + 1)^{-1} \in \mathcal{L}(H_{\text{rad}}^{-1}(\mathcal{G}, \mathbb{R}))$  compact, its range is closed. Then  $\text{Ran}(L_-) = \ker(L_-)^\perp$  and the second statement of the proposition follows.

The first statement about  $L_+$  is similar.  $\square$

Next, we give some useful integral identities.

**Lemma 3.5.** *Let  $Q$ ,  $\Lambda Q$  and  $\rho$  be as in Lemma 3.4. Then:*

- (i)  $\int_{\mathcal{G}} y^2 Q \Lambda Q \, dy = - \int_{\mathcal{G}} y^2 Q^2 \, dy$ ;
- (ii)  $\int_{\mathcal{G}} Q \Lambda Q \, dy = 0$ ;
- (iii)  $\int_{\mathcal{G}} Q \rho \, dy = \frac{1}{2} \int_{\mathcal{G}} y^2 Q^2 \, dy$ .

*Proof.* (i) For real parameters  $\mu > -1$  and  $r \geq 1$ , we will show that

$$\int_{\mathcal{G}} y^\mu Q^r \Lambda Q \, dy = \left( \frac{1}{2} - \frac{\mu + 1}{r + 1} \right) \int_{\mathcal{G}} y^\mu Q^{r+1} \, dy, \quad (3.3)$$

from which (i) follows. Now, to prove (3.3), we only need to show that

$$\int_{\mathcal{G}} y^\mu Q^r y Q_y \, dy = - \frac{\mu + 1}{r + 1} \int_{\mathcal{G}} y^\mu Q^{r+1} \, dy. \quad (3.4)$$

Integrating by parts, we have

$$\begin{aligned} \int_{\mathcal{G}} y^{\mu+1} Q^r Q_y \, dy &= - \int_{\mathcal{G}} [(\mu + 1) Q^r y^\mu + r y^{\mu+1} Q^{r-1} Q_y] Q \, dy \\ &= -(\mu + 1) \int_{\mathcal{G}} y^\mu Q^{r+1} \, dy - r \int_{\mathcal{G}} Q^r Q_y y^{\mu+1} \, dy, \end{aligned}$$

which is equivalent to (3.4). This completes the proof of (3.3).

(ii) follows directly from (3.3) with  $\mu = 0$  and  $r = 1$ , but the following argument is more instructive. Since the  $L^2$  scaling  $Q_\lambda(y) = \lambda^{\frac{1}{2}} Q(\lambda y)$  leaves the  $L^2$  norm invariant, we have that

$$0 = \frac{d}{d\lambda} \|Q_\lambda\|_{L^2}^2 = 2 \int_{\mathcal{G}} Q_\lambda \frac{\partial Q_\lambda}{\partial \lambda} \, dy, \quad \forall \lambda > 0.$$

The result follows by letting  $\lambda = 1$ .

(iii) Using Lemma 3.3 and (i), we have the identities

$$\int_{\mathcal{G}} Q \rho \, dy = - \frac{1}{2} \int_{\mathcal{G}} L_+ \Lambda Q \rho \, dy = - \frac{1}{2} \int_{\mathcal{G}} \Lambda Q L_+ \rho \, dy = - \frac{1}{2} \int_{\mathcal{G}} \Lambda Q y^2 Q \, dy = \frac{1}{2} \int_{\mathcal{G}} y^2 Q^2 \, dy.$$

The proof is complete.  $\square$

We now state well-known coercivity properties of the operators  $L_{\pm}$ , which we prove for the reader's convenience. We start with positivity properties.

**Lemma 3.6.**  $L_{\pm}$  satisfy the following positivity relations in  $H_{\text{rad}}^1$ , where  $\perp$  denotes orthogonality in  $L^2$ :

$$\langle L_- v, v \rangle_{H^{-1} \times H^1} \gtrsim \|v\|_{H^1}^2 \quad \text{on } \{\rho\}^{\perp}, \quad (3.5)$$

$$\langle L_+ v, v \rangle_{H^{-1} \times H^1} \gtrsim \|v\|_{H^1}^2 \quad \text{on } \{Q, y^2 Q\}^{\perp}. \quad (3.6)$$

*Proof.* To prove (3.5), we first observe that Lemma 3.3 implies

$$\langle L_- w, w \rangle_{H^{-1} \times H^1} \geq \|w\|_{L^2}^2, \quad \forall w \in \{Q\}^{\perp}. \quad (3.7)$$

From now on, we denote by  $\|\cdot\|$  and  $(\cdot, \cdot)$  the  $L^2$  norm and inner product, and  $\langle \cdot, \cdot \rangle$  the duality product.

Let  $v \in \{\rho\}^{\perp}$ . Let  $w \in \{Q\}^{\perp}$  and  $t \in \mathbb{R}$  such that  $v = w + tQ$ . Since  $(Q, \rho) \neq 0$  (see Lemma 3.5), we necessarily have

$$t = -\frac{(w, \rho)}{(Q, \rho)},$$

and hence

$$\|v\|_{L^2}^2 = \|w\|^2 + 2t(w, Q) + t^2\|Q\|_{L^2}^2 = \|w\|_{L^2}^2 + \frac{(w, \rho)^2}{(Q, \rho)^2} \|Q\|_{L^2}^2 \leq \|w\|_{L^2}^2 \left(1 + \frac{\|\rho\|_{L^2}^2 \|Q\|_{L^2}^2}{(Q, \rho)^2}\right).$$

Setting

$$C_1 = \left(1 + \frac{\|\rho\|_{L^2}^2 \|Q\|_{L^2}^2}{(Q, \rho)^2}\right)^{-1} > 0,$$

it then follows by (3.7) that

$$\langle L_- v, v \rangle = \langle L_- w, w \rangle \geq \|w\|_{L^2}^2 \geq C_1 \|v\|_{L^2}^2. \quad (3.8)$$

To deduce (3.5), we argue by contradiction. Suppose there exists a sequence  $\{v_n\}$  in  $H_{\text{rad}}^1 \cap \{\rho\}^{\perp}$  such that  $\|v_n\|_{H^1} = 1$  for all  $n$  and  $\langle L_- v_n, v_n \rangle \rightarrow 0$  as  $n \rightarrow \infty$ . Then (3.8) implies  $\|v_n\|_{L^2} \rightarrow 0$ , so  $\|\partial_x v_n\|_{L^2} \rightarrow 1$ . Then

$$\begin{aligned} \langle L_- v_n, v_n \rangle &= \|\partial_x v_n\|_{L^2}^2 + \|v_n\|_{L^2}^2 - \int_{\mathcal{G}} Q^{p-1} v_n^2 \, dy \\ &\geq \|\partial_x v_n\|_{L^2}^2 + \|v_n\|_{L^2}^2 (1 - \|Q\|_{L^\infty}^{p-1}) \rightarrow 1. \end{aligned}$$

This contradiction concludes the proof of (3.5).

The proof of (3.6) follows in the same way from

$$\langle L_+ v, v \rangle \gtrsim \|v\|_{L^2}^2 \quad \text{on } \{Q, y^2 Q\}^{\perp}.$$

However, the proof of this inequality is much more involved than that of (3.8), see [53].  $\square$

**Lemma 3.7.** There exist  $\mu_-, \mu_+ > 0$  such that, for all  $v \in H_{\text{rad}}^1$ ,

$$\langle L_- v, v \rangle_{H^{-1} \times H^1} \geq \mu_- \|v\|_{H^1}^2 - \mu_-^{-1} (v, \rho)_{L^2}^2 \quad (3.9)$$

and

$$\langle L_+ v, v \rangle_{H^{-1} \times H^1} \geq \mu_+ \|v\|_{H^1}^2 - \mu_+^{-1} [(v, Q)_{L^2}^2 + (v, y^2 Q)_{L^2}^2]. \quad (3.10)$$

There exists  $\mu > 0$  such that

$$\langle L_+ v, v \rangle_{H^{-1} \times H^1} + \langle L_- v, v \rangle_{H^{-1} \times H^1} \geq \mu \|v\|_{H^1}^2 - \mu^{-1} [(v, Q)_{L^2}^2 + (v, y^2 Q)_{L^2}^2 + (v, \rho)_{L^2}^2]. \quad (3.11)$$



*Proof.* We prove (3.10) and leave the proof of (3.9), which is very similar, to the reader. Estimate (3.11) will then be a consequence of (3.9) and (3.10).

Any  $v \in H_{\text{rad}}^1$  can be written as

$$v = w + sQ + ty^2Q, \quad w \in \{Q, y^2Q\}^\perp$$

with

$$s = \frac{(v, Q)_{L^2} \|y^2Q\|^2 - (v, y^2Q)_{L^2} (Q, y^2Q)_{L^2}}{\|Q\|^2 \|y^2Q\|^2 - (Q, y^2Q)_{L^2}^2}, \quad t = \frac{(v, y^2Q)_{L^2} \|Q\|^2 - (v, Q)_{L^2} (Q, y^2Q)_{L^2}}{\|Q\|^2 \|y^2Q\|^2 - (Q, y^2Q)_{L^2}^2}.$$

Then

$$\begin{aligned} \|v\|_{H^1}^2 &= \|w\|_{H^1}^2 + s^2 \|Q\|_{H^1}^2 + t^2 \|y^2Q\|_{H^1}^2 + 2st \langle Q, y^2Q \rangle_{H^1} + 2s \langle w, Q \rangle_{H^1} + 2t \langle w, y^2Q \rangle_{H^1} \\ &\leq \|w\|_{H^1}^2 + s^2 \|Q\|_{H^1}^2 + t^2 \|y^2Q\|_{H^1}^2 + (s^2 + t^2) |\langle Q, y^2Q \rangle_{H^1}| \\ &\quad + 2s \|w\|_{H^1} \|Q\|_{H^1} + 2t \|w\|_{H^1} \|y^2Q\|_{H^1} \\ &\leq \|w\|_{H^1}^2 + s^2 \|Q\|_{H^1}^2 + t^2 \|y^2Q\|_{H^1}^2 + (s^2 + t^2) |\langle Q, y^2Q \rangle_{H^1}| \\ &\quad + s^2 + \|w\|_{H^1}^2 \|Q\|_{H^1}^2 + t^2 + \|w\|_{H^1}^2 \|y^2Q\|_{H^1}^2 \\ &\leq (1 + \|Q\|_{H^1}^2 + \|y^2Q\|_{H^1}^2) \|w\|_{H^1}^2 + (1 + \|Q\|_{H^1}^2 + |\langle Q, y^2Q \rangle_{H^1}|) s^2 \\ &\quad + (1 + \|y^2Q\|_{H^1}^2 + |\langle Q, y^2Q \rangle_{H^1}|) t^2. \end{aligned}$$

Hence, there exist constants  $A, B, C > 0$  such that

$$\|w\|_{H^1}^2 \geq A \|v\|_{H^1}^2 - B \frac{(v, Q)^2}{\|Q\|^4} - C \frac{(v, y^2Q)^2}{\|y^2Q\|^4}. \quad (3.12)$$

Using similar calculations, (3.6) yields a constant  $K > 0$  such that, for any  $\varepsilon > 0$ ,

$$\begin{aligned} &\langle L_+ v, v \rangle \\ &= \langle L_+ w, w \rangle + s^2 \langle L_+ Q, Q \rangle + t^2 \langle L_+ y^2Q, y^2Q \rangle + 2st \langle L_+ Q, y^2Q \rangle + 2s \langle w, L_+ Q \rangle + 2t \langle w, L_+ y^2Q \rangle \\ &\geq K \|w\|_{H^1}^2 - (|\langle L_+ Q, Q \rangle| + |\langle L_+ y^2Q, y^2Q \rangle|) s^2 - (|\langle L_+ y^2Q, y^2Q \rangle| + |\langle L_+ y^2Q, y^2Q \rangle|) t^2 \\ &\quad - 2\varepsilon^{-1} |s| \varepsilon \|w\|_{L^2} \|L_+ Q\|_{L^2} - 2\varepsilon^{-1} |t| \varepsilon \|w\|_{L^2} \|L_+ y^2Q\|_{L^2} \\ &\geq K \|w\|_{H^1}^2 - (|\langle L_+ Q, Q \rangle| + |\langle L_+ Q, y^2Q \rangle|) s^2 - (|\langle L_+ y^2Q, y^2Q \rangle| + |\langle L_+ Q, y^2Q \rangle|) t^2 \\ &\quad - (\varepsilon^{-2} s^2 + \varepsilon^2 \|L_+ Q\|_{L^2}^2 \|w\|_{H^1}^2) - (\varepsilon^{-2} t^2 + \varepsilon^2 \|L_+ y^2Q\|_{L^2}^2 \|w\|_{H^1}^2) \\ &\geq \left[ K - \varepsilon^2 (\|L_+ Q\|_{L^2}^2 + \|L_+ y^2Q\|_{L^2}^2) \right] \|w\|_{H^1}^2 \\ &\quad - (\varepsilon^{-2} + |\langle L_+ Q, Q \rangle| + |\langle L_+ Q, y^2Q \rangle|) s^2 - (\varepsilon^{-2} + |\langle L_+ y^2Q, y^2Q \rangle| + |\langle L_+ Q, y^2Q \rangle|) t^2. \end{aligned}$$

Therefore, choosing  $\varepsilon > 0$  small enough, there exist constants  $K_1, K_2, K_3 > 0$  such that

$$\langle L_+ v, v \rangle \geq K_1 \|w\|_{H^1}^2 - K_2 \frac{(v, Q)^2}{\|Q\|^4} - K_3 \frac{(v, y^2Q)^2}{\|y^2Q\|^4}.$$

Combining this with (3.12) concludes the proof of (3.10).  $\square$

#### 4. CONSTRUCTION OF THE PROFILE

In this section we prove Proposition 2.1. For  $b \in C^1(J, \mathbb{R})$ ,  $\lambda \in C^1(J, \mathbb{R}_+^*)$  and  $\alpha \in C^1(J, \mathbb{R})$ , we denote by  $\mathcal{O}_{C_{\text{exp}}^1}(b, \lambda, \alpha)$  any function  $u : J \rightarrow C_{\text{exp}}^1$  such that we have, for some  $C > 0$

$$\forall s \in J, \quad \|u(s)\|_{C_{\text{exp}}^1} \leq C (|\lambda b + \lambda_s| + \lambda |b_s + b^2 - \alpha|) + C(b^2 + \lambda)^\kappa.$$

Then we have to construct  $\alpha$  and  $P$  of the form (2.6) and (2.5) such that  $\Psi_\kappa = \mathcal{O}_{C_{\text{exp}}^1}(b, \lambda, \alpha)$ , where  $\Psi_\kappa$  is defined by (2.7).

**Lemma 4.1.** *Let  $g \in C_{\text{exp}}^1$ ,  $\eta \in \mathbb{R}$  and  $\mu \in \mathbb{R}$ . Suppose that  $u \in H_{\text{rad}}^1(\mathcal{G})$  is a solution of*

$$-u'' + u + \mu Q^4 u = \eta \delta + g.$$

*Then we have  $u \in C_{\text{exp}}^1$ .*

*Proof.* By elliptic regularity we have  $u \in H^2(\mathcal{G})$ , so  $u$  is of class  $C^1$  and then of class  $C^3$  on each edge. For  $y \in \mathcal{G}$  we have

$$u(y) = \frac{1}{2} \int_0^{+\infty} e^{-|y-z|} (g(z) - \kappa Q(z)^4 u(z)) dz + e^{-y} \left( \frac{\eta}{N} + \int_0^{+\infty} e^{-z} (g(z) - \mu Q(z)^4 u(z)) dz \right).$$

Since  $z \mapsto g(z) - \mu Q(z)^4 u(z)$  decays at least like  $e^{-\frac{z}{2}}$ , the conclusion follows.  $\square$

*Proof of Proposition 2.1.* Let  $P_{j,k} \in C_{\text{exp}}^1$  for  $(j, k) \in \Sigma_\kappa$  and  $\alpha_{j,k} \in \mathbb{R}$  for  $(j, k) \in \Sigma_\kappa$  with  $j$  even. Let  $P$  and  $\alpha$  be defined by (2.6) and (2.5). For  $(j, k) \in \Sigma_\kappa$  we have

$$\begin{aligned} i\partial_s((ib)^j \lambda^k P_{j,k}) &= -j(ib)^{j-1} b_s \lambda^k P_{j,k} + i(ib)^j k \lambda^{k-1} \lambda_s P_{j,k} \\ &= -j(ib)^{j+1} \lambda^k P_{j,k} - j(ib)^{j-1} \alpha \lambda^k P_{j,k} - (ib)^{j+1} k \lambda^k P_{j,k} + \mathcal{O}_{C_{\text{exp}}^1}(b, \lambda, \alpha) \\ &= -(j+k)(ib)^{j+1} \lambda^k P_{j,k} - \sum_{\substack{(p,q) \in \Sigma_\kappa \\ p \text{ even}}} j \alpha_{p,q} (ib)^{j-1+p} \lambda^{k+q} P_{j,k} + \mathcal{O}_{C_{\text{exp}}^1}(b, \lambda, \alpha). \end{aligned}$$

Notice that some terms in the sum are actually in  $\mathcal{O}_{C_{\text{exp}}^1}(b, \lambda, \alpha)$ . On the other hand there exists a family  $(\Phi_{j,k})_{(j,k) \in \Sigma_\kappa}$  in  $C_{\text{exp}}^1$  which only depends on  $Q$  and the  $P_{j,k}$ ,  $(j, k) \in \Sigma_\kappa$ , such that

$$|P|^4 P = Q^5 + \sum_{(j,k) \in \Sigma_\kappa} (ib)^j \lambda^k \Phi_{j,k} + \mathcal{O}_{C_{\text{exp}}^1}(b, \lambda, \alpha).$$

In particular  $\Phi_{0,1} = 5Q^4 P_{0,1}$ . Then we have

$$\Psi_\kappa = \sum_{(j,k) \in \Sigma_\kappa} (ib)^j \lambda^k \Psi_{j,k} + \mathcal{O}_{C_{\text{exp}}^1}(b, \lambda, \alpha)$$

where, for  $(j, k) \in \Sigma_\kappa$ ,

$$\begin{aligned} \Psi_{j,k} &= -(j-1+k)P_{j-1,k} + \sum_{\substack{(p,q) \in \Sigma_\kappa \\ p \text{ even}}} (j+1-p)\alpha_{p,q} P_{j+1-p, k-q} \\ &\quad + \partial_{yy} P_{j,k} - P_{j,k} - \gamma \delta P_{j,k-1} + \Phi_{j,k} + \sum_{\substack{p_1+p_2=j \\ q_1+q_2=k \\ p_1 \text{ even}}} \alpha_{p_1, q_1} \frac{y^2}{4} P_{p_2, q_2}. \end{aligned}$$

We have used the convention that

$$P_{p,q} = \begin{cases} Q & \text{if } p = 0 \text{ and } q = 0, \\ 0 & \text{if } p < 0 \text{ or } q \leq 0. \end{cases}$$

We now show that we can choose the  $P_{j,k}$  and  $\alpha_{j,k}$  in such a way that  $\Psi_{j,k} = 0$  for all  $(j, k) \in \Sigma_\kappa$ . For  $(j_1, k_1), (j_2, k_2) \in \mathbb{Z}^2$  we say that  $(j_1, k_1) < (j_2, k_2)$  if  $k_1 < k_2$  or  $(k_1 = k_2 \text{ and } j_1 < j_2)$ .

Let  $(m, k) \in \mathbb{N} \times \mathbb{N}^*$  with  $m + k < \kappa$ . Assume that for all  $(j', k') \in \Sigma_\kappa$  with  $(j', k') < (2m, k)$  we have defined  $\alpha_{j', k'}$  (if  $j'$  is even) and  $P_{j', k'}$  in such a way that  $\Psi_{\tilde{j}, \tilde{k}} = 0$  for all  $(\tilde{j}, \tilde{k}) \in \Sigma_\kappa$  with  $(\tilde{j}, \tilde{k}) < (2m, k)$ . For  $j \in \{2m, 2m + 1\}$  we have

$$\Phi_{j,k} = (3 + 2(-1)^j)Q^4 P_{j,k} + \tilde{\Phi}_{j,k},$$

for some  $\tilde{\Phi}_{j,k} \in C_{\text{exp}}^1$  which only depends on  $P_{j', k'}$  with  $(j', k') < (j, k)$ . Then for some  $\tilde{\Psi}_{2m,k}$  and  $\tilde{\Psi}_{2m+1,k}$  in  $C_{\text{exp}}^1$  (which depend on  $P_{j', k'}$  and  $\alpha_{j', k'}$  for  $(j', k') < (2m, k)$ ) we have

$$\begin{aligned} \Psi_{2m,k} &= -L_+ P_{2m,k} - \gamma \delta P_{2m,k-1} + \alpha_{2m,k} \frac{y^2}{4} Q + \tilde{\Psi}_{2m,k}, \\ \Psi_{2m+1,k} &= -L_- P_{2m+1,k} - \gamma \delta P_{2m+1,k-1} - (2m+k)P_{2m,k} + \tilde{\Psi}_{2m+1,k}. \end{aligned}$$

By Proposition 3.4 there exists  $P_{2m,k} \in H^1(\mathcal{G})$  such that  $\Psi_{2m,k} = 0$  for any choice of  $\alpha_{2m,k} \in \mathbb{R}$ . Moreover,  $P_{2m,k} \in C_{\text{exp}}^1$  by Lemma 4.1. We choose  $\alpha_{2m,k}$  in such a way that (see Proposition 3.4 again)

$$(2m+k)P_{2m,k} + \gamma \delta P_{2m+1,k-1} - \tilde{\Psi}_{2m+1,k} \in \text{span}\{Q\}^\perp = \text{Ran}(L_-). \quad (4.1)$$

Using  $L_+ \Lambda Q = -2Q$  and the selfadjointness of  $L_+$ , this condition reads

$$(2m+k) \langle L_+ P_{2m,k}, \Lambda Q \rangle + \left\langle \gamma \delta P_{2m+1,k-1} - \tilde{\Psi}_{2m+1,k}, L_+ \Lambda Q \right\rangle = 0,$$

that is,

$$\alpha_{2m,k} = \frac{4}{\|yQ\|_{L^2}^2} \left( \left\langle -\gamma \delta P_{2m,k-1} + \tilde{\Psi}_{2m,k}, \Lambda Q \right\rangle + \frac{1}{2m+k} \left\langle \gamma \delta P_{2m+1,k-1} - \tilde{\Psi}_{2m+1,k}, L_+ \Lambda Q \right\rangle \right).$$

By (4.1), we can then choose  $P_{2m+1,k} \in H^1(\mathcal{G})$  (defined up to a multiple of  $Q$ ) such that  $\Psi_{2m+1,k} = 0$ . Again, by Lemma 4.1 we have  $P_{2m+1,k} \in C_{\text{exp}}^1$ . The base case of the above induction process is given by the equations

$$\begin{aligned} 0 &= -L_+ P_{0,1} - \gamma \delta Q + \alpha_{0,1} \frac{y^2}{4} Q, \\ 0 &= -L_- P_{1,1} - P_{0,1}, \end{aligned}$$

from which we can compute explicitly  $\alpha_{0,1}$  using the results of Section 3.3:

$$\begin{aligned} 0 &= \langle P_{0,1}, Q \rangle = \left\langle P_{0,1}, -\frac{1}{2} L_+ \Lambda Q \right\rangle = -\frac{1}{2} \langle L_+ P_{0,1}, \Lambda Q \rangle \\ &= -\frac{1}{2} \left\langle -\gamma \delta Q + \alpha_{0,1} \frac{y^2}{4} Q, \Lambda Q \right\rangle = \frac{1}{2} \gamma \langle \delta Q, \Lambda Q \rangle - \frac{\alpha_{0,1}}{8} \langle y^2 Q, \Lambda Q \rangle \\ \implies \alpha_{0,1} &= \frac{4\gamma \langle \delta Q, \Lambda Q \rangle}{\langle y^2 Q, \Lambda Q \rangle} = \frac{4\gamma Q(0) \Lambda Q(0)}{\int_{\mathcal{G}} y^2 Q \Lambda Q} = -2\gamma \frac{Q(0)^2}{\|yQ\|_{L^2}^2}. \end{aligned} \quad (4.2)$$

This gives the formula for the coefficient  $\beta$  in the model dynamical system (2.13), and completes the proof of (2.5)-(2.8).

Formula (2.9), (2.10) and (2.11) can be proved in the same way as in [37, 52].  $\square$

## 5. MODULATION

In this section we prove Proposition 2.2. The strategy is classical but for the reader's convenience we provide a proof adapted to our context.

For  $\pi = (\theta, b, \lambda) \in \mathbb{S}^1 \times \mathbb{R} \times \mathbb{R}_+$  and  $u = (u_j) \in L^2(\mathcal{G})$  we define  $\Theta_\pi v \in \mathcal{L}(L^2(\mathcal{G}))$  by

$$(\Theta_\pi u)_j(x) = \frac{1}{\sqrt{\lambda}} e^{i\theta} e^{-\frac{ibx^2}{4\lambda^2}} u_j\left(\frac{x}{\lambda}\right).$$

This defines a unitary operator  $\Theta_\pi$  on  $L^2(\mathcal{G})$ .

For  $\delta > 0$  we set

$$\mathcal{Q}_\delta = \bigcup_{\theta \in \mathbb{R}, \lambda \in ]0, \delta[} B_{L^2(\mathcal{G})}(\Theta_{\theta,0,\lambda} Q, \delta).$$

It is endowed with the topology inherited from  $L^2(\mathcal{G})$ . For  $\eta > 0$  we also set

$$\Omega_\eta = \mathbb{S}^1 \times (-\eta, \eta) \times (0, \eta).$$

**Proposition 5.1.** *Let  $\eta > 0$ . There exist  $\delta > 0$  and a function  $\pi = (\theta, b, \lambda) \in C^1(\mathcal{Q}_\delta, \Omega_\eta)$  such that for any  $u \in \mathcal{Q}_\delta$  we have in  $L^2(\mathcal{G})$*

$$\Theta_{\pi(u)}^{-1} u - P(b(u), \lambda(u)) \in \{y^2 P(b(u), \lambda(u)), i\Lambda P(b(u), \lambda(u)), i\rho\}^\perp,$$

where  $P(b, \lambda)$  is as defined in Proposition 2.1.

*Proof.* For  $\pi = (\theta, b, \lambda) \in \mathbb{S}^1 \times \mathbb{R} \times \mathbb{R}_+$ ,  $\beta \in \mathbb{R}$  and  $v \in L^2(\mathcal{G})$  we set

$$h(\pi; v, \beta) = \Theta_\pi^{-1} v - P_{b,\beta\lambda} \quad \text{and} \quad F(\pi; v, \beta) = \begin{pmatrix} (h(\pi, v, \beta), y^2 P_{b,\beta\lambda})_{L^2(\mathcal{G})} \\ (h(\pi, v, \beta), i\Lambda P_{b,\beta\lambda})_{L^2(\mathcal{G})} \\ (h(\pi, v, \beta), i\rho)_{L^2(\mathcal{G})} \end{pmatrix}.$$

This defines functions of class  $C^1$  on  $\mathbb{S}^1 \times \mathbb{R} \times \mathbb{R}_+ \times L^2(\mathcal{G}) \times \mathbb{R}$ . Moreover we have  $h(0, 0, 1; Q, 0) = 0$  and  $F(0, 0, 1; Q, 0) = 0$  (the interest of the extra parameter  $\beta$  is that we can start the analysis around  $\lambda = 1$  and  $Q = P(b = 0, \lambda = 0)$ ). We have  $(\partial_b P_{b,\beta\lambda}, \partial_\lambda P_{b,\beta\lambda})|_{b=0, \beta=0, \lambda=1} = (0, 0)$ , so

$$\nabla_{\theta,b,\lambda} h(0, 0, 1; Q, 0) = \begin{pmatrix} -iQ \\ \frac{iy^2}{4} Q \\ \Lambda Q \end{pmatrix}.$$

By Lemma 3.5 we have  $(Q, \Lambda Q) = 0$ , so

$$\text{Jac}_{\theta,b,\lambda} F(0, 0, 1; Q, 0) = \begin{pmatrix} 0 & 0 & (\Lambda Q, y^2 Q) \\ 0 & \frac{1}{4}(y^2 Q, \Lambda Q) & 0 \\ -(Q, \rho) & \frac{1}{4}(y^2 Q, \rho) & 0 \end{pmatrix}.$$

We also have  $(y^2 Q, \Lambda Q) \neq 0$  and  $(Q, \rho) \neq 0$ , so this partial jacobian is invertible. By the Implicit Function Theorem, there exist a neighborhood  $\mathcal{U} \subset \mathbb{S}^1 \times (-\eta, \eta) \times (0, 2)$  of  $(0, 0, 1)$ , a neighborhood  $\mathcal{V}$  of  $(Q, 0)$  in  $L^2(\mathcal{G}) \times \mathbb{R}$  and a function  $\Pi_0 = (\theta_0, b_0, \lambda_0) : \mathcal{V} \rightarrow \mathcal{U}$  of class  $C^1$  such that for all  $\pi \in \mathcal{U}$  and  $(v, \beta) \in \mathcal{V}$  we have

$$F(\pi; v, \beta) = 0 \quad \iff \quad \pi = \Pi_0(v, \beta).$$

We fix  $\delta > 0$  so small that  $B(Q, \delta) \times (-\delta, \delta) \subset \mathcal{V}$ .

Let  $u \in \mathcal{Q}_\delta$ . Let  $\theta_1, \theta_2 \in \mathbb{S}^1$  and  $\lambda_1, \lambda_2 \in ]0, \delta[$  be such that  $v_1 = \Theta_{\theta_1, 0, \lambda_1}^{-1} u$  and  $v_2 = \Theta_{\theta_2, 0, \lambda_2}^{-1} u$  belong to  $B(Q, \delta)$ . By definition we have

$$\Theta_{\theta_0(v_1, \lambda_1), b_0(v_1, \lambda_1), \lambda(v_1, \lambda_1)}^{-1} u - \tilde{P} \in \{y^2 \tilde{P}, i\Lambda \tilde{P}, i\rho\}^\perp,$$

where we have set  $\tilde{P} = P(b_0(v_1, \lambda_1), \lambda_1 \lambda_0(v_1, \lambda_1))$ . Since  $v_1 = \Theta_{\theta_1 - \theta_2, 0, \lambda_1 / \lambda_2}^{-1} v_2$  we also have

$$\Theta_{\theta_0(v_1, \lambda_1) + \theta_1 - \theta_2, b_0(v_1, \lambda_1), \lambda(v_1, \lambda_1) \lambda_1 / \lambda_2} - \tilde{P} \in \{y^2 \tilde{P}, i\Lambda \tilde{P}, i\rho\}^\perp,$$

where we can also write  $\tilde{P} = P(b_0(v_1, \lambda_1), \lambda_2 \lambda_0(v_1, \lambda_1) \lambda_1 / \lambda_2)$ . This proves that

$$\theta_0(v_2, \lambda_2) + \theta_2 = \theta_0(v_1, \lambda_1) + \theta_1, \quad b_0(v_2, \lambda_2) = b_0(v_1, \lambda_1), \quad \lambda_0(v_2, \lambda_2) \lambda_2 = \lambda_0(v_1, \lambda_1) \lambda_1.$$

Thus we can set

$$\theta(u) = \theta_0(v_1, \lambda_1) + \theta_1, \quad b(u) = b_0(v_1, \lambda_1), \quad \lambda(u) = \lambda_0(v_1, \lambda_1) \lambda_1, \quad (5.1)$$

and this definition does not depend on the choice of  $\theta_1$  or  $\lambda_1$ . This defines a function  $\pi = (\theta, b, \lambda) \in C^1(\mathcal{Q}_\delta, \Omega_\eta)$ . Moreover we have

$$\begin{aligned} \Theta_{\theta(u), b(u), \lambda(u)}^{-1} u - P(b(u), \lambda(u)) &= \Theta_{\theta_0(v_1, \lambda_1), 0, \lambda_0(v_1, \lambda_1)}^{-1} v_1 - P(b_0(v_1, \lambda_1), \lambda_1 \lambda_0(v_1, \lambda_1)) \\ &\in \{y^2 P(b(u), \lambda(u)), i\Lambda P(b(u), \lambda(u)), i\rho\}^\perp, \end{aligned}$$

which completes the proof.  $\square$

The function  $\pi = (\theta, b, \lambda)$  defined on  $\mathcal{Q}_\delta$  in the previous proposition is of class  $C^1$  if  $\mathcal{Q}_\delta$  is endowed with the topology of  $L^2(\mathcal{G})$ . However, a typical solution  $u(t)$  of (1.1) is only of class  $C^1$  in  $H^{-1}(\mathcal{G})$ , and continuous in  $H^1(\mathcal{G})$  (hence in  $L^2(\mathcal{G})$ ). To prove Proposition 2.2, we will use the fact that a solution of (1.1) can be approximated by a regular solution, of class  $C^1$  in  $L^2(\mathcal{G})$ .

**Proposition 5.2.** *Let  $I$  be an interval of  $\mathbb{R}$ . Let  $u \in C^0(I, L^2(\mathcal{G})) \cap C^1(I, H^{-1}(\mathcal{G}))$ . Assume that there exists a sequence  $(u_k)_{k \in \mathbb{N}}$  in  $C^1(I, L^2(\mathcal{G}))$  which goes to  $u$  in  $C^0(I, L^2(\mathcal{G})) \cap C^1(I, H^{-1}(\mathcal{G}))$ . Then the map  $\pi \circ u$  (with  $\pi$  given by Proposition 5.1) is of class  $C^1$  on  $I$ .*

*Proof.* We use the notation introduced in the proof of Proposition 5.1.

For  $t \in I$  we set  $\pi(t) = (\theta(t), b(t), \lambda(t)) = \pi(u(t))$ . We fix  $\tau_1 \in I$  and prove that  $\pi \circ u$  is of class  $C^1$  on a neighborhood of  $\tau_1$ . Let  $\theta_1 \in \mathbb{S}^1$  and  $\lambda_1 \in ]0, \delta[$  be such that  $\Theta_{\theta_1, 0, \lambda_1}^{-1} u(\tau_1) \in B(Q, \delta)$ . For  $t \in I$  we set  $v(t) = \Theta_{\theta_1, 0, \lambda_1}^{-1} u(t)$ , and for  $k \in \mathbb{N}$  we set  $v_k(t) = \Theta_{\theta_1, 0, \lambda_1}^{-1} u_k(t)$ . Let  $I_1$  be a neighborhood of  $\tau_1$  in  $I$  such that  $v(t) \in B(Q, \delta)$  and  $v_k(t) \in B(Q, \delta)$  for all  $t \in I_1$  and  $k \in \mathbb{N}$ . For  $t \in I_1$  we have by (5.1)

$$\theta(t) = \theta_0(v(t), \lambda_1) + \theta_1, \quad b(t) = b_0(v(t), \lambda_1), \quad \lambda(t) = \lambda_0(v(t), \lambda_1) \lambda_1,$$

so it is enough to prove that  $t \mapsto \Pi_0(v(t), \lambda_1)$  is of class  $C^1$  on  $I_1$ . Notice that  $v_k$  belongs to  $C^1(I_1, L^2(\mathcal{G}))$  and goes to  $v$  in  $C^0(I_1, L^2(\mathcal{G})) \cap C^1(I_1, H^{-1}(\mathcal{G}))$  as  $k \rightarrow \infty$ .

For  $t \in I_1$  we set  $\tilde{\pi}(t) = \Pi_0(v(t), \lambda_1)$ , and for  $k \in \mathbb{N}$  we set  $\tilde{\pi}_k(t) = (\tilde{\theta}_k(t), \tilde{b}_k(t), \tilde{\lambda}_k(t)) = \Pi_0(v_k(t), \lambda_1)$ . We have  $\tilde{\pi} \in C^0(I_1, \Omega_\eta)$  and  $\tilde{\pi}_k \in C^1(I_1, \Omega_\eta)$ . Let  $k \in \mathbb{N}$ . For all  $t \in I_1$  we have  $F(\tilde{\pi}_k(t); v_k(t), \lambda_1) = 0$ . After differentiation we get for  $t \in I_1$

$$\tilde{\pi}'_k(t) = -D_\pi F(\tilde{\pi}_k(t), v_k(t), \lambda_1)^{-1} \cdot D_v F(\tilde{\pi}_k(t), v_k(t), \lambda_1) \cdot v'_k(t).$$

For  $t \in J$  we set  $P_k(t) = P_{b_k(t), \lambda_1 \tilde{\lambda}_k(t)}$ . Then

$$D_v F(\tilde{\pi}(t), v_k(t), \lambda_1) \cdot v'_k(t) = \begin{pmatrix} (\Theta_{\tilde{\pi}_k(t)}^{-1} v'_k(t), y^2 P_k(t)) \\ (\Theta_{\tilde{\pi}_k(t)}^{-1} v'_k(t), i\Lambda P_k(t)) \\ (\Theta_{\tilde{\pi}_k(t)}^{-1} v'_k(t), i\rho) \end{pmatrix} = \begin{pmatrix} \langle v'_k(t), \Theta_{\tilde{\pi}_k(t)}(y^2 P_k(t)) \rangle \\ \langle v'_k(t), \Theta_{\tilde{\pi}_k(t)}(i\Lambda P_k(t)) \rangle \\ \langle v'_k(t), \Theta_{\tilde{\pi}_k(t)}(i\rho) \rangle \end{pmatrix}.$$

Then  $\pi'_k$  is continuous on  $I_1$ . For all  $t \in I_1$  we have

$$\tilde{\pi}_k(t) = \tilde{\pi}_k(\tau_1) + \int_{\tau_1}^t \tilde{\pi}'_k(\tau) d\tau.$$

Taking the limit  $k \rightarrow \infty$  gives for all  $t \in J$

$$\tilde{\pi}(t) = \tilde{\pi}(\tau_1) - \int_{\tau_1}^t D_\pi F(\tilde{\pi}(t), v(t), \lambda_1)^{-1} \begin{pmatrix} \langle v'(t), \Theta_{\tilde{\pi}(t)}(y^2 P(t)) \rangle \\ \langle v'(t), \Theta_{\tilde{\pi}(t)}(i\Lambda P(t)) \rangle \\ \langle v'(t), \Theta_{\tilde{\pi}(t)}(i\rho) \rangle \end{pmatrix} d\tau,$$

where  $P(t) = P_{\tilde{b}(t), \lambda_1 \tilde{\lambda}(t)}$ . The integrand is a continuous functions of  $\tau$ , so  $\tilde{\pi}$  is of class  $C^1$  on  $I_1$ .  $\square$

Now we have all the ingredients for the proof of Proposition 2.2.

*Proof of Proposition 2.2.* Let  $\tau_1 \in I$  and  $u_1 = u(\tau_1)$ . Let  $K$  be a compact neighborhood of  $\tau_1$  in  $I$ . Since  $D(H_\gamma)$  is dense in  $L^2(\mathcal{G})$  there exists a sequence  $(u_{1,k})_{n \in \mathbb{N}}$  in  $D(H_\gamma)$  which goes to  $u_1$  in  $L^2(\mathcal{G})$ . For  $k \in \mathbb{N}$  we denote by  $u_k \in C^0(I_k, D(H_\gamma)) \cap C^1(I_k, L^2(\mathcal{G}))$  the maximal solution (defined on the interval  $I_k$ ) of (1.1) such that  $u_k(\tau_1) = u_{1,k}$ . Removing a finite number of terms if necessary, we can assume that  $K \subset I_k$  for all  $k \in \mathbb{N}$ , and  $u_k$  goes to  $u$  in  $C^0(K, H^1(\mathcal{G})) \cap C^1(K, H^{-1}(\mathcal{G}))$ . Then we can apply Propositions 5.1 and 5.2, which gives Proposition 2.2.  $\square$

## 6. PROOF OF THE UNIFORM ESTIMATES

Using  $\tilde{\lambda}$ , we can now define precisely the rescaled time variable  $s$  that appears in the formal change of variables (2.1)–(2.2). We let

$$s := s(t) = s_1 - \int_t^{t_1} \frac{1}{\tilde{\lambda}(\tau)^2} d\tau \quad (6.1)$$

where the final time  $s_1$  is computed as follows by means of the solution

$$\lambda_{\text{mo}}(s) = \frac{2}{\beta} \frac{1}{s^2}$$

of the model dynamical system (2.13):

$$dt = \lambda_{\text{mo}}^2(s) ds \implies |t_1| = -t_1 = \frac{4}{\beta^2} \int_{s_1}^{+\infty} s^{-4} ds \implies s_1 = \left( \frac{4}{3\beta^2} \right)^{1/3} |t_1|^{-1/3}.$$

Observe that  $s$  is a strictly increasing function of  $t$ . Hence,  $t$  may in turn be expressed as a function of  $s$ . This will allow us to obtain Proposition 2.6 as a consequence of the uniform estimates in variable  $s$  which are stated below, in Proposition 6.1.

We then express the modulation parameters  $\tilde{b}$ ,  $\tilde{\lambda}$  and  $\tilde{\theta}$  as functions of the variable  $s$  by setting

$$b(s) := \tilde{b}(t(s)), \quad \lambda(s) = \tilde{\lambda}(t(s)), \quad \theta(s) = \tilde{\theta}(t(s)). \quad (6.2)$$

Next, in the change of variables (2.1)–(2.2), we write

$$w = P + h,$$

where  $P = P_{b,\lambda}$  is defined in (2.5). That is, in the original variables,

$$u = \lambda^{-1/2} e^{i(\theta - b\frac{y^2}{4})} (P + h).$$

For  $v \in \Sigma$ , we let

$$\widetilde{\text{Mod}}_{\text{op}}(s)v := (1 - \theta_s)v + (b_s + b^2)\frac{y^2}{4}v - i\left(b + \frac{\lambda_s}{\lambda}\right)\Lambda v - b\left(b + \frac{\lambda_s}{\lambda}\right)\frac{y^2}{2}v$$

and

$$\text{Mod}_{\text{op}}(s)v := (1 - \theta_s)v + (b_s + b^2 - \alpha)\frac{y^2}{4}v - i\left(b + \frac{\lambda_s}{\lambda}\right)\Lambda v.$$

Hence,

$$\widetilde{\text{Mod}}_{\text{op}}(s)v = \text{Mod}_{\text{op}}(s)v + \alpha\frac{y^2}{4}v - b\left(b + \frac{\lambda_s}{\lambda}\right)\frac{y^2}{2}v.$$

With this notation, (2.3) becomes

$$ih_s + h_{yy} - h - \gamma\lambda\delta h + f(P+h) - f(P) + \widetilde{\text{Mod}}_{\text{op}}(s)h = -\Psi_K - \text{Mod}_{\text{op}}(s)P + b\left(b + \frac{\lambda_s}{\lambda}\right)\frac{y^2}{2}P. \quad (6.3)$$

For any fixed  $\lambda > 0$ , we define the norm  $\|\cdot\|_\lambda$  on  $\Sigma$ , equivalent to the usual norm  $\|\cdot\|_1$ , by

$$\|v\|_\lambda^2 := \|v\|_{H^1}^2 + \lambda\|yv\|_{L^2}^2, \quad v \in \Sigma. \quad (6.4)$$

We now let  $s_\star$  be the infimum of  $\sigma \in [s_0, s_1]$  such that, for all  $s \in [\sigma, s_1]$ ,

$$\|h(s)\|_\lambda \lesssim s^{-(\kappa-2)}, \quad \left|\frac{\lambda(s)^{1/2}}{\lambda_{\text{mo}}(s)^{1/2}} - 1\right| \lesssim \frac{1}{s^{1/2}}, \quad \left|\frac{b(s)}{b_{\text{mo}}(s)} - 1\right| \lesssim \frac{1}{s^{1/2}}, \quad (6.5)$$

for some  $\kappa \in \mathbb{N}$ ,  $\kappa \geq 7$ . Since, by construction,  $h(s_1) = 0$ , it follows from Proposition 2.5 that (6.5) is satisfied at  $s = s_1$ . Hence, by continuity,  $s_\star \in [s_0, s_1]$ .

We will use a bootstrap argument involving (6.5) to prove the following uniform estimates in  $s$ , from which Proposition 2.6 will follow using the change of variables (6.1).

**Proposition 6.1** (Uniform estimates in the  $s$  variable). *There exists  $s_0$  independent of  $s_1$  such that the solution  $u_1$  of (1.1) defined by (2.17) exists and, under the change of variables (6.1), satisfies (2.12) on  $[s_0, s_1]$ . Moreover, the corresponding functions  $b(s)$ ,  $\lambda(s)$  and  $h(s)$  given by Proposition 2.2 satisfy*

$$\|h(s)\|_{\lambda(s)} \lesssim s^{-(\kappa-1)}, \quad (6.6)$$

$$\left|\frac{\lambda(s)^{1/2}}{\lambda_{\text{mo}}(s)^{1/2}} - 1\right| \lesssim \frac{1}{s^2}, \quad \left|\frac{b(s)}{b_{\text{mo}}(s)} - 1\right| \lesssim \frac{1}{s^2}, \quad s \in [s_0, s_1]. \quad (6.7)$$

*Remark 6.2.* The estimates (6.7) can be improved to

$$\left|\frac{\lambda(s)^{1/2}}{\lambda_{\text{mo}}(s)^{1/2}} - 1\right| \lesssim \frac{1}{s^4}, \quad \left|\frac{b(s)}{b_{\text{mo}}(s)} - 1\right| \lesssim \frac{1}{s^4}, \quad s \in [s_0, s_1], \quad (6.8)$$

by shifting the energy level in the definition of  $\mathcal{F}$  in (2.15) to  $\underline{\mathcal{E}}^\star = \mathcal{E}^\star - (\varepsilon_{0,1} + 2\beta\varepsilon_{2,0})$  (see the proof of Proposition 6.1).

**6.1. Modulation estimates.** We now justify quantitatively that, for large times, the modulation parameters are approximate solutions of the model dynamical system (2.13).

Let

$$\text{Mod}(s) := \begin{bmatrix} b + \lambda_s/\lambda \\ b_s + b^2 - \alpha \\ 1 - \theta_s \end{bmatrix}, \quad s \in [s_0, s_1].$$

**Lemma 6.3.** *For all  $s \in [s_\star, s_1]$ , there holds*

$$|\text{Mod}(s)| \lesssim s^{-\kappa} \quad (6.9)$$

$$|(h(s), Q)_{L^2}| \lesssim s^{-\kappa}. \quad (6.10)$$

*Proof.* Let us first define

$$s_{**} := \inf \left\{ \sigma \in [s_*, s_1] : |(h(s), P_{b,\lambda})_{L^2}| < s^{-\kappa} \ \forall s \in [\sigma, s_1] \right\}.$$

Since  $h(s_1) = 0$ , we have  $s_{**} \in [s_*, s_1]$ . We will show by a bootstrap argument that  $s_{**} = s_*$ , from which (6.10) easily follows. This will come as a by-product of (6.9), which we now prove using (6.3) and the orthogonality conditions from Proposition 2.2 on the interval  $[s_{**}, s_1]$ .

We start by differentiating  $(h(s), i\Lambda P) = 0$  with respect to  $s$ :

$$\langle h_s, i\Lambda P \rangle + \langle h, i\Lambda P_s \rangle = 0, \quad s \in [s_*, s_1]. \quad (6.11)$$

Using (2.5), (2.6) and (6.5), we have that

$$|(h, i\Lambda P_s)| \lesssim \|h\|_{L^2} (|\text{Mod}(s)| + s^{-2}) \lesssim s^{-(\kappa-2)} |\text{Mod}(s)| + s^{-\kappa} \lesssim s^{-2} |\text{Mod}(s)| + s^{-\kappa}.$$

On the other hand,  $\langle h_s, i\Lambda P \rangle = -\langle ih_s, \Lambda P \rangle$ . Since

$$f(P+h) - f(P) = df(P)h + O(|h|^2) = df(P)h + O(s^{-(\kappa-2)}|h|) = df(Q)h + O(s^{-2}|h|),$$

the equation for  $h$  reads

$$\begin{aligned} ih_s &= L_+ h_1 + iL_- h_2 + \gamma\lambda\delta h + O(s^{-2}|h|) \\ &\quad - \left[ \widetilde{\text{Mod}}_{\text{op}}(s)h + \text{Mod}_{\text{op}}(s)P - b\left(b + \frac{\lambda_s}{\lambda}\right)\frac{y^2}{2}P + \Psi_\kappa \right]. \end{aligned} \quad (6.12)$$

Now, the definition of  $s_{**}$  yields

$$\begin{aligned} \langle L_+ h_1 + iL_- h_2 + \gamma\lambda\delta h + O(s^{-2}|h|), \Lambda P \rangle &= \langle L_+ h_1 + iL_- h_2 + \gamma\lambda\delta h, \Lambda Q \rangle + O(s^{-2}\|h\|_{L^2}) \\ &= \langle h_1, L_+ \Lambda Q \rangle - \langle L_- h_2, i\Lambda Q \rangle + \gamma\lambda h(0)\Lambda Q(0) + O(s^{-2}\|h\|_{L^2}) \\ &= -2\langle h, Q \rangle + \gamma\lambda h(0)\Lambda Q(0) + O(s^{-2}\|h\|_{L^2}) \\ &= -2\langle h, P \rangle + \gamma\lambda h(0)\Lambda Q(0) + O(s^{-2}\|h\|_{L^2}) \\ &= O(s^{-\kappa}), \quad \forall s \in [s_{**}, s_1]. \end{aligned}$$

The second line of (6.12) has three terms. Firstly,

$$\begin{aligned} \left\langle \widetilde{\text{Mod}}_{\text{op}}(s)h, \Lambda P \right\rangle &= \left\langle \text{Mod}_{\text{op}}(s)h + \alpha\frac{y^2}{4}h - b\left(b + \frac{\lambda_s}{\lambda}\right)\frac{y^2}{2}h, \Lambda P \right\rangle \\ &= O(|\text{Mod}(s)|\|h\|_{L^2}) + O(b|\text{Mod}(s)|\|h\|_{L^2}) + O(|\alpha(s)|\|h\|_{L^2}) \\ &= O(|\text{Mod}(s)|\|h\|_{L^2}) + O(s^{-\kappa}). \end{aligned}$$

Next, by Lemma 3.5,

$$\begin{aligned} &\langle \text{Mod}_{\text{op}}(s)P, \Lambda P \rangle - \left\langle b\left(b + \frac{\lambda_s}{\lambda}\right)\frac{y^2}{2}P, \Lambda P \right\rangle \\ &= \langle \text{Mod}_{\text{op}}(s)Q, \Lambda Q \rangle - \left\langle b\left(b + \frac{\lambda_s}{\lambda}\right)\frac{y^2}{2}Q, \Lambda Q \right\rangle + O(s^{-2}|\text{Mod}(s)|) \\ &= -\frac{1}{4}(b_s + b^2 - \alpha)\|yQ\|_{L^2}^2 - \left\langle \left(b + \frac{\lambda_s}{\lambda}\right)\Lambda Q, \Lambda Q \right\rangle + \frac{1}{2}b\left(b + \frac{\lambda_s}{\lambda}\right)\|yQ\|_{L^2}^2 + O(s^{-2}|\text{Mod}(s)|) \\ &= \frac{1}{4}\|yQ\|_{L^2}^2 \left[ 2b\left(b + \frac{\lambda_s}{\lambda}\right) - (b_s + b^2 - \alpha) \right] + O(s^{-2}|\text{Mod}(s)|). \end{aligned}$$

Finally,

$$\langle \Psi_\kappa, \Lambda P \rangle = O(s^{-2}|\text{Mod}(s)|) + O(s^{-2\kappa}).$$



All in all, we find that

$$\langle h_s, i\Lambda P \rangle = \frac{1}{4} \|yQ\|_{L^2}^2 \left[ 2b \left( b + \frac{\lambda_s}{\lambda} \right) - (b_s + b^2 - \alpha) \right] + O(s^{-2} |\text{Mod}(s)|) + O(s^{-\kappa}).$$

Hence, the restriction of (6.11) to  $[s_{**}, s_1]$  gives

$$(b_s + b^2 - \alpha) - 2b \left( b + \frac{\lambda_s}{\lambda} \right) = O(s^{-2} |\text{Mod}(s)|) + O(s^{-\kappa}). \quad (6.13)$$

We next differentiate  $(h(s), y^2 P) = 0$  with respect to  $s$ :

$$\langle h_s, y^2 P \rangle + \langle h, y^2 P_s \rangle = 0, \quad s \in [s_*, s_1]. \quad (6.14)$$

As above, we obtain

$$|\langle h, y^2 P_s \rangle| \lesssim \|h\|_{L^2} (|\text{Mod}(s)| + b^2 + \lambda) \lesssim s^{-2} |\text{Mod}(s)| + s^{-\kappa}.$$

On the other hand,  $\langle h_s, y^2 P \rangle = \langle ih_s, iy^2 P \rangle$ . We shall again use (6.12) on  $[s_{**}, s_1]$ . It follows by Proposition 2.2 and (6.5) that

$$\begin{aligned} \langle L_+ h_1 + iL_- h_2 + \gamma\lambda\delta h + O(s^{-2}|h|), iy^2 P \rangle &= \langle L_+ h_1 + iL_- h_2 + \gamma\lambda\delta h, iy^2 Q \rangle + O(s^{-2} \|h\|_{L^2}) \\ &= \langle h_2, L_- y^2 Q \rangle + O(s^{-2} \|h\|_{L^2}) \\ &= -4 \langle h_2, \Lambda Q \rangle + O(s^{-2} \|h\|_{L^2}) \\ &= -4 \langle h, i\Lambda P \rangle + O(s^{-2} \|h\|_{L^2}) \\ &= O(s^{-\kappa}), \quad \forall s \in [s_{**}, s_1]. \end{aligned}$$

Using Lemma 3.5 and (6.5), similar calculations as above yield

$$\begin{aligned} \left\langle \widetilde{\text{Mod}}_{\text{op}}(s)h + \text{Mod}_{\text{op}}(s)P - b \left( b + \frac{\lambda_s}{\lambda} \right) \frac{y^2}{2} P + \Psi_\kappa, iy^2 P \right\rangle \\ = \|yQ\|_{L^2}^2 \left( b + \frac{\lambda_s}{\lambda} \right) + O(s^{-2} |\text{Mod}(s)|) + O(s^{-\kappa}). \end{aligned}$$

Hence, gathering all terms of (6.14), we find that

$$\left| b + \frac{\lambda_s}{\lambda} \right| \lesssim s^{-2} |\text{Mod}(s)| + s^{-\kappa}, \quad \forall s \in [s_{**}, s_1]. \quad (6.15)$$

It then follows from (6.13) that

$$|b_s + b^2 - \alpha| \lesssim s^{-2} |\text{Mod}(s)| + s^{-\kappa}, \quad \forall s \in [s_{**}, s_1]. \quad (6.16)$$

Finally, differentiating  $(h(s), i\rho) = 0$  with respect to  $s$  yields

$$\langle ih_s, \rho \rangle = 0, \quad s \in [s_*, s_1].$$

Using again Proposition 2.2 and (6.5), we have

$$\begin{aligned} \langle L_+ h_1 + iL_- h_2 + \gamma\lambda\delta h + O(s^{-2}|h|), \rho \rangle &= \langle h_1, L_+ \rho \rangle + O(s^{-2} \|h\|_{L^2}) \\ &= \langle h_1, y^2 Q \rangle + O(s^{-2} \|h\|_{L^2}) \\ &= O(s^{-\kappa}), \quad \forall s \in [s_{**}, s_1]. \end{aligned}$$

On the other hand, Lemma 3.5 and (6.5) yield

$$\begin{aligned} & \left\langle \widetilde{\text{Mod}}_{\text{op}}(s)h + \text{Mod}_{\text{op}}(s)P - b\left(b + \frac{\lambda_s}{\lambda}\right)\frac{y^2}{2}P + \Psi_{K,\rho} \right\rangle \\ &= \frac{1}{4}(y^2Q, \rho)(b_s + b^2 - \alpha) + \frac{1}{2}\|yQ\|_{L^2}^2(1 - \theta_s) - \frac{1}{2}(y^2Q, \rho)b\left(b + \frac{\lambda_s}{\lambda}\right) + O(s^{-2}|\text{Mod}(s)|) + O(s^{-\kappa}). \end{aligned}$$

It follows that

$$|1 - \theta_s| = O(|b_s + b^2 - \alpha|) + O\left(b\left|b + \frac{\lambda_s}{\lambda}\right|\right) + O(s^{-2}|\text{Mod}(s)|) + O(s^{-\kappa}).$$

Thus, by (6.15) and (6.16),

$$|1 - \theta_s| \lesssim s^{-2}|\text{Mod}(s)| + s^{-\kappa}, \quad \forall s \in [s_{**}, s_1]. \quad (6.17)$$

Gathering (6.15), (6.16) and (6.17), we conclude that

$$|\text{Mod}(s)| \lesssim s^{-2}|\text{Mod}(s)| + s^{-\kappa},$$

whence

$$|\text{Mod}(s)| \lesssim s^{-\kappa}, \quad \forall s \in [s_{**}, s_1]. \quad (6.18)$$

Since  $h(s_1) = 0$ ,  $(b, \lambda)(s_1) = (b_1, \lambda_1)$ , by conservation of the mass and  $L^2$  scaling, we now have

$$\begin{aligned} \|P_{b_1, \lambda_1}\|_{L_y^2}^2 &= \|u(t_1)\|_{L_x^2}^2 = \|u(t)\|_{L_x^2}^2 = \|P_{b, \lambda} + h\|_{L_y^2}^2 \\ &= \|P_{b, \lambda}\|_{L^2}^2 + 2(P_{b, \lambda}, h) + \|h\|_{L^2}^2 \\ \implies (P_{b, \lambda}, h) &= -\frac{1}{2}\|h\|_{L^2}^2 + \frac{1}{2}(\|P_{b_1, \lambda_1}\|_{L^2}^2 - \|P_{b, \lambda}\|_{L^2}^2). \end{aligned}$$

Furthermore, using (2.7), (2.8) and (6.18), we find that

$$\begin{aligned} \left| \frac{d}{ds} \|P_{b, \lambda}\|_{L^2}^2 \right| &= 2|(P_{b, \lambda}, \partial_s P_{b, \lambda})| = 2 \text{Re} \left| \int_{\mathcal{G}} \overline{P_{b, \lambda}} \partial_s P_{b, \lambda} dy \right| \\ &= 2 \text{Im} \left| \int_{\mathcal{G}} \overline{P_{b, \lambda}} i \partial_s P_{b, \lambda} dy \right| = 2 \text{Im} \left| \int_{\mathcal{G}} \overline{P_{b, \lambda}} \Psi_{\kappa} dy \right| \lesssim \|\Psi_{\kappa}\|_{C_{\text{exp}}^1} = O(s^{-(\kappa+2)}). \end{aligned}$$

Hence, integrating from  $s$  to  $s_1$ ,

$$\left| \|P_{b_1, \lambda_1}\|_{L^2}^2 - \|P_{b, \lambda}\|_{L^2}^2 \right| \lesssim s^{-(\kappa+1)}$$

and so, by (6.5),

$$|(P_{b, \lambda}, h)| \lesssim \|h\|_{L^2}^2 + s^{-(\kappa+1)} \lesssim s^{-(\kappa+1)}, \quad \forall s \in [s_{**}, s_1].$$

Therefore,  $s_{**} = s_*$  and

$$|(P_{b, \lambda}, h)| \lesssim s^{-(\kappa+1)}, \quad \forall s \in [s_*, s_1].$$

Finally, (6.10) follows from (6.5) by observing that

$$(Q, h) = (P_{b, \lambda}, h) - O(\lambda \|h\|_{L^2}) = (P_{b, \lambda}, h) - O(s^{-2} \|h\|_{L^2}).$$

We thus conclude that (6.9) holds on the whole interval  $[s_*, s_1]$ , which completes the proof.  $\square$

**6.2. A monotone energy-virial functional.** Our main tool to bootstrap the estimate on  $h$  in (6.5) will be the functional

$$S(s, h) := \frac{H(s, h)}{\lambda^m(s)}, \quad (6.19)$$

where

$$H(s, h) := \frac{1}{2}\|h\|_{H^1}^2 + \frac{\lambda}{2}\|yh\|_{L^2}^2 + \frac{\gamma\lambda}{2}|h(0)|^2 - \int_{\mathcal{G}} [F(P_{b,\lambda} + h) - F(P_{b,\lambda}) - \mathrm{d}F(P_{b,\lambda})h] \, \mathrm{d}y, \quad (6.20)$$

and  $m$  is a positive integer which will be determined later.

We start with a simple upper bound for  $H$ .

**Lemma 6.4.** *For  $s \in [s_*, s_1]$ ,*

$$H(s, h) \leq \frac{1}{2}\|h\|_{\lambda}^2 + O(s^{-3(\kappa-2)}).$$

*Proof.* We have

$$F(P+h) - F(P) - \mathrm{d}F(P)h = \frac{1}{6}|P+h|^6 - \frac{1}{6}|P|^6 - \mathrm{Re}|P|^4 P \bar{h} = \frac{5}{2}Q^4 h_1^2 + \frac{1}{2}Q^4 h_2^2 + O(\|h\|^3) \quad (6.21)$$

and the result follows from (6.5) and the Sobolev embedding  $H^1 \hookrightarrow L^3$ .  $\square$

**Proposition 6.5.** *For  $s \in [s_*, s_1]$ , there holds*

$$S(s, h) \gtrsim \frac{1}{\lambda^m} (\|h\|_{\lambda}^2 + O(s^{-2\kappa})).$$

*Proof.* By (6.21), the embedding  $H^1 \hookrightarrow L^3$ , Lemma 3.7 and Proposition 2.2, we have

$$\begin{aligned} \frac{1}{2}\|h\|_{H^1}^2 - \int_{\mathcal{G}} [F(P_{b,\lambda} + h) - F(P_{b,\lambda}) - \mathrm{d}F(P_{b,\lambda})h] \, \mathrm{d}y \\ &= \frac{1}{2} \left[ \|h\|_{H^1}^2 - \int_{\mathcal{G}} (5Q^4 h_1^2 + Q^4 h_2^2) \, \mathrm{d}y \right] + O(\|h\|_{H^1}^3) \\ &= \frac{1}{2} \left[ \langle L_+ h_1, h_1 \rangle + \langle L_- h_2, h_2 \rangle \right] + O(\|h\|_{H^1}^3) \\ &\geq \frac{\mu}{2} \|h\|_{H^1}^2 - \frac{1}{2\mu} ((h, Q)_{L^2}^2 + O(s^{-4} \|h\|_{L^2}^2)) + O(\|h\|_{H^1}^3). \end{aligned}$$

By (6.5) and (6.10), we conclude that

$$\frac{1}{2}\|h\|_{H^1}^2 - \int_{\mathcal{G}} [F(P_{b,\lambda} + h) - F(P_{b,\lambda}) - \mathrm{d}F(P_{b,\lambda})h] \, \mathrm{d}y \geq \frac{\mu}{2}\|h\|_{H^1}^2 + O(s^{-2\kappa}) + O(s^{-3(\kappa-2)}).$$

Let  $k_0 = \frac{1}{2} \min\{\mu, 1\}$ . Since  $\kappa \geq 6$  and  $\frac{\gamma\lambda}{2}|h(0)|^2 > 0$ , it follows that

$$H(s, h) \geq k_0 \|h\|_{\lambda}^2 + O(s^{-2\kappa}), \quad \forall s \in [s_*, s_1], \quad (6.22)$$

which completes the proof.  $\square$

We now estimate the time derivative of  $H(s, h(s))$ :

$$\frac{\mathrm{d}}{\mathrm{d}s} H(s, h(s)) = D_s H(s, h(s)) + \langle D_h H(s, h(s)), h_s \rangle. \quad (6.23)$$

**Proposition 6.6.** *There exists a constant  $k > 0$  such that, for all  $s \in [s_*, s_1]$ ,*

$$\frac{\mathrm{d}H}{\mathrm{d}s} \geq b (-k \|h\|_{\lambda}^2 + O(s^{-2(\kappa-1)})). \quad (6.24)$$

The proof of Proposition 6.6 breaks down into several lemmas. In a number of arguments below, it is understood implicitly that an estimate holds *provided*  $s_0$  is chosen sufficiently large.

*Remark 6.7.* By inspecting closely the estimates involved, one observes that it suffices to choose

$$k > 1 + \max\{1, 4/\beta\} + \max\{2, \beta/2\}.$$

**Lemma 6.8.** *There exists  $k_1 > 0$  such that, for all  $s \in [s_*, s_1]$ ,*

$$D_s H(s, h(s)) \geq -b k_1 \|h\|_\lambda^2.$$

*Proof.* We have

$$D_s H = \frac{\lambda_s}{2} \|yh\|_{L^2}^2 + \frac{\gamma\lambda_s}{2} |h(0)|^2 - \partial_s \int_{\mathcal{G}} [F(P_{b,\lambda} + h) - F(P_{b,\lambda}) - dF(P_{b,\lambda})h] dy.$$

First, by (6.9),

$$\lambda_s = -b\lambda + O(s^{-(\kappa+2)}) = O(s^{-3}).$$

Hence,

$$\left| \frac{\lambda_s}{2} \|yh\|_{L^2}^2 + \frac{\gamma\lambda_s}{2} |h(0)|^2 \right| \lesssim s^{-3} \|yh\|_{L^2}^2 + s^{-3} \|h\|_{H^1}^2.$$

On the other hand, writing

$$f(P_{b,\lambda} + h) - f(P_{b,\lambda}) = df(P_{b,\lambda})h + R_Q(h),$$

where

$$|R_Q(h)| \lesssim |h|^2,$$

we have that

$$\begin{aligned} \partial_s [F(P_{b,\lambda} + h) - F(P_{b,\lambda}) - dF(P_{b,\lambda})h] &= \operatorname{Re} [(df(P_{b,\lambda})h + R_Q(h))\partial_s \overline{P_{b,\lambda}} - \partial_s f(P_{b,\lambda})\bar{h}] \\ &= \operatorname{Re} R_Q(h)\partial_s \overline{P_{b,\lambda}}. \end{aligned}$$

Therefore,

$$\partial_s \int_{\mathcal{G}} [F(P_{b,\lambda} + h) - F(P_{b,\lambda}) - dF(P_{b,\lambda})h] dy = \operatorname{Re} \int_{\mathcal{G}} R_Q(h)\partial_s \overline{P_{b,\lambda}} dy \lesssim b\lambda \|h\|_{L^2}^2 \lesssim s^{-3} \|h\|_{H^1}^2.$$

Thus,

$$|D_s H| \lesssim s^{-3} \|yh\|_{L^2}^2 + s^{-3} \|h\|_{H^1}^2 \lesssim s^{-1} (\|h\|_{H^1}^2 + s^{-2} \|yh\|_{L^2}^2).$$

The result now follows from the asymptotics  $b = O(s^{-1})$  and  $\lambda = O(s^{-2})$  for large  $s$ .  $\square$

We next compute the second term in (6.23) as follows:

$$\langle D_h H, h_s \rangle = \langle iD_h H, ih_s \rangle, \tag{6.25}$$

where

$$\begin{aligned} D_h H &= -\partial_y^2 h + h + \lambda y^2 h + \gamma\lambda\delta h - [f(P+h) - f(P)] \\ &= -\partial_y^2 h + h + \lambda y^2 h + \gamma\lambda\delta h - df(P_{b,\lambda})h - R_Q(h), \end{aligned}$$

and (6.3) reads

$$ih_s = D_h H - \lambda y^2 h - \widetilde{\operatorname{Mod}_{\text{op}}(s)} h - \operatorname{Mod}_{\text{op}}(s)P + \Phi_\kappa,$$

where, by (2.8) and (6.9),

$$\Phi_\kappa := b \left( b + \frac{\lambda_s}{\lambda} \right) \frac{y^2}{2} P - \Psi_\kappa(s) = O_\Sigma(s^{-(\kappa+1)}). \tag{6.26}$$

Now, using (6.25) and the relation  $\langle iD_h H, D_h H \rangle = 0$ , we have

$$\langle D_h H, h_s \rangle = -\lambda \langle iD_h H, y^2 h \rangle - \langle iD_h H, \widetilde{\text{Mod}}_{\text{op}}(s)h \rangle - \langle iD_h H, \text{Mod}_{\text{op}}(s)P \rangle + \langle iD_h H, \Phi_\kappa \rangle. \quad (6.27)$$

The higher order terms in the right-hand side of (6.27) is the first one and the term

$$\frac{1}{4} \left( b_s - b^2 - 2b \frac{\lambda_s}{\lambda} \right) \langle iD_h H, y^2 h \rangle$$

coming from the second one. We now show that they both are of order  $s^{-1} \|h\|_\lambda^2$ .

**Lemma 6.9.** *For  $s \in [s_*, s_1]$ , we have*

$$|\lambda \langle iD_h H, y^2 h \rangle| \lesssim s^{-1} \|h\|_\lambda^2 \quad (6.28)$$

and

$$\left| \frac{1}{4} \left( b_s - b^2 - 2b \frac{\lambda_s}{\lambda} \right) \langle iD_h H, y^2 h \rangle \right| \lesssim s^{-1} \|h\|_\lambda^2. \quad (6.29)$$

*Proof.* We only prove (6.28). (6.29) follows by the same arguments since

$$b_s - b^2 - 2b \frac{\lambda_s}{\lambda} = O(\lambda)$$

for large  $s$ .

Discarding duality products whose real part is zero, we have that

$$\begin{aligned} \langle iD_h H, y^2 h \rangle &= -\langle -\partial_y^2 h + h + \lambda y^2 h + \gamma \lambda \delta h - \text{d}f(P)h - R_Q(h), iy^2 h \rangle \\ &= -\langle -\partial_y^2 h - \text{d}f(P)h - R_Q(h), iy^2 h \rangle. \end{aligned}$$

An integration by parts yields

$$\langle \partial_y^2 h, iy^2 h \rangle = 2 \text{Im} \int_{\mathcal{G}} y h_y \bar{h} \, dy.$$

Now,

$$\left| 2 \text{Im} \int_{\mathcal{G}} y h_y \bar{h} \, dy \right| \leq 2s^{1/2} \|h_y\|_{L^2} s^{-1/2} \|yh\|_{L^2} \leq s \|h_y\|_{L^2}^2 + s^{-1} \|yh\|_{L^2}^2.$$

On the other hand,

$$|\langle \text{d}f(P)h, y^2 h \rangle| \lesssim \|h\|_{L^2}^2$$

and

$$|\langle R_Q(h), y^2 h \rangle| \lesssim \int_{\mathcal{G}} Q^3 y^2 |h|^3 \, dy \leq \left( \int_{\mathcal{G}} Q^6 y^4 \, dy \right)^{1/2} \left( \int_{\mathcal{G}} |h|^6 \, dy \right)^{1/2} \lesssim \|h\|_{H^1}^3,$$

thanks to the continuous embedding  $H^1 \hookrightarrow L^6$ . It follows that

$$\begin{aligned} |\lambda \langle iD_h H, y^2 h \rangle| &\lesssim \lambda (s \|h_y\|_{L^2}^2 + s^{-1} \|yh\|_{L^2}^2 + \|h\|_{L^2}^2 + \|h\|_{H^1}^3) \\ &\lesssim s^{-1} (\|h_y\|_{L^2}^2 + \lambda \|yh\|_{L^2}^2 + \|h\|_{L^2}^2) + s^{-2} \|h\|_{H^1}^3 \\ &\lesssim s^{-1} (\|h\|_{H^1}^2 + \lambda \|yh\|_{L^2}^2), \end{aligned}$$

which concludes the proof.  $\square$

**Lemma 6.10.** *For  $s \in [s_*, s_1]$ ,*

$$|\langle iD_h H, \widetilde{\text{Mod}}_{\text{op}}(s)h \rangle| \lesssim s^{-1} \|h\|_\lambda^2.$$

*Proof.* We have

$$\langle iD_h H, \widetilde{\text{Mod}}_{\text{op}}(s)h \rangle = (1 - \theta_s) \langle iD_h H, h \rangle + \frac{1}{4} \left( b_s - b^2 - 2b \frac{\lambda_s}{\lambda} \right) \langle iD_h H, y^2 h \rangle - \left( b + \frac{\lambda_s}{\lambda} \right) \langle iD_h H, i\Lambda h \rangle$$

and we have already estimated the second term in Lemma 6.9.

By (6.9),

$$\begin{aligned} \langle iD_h H, h \rangle &= -\langle D_h H, ih \rangle = -\langle -\partial_y^2 h + h + \lambda y^2 h + \gamma \lambda \delta h - \text{d}f(P)h - R_Q(h), ih \rangle \\ &= \langle \gamma \lambda \delta h + \text{d}f(P)h + R_Q(h), ih \rangle \end{aligned}$$

$$\implies |(1 - \theta_s) \langle iD_h H, h \rangle| \lesssim s^{-\kappa} (\lambda |h(0)|^2 + \|h\|_{L^2}^2 + \|h\|_{H^1}^3) \lesssim s^{-\kappa} \|h\|_{H^1}^2. \quad (6.30)$$

Next, integrating by parts,

$$\begin{aligned} \langle iD_h H, i\Lambda h \rangle &= \langle D_h H, \Lambda h \rangle = \langle -\partial_y^2 h + h + \lambda y^2 h + \gamma \lambda \delta h - [f(P+h) - f(P)], \Lambda h \rangle \\ &= \langle -\partial_y^2 h + h, \Lambda h \rangle + \lambda \langle y^2 h, \Lambda h \rangle + \gamma \lambda \text{Re } h(0) \overline{\Lambda h}(0) - \langle f(P+h) - f(P), \Lambda h \rangle \\ &= \|h_y\|_{L^2}^2 - \lambda \int_{\mathcal{G}} y^2 |h|^2 \, dy + \frac{\gamma \lambda}{2} |h(0)|^2 - \langle f(P+h) - f(P), \Lambda h \rangle. \end{aligned}$$

Hence,

$$|\langle iD_h H, i\Lambda h \rangle| \lesssim \|h\|_{\lambda}^2 + |\langle f(P+h) - f(P), \Lambda h \rangle|.$$

Now,

$$\begin{aligned} \langle f(P+h) - f(P), \Lambda h \rangle &= \text{Re} \int_{\mathcal{G}} (f(P+h) - f(P)) \overline{\Lambda h} \, dy \\ &= \frac{1}{2} \text{Re} \int_{\mathcal{G}} (f(P+h) - f(P)) \bar{h} \, dy + \text{Re} \int_{\mathcal{G}} (f(P+h) - f(P)) y \bar{h}_y \, dy \end{aligned}$$

and

$$\int_{\mathcal{G}} (f(P+h) - f(P)) y \bar{h}_y \, dy = - \int_{\mathcal{G}} (f(P+h) - f(P)) \bar{h} \, dy - \int_{\mathcal{G}} y (f(P+h) - f(P))_y \bar{h} \, dy,$$

so that

$$\langle f(P+h) - f(P), \Lambda h \rangle = -\frac{1}{2} \text{Re} \int_{\mathcal{G}} (f(P+h) - f(P)) \bar{h} \, dy - \text{Re} \int_{\mathcal{G}} y (f(P+h) - f(P))_y \bar{h} \, dy.$$

On the other hand,

$$\begin{aligned} \int_{\mathcal{G}} y (f(P+h) - f(P))_y \bar{h} \, dy &= \int_{\mathcal{G}} y (\text{d}f(P+h)(P+h)_y - \text{d}f(P)P_y) \bar{h} \, dy \\ &= \int_{\mathcal{G}} y (\text{d}f(P+h) - \text{d}f(P)) P_y \bar{h} \, dy + \int_{\mathcal{G}} \text{d}f(P+h) h_y \bar{h} \, dy. \end{aligned}$$

Using Sobolev embeddings, we have

$$\left| \int_{\mathcal{G}} (f(P+h) - f(P)) \bar{h} \, dy \right| \leq \|f(P+h) - f(P)\|_{L^2} \|h\|_{L^2} \lesssim \|h\|_{H^1}^2$$

and

$$\begin{aligned} \left| \int_{\mathcal{G}} y(f(P+h) - f(P))_y \bar{h} \, dy \right| &\leq \|yP_y\|_{L^\infty} \|df(P+h) - df(P)\|_{L^2} \|h\|_{L^2} \\ &\quad + \|df(P+h)\|_{L^\infty} \|h_y\|_{L^2} \|yh\|_{L^2} \\ &\lesssim \|h\|_{H^1}^2 + \|yh\|_{L^2}^2. \end{aligned}$$

We conclude that

$$|\langle f(P+h) - f(P), \Lambda h \rangle| \lesssim s^2 \|h\|_\lambda^2.$$

Hence, by (6.9), we have

$$\left| \left( b + \frac{\lambda s}{\lambda} \right) \langle iD_h H, i\Lambda h \rangle \right| \lesssim s^{-(\kappa-2)} \|h\|_\lambda^2. \quad (6.31)$$

The result now follows from estimates (6.29), (6.30) and (6.31).  $\square$

**Lemma 6.11.** For  $s \in [s_*, s_1]$ ,

$$|\langle iD_h H, \text{Mod}_{\text{op}}(s)P \rangle| \lesssim s^{-(2\kappa-1)}.$$

*Proof.* It will be convenient to write  $D_h H$  as

$$D_h H = L_+ h_1 + iL_- h_2 + \lambda y^2 h + \gamma \lambda \delta h - R_Q(h),$$

for  $h = h_1 + ih_2 \in \Sigma$ . It then follows from (6.5) and (6.9) that

$$\begin{aligned} \langle iD_h H, \text{Mod}_{\text{op}}(s)P \rangle &= \langle iD_h H, \text{Mod}_{\text{op}}(s)Q \rangle + O(s^{-(\kappa+2)} \|h\|_\Sigma) \\ &= \langle iD_h H, \text{Mod}_{\text{op}}(s)Q \rangle + O(s^{-(\kappa+1)} \|h\|_\lambda) \\ &= \langle iD_h H, \text{Mod}_{\text{op}}(s)Q \rangle + O(s^{-(2\kappa-1)}). \end{aligned}$$

Now,

$$\langle iD_h H, \text{Mod}_{\text{op}}(s)Q \rangle = (1-\theta_s) \langle iD_h H, Q \rangle + \frac{1}{4} \left( b_s - b^2 - 2b \frac{\lambda s}{\lambda} - \alpha \right) \langle iD_h H, y^2 Q \rangle - \left( b + \frac{\lambda s}{\lambda} \right) \langle iD_h H, i\Lambda Q \rangle.$$

First, since  $L_- Q = 0$ , we have

$$\begin{aligned} \langle iD_h H, Q \rangle &= -\langle D_h H, iQ \rangle = -\langle L_+ h_1 + iL_- h_2 + \lambda y^2 h + \gamma \lambda \delta h - R_Q(h), iQ \rangle \\ &= -\langle L_- h_2, Q \rangle - \lambda \text{Re} \int_{\mathcal{G}} y^2 h i \bar{Q} \, dy - \gamma \lambda \text{Re} h(0) i \bar{Q}(0) + O(\|h\|_{L^2}^2) \\ &= -\langle h_2, L_- Q \rangle - \lambda \text{Im} \int_{\mathcal{G}} y^2 h Q \, dy - \gamma \lambda \text{Im} h(0) Q(0) + O(\|h\|_{L^2}^2) \\ &= -\lambda \text{Im} \int_{\mathcal{G}} y^2 h Q \, dy - \gamma \lambda \text{Im} h(0) Q(0) + O(\|h\|_{L^2}^2). \end{aligned}$$

Hence, by (6.5),

$$\begin{aligned} |(1-\theta_s) \langle iD_h H, Q \rangle| &\lesssim s^{-\kappa} (s^{-2} \|h\|_{H^1} + s^{-2} \|yh\|_{L^2} + \|h\|_{L^2}^2) \lesssim s^{-(\kappa+1)} \|h\|_\lambda + s^{-\kappa} \|h\|_{L^2}^2 \\ &\lesssim s^{-(2\kappa-1)}. \end{aligned} \quad (6.32)$$

Next,

$$\begin{aligned}
\langle iD_h H, y^2 Q \rangle &= -\langle D_h H, iy^2 Q \rangle = -\langle L_+ h_1 + iL_- h_2 + \lambda y^2 h + \gamma \lambda \delta h - R_Q(h), iy^2 Q \rangle \\
&= -\langle L_- h_2, y^2 Q \rangle - \lambda \operatorname{Re} \int_{\mathcal{G}} y^2 h \overline{iy^2 Q} \, dy + O(\|h\|_{L^2}^2) \\
&= -\langle h_2, L_- y^2 Q \rangle - \lambda \operatorname{Im} \int_{\mathcal{G}} y^2 h y^2 Q \, dy + O(\|h\|_{L^2}^2) \\
&= 4\langle h_2, \Lambda Q \rangle - \lambda \operatorname{Im} \int_{\mathcal{G}} y h y^3 Q \, dy + O(\|h\|_{L^2}^2).
\end{aligned}$$

Since  $\langle h_2, \Lambda Q \rangle = O(s^{-2}\|h\|_{L^2})$  by Proposition 2.2, it follows that

$$\begin{aligned}
\left| \frac{1}{4} \left( b_s - b^2 - 2b \frac{\lambda_s}{\lambda} - \beta \lambda \right) \langle iD_h H, y^2 Q \rangle \right| &\lesssim s^{-\kappa} (s^{-2} \|yh\|_{L^2} + \|h\|_{L^2}^2) \\
&\lesssim s^{-(\kappa+2)} \|h\|_{\Sigma} + s^{-\kappa} \|h\|_{L^2}^2 \\
&\lesssim s^{-(2\kappa-1)}.
\end{aligned} \tag{6.33}$$

Finally, by (6.5) and (6.10),

$$\begin{aligned}
\langle iD_h H, i\Lambda Q \rangle &= \langle D_h H, \Lambda Q \rangle \\
&= \langle L_+ h_1 + iL_- h_2 + \lambda y^2 h + \gamma \lambda \delta h - R_Q(h), \Lambda Q \rangle \\
&= \langle h_1, L_+ \Lambda Q \rangle + \lambda \operatorname{Re} \int_{\mathcal{G}} y^2 h \Lambda Q \, dy + \gamma \lambda \operatorname{Re} h(0) \Lambda Q(0) + O(\|h\|_{L^2}^2) \\
&= -2(h, Q)_{L^2} + \lambda \operatorname{Re} \int_{\mathcal{G}} y h y \Lambda Q \, dy + \gamma \lambda \operatorname{Re} h(0) \Lambda Q(0) + O(\|h\|_{L^2}^2) \\
\implies \left| \left( b + \frac{\lambda_s}{\lambda} \right) \langle iD_h H, i\Lambda Q \rangle \right| &\lesssim s^{-\kappa} (|(h, Q)_{L^2}| + s^{-2} \|yh\|_{L^2} + s^{-2} \|h\|_{H^1} + \|h\|_{L^2}^2) \\
&\lesssim s^{-2\kappa} + s^{-(\kappa+1)} \|h\|_{\lambda} + s^{-\kappa} \|h\|_{L^2}^2 \\
&\lesssim s^{-(2\kappa-1)}.
\end{aligned} \tag{6.34}$$

The result now follows by combining estimates (6.32), (6.33) and (6.34).  $\square$

**Lemma 6.12.** For  $s \in [s_*, s_1]$ ,

$$|\langle iD_h H, \Phi_{\kappa} \rangle| \lesssim s^{-(2\kappa-1)}.$$

*Proof.* We have

$$\langle iD_h H, \Phi_{\kappa} \rangle = -\langle D_h H, i\Phi_{\kappa} \rangle = -\langle -\partial_y^2 h + h + \lambda y^2 h + \gamma \lambda \delta h - \operatorname{df}(P_{b,\lambda})h - R_Q(h), i\Phi_{\kappa} \rangle.$$

Hence,

$$\begin{aligned}
|\langle iD_h H, \Phi_{\kappa} \rangle| &\leq \|h\|_{H^1} \|\Phi_{\kappa}\|_{H^1} + \lambda \int_{\mathcal{G}} y^2 |h| |\Phi_{\kappa}| \, dy + \gamma \lambda |h(0)| |\Phi_{\kappa}(0)| + |\langle \operatorname{df}(P)h, \Phi_{\kappa} \rangle| + |\langle R_Q(h), \Phi_{\kappa} \rangle| \\
&\lesssim \|h\|_{H^1} \|\Phi_{\kappa}\|_{H^1} + s^{-2} \left( \int_{\mathcal{G}} y^2 |h|^2 \, dy \right)^{1/2} \left( \int_{\mathcal{G}} y^2 |\Phi_{\kappa}|^2 \, dy \right)^{1/2} \\
&\quad + s^{-2} \|h\|_{H^1} \|\Phi_{\kappa}\|_{H^1} + \|h\|_{L^2} \|\Phi_{\kappa}\|_{L^2}.
\end{aligned}$$

Therefore, by (6.5) and (6.26),

$$|\langle iD_h H, \Phi_{\kappa} \rangle| \lesssim s^{-(\kappa+1)} \|h\|_{H^1} + s^{-(\kappa+3)} \|yh\|_{L^2} \lesssim s^{-(\kappa+1)} \|h\|_{\lambda} \lesssim s^{-(2\kappa-1)},$$



as claimed.  $\square$

**Lemma 6.13.** *There exists  $k_2 > 0$  such that, for all  $s \in [s_*, s_1]$ ,*

$$\langle D_h H(s, h(s)), h_s \rangle \geq -b k_2 \|h\|_\lambda^2 + O(s^{-(2\kappa-1)}).$$

*Proof.* Using the fact that  $b \sim s^{-1}$ , this is a direct consequence of (6.27) and Lemmas 6.9 to 6.12.  $\square$

*Proof of Proposition 6.6.* Proposition 6.6 now follows from Lemmas 6.8 and 6.13.  $\square$

**Proposition 6.14.** *Let  $k_0$  and  $k$  be as in (6.22) and Proposition 6.6. Choose  $m \in \mathbb{N}$  such that  $m \geq 2k/k_0$ . Then, for all  $s \in [s_*, s_1]$ ,*

$$\frac{dS}{ds} \gtrsim \frac{b}{\lambda^m} (\|h\|_\lambda^2 + O(s^{-2(\kappa-1)})).$$

*Proof.* We have

$$\frac{dS}{ds} = \frac{1}{\lambda^m} \left( -m \frac{\lambda_s}{\lambda} H + \frac{dH}{ds} \right).$$

Furthermore, for  $s$  large enough,  $-\lambda_s/\lambda \geq b/2$ . Hence, in view of (6.22) and Proposition 6.6,

$$\begin{aligned} \frac{dS}{ds} &\geq \frac{b}{\lambda^m} \left[ \frac{m}{2} (k_0 \|h\|_\lambda^2 + O(s^{-2\kappa})) + (-k \|h\|_\lambda^2 + O(s^{-2(\kappa-1)})) \right] \\ &\gtrsim \frac{b}{\lambda^m} (\|h\|_\lambda^2 + O(s^{-2(\kappa-1)})), \quad \forall s \in [s_*, s_1], \end{aligned}$$

as claimed.  $\square$

**6.3. Proof of Proposition 2.6.** We first prove the uniform estimates in  $s$ .

*Proof of Proposition 6.1.* We will prove that estimates (6.5) can be improved to (6.6) and (6.7) on  $[s_*, s_1]$ . Then, choosing  $s_0$  large enough, it follows by continuity that, in fact,  $s_* = s_0$ , so that (6.6) and (6.7) hold on  $[s_0, s_1]$ .

We first prove (6.6). By Proposition 6.5 and the definition of  $S$ , there exists a constant  $a > 1$  such that

$$\frac{1}{a} \frac{1}{\lambda^m} (\|h\|_\lambda^2 - a^2 s^{-2(\kappa-1)}) \leq S(s, h) \leq \frac{a}{\lambda^m} \|h\|_\lambda^2, \quad s \in [s_*, s_1]. \quad (6.35)$$

Choosing  $a$  large enough, Proposition 6.14 yields

$$\frac{dS}{ds} \geq \frac{1}{a} \frac{b}{\lambda^m} (\|h\|_\lambda^2 - a^2 s^{-2(\kappa-1)}), \quad s \in [s_*, s_1]. \quad (6.36)$$

Let

$$s_\dagger := \inf\{s \in [s_*, s_1] : \|h(\sigma)\|_\lambda(\sigma) \leq 2a^2 \sigma^{-(\kappa-1)} \forall \sigma \in [s_\dagger, s_1]\}.$$

Since  $h(s_1) = 0$ , it follows by continuity that  $s_\dagger \in [s_*, s_1]$ . We will prove that  $s_\dagger = s_*$ .

Suppose by contradiction that  $s_\dagger > s_*$ . Then, in particular,  $\|h(s_\dagger)\|_\lambda(s_\dagger) = 2a^2 s_\dagger^{-(\kappa-1)}$ . Defining

$$s_\ddagger := \sup\{s \in [s_\dagger, s_1] : \|h(\sigma)\|_\lambda(\sigma) \geq a \sigma^{-(\kappa-1)} \forall \sigma \in [s_\dagger, s]\},$$

we have  $s_\star < s_\dagger < s_\ddagger < s_1$  and  $\|h(s_\ddagger)\|_{\lambda(s_\ddagger)} = a s_\ddagger^{-(\kappa-1)}$ . Furthermore, by (6.36),  $S$  is non-decreasing on  $[s_\dagger, s_\ddagger]$ . Hence, using (6.35) and our bootstrap assumption on  $\lambda$  in (6.5), we find that

$$\begin{aligned} \|h(s_\dagger)\|_{\lambda(s_\dagger)}^2 - a^2 s_\dagger^{-2(\kappa-1)} &\leq a \lambda^m(s_\dagger) S(s_\dagger, h(s_\dagger)) \leq a \lambda^m(s_\dagger) S(s_\ddagger, h(s_\ddagger)) \\ &\leq a^2 \frac{\lambda^m(s_\dagger)}{\lambda^m(s_\ddagger)} \|h(s_\ddagger)\|_{\lambda(s_\ddagger)}^2 = a^4 \frac{\lambda^m(s_\dagger)}{\lambda^m(s_\ddagger)} s_\ddagger^{-2(\kappa-1)} \\ &\leq 2a^4 \left(\frac{s_\ddagger}{s_\dagger}\right)^{4m} s_\ddagger^{-2(\kappa-1)} \leq 2a^4 s_\dagger^{-2(\kappa-1)}. \end{aligned}$$

It follows that

$$\|h(s_\dagger)\|_{\lambda(s_\dagger)}^2 \leq a^2 s_\dagger^{-2(\kappa-1)} + 2a^4 s_\dagger^{-2(\kappa-1)} \leq 3a^4 s_\dagger^{-2(\kappa-1)},$$

a contradiction.

We now prove (6.7). Let  $E^\star \in \mathbb{R}$ ,  $\mathcal{E}^\star = C_Q^{-1} E^\star$  and  $b_1, \lambda_1$  be given by Proposition 2.5. It follows from (2.10) that

$$|E(\tilde{P}(b_1, \lambda_1, \theta_1)) - E^\star| \lesssim \frac{(b_1^2 + \lambda_1)^\kappa}{\lambda_1^2} \lesssim s_1^{4-2\kappa}. \quad (6.37)$$

Now, the energy estimate (2.9) and the modulation estimate (6.9) yield

$$\begin{aligned} |E(\tilde{P}(b(s), \lambda(s), \theta(s))) - E(\tilde{P}(b_1, \lambda_1, \theta_1))| &= \left| \int_s^{s_1} \frac{d}{d\sigma} E(\tilde{P}(b(\sigma), \lambda(\sigma), \theta(\sigma))) d\sigma \right| \\ &\lesssim \int_s^{s_1} \sigma^{4-\kappa} d\sigma \lesssim s^{5-\kappa}, \quad s \in [s_\star, s_1]. \end{aligned} \quad (6.38)$$

It then follows by (6.37) and (6.38) that

$$|E(\tilde{P}(b(s), \lambda(s), \theta(s))) - E^\star| \lesssim s^{5-\kappa}, \quad s \in [s_\star, s_1]. \quad (6.39)$$

Hence, using (2.10) at time  $s$ , we find

$$\begin{aligned} |\mathcal{E}(b(s), \lambda(s)) - \mathcal{E}^\star| &\lesssim \left| \mathcal{E}(b, \lambda) - \frac{E(\tilde{P}(b, \lambda, \theta))}{C_Q} \right| + \left| \frac{E(\tilde{P}(b, \lambda, \theta))}{C_Q} - \mathcal{E}^\star \right| \\ &\lesssim s^{4-2\kappa} + s^{5-\kappa} \lesssim s^{5-\kappa}, \quad s \in [s_\star, s_1]. \end{aligned} \quad (6.40)$$

Next, the formula (2.11) defining  $\mathcal{E}$  yields

$$\lambda^2 \mathcal{E}(b, \lambda) = b^2 - 2\beta\lambda + O(\lambda^2).$$

Thus, by (6.40),

$$|b^2 - 2\beta\lambda - \mathcal{E}^\star \lambda^2| \lesssim \lambda^2(s) + s^{5-\kappa} \lesssim s^{-4}. \quad (6.41)$$

It follows that

$$|b - \sqrt{2\beta\lambda + \mathcal{E}^\star \lambda^2}| |b + \sqrt{2\beta\lambda + \mathcal{E}^\star \lambda^2}| \lesssim s^{-4}.$$

Hence, by (6.5),

$$|b - \sqrt{2\beta\lambda + \mathcal{E}^\star \lambda^2}| \lesssim s^{-3}, \quad s \in [s_\star, s_1]. \quad (6.42)$$

From (6.9), we have  $b = -\lambda_s/\lambda + O(s^{-\kappa})$ , hence

$$\left| \frac{\lambda_s}{\lambda} + \sqrt{2\beta\lambda + \mathcal{E}^\star \lambda^2} \right| \lesssim s^{-3}$$

and we deduce from the definition of  $\mathcal{F}$  in (2.15) that

$$\left| \frac{d}{ds} \mathcal{F}(\lambda(s)) - 1 \right| = \left| \frac{\lambda_s}{\lambda \sqrt{2\beta\lambda + \mathcal{E}^\star \lambda^2}} + 1 \right| = \frac{1}{\sqrt{2\beta\lambda + \mathcal{E}^\star \lambda^2}} \left| \frac{\lambda_s}{\lambda} + \sqrt{2\beta\lambda + \mathcal{E}^\star \lambda^2} \right| \lesssim s^{-2}.$$

Integrating from  $s$  to  $s_1$  and using  $\mathcal{F}(\lambda(s_1)) = s_1$  yields

$$\begin{aligned} \left| \mathcal{F}(\lambda(s)) - s \right| &\leq \left| \int_s^{s_1} \left( \frac{d}{d\sigma} \mathcal{F}(\lambda(\sigma)) - 1 \right) d\sigma \right| \lesssim s^{-1} \\ \implies \mathcal{F}(\lambda(s)) &= s + O(s^{-1}), \quad s \in [s_*, s_1]. \end{aligned}$$

On the other hand, it follows from (7.6) that

$$\left| \frac{\lambda(s)^{1/2}}{\lambda_{\text{mo}}(s)^{1/2}} - 1 \right| \lesssim \frac{1}{s^2}, \quad s \in [s_*, s_1].$$

Finally, returning to (6.42) and using again (6.5),

$$\begin{aligned} b - b_{\text{mo}} &= \sqrt{2\beta\lambda + \mathcal{E}^*\lambda^2} - \sqrt{2\beta\lambda_{\text{mo}}} + O(s^{-3}) = \frac{2\beta\lambda + \mathcal{E}^*\lambda^2 - 2\beta\lambda_{\text{mo}}}{\sqrt{2\beta\lambda + \mathcal{E}^*\lambda^2} + \sqrt{2\beta\lambda_{\text{mo}}}} + O(s^{-3}) \\ &= O(b_{\text{mo}})[\mathcal{E}^*\lambda^2 + 2\beta(\lambda^{1/2} - \lambda_{\text{mo}}^{1/2})(\lambda^{1/2} + \lambda_{\text{mo}}^{1/2})] + O(s^{-3}) \\ \implies \left| \frac{b(s)}{b_{\text{mo}}(s)} - 1 \right| &\lesssim \frac{1}{s^2}, \quad s \in [s_*, s_1]. \end{aligned}$$

This concludes the proof.  $\square$

*Remark 6.15.* We observe here that the estimates (6.7) can be improved by a closer inspection of the energy expansion  $\mathcal{E}$ . Indeed, (2.11) yields

$$\lambda^2 \mathcal{E}(b, \lambda) = b^2 - 2\beta\lambda + e_0\lambda^2 + O(s^{-6}), \quad e_0 := \varepsilon_{0,1} + 2\beta\varepsilon_{2,0}.$$

Hence, using (6.40) and choosing  $\kappa \geq 11$ , estimate (6.41) improves to

$$|b^2 - 2\beta\lambda - (\mathcal{E}^* - e_0)\lambda^2| \lesssim s^{-6} + s^{5-\kappa} \lesssim s^{-6}.$$

Then, replacing  $\mathcal{E}^*$  by  $\mathcal{E}^* - e_0$  in Proposition 2.5 and using this improved estimate in the rest of the proof yields (6.8).

We conclude this section with the

*Proof of Proposition 2.6.* The change of variables (6.1) yields

$$dt = \tilde{\lambda}(t(s))^2 ds \implies t_1 - t = \int_s^{s_1} \tilde{\lambda}(\sigma)^2 d\sigma.$$

From (6.7), we have

$$\lambda(\sigma)^2 = \lambda_{\text{mo}}(\sigma)^2 + O(\sigma^{-6}).$$

Noting that

$$t_1 = - \int_{s_1}^{\infty} \lambda_{\text{mo}}(\sigma)^2 d\sigma,$$

we find

$$t(s) = - \int_s^{\infty} \lambda_{\text{mo}}(\sigma)^2 d\sigma + O(s^{-5}) = -\frac{4}{3\beta^2} s^{-3} + O(s^{-5}).$$

Hence,

$$s \sim \left( \frac{3\beta^2}{4} |t| \right)^{-1/3}, \quad t \rightarrow 0^-.$$

It follows that

$$\tilde{b}_{\text{mo}}(t) := b_{\text{mo}}(s(t)) = \frac{2}{s(t)} \sim 2 \left( \frac{3\beta^2}{4} |t| \right)^{1/3}, \quad t \rightarrow 0^-$$

and

$$\tilde{\lambda}_{\text{mo}}(t) := \lambda_{\text{mo}}(s(t)) = \frac{2}{\beta s(t)^2} \sim \frac{2}{\beta} \left( \frac{3\beta^2}{4} |t| \right)^{2/3}, \quad t \rightarrow 0^-.$$

Estimates (2.18) now follow directly from (6.7). Recalling the definition of the norm  $\|\cdot\|_\lambda$  in (6.4), the estimates (2.19) then follow from (2.18) and (6.6), while (2.20) follows from (6.39). This finishes the proof.  $\square$

## 7. APPENDIX

In this appendix, we consider the model nonlinear dynamical system

$$b_s + b^2 - \beta\lambda = 0, \quad \frac{\lambda_s}{\lambda} + b = 0, \quad 1 - \theta_s = 0$$

appearing for the parameters in the derivation of the profile. We have kept here only the first term of  $\alpha$ , where we recall that

$$\beta = -2 \frac{\gamma Q(0)^2}{\|yQ\|_{L^2}} > 0.$$

We denote the parameters depending on the variable  $t$  by  $(\tilde{b}, \tilde{\lambda}, \tilde{\theta})$  and the parameters depending on the variable  $s$  by  $(b, \lambda, \theta)$ , related by

$$\tilde{b}(t) = b(s(t)), \quad \tilde{\lambda}(t) = \lambda(s(t)), \quad \tilde{\theta}(t) = \theta(s(t)).$$

The dynamical system for the parameters is given in  $t$  by

$$\tilde{b} + \tilde{\lambda}_t \tilde{\lambda} = 0, \quad \tilde{b}_t + \left( \frac{\tilde{b}}{\tilde{\lambda}} \right)^2 - \frac{\tilde{\beta}}{\tilde{\lambda}} = 0, \quad \tilde{\theta}_t - \frac{1}{\tilde{\lambda}^2} = 0.$$

With

$$\frac{ds}{dt} = \frac{1}{\lambda^2(t)},$$

$(b, \lambda, \theta)$  satisfy the following equivalent system in the variable  $s$ :

$$b + \frac{\lambda_s}{\lambda} = 0, \quad b_s + b^2 - \beta\lambda = 0, \quad \theta_s - 1 = 0.$$

Since the equation for  $\theta$  is independent, we shall focus our analysis on the first two equations.

We thus consider the system

$$\begin{cases} b + \frac{\lambda_s}{\lambda} = 0, \\ b_s + b^2 - \beta\lambda = 0, \end{cases} \quad (7.1)$$

with initial data

$$b(s_1) = b_1, \quad \lambda(s_1) = \lambda_1. \quad (7.2)$$

This is a Hamiltonian system, with conserved energy

$$\mathcal{E}_{\text{mo}}(b, \lambda) = \left( \frac{b}{\lambda} \right)^2 - \frac{2\beta}{\lambda}.$$

An exact solution with energy  $\mathcal{E}_{\text{mo}} = 0$  is given by

$$b_{\text{mo}}(s) = \frac{2}{s}, \quad \lambda_{\text{mo}}(s) = \frac{2}{\beta s^2}. \quad (7.3)$$

**Proposition 7.1.** *For any data  $b_1, \lambda_1 > 0$ , the solution of the Cauchy problem (7.1)-(7.2) satisfies*

$$b(s) = b_{\text{mo}}(s) + O(s^{-2}), \quad \lambda(s) = \lambda_{\text{mo}}(s) + O(s^{-4}).$$

*Proof.* Defining the auxiliary unknown  $\mu$  by

$$\mu = \frac{1}{\lambda},$$

direct calculations using (7.1) yield

$$\mu_s = -\frac{\lambda_s/\lambda}{\lambda} = \frac{b}{\lambda}, \quad \mu_{ss} = \left(\frac{b}{\lambda}\right)_s = \frac{b_s - b\lambda_s/\lambda}{\lambda} = \frac{b_s + b^2}{\lambda} = \beta.$$

Integrating in  $s$ , we get

$$\mu_s(s) = \beta(s - s_1) + \frac{b_1}{\lambda_1}, \quad \mu(s) = \frac{1}{2}\beta(s - s_1)^2 + \frac{b_1}{\lambda_1}(s - s_1) + \frac{1}{\lambda_1}.$$

Hence,  $b$  and  $\lambda$  are given by

$$b(s) = \left(\beta(s - s_1) + \frac{b_1}{\lambda_1}\right) \left(\frac{1}{2}\beta(s - s_1)^2 + \frac{b_1}{\lambda_1}(s - s_1) + \frac{1}{\lambda_1}\right)^{-1},$$

$$\lambda(s) = \left(\frac{1}{2}\beta(s - s_1)^2 + \frac{b_1}{\lambda_1}(s - s_1) + \frac{1}{\lambda_1}\right)^{-1}.$$

Letting  $s \rightarrow \infty$ , we obtain

$$b(s) = b_{\text{mo}}(s) + O(s^{-2}), \quad \lambda(s) = \lambda_{\text{mo}}(s) + O(s^{-4}).$$

□

*Proof of Proposition 2.5.* We first explain how the function  $\mathcal{F}$  comes into play. In order to integrate (7.1), one can use the conservation of  $\mathcal{E}_{\text{mo}}$ . Considering a solution of (7.1) with energy  $\mathcal{E}^*$ , one has, for  $\lambda \ll 1$ ,

$$b = \sqrt{\mathcal{E}^*\lambda^2 + 2\beta\lambda}, \quad b + \frac{\lambda_s}{\lambda} = 0. \quad (7.4)$$

Hence, assuming that  $\lambda_0 = \lambda(s_0) > 0$  is such that

$$\mathcal{E}^*\lambda_0 + 2\beta > 0,$$

we obtain that

$$\int_{s_0}^s \frac{\lambda_\sigma(\sigma)}{\lambda(\sigma)^{3/2} \sqrt{\mathcal{E}^*\lambda(\sigma) + 2\beta}} d\sigma = s_0 - s, \quad s \geq s_0,$$

i.e.

$$\mathcal{F}(\lambda) = s - s_0, \quad s \geq s_0. \quad (7.5)$$

Since

$$\frac{1}{\mu^{3/2} \sqrt{\mathcal{E}^*\mu + 2\beta}} \sim \frac{1}{\sqrt{2\beta}\mu^{3/2}}, \quad \mu \rightarrow 0^+,$$

we deduce that  $\mathcal{F}(\lambda)$  is strictly decreasing with

$$\mathcal{F}(\lambda_0) = 0 \quad \text{and} \quad \lim_{\lambda \rightarrow 0^+} \mathcal{F}(\lambda) = +\infty.$$

Therefore, (7.5) can be solved to find  $\lambda(s)$  and the solution of (7.1) then follows by returning to the first equation in (7.4).

We will now use  $\mathcal{F}$  to construct the final data  $(b_1, \lambda_1)$  of the exact modulation parameters  $b(s), \lambda(s)$  given by Proposition 2.2. Firstly, for any  $s_1 > 0$ , there exists a unique  $\lambda_1 > 0$  such that  $\mathcal{F}(\lambda_1) = s_1$ . For the model system, (7.3) yields

$$s_1 = \sqrt{\frac{2}{\beta \lambda_{\text{mo}}(s_1)}}.$$

On the other hand,

$$\begin{aligned} \left| \mathcal{F}(\lambda) - \sqrt{\frac{2}{\beta \lambda}} \right| &= \left| \int_{\lambda}^{\lambda_0} \frac{d\mu}{\mu^{3/2} \sqrt{\mathcal{E}^* \mu + 2\beta}} - \sqrt{\frac{2}{\beta}} \lambda^{-1/2} \right| \\ &\leq \left| \int_{\lambda}^{\lambda_0} \frac{d\mu}{\mu^{3/2} \sqrt{\mathcal{E}^* \mu + 2\beta}} - \sqrt{\frac{2}{\beta}} (\lambda^{-1/2} - \lambda_0^{-1/2}) \right| + \sqrt{\frac{2}{\beta}} \lambda_0^{-1/2} \\ &\leq \left| \int_{\lambda}^{\lambda_0} \frac{1}{\mu^{3/2}} \left( \frac{1}{\sqrt{\mathcal{E}^* \mu + 2\beta}} - \frac{1}{\sqrt{2\beta}} \right) d\mu \right| + \sqrt{\frac{2}{\beta}} \lambda_0^{-1/2} \\ &\lesssim \int_{\lambda}^{\lambda_0} \frac{d\mu}{\mu^{1/2}} + 1 = O_{\lambda \rightarrow 0}(1). \end{aligned} \tag{7.6}$$

Hence, with  $\lambda = \lambda_1$ , we get

$$\left| s_1 - \sqrt{\frac{2}{\beta \lambda_1}} \right| = O(1), \quad \text{i.e.} \quad \left| \sqrt{\frac{2}{\beta \lambda_{\text{mo}}(s_1)}} - \sqrt{\frac{2}{\beta \lambda_1}} \right| = O(1).$$

After some algebra, this yields

$$\left| \frac{\lambda_1^{1/2}}{\lambda_{\text{mo}}(s_1)^{1/2}} - 1 \right| \lesssim \lambda_{\text{mo}}(s_1)^{1/2} \lesssim \frac{1}{s_1},$$

or

$$|\lambda_1^{1/2} - \lambda_{\text{mo}}(s_1)^{1/2}| \lesssim \lambda_{\text{mo}}(s_1) \lesssim \frac{1}{s_1^2}.$$

To find  $b_1$ , we seek a solution of

$$h(b) := \lambda_1^2 \mathcal{E}(b, \lambda_1) = \lambda_1^2 \mathcal{E}^*$$

close to  $b_{\text{mo}}(s_1) = 2/s_1$ . Using the expression of  $\mathcal{E}$  in (2.11) and  $\mathcal{E}_{\text{mo}}(b_{\text{mo}}, \lambda_{\text{mo}}) = 0$ , we find

$$\begin{aligned} h(b) &= b^2 - b_{\text{mo}}(s_1)^2 + b_{\text{mo}}(s_1)^2 - 2\beta \lambda_1 + O(s_1^{-4}) \\ &= b^2 - b_{\text{mo}}(s_1)^2 + 2\beta (\lambda_{\text{mo}}(s_1)^{1/2} - \lambda_1^{1/2}) (\lambda_{\text{mo}}(s_1)^{1/2} + \lambda_1^{1/2}) + O(s_1^{-4}) \\ &= b^2 - \left( \frac{2}{s_1} \right)^2 + 2\beta O(s_1^{-2}) O(s_1^{-1}) + O(s_1^{-4}) \\ &= b^2 - \left( \frac{2}{s_1} \right)^2 + O(s_1^{-3}). \end{aligned}$$

In particular,

$$h(b_{\text{mo}}(s_1)) = O(s_1^{-3}).$$

Furthermore, by direct calculation,

$$h'(b_{\text{mo}}(s_1)) = 2b_{\text{mo}}(s_1) + O(s_1^{-3}) \gtrsim s_1^{-1}.$$

Since  $\lambda_1^2 \mathcal{E}^* = O(s_1^{-4})$ , expanding  $h$  around  $b_{\text{mo}}(s_1)$  shows that, if  $s_1$  is sufficiently large then there exists a unique  $b_1 > 0$  such that

$$h(b_1) = \lambda_1^2 \mathcal{E}^* \quad \text{and} \quad |b_1 - b_{\text{mo}}(s_1)| \lesssim \frac{s_1^{-3}}{s_1^{-1}} = s_1^{-2},$$

hence

$$\left| \frac{b_1}{b_{\text{mo}}(s_1)} - 1 \right| \lesssim \frac{1}{s_1},$$

which finishes the proof.  $\square$

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