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# Finite Volume Approximations for Non-Linear Parabolic Problems with Stochastic Forcing

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**Abstract.** We propose a two-point flux approximation finite-volume scheme for a stochastic non-linear parabolic equation with a multiplicative noise. The time discretization is implicit except for the stochastic noise term in order to be compatible with stochastic integration in the sense of Itô. We show existence and uniqueness of solutions to the scheme and the appropriate measurability for stochastic integration follows from the uniqueness of approximate solutions.

**Keywords:** stochastic non-linear parabolic equation, multiplicative Lipschitz noise, finite-volume method, upwind scheme, diffusion-convection equation, variational approach

## 1 Introduction

Let  $\Lambda$  be a bounded, open, connected and polygonal set of  $\mathbb{R}^2$ . Moreover let  $(\Omega, \mathcal{A}, \mathbb{P})$  be a probability space endowed with a right-continuous, complete filtration  $(\mathcal{F}_t)_{t \geq 0}$  and let  $(W(t))_{t \geq 0}$  be a standard, one-dimensional Brownian motion with respect to  $(\mathcal{F}_t)_{t \geq 0}$  on  $(\Omega, \mathcal{A}, \mathbb{P})$ .

For  $T > 0$ , we consider the following non-linear parabolic problem forced by a multiplicative stochastic noise:

$$\begin{aligned} du - \Delta u \, dt + \operatorname{div}(\mathbf{v}f(u)) \, dt &= g(u) \, dW(t) + \beta(u) \, dt, & \text{in } \Omega \times (0, T) \times \Lambda; \\ u(0, \cdot) &= u_0, & \text{in } \Omega \times \Lambda; \\ \nabla u \cdot \mathbf{n} &= 0, & \text{on } \Omega \times (0, T) \times \partial\Lambda; \end{aligned} \tag{1}$$

where  $\operatorname{div}$  is the divergence operator with respect to the space variable and  $\mathbf{n}$  denotes the unit normal vector to  $\partial\Lambda$  outward to  $\Lambda$ . After setting  $L_f, L_\beta$  and  $L_g$  in  $\mathbb{R}_+^*$ , we assume the following hypotheses on the data:

- $H_1$ :  $u_0 \in L^2(\Omega; H^1(\Lambda))$  is  $\mathcal{F}_0$ -measurable.  
 $H_2$ :  $f : \mathbb{R} \rightarrow \mathbb{R}$  is nondecreasing,  $L_f$ -Lipschitz continuous with  $f(0) = 0$ .  
 $H_3$ :  $g : \mathbb{R} \rightarrow \mathbb{R}$  is a  $L_g$ -Lipschitz continuous function.  
 $H_4$ :  $\beta : \mathbb{R} \rightarrow \mathbb{R}$  is  $L_\beta$ -Lipschitz continuous with  $\beta(0) = 0$ .  
 $H_5$ :  $\mathbf{v} \in \mathcal{C}^1([0, T] \times \Lambda; \mathbb{R}^2)$  such that  $\operatorname{div}(\mathbf{v}) = 0$  in  $[0, T] \times \Lambda$  and  $\mathbf{v} \cdot \mathbf{n} = 0$  on  $[0, T] \times \partial\Lambda$ .

### 1.1 Concept of Solution and Main Result

We will be interested in the concept of solution as defined below, which we will call a variational solution:

**Definition 1.** *A predictable stochastic process  $u$  is a variational solution to Problem (1) if it belongs to*

$$L^2(\Omega; \mathcal{C}([0, T]; L^2(\Lambda))) \cap L^2(\Omega; L^2(0, T; H^1(\Lambda)))$$

and satisfies, for all  $t \in [0, T]$ , in  $L^2(\Lambda)$ , and  $\mathbb{P}$ -a.s. in  $\Omega$

$$\begin{aligned}
 u(t) - u_0 - \int_0^t \Delta u(s) ds + \int_0^t \operatorname{div}(\mathbf{v}(s, \cdot) f(u(s))) ds \\
 = \int_0^t g(u(s)) dW(s) + \int_0^t \beta(u(s)) ds.
 \end{aligned}$$

Existence, uniqueness and regularity of this variational solution is well-known in the literature, see, e.g., [2].

### 1.2 Outline

In this contribution, we propose a finite-volume approximation scheme for the solution of (1) in the sense of Definition 1. We show existence and uniqueness of solutions to the scheme. In Section 2, we introduce the notation for our finite-volume framework. In Section 3, we introduce our finite-volume scheme. The main result is contained in Section 4.

## 2 Admissible Finite-Volume Meshes and Notations

In order to perform a finite-volume approximation of the variational solution of Problem (1) on  $[0, T] \times \Lambda$  we need first of all to set a choice for the temporal and spatial discretization. For the time-discretization, let  $N \in \mathbb{N}^*$  be given. We define the fixed time step  $\Delta t = \frac{T}{N}$  and divide the interval  $[0, T]$  in  $0 = t_0 < t_1 < \dots < t_N = T$  equidistantly with  $t_n = n\Delta t$  for all  $n \in \{0, \dots, N\}$ . For the space discretization, we refer to [1] and consider finite-volume admissible meshes in the sense of the following definition.

**Definition 2 (Admissible finite-volume mesh).** *An admissible finite-volume mesh  $\mathcal{T}$  of  $\Lambda$  (see Fig. 1) is given by a family of open polygonal and convex subsets  $K$ , called control volumes of  $\mathcal{T}$ , satisfying the following properties:*

- $\bar{\Lambda} = \bigcup_{K \in \mathcal{T}} \bar{K}$ .
- If  $K, L \in \mathcal{T}$  with  $K \neq L$  then  $\text{int } K \cap \text{int } L = \emptyset$ .
- If  $K, L \in \mathcal{T}$ , with  $K \neq L$  then either the 1-dimensional Lebesgue measure of  $\bar{K} \cap \bar{L}$  is 0 or  $\bar{K} \cap \bar{L}$  is the edge, denoted by  $\sigma = K|L$ , separating the control volumes  $K$  and  $L$ .
- To each control volume  $K \in \mathcal{T}$ , we associate a point  $x_K \in \bar{K}$  (called the center of  $K$ ) such that: If  $K, L \in \mathcal{T}$  are two neighbouring control volumes the straight line between the centers  $x_K$  and  $x_L$  is orthogonal to the edge  $\sigma = K|L$ .

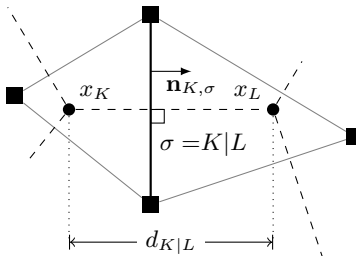


Fig. 1. Notations of the mesh  $\mathcal{T}$  associated with  $\Lambda$

Once an admissible finite-volume mesh  $\mathcal{T}$  of  $\Lambda$  is fixed, we will use the following notations.

### 2.1 Notation

- $h = \text{size}(\mathcal{T}) = \sup\{\text{diam}(K) : K \in \mathcal{T}\}$ , the mesh size.
- $d_h \in \mathbb{N}$  the number of control volumes  $K \in \mathcal{T}$  with  $h = \text{size}(\mathcal{T})$ .
- $\mathcal{E}_{\text{int}} := \{\sigma : \sigma \not\subseteq \partial\Lambda\}$  is the set of interior edges of the mesh  $\mathcal{T}$ .
- For  $K \in \mathcal{T}$ ,  $\mathcal{E}_K$  is the set of edges of  $K$ ,  $\mathcal{E}_{K,\text{int}} = \mathcal{E}_K \cap \mathcal{E}_{\text{int}}$  and  $m_K$  is the Lebesgue measure of  $K$ .
- For  $K \in \mathcal{T}$  and  $\sigma \in \mathcal{E}_K$ ,  $\mathbf{n}_{K,\sigma}$  is the unit normal vector to  $\sigma$  outward to  $K$ .
- Let  $K, L \in \mathcal{T}$  be two neighbouring control volumes. For  $\sigma = K|L \in \mathcal{E}_{\text{int}}$ , let  $m_\sigma$  be the length of  $\sigma$  and  $d_{K|L}$  the distance between  $x_K$  and  $x_L$ .
- For any vector  $u_h = (u_K)_{K \in \mathcal{T}} \in \mathbb{R}^{d_h}$ , we define the  $L^2$ -norm on  $\Lambda$  by

$$\|u_h\|_{L^2(\Lambda)} = \left( \sum_{K \in \mathcal{T}} m_K |u_K|^2 \right)^{\frac{1}{2}}.$$

In the sequel, we note  $|x|$  the euclidian norm of  $x \in \mathbb{R}^d$  with  $d \geq 1$ .

## 3 The Finite-Volume Scheme

Firstly, we define the vector  $u_h^0 = (u_K^0)_{K \in \mathcal{T}} \in \mathbb{R}^{d_h}$  by the discretization of the initial condition  $u_0$  of Problem (1) over each control volume:

$$u_K^0 := \frac{1}{m_K} \int_K u_0(x) dx, \quad \forall K \in \mathcal{T}. \quad (2)$$

The finite-volume scheme we propose reads, for this given initial  $\mathcal{F}_0$ -measurable random vector  $u_h^0 \in \mathbb{R}^{d_h}$ :

For any  $n \in \{0, \dots, N-1\}$ , knowing  $u_h^n = (u_K^n)_{K \in \mathcal{T}} \in \mathbb{R}^{d_h}$  we search for  $u_h^{n+1} = (u_K^{n+1})_{K \in \mathcal{T}} \in \mathbb{R}^{d_h}$  such that, for almost every  $\omega \in \Omega$ , the vector  $u_h^{n+1}$  is solution to the following random equations

$$\begin{aligned} & \frac{m_K}{\Delta t} (u_K^{n+1} - u_K^n) + \sum_{\sigma=K|L \in \mathcal{E}_{K,\text{int}}} m_\sigma v_{K,\sigma}^{n+1} f(u_\sigma^{n+1}) \\ & + \sum_{\sigma=K|L \in \mathcal{E}_{K,\text{int}}} \frac{m_\sigma}{d_{K|L}} (u_K^{n+1} - u_L^{n+1}) \\ & = \frac{m_K}{\Delta t} g(u_K^n) (W^{n+1} - W^n) + m_K \beta(u_K^{n+1}), \quad \forall K \in \mathcal{T}, \end{aligned} \quad (3)$$

where, by denoting  $\gamma$  the  $(d-1)$ -dimensional Lebesgue measure,

$$v_{K,\sigma}^{n+1} = \frac{1}{\Delta t m_\sigma} \int_{t_n}^{t_{n+1}} \int_\sigma \mathbf{v}(t, x) \cdot \mathbf{n}_{K,\sigma} d\gamma(x) dt,$$

and  $u_\sigma^{n+1}$  denotes the upstream value at time  $t_{n+1}$  with respect to  $\sigma$  defined as follows: If  $\sigma = K|L \in \mathcal{E}_{K,\text{int}}$  is the interface between the control volumes  $K$  and  $L$ ,  $u_\sigma^{n+1}$  is equal to  $u_K^{n+1}$  if  $v_{K,\sigma}^{n+1} \geq 0$  and to  $u_L^{n+1}$  if  $v_{K,\sigma}^{n+1} < 0$ . Note also that  $W^{n+1} - W^n = W(t_{n+1}) - W(t_n)$  for  $n \in \{0, \dots, N-1\}$ .

*Remark 1.* Since  $\text{div}(\mathbf{v}) = 0$  in  $[0, T] \times \Lambda$ , for any  $n \in \{0, \dots, N-1\}$  and  $K \in \mathcal{T}$  one has  $\sum_{\sigma=K|L \in \mathcal{E}_{K,\text{int}}} m_\sigma v_{K,\sigma}^{n+1} = 0$ . Thus, using that  $v_{K,\sigma}^{n+1} = (v_{K,\sigma}^{n+1})^+ - (v_{K,\sigma}^{n+1})^-$  (where, for  $r \in \mathbb{R}$ ,  $r^+ := \max\{r, 0\}$  and  $r^- := -\min\{0, r\}$ ) an equivalent formulation of the scheme (3) is given by

$$\begin{aligned} & \frac{m_K}{\Delta t} (u_K^{n+1} - u_K^n) + \sum_{\sigma=K|L \in \mathcal{E}_{K,\text{int}}} m_\sigma (v_{K,\sigma}^{n+1})^- (f(u_K^{n+1}) - f(u_L^{n+1})) \\ & + \sum_{\sigma=K|L \in \mathcal{E}_{K,\text{int}}} \frac{m_\sigma}{d_{K|L}} (u_K^{n+1} - u_L^{n+1}) \\ & = \frac{m_K}{\Delta t} g(u_K^n) (W^{n+1} - W^n) + m_K \beta(u_K^{n+1}), \quad \forall K \in \mathcal{T}. \end{aligned} \quad (4)$$

## 4 Main Result

**Proposition 1 (Existence of a discrete solution).** *Assume that hypotheses  $H_1$  to  $H_5$  hold. Let  $\mathcal{T}$  be an admissible finite-volume mesh of  $\Lambda$  in the sense of Definition 2 with a mesh size  $h$  and  $N \in \mathbb{N}^*$ . Then, there exists a unique solution  $(u_h^n)_{1 \leq n \leq N} \in (\mathbb{R}^{d_h})^N$  to Problem (3) associated with the initial vector  $u_h^0$  defined by (2). Additionally, for any  $n \in \{0, \dots, N\}$ ,  $u_h^n$  is a  $\mathcal{F}_{t_n}$ -measurable random vector.*

*Proof.* We fix  $n \in \{0, \dots, N-1\}$  and choose an arbitrary vector  $u_h^n = (u_K^n)_{K \in \mathcal{T}} \in \mathbb{R}^{d_h}$ . Firstly, we will show that there exists at least one random vector  $u_h^{n+1} =$

$(u_K^{n+1})_{K \in \mathcal{T}} \in \mathbb{R}^{d_h}$  such that (4) holds true  $\mathbb{P}$ -a.s in  $\Omega$ . To this end, we define the mapping  $\mathbf{P}^n : \mathbb{R}^{d_h} \rightarrow \mathbb{R}^{d_h}$ ,  $\mathbf{P}^n = (P_1^n, \dots, P_{d_h}^n)$  such that for any  $i \in \{1, \dots, d_h\}$

$$\begin{aligned} P_i^n(w_{K_1}, \dots, w_{K_{d_h}}) &= \frac{m_{K_i}}{\Delta t} w_{K_i} - m_{K_i} \beta(w_{K_i}) \\ &\quad + \sum_{\sigma=K_i | K_j \in \mathcal{E}_{K_i, \text{int}}} m_\sigma (v_{K_i, \sigma}^{n+1})^- (f(w_{K_i}) - f(w_{K_j})) \\ &\quad + \sum_{\sigma=K_i | K_j \in \mathcal{E}_{K_i, \text{int}}} \frac{m_\sigma}{d_{K_i | K_j}} (w_{K_i} - w_{K_j}) - \frac{m_{K_i}}{\Delta t} \xi_i^n \end{aligned}$$

where  $\xi_i^n := u_{K_i}^n + g(u_{K_i}^n)(W^{n+1} - W^n)$ .

Obviously,  $\mathbf{P}^n$  is a continuous mapping. Next, we show that there exists  $\varrho > 0$  such that for all  $w_h = (w_{K_i})_{1 \leq i \leq d_h} \in \mathbb{R}^{d_h}$  such that  $|w_h| = \varrho$ ,

$$(\mathbf{P}^n(w_h), w_h)_{\mathbb{R}^{d_h}} := \sum_{i=1}^{d_h} P_i^n(w_h) w_{K_i} \geq 0.$$

In this case, from [3, Lemma 4.3] it follows that there exists at least one  $\bar{w}_h \in \mathbb{R}^{d_h}$  such that  $|\bar{w}_h| \leq \varrho$  and  $\mathbf{P}^n(\bar{w}_h) = 0$ . We have

$$\begin{aligned} \sum_{i=1}^{d_h} P_i^n(w_h) w_{K_i} &= \sum_{i=1}^{d_h} \frac{m_{K_i}}{\Delta t} w_{K_i}^2 - \sum_{i=1}^{d_h} m_{K_i} \beta(w_{K_i}) w_{K_i} - \sum_{i=1}^{d_h} \frac{m_{K_i}}{\Delta t} \xi_i^n w_{K_i} \\ &\quad + \sum_{i=1}^{d_h} \sum_{\sigma=K_i | K_j \in \mathcal{E}_{K_i, \text{int}}} m_\sigma (v_{K_i, \sigma}^{n+1})^- (f(w_{K_i}) - f(w_{K_j})) w_{K_i} \\ &\quad + \sum_{i=1}^{d_h} \sum_{\sigma=K_i | K_j \in \mathcal{E}_{K_i, \text{int}}} \frac{m_\sigma}{d_{K_i | K_j}} (w_{K_i} - w_{K_j}) w_{K_i} \\ &=: I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned}$$

Since  $\beta$  is Lipschitz continuous, the term  $I_2$  satisfies

$$I_2 \geq -L_\beta \|w_h\|_{L^2(\Lambda)}^2. \quad (5)$$

Moreover, by discrete partial integration,

$$I_5 = \sum_{\sigma=K_i | K_j \in \mathcal{E}_{K_i, \text{int}}} \frac{m_\sigma}{d_{K_i | K_j}} |w_{K_i} - w_{K_j}|^2 \geq 0. \quad (6)$$

Now, we focus on the term  $I_4$ . Since  $f$  is Lipschitz continuous and nondecreasing, thanks to [1, Lemma 18.5], for any  $r \in \mathbb{R}$ , using the notation  $\Phi(r) = \int_0^r f'(s) s ds$ , for any  $a, b \in \mathbb{R}$ , one has

$$\begin{aligned} b(f(b) - f(a)) &= \int_a^b (sf(s))' ds - (b-a)f(a) = \int_a^b \Phi'(s) ds + \int_a^b (f(s) - f(a)) ds \\ &\geq (\Phi(b) - \Phi(a)) + \frac{1}{2L_f} (f(b) - f(a))^2. \end{aligned}$$

Thus, since  $\operatorname{div} \mathbf{v} = 0$  in  $[0, T] \times \Lambda$  and  $v_{K_i, \sigma}^{n+1} = -v_{K_j, \sigma}^{n+1}$ , we obtain

$$\begin{aligned} I_4 &\geq \sum_{i=1}^{d_h} \sum_{\sigma=K_i|K_j \in \mathcal{E}_{K_i, \text{int}}} m_\sigma (v_{K_i, \sigma}^{n+1})^- (\Phi(w_{K_i}) - \Phi(w_{K_j})) \\ &= \sum_{i=1}^{d_h} \Phi(w_{K_i}) \sum_{\sigma=K_i|K_j \in \mathcal{E}_{K_i, \text{int}}} m_\sigma (v_{K_i, \sigma}^{n+1}) = 0. \end{aligned} \quad (7)$$

For the term  $I_3$ , since  $-ab \geq -\frac{1}{2}(a^2 + b^2)$  one has

$$I_3 \geq -\frac{1}{2\Delta t} \left( \|w_h\|_{L^2(\Lambda)}^2 + \|\xi_h^n\|_{L^2(\Lambda)}^2 \right). \quad (8)$$

From (5), (6), (7) and (8) for some  $\alpha > 0$ , choosing  $\Delta t \leq \frac{1}{2(\alpha + L_\beta)}$  we now get

$$\begin{aligned} \sum_{i=1}^{d_h} P_i^n(w_h) w_{K_i} &\geq \frac{1}{2\Delta t} \|w_h\|_{L^2(\Lambda)}^2 - L_\beta \|w_h\|_{L^2(\Lambda)}^2 - \frac{1}{2\Delta t} \|\xi_h^n\|_{L^2(\Lambda)}^2 \\ &\geq \alpha (\min_{K \in \mathcal{T}} m_K) |w_h|^2 - \frac{1}{2\Delta t} \|\xi_h^n\|_{L^2(\Lambda)}^2. \end{aligned} \quad (9)$$

Then, setting

$$\varrho := \sqrt{\frac{1}{2\alpha (\min_{K \in \mathcal{T}} m_K) \Delta t}} \|\xi_h^n\|_{L^2(\Lambda)} > 0$$

we get  $(\mathbf{P}^n(w_h), w_h)_{\mathbb{R}^{d_h}} \geq 0$  from (9) for all  $w_h \in \mathbb{R}^{d_h}$  such that  $|w_h| = \varrho$ . Hence, there exists at least one element  $\bar{w}_h$  such that  $\mathbf{P}^n(\bar{w}_h) = 0$ . Thus,  $u_h^{n+1} := \bar{w}_h \in \mathbb{R}^{d_h}$  is solution to the numerical scheme (4).

Next, we will prove the uniqueness of the solution. Therefore, we assume that there exist  $w_h = (w_{K_i})_{1 \leq i \leq d_h} \in \mathbb{R}^{d_h}$  and  $z_h = (z_{K_i})_{1 \leq i \leq d_h} \in \mathbb{R}^{d_h}$  satisfying  $\mathbf{P}^n(w_h) = \mathbf{P}^n(z_h) = 0$ . Taking  $P_i^n(w_h) - P_i^n(z_h)$ , and using the initial formulation of the scheme (3), for any  $i = 1, \dots, d_h$  we obtain

$$\begin{aligned} &\frac{m_{K_i}}{\Delta t} (w_{K_i} - z_{K_i}) - m_{K_i} (\beta(w_{K_i}) - \beta(z_{K_i})) \\ &+ \sum_{\sigma=K_i|K_j \in \mathcal{E}_{K_i, \text{int}}} m_\sigma v_{K_i, \sigma}^{n+1} (f(w_\sigma) - f(z_\sigma)) \\ &+ \sum_{\sigma=K_i|K_j \in \mathcal{E}_{K_i, \text{int}}} \frac{m_\sigma}{d_{K_i|K_j}} ((w_{K_i} - w_{K_j}) - (z_{K_i} - z_{K_j})) = 0, \end{aligned}$$

where  $w_\sigma$  and  $z_\sigma$  are the upstream value with respect to  $\sigma$ .

Now, we adjust the method developed in the proof of [1, Proposition 26.1]: Using the monotonicity of  $f$ , the fact that  $v_{K_i, \sigma}^{n+1} = (v_{K_i, \sigma}^{n+1})^+ - (v_{K_i, \sigma}^{n+1})^-$  and

taking the absolute value, one has

$$\begin{aligned}
 & \frac{m_{K_i}}{\Delta t} |w_{K_i} - z_{K_i}| + \sum_{\sigma=K_i|K_j \in \mathcal{E}_{K_i, \text{int}}} \frac{m_\sigma}{d_{K_i|K_j}} |w_{K_i} - z_{K_i}| \\
 & + \sum_{\sigma=K_i|K_j \in \mathcal{E}_{K_i, \text{int}}} m_\sigma (v_{K_i, \sigma}^{n+1})^+ |f(w_{K_i}) - f(z_{K_i})| \\
 & \leq m_{K_i} |\beta(w_{K_i}) - \beta(z_{K_i})| + \sum_{\sigma=K_i|K_j \in \mathcal{E}_{K_i, \text{int}}} \frac{m_\sigma}{d_{K_i|K_j}} |w_{K_j} - z_{K_j}| \\
 & + \sum_{\sigma=K_i|K_j \in \mathcal{E}_{K_i, \text{int}}} m_\sigma (v_{K_i, \sigma}^{n+1})^- |f(w_{K_j}) - f(z_{K_j})|.
 \end{aligned} \tag{10}$$

For  $\eta > 0$ ,  $x \in \mathbb{R}^2$  we define  $\varphi(x) = \exp(-\eta|x|)$  and for  $K_i \in \mathcal{T}$ ,  $i = 1, \dots, d_h$  let

$$\varphi_{K_i} := \frac{1}{m_{K_i}} \int_{K_i} \varphi(x) dx.$$

Multiplying (10) by  $\varphi_{K_i}$ , taking the sum over  $i = 1, \dots, d_h$  and rearranging the sums on the right-hand side by fixing  $j$  and varying over  $i$  we obtain

$$\begin{aligned}
 & \sum_{i=1}^{d_h} \frac{m_{K_i}}{\Delta t} \varphi_{K_i} |w_{K_i} - z_{K_i}| + \sum_{i=1}^{d_h} \varphi_{K_i} \sum_{\sigma=K_i|K_j \in \mathcal{E}_{K_i, \text{int}}} \frac{m_\sigma}{d_{K_i|K_j}} |w_{K_i} - z_{K_i}| \\
 & + \sum_{i=1}^{d_h} \varphi_{K_i} \sum_{\sigma=K_i|K_j \in \mathcal{E}_{K_i, \text{int}}} m_\sigma (v_{K_i, \sigma}^{n+1})^+ |f(w_{K_i}) - f(z_{K_i})| \leq I_1 + I_2 + I_3
 \end{aligned} \tag{11}$$

where

$$\begin{aligned}
 I_2 & \leq \sum_{i=1}^{d_h} |w_{K_i} - z_{K_i}| \sum_{\sigma=K_i|K_j \in \mathcal{E}_{K_i, \text{int}}} \frac{m_\sigma}{d_{K_i|K_j}} |\varphi_{K_i} - \varphi_{K_j}| \\
 & + \sum_{i=1}^{d_h} |w_{K_i} - z_{K_i}| \sum_{\sigma=K_i|K_j \in \mathcal{E}_{K_i, \text{int}}} \frac{m_\sigma}{d_{K_i|K_j}} \varphi_{K_i}
 \end{aligned} \tag{12}$$

and similarly, since  $(v_{K_j, \sigma}^{n+1})^- = (-v_{K_i, \sigma}^{n+1})^- = (v_{K_i, \sigma}^{n+1})^+$  for  $\sigma = K_i|K_j$ , using the Lipschitz continuity of  $f$

$$\begin{aligned}
 I_3 & \leq \sum_{i=1}^{d_h} L_f |w_{K_i} - z_{K_i}| \sum_{\sigma=K_i|K_j \in \mathcal{E}_{K_i, \text{int}}} m_\sigma (v_{K_j, \sigma}^{n+1})^- |\varphi_{K_i} - \varphi_{K_j}| \\
 & + \sum_{i=1}^{d_h} |f(w_{K_i}) - f(z_{K_i})| \sum_{\sigma=K_i|K_j \in \mathcal{E}_{K_i, \text{int}}} m_\sigma (v_{K_i, \sigma}^{n+1})^+ \varphi_{K_i}.
 \end{aligned} \tag{13}$$



Now, plugging (12) and (13) into (11) and using the Lipschitz continuity of  $\beta$  we obtain for all  $i = 1, \dots, d_h$

$$\sum_{i=1}^{d_h} a_i |w_{K_i} - z_{K_i}| \leq \sum_{i=1}^{d_h} b_i |w_{K_i} - z_{K_i}| \quad (14)$$

with

$$a_i := m_{K_i} \left( \frac{1}{\Delta t} - L_\beta \right) \varphi_{K_i}$$

$$b_i := \sum_{\sigma=K_i | K_j \in \mathcal{E}_{K_i, \text{int}}} \left( \frac{m_\sigma}{d_{K_i | K_j}} + m_\sigma (v_{K_j, \sigma}^{n+1})^- \right) |\varphi_{K_i} - \varphi_{K_j}|.$$

Now, taking  $\Delta t \leq \frac{1}{2L_\beta}$  using the same arguments as in the proof of [1, Proposition 26.1], we may choose  $\eta > 0$  small enough such that  $a_i > b_i$  for all  $i = 1, \dots, d_h$ . Thus  $w_{K_i} = z_{K_i}$  then follows from (14) for all  $i = 1, \dots, d_h$ . Since the initial vector  $u_h^0$  is given, the existence of a unique solution  $(u_h^n)_{1 \leq n \leq N} \in \mathbb{R}^{d_h}$  follows by iteration.

It is left to prove that  $u_h^n$  is a  $\mathcal{F}_{t_n}$ -measurable random vector for all  $n = 1, \dots, N$ . We have already shown that for any given  $\xi_h \in \mathbb{R}^{d_h}$  there exists a unique  $w_h = (w_{K_i})_{1 \leq i \leq d_h} \in \mathbb{R}^{d_h}$  such that  $\mathbf{Q}(w_h) = \xi_h$  where  $\mathbf{Q} : \mathbb{R}^{d_h} \rightarrow \mathbb{R}^{d_h}$ ,  $\mathbf{Q}(w_h) = (Q_1, \dots, Q_{d_h})(w_h)$  is defined by  $Q_i(w_h) = P_i(w_h) - \frac{m_{K_i}}{\Delta t} \xi_i^n$  for all  $i = 1, \dots, d_h$ . Thus,  $u_h^{n+1} = \mathbf{Q}^{-1}(\xi_h^n)$  P-a.s. in  $\Omega$ , where  $\xi_h^n = (\xi_1^n, \dots, \xi_{d_h}^n)$ . Since  $\mathbf{Q}^{-1}$  is continuous, if  $\xi_h^n$  is  $\mathcal{F}_{t_{n+1}}$ -measurable the same holds true for  $u_h^{n+1}$ . Indeed, let  $(\zeta^k)_k \subset \mathbb{R}^{d_h}$  be a sequence such that  $\zeta^k \rightarrow \zeta$  for some  $\zeta \in \mathbb{R}^{d_h}$  for  $k \rightarrow \infty$ . Then, for  $w^k := \mathbf{Q}^{-1}(\zeta^k)$  from (9) and from the theorem of Bolzano-Weierstrass it follows that there exists  $w \in \mathbb{R}^{d_h}$  such that, passing to an unlabelled subsequence if necessary,  $w^k \rightarrow w$  for  $k \rightarrow \infty$ . This strong convergence is enough to pass to the limit in  $\mathbf{Q}(w^k)$ , and therefore  $\mathbf{Q}(w) = \zeta$ . Thanks to uniqueness, we get convergence of the whole sequence  $(w^k)_k$ , hence  $\lim_{k \rightarrow \infty} \mathbf{Q}^{-1}(\zeta^k) = \mathbf{Q}^{-1}(\zeta)$  and  $\mathbf{Q}^{-1}$  is continuous.  $\square$

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