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Abstract. We propose a two-point flux approximation finite-volume scheme for a stochastic non-linear parabolic equation with a multiplicative noise. The time discretization is implicit except for the stochastic noise term in order to be compatible with stochastic integration in the sense of Itô. We show existence and uniqueness of solutions to the scheme and the appropriate measurability for stochastic integration follows from the uniqueness of approximate solutions.

Keywords: stochastic non-linear parabolic equation, multiplicative Lipschitz noise, finite-volume method, upwind scheme, diffusion-convection equation, variational approach

1 Introduction

Let Λ be a bounded, open, connected and polygonal set of $\mathbb{R}^2$. Moreover let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space endowed with a right-continuous, complete filtration $(\mathcal{F}_t)_{t\geq 0}$ and let $(W(t))_{t\geq 0}$ be a standard, one-dimensional Brownian motion with respect to $(\mathcal{F}_t)_{t\geq 0}$ on $(\Omega, \mathcal{A}, \mathbb{P})$.
For $T > 0$, we consider the following non-linear parabolic problem forced by a multiplicative stochastic noise:

$$\begin{aligned} du - \Delta u\, dt + \operatorname{div}\big(\mathbf{v} f(u)\big)\, dt &= g(u)\, dW(t) + \beta(u)\, dt, && \text{in } \Omega \times (0,T) \times \Lambda; \\ u(0,.) &= u_0, && \text{in } \Omega \times \Lambda; \\ \nabla u \cdot \mathbf{n} &= 0, && \text{on } \Omega \times (0,T) \times \partial\Lambda; \end{aligned} \tag{1}$$

where div is the divergence operator with respect to the space variable and $\mathbf{n}$ denotes the unit normal vector to $\partial\Lambda$ outward to Λ. After setting L_f, L_β and L_g in $\mathbb{R}_+^*$, we assume the following hypotheses on the data:

H_1: $u_0 \in L^2(\Omega; H^1(\Lambda))$ is $\mathcal{F}_0$-measurable.
H_2: $f : \mathbb{R} \to \mathbb{R}$ is nondecreasing, L_f-Lipschitz continuous with $f(0) = 0$.
H_3: $g : \mathbb{R} \to \mathbb{R}$ is a L_g-Lipschitz continuous function.
H_4: $\beta : \mathbb{R} \to \mathbb{R}$ is L_β-Lipschitz continuous with $\beta(0) = 0$.
H_5: $\mathbf{v} \in \mathscr{C}^1([0,T] \times \Lambda; \mathbb{R}^2)$ such that $\mathrm{div}(\mathbf{v}) = 0$ in $[0,T] \times \Lambda$ and $\mathbf{v} \cdot \mathbf{n} = 0$ on $[0,T] \times \partial\Lambda$.

1.1 Concept of Solution and Main Result

We will be interested in the concept of solution as defined below, which we will call a variational solution:

Definition 1. *A predictable stochastic process u is a variational solution to Problem (1) if it belongs to*

$$L^2(\Omega; \mathscr{C}([0,T]; L^2(\Lambda))) \cap L^2(\Omega; L^2(0,T; H^1(\Lambda)))$$

and satisfies, for all $t \in [0,T]$, in $L^2(\Lambda)$, and $\mathbb{P}$-a.s. in Ω

$$\begin{aligned} &u(t) - u_0 - \int_0^t \Delta u(s)\, ds + \int_0^t \mathrm{div}\left(\mathbf{v}(s,.) f(u(s))\right) ds \\ &= \int_0^t g(u(s))\, dW(s) + \int_0^t \beta(u(s))\, ds. \end{aligned}$$

Existence, uniqueness and regularity of this variational solution is well-known in the literature, see, e.g., [2].

1.2 Outline

In this contribution, we propose a finite-volume approximation scheme for the solution of (1) in the sense of Definition 1. We show existence and uniqueness of solutions to the scheme. In Section 2, we introduce the notation for our finite-volume framework. In Section 3, we introduce our finite-volume scheme. The main result is contained in Section 4.

2 Admissible Finite-Volume Meshes and Notations

In order to perform a finite-volume approximation of the variational solution of Problem (1) on $[0,T] \times \Lambda$ we need first of all to set a choice for the temporal and spatial discretization. For the time-discretization, let $N \in \mathbb{N}^*$ be given. We define the fixed time step $\Delta t = \frac{T}{N}$ and divide the interval $[0,T]$ in $0 = t_0 < t_1 < ... < t_N = T$ equidistantly with $t_n = n\Delta t$ for all $n \in \{0, ..., N\}$. For the space discretization, we refer to [1] and consider finite-volume admissible meshes in the sense of the following definition.

Definition 2 (Admissible finite-volume mesh). *An admissible finite-volume mesh $\mathcal{T}$ of Λ (see Fig. 1) is given by a family of open polygonal and convex subsets K, called control volumes of $\mathcal{T}$, satisfying the following properties:*

- $\overline{\Lambda} = \bigcup_{K\in\mathcal{T}} \overline{K}$.
- *If* $K, L \in \mathcal{T}$ *with* $K \neq L$ *then* $\operatorname{int} K \cap \operatorname{int} L = \emptyset$.
- *If* $K, L \in \mathcal{T}$, *with* $K \neq L$ *then either the* 1*-dimensional Lebesgue measure of* $\overline{K} \cap \overline{L}$ *is* 0 *or* $\overline{K} \cap \overline{L}$ *is the edge, denoted by* $\sigma = K|L$, *separating the control volumes* K *and* L.
- *To each control volume* $K \in \mathcal{T}$, *we associate a point* $x_K \in \overline{K}$ *(called the center of* K*) such that: If* $K, L \in \mathcal{T}$ *are two neighbouring control volumes the straight line between the centers* x_K *and* x_L *is orthogonal to the edge* $\sigma = K|L$.

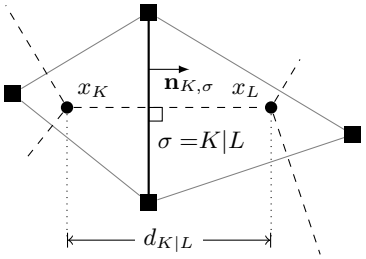

Fig. 1. Notations of the mesh $\mathcal{T}$ associated with Λ

Once an admissible finite-volume mesh $\mathcal{T}$ of Λ is fixed, we will use the following notations.

2.1 Notation

- $h = \operatorname{size}(\mathcal{T}) = \sup\{\operatorname{diam}(K) : K \in \mathcal{T}\}$, the mesh size.
- $d_h \in \mathbb{N}$ the number of control volumes $K \in \mathcal{T}$ with $h = \operatorname{size}(\mathcal{T})$.
- $\mathcal{E}_{\text{int}} := \{\sigma : \sigma \nsubseteq \partial\Lambda\}$ is the set of interior edges of the mesh $\mathcal{T}$.
- For $K \in \mathcal{T}$, $\mathcal{E}_K$ is the set of edges of K, $\mathcal{E}_{K,\text{int}} = \mathcal{E}_K \cap \mathcal{E}_{\text{int}}$ and m_K is the Lebesgue measure of K.
- For $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}_K$, $\mathbf{n}_{K,\sigma}$ is the unit normal vector to σ outward to K.
- Let $K, L \in \mathcal{T}$ be two neighbouring control volumes. For $\sigma = K|L \in \mathcal{E}_{\text{int}}$, let m_σ be the length of σ and $d_{K|L}$ the distance between x_K and x_L.
- For any vector $u_h = (u_K)_{K\in\mathcal{T}} \in \mathbb{R}^{d_h}$, we define the L^2-norm on Λ by

$$\|u_h\|_{L^2(\Lambda)} = \left(\sum_{K\in\mathcal{T}} m_K |u_K|^2\right)^{\frac{1}{2}}.$$

In the sequel, we note $|x|$ the euclidian norm of $x \in \mathbb{R}^d$ with $d \geq 1$.

3 The Finite-Volume Scheme

Firstly, we define the vector $u_h^0 = (u_K^0)_{K\in\mathcal{T}} \in \mathbb{R}^{d_h}$ by the discretization of the initial condition u_0 of Problem (1) over each control volume:

$$u_K^0 := \frac{1}{m_K} \int_K u_0(x)\, dx, \quad \forall K \in \mathcal{T}. \tag{2}$$

The finite-volume scheme we propose reads, for this given initial $\mathcal{F}_0$-measurable random vector $u_h^0 \in \mathbb{R}^{d_h}$:
For any $n \in \{0, \dots, N-1\}$, knowing $u_h^n = (u_K^n)_{K\in\mathcal{T}} \in \mathbb{R}^{d_h}$ we search for $u_h^{n+1} = (u_K^{n+1})_{K\in\mathcal{T}} \in \mathbb{R}^{d_h}$ such that, for almost every $\omega \in \Omega$, the vector u_h^{n+1} is solution to the following random equations

$$\begin{aligned}
&\frac{m_K}{\Delta t}(u_K^{n+1} - u_K^n) + \sum_{\sigma=K|L\in\mathcal{E}_{K,\mathrm{int}}} m_\sigma v_{K,\sigma}^{n+1} f(u_\sigma^{n+1}) \\
&\quad + \sum_{\sigma=K|L\in\mathcal{E}_{K,\mathrm{int}}} \frac{m_\sigma}{d_{K|L}}(u_K^{n+1} - u_L^{n+1}) \qquad (3)\\
&= \frac{m_K}{\Delta t} g(u_K^n)(W^{n+1} - W^n) + m_K \beta(u_K^{n+1}), \quad \forall K \in \mathcal{T},
\end{aligned}$$

where, by denoting γ the $(d-1)$-dimensional Lebesgue measure,

$$v_{K,\sigma}^{n+1} = \frac{1}{\Delta t m_\sigma} \int_{t_n}^{t_{n+1}} \int_\sigma \mathbf{v}(t,x)\cdot \mathbf{n}_{K,\sigma}\, d\gamma(x) dt,$$

and u_σ^{n+1} denotes the upstream value at time t_{n+1} with respect to σ defined as follows: If $\sigma = K|L \in \mathcal{E}_{K,\mathrm{int}}$ is the interface between the control volumes K and L, u_σ^{n+1} is equal to u_K^{n+1} if $v_{K,\sigma}^{n+1} \geqslant 0$ and to u_L^{n+1} if $v_{K,\sigma}^{n+1} < 0$. Note also that $W^{n+1} - W^n = W(t_{n+1}) - W(t_n)$ for $n \in \{0, \dots, N-1\}$.

Remark 1. Since $\mathrm{div}(\mathbf{v}) = 0$ in $[0,T] \times \Lambda$, for any $n \in \{0, \cdots, N-1\}$ and $K \in \mathcal{T}$ one has $\sum_{\sigma=K|L\in\mathcal{E}_{K,\mathrm{int}}} m_\sigma v_{K,\sigma}^{n+1} = 0$. Thus, using that $v_{K,\sigma}^{n+1} = (v_{K,\sigma}^{n+1})^+ - (v_{K,\sigma}^{n+1})^-$ (where, for $r \in \mathbb{R}$, $r^+ := \max\{r, 0\}$ and $r^- := -\min\{0, r\}$) an equivalent formulation of the scheme (3) is given by

$$\begin{aligned}
&\frac{m_K}{\Delta t}(u_K^{n+1} - u_K^n) + \sum_{\sigma=K|L\in\mathcal{E}_{K,\mathrm{int}}} m_\sigma (v_{K,\sigma}^{n+1})^- \Big(f(u_K^{n+1}) - f(u_L^{n+1})\Big) \\
&\quad + \sum_{\sigma=K|L\in\mathcal{E}_{K,\mathrm{int}}} \frac{m_\sigma}{d_{K|L}}(u_K^{n+1} - u_L^{n+1}) \qquad (4)\\
&= \frac{m_K}{\Delta t} g(u_K^n)\left(W^{n+1} - W^n\right) + m_K \beta(u_K^{n+1}), \quad \forall K \in \mathcal{T}.
\end{aligned}$$

4 Main Result

Proposition 1 (Existence of a discrete solution). *Assume that hypotheses H_1 to H_5 hold. Let $\mathcal{T}$ be an admissible finite-volume mesh of Λ in the sense of Definition 2 with a mesh size h and $N \in \mathbb{N}^*$. Then, there exists a unique solution $(u_h^n)_{1\leq n\leq N} \in (\mathbb{R}^{d_h})^N$ to Problem (3) associated with the initial vector u_h^0 defined by (2). Additionally, for any $n \in \{0, \dots, N\}$, u_h^n is a $\mathcal{F}_{t_n}$-measurable random vector.*

Proof. We fix $n \in \{0, \dots, N-1\}$ and choose an arbitrary vector $u_h^n = (u_K^n)_{K\in\mathcal{T}} \in \mathbb{R}^{d_h}$. Firstly, we will show that there exists at least one random vector $u_h^{n+1} =$

$(u_K^{n+1})_{K\in\mathcal{T}} \in \mathbb{R}^{d_h}$ such that (4) holds true $\mathbb{P}$-a.s in Ω. To this end, we define the mapping $\mathbf{P}^n : \mathbb{R}^{d_h} \to \mathbb{R}^{d_h}$, $\mathbf{P}^n = (P_1^n, \dots, P_{d_h}^n)$ such that for any $i \in \{1, \dots, d_h\}$

$$\begin{aligned} P_i^n(w_{K_1}, \dots, w_{K_{d_h}}) =& \frac{m_{K_i}}{\Delta t} w_{K_i} - m_{K_i}\beta(w_{K_i}) \\ &+ \sum_{\sigma=K_i|K_j\in\mathcal{E}_{K_i,\mathrm{int}}} m_\sigma (v_{K_i,\sigma}^{n+1})^- (f(w_{K_i}) - f(w_{K_j})) \\ &+ \sum_{\sigma=K_i|K_j\in\mathcal{E}_{K_i,\mathrm{int}}} \frac{m_\sigma}{d_{K_i|K_j}} (w_{K_i} - w_{K_j}) - \frac{m_{K_i}}{\Delta t}\xi_i^n \end{aligned}$$

where $\xi_i^n := u_{K_i}^n + g(u_{K_i}^n)(W^{n+1} - W^n)$.

Obviously, $\mathbf{P}^n$ is a continuous mapping. Next, we show that there exists $\varrho > 0$ such that for all $w_h = (w_{K_i})_{1\le i\le d_h} \in \mathbb{R}^{d_h}$ such that $|w_h| = \varrho$,

$$(\mathbf{P}^n(w_h), w_h)_{\mathbb{R}^{d_h}} := \sum_{i=1}^{d_h} P_i^n(w_h) w_{K_i} \ge 0.$$

In this case, from [3, Lemma 4.3] it follows that there exists at least one $\overline{w}_h \in \mathbb{R}^{d_h}$ such that $|\overline{w}_h| \le \varrho$ and $\mathbf{P}^n(\overline{w}_h) = 0$. We have

$$\begin{aligned} \sum_{i=1}^{d_h} P_i^n(w_h) w_{K_i} =& \sum_{i=1}^{d_h} \frac{m_{K_i}}{\Delta t} w_{K_i}^2 - \sum_{i=1}^{d_h} m_{K_i}\beta(w_{K_i}) w_{K_i} - \sum_{i=1}^{d_h} \frac{m_{K_i}}{\Delta t}\xi_i^n w_{K_i} \\ &+ \sum_{i=1}^{d_h} \sum_{\sigma=K_i|K_j\in\mathcal{E}_{K_i,\mathrm{int}}} m_\sigma (v_{K_i,\sigma}^{n+1})^- (f(w_{K_i}) - f(w_{K_j})) w_{K_i} \\ &+ \sum_{i=1}^{d_h} \sum_{\sigma=K_i|K_j\in\mathcal{E}_{K_i,\mathrm{int}}} \frac{m_\sigma}{d_{K_i|K_j}} (w_{K_i} - w_{K_j}) w_{K_i} \\ =:& I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned}$$

Since β is Lipschitz continuous, the term I_2 satisfies

$$I_2 \ge -L_\beta \|w_h\|_{L^2(\Lambda)}^2. \tag{5}$$

Moreover, by discrete partial integration,

$$I_5 = \sum_{\sigma=K_i|K_j\in\mathcal{E}_{K_i,\mathrm{int}}} \frac{m_\sigma}{d_{K_i|K_j}} |w_{K_i} - w_{K_j}|^2 \ge 0. \tag{6}$$

Now, we focus on the term I_4. Since f is Lipschitz continuous and nondecreasing, thanks to [1, Lemma 18.5], for any $r \in \mathbb{R}$, using the notation $\Phi(r) = \int_0^r f'(s)s\,ds$, for any $a, b \in \mathbb{R}$, one has

$$\begin{aligned} b(f(b) - f(a)) =& \int_a^b (sf(s))' ds - (b-a)f(a) = \int_a^b \Phi'(s) ds + \int_a^b (f(s) - f(a)) ds \\ &\ge (\Phi(b) - \Phi(a)) + \frac{1}{2L_f}(f(b) - f(a))^2. \end{aligned}$$

Thus, since $\operatorname{div} \mathbf{v} = 0$ in $[0,T] \times \Lambda$ and $v^{n+1}_{K_i,\sigma} = -v^{n+1}_{K_j,\sigma}$, we obtain

$$\begin{aligned} I_4 &\geq \sum_{i=1}^{d_h} \sum_{\sigma = K_i | K_j \in \mathcal{E}_{K_i,\mathrm{int}}} m_\sigma (v^{n+1}_{K_i,\sigma})^- (\Phi(w_{K_i}) - \Phi(w_{K_j})) \\ &= \sum_{i=1}^{d_h} \Phi(w_{K_i}) \sum_{\sigma = K_i | K_j \in \mathcal{E}_{K_i,\mathrm{int}}} m_\sigma (v^{n+1}_{K_i,\sigma}) = 0. \end{aligned} \tag{7}$$

For the term I_3, since $-ab \geq -\frac{1}{2}(a^2 + b^2)$ one has

$$I_3 \geq -\frac{1}{2\Delta t} \left(\|w_h\|^2_{L^2(\Lambda)} + \|\xi^n_h\|^2_{L^2(\Lambda)} \right). \tag{8}$$

From (5), (6), (7) and (8) for some $\alpha > 0$, choosing $\Delta t \leq \frac{1}{2(\alpha + L_\beta)}$ we now get

$$\begin{aligned} \sum_{i=1}^{d_h} P^n_i(w_h) w_{K_i} &\geq \frac{1}{2\Delta t} \|w_h\|^2_{L^2(\Lambda)} - L_\beta \|w_h\|^2_{L^2(\Lambda)} - \frac{1}{2\Delta t} \|\xi^n_h\|^2_{L^2(\Lambda)} \\ &\geq \alpha (\min_{K \in \mathcal{T}} m_K) |w_h|^2 - \frac{1}{2\Delta t} \|\xi^n_h\|^2_{L^2(\Lambda)}. \end{aligned} \tag{9}$$

Then, setting

$$\varrho := \sqrt{\frac{1}{2\alpha (\min_{K \in \mathcal{T}} m_K) \Delta t}} \|\xi^n_h\|_{L^2(\Lambda)} > 0$$

we get $(\mathbf{P}^n(w_h), w_h)_{\mathbb{R}^{d_h}} \geq 0$ from (9) for all $w_h \in \mathbb{R}^{d_h}$ such that $|w_h| = \varrho$. Hence, there exists at least one element $\overline{w}_h$ such that $\mathbf{P}^n(\overline{w}_h) = 0$. Thus, $u^{n+1}_h := \overline{w}_h \in \mathbb{R}^{d_h}$ is solution to the numerical scheme (4).

Next, we will prove the uniqueness of the solution. Therefore, we assume that there exist $w_h = (w_{K_i})_{1 \leq i \leq d_h} \in \mathbb{R}^{d_h}$ and $z_h = (z_{K_i})_{1 \leq i \leq d_h} \in \mathbb{R}^{d_h}$ satisfying $\mathbf{P}^n(w_h) = \mathbf{P}^n(z_h) = 0$. Taking $P^n_i(w_h) - P^n_i(z_h)$, and using the initial formulation of the scheme (3), for any $i = 1, \ldots, d_h$ we obtain

$$\begin{aligned} &\frac{m_{K_i}}{\Delta t} (w_{K_i} - z_{K_i}) - m_{K_i} (\beta(w_{K_i}) - \beta(z_{K_i})) \\ &\quad + \sum_{\sigma = K_i | K_j \in \mathcal{E}_{K_i,\mathrm{int}}} m_\sigma v^{n+1}_{K_i,\sigma} (f(w_\sigma) - f(z_\sigma)) \\ &\quad + \sum_{\sigma = K_i | K_j \in \mathcal{E}_{K_i,\mathrm{int}}} \frac{m_\sigma}{d_{K_i|K_j}} \big((w_{K_i} - w_{K_j}) - (z_{K_i} - z_{K_j}) \big) = 0, \end{aligned}$$

where w_σ and z_σ are the upstream value with respect to σ.

Now, we adjust the method developed in the proof of [1, Proposition 26.1]: Using the monotonicity of f, the fact that $v^{n+1}_{K_i,\sigma} = (v^{n+1}_{K_i,\sigma})^+ - (v^{n+1}_{K_i,\sigma})^-$ and

taking the absolute value, one has

$$\begin{aligned}
&\frac{m_{K_i}}{\Delta t}|w_{K_i}-z_{K_i}|+\sum_{\sigma=K_i|K_j\in\mathcal{E}_{K_i,\mathrm{int}}}\frac{m_\sigma}{d_{K_i|K_j}}|w_{K_i}-z_{K_i}|\\
&\quad+\sum_{\sigma=K_i|K_j\in\mathcal{E}_{K_i,\mathrm{int}}}m_\sigma(v^{n+1}_{K_i,\sigma})^+|f(w_{K_i})-f(z_{K_i})|\\
&\le m_{K_i}|\beta(w_{K_i})-\beta(z_{K_i})|+\sum_{\sigma=K_i|K_j\in\mathcal{E}_{K_i,\mathrm{int}}}\frac{m_\sigma}{d_{K_i|K_j}}|w_{K_j}-z_{K_j}|\\
&\quad+\sum_{\sigma=K_i|K_j\in\mathcal{E}_{K_i,\mathrm{int}}}m_\sigma(v^{n+1}_{K_i,\sigma})^-|f(w_{K_j})-f(z_{K_j})|.
\end{aligned}\tag{10}$$

For $\eta>0$, $x\in\mathbb{R}^2$ we define $\varphi(x)=\exp(-\eta|x|)$ and for $K_i\in\mathcal{T}$, $i=1,\dots,d_h$ let

$$\varphi_{K_i}:=\frac{1}{m_{K_i}}\int_{K_i}\varphi(x)\,dx.$$

Multiplying (10) by φ_{K_i}, taking the sum over $i=1,\dots,d_h$ and rearranging the sums on the right-hand side by fixing j and varying over i we obtain

$$\begin{aligned}
&\sum_{i=1}^{d_h}\frac{m_{K_i}}{\Delta t}\varphi_{K_i}|w_{K_i}-z_{K_i}|+\sum_{i=1}^{d_h}\varphi_{K_i}\sum_{\sigma=K_i|K_j\in\mathcal{E}_{K_i,\mathrm{int}}}\frac{m_\sigma}{d_{K_i|K_j}}|w_{K_i}-z_{K_i}|\\
&+\sum_{i=1}^{d_h}\varphi_{K_i}\sum_{\sigma=K_i|K_j\in\mathcal{E}_{K_i,\mathrm{int}}}m_\sigma(v^{n+1}_{K_i,\sigma})^+|f(w_{K_i})-f(z_{K_i})|\le I_1+I_2+I_3
\end{aligned}\tag{11}$$

where

$$\begin{aligned}
I_2\le&\sum_{i=1}^{d_h}|w_{K_i}-z_{K_i}|\sum_{\sigma=K_i|K_j\in\mathcal{E}_{K_i,\mathrm{int}}}\frac{m_\sigma}{d_{K_i|K_j}}|\varphi_{K_i}-\varphi_{K_j}|\\
&+\sum_{i=1}^{d_h}|w_{K_i}-z_{K_i}|\sum_{\sigma=K_i|K_j\in\mathcal{E}_{K_i,\mathrm{int}}}\frac{m_\sigma}{d_{K_i|K_j}}\varphi_{K_i}
\end{aligned}\tag{12}$$

and similarly, since $(v^{n+1}_{K_j,\sigma})^-=(-v^{n+1}_{K_i,\sigma})^-=(v^{n+1}_{K_i,\sigma})^+$ for $\sigma=K_i|K_j$, using the Lipschitz continuity of f

$$\begin{aligned}
I_3\le&\sum_{i=1}^{d_h}L_f|w_{K_i}-z_{K_i}|\sum_{\sigma=K_i|K_j\in\mathcal{E}_{K_i,\mathrm{int}}}m_\sigma(v^{n+1}_{K_j,\sigma})^-|\varphi_{K_i}-\varphi_{K_j}|\\
&+\sum_{i=1}^{d_h}|f(w_{K_i})-f(z_{K_i})|\sum_{\sigma=K_i|K_j\in\mathcal{E}_{K_i,\mathrm{int}}}m_\sigma(v^{n+1}_{K_i,\sigma})^+\varphi_{K_i}.
\end{aligned}\tag{13}$$

Now, plugging (12) and (13) into (11) and using the Lipschitz continuity of β we obtain for all $i = 1, \dots, d_h$

$$\sum_{i=1}^{d_h} a_i |w_{K_i} - z_{K_i}| \leq \sum_{i=1}^{d_h} b_i |w_{K_i} - z_{K_i}| \tag{14}$$

with

$$a_i := m_{K_i} \left(\frac{1}{\Delta t} - L_\beta \right) \varphi_{K_i}$$

$$b_i := \sum_{\sigma = K_i | K_j \in \mathcal{E}_{K_i, \mathrm{int}}} \left(\frac{m_\sigma}{d_{K_i | K_j}} + m_\sigma (v^{n+1}_{K_j, \sigma})^- \right) |\varphi_{K_i} - \varphi_{K_j}|.$$

Now, taking $\Delta t \leq \frac{1}{2L_\beta}$ using the same arguments as in the proof of [1, Proposition 26.1], we may choose $\eta > 0$ small enough such that $a_i > b_i$ for all $i = 1, \dots d_h$. Thus $w_{K_i} = z_{K_i}$ then follows from (14) for all $i = 1, \dots, d_h$. Since the initial vector u_h^0 is given, the existence of a unique solution $(u_h^n)_{1 \leq n \leq N} \in \mathbb{R}^{d_h}$ follows by iteration.

It is left to prove that u_h^n is a $\mathcal{F}_{t_n}$-measurable random vector for all $n = 1, \dots, N$. We have already shown that for any given $\xi_h \in \mathbb{R}^{d_h}$ there exists a unique $w_h = (w_{K_i})_{1 \leq i \leq d_h} \in \mathbb{R}^{d_h}$ such that $\mathbf{Q}(w_h) = \xi_h$ where $\mathbf{Q} : \mathbb{R}^{d_h} \to \mathbb{R}^{d_h}$, $\mathbf{Q}(w_h) = (Q_1, \dots, Q_{d_h})(w_h)$ is defined by $Q_i(w_h) = P_i(w_h) - \frac{m_{K_i}}{\Delta t} \xi_i^n$ for all $i = 1, \dots, d_h$. Thus, $u_h^{n+1} = \mathbf{Q}^{-1}(\xi_h^n)$ $\mathbb{P}$-a.s. in Ω, where $\xi_h^n = (\xi_1^n, \dots, \xi_{d_h}^n)$. Since $\mathbf{Q}^{-1}$ is continuous, if ξ_h^n is $\mathcal{F}_{t_{n+1}}$-measurable the same holds true for u_h^{n+1}. Indeed, let $(\zeta^k)_k \subset \mathbb{R}^{d_h}$ be a sequence such that $\zeta^k \to \zeta$ for some $\zeta \in \mathbb{R}^{d_h}$ for $k \to \infty$. Then, for $w^k := \mathbf{Q}^{-1}(\zeta^k)$ from (9) and from the theorem of Bolzano-Weierstrass it follows that there exists $w \in \mathbb{R}^{d_h}$ such that, passing to an unlabelled subsequence if necessary, $w^k \to w$ for $k \to \infty$. This strong convergence is enough to pass to the limit in $\mathbf{Q}(w^k)$, and therefore $\mathbf{Q}(w) = \zeta$. Thanks to uniqueness, we get convergence of the whole sequence $(w^k)_k$, hence $\lim_{k \to \infty} \mathbf{Q}^{-1}(\zeta^k) = \mathbf{Q}^{-1}(\zeta)$ and $\mathbf{Q}^{-1}$ is continuous. □

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