Asymptotic behavior of the two-dimensional Vlasov-Poisson-Fokker-Planck equation with a strong external magnetic field

Mihaï Bostan, Anh-Tuan Vu

Asymptotic behavior of the two-dimensional Vlasov-Poisson-Fokker-Planck equation with a strong external magnetic field

Mihaï BOSTAN [*], Anh-Tuan VU [†]

(March 24, 2023)

Abstract

The subject matter of the paper concerns the Vlasov-Poisson-Fokker-Planck (VPFP) equations in the context of magnetic confinement. We study the long-time behavior of the VPFP system with an intense external magnetic field, when neglecting the curvature of the magnetic lines. When the intensity of the magnetic field tends to infinity, the long-time behavior of the particle concentration is described by a first-order nonlinear hyperbolic equation of the Euler type for fluid mechanics. More exactly, when the magnetic field is uniform, we find the vorticity formulation of the incompressible Euler equations in two-dimensional space. Our proofs rely on the modulated energy method.

Keywords: Vlasov-Poisson-Fokker-Planck equations, Guiding center approximation, Modulated energy.

AMS classification: 35Q75, 78A35, 82D10

1 Introduction

?⟨Intro⟩? We consider $f = f(t, x, v)$ the density of a population of charged particles of mass m, charge q depending on time t, position x and velocity v. We are interested in the Vlasov-Poisson system, in the presence of an external magnetic field, taking into account the collisions between charged particles. Neglecting the curvature of the magnetic lines, we assume that the external magnetic field has a constant direction orthogonal to (Ox_1, Ox_2) but a variable amplitude $B(x)$. In dimension two, we set $x = (x_1, x_2)$, $v = (v_1, v_2)$. The Vlasov-Poisson-Fokker-Planck equation is written in the form

$$\partial_t f + v \cdot \nabla_x f + \frac{q}{m} \left\{ E\left[f(t)\right](x) + B(x)^{\perp} v \right\} \cdot \nabla_v f = Q_{\mathrm{FP}}(f), \ (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2 \tag{1}$$

VPFP2D-nonScal

[*] Aix Marseille Université, CNRS, Centrale Marseille, Institut de Mathématiques de Marseille, UMR 7373, Château Gombert 39 rue F. Joliot Curie, 13453 Marseille FRANCE. E-mail : mihai.bostan@univ-amu.fr

[†] Aix Marseille Université, CNRS, Centrale Marseille, Institut de Mathématiques de Marseille, UMR 7373, Château Gombert 39 rue F. Joliot Curie, 13453 Marseille FRANCE. E-mail : anh-tuan.vu@univ-amu.fr

where the notation ${}^{\perp}(\cdot)$ stands for the rotation of angle $-\pi/2$, *i.e.*, ${}^{\perp}v = (v_2, -v_1)$ and the magnetic field $\mathbf{B}(x) = (0, 0, B(x))$, $x \in \mathbb{R}^2$. The potential $\Phi[f]$ satisfies the Poisson equation

$$-\varepsilon_0 \Delta_x \Phi\left[f\left(t\right)\right](x) = q\left(\int_{\mathbb{R}^2} f\left(t,x,v\right)\ \mathrm{d}v - D(x)\right),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2.$$

whose fundamental solution is $z \to -\frac{1}{2\pi}\ln|z|, z \in \mathbb{R}^2 \backslash \{0\}$. Here, the function $D = D(x)$ is the concentration of a background of positive charges and is assumed to be given. The constant ϵ_0 represents the electric permittivity of the vacuum. For any particle density $f = f(t,x,v)$, the notation $E[f(t)](x)$ represents the Poisson electric field which derives from the potential $\Phi[f(t)](x)$ given by

$$E\left[f(t)\right](x) = \frac{q}{2\pi\varepsilon_0}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2} f\left(t,x',v'\right)\ \mathrm{d}v' - D(x')\right)\frac{x-x'}{|x-x'|^2}\ \mathrm{d}x' \tag{2}$$?ElecField-nonS

and $n[f(t)]$, $j[f(t)]$ stand for the concentration and the current density respectively

$$n\left[f(t)\right] = \int_{\mathbb{R}^2} f\left(t,\cdot,v\right)\ \mathrm{d}v,\quad j\left[f(t)\right] = \int_{\mathbb{R}^2} v f\left(t,\cdot,v\right)\ \mathrm{d}v.$$

In the equation (1), the operator Q_{FP} is the linear Fokker-Planck operator acting on velocities

$$Q_{\mathrm{FP}}\left(f\right) = \mathrm{div}_v\left(\sigma\nabla_v f + vf\right),$$

where σ is the velocity diffusion, see [9] for the introduction of this operator, based on the principle of Brownian motion. We complete the above system by the initial condition

$$f\left(0,x,v\right) = f_{\mathrm{in}}\left(x,v\right),\ (x,v) \in \mathbb{R}^2 \times \mathbb{R}^2. \tag{3}$$?Initial-nonSca

In this work, we analyze the evolution of the distribution density f over a long time, in the regime of an intense magnetic field (gyro-kinetic), in order to observe the drift phenomenon in the directions orthogonal to the magnetic field. Indeed, it is well known that the velocities of electric cross field drift and the magnetic gradient drift are proportional to $\frac{1}{B}$ and consequently it is necessary to observe the drift movements over a large time proportional to B. In other words, we consider

$$f\left(t,x,v\right) = f^{\varepsilon}\left(\bar{t},x,v\right),\ B^{\varepsilon}(x) = \frac{B(x)}{\varepsilon},\ \bar{t} = \varepsilon t.$$

Here $\varepsilon > 0$ is a small parameter related to the ratio between the cyclotronic period and the advection time scale. Hence $\partial_t f = \varepsilon\partial_{\bar{t}} f^{\varepsilon}$. Then in the equation (1), the term ∂_t is to be replaced by $\varepsilon\partial_{\bar{t}}$ or by $\varepsilon\partial_t$ to simplify our notation, and the Vlasov-Poisson-Fokker-Planck system becomes

$$\varepsilon\partial_t f^{\varepsilon} + v\cdot\nabla_x f^{\varepsilon} + \frac{q}{m}E\left[f^{\varepsilon}(t)\right]\cdot\nabla_v f^{\varepsilon} + \frac{\omega_c(x)}{\varepsilon}{}^{\perp}v\cdot\nabla_v f^{\varepsilon} = Q_{FP}(f^{\varepsilon}), \tag{4}$$ VPFP2d-Scale

$$E\left[f^{\varepsilon}\right] = -\nabla_x\Phi[f^{\varepsilon}],\ -\varepsilon_0\Delta_x\Phi[f^{\varepsilon}] = q\left(n^{\varepsilon} - D\right) = q\left(\int_{\mathbb{R}^2} f^{\varepsilon}\left(t,\cdot,v\right)\ \mathrm{d}v - D\right), \tag{5}$$ Poisson2D-Scal

where $\omega_c(x) = \frac{qB(x)}{m}$ stands for the cyclotron frequency. We complete with an initial condition

$$f^{\varepsilon}\left(0,x,v\right) = f^{\varepsilon}_{\mathrm{in}}\left(x,v\right),\ (x,v) \in \mathbb{R}^2 \times \mathbb{R}^2. \tag{6}$$ Initial2D-Scal

The existence theory of the weak and classical solution of the VPFP system is now well developed and understood. Let us summarize the literature concerning existence results for

this problem. In the absence of the external magnetic field *i.e.*, $B(x) = 0$, several existence results for the VPFP system are known. The classic solutions have been studied by Degond in [11] which showed the global/local existence and the uniqueness of the strong solution in one and two/three dimensions respectively, without friction term *i.e.*, $Q_{\mathrm{FP}} = \sigma\Delta_v$. Victory and O'Dwyer obtained in [10] the same result of existence of classical solution using the fundamental solution of the operator $\partial_t + v \cdot \nabla_x - \nabla_v \cdot (\sigma\nabla_v + v)$. In [32], G. Rein and J. Weckler gave sufficient conditions to show the global existence of classical solutions in three dimensions. Regarding weak solutions, we can mention the works of Victory in [14], J. A. Carrillo and J. Soler in [12] with an initial data in the space L^p. With the magnetized VPFP system, when the external magnetic field is uniform *i.e.*, $\nabla_x B(x) = 0, x \in \mathbb{R}^2$, it seems that the methods used in the articles above, also apply. We followed the method of [11] to show the existence and uniqueness of the global classical solution in time. We present the detailed proof in Section 6.3 of the Appendix B. In the case $B(x)$ is general, we show the global existence in time of weak solutions, in the sense of Definition 2.1. The detailed proof is provided in Section 2.

We study the asymptotic behavior of the solutions $(f^\varepsilon)_{\varepsilon>0}$ of the problem (4), (5), (6) when ε tends to 0. By investigating the balance of free energy associated with the VPFP system, we show formally in Section 4 that the family $(f^\varepsilon)_{\varepsilon>0}$ converges to the limit distribution function $f(t,x,v) = n(t,x)\frac{1}{2\pi\sigma}e^{\frac{-|v|^2}{2\sigma}}$, where the limit concentration n verifies the first-order nonlinear hyperbolic equation

$$\partial_t n + \mathrm{div}_x \left[n \left(\frac{{}^\perp E[n]}{B(x)} - \sigma \frac{{}^\perp \nabla \omega_c(x)}{\omega_c^2(x)} \right) \right] = 0, \ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2 \tag{7}$$ equ:LimMod2D

coupled to the Poisson equation

$$E[n] = -\nabla_x \Phi[n], \ -\varepsilon_0 \Delta_x \Phi[n] = q(n - D) \tag{8}$$ LimPoisson2D

with the initial condition

$$n(0,x) = n_{\mathrm{in}}(x) = \int_{\mathbb{R}^2} f(0,x,v)\, \mathrm{d}v. \tag{9}$$ LimInit2D

Let us observe the limit equation (7), we see that the concentration n is advected along the vector field $\left(\frac{{}^\perp E}{B(x)} - \sigma \frac{{}^\perp \nabla \omega_c}{\omega_c^2(x)} \right)$ which is the drift velocity respectively to the sum of the electric cross field drift $\frac{{}^\perp E}{B}$ and the magnetic gradient drift $\sigma \frac{{}^\perp \nabla \omega_c(x)}{\omega_c(x)^2}$. These drift velocities were mentioned in the limit model of M. Herda, L.M. Rodrigues [21] and P. Degond, F. Filbet [15]. In the case of the uniform magnetic field, the above model becomes

$$\begin{aligned}
&\partial_t n + \frac{{}^\perp E[n]}{B} \cdot \nabla_x n = 0, \ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2 \\
&E[n] = -\nabla_x \Phi[n], \ -\epsilon_0 \Delta_x \Phi[n] = q\,(n - D)\,, \ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2 \\
&n\,(0,x) = n_{\mathrm{in}}(x), \ x \in \mathbb{R}^2
\end{aligned}$$

that is to say, the vorticity formulation of the two-dimensional incompressible Euler equations, with the cross electric field drift velocity $\frac{{}^\perp E}{B}$ and the vorticity $\mathrm{rot}_x^\perp E = -\frac{q}{\varepsilon_0}(n - D)$. Notice that the same model was obtained by F. Golse, L. Saint-Raymond in [17], L. Saint-Raymond [30] and E. Miot [25] from the two-dimensional Vlasov-Poisson system without collisions. The authors justified rigorously the convergence towards the two-dimensional Euler equation of incompressible fluids in the other approach. Concerning the collisions between charged particles, we can mention the work of M. Herda and L.M. Rodrigues in [21]. In this paper,

the authors are interested in the limit $\varepsilon \searrow 0$ of the VPFP system (4), (5), (6) in three-dimensional version $(t,x,v) \in \mathbb{R}_+ \times \mathbb{T}^3 \times \mathbb{R}^3$ (where $\mathbb{T} = \mathbb{R}/Z$ is a torus one-dimensional). They formally show that the family $(f^\varepsilon)_{\varepsilon>0}$ converges to the limit distribution function f and the limit electric potential ϕ which have reached an adiabatic regime along the magnetic field

$$f(t,x,v) = n(t,x)\frac{1}{(2\pi)^{3/2}}e^{-\frac{|v|^2}{2}} = N(t,x_\perp)\frac{e^{-q\phi(t,x)}}{\int_{\mathbb{T}} e^{-q\phi(t,x_\perp,x_\parallel)}\mathrm{d}x_\parallel}M(v),\ (t,x,v) \in \mathbb{R}_+ \times \mathbb{T}^3 \times \mathbb{R}^3$$

where $x = (x_\perp, x_\parallel) \in \mathbb{T}^2 \times \mathbb{T}$ and the concentration n is the anisotropic Boltzmann-Gibbs density. The limit model is derived by the reduced macroscopic density $N : \mathbb{R}_+ \times \mathbb{T}^2 \to \mathbb{R}_+$ in the perpendicular direction, satisfying

$$\partial_t N - \mathrm{div}_{x_\perp}\left(N^\perp\left(\nabla_{x_\perp}\tilde{\phi}\right)\right) = 0$$

with the initial condition

$$N(0,x_\perp) = N_{\mathrm{in}}(x_\perp) = \int_{\mathbb{T}}\int_{\mathbb{R}^2} f_0(x_\perp, x_\parallel, v)\mathrm{d}x_\parallel\ \mathrm{d}v$$

where $\tilde{\phi} : \mathbb{R}_+ \times \mathbb{T}^2 \to \mathbb{R}$ is the average potential

$$\tilde{\phi}(t,x_\perp) = -q\ln\left(\int_{\mathbb{T}} e^{-q\phi(t,x_\perp,x_\parallel)}\mathrm{d}x_\parallel\right).$$

Their results of passing to the limit concerned a linear model where the electric field is given *i.e.*, $E[f^\varepsilon] = E = -\nabla_x\phi$, for a given potential ϕ. However, in the non-linear case of the VPFP type, they do not completely justify the passage to the limit model from the kinetic equation.

To the best of our knowledge, there has been no result on the asymptotic regime when the magnetic field is non-uniform. In the current work, the asymptotic behavior will be investigated by appealing to the relative entropy or modulated energy method, as introduced in [33]. This relative entropy method relies on the smooth solution of the limit system. By this technique, one gets strong convergences. Many asymptotic regimes were obtained using this technique, see [7, 8, 18, 29] for quasineutral regimes in collisionless plasma physics, [31, 1] for hydrodynamic limits in gaz dynamics, [19] for fluid-particle interaction, [5, 4, 20] for high electric or magnetic field limits in plasma physics.

Before establishing our main result, we define the modulated energy $\mathcal{E}[n^\varepsilon(t)|n(t)]$ by

$$\mathcal{E}[n^\varepsilon(t)|n(t)] = \sigma\int_{\mathbb{R}^2} n(t)h\left(\frac{n^\varepsilon(t)}{n(t)}\right)\ \mathrm{d}x + \frac{\varepsilon_0}{2m}\int_{\mathbb{R}^2}|\nabla_x\Phi[n^\varepsilon] - \nabla_x\Phi[n]|^2\ \mathrm{d}x$$

where $h : \mathbb{R}_+ \to \mathbb{R}_+$ is the convex function defined by $h(s) = s\ln s - s + 1, s \in \mathbb{R}_+$. This quantity splits into the standard L^2 norm of the electric field plus the relative entropy between the particle density n^ε of (4), (5), (6) and the particle concentration n of the limit model (7), (8), (9). The main result of this paper is the following

Theorem 1.1

⟨MainThm2D⟩ *Let $T > 0$. Let $B \in C_b^3(\mathbb{R}^2)$ be a smooth magnetic field, such that $\inf_{x\in\mathbb{R}^2} B(x) = B_0 > 0$ and D be a fixed background density verifying $|x|D \in L^1(\mathbb{R}^2)$, $D \in W^{1,1}(\mathbb{R}^2) \cap W^{2,\infty}(\mathbb{R}^2)$. Assume that the initial particle densities $(f^\varepsilon_{\mathrm{in}})_{\varepsilon>0}$ satisfy the hypotheses H1, H2, H3 (see Section 2 below) and $M_{\mathrm{in}} := \sup_{\varepsilon>0} M^\varepsilon_{\mathrm{in}} < +\infty$, $U_{\mathrm{in}} := \sup_{\varepsilon>0} U^\varepsilon_{\mathrm{in}} < +\infty$ where*

$$M^\varepsilon_{\mathrm{in}} := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f^\varepsilon(x,v)\ \mathrm{d}v\mathrm{d}x,\ U^\varepsilon_{\mathrm{in}} := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|v|^2}{2}f^\varepsilon_{\mathrm{in}}(x,v)\ \mathrm{d}v\mathrm{d}x + \frac{\varepsilon_0}{2m}\int_{\mathbb{R}^2}|\nabla_x\Phi[f^\varepsilon_{\mathrm{in}}]|^2\ \mathrm{d}x.$$

Let f^ε be the weak solutions of the VPFP system (4), (5), (6) *with initial data $f^\varepsilon_{\rm in}$ provided by Theorem 2.2. We also assume that the initial concentration $n_{\rm in}$ verifies the hypotheses $H4$, $H5$ (see Section 5 below) and let n be the unique smooth solution of the limit system* (7), (8), (9) *with initial condition $n_{\rm in}$ constructed in Proposition 5.1. We suppose that*

$$\lim_{\varepsilon\searrow 0}\sigma\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} n^\varepsilon_{\rm in}M(v)h\left(\frac{f^\varepsilon_{\rm in}}{n^\varepsilon_{\rm in}M}\right)\,{\rm d}v{\rm d}x=0,\ \lim_{\varepsilon\searrow 0}\mathcal{E}[n^\varepsilon_{\rm in}|n_{\rm in}]=0$$

where $n^\varepsilon_{\rm in}=\int_{\mathbb{R}^2}f^\varepsilon_{\rm in}\,{\rm d}v$, $\varepsilon>0$. Then we have

$$\lim_{\varepsilon\searrow 0}\sup_{0\le t\le T}\sigma\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} n^\varepsilon M(v)h\left(\frac{f^\varepsilon}{n^\varepsilon M}\right)\,{\rm d}v{\rm d}x=0,\ \lim_{\varepsilon\searrow 0}\sup_{0\le t\le T}\mathcal{E}[n^\varepsilon(t)|n(t)]=0$$

$$\lim_{\varepsilon\searrow 0}\frac{1}{\varepsilon}\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|\sigma\nabla_v f^\varepsilon+vf^\varepsilon|^2}{f^\varepsilon}\,{\rm d}v{\rm d}x{\rm d}t=0.$$

In particular we have the convergences $\lim_{\varepsilon\searrow 0}f^\varepsilon=nM$ in $L^\infty(]0,T[;L^1(\mathbb{R}^2\times\mathbb{R}^2))$ and $\lim_{\varepsilon\searrow 0}\nabla_x\Phi[f^\varepsilon]=\nabla_x\Phi[n]$ in $L^\infty(]0,T[;L^2(\mathbb{R}^2))$.

Remark 1.1

?⟨RemElecL22D⟩? *In two dimensional setting, the initial potential energy $\frac{\varepsilon_0}{2m}\int_{\mathbb{R}^2}|\nabla_x\Phi[f^\varepsilon_{\rm in}]|^2\,{\rm d}x$ may not be finite (or the electric field $E[f^\varepsilon_{\rm in}]$ cannot belong to $L^2(\mathbb{R}^2)$) even if the initial datum $f^\varepsilon_{\rm in}$ lies in $C^\infty_0(\mathbb{R}^2\times\mathbb{R}^2)$. This is due to the fact that the kernel $x/|x|^d$ does not belong to $L^2(\mathbb{R}^2)$ at infinity, see the Appendix C for a disscusion. For these reasons one needs to slightly modify the Poisson equation adding a fixed background density D satisfying the global neutrality relation $H3$, see Section 2 below.*

The paper is structured as follows. Section 2 is devoted to establish the global existence of weak solutions to the VPFP system with external magnetic field. In Section 3, we derive a priori estimates with respect to the small parameter $\varepsilon>0$ on the weak solutions from the evolution of physical quantities associated to the VPFP system. Section 4 is devoted to the formal derivation of the limit model. The well-posedness of the limit model is studied in the next section. We establish existence and uniqueness results for the strong solution. The convergence towards the limit model is justified rigorously in Section 5. We obtain strong convergence for well prepared initial conditions.

2 Global existence of weak solutions of the VPFP equations

In this section we will study the global existence of weak solution for the VPFP equation in the presence of an external magnetic field for fixed $\varepsilon>0$. In order to simplify the proofs of existence of the solution, as we do not want any uniform estimate with respect to ε, we will take $\varepsilon=1$ and omit all the subscripts. Thus we first consider the following problem

$$\partial_t f+v\cdot\nabla_x f+E[f]\cdot\nabla_v f+B(x)^\perp v\cdot\nabla_v f=\mathrm{div}_v(\sigma\nabla_v f+vf),\tag{10}$$ eq:VPFP-NonEps

$$E[f]=-\nabla_x\Phi\,[f]\,,\ -\Delta_x\Phi[f]=\int_{\mathbb{R}^2}f\,(t,\cdot,v)\ {\rm d}v-D\tag{11}$$ eq:Poi-NonEps

$$f(0,x,v)=f_{\rm in}\,(x,v)\,,\ (x,v)\in\mathbb{R}^2\times\mathbb{R}^2.\tag{12}$$ eq:Init-NonEps

The dependency on the small parameter $\varepsilon>0$ will be taken into account when establishing a priori estimates uniform in ε in the next section.
The idea of the proof is as follows: we will first linearize the VPFP system (10), (11), (12) by an iterative method, based on the resolution of the linear Vlasov-Fokker-Planck equation

with free transport thanks to the technique introduced by P. Degond in [11]. We adapt this method, also taking into account the velocity transport, generated by the external magnetic field. Then we will pass to the limit to obtain a weak solution by appealing to the velocity averaging lemma, cf. [22]. We will suppose the initial data $f_{\rm in}$ also satisfies the hypotheses

?⟨Hypothesis1⟩?
H1) $f_{\rm in} \geq 0,\ f_{\rm in} \in (L^1 \cap L^\infty)(\mathbb{R}^2 \times \mathbb{R}^2),\ (|x| + |v|^2 + |\ln f_{\rm in}|) f_{\rm in} \in L^1(\mathbb{R}^2 \times \mathbb{R}^2)$

?⟨Hypothesis2⟩?
H2) $(1 + |v|^2) f_{\rm in}^{\gamma/2} \in L^\infty(\mathbb{R}^2 \times \mathbb{R}^2),\ \gamma > 2$

H3) $\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f_{\rm in}\, {\rm d}v{\rm d}x = \int_{\mathbb{R}^2} D(x)\, {\rm d}x.$

2.1 The linear Vlasov-Fokker-Planck (VFP) equation

We consider the Vlasov-Fokker-Planck equation with a given electric field $E(x) = -\nabla_x \Phi(x)$

$$\left\{ \begin{array}{l} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f + {\rm div}_v(B(x)^\perp v f) = \sigma \Delta_v f + {\rm div}_v(v f), \\ f(0,x,v) = f_{\rm in}(x,v). \end{array} \right. \tag{13}$$ equ:VFP2D

We notice that the global existence and uniqueness of a weak solution in the distribution sense of the VFP equation (13) is demonstrated following by the standard theory for linear kinetic equations in [11]. We have the following result, see Appendix A for the main lines of the proof.

Theorem 2.1
⟨ExiVFP2D⟩ *For a given $T \in]0,\infty[$. Let $f_{\rm in}$ be an initial data verifying* H1, H2 *and $E(x)$ be an external electric field belongs to $(L^\infty(\mathbb{R}^2))^2$. Then there exists a unique positive weak solution of the equation* (13) *on the interval $[0,T]$ in the sense of Definition 2.1 provided by Proposition 6.2 such that $f \in L^\infty([0,T]; L^\infty \cap L^1(\mathbb{R}^2 \times \mathbb{R}^2))$. Furthermore, f belongs to $L^2([0,T] \times \mathbb{R}^2_x; H^1(\mathbb{R}^2_v))$ and verifies the following estimates*

$$\|f\|_{L^\infty([0,T];L^p(\mathbb{R}^2\times\mathbb{R}^2))} \leq {\rm e}^{\frac{p-1}{p} dT} \|f_{\rm in}\|_{L^p(\mathbb{R}^2\times\mathbb{R}^2)},\ p \in]1,\infty[\tag{14}$$?InegLp?

$$\|f\|_{L^\infty(0,T;L^1(\mathbb{R}^2\times\mathbb{R}^2))} = \|f_{\rm in}\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)},\ \|f\|_{L^\infty([0,T]\times\mathbb{R}^2\times\mathbb{R}^2)} \leq {\rm e}^{dT} \|f_{\rm in}\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}$$

$$\sup_{[0,T]} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(t,x,v) \frac{|v|^2}{2}\, {\rm d}v{\rm d}x < C(\|E\|_{L^\infty}, T, \sigma) \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f_{\rm in}(x,v) \frac{|v|^2}{2}\, {\rm d}v{\rm d}x. \tag{15}$$?InegKinEner?

$$\sup_{[0,T]} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(t,x,v)|x|\, {\rm d}v{\rm d}x < C(T) \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f_{\rm in}(x,v)|x|\, {\rm d}v{\rm d}x. \tag{16}$$?InegPosition?

$$\|\nabla_v f^{1/2}\|_{L^2([0,T];L^2(\mathbb{R}^2\times\mathbb{R}^2))} \leq C(\|E\|_{L^\infty}, T, f_{\rm in}, \sigma) + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \sigma f_{\rm in} |\ln f_{\rm in}|\, {\rm d}v{\rm d}x. \tag{17}$$?InegDissipati

We next provide an auxiliary lemma showing some relationship between the local density $n[f] = \int_{\mathbb{R}^2} f\, {\rm d}v$, the current $j[f] = \int_{\mathbb{R}^2} v f\, {\rm d}v$ and the kinetic energy $\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |v|^2 f\, {\rm d}v{\rm d}x$.

⟨IneqDensity2D⟩ **Lemma 2.1**
Assume that $f \in L^1 \cap L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ and $|v|^2 f \in L^1(\mathbb{R}^2 \times \mathbb{R}^2)$. Then there exists a constant $C > 0$ such that

$$\|n[f]\|_{L^p(\mathbb{R}^2)} \leq C,\ p \in [1,2],\ \|j[f]\|_{L^p(\mathbb{R}^2)} \leq C,\ p \in [1,4/3].$$

Proof.
We first estimate the L^p norm for the density $n[f]$. It is obvious when $p = 1$. Let $p \in]1, 2[$ and q such that $1/p + 1/q = 1$. Observer that

$$n[f](x) = \int_{\mathbb{R}^2} f(x,v)\,\mathrm{d}v = \int_{\mathbb{R}^d} (1+|v|)^{2/p} f^{1/p}(x,v) \frac{f(x,v)^{1/q}}{(1+|v|)^{2/p}}\,\mathrm{d}v$$

we deduce that

$$\begin{aligned} n[f](x) &\le \left(\int_{\mathbb{R}^d} (1+|v|)^2 f(v)\,\mathrm{d}v\right)^{1/p} \left(\int_{\mathbb{R}^d} \frac{f(v)}{(1+|v|)^{2q/p}}\,\mathrm{d}v\right)^{1/q} \\ &\le \|f\|^{1/q}_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \left(\int_{\mathbb{R}^2} \frac{1}{(1+|v|)^{2q/p}}\,\mathrm{d}v\right)^{1/q} \left(\int_{\mathbb{R}^d} (1+|v|)^2 f(v)\,\mathrm{d}v\right)^{1/p}. \end{aligned}$$

Since $p \in]1, 2[$ we have $\frac{2q}{p} > 2$ therefore $\int_{\mathbb{R}^2} \frac{1}{(1+|v|)^{2q/p}}\,\mathrm{d}v = C < +\infty$. Then we obtain

$$\|n[f]\|_{L^p(\mathbb{R}^2)} \le C\|f\|^{1/q}_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} (1+|v|)^2 f(v)\,\mathrm{d}v\mathrm{d}x\right)^{1/p}.$$

When $p = 2$, for any $R > 0$, we have

$$\begin{aligned} n[f](x) = \int_{\mathbb{R}^2} f(x,v)\,\mathrm{d}v &= \int_{\mathbb{R}^2} f(x,v) 1_{\{|v|\le R\}}\,\mathrm{d}v + \int_{\mathbb{R}^2} f(x,v) 1_{\{|v|>R\}}\,\mathrm{d}v \\ &\le \|f\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \pi R^2 + \frac{1}{R^2}\int_{\mathbb{R}^2} f(x,v)|v|^2\,\mathrm{d}v. \end{aligned}$$

We now take $R = \left(\frac{\int_{\mathbb{R}^2} f|v|^2\,\mathrm{d}v}{\|f\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}}\right)^{1/4}$ to obtain

$$n[f](x) = \int_{\mathbb{R}^2} f(x,v)\,\mathrm{d}v \le (1+\pi)\,\|f\|^{1/2}_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \left(\int_{\mathbb{R}^2} f|v|^2\,\mathrm{d}v\right)^{1/2},$$

then raising each side of the inequality to the power 2 and integrating in the variable x gives the result. By combining these estimates, we obtain the bound of the norm $L^p, p \in [1, 2]$ for the density particle $n[f]$. For the current $j[f]$, we use the same argument as above to obtain the desired estimate. □

The next result will be useful in order to estimate the L^∞ norm of the density particle $n[f]$, so as to control electric field. However, we cannot obtain L^∞ estimate for the such hypothesis in Lemma 2.1. The key is the decay of the solution f when the velocity goes to infinity.

Lemma 2.2
If $(1+|v|^2)^\gamma f \in L^\infty(\mathbb{R}^2\times\mathbb{R}^2)$, with $\gamma > 2$ we have the bound of the L^∞ norm
⟨LinftyDensity⟩

$$\|n[f]\|_{L^\infty(\mathbb{R}^2)} \le C(\gamma)\|f\|^{(\gamma-2)/\gamma}_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \|(1+|v|^2)^{\gamma/2} f\|^{2/\gamma}_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}.$$

In particular, if $|v|^2 f \in L^1(\mathbb{R}^2\times\mathbb{R}^2))$ then the current $j[f]$ belongs to $L^2(\mathbb{R}^2)$ and satisfies

$$\|j[f]\|_{L^2(\mathbb{R}^2)} \le \|n[f]\|_{L^\infty(\mathbb{R}^2)} \||v|^2 f\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)}.$$

Proof.
For any $R > 0$, we have

$$n[f](x) = \int_{\mathbb{R}^2} f(x,v)\,\mathrm{d}v = \int_{\mathbb{R}^2} f 1_{\{|v|<R\}}\,\mathrm{d}v + \int_{\mathbb{R}^d} f 1_{\{|v|\geq R\}}\,\mathrm{d}v \leq 2\pi R^2 \|f\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}$$
$$+ \int_{\mathbb{R}^2} 1_{\{|v|\geq R\}} \frac{1}{(1+|v|^2)^{\gamma/2}}\,\mathrm{d}v \|(1+|v|^2)^{\gamma/2} f\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}$$
$$\leq 2\pi R^2 \|f\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} + \frac{2\pi}{\gamma-2} R^{2-\gamma} |(1+|v|^2)^{\gamma/2} f\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}.$$

We take $R = \left(\frac{1}{\gamma-2} \frac{\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}}{\|f\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}} \right)^{1/\gamma}$, we complete the bound estimate of the L^∞ norm for $n[f]$. As a consequence, we can give the estimate in L^2 norm of $j[f]$. Indeed, we have

$$j[f](x) = \int_{\mathbb{R}^2} vf\,\mathrm{d}v \leq \left(\int_{\mathbb{R}^2} |v|^2 f\,\mathrm{d}v \right)^{1/2} \left(\int_{\mathbb{R}^d} f\,\mathrm{d}v \right)^{1/2}.$$

This implies that

$$\int_{\mathbb{R}^2} |j[f](x)|^2\,\mathrm{d}x \leq \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} |v|^2 f\,\mathrm{d}v \right) \left(\int_{\mathbb{R}^2} f\,\mathrm{d}v \right)\,\mathrm{d}x$$
$$\leq \|n[f]\|_{L^\infty(\mathbb{R}^2)} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |v|^2 f\,\mathrm{d}v\mathrm{d}x$$

which concludes the desired estimate of $j[f]$. □

We then show the bound estimate on the electric energy in the lemma below.

Lemma 2.3
Let $\rho(x) = \int_{\mathbb{R}^2} f(x,v)\,\mathrm{d}v$ be a function which belongs to $L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and let $E(x)$ be such that
⟨BoundElecEner2D⟩

$$E(x) = \frac{1}{2\pi} \nabla_x \ln|\cdot| \star \rho.$$

Then we have

$$\|E\|_{L^\infty(\mathbb{R}^2)} \leq \frac{1}{2\pi} \left(\|f\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)} \right)^{1/2} \left(\|f\|^{\frac{\gamma-2}{\gamma}}_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \|(1+|v|^2)^{\gamma/2} f\|^{2/\gamma}_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \right)^{1/2}.$$

Proof.
We first recall the classical inequality

$$\|E\|_{L^\infty(\mathbb{R}^2)} \leq C \|\rho\|^{1/2}_{L^1(\mathbb{R}^2)} \|\rho\|^{1/2}_{L^\infty(\mathbb{R}^2)}$$

for some constant $C > 0$. Together with the Lemma 2.2 yields the desired result.

□

2.2 The Vlasov-Poisson-Fokker-Planck equation

We first introduce the concept of weak solution to the problem (10), (11), (12).

Definition 2.1
For a given $T \in]0,\infty[$. We say that the pair $(f, E[f])$ is a weak solution to the system (10), (11), (12) *if and only if the following conditions are satisfied*
⟨DefWeakSol⟩

(i)

$$f \geq 0,\ f \in L^\infty([0,T[;L^1 \cap L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)),\quad |E[f]|f \in L^1_{\rm loc}([0,T[\times\mathbb{R}^2 \times \mathbb{R}^2).$$

(ii) For any $\varphi \in C_0^\infty([0,T[\times\mathbb{R}^2 \times \mathbb{R}^2)$*, we have*

$$\int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f\left(\partial_t\varphi + v\cdot\nabla_x\varphi + (E[f]+B(x)^\perp v)\cdot\nabla_v\varphi\right)\,\mathrm{d}v\mathrm{d}x\mathrm{d}t$$
$$+\int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f\left(\sigma\Delta_v\varphi - v\cdot\nabla_v\varphi\right)\,\mathrm{d}v\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f_{\rm in}(x,v)\varphi(0,x,v)\,\mathrm{d}v\mathrm{d}x = 0.$$

Now, we provide the global existence of the weak solution to the VPFP system (10), (11), (12) based on a compactness argument. For this purpose, we need the following velocity averaging lemma obtained in [22], see also [27]. The averaging lemma allows to pass to the limit in the VPFP equation including the nonlinear term $E[f]f$ in the sense of distribution, see [28].

Lemma 2.4

⟨VelAver⟩ *Let* $(g^k)_k$ *be bounded in* $L^p_{\rm loc}([0,T]\times\mathbb{R}^2\times\mathbb{R}^2)$ *with* $1<p<\infty$*, and* $(G^k)_k$ *be bounded in* $L^p_{\rm loc}([0,T]\times\mathbb{R}^2\times\mathbb{R}^2)$*. If for any* k*,* g^k *and* G^k *satisfy the equation*

$$\partial_t g^k + v\cdot\nabla_x g^k = \nabla_v G^k,\ \ g^k(t=0) = g_0 \in L^p(\mathbb{R}^2\times\mathbb{R}^2),$$

then for any $\psi \in C_c^1(\mathbb{R}^2\times\mathbb{R}^2)$ *we have* $\left(\int_{\mathbb{R}^d} f^k\psi\,\mathrm{d}v\right)_k$ *is relatively compact in* $L^p_{\rm loc}([0,T]\times\mathbb{R}^2)$*.*

We then use the previous lemma to show the following result, see Lemma 2.8 in [22]

Lemma 2.5

⟨Compactness⟩ *Let* $(g^k)_k$ *and* $(G^k)_k$ *be as in the Lemma 2.4 and we assume that*

$$g^k \text{ is bounded in } L^p([0,T]\times\mathbb{R}^2\times\mathbb{R}^2),$$
$$(|v|^2+|x|)g^k \text{ is bounded in } L^\infty(0,T;L^1(\mathbb{R}^2\times\mathbb{R}^2)).$$

Then for any $\psi(v)$ *such that* $|\psi(v)| \leq c|v|$ *and* $1<q<\frac{4}{3}$*, the sequence* $\left(\int_{\mathbb{R}^d} g^k\psi\,\mathrm{d}v\right)_k$ *is compact in* $L^q([0,T]\times\mathbb{R}^2)$*.*

We state the following result

Theorem 2.2

⟨main_weak_sol⟩ *Let* $T>0$*. Let* $B \in C_b^1(\mathbb{R}^2)$ *be a smooth magnetic field and* D *be a fixed background density verifying* $|x|D \in L^1(\mathbb{R}^2)$*,* $D \in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$*. Assume that the initial condition* $f_{\rm in}$ *satisfies the hypotheses* H1*,* H2 *and* H3*. Then there exists a weak solution to the problem* (10)*,* (11)*,* (12) *in the sense of Definition 2.1, satisfying*

$$f \geq 0,\ f \in L^\infty([0,T];L^1\cap L^\infty(\mathbb{R}^2\times\mathbb{R}^2)),\ \left(1+|v|^2\right)^{\gamma/2} f \in L^\infty([0,T]\times\mathbb{R}^2\times\mathbb{R}^2)$$
$$(|x|+|v|^2+|\ln f|)f \in L^\infty([0,T];L^1(\mathbb{R}^2\times\mathbb{R}^2))$$
$$E[f] \in L^\infty([0,T]\times\mathbb{R}^2),\ E[f] \in L^\infty([0,T];L^2(\mathbb{R}^2)). \qquad (18)$$ PropWeakSol

Furthermore, we have $f \in L^2([0,T]\times\mathbb{R}^2_x, H^1(\mathbb{R}^2_v))$*.*

The proof of Theorem 2.2 will be devided in 4 steps. The first is devoted to a construction of an iterative sequence, the second to a convergence of the sequence, the third passes to the limit and the last step studies the properties of the solution.

Step 1: Construction of an iterative sequence

We first construct a sequence $(f^k)_{k\geq 0}$ as follows: we start with $E^0(t,x)=0$ and $f^0(t,x,v)=f_{\rm in}(x,v)$. For a give electric field $E^k(t,x)$ belongs to $(L^\infty([0,T]\times\mathbb{R}^2))^2$, we consider f^{k+1} the unique weak solution of the following linear transport equation, cf. Theorem 2.1

$$\partial_t f^{k+1}+v\cdot\nabla_x f^{k+1}+E^k\cdot\nabla_v f^{k+1}+B(x)^\perp v\cdot\nabla_v f^{k+1}=\sigma\Delta_v f^{k+1}+\mathrm{div}_v(vf^{k+1}), \quad (19)$$
$$f^{k+1}(0,x,v)=f_{\rm in}(x,v),\ (x,v)\in\mathbb{R}^2\times\mathbb{R}^2.$$

Then the density n^{k+1} and the electric field E^{k+1} are defined by

$$n^{k+1}(t,x)=\int_{\mathbb{R}^2} f^{k+1}(t,x,v)\,\mathrm{d}v,\ \ E^{k+1}(t,x)=-\frac{q}{2\pi\epsilon_0}\nabla_x\ln|\cdot|\star\left(n^{k+1}-D\right).$$

Thanks to Theorem 2.1, we obtain the following estimations

$$f^{k+1}\geq 0,\ \sup_{[0,T]}\|f^{k+1}(t)\|_{L^p(\mathbb{R}^2\times\mathbb{R}^2)}\leq C(T)\|f_{\rm in}\|_{L^p(\mathbb{R}^2\times\mathbb{R}^2)},\ p\in[1,\infty] \quad (20)$$

$$\sup_{[0,T]}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f^{k+1}\frac{|v|^2}{2}\,\mathrm{d}v\mathrm{d}x<C(T,\|E^k\|_{L^\infty})\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f_{\rm in}|v|^2\,\mathrm{d}v\mathrm{d}x \quad (21)$$

$$\sup_{[0,T]}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f^{k+1}|x|\,\mathrm{d}v\mathrm{d}x<C(T)\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f_{\rm in}|x|\,\mathrm{d}v\mathrm{d}x \quad (22)$$

$$\|\sigma\nabla_v f^{k+1}/\sqrt{f^{k+1}}\|_{L^2([0,T];L^2(\mathbb{R}^2\times\mathbb{R}^2))}\leq C(\|E^k\|_{L^\infty},T,f_{\rm in},\sigma)+\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\sigma f_{\rm in}|\ln f_{\rm in}|\,\mathrm{d}v\mathrm{d}x. \quad (23)$$

We will now establish the uniform estimates with respect to k of the electric field E^k, that means $\sup_{[0,T]}\|E^k\|_{L^\infty(\mathbb{R}^2)}<C$ for some contant $C>0$, not depending on k, which imply clearly that the sequence $(f^k)_{k\in\mathbb{N}}$ is well-defined. Thanks to Lemma 2.2 and (20), it suffices to show that for all $k\in\mathbb{N}$, the following inequality

$$\|Y^k(t)\|_{L^\infty(\mathbb{R}^2)}<C$$

where we denote $Y^k(t)=(1+|v|^2)^{\gamma/2}f^k(t,x,v)$.

Lemma 2.6

⟨LInftyNormVeloc⟩ *Let $f_{\rm in}$ be an initial data verifying the hypothesis*

$$\|Y^0\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}=\|(1+|v|^2)^{\gamma/2}f_{\rm in}\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}<\infty,\ \gamma>2.$$

Then there exists a function $\alpha(t)$ independent of k such that $\alpha\in L^\infty_{\rm loc}([0,\infty[)$, satisfying for every $k\in\mathbb{N}$

$$\|Y^{k+1}(t)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}\leq\alpha(t).$$

Proof.

The proof relies on the maximum principle for the linear Vlasov-Fokker-Planck equation, which is stated in Remark 6.1 of Appendix A. We apply it to the equations solved by $Y^k(t)$. First, we multiply equation (19) by $(1+|v|^2)^{\gamma/2}$ we easily get

$$\begin{aligned}\partial_t Y^{k+1}+v\cdot\nabla_x Y^{k+1}+E^k\cdot\nabla_v Y^{k+1}+B(x)^\perp v\cdot\nabla_v Y^{k+1}-\sigma\Delta_v Y^{k+1}-\mathrm{div}_v(vY^{k+1}) \quad (24)\\ =\gamma(1+|v|^2)^{(\gamma-2)/2}(v\cdot E^k)f^{k+1}-2\sigma\gamma(1+|v|^2)^{(\gamma-2)/2}v\cdot\nabla_v f^{k+1}\\ -\sigma\gamma(\gamma-2)(1+|v|^2)^{(\gamma-4)/2}|v|^2f^{k+1}-2\sigma\gamma(1+|v|^2)^{(\gamma-2)/2}f^{k+1}\\ -\gamma(1+|v|^2)^{(\gamma-2)/2}|v|^2f^{k+1}.\end{aligned}$$

But we have

$$-2\sigma\gamma(1+|v|^2)^{(\gamma-2)/2}v\cdot\nabla_v f^{k+1} = -\frac{2\sigma\gamma}{1+|v|^2}(1+|v|^2)^{\gamma/2}v\cdot\nabla_v f^{k+1}$$
$$= -\frac{2\sigma\gamma}{1+|v|^2}v\cdot\nabla_v Y^{k+1} + 2\sigma\gamma^2\frac{|v|^2}{1+|v|^2}(1+|v|^2)^{(\gamma-2)/2}f^{k+1}$$

so that the equation (24) can be rewritten

$$\partial_t Y^{k+1} + v\cdot\nabla_x Y^{k+1} + \left(E^k + 2\sigma\gamma\frac{v}{1+|v|^2}\right)\cdot\nabla_v Y^{k+1} + B(x)^\perp v\cdot\nabla_v Y^{k+1}$$
$$-\sigma\Delta_v Y^{k+1} - \mathrm{div}_v(vY^{k+1}) = R_1 + R_2 \qquad (25)$$ equaYBis

with

$$R_1 = \gamma(1+|v|^2)^{(\gamma-2)/2}(v\cdot E^k)f^{k+1}$$
$$R_2 = \frac{\sigma\gamma^2|v|^2}{1+|v|^2}(1+|v|^2)^{(\gamma-2)/2}f^{k+1} - 2\sigma\gamma(1+|v|^2)^{(\gamma-2)/2}f^{k+1} - \gamma(1+|v|^2)^{(\gamma-2)/2}|v|^2 f^{k+1}.$$

Now, thanks to the hypotheses on the initial data, we apply the L^∞ estimate in the Remark 6.1 to (25), and therefore we obtain

$$\|Y^{k+1}\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \le e^{2T}\|Y^0\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} + \int_0^t \left(\|R_1(s)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} + \|R_2(s)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}\right)\mathrm{d}s. \qquad (26)$$ InegInftyY

But

$$\|R_2(s)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \le (\sigma\gamma^2 + (2\sigma+1)\gamma)\|Y^{k+1}(s)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}$$
$$= C_1(\sigma,\gamma)\|Y^{k+1}(s)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)},$$

and

$$\|R_1(s)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \le \gamma\|E^k(s)\|_{L^\infty(\mathbb{R}^2)}\|(1+|v|^2)^{(\gamma-1)/2}f^{k+1}\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}.$$

Then we use the Lemma 2.3 for the electric field E^k and by combining with the bound of L^p norm (20) we get

$$\|E^k(s)\|_{L^\infty(\mathbb{R}^2)} \le C(D,f_{\mathrm{in}})\left(1 + \|(1+|v|^2)^{\gamma/2}f^k(s)\|^{1/\gamma}_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}\right)$$
$$= C(D,f_{\mathrm{in}})\left(1 + \|Y^k(s)\|^{1/\gamma}_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}\right),$$

where $C(D,f_{\mathrm{in}})$ stands for the constant, depending only on D and f_{in}. On the other hand, thanks to the result of the elementary interpolation and (20) we have

$$\|(1+|v|^2)^{(\gamma-1)/2}f^{k+1}(s)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \le C(\gamma)\|f^{k+1}(s)\|^{1/\gamma}_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}\|Y^{k+1}(s)\|^{1-1/\gamma}_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}$$
$$\le C(\gamma,f_{\mathrm{in}})\|Y^{k+1}(s)\|^{1-1/\gamma}_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}.$$

Therefore the previous bound estimate of R_1 becomes

$$\|R_1(s)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \le C\left(\|Y^{k+1}(s)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} + \|Y^k(s)\|^{1/\gamma}_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}\|Y^{k+1}(s)\|^{1-1/\gamma}_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}\right)$$

for some positive constant C depending only on D, f_{in}, γ. Together the estimates of R_1 and R_2, the inequality (26) becomes

$$\|Y^{k+1}(t)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \leq e^{2T}\|Y^0\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} + C_1\int_0^t \|Y^{k+1}(s)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}\mathrm{d}s + C_2\int_0^t \|Y^k(s)\|^{1/\gamma}_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}\|Y^{k+1}(s)\|^{1-1/\gamma}_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}\mathrm{d}s. \quad (27)$$ InegInftyYBis

for some positive constants C_1, C_2, not depending on k.
Now, let $\alpha(t)$ be the solution of the linear equation which corresponds to the inequality (27)

$$\dot\alpha(t) = (C_1+C_2)\alpha(t), \quad \alpha(0) = e^{2T}\|Y^0\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}.$$

Then we prove by induction on k that we have

$$\|Y^k(t)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \leq \alpha(t),\ \forall t\in[0,T],\ \forall k\in\mathbb{N}. \quad (28)$$ BoundNormY

Indeed, denoting by Ψ^{k+1} the right hand side of (28) then we shall prove that an upper bound T for the set

$$T = \sup\left\{t\in\mathbb{R}_+ | \Psi^{k+1}(s)\leq\alpha(s),\ \forall s\in[0,t[\right\}$$

does not exist. If the converse were true, there exists $k_0\in\mathbb{N}$ such that $\Psi^{k_0+1}(t)\leq\alpha(t)$, for every $t\in[0,T[$ and $\Psi^{k_0+1}(T) > \alpha(T)$. Since for any $k\in\mathbb{N}$ we have

$$\begin{aligned}\dot\Psi^{k+1}(t) &= C_1\|Y^{k+1}(t)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} + C_2\|Y^k(t)\|^{1/\gamma}_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}\|Y^{k+1}(t)\|^{1-1/\gamma}_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}\\ &\leq C_1\alpha(t) + C_2\alpha(t) = \dot\alpha(t),\ t\in[0,T[\end{aligned}$$

thus this implies that $\Psi^{k+1}(t)\leq\alpha(t),\ \forall t\in[0,T[$ for any $k\in\mathbb{N}$. As $\Psi^{k+1}(t)$ is an increasing function and $\alpha(t)$ is continuous hence one gets

$$\Psi^{k+1}(T) = \limsup_{t\nearrow T}\Psi^{k+1}(t) \leq \limsup_{t\nearrow T}\alpha(t) = \alpha(T)$$

which shows a contradiction. Therefore, (28) is the desired estimate for Y^{k+1}. □

By Lemma 2.6, we deduce that the constants in the inequalities (21), (23) respectively are independent with respect to k. Together with the Lemma 2.2 and (20) yields the uniform bound of the sequence $(n^k)_{k\in\mathbb{N}}$ in $L^\infty([0,T];L^p(\mathbb{R}^2))$, for any $p\in[1,\infty]$.

Step 2: Compactness and convergence

It follows from the uniform bound of the sequences that there exist a limit (f,n,E) such that up to extraction of a subsequence, it holds as $k\to\infty$ that

$$\begin{aligned} f^{k+1} &\rightharpoonup f \ \text{ weak } \star \text{ in } L^\infty([0,T];L^p(\mathbb{R}^2\times\mathbb{R}^2)),\ p\in]1,\infty],\\ n^{k+1} &\rightharpoonup n \ \text{ weak } \star \text{ in } L^\infty([0,T];L^p(\mathbb{R}^2)),\ p\in]1,\infty],\\ E^{k+1} &\rightharpoonup E \ \text{ weak } \star \text{ dans } L^\infty([0,T]\times\mathbb{R}^2).\end{aligned}$$

Furthermore, by using the Lemma 2.5 with $\psi(v)=1$ we get the strong convergence

$$n^{k+1}\to n \text{ in } L^q([0,T]\times\mathbb{R}^2),\ \ q\in]1,4/3[. \quad (29)$$?strongconv?

Indeed, by uniform estimates (20), (21) and (22) the conditions in the Lemma 2.5 are verified. Let us write

$$G^{k+1} := \sigma\nabla_v f^{k+1} + vf^{k+1} - E^k f^{k+1} - B(x)^\perp v f^{k+1}.$$

Then the equation (19) can be written as

$$\partial_t f^{k+1} + v \cdot \nabla_x f^{k+1} = \nabla_v G^{k+1}.$$

We now claim that the sequence $(G^k)_{k\in\mathbb{N}}$ is bounded in $L^q([0,T]\times\mathbb{R}^2\times\mathbb{R}^2)$ to apply the averaging lemma, Lemma 2.5. Hence, we need to prove the following lemma.

Lemma 2.7

⟨BoundGk⟩ *For any $q\in[1,2]$, there exists a constant C independent of k such that for every $n\in\mathbb{N}$ we have*

$$\|G^{k+1}\|_{L^q([0,T]\times\mathbb{R}^2\times\mathbb{R}^2)} \le C, \;\; q\in[1,2].$$

Proof.

As the sequence of electric fields E^k is bounded in $L^\infty([0,T]\times\mathbb{R}^2)$ and the magnetic field B belongs to $L^\infty(\mathbb{R}^2)$ we obtain

$$\|G^{k+1}\|_{L^q} \le \|\sigma\nabla_v f^{k+1}\|_{L^q} + (1+\|B\|_{L^\infty})\|vf^{k+1}\|_{L^q([0,T]\times\mathbb{R}^2\times\mathbb{R}^2)} + C\|f^{k+1}\|_{L^q}$$

for some positive constant C not depending on k.

From (20) it is easily seen that $\|f^{k+1}\|_{L^q([0,T]\times\mathbb{R}^2\times\mathbb{R}^2)} \le T^{1/q}\|f_{\text{in}}\|_{L^q(\mathbb{R}^2\times\mathbb{R}^2)}$. On the other hand, since $\|vf^{k+1}\|_{L^q([0,T]\times\mathbb{R}^2\times\mathbb{R}^2)} \le T^{1/q}\sup_{[0,T]}\|vf^{k+1}\|_{L^q(\mathbb{R}^2\times\mathbb{R}^2)}$ and thanks to Hölder's inquality for $q\in[1,2[$ we have

$$\begin{aligned}
\|vf^{k+1}\|_{L^q(\mathbb{R}^2\times\mathbb{R}^2)} &= \left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f|v|^q \,\mathrm{d}v\mathrm{d}x\right)^{1/q} = \left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |v|^q f^{q/2} f^{q/2} \,\mathrm{d}v\mathrm{d}x\right)^{1/q} \\
&\le \left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |v|^2 f \,\mathrm{d}v\mathrm{d}x\right)^{1/2} \left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f^{q/(2-q)} \,\mathrm{d}v\mathrm{d}x\right)^{(2-q)/2} \\
&\le \left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |v|^2 f \,\mathrm{d}v\mathrm{d}x\right)^{1/2} \|f\|^{q/2}_{L^{q/(2-q)}(\mathbb{R}^2\times\mathbb{R}^2)}.
\end{aligned}$$

When $q=2$ we also get

$$\|vf^{k+1}\|_{L^2(\mathbb{R}^2\times\mathbb{R}^2)} \le \|f\|^{1/2}_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |v|^2 f \,\mathrm{d}v\mathrm{d}x\right)^{1/2}.$$

Consequently, the sequence $(vf^k)_k$ is bounded in $L^q([0,T]\times\mathbb{R}^2\times\mathbb{R}^2)$, for any $q\in[1,2]$. It remains to uniformly bound the sequence $\|\sigma\nabla_v f^{k+1}\|_{L^q([0,T]\times\mathbb{R}^2\times\mathbb{R}^2)}$. Using Hölder's inequality again for $q\in[1,2[$, we have

$$\begin{aligned}
\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\nabla_v f^{k+1}|^q \,\mathrm{d}v\mathrm{d}x\mathrm{d}t &= \int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left(f^{k+1}\right)^{\frac{q}{2}} \frac{|\nabla_v f^{k+1}|^q}{(f^{k+1})^{\frac{q}{2}}} \,\mathrm{d}v\mathrm{d}x\mathrm{d}t \\
&\le \int_0^T \|f^{k+1}\|^{\frac{q}{2}}_{L^{\frac{q}{2-q}}} \left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|\nabla_v f^{k+1}|^2}{f^{k+1}} \,\mathrm{d}v\mathrm{d}x\right)^{\frac{q}{2}} \mathrm{d}t \\
&\le C(T)\|f^{k+1}\|^{\frac{q}{2}}_{L^\infty[(0,T];L^{\frac{q}{2-q}})} \left(\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|\nabla_v f^{k+1}|^2}{f^{k+1}} \,\mathrm{d}v\mathrm{d}x\mathrm{d}t\right)^{\frac{q}{2}}
\end{aligned}$$

and when $p=2$ we also get

$$\begin{aligned}
\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\nabla_v f^{k+1}|^2 \,\mathrm{d}v\mathrm{d}x\mathrm{d}t &= \int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f^{k+1} \frac{|\nabla_v f^{k+1}|^2}{f^{k+1}} \,\mathrm{d}v\mathrm{d}x\mathrm{d}t \\
&\le \|f^{k+1}\|_{L^\infty([0,T]\times\mathbb{R}^2\times\mathbb{R}^2)} \int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|\nabla_v f^{k+1}|^2}{f^{k+1}} \,\mathrm{d}v\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Thanks to Lemma 2.6 and (23), we deduce that the sequence $(|\nabla_v f^k|^2/f^k)_{k\in\mathbb{N}}$ is bounded in $L^1([0,T]\times\mathbb{R}^2\times\mathbb{R}^2)$. Therefore, the sequence $(\nabla_v f^k)_{k\in\mathbb{N}}$ is bounded in $(L^q([0,T]\times\mathbb{R}^2\times\mathbb{R}^2))^2$ with $q\in[1,2]$. Altogether the above estimates we conclude the result of Lemma 2.7. □

Step 3: Passing to the limit
Thanks to the weak convergences obtained in Step 2, we see that to pass to the limit in the weak formulation of equation (19) it suffices to show convergence towards 0 for any test function $\varphi \in C_c^\infty([0,T)\times\mathbb{R}^2\times\mathbb{R}^2)$ of the non-linear contribution

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\Big[(\nabla_x \ln(|\cdot|)\star n^k)f^{k+1} - (\nabla_x \ln(|\cdot|)\star n)f\Big]\,\varphi\,\mathrm{d}v\mathrm{d}x\mathrm{d}t\\
&= \int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\Big[(\nabla_x \ln(|\cdot|)\star (n^k-n))f^{k+1}\Big]\,\varphi\,\mathrm{d}v\mathrm{d}x\mathrm{d}t\\
&+ \int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(\nabla_x \ln(|\cdot|)\star n)\varphi(f^{k+1}-f)\,\mathrm{d}v\mathrm{d}x\mathrm{d}t. \qquad (30)
\end{aligned}$$

NonLinearEner

For the first term in (30) we write

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\Big[(\nabla_x \ln(|\cdot|)\star (n^k-n))f^{k+1}\Big]\,\varphi\,\mathrm{d}v\mathrm{d}x\mathrm{d}t\\
&= \int_0^T\int_{\mathbb{R}^2}\big[\nabla_x \ln(|\cdot|)1_{\{|\cdot|\le 1\}}\big]\star (n^k-n))\left(\int_{\mathbb{R}^d} f^{k+1}\varphi\,\mathrm{d}v\right)\,\mathrm{d}x\mathrm{d}t\\
&+ \int_0^T\int_{\mathbb{R}^2}\big[\nabla_x \ln(|\cdot|)1_{\{|\cdot|>1\}}\big]\ast (n^k-n))\left(\int_{\mathbb{R}^d} f^{k+1}\varphi\,\mathrm{d}v\right)\,\mathrm{d}x\mathrm{d}t\\
&=: I_1+I_2.
\end{aligned}$$

Estimating now I_1. Notice that $\nabla_x \ln(|x|)1_{\{|\cdot|\le 1\}}\in L^1(\mathbb{R}^2)$ we have

$$I_1 \le \int_0^T \|\nabla_x \ln(|x|)1_{\{|\cdot|\le 1\}}\|_{L^1(\mathbb{R}^2)}\|n^k-n\|_{L^q(\mathbb{R}^2)}\left\|\int_{\mathbb{R}^d} f^{k+1}\varphi\,\mathrm{d}v\right\|_{L^{q'}(\mathbb{R}^2)}$$

where $q\in]1,4/3[$ and q' is the Hölder conjugate of q. Here we have used Young's inequality for the convolutions

$$\int_{\mathbb{R}^2}(f\ast g)h\,\mathrm{d}x \le \|f\|_{L^p}\|g\|_{L^q}\|h\|_{L^r},\ \ \frac{1}{p}+\frac{1}{q}+\frac{1}{r}=2.$$

Then by the Hölder inequality in variable t we have

$$I_1 \le \|\nabla_x \ln(|x|)1_{\{|\cdot|\le 1\}}\|_{L^1(\mathbb{R}^2)}\|n^k-n\|_{L^q([0,T]\times\mathbb{R}^2)}\left\|\int_{\mathbb{R}^2} f^{k+1}\varphi\,\mathrm{d}v\right\|_{L^{q'}([0,T]\times\mathbb{R}^2)}.$$

Notice that the sequence $(f^k)_{k\in\mathbb{N}}$ is bounded in $L^\infty([0,T]\times\mathbb{R}^2\times\mathbb{R}^2)$ and $\varphi\in C_0^\infty([0,T)\times\mathbb{R}^2\times\mathbb{R}^2)$ imply $(f^{k+1}\varphi)_{k\in\mathbb{N}}$ is bounded in $L^{q'}([0,T]\times\mathbb{R}^2)$. Since n^k converges strongly to n in $L^q([0,T]\times\mathbb{R}^2)$ therefore we get $I_1\to 0$ as $k\to\infty$.
Estimating now I_2. Notice that $\nabla_x \ln(|x|)1_{\{|\cdot|\ge 1\}}\in L^p(\mathbb{R}^2)$ for any $p\in]2,\infty[$. Using Young's inequality again for the convolutions then we have

$$I_2 \le \int_0^T \|\nabla_x \ln(|x|)1_{\{|\cdot|\ge 1\}}\|_{L^{p'}(\mathbb{R}^2)}\|n^k-n\|_{L^p(\mathbb{R}^2)}\left\|\int_{\mathbb{R}^d} f^{k+1}\varphi\,\mathrm{d}v\right\|_{L^1(\mathbb{R}^2)},$$

where $p\in]1,4/3[$ and p' is the Hölder conjugate of p ($p'>4$). Performing in the same way as I_1, we also have $I_2\to 0$ as $k\to+\infty$. Combining the convergences of I_1 and I_2 we deduce that the first term in (30) converges to 0, as $k\to+\infty$.

For the second term in (30) we write

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (\nabla_x \ln(|\cdot|) \star n)\varphi(f^{k+1} - f)\, \mathrm{d}v\mathrm{d}x\mathrm{d}t \\
&= \int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left[\nabla_x \ln(|\cdot|)1_{\{|\cdot|<1\}}\right] \star n\, \varphi(f^{k+1} - f)\, \mathrm{d}v\mathrm{d}x\mathrm{d}t \\
&+ \int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left[\nabla_x \ln(|\cdot|)1_{\{|\cdot|\geq 1\}}\right] \star n\, \varphi(f^{k+1} - f)\, \mathrm{d}v\mathrm{d}x\mathrm{d}t \\
&=: K_1 + K_2.
\end{aligned}$$

Estimating now K_1. Notice that $\nabla_x \ln(|\cdot|)1_{\{|\cdot|<1\}} \star n\varphi$ belongs to $(L^1([0,T]; L^p(\mathbb{R}^2 \times \mathbb{R}^2)))^2$ avec $p \in]1,2]$. Indeed, since

$$\begin{aligned}
&\int_0^T \left(\int_{\mathbb{R}^2} \left(\left[\nabla_x \ln(|\cdot|)1_{\{|\cdot|<1\}}\right] \star n\right)^p \left(\int_{\mathbb{R}^2} |\varphi|^p \, \mathrm{d}v \right) \mathrm{d}x \right)^{\frac{1}{p}} \mathrm{d}t \\
&\leq \int_0^T \left\| \left(\int_{\mathbb{R}^2} |\varphi|^p \, \mathrm{d}v \right)^{1/p} \right\|_{L^\infty(\mathbb{R}^2)} \mathrm{d}t \sup_{[0,T]} \left(\int_{\mathbb{R}^2} \left(\left[\nabla_x \ln(|\cdot|)1_{\{|\cdot|<1\}}\right] \star n\right)^p \, \mathrm{d}x \right)^{1/p}
\end{aligned}$$

and

$$\sup_{[0,T]} \left(\int_{\mathbb{R}^2} \left(\left[\nabla_x \ln(|\cdot|)1_{\{|\cdot|<1\}}\right] \star n\right)^p \, \mathrm{d}x \right)^{1/p} \leq \|\nabla_x \ln(|\cdot|)1_{\{|\cdot|<1\}}\|_{L^1(\mathbb{R}^2)} \|n\|_{L^\infty(0,T;L^p(\mathbb{R}^2))}$$

where we have used the convolution inequality

$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}, \ \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$$

we deduce that $\nabla_x \ln(|\cdot|)1_{\{|\cdot|<1\}} \star n\varphi \in L^1([0,T]; L^p(\mathbb{R}^2 \times \mathbb{R}^2))$ with $p \in]1,2]$. As the sequence $(f^k)_{k\in\mathbb{N}}$ converges weakly-$\star$ to f in $L^\infty([0,T]; L^q(\mathbb{R}^2 \times \mathbb{R}^2))$ with $q \in]1,\infty]$ thus $K_1 \to 0$ when $k \to +\infty$.
Estimating now K_2. Since $\nabla_x \ln(|\cdot|)1_{\{|\cdot|\geq 1\}} \in L^p(\mathbb{R}^2)$ with $p > 2$, and $n \in L^\infty([0,T]; L^q(\mathbb{R}^2))$ with $q \in]1,\infty]$ and by using the convolution inequality we get $\nabla_x \ln(|\cdot|)1_{\{|\cdot|\geq 1\}} \star n \in L^\infty([0,T]; L^r(\mathbb{R}^2))$ with $r > 2$. This implies that $\nabla_x \ln(|\cdot|)1_{\{|\cdot|\geq 1\}} \star n\varphi$ lies in $L^1([0,T]; L^p(\mathbb{R}^2 \times \mathbb{R}^2))$ with $p \in]2,\infty[$. Thus we also have $K_2 \to 0$ as $k \to +\infty$.

Finally, the contribution (30) converges to 0 as $k \to +\infty$. Therefore we obtain f is the weak solution of VPFP system (10), (11), (12) with the electric field E satisfying $E = -\frac{q}{2\pi\epsilon_0} \nabla_x \ln|\cdot| \star (n - D)$. Furthermore, since the sequence $(f^k)_{k\in\mathbb{N}}$ belongs to $L^2([0,T] \times \mathbb{R}^2_x, H^1(\mathbb{R}^2_v))$ it is easily check that $f \in L^2([0,T] \times \mathbb{R}^2_x, H^1(\mathbb{R}^2_v))$ by using the Theorem 6.1.

Step 4: Properties (18) **of solutions**

The nonegative limit function f is a direct consequence of the weak-$\star$ convergence of the nonegative sequence $(f^k)_{k\in\mathbb{N}}$ in $L^\infty([0,T] \times \mathbb{R}^2 \times \mathbb{R}^2)$. In particular, $f \in L^\infty([0,T] \times \mathbb{R}^2 \times \mathbb{R}^2)$. Moreover, we also have $(1 + |v|^2)^{\gamma/2} f \in L^\infty([0,T] \times \mathbb{R}^2 \times \mathbb{R}^2)$ since the sequence $((1 + |v|^2)^{\gamma/2} f^k)_{k\in\mathbb{N}}$ is bounded in $L^\infty([0,T] \times \mathbb{R}^2 \times \mathbb{R}^2)$. Now, let φ be any nonnegative function in $C_0^\infty([0,T[)$ and $R > 0$ be a constant. To prove $f \in L^\infty([0,T]; L^1(\mathbb{R}^2 \times \mathbb{R}^2))$ we use the function $\psi_R(t,x,v) = \varphi(t)1_{\{|x|\leq R, |v|\leq R\}}$. Hence by the weak-$\star$ convergence of $(f^k)_{k\in\mathbb{N}}$ to f we deduce that

$$\int_0^T \varphi(t) \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(t,x,v) 1_{\{|x|\leq R, |v|\leq R\}} \, \mathrm{d}v\mathrm{d}x\mathrm{d}t \leq \limsup_{k\to\infty} \int_0^T \varphi(t) \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(t,x,v) \, \mathrm{d}v\mathrm{d}x\mathrm{d}t.$$

Taking now the limit $R \to \infty$ and apply the dominated convergence theorem to get $f \in L^\infty([0,T];L^1(\mathbb{R}^2\times\mathbb{R}^2))$. Similarly, if we choose the test function $\psi_R(t,x,v) = \varphi(t)(|x| + |v|^2)1_{\{|x|\le R,|v|\le R\}}$ then we can show that $(|x|+|v|^2)f \in L^\infty([0,T];L^1(\mathbb{R}^2\times\mathbb{R}^2))$. We complete the property of the solution by showing that $f\ln f \in L^\infty([0,T];L^1(\mathbb{R}^2\times\mathbb{R}^2))$. Indeed, we have the identity

$$f|\ln f| = f\ln f\chi_{\{f\ge 1\}} - f\ln f\chi_{\{0\le f\le 1\}}.$$

Since $f\ln f\chi_{\{f\ge1\}} \le f^2$ and $f\ln f\chi_{\{0\le f\le1\}} \le Ce^{-(|x|+|v|^2)} + (|x|+|v|^2)f$, for some constant $C>0$ together with $f \in L^\infty(0,T;L^2(\mathbb{R}^2\times\mathbb{R}^2))$ and $(|x|+|v|^2)f \in L^\infty(0,T;L^1(\mathbb{R}^2\times\mathbb{R}^2))$, we deduce that $f\ln f \in L^\infty(0,T;L^1(\mathbb{R}^2\times\mathbb{R}^2))$.

The following lemma provides the property on the potential $\Phi[f]$ and the electric field $E[f] = -\nabla_x\Phi[f]$ of the Poisson equation on $\mathbb{R}^2$, so as to control the potential energy. We refer to Lemma 3 in [20].

Lemma 2.8

⟨PropSolPoi⟩ *Let $\rho \in L^p(\mathbb{R}^2)$ with any $p\in[1,\infty]$ be such that*

$$\int_{\mathbb{R}^2}(1+|x|)|\rho(x)|\,\mathrm{d}x < +\infty,\ \int_{\mathbb{R}^2}\rho(x)\,\mathrm{d}x = 0.$$

Consider the potential Φ given by $\Phi(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2}\ln|x-y|\rho(y)\mathrm{d}y$. Then, Φ is a continuous and bounded function such that $\lim_{|x|\to\infty}\Phi(x) = 0$. Furthermore, we also have $\Phi \in L^2(\mathbb{R}^2)$ and $\nabla\phi \in (L^2(\mathbb{R}^2))^2$.

Proof.

Since $-\frac{1}{2\pi}\ln|x|$ is the fundamental solution of $-\Delta_x$ on $\mathbb{R}^2$, we have $|\xi|^2\hat\Phi(\xi) = \hat\rho(\xi)$ by using the Fourier transform. Then the integral of ρ vanishes so $\hat\rho(0) = 0$ which implies that

$$|\hat\Phi(\xi)| = \left|\frac{\hat\rho(\xi)}{|\xi|^2}\right| \le \left|\frac{\hat\rho(\xi)-\hat\rho(0)}{|\xi|^2}\right|1_{\{|\xi|\le1\}} + \left|\frac{\hat\rho(\xi)}{|\xi|^2}\right|1_{\{|\xi|>1\}}.$$

On the other hand, $|\hat\rho(\xi)-\hat\rho(0)| \le |\xi|\|\nabla\hat\rho\|_{L^\infty(\mathbb{R}^2)} \le |\xi|\|\widehat{\nabla\rho}\|_{L^1(\mathbb{R}^2)} = |\xi|\|x\rho\|_{L^1(\mathbb{R}^2)}$. Hence

$$|\hat\Phi(\xi)| \le \|x\rho\|_{L^1(\mathbb{R}^2)}\frac{1}{|\xi|}1_{\{|\xi|\le1\}} + \left|\frac{\hat\rho(\xi)}{|\xi|^2}\right|1_{\{|\xi|>1\}}. \tag{31}$$ EstmFourier

Since $1/|\xi| \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ and $\hat\rho \in L^2(\mathbb{R}^2)$, it is easily obtain from (31) that $\hat\Phi \in L^1(\mathbb{R}^2)$. It follows that $x\mapsto\Phi(x)$ is a continuous and bounded function which tends to 0 at infinity. Hence, $\Phi\in L^\infty(\mathbb{R}^2)$. Furthermore, we can show that $\Phi\in L^q(\mathbb{R}^2)$ with any $q\in]1,2[$ such that q is the Hölder conjugate of p. Indeed, first we observer that $1/|\xi| \in L^q_{\mathrm{loc}}(\mathbb{R}^2)$ with $1<q<2$. Then since $\rho\in L^p(\mathbb{R}^2)$, $1<p<2$ we deduce that $\hat\rho\in L^q(\mathbb{R}^2)$ with $1/p+1/q=1$. Applying the inequality $(a+b)^r \le 2^r(a^r+b^r)$, for $a,b,r\ge0$ we conclude that $\Phi \in Ł^q(\mathbb{R}^2), q\in]1,2[$. Together with $\Phi\in L^\infty(\mathbb{R}^2)$, we get $\Phi\in L^2(\mathbb{R}^2)$. Similar to the derivative of Φ, we have

$$|\widehat{\nabla\Phi}(\xi)| = \left|\frac{\xi}{|\xi|^2}\hat\rho(\xi)\right| \le \|x\rho\|_{L^1(\mathbb{R}^2)}1_{\{|\xi|\le1\}} + \left|\frac{\hat\rho(\xi)}{|\xi|^2}\right|1_{\{|\xi|>1\}}.$$

It is easily deduce that $\nabla\Phi\in(L^2(\mathbb{R}^2))^2$ by $\rho\in L^2(\mathbb{R}^2)$ and the Plancherel theorem. Therefore, the potential energy is finite and we have the the identity

$$\int_{\mathbb{R}^2}\rho\Phi\,\mathrm{d}x = \int_{\mathbb{R}^2}|\nabla\Phi|^2\,\mathrm{d}x.$$

via standard approximation and truncation arguments. □

3 A priori estimates

The aim of this section is the derivation of a priori estimates, uniform with respect to ε, on the weak solution f^ε provided by Theorem 2.2. These estimates are deduced from the conservation properties of the system and from the dissipation mechanism due to the collisions. We recall that $(f^\varepsilon, E[f^\varepsilon])$ is a weak solution to the problem (4), (5), (6) on $[0,T]$ with any $T>0$, if for any the test function $\varphi \in C_0^\infty([0,T[\times\mathbb{R}^2\times\mathbb{R}^2)$ we have

$$\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f^\varepsilon\left(\varepsilon\partial_t\varphi + v\cdot\nabla_x\varphi + \frac{q}{m}(E[f^\varepsilon] + \frac{B(x)}{\varepsilon}{}^\perp v)\cdot\nabla_v\varphi\right)\,\mathrm{d}v\mathrm{d}x\mathrm{d}t \qquad (32)$$
$$+\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f^\varepsilon\left(\sigma\Delta_v\varphi - v\cdot\nabla_v\varphi\right)\,\mathrm{d}v\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\varepsilon f^\varepsilon_{\rm in}(x,v)\varphi(0,x,v)\,\mathrm{d}v\mathrm{d}x = 0.$$

Let us define the free energy of the VPFP system (4), (5), (6) as

$$\mathcal{E}[f^\varepsilon] = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(\sigma f^\varepsilon\ln f^\varepsilon + f^\varepsilon\frac{|v|^2}{2})\,\mathrm{d}v\mathrm{d}x + \frac{\varepsilon_0}{2m}\int_{\mathbb{R}^2}|E[f^\varepsilon]|^2\,\mathrm{d}x.$$

Proposition 3.1

⟨WeakFreeEnergy2D⟩ *Let $(f^\varepsilon, E[f^\varepsilon])$ be a weak solution of the system (4), (5), (6) provided by Theorem 2.2. Then, we have the mass conservation and the balance of the free energy*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} f^\varepsilon(t)\,\mathrm{d}x = 0,\ \ \varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}[f^\varepsilon(t)] = -\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|\sigma\nabla_v f^\varepsilon + v f^\varepsilon|^2}{f^\varepsilon}\,\mathrm{d}v\mathrm{d}x.$$

The mass conservation follows formally by integrating (4) in v, which gives the continuity equation for the mass density, and then integrating in x. On the other hand, the law for the balance of the total energy is derived formally by summing up these relations below. First, multiplying the equation (4) by $\frac{|v|^2}{2}$ to obtain the balance of kinetic energy

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|v|^2}{2}f^\varepsilon\,\mathrm{d}v\mathrm{d}x = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{q}{m}E[f^\varepsilon]\cdot v f^\varepsilon\,\mathrm{d}v\mathrm{d}x - \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(\sigma\nabla_v f^\varepsilon + v f^\varepsilon)\cdot v\,\mathrm{d}v\mathrm{d}x.$$

Then, thanks to the continuty equation $\partial_t n[f^\varepsilon] + \mathrm{div}_x\int_{\mathbb{R}^2} v f^\varepsilon\ \mathrm{d}v = 0$, we multiply this equation by $\Phi[f^\varepsilon]$ and use the Poisson equation to find the balance of potential energy

$$\frac{\varepsilon_0\varepsilon}{2m}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}|E[f^\varepsilon]|^2\,\mathrm{d}x = -\frac{q}{m}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}E[f^\varepsilon]\cdot v f^\varepsilon\,\mathrm{d}v\mathrm{d}x.$$

Finally, multiplying the equation (4) by $\sigma(1+\ln f^\varepsilon)$ to get the balance of entropy

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\sigma f^\varepsilon\ln f^\varepsilon\,\mathrm{d}v\mathrm{d}x = -\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(\sigma\nabla_v f^\varepsilon + v f^\varepsilon)\cdot\frac{\sigma\nabla_v f^\varepsilon}{f^\varepsilon}\,\mathrm{d}v\mathrm{d}x.$$

As for weak solutions, we shall follow the same scheme. We find relations analogous to previous relations in the Lemmas below. The difficulty is in overcoming the lack of regularity and the need to justify operations that are taken for granted when the solutions are smooth. We will prove these properties of solutions by combining the formal arguments above with the choice of an appropriate sequence of test functions in (32) for every studied property. A similar rigorous approach that the one given in Refs. [2] and [6] can be easily adapted for the properties studied in our weak solution.

We start with the balance of kinetic energy.

Lemma 3.1

⟨BalanceKin2D⟩ *Let f^ε be the weak solution of the problem (4), (5), (6) provided by Theorem 2.2. Then we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|v|^2}{2}f^\varepsilon\,\mathrm{d}v\mathrm{d}x=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{q}{m}E[f^\varepsilon]\cdot vf^\varepsilon\,\mathrm{d}v\mathrm{d}x-\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(\sigma\nabla_v f^\varepsilon+vf^\varepsilon)\cdot v\,\mathrm{d}v\mathrm{d}x.$$

Proof.

Let χ be a nonegative function of class $C_0^\infty(\mathbb{R})$ such that

$$\chi(s)=1,\ \text{ on } |s|\le 1,\ \ \chi(s)=0 \text{ on } |s|\ge 2,$$

we define the function χ_R as $\chi_R(z)=\chi\left(\frac{|z|}{R}\right)$. Then $\chi_R(z)=1$ on $|z|\le R$, $\chi_R(z)=0$ on $|z|\ge 2R$ and $\|\nabla_z\chi_R\|_{L^\infty}\le\frac{\|\chi'\|_\infty}{R}$.

By using the test functions $\varphi(t,x,v)=\phi(t)\chi_R(x)\chi_R(v)\frac{|v|^2}{2}$ with $\phi\in C_0^\infty([0,T[)$ in the definition of weak solution (32), we obtain

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}f^\varepsilon\left[\varepsilon\partial_t\phi(t)\chi_R(x)+v\cdot\nabla_x\chi_R(x)\phi(t)\right]\chi_R(v)\frac{|v|^2}{2}\,\mathrm{d}v\mathrm{d}x\mathrm{d}t\\
&+\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}f^\varepsilon\left(E[f^\varepsilon]+\frac{B(x)}{\varepsilon}{}^\perp v\right)\cdot\nabla_v\left(\chi_R(v)\frac{|v|^2}{2}\right)\phi(t)\chi_R(x)\,\mathrm{d}v\mathrm{d}x\mathrm{d}t\\
&+\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}f^\varepsilon\left(\sigma\Delta_v-v\cdot\nabla_v\right)\left(\chi_R(v)\frac{|v|^2}{2}\right)\phi(t)\chi_R(x)\,\mathrm{d}v\mathrm{d}x\mathrm{d}t\\
&+\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\varepsilon f_{\mathrm{in}}^\varepsilon(x,v)\phi(0)\chi_R(x)\chi_R(v)\frac{|v|^2}{2}\,\mathrm{d}v\mathrm{d}x=0.
\end{aligned}$$

A simple computation shows that

$$\begin{aligned}
\nabla_v\left(\chi_R(v)\frac{|v|^2}{2}\right)&=\frac{v}{R|v|}\chi'\left(\frac{|v|}{R}\right)\frac{|v|^2}{2}+v\chi_R(v),\\
\Delta_v\left(\chi_R(v)\frac{|v|^2}{2}\right)&=\mathrm{div}_v\left(\frac{v}{R|v|}\chi'\left(\frac{|v|}{R}\right)\frac{|v|^2}{2}\right)+\mathrm{div}_v\left(v\chi_R(v)\right)\\
&=\frac{1}{R}\left[\chi'\left(\frac{|v|}{R}\right)\left(\frac{|v|}{2}+\frac{|v|}{R}\right)+\chi''\left(\frac{|v|}{R}\right)\frac{|v|^2}{2R}\right]+2\chi_R(v)+\chi'\left(\frac{|v|}{R}\right)\frac{|v|}{R}.
\end{aligned}$$

For each $\varepsilon>0$, using the Theorem 2.2 on the solution, we have $(1+|v|^2)f^\varepsilon\in L^\infty([0,T];L^1(\mathbb{R}^2\times\mathbb{R}^2))$ and $E[f^\varepsilon]\in L^\infty([0,T]\times\mathbb{R}^2)$. Letting $R\to\infty$, one gets, by the dominated convergence theorem, the following relation for any $\phi\in C_0^\infty([0,T[)$

$$\begin{aligned}
&\int_0^T\partial_t\phi(t)\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\varepsilon\frac{|v|^2}{2}f^\varepsilon(t,x,v)\,\mathrm{d}v\mathrm{d}x\mathrm{d}t+\int_0^T\phi(t)\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}E[f^\varepsilon]\cdot vf^\varepsilon\,\mathrm{d}v\mathrm{d}x\mathrm{d}t\\
&+\int_0^T\phi(t)\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(2\sigma-|v|^2)f^\varepsilon(t,x,v)\,\mathrm{d}v\mathrm{d}x\mathrm{d}t+\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\varepsilon\frac{|v|^2}{2}f_{\mathrm{in}}^\varepsilon(x,v)\phi(0)\,\mathrm{d}v\mathrm{d}x=0.
\end{aligned}$$

On the other hand, by Proposition 6.2, our weak solution f^ε belongs to $L^2([0,T]\times\mathbb{R}^2_x,H^1(\mathbb{R}^2_v))$ and tends to 0 at infinity since $(1+|v|^2)^{\gamma/2}f^\varepsilon\in L^\infty$, thus by the divergence theorem we have

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(2\sigma-|v|^2)f^\varepsilon(t,x,v)\,\mathrm{d}v\mathrm{d}x=-\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(\sigma\nabla_v f^\varepsilon+vf^\varepsilon)\cdot v\,\mathrm{d}v\mathrm{d}x.$$

Substituting into the previous relation, we easily deduce the assertions on the lemma. □

In the following lemma we obtain the balance of the potential energy.

Lemma 3.2

⟨BalancePot2D⟩ *Let f^ε be the weak solution of the problem (4), (5), (6) provided by Theorem 2.2. Then we have*

$$\frac{\varepsilon_0\varepsilon}{2m}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}|E[f^\varepsilon]|^2\,\mathrm{d}x = -\frac{q}{m}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}E[f^\varepsilon]\cdot v f^\varepsilon\,\mathrm{d}v\mathrm{d}x = -\frac{q}{m}\int_{\mathbb{R}^2}E[f^\varepsilon]\cdot j[f^\varepsilon]\,\mathrm{d}x.$$

Proof.

First, we show that $\Phi[f^\varepsilon]$, $E[f^\varepsilon]$ and $\partial_t E[f^\varepsilon]$ belong to $L^\infty([0,T];L^2(\mathbb{R}^2))$. We will apply the Lemma 2.8. The conditions in Lemma 2.8 are fulfilled by the properties on the solution f^ε and the background densities $D(x)$ by assumption H3. Hence one gets $\Phi[f^\varepsilon]$ and $E[f^\varepsilon]$ lie in $L^\infty([0,T];L^2(\mathbb{R}^2))$. It remains to prove that $\partial_t E[f^\varepsilon]$ belong to $L^\infty([0,T];L^2(\mathbb{R}^2))^2$. Thanks to the continuity equation on $[0,T[\times\mathbb{R}^2$ in the sense of distributions

$$\partial_t n[f^\varepsilon] + \mathrm{div}_x\int_{\mathbb{R}^2} v f^\varepsilon\,\mathrm{d}v = 0$$

see Lemma 4.1 below, together with the Poisson equation (5), we deduce that

$$\partial_t E[f^\varepsilon(t)](x) = -\frac{q}{2\pi\varepsilon_0}\nabla\ln|\cdot|\star\partial_t(n[f^\varepsilon]-D) = \frac{1}{2\pi\varepsilon_0}\nabla\ln|\cdot|\star(\mathrm{div}_x j[f^\varepsilon]).$$

In order to estimate $\partial_t E[f^\varepsilon(t)]$, we will use the Calderon-Zygmund inequality, see Lemma 6.7 below in the Appendix D, but in the dual version. Let η be a test function in $C_0^\infty([0,T[\times\mathbb{R}^2)$. We have

$$\left\langle\frac{1}{2\pi\varepsilon_0}\nabla\ln|\cdot|\star(\mathrm{div}_x j[f^\varepsilon]),\eta\right\rangle = \int_{\mathbb{R}^2}\frac{q}{2\pi\varepsilon_0}D^2\ln|\cdot|\star\eta(x)\cdot j[f^\varepsilon]\,\mathrm{d}x.$$

By Lemma 2.2 one gets $j[f^\varepsilon]\in L^\infty([0,T];L^2(\mathbb{R}^2\times\mathbb{R}^2))$. Therefore we deduce that

$$\begin{aligned}\left|\left\langle\frac{1}{2\pi\varepsilon_0}\nabla\ln|\cdot|\star(\mathrm{div}_x j[f^\varepsilon]),\eta\right\rangle\right| &\le \frac{q}{2\pi\varepsilon_0}\|D^2\ln|\cdot|\star\eta\|_{L^2}\|j[f^\varepsilon]\|_{L^2}\\ &\le C\|\eta\|_{L^2}\|j[f^\varepsilon]\|_{L^2}.\end{aligned}$$

It allows to conclude that $\partial_t E[f^\varepsilon]$ belongs to $L^\infty([0,T];L^2(\mathbb{R}^2))$.

Now let $\nu>0$ and let $\kappa\in C_0^\infty(\mathbb{R}^2)$ be a standard mollifier. Define the regularization kernel $\kappa_\nu := \frac{1}{\nu^2}\kappa(\frac{x}{\nu})$. Convoluting with κ_ν in the equation $\mathrm{div}_x(\partial_t E[f^\varepsilon]+\frac{q}{\varepsilon_0\varepsilon}j[f^\varepsilon])=0$ we obtain

$$\mathrm{div}_x(\partial_t E^\nu[f^\varepsilon]+\frac{q}{\varepsilon_0\varepsilon}j^\nu[f^\varepsilon])=0$$

where $E^\nu[f^\varepsilon]=E[f^\varepsilon]\star\kappa_\nu$, $j^\nu[f^\varepsilon]=j[f^\varepsilon]\star\kappa_\nu$. Multiplying the previous equation by $\Phi^\nu[f^\varepsilon]\chi_R(x)$ and integrate by parts to find that

$$\begin{aligned}\int_{\mathbb{R}^2}\partial_t E^\nu[f^\varepsilon]\cdot E^\nu[f^\varepsilon]\chi_R(x)\,\mathrm{d}x + \frac{q}{\varepsilon_0\varepsilon}\int_{\mathbb{R}^2}E^\nu[f^\varepsilon]\cdot j^\nu[f^\varepsilon]\chi_R(x)\,\mathrm{d}x\\ +\int_{\mathbb{R}^2}(\partial_t E^\nu[f^\varepsilon]+\frac{q}{\varepsilon_0\varepsilon}j^\nu[f^\varepsilon])\cdot\Phi^\nu[f^\varepsilon]\nabla\chi_R\,\mathrm{d}x=0.\end{aligned}$$

where $\Phi^\nu[f^\varepsilon]=\Phi[f^\varepsilon]\star\kappa_\nu$ and χ_R stands for the family of smooth cut-off functions, defined in Lemma 3.1. Let $\nu\to 0$. The terms on the left side converge as a consequence of the theorem of smooth approximations from the first arguments on $\Phi[f^\varepsilon], E[f^\varepsilon], \partial_t E[f^\varepsilon]$. Then we obtain

$$\begin{aligned}\int_{\mathbb{R}^2}\partial_t E[f^\varepsilon]\cdot E[f^\varepsilon]\chi_R(x)\,\mathrm{d}x + \frac{q}{\varepsilon_0\varepsilon}\int_{\mathbb{R}^2}E[f^\varepsilon]\cdot j[f^\varepsilon]\chi_R(x)\,\mathrm{d}x\\ +\int_{\mathbb{R}^2}(\partial_t E[f^\varepsilon]+\frac{q}{\varepsilon_0\varepsilon}j[f^\varepsilon])\cdot\Phi[f^\varepsilon]\nabla\chi_R\,\mathrm{d}x=0.\end{aligned}$$

Letting $R \to \infty$, the dominated convergence theorem yields

$$\int_{\mathbb{R}^2} \partial_t E[f^\varepsilon] \cdot E[f^\varepsilon] \, \mathrm{d}x + \frac{q}{\varepsilon_0 \varepsilon} \int_{\mathbb{R}^2} E[f^\varepsilon] \cdot j[f^\varepsilon] \, \mathrm{d}x = 0$$

which gives the result in the lemma. □

Finally, let us deduce the balance of entropy.

Lemma 3.3

⟨BalanceEntropy⟩ *Let f^ε be a weak solution of the problem (4), (5), (6) provided by Theorem 2.2. Then we have*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \sigma f^\varepsilon \ln f^\varepsilon \, \mathrm{d}v\mathrm{d}x = - \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (\sigma \nabla_v f^\varepsilon + v f^\varepsilon) \cdot \frac{\sigma \nabla_v f^\varepsilon}{f^\varepsilon} \, \mathrm{d}v\mathrm{d}x.$$

Proof.
First, we will show that for any $\Psi \in C^2(\mathbb{R})$ such that $\Psi'' \in L^\infty(\mathbb{R})$ and $\Psi(0) = 0$, $\Psi(f)$ solves the following equation in the sense of distribution on $[0, T[\times \mathbb{R}^2 \times \mathbb{R}^2$

$$\begin{aligned} \varepsilon \partial_t \Psi(f^\varepsilon) + v \cdot \nabla_x \Psi(f^\varepsilon) + \frac{q}{m} E[f^\varepsilon] \cdot \nabla_v \Psi(f^\varepsilon) + \frac{q}{m} \frac{B(x)}{\varepsilon}^\perp v \cdot \nabla_v \Psi(f^\varepsilon) \\ - v \cdot \nabla_v \Psi(f^\varepsilon) - \sigma \Delta_v \Psi(f^\varepsilon) = 2 f^\varepsilon \Psi'(f^\varepsilon) - \sigma \Psi''(f^\varepsilon) \left| \nabla_v f^\varepsilon \right|^2 . \end{aligned} \tag{33}$$

NonlinearEqu

Let us consider a sequence of mollifiers $\kappa_\nu := \kappa_\nu^{t,x,v} = \kappa_\nu(t) \star \kappa_\nu(x) \star \kappa_\nu(v)$ which approximates the Dirac delta function, and $\kappa_\nu(t)$ is supposed to have its support in the negative real axis. Then we define $f^{\varepsilon,\nu} = f^\varepsilon \star_{t,x,v} \kappa_\nu$. It is well known that f^ε belongs to $L^2([0,T] \times \mathbb{R}^2_x; H^1(\mathbb{R}^2_v))$ then $f^{\varepsilon,\kappa} \in C^\infty(0,T; H^m(\mathbb{R}^2 \times \mathbb{R}^2))$, for every $\nu > 0$, $m \geq 1$. Moreover, by the theorem of smooth approximations, we also have

$$f^{\varepsilon,\nu} \to f^\varepsilon \text{ in } L^2([0,T]; L^2(\mathbb{R}^2 \times \mathbb{R}^2)), \ \nabla_v f^{\varepsilon,\nu} \to \nabla_v f^\varepsilon \text{ in } L^2([0,T]; L^2(\mathbb{R}^2 \times \mathbb{R}^2)).$$

We convolute with κ_ν in the equation (4), we obtain

$$\varepsilon \partial_t f^{\varepsilon,\nu} + v \cdot \nabla_x f^{\varepsilon,\nu} + \frac{q}{m} E[f^\varepsilon] \cdot \nabla_v f^{\varepsilon,\nu} + \frac{q}{m} \frac{B(x)}{\varepsilon}^\perp v \cdot \nabla_v f^{\varepsilon,\nu} - \mathrm{div}_v(v f^{\varepsilon,\nu}) - \sigma \Delta_v f^{\varepsilon,\nu} = \sum_{i=1}^4 h_i^{\varepsilon,\nu} \tag{34}$$

equ:ConvolVPFP

where the functions $h_i^{\varepsilon,\nu}$ are defined by

$$\begin{aligned} h_1^{\varepsilon,\nu} &= v \cdot \nabla_x (\kappa_\nu \star f^\varepsilon) - (v \cdot \nabla_x f^\varepsilon) \star \kappa_\nu \\ h_2^{\varepsilon,\nu} &= \frac{q}{m} E[f^\varepsilon] \cdot \nabla_v (\kappa_\nu \star f^\varepsilon) - (E[f^\varepsilon] \cdot \nabla_v f^\varepsilon) \star \kappa_\nu \\ h_3^{\varepsilon,\nu} &= \frac{q}{m} \frac{B(x)}{\varepsilon}^\perp v \cdot \nabla_v (\kappa_\nu \star f^\varepsilon) - (B(x)^\perp v \cdot \nabla_v f^\varepsilon) \star \kappa_\nu \\ h_4^{\varepsilon,\nu} &= -[v \cdot \nabla_v (\kappa_\nu \star f^\varepsilon) - (v \cdot \nabla_v f^\varepsilon) \star \kappa_\nu]. \end{aligned}$$

For each $\varepsilon > 0$, it is obviously that $h_2^{\varepsilon,\nu}$ tends to 0 in $L^2([0,T]; L^2(\mathbb{R}^2 \times \mathbb{R}^2))$ as $\nu \searrow 0$, since $\nabla_v f^{\varepsilon,\nu} = \nabla_v (\kappa_\nu \star f^\varepsilon)$ converges to $\nabla_v f^\varepsilon$ in $L^2([0,T]; L^2(\mathbb{R}^2 \times \mathbb{R}^2))$ and $E[f^\varepsilon] \in (L^\infty([0,T] \times \mathbb{R}^2))^2$ hence $E[f^\varepsilon] \cdot \nabla_v (\kappa_\nu \star f^\varepsilon)$ tends to $E[f^\varepsilon] \cdot \nabla_v f^\varepsilon$ and $(E[f^\varepsilon] \cdot \nabla_v f^\varepsilon) \star \kappa_\nu$ also tends to $E[f^\varepsilon]) \cdot \nabla_v f^\varepsilon$ in $L^2([0,T]; L^2(\mathbb{R}^2 \times \mathbb{R}^2))$. For other terms $h_i^{\varepsilon,\nu}$, we also obtain that $h_i^{\varepsilon,\nu} \to 0$ in $L^1([0,T]; L^\beta_{\mathrm{loc}}(\mathbb{R}^2 \times \mathbb{R}^2))$. Here, we have used the following property, see [16] Lemma II.1: Let $W \in L^1(0,T; (W^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^d))^d), g \in L^\infty(0,T; L^p_{\mathrm{loc}}(\mathbb{R}^d))$, then we have

$$(W \cdot \nabla g) \star \delta_\varepsilon - W \cdot \nabla (g \star \delta_\varepsilon) \to 0 \quad \text{in } L^1(0,T; L^\beta(\mathbb{R}^d)),$$

where $\dfrac{1}{\beta}=\dfrac{1}{\alpha}+\dfrac{1}{p}$ if α or $p<\infty$, $\beta<\infty$ is arbitrary if $\alpha=p=\infty$.

Now we establish the equation (33). Since $\Psi\in C^2(\mathbb{R})$ then $\Psi(f^{\varepsilon,\nu})\in C^2(\mathbb{R})$. Multiplying by $\Psi'(f^{\varepsilon,\nu})$ in (34), one gets

$$\partial_t\Psi(f^{\varepsilon,\nu})+v\cdot\nabla_x\Psi(f^{\varepsilon,\nu})+\frac{q}{m}E[f^\varepsilon]\cdot\nabla_v\Psi(f^{\varepsilon,\nu})+\frac{q}{m}\frac{B(x)}{\varepsilon}{}^{\perp}v\cdot\nabla_v\Psi(f\varepsilon,\nu)-v\cdot\nabla_v\Psi(f^{\varepsilon,\nu})$$
$$-\sigma\Delta_v\Psi(f^{\varepsilon,\nu})=2f^{\varepsilon,\nu}\Psi'(f^{\varepsilon,\nu})-\sigma\Psi''(f^{\varepsilon,\nu})|\nabla_v f^{\varepsilon,\nu}|^2+\sum_{i=1}^{4}h_i^{\varepsilon,\nu}\Psi'(f^{\varepsilon,\nu}).$$

Since $\Psi\in C^2(\mathbb{R}^2)$ with $\Psi''\in L^\infty(\mathbb{R}^2)$ thus Ψ is at the most quadratic at infinity, we have $\Psi(f^{\varepsilon,\nu})\to\Psi(f^\varepsilon)$ in $L^1_{\rm loc}$, $\Psi'(f^{\varepsilon,\nu})\to\Psi'(f^\varepsilon)$ in $L^2_{\rm loc}$. Passing to the limit we obtain the equation (33) in the sense of distribution.

Now we are ready to establish the balance of the entropy identity. We shall apply (33) for the following function

$$\psi_\delta(f^\varepsilon)=(\delta+f^\varepsilon)\ln\left(1+\frac{f^\varepsilon}{\delta}\right)+f^\varepsilon\ln\delta,\ \delta>0.$$

A simple computations show that

$$\psi'_\delta(f)=\ln\left(1+\frac{f}{\delta}\right)+\ln\delta+1,\ \psi''_\delta(f)=\frac{1}{\delta+f}\le\frac{1}{\delta}.$$

Thus, the function ψ_δ belongs to $C^2(\mathbb{R})$ and satisfies $\psi''_\delta\in L^\infty$ with $\psi_\delta(0)=0$. Moreover, $\psi_\delta\in L^\infty([0,T];L^1(\mathbb{R}^2\times\mathbb{R}^2))$, since $(\delta+f^\varepsilon)\ln\left(1+\frac{f^\varepsilon}{\delta}\right)\le(\delta+f^\varepsilon)\frac{f^\varepsilon}{\delta}=f^\varepsilon+\frac{(f^\varepsilon)2}{\delta}$. Therefore, $\psi_\delta(f^\varepsilon)$ satisfies the following equation in the sense of distribution

$$\partial_t\psi_\delta(f^\varepsilon)+v\cdot\nabla_x\psi_\delta(f^\varepsilon)+\frac{q}{m}E[f^\varepsilon]\cdot\nabla_v\psi_\delta(f^\varepsilon)+\frac{q}{m}\frac{B(x)}{\varepsilon}{}^{\perp}v\cdot\nabla_v\psi_\delta(f^\varepsilon)$$
$$-v\cdot\nabla_v\psi_\delta(f^\varepsilon)-\sigma\Delta_v\psi_\delta(f^\varepsilon)=2f^\varepsilon\psi'_\delta(f^\varepsilon)-\sigma\psi''_\delta(f^\varepsilon)|\nabla_v f^\varepsilon|^2. \tag{35}$$ equ:NonlinearV

We consider the test function $\varphi(t,x,v)=\phi(t)\chi_R(x)\chi_R(v)$, where the function χ_R was defined in the Lemma 3.1. For each $\varepsilon>0$, we have $\psi_\delta(f^\varepsilon)\in L^\infty([0,T];L^1(\mathbb{R}^2\times\mathbb{R}^2))$ and $E[f^\varepsilon]\in L^\infty([0,T]\times\mathbb{R}^2)$. Passing to the limit as $R\to\infty$, we easily deduce from (35) the following relation

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\psi_\delta(f^\varepsilon(t))\,\mathrm{d}v\mathrm{d}x+\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{\sigma|\nabla_v f^\varepsilon|^2}{\delta+f^\varepsilon}\,\mathrm{d}v\mathrm{d}x+2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\psi_\delta(f^\varepsilon)\,\mathrm{d}v\mathrm{d}x\mathrm{d}\tau$$
$$=2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}f^\varepsilon\left(\ln\left(1+\frac{f^\varepsilon}{\delta}\right)+\ln\delta+1\right)\,\mathrm{d}v\mathrm{d}x$$

which is equivalent to

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\psi_\delta(f^\varepsilon(t))\,\mathrm{d}v\mathrm{d}x+\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{\sigma|\nabla_v f^\varepsilon|^2}{\delta+f^\varepsilon}\,\mathrm{d}v\mathrm{d}x=2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(f^\varepsilon-\delta\ln\left(1+\frac{f^\varepsilon}{\delta}\right)\right)\,\mathrm{d}v\mathrm{d}x \tag{36}$$ equ:LimNonLine

Next we will study the limit of (36) as $\delta\to0$. Let us recall that the solution f^ε satisfies the properties: $f^\varepsilon\in L^\infty([0,T];L^1\cap L^\infty)$, $f^\varepsilon\ln f^\varepsilon\in L^\infty([0,T];L^1(\mathbb{R}^2\times\mathbb{R}^2))$. Since

$$\psi_\delta(f^\varepsilon)-f^\varepsilon\ln f^\varepsilon=\delta\ln\left(1+\frac{f^\varepsilon}{\delta}\right)+f^\varepsilon\ln\left(1+\frac{\delta}{f^\varepsilon}\right)$$

we deduce that this term tends to 0 a.e in $\mathbb{R}^2$ as $\delta \to 0$ and it is uniform bounded with respect to δ by

$$\begin{aligned}|\psi_\delta(f^\varepsilon) - f^\varepsilon \ln f^\varepsilon| &\le \delta \frac{f^\varepsilon}{\delta} + f^\varepsilon |\ln(\delta + f^\varepsilon)| + |f^\varepsilon \ln f^\varepsilon| \\ &\le f^\varepsilon + f^\varepsilon \ln(1 + f^\varepsilon) + |f^\varepsilon \ln f^\varepsilon| \\ &\le f^\varepsilon + (f^\varepsilon)^2 + |f^\varepsilon \ln f^\varepsilon|\end{aligned}$$

which belongs to $L^1(\mathbb{R}^2)$. Therefore, by the dominated convergece theorem, we get

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\psi_\delta(f^\varepsilon) - f^\varepsilon \ln f^\varepsilon| \,\mathrm{d}v\mathrm{d}x \to 0, \;\; \delta \searrow 0.$$

Using the same arguments for the integral in the right hand side of (36), we also have

$$2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left(f^\varepsilon - \delta \ln\left(1 + \frac{f^\varepsilon}{\delta}\right)\right) \,\mathrm{d}v\mathrm{d}x \to 2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f^\varepsilon \,\mathrm{d}v\mathrm{d}x, \text{ as } \delta \searrow 0.$$

On the other hand, integrating (36) between 0 and T yields

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\sigma|\nabla_v f^\varepsilon|^2}{\delta + f^\varepsilon} \,\mathrm{d}v\mathrm{d}x\mathrm{d}s \\ &= -\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \psi(f^\varepsilon)|_0^T \,\mathrm{d}v\mathrm{d}x + 2\int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left(f^\varepsilon - \delta \ln\left(1 + \frac{f^\varepsilon}{\delta}\right)\right) \,\mathrm{d}v\mathrm{d}x\mathrm{d}s\end{aligned}$$

which shows that the sequence $(|\nabla_v f^\varepsilon|^2/(\delta + f^\varepsilon))_{\delta>0}$ is bounded in $L^1([0,T]\times\mathbb{R}^2\times\mathbb{R}^2)$. Thanks to Fatou's lemma, one gets $\nabla_v \sqrt{f^\varepsilon} \in L^2([0,T]\times\mathbb{R}^2\times\mathbb{R}^2)$. By using the dominated convergence theorem together with the previous arguments, we obtain

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f^\varepsilon \ln f^\varepsilon|_0^T \,\mathrm{d}v\mathrm{d}x + \int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\sigma|\nabla_v f^\varepsilon|^2}{f^\varepsilon} \,\mathrm{d}v\mathrm{d}x\mathrm{d}s = 2\int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f^\varepsilon \,\mathrm{d}v\mathrm{d}x\mathrm{d}s$$

which can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f^\varepsilon \ln f^\varepsilon \,\mathrm{d}v\mathrm{d}x = -\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{\sigma|\nabla_v f^\varepsilon|^2}{f^\varepsilon} \,\mathrm{d}v\mathrm{d}x - \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} v\cdot\nabla_v f^\varepsilon \,\mathrm{d}v\mathrm{d}x$$

since $f^\varepsilon \in L^2([0,T]\times\mathbb{R}^2_x; H^1(\mathbb{R}^2_v))$. So we complete the proof of lemma. □

Proof. (of Proposition 3.1)
The mass conservation can be deduced by testing the test function $\varphi(t,x,v) = \phi(t)\chi_R(|x|)\chi_R(|v|)$ in (32). On the other hand, using the Lemmas 3.1, 3.2 and 3.3, we imply the desired result for the balance of energy $\mathcal{E}[f^\varepsilon]$. □

We establish now uniform bounds for the kinetic energy.

Lemma 3.4

⟨BoundKinEner2D⟩ *Assume that the initial particle densities $(f^\varepsilon_{\mathrm{in}})$ satisfy $f^\varepsilon_{\mathrm{in}} \ge 0$, $M_{\mathrm{in}} := \sup_{\varepsilon>0} M^\varepsilon_{\mathrm{in}} < +\infty$, $U_{\mathrm{in}} := \sup_{\varepsilon>0} U^\varepsilon_{\mathrm{in}} < +\infty$, where for any $\varepsilon > 0$*

$$M^\varepsilon_{\mathrm{in}} := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f^\varepsilon_{\mathrm{in}}(x,v) \,\mathrm{d}v\mathrm{d}x, \; U^\varepsilon_{\mathrm{in}} := \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|v|^2}{2} f^\varepsilon_{\mathrm{in}}(x,v) \,\mathrm{d}v\mathrm{d}x + \frac{\varepsilon_0}{2m}\int_{\mathbb{R}^2} |\nabla_x \Phi[f^\varepsilon_{\mathrm{in}}]|^2 \,\mathrm{d}x.$$

We assume that $(f^\varepsilon)_{\varepsilon>0}$ are smooth solutions of (4), (5), (6). Then we have

$$\varepsilon \sup_{0\le t\le T}\left\{ \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|v|^2}{2} f^\varepsilon(t,x,v) \,\mathrm{d}v\mathrm{d}x + \frac{\varepsilon_0}{2m}\int_{\mathbb{R}^2} |\nabla_x \Phi[f^\varepsilon]|^2 \,\mathrm{d}x\right\} \le \varepsilon U_{\mathrm{in}} + 2\sigma T M_{\mathrm{in}}$$

and

$$\int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |v|^2 f^\varepsilon(t,x,v) \,\mathrm{d}v\mathrm{d}x\mathrm{d}t \le \varepsilon U_{\mathrm{in}} + 2\sigma T M_{\mathrm{in}}.$$

Proof.
Using the Lemmas 3.1 and 3.2 yields

$$\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|v|^2}{2}f^\varepsilon(t,x,v)\,\mathrm{d}v\mathrm{d}x+\frac{\varepsilon_0}{2m}\int_{\mathbb{R}^2}|\nabla_x\Phi[f^\varepsilon]|^2\,\mathrm{d}x\right\}=2\sigma M_{\mathrm{in}}^\varepsilon-\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|v|^2f^\varepsilon\,\mathrm{d}v\mathrm{d}x$$

and therefore we obtain

$$\begin{aligned}&\varepsilon\left\{\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|v|^2}{2}f^\varepsilon(t,x,v)\,\mathrm{d}v\mathrm{d}x+\frac{\varepsilon_0}{2m}\int_{\mathbb{R}^2}|\nabla_x\Phi[f^\varepsilon]|^2\,\mathrm{d}x\right\}+\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|v|^2f^\varepsilon\,\mathrm{d}v\mathrm{d}x\mathrm{d}s\\&=\varepsilon U_{\mathrm{in}}^\varepsilon+2\sigma tM_{\mathrm{in}}^\varepsilon\end{aligned}$$

which yields the results. □

4 Formal derivation of the limit model

The asymptotic behavior of the Vlasov-Fokker-Planck-Poisson equation (4) when ε becomes small comes from the balance of the free energy functional $\mathcal{E}[f^\varepsilon]$. Thanks to the Proposition 3.1, we deduce that

$$\varepsilon\mathcal{E}[f^\varepsilon(t)]+\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|\sigma\nabla_vf^\varepsilon+vf^\varepsilon|^2}{f^\varepsilon}\,\mathrm{d}v\mathrm{d}x\mathrm{d}s=\varepsilon\mathcal{E}[f^\varepsilon(0)].$$

Since the dissipation term can rewrite as

$$\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|\sigma M\nabla_v(f^\varepsilon/M)|^2}{f^\varepsilon}\,\mathrm{d}v\mathrm{d}x\mathrm{d}s$$

where M stands for the Maxwellian equilibrium $M(v)=(2\pi\sigma)^{-1}\exp\left(-\frac{|v|^2}{2\sigma}\right), v\in\mathbb{R}^2$. Therefore, at least formally, we deduce that $f^\varepsilon=f+\mathcal{O}(\varepsilon)$, as $\varepsilon\searrow 0$, where the leading order distribution function f satisfies

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|\sigma M\nabla_v(f/M)|^2}{f}\,\mathrm{d}v\mathrm{d}x=0,\ t\in\mathbb{R}_+.$$

Hence, we obtain $f(t,x,v)=n(t,x)M(v), (t,x,v)\in\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{R}^2$. Then, the question is to determine the evolution equation satisfied by the concentration $n(t,x)=\int_{\mathbb{R}^2}f(t,x,v)\,\mathrm{d}v$.
We are looking the model for the concentration $n[f^\varepsilon]=\int_{\mathbb{R}^2}f^\varepsilon\,\mathrm{d}v$. First, by integrating the equation (4) with 1 and v, we straightforwardly get the local conservation laws satisfied by the first two moments.

Lemma 4.1
⟨ConservationLaw⟩ *Let $\varepsilon>0$. Let f^ε be a weak solution of the system* (4), (5), (6) *provided by Theorem 2.2. Then the following conservation laws hold in the distributional sense*

$$\partial_tn[f^\varepsilon]+\frac{1}{\varepsilon}\mathrm{div}_xj[f^\varepsilon]=0. \tag{37}$$ ContinuLaw

$$\varepsilon\partial_tj[f^\varepsilon]+\mathrm{div}_x\int_{\mathbb{R}^2}v\otimes vf^\varepsilon\,\mathrm{d}v-\frac{q}{m}E[f^\varepsilon]n[f^\varepsilon]-\frac{qB(x)}{m}\frac{{}^\perp j[f^\varepsilon]}{\varepsilon}=-j[f^\varepsilon]. \tag{38}$$ MomentLaw

Proof.
For each $\varepsilon > 0$, $(f^\varepsilon, E[f^\varepsilon])$ solves (4) in the sense of distribution given by equation (32) and satisfies $(1+|v|^2)f^\varepsilon \in L^\infty([0,T]; L^1(\mathbb{R}^2\times\mathbb{R}^2))$, $E[f^\varepsilon] \in L^\infty([0,T]\times\mathbb{R}^2)$. Then, we test (32) on the test functions of the form $\varphi(t,x,v) = \phi(t)\chi_R(x)\chi_R(v)$ and $\varphi(t,x,v) = \phi(t)\chi_R(x)\chi_R(v)v$, where the function χ_R was defined in Lemma 3.1, and $\phi \in C_0^\infty([0,T[)$. Letting $R \to \infty$, one gets, by dominated convergence theorem, the relations (37) and (38) which hold in the distribution sense on $R_+^\star \times \mathbb{R}^2$ and are respectively the continuity equation and the momentum equation. □

Then, we apply the rotation $v \mapsto {}^\perp v$ to the equation (38) and eliminating $\frac{1}{\varepsilon} j[f^\varepsilon]$ between the resulting equation and (37) leads to the new equation for the concentration $n[f^\varepsilon]$.

Corollary 4.1
Let $\varepsilon > 0$. Let f^ε be a weak solution of the system (4), (5), (6) *provided by Theorem 2.2.* ?⟨NewConcen⟩? *Then the concentration $n[f^\varepsilon]$ satisfies the following equation*

$$\partial_t n[f^\varepsilon] + \mathrm{div}_x \left[n[f^\varepsilon] \left(\frac{{}^\perp E[f^\varepsilon]}{B(x)} - \sigma \frac{{}^\perp \nabla_x \omega_c(x)}{\omega_c(x)^2} \right) \right] = \mathrm{div}_x F^\varepsilon \tag{39}$$ ModConcen2D

where we denote

$$F^\varepsilon = \frac{\varepsilon \partial_t {}^\perp j[f^\varepsilon] + {}^\perp j[f^\varepsilon]}{\omega_c(x)} + \frac{1}{\omega_c(x)} {}^\perp \mathrm{div}_x \int_{\mathbb{R}^2} (v\otimes v - \sigma I_2) f^\varepsilon \, \mathrm{d}v.$$

Proof.
The proof of the result is obviously by observing that the momentum flux tensor can be decomposed as

$$\int_{\mathbb{R}^2} v\otimes v f^\varepsilon \, \mathrm{d}v = \int_{\mathbb{R}^2} (v\otimes v - \sigma I_2) f^\varepsilon \, \mathrm{d}v + \sigma I_2 n[f^\varepsilon].$$

□

Passing formal to the limit in (39), as $\varepsilon \searrow 0$, we get

$$\partial_t n[f] + \mathrm{div}_x \left[n[f] \left(\frac{{}^\perp E[f]}{B(x)} - \sigma \frac{{}^\perp \nabla \omega_c(x)}{\omega_c^2(x)} \right) \right] = 0,$$

where we have used that f^ε tends to $f = n(t,x)M(v)$ leading to $n[f^\varepsilon] \to n[f]$, $j[f^\varepsilon] \to j[f] = 0$ and $\int_{\mathbb{R}^2}(v\otimes v - \sigma I_2) f^\varepsilon \, \mathrm{d}v \to \int_{\mathbb{R}^2}(v\otimes v - \sigma I_2) f \, \mathrm{d}v = 0$. Therefore the limit model is

$$\partial_t n + \mathrm{div}_x \left[n \left(\frac{{}^\perp E[n]}{B(x)} - \sigma \frac{{}^\perp \nabla \omega_c(x)}{\omega_c^2(x)} \right) \right] = 0, \ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2 \tag{40}$$ LimitMod2D

$$E[n] = -\nabla_x \Phi[n], \ -\varepsilon_0 \Delta_x \Phi[n] = q(n - D) \tag{41}$$ PoissonLim2D

with the initial condition

$$n(0,x) = n_{\mathrm{in}}(x), \ x \in \mathbb{R}^2. \tag{42}$$ LimitInitial2D

We have the following balances for the previous limit model

Proposition 4.1
Any smooth solution of the limit model (40), (41), (42) *verifies the mass and free energy* ⟨ConserveEnerLim⟩ *conservations*

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_{\mathbb{R}^2} n(t,x) \, \mathrm{d}x = 0, \ \frac{\mathrm{d}}{\mathrm{dt}} \int_{\mathbb{R}^2} \left\{ \sigma n \ln n + \frac{\varepsilon_0}{2m} |\nabla_x \Phi[n]|^2 \right\} \, \mathrm{d}x = 0.$$

Proof.
Clearly we have the total mass conservation. For the energy conservation, a straightforward computation, the evolution in time of the energy for the limit model can be written as

$$\int_{\mathbb{R}^2} \sigma \partial_t n (1 + \ln n) \, \mathrm{d}x + \int_{\mathbb{R}^2} \frac{\varepsilon_0}{m} E[n] \cdot \partial_t E[n] \, \mathrm{d}x.$$

Using the equation (40) for the first integral in the previous equality, we have

$$\int_{\mathbb{R}^2} \sigma \partial_t n (1 + \ln n) \, \mathrm{d}x = \int_{\mathbb{R}^2} \sigma \left(\frac{{}^{\perp}E[n]}{B(x)} - \sigma \frac{{}^{\perp}\nabla \omega_c(x)}{\omega_c^2(x)} \right) \cdot \nabla n \, \mathrm{d}x = \sigma \int_{\mathbb{R}^2} \frac{{}^{\perp}E[n] \cdot \nabla B}{B^2(x)} \, \mathrm{d}x.$$

Thanks to Poisson's equation (41), then using again (40) for the second integral, we get

$$\begin{aligned} \int_{\mathbb{R}^2} \frac{\varepsilon_0}{m} E[n] \cdot \partial_t E[n] \, \mathrm{d}x = \int_{\mathbb{R}^2} \frac{q}{m} \Phi[n] \partial_t n \, \mathrm{d}x &= -\frac{q}{m} \int_{\mathbb{R}^2} E[n] \cdot n \left(\frac{{}^{\perp}E[n]}{B(x)} - \sigma \frac{{}^{\perp}\nabla \omega_c(x)}{\omega_c^2(x)} \right) \, \mathrm{d}x \\ &= -\sigma \int_{\mathbb{R}^2} n \frac{{}^{\perp}E[n] \cdot \nabla B}{B^2(x)} \, \mathrm{d}x. \end{aligned}$$

Combining these equalities we obtain the balance of the energy. □

5 Well-posedness of the limit model

In this section we focus on the existence, uniqueness and the properties of the solution for the limit model (40), (41), (42). We will construct smooth solution on any time interval $[0, T]$, $T \in \mathbb{R}_+$, following the same arguments as in the well posedness proof for the Vlasov–Poisson problem with external magnetic field, cf. [3]. We assume that the initial condition n_{in} satisfies the hypotheses

H4) $n_{\mathrm{in}} \geq 0$, $|x| n_{\mathrm{in}} \in L^1(\mathbb{R}^2)$, $n_{\mathrm{in}} \in W^{1,1}(\mathbb{R}^2) \cap W^{1,\infty}(\mathbb{R}^2)$

H5) $\int_{\mathbb{R}^2} n_{\mathrm{in}}(x) \, \mathrm{d}x = \int_{\mathbb{R}^2} D(x) \, \mathrm{d}x$.

and the external magnetic field $B(x)$ verifies

$$B \in C_b^2(\mathbb{R}^2), \quad \inf_{x \in \mathbb{R}^2} |B(x)| = B_0 > 0.$$

Solution integrated along the characteristics
First, a standard computation, the equation (40) can be rewritten for the unknown n/B as

$$\partial_t \left(\frac{n}{B} \right) + \left(\frac{{}^{\perp}E[n]}{B} - \sigma \frac{{}^{\perp}\nabla \omega_c(x)}{\omega_c^2(x)} \right) \cdot \nabla_x \left(\frac{n}{B} \right) = 0. \tag{43}$$ EquivLimMo2D

For any smooth field $E \in L^\infty([0,T]; W^{1,\infty}(\mathbb{R}^2))^2$, we consider the associated characteristics flow of (43) given by

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} X(t; s, x) = \frac{{}^{\perp} E(t, X(t; s, x))}{B(X(t; s, x))} - \sigma \frac{{}^{\perp}\nabla \omega_c(X(t; s, x))}{\omega_c^2(X(t; s, x))}, && t, s \in [0, T] \\ &X(s; s, x) = x, && s \in [0, T], \ x \in \mathbb{R}^2 \end{aligned} \tag{44}$$ equ:CharLimMo2

where $X(t; s, x)$ is the solution of the equation (44), t represents the time variable, s is the initial time and x is the initial position. $X(s; s, x) = x$ is our initial condition. Notice that by

the hypothesis on the magnetic field $B(x)$, the vector field $\sigma\frac{{}^{\perp}\nabla\omega_c}{\omega_c^2(x)}$ is also smooth with respect to x and we have

$$\left\|\sigma\frac{{}^{\perp}\nabla\omega_c}{\omega_c^2(x)}\right\|_{W^{1,\infty}(\mathbb{R}^2)} \leq C(\sigma, B, B_0).$$

Therefore, thanks to Cauchy-Lipschitz theorem, the characteristics in (44) are well defined for any $(s,x) \in [0,T]\times\mathbb{R}^2$ and $X(t;s,x) \in W^{1,\infty}\left([0,T]\times[0,T]\times\mathbb{R}^2\right)^2$. Then the equation (43) can be written as

$$\frac{\mathrm{d}}{\mathrm{dt}}\left[\frac{n\left(t, X\left(t;s,x\right)\right)}{B(X(t;s,x))}\right] = 0$$

which yields the solution of the transport equation (43) given by

$$n\left(t,x\right) = B(x)\frac{n\left(0, X\left(0;t,x\right)\right)}{B(X(0;t,x))} = B(x)\frac{n_{\mathrm{in}}\left(X\left(0;t,x\right)\right)}{B\left(X\left(0;t,x\right)\right)},\ t\in[0,T]. \tag{45}$$ SolCharac2D

Conservation law on a volume
We have the following result

$$\int_{\mathbb{R}^2}|n(t,x)|\,\mathrm{d}x = \int_{\mathbb{R}^2} n_{\mathrm{in}}(x)\,\mathrm{d}x,\ t\in[0,T]. \tag{46}$$ ConserLaw

Indeed, we denote $J(t;s,x)$ is the Jacobian matrix of $X(t;s,x)$ with respect to x at $(t;s,x)$. Then the evolution of determinant for the Jacobian matrix $J(t;s,x)$ is given by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\mathrm{det}J(t;s,x) &= \mathrm{div}_x\left(\frac{{}^{\perp}E}{B} - \sigma\frac{{}^{\perp}\nabla\omega_c}{\omega_c^2(x)}\right)(X(t;s,x))\mathrm{det}J(t;s,x)\\ \mathrm{det}J(s;s,x) &= 1\end{aligned}$$

which is equivalent to

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathrm{det}J(t;s,x) = -\frac{{}^{\perp}E(t,X(t;s,x)\cdot\nabla B(X(t;s,x))}{B^2(X(t;s,x))}\mathrm{det}J(t;s,x). \tag{47}$$ JacobDeter2D

On the other hand, using the equation (44) we deduce that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{dt}}\ln\left|B\left(X\left(t;s,x\right)\right)\right|\\ &= \frac{B(X(t;s,x))}{|B(X(t;s,x))|}\frac{\nabla B\left(X\left(t;,x\right)\right)}{\left|B\left(X\left(t;s,x\right)\right)\right|}\cdot\left(\frac{{}^{\perp}E\left(t,X\left(t;s,x\right)\right)}{B\left(X\left(t;s,x\right)\right)} - \frac{{}^{\perp}\nabla\omega_c\left(X\left(t;s,x\right)\right)}{\omega_c(X\left(t;s,x\right))^2}\right)\\ &= \frac{\nabla B\left(X\left(t;s,x\right)\right)\cdot{}^{\perp}E\left(t,X\left(t;s,x\right)\right)}{B(X\left(t;s,x\right))^2}.\end{aligned} \tag{48}$$ LnBTrajec

Combining (47) and (48) yields

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathrm{det}J(t;s,x) = -\frac{\mathrm{d}}{\mathrm{dt}}\ln\left|B\left(X\left(t;s,x\right)\right)\right|\mathrm{det}J(t;s,x)$$

together with $\mathrm{det}J(s;s,x) = 1$ one gets $\left|B\left(X\left(t;s,x\right)\right)\right|\mathrm{det}J(t;s,x) = |B(x)|$. Therefore, integrating the equality (45) with respect to x and then changing the variable x to $X(t;0,x)$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2}|n(t,x)|\,\mathrm{d}x &= \int_{\mathbb{R}^2}|B\left(x\right)|\,\frac{\left|n_{\mathrm{in}}\left(X\left(0;t,x\right)\right)\right|}{\left|B\left(X\left(0;t,x\right)\right)\right|}\,\mathrm{d}x\\ &= \int_{\mathbb{R}^2}\left|B\left(X\left(t;0,x\right)\right)\right|\frac{n_{\mathrm{in}}\left(x\right)}{|B\left(x\right)|}\mathrm{det}J\left(t;0,x\right)\,\mathrm{d}x\\ &= \int_{\mathbb{R}^2} n_{\mathrm{in}}(x)\,\mathrm{d}x\end{aligned}$$

which completes the proof of the inequality (46).

A priori estimates

We establish here a priori estimates on the solution $n(t,x)$ provided by (45).

The bound in $L^\infty([0,T];W^{1,\infty}(\mathbb{R}^2))$

Lemma 5.1

Let $n(t,x)$ be a solution of (43) given by (45). Then we have

⟨BoInftyLimMod2D⟩

$$\sup_{t\in[0,T]}\|n(t)\|_{L^\infty(\mathbb{R}^2)}\leq C(B,B_0)\|n_{\text{in}}\|_{L^\infty(\mathbb{R}^2)} \tag{49}$$ InftyNorm2D

$$\sup_{t\in[0,T]}\|\nabla n(t)\|_{L^\infty(\mathbb{R}^2)}\leq C(q,m,n_{\text{in}},T,B,B_0)(1+\exp\left(\int_0^t\|E(s,\cdot)\|_{W^{1,\infty}(\mathbb{R}^2)}\mathrm{d}s\right)). \tag{50}$$ InftyNormGrad2

Proof.

The bound (49) is obviouly from the formula (45) and the hypothesis of the magnetic field. For the estimate (50), taking the derivative with respecr to x in (45), we have

$$\begin{aligned}\nabla_x n(t,x) = {}^t\left(\partial_x X\right)(0;t,x)\left[\frac{\nabla n_{\text{in}}\left(X\left(0;t,x\right)\right)}{B\left(X\left(0;t,x\right)\right)}-\frac{n_{\text{in}}\left(X\left(0;t,x\right)\right)\nabla B\left(X\left(0;t,x\right)\right)}{B^2(X\left(0;t,x\right))}\right]B\left(x\right)\\ +\nabla B(x)\frac{n_{\text{in}}\left(X\left(0;t,x\right)\right)}{B\left(X\left(0;t,x\right)\right)}\end{aligned} \tag{51}$$ GradSolChar

which implies that

$$|\nabla n(t,x)|\leq C(n_{\text{in}},B,B_0)(1+|\left(\partial_x X\right)(0;t,x)|) \tag{52}$$ IneqGradSolCha

where $C(n_{\text{in}},B,B_0)$ is the constant depending on n_{in},B,B_0. Then we have to estimate the derivative of $(\partial_x X)(0;t,x)$. Taking the derivative with respect to x in (44), we deduce that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\left(\partial_x X\right)(t) = \frac{\left(\partial_x{}^\perp E\right)\left(t,X\left(t\right)\right)\partial_x X\left(t\right)}{B\left(X\left(t\right)\right)}-\frac{{}^\perp E\left(t,X\left(t\right)\right)\otimes\left(\nabla B\left(X\left(t\right)\right)\partial_x X\left(t\right)\right)}{B(X\left(t\right))^2}\\ +\frac{{}^\perp\nabla\omega_c\left(X\left(t\right)\right)\otimes\left(\nabla\omega_c\left(X\left(t\right)\right)\partial_x X\left(t\right)\right)}{\omega_c(X\left(t\right))^3}-\frac{\left(\partial_x{}^\perp\nabla\omega_c\right)\left(X\left(t\right)\right)\partial_x X\left(t\right)}{\omega_c(X\left(t\right))^2}\end{aligned}$$

and after integrating in time between s and t we find

$$|(\partial_x X)(t)|\leq 1+\int_s^t(\|E(\tau,\cdot)\|_{W^{1,\infty}(\mathbb{R}^2)}+C(q,m,B,B_0))|(\partial_x X)(\tau)|\mathrm{d}\tau$$

where we have written $X(t)$ instead of $X(t;s,x)$ for simplicity, and $C(q,m,B,B_0)$ stands for the constant depending only on q,m,B,B_0. Thanks to Gronwall's inequality we deduce that

$$|(\partial_x X)(t;s,x)|\leq C(q,m,T,B,B_0)\exp\left(\int_s^t\|E(\tau,\cdot)\|_{W^{1,\infty}(\mathbb{R}^2)}\mathrm{d}\tau\right). \tag{53}$$ EstGradCharac

Therefore, substituting (53) into (52) we get

$$|\nabla n(t,x)|\leq C(q,m,n_{\text{in}},T,B,B_0)(1+\exp\left(\int_0^t\|E(s,\cdot)\|_{W^{1,\infty}(\mathbb{R}^2)}\mathrm{d}s\right))$$

which yields the desired estimate. □

The bound in $L^\infty([0,T];W^{1,1}(\mathbb{R}^2))$

Lemma 5.2

⟨BoL1LimMod2D⟩ *Let $n(t,x)$ be a solution of (43) given by (45). Then we have*

$$\|n(t)\|_{L^1(\mathbb{R}^2)} = \|n_{\text{in}}\|_{L^1(\mathbb{R}^2)},\ t\in[0,T] \tag{54}$$ L1Norm2D

$$\sup_{t\in[0,T]}\|\nabla n(t)\|_{L^1(\mathbb{R}^2)} \le C(q,m,T,B,B_0)(1+\|E\|_{L^\infty([0,T];W^{1,\infty}(\mathbb{R}^2))})\|n_{\text{in}}\|_{W^{1,1}(\mathbb{R}^2)}. \tag{55}$$ L1NormGrad2D

Proof.

(54) is clearly. For the estimate (55), taking the absolute value on both sides in (51) then integrating with respect to x and changing the variable x to $X(t;0,x)$, we get

$$\begin{aligned}\int_{\mathbb{R}^2}|\nabla n(t,x)|\,\mathrm{d}x \le \int_{\mathbb{R}^2}|(\partial_x X)(0;t,\cdot)|\left(|\nabla n_{\text{in}}(x)| + \frac{|\nabla B(x)|}{B(x)}n_{\text{in(x)}}\right)\,\mathrm{d}x\\ + \int_{\mathbb{R}^2}\frac{|\nabla B(X(t;0,x))|}{B(X(t;0,x))}n_{\text{in}}(x)\,\mathrm{d}x\end{aligned}$$

which implies that

$$\int_{\mathbb{R}^2}|\nabla n(t,x)|\,\mathrm{d}x \le (\sup_{t\in[0,T]}|\partial_x X(0;t,\cdot)| + C(B,B_0))\|n_{\text{in}}\|_{W^{1,1}(\mathbb{R}^2)}.$$

Using the inequality (53) we get the estimate (55). □

Global existence of smooth solutions

We define the following set of electric vector field

$$\Sigma = \left\{E\in L^\infty\left([0,T];W^{1,\infty}\left(\mathbb{R}^2\right)\right)^2 : \|E(t)\|_{L^\infty_{t,x}} \le M,\ \|\partial_x E\,(t)\|_{L^\infty_x} \le \alpha\,(t)\,,\ t\in[0,T]\right\}$$

where the constant $M>0$ and the function $\alpha(t):[0,T]\to\mathbb{R}_+$ will be determined later. Given an electric field E in Σ. Considering the charecteristic solution of (43) on $\mathbb{R}^2$, corresponding to the electric field E, denoted by n^E which is given by the formula (45). We then construct the following map $\mathcal{F}$ on Σ, whose fixed point gives the solution of the system (43), (41), (42)

$$E\to\mathcal{F}\,(E)\,(x) = -\frac{q}{2\pi\varepsilon_0}\left(\nabla\ln|\cdot|\right)*_x\left(n^E - D\right)(x). \tag{56}$$?MapFixed2D?

We will show that the map $\mathcal{F}$ is left invariant on the set Σ for a convenient choice of the positive constant M and the function $\alpha(t)$, then we want to establish an estimate like

$$\|\mathcal{F}\,(E) - \mathcal{F}(\tilde{E})(t)\|_{L^\infty(\mathbb{R}^2)} \le C_T\int_0^T\|(E-\tilde{E})(t)\|_{L^\infty(\mathbb{R}^2)}\mathrm{d}t,\ \forall t\in[0,T] \tag{57}$$ Mapconstract2D

for some constant C_T, not depending on $E,\tilde{E}$. After that, the existence of the solution of the system (43), (41), (42) immediately, based on the construction of an iterative method for $\mathcal{F}$. Before starting, let us recall the following classical inequality

Lemma 5.3

⟨ClassIneq⟩ *Let $\rho(x)$ be a function which belongs to $L^1(\mathbb{R}^2)\cap W^{1,\infty}(\mathbb{R}^2)$ and let $E(x)$ such that*

$$E(x) = \int_{\mathbb{R}^2}\frac{x-y}{|x-y|^2}\rho(y)\mathrm{d}y.$$

Then we have the following estimates

$$\|E\|_{L^\infty(\mathbb{R}^2)} \le C\|\rho\|^{1/2}_{L^1(\mathbb{R}^2)}\|\rho\|^{1/2}_{L^\infty(\mathbb{R}^2)}, \tag{58}$$ LemmaE

$$\|\nabla_x E\|_{L^\infty(\mathbb{R}^2)} \le C(1+\|\rho\|_{L^\infty(\mathbb{R}^2)}(1+\ln^+\|\nabla_x\rho\|_{L^\infty(\mathbb{R}^2)}) + \|\rho\|_{L^1(\mathbb{R}^2)}) \tag{59}$$ LemmaGradE

here the notation $\ln^+$ stands for the positive part of ln.

Lemma 5.4
There exists a positive constant M and a function $\alpha(t)$ such that $\mathcal{F}(\Sigma) \subset \Sigma$.

?⟨ClosedSet2D⟩?

Proof.
Let $E \in \Sigma$. Thanks to (58), (49) and (54) we have

$$\|\mathcal{F}(E)(t,\cdot)\|_{L^\infty(\mathbb{R}^2)} \leq C(q,\varepsilon_0,B,B_0)\left(\|n_{\rm in}\|_{L^1(\mathbb{R}^2)} + \|D\|_{L^1(\mathbb{R}^2)}\right)^{1/2}\left(\|n_{\rm in}\|_{L^\infty(\mathbb{R}^2)} + \|D\|_{L^\infty(\mathbb{R}^2)}\right)^{1/2}.$$

We choose here the constant M in the set Σ by

$$M = C(q,\varepsilon_0,B,B_0)\left(\|n_{\rm in}\|_{L^1(\mathbb{R}^2)} + \|D\|_{L^1(\mathbb{R}^2)}\right)^{1/2}\left(\|n_{\rm in}\|_{L^\infty(\mathbb{R}^2)} + \|D\|_{L^\infty(\mathbb{R}^2)}\right)^{1/2}$$

hence we have $\sup_{t\in[0,T]}\|\mathcal{F}(E)(t,\cdot)\|_{L^\infty(\mathbb{R}^2)} \leq M$, for any $E \in \Sigma$.
We estimate now $\|\partial\mathcal{F}(E)(t,\cdot)\|_{L^\infty(\mathbb{R}^2)}$. Thanks to (59), (49) and (54) we obtain

$$\|\partial\mathcal{F}(E)(t,\cdot)\|_{L^\infty(\mathbb{R}^2)} \leq C_0(1+\ln^+(\|\nabla n(t)\|_{L^\infty(\mathbb{R}^2)} + \|\nabla D\|_{L^\infty(\mathbb{R}^2)}))$$

where $C_0 = C(n_{\rm in}, D, B, B_0)$ which leads to estimate $\ln^+(\|\nabla n(t)\|_{L^\infty(\mathbb{R}^2)} + \|\nabla D\|_{L^\infty(\mathbb{R}^2)})$. By inequality (50), we have

$$\|\nabla n(t)\|_{L^\infty(\mathbb{R}^2)} \leq C(q,m,n_{\rm in},T,B,B_0)(1+\exp\left(\int_0^t \|\partial_x E(s,\cdot)\|_{L^\infty(\mathbb{R}^2)}\mathrm{d}s\right))$$

which yields

$$\ln^+(\|\nabla n(t)\|_{L^\infty(\mathbb{R}^2)} + \|\nabla D\|_{L^\infty(\mathbb{R}^2)}) \leq C_1(1+\int_0^t \|\partial_x E(s,\cdot)\|_{L^\infty(\mathbb{R}^2)}\mathrm{d}s)$$

where $C_1 = C(q,m,n_{\rm in},D,T,B,B_0)$. Using the standard inequality $1+e^x \leq e^{x+1}$ holds for any $x \geq 0$, we deduce that

$$\ln^+(\|\nabla n(t)\|_{L^\infty(\mathbb{R}^2)} + \|\nabla D\|_{L^\infty(\mathbb{R}^2)}) \leq (\ln^+ C_1 + 1) + \int_0^t \|\partial_x E(s,\cdot)\|_{L^\infty(\mathbb{R}^2)}\mathrm{d}s.$$

Finally, denoting by $C_2 = \ln^+ C_1 + 1$ we have

$$\|\partial\mathcal{F}(E)(t,\cdot)\|_{L^\infty(\mathbb{R}^2)} \leq C_0C_2 + C_0\int_0^t \|\partial_x E(s,\cdot)\|_{L^\infty(\mathbb{R}^2)}\mathrm{d}s.$$

Denote by $\alpha(t)$ the solution on $[0,T]$ of the linear equation $d\alpha/dt = C_0\alpha(t)$ with the initial condition $\alpha(0) = C_0C_2$. We choose here the function $\alpha(t) = C_0C_2e^{C_0t}$ in the set Σ, then we have $\|\partial\mathcal{F}(E)(t,\cdot)\|_{L^\infty(\mathbb{R}^2)} \leq \alpha(t), t \in [0,T]$ for any $E \in \Sigma$. □

Now we will establish the inequality (57). Let us consider $E, \tilde{E} \in \Sigma$ and denote by $n^E, \tilde{n}^{\tilde{E}}$ the characteristics solutions of (43) and (44) corresponding to the electric fields $E, \tilde{E}$ respectively. It is easily seen by (58) that

$$\|\mathcal{F}(E) - \mathcal{F}(\tilde{E})(t)\|_{L^\infty(\mathbb{R}^2)} \leq C_T\int_0^t \|n^E(s) - \tilde{n}^{\tilde{E}}(s)\|^{1/2}_{L^\infty(\mathbb{R}^2)}\|n^E(s) - \tilde{n}^{\tilde{E}}(s)\|^{1/2}_{L^1(\mathbb{R}^2)}\mathrm{d}s \quad (60)$$ MapcontractBis

where C_T is the positive constant, not depending on $E, \tilde{E}$. Then, the inequality (57) is derived from the inequality (60) and the Lemmas 5.5 and 5.6 below.

Lemma 5.5
⟨DiffNormInfty2D⟩ *We have*

$$\|n^E - \tilde{n}^{\tilde{E}}\|_{L^\infty(\mathbb{R}^2)} \leq C_T \int_0^t \|E(s,\cdot) - \tilde{E}(s,\cdot)\|_{L^\infty(\mathbb{R}^2)} \mathrm{d}s$$

for some constant $C_T > 0$, not depending on $E, \tilde{E}$.

Proof.
Let us denote $X^E, X^{\tilde{E}}$ the characteristic solutions of (44) corresponding to $E, \tilde{E}$ respectively. Thanks to the formula (45) we have

$$\begin{aligned}|n^E(t,x) - \tilde{n}^{\tilde{E}}(t,x)| \leq |B(x)| &\frac{n_{\mathrm{in}}(X^E(0;t,x)) - n_{\mathrm{in}}(X^{\tilde{E}}(0;t,x))}{B(X^E(0;t,x))} \\ &+ |B(x)| n_{\mathrm{in}}(X^{\tilde{E}}(0;t,x)) \left| \frac{1}{B(X^E(0;t,x))} - \frac{1}{B(X^{\tilde{E}}(0;t,x))} \right|\end{aligned}$$

which implies that

$$|n^E(t,x) - \tilde{n}^{\tilde{E}}(t,x)| \leq C(n_{\mathrm{in}}, B, B_0) |X^E(0;t,x) - X^{\tilde{E}}(0;t,x)| \tag{61}$$ DiffDensity2D

On the other hand, from the characteristic equation (44) we deduce that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}} \left(X^E - X^{\tilde{E}} \right)(t;s,x) = &\frac{{}^\perp E\left(t, X^E(t;s,x)\right)}{B\left(X^E(t;s,x)\right)} - \frac{{}^\perp \tilde{E}(t, X^{\tilde{E}}(t;s,x))}{B(X^{\tilde{E}}(t;s,x))} \\ &- \sigma \frac{{}^\perp \nabla \omega_c \left(X^E(t;s,x)\right)}{\omega_c^2\left(X^E(t;s,x)\right)} + \sigma \frac{{}^\perp \nabla \omega_c(X^{\tilde{E}}(t;s,x))}{\omega_c^2(X^{\tilde{E}}(t;s,x))}, \\ \left(X^E - X^{\tilde{E}}\right)(s;s,x) = 0.\end{aligned}$$

The first term in the right hand side of the previous equality can be estimated by

$$\begin{aligned}\left| \frac{{}^\perp E\left(t, X^E(t)\right)}{B\left(X^E(t)\right)} - \frac{{}^\perp \tilde{E}(t, X^{\tilde{E}}(t))}{B(X^{\tilde{E}}(t))} \right| &\leq \left| \frac{{}^\perp E(t, X^E(t)) - {}^\perp \tilde{E}(t, X^E(t))}{B(X^E(t))} \right| \\ &+ \left| \frac{{}^\perp \tilde{E}(t, X^E(t)) - {}^\perp \tilde{E}\left(t, X^{\tilde{E}}(t)\right)}{B\left(X^E(t)\right)} \right| \\ &+ \left| {}^\perp \tilde{E}(t, X^{\tilde{E}}(t)) \left(\frac{1}{B(X^E(t))} - \frac{1}{B(X^{\tilde{E}}(t))} \right) \right| \\ &\leq \frac{\|E(t) - \tilde{E}(t)\|_{L^\infty(\mathbb{R}^2)}}{B_0} + C(B, B_0, M) \left| \tilde{X}^E(t) - \tilde{X}^{\tilde{E}}(t) \right|,\end{aligned}$$

since $\tilde{E} \in \Sigma$ while the second term can be bounded by

$$\begin{aligned}\left| \frac{{}^\perp \nabla \omega_c \left(X^E(t)\right)}{\omega_c^2\left(X^E(t)\right)} - \frac{{}^\perp \nabla \omega_c \left(X^{\tilde{E}}(t)\right)}{\omega_c^2\left(X^{\tilde{E}}(t)\right)} \right| &\leq \left| \frac{\nabla \omega_c \left(X^E(t)\right) - \nabla \omega_c \left(X^{\tilde{E}}(t)\right)}{\omega_c^2\left(X^E(t)\right)} \right| \\ &+ \left| \nabla \omega_c \left(X^{\tilde{E}}(t)\right) \left(\frac{1}{\omega_c^2\left(X^E(t)\right)} - \frac{1}{\omega_c^2\left(X^{\tilde{E}}(t)\right)} \right) \right| \\ &\leq \left(\left\| \partial_x^2 \omega_c \right\|_{L^\infty(\mathbb{R}^2)} + \frac{2\|\nabla \omega_c\|_\infty}{\|\omega_c\|_\infty^3} \right) \left| X^E(t) - X^{\tilde{E}}(t) \right|\end{aligned}$$

where we denote $(X^E(t), X^{\tilde{E}}(t) = X^E(t;s,x), X^{\tilde{E}}(t;s,x)$. Integrating between s and t together with previous estimates we find

$$|X_1(t) - X_2(t)| \leq \int_s^t \frac{1}{B_0}\|E_1(\tau) - E_2(\tau)\|_{L^\infty(\mathbb{R}^2)}\mathrm{d}\tau + C(\sigma, B, B_0, M)\int_s^t |X_1(\tau) - X_2(\tau)|\mathrm{d}\tau.$$

Thanks to Gronwall's inequality one gets

$$\left|X^E(t;s,x) - X^{\tilde{E}}(t;s,x)\right| \leq e^{C(\sigma,B,B_0,M)|t-s|}\frac{1}{B_0}\int_s^t \|E_1(\tau) - E_2(\tau)\|_{L^\infty} d\tau$$

which together with (61) yields the desired estimate of the lemma. □

Lemma 5.6

⟨DiffNormL12D⟩ *We have*

$$\|n^E - \tilde{n}^{\tilde{E}}\|_{L^1(\mathbb{R}^2)} \leq C_T\int_0^t \|E(s,\cdot) - \tilde{E}(s,\cdot)\|_{L^1(\mathbb{R}^2)}\mathrm{d}s$$

for some constant $C_T > 0$, not depending on $E, \tilde{E}$.

Proof.
Since $n^E, \tilde{n}^{\tilde{E}}$ are solutions of (43) corresponding to $E, \tilde{E}$ thus we deduce that

$$\partial_t\left(n^E - n^{\tilde{E}}\right) + B\left(\frac{{}^\perp E}{B} - \sigma\frac{{}^\perp\nabla\omega_c}{\omega_c^2}\right)\cdot\nabla_x\left(\frac{n^E - n^{\tilde{E}}}{B}\right) + \left({}^\perp E - {}^\perp\tilde{E}\right)\cdot\nabla_x\left(\frac{n^{\tilde{E}}}{B}\right) = 0,$$
$$\left(n^E - n^{\tilde{E}}\right)(0,x) = 0.$$

Multiplying this equation by $\mathrm{sign}(n^E - n^{\tilde{E}})$ and then integrating with respect to x we find

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathbb{R}^2}\left|n^E(t) - n^{\tilde{E}}(t)\right|\mathrm{d}x + \int_{\mathbb{R}^2} B\left(\frac{{}^\perp E}{B} - \frac{{}^\perp\nabla\omega_c}{\omega_c^2}\right)\cdot\nabla_x\left|\frac{n^E - n^{\tilde{E}}}{B}\right|\mathrm{d}x$$
$$+\int_{\mathbb{R}^2}\mathrm{sign}\left(n^E - n^{\tilde{E}}\right)\left({}^\perp E - {}^\perp\tilde{E}\right)\cdot\nabla_x\left(\frac{n^{\tilde{E}}}{B}\right)\mathrm{d}x = 0. \qquad (62)$$ DeriDiffNormL1

Thanks to Lemma 5.2 we have $n^E, \tilde{n}^{\tilde{E}} \in W^{1,1}(\mathbb{R}^2)$ a.e $t \in [0,T]$ and since $\mathrm{div}_x\left[B\left(\frac{{}^\perp E}{B} - \frac{{}^\perp\nabla\omega_c}{\omega_c^2}\right)\right] = 0$ so by the divergence theorem, we obtain that

$$\int_{\mathbb{R}^2} B\left(\frac{{}^\perp E}{B} - \frac{{}^\perp\nabla\omega_c}{\omega_c^2}\right)\cdot\nabla_x\left|\frac{n^E - n^{\tilde{E}}}{B}\right|\mathrm{d}x = 0.$$

Then, from (62) we imply

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathbb{R}^2}\left|n^E(t) - n^{\tilde{E}}(t)\right|\,\mathrm{d}x \leq C(B,B_0)\|E(t,\cdot) - \tilde{E}(t,\cdot)\|_{L^\infty(\mathbb{R}^2)}\|n^{\tilde{E}}(t,\cdot)\|_{W^{1,1}(\mathbb{R}^2)}.$$

Integrating between 0 and t of this inequality leads to

$$\|n^E(t) - \tilde{n}^{\tilde{E}}(t)\|_{L^1(\mathbb{R}^2)} \leq C(B,B_0)\sup_{t\in[0,T]}\|n^{\tilde{E}}(t,\cdot)\|_{W^{1,1}(\mathbb{R}^2)}\int_0^t \|E(s,\cdot) - \tilde{E}(s,\cdot)\|_{L^\infty(\mathbb{R}^2)}\mathrm{d}s.$$

Finally, by estimate (55) we get

$$\left\|n^E(t) - n^{\tilde{E}}(t)\right\|_{L^1(\mathbb{R}^2)} \leq C_T\int_0^t \|E(s) - \tilde{E}(s)\|_{L^\infty(\mathbb{R}^2)}\mathrm{d}s,\ \forall t \in [0,T]$$

for some constant $C_T > 0$, not depending on $E, \tilde{E}$. □

Now, we shall prove that the sequence of iterative method by map $\mathcal{F}$ converges to a solution of the original problem. First, we consider $E_0 = 0$, then we put $E_1 = \mathcal{F}(E_0), ..., E_{k+1} = \mathcal{F}(E_k)$ for each $k \in \mathbb{N}$. Applying (57) we have

$$\|E_{k+1}(t) - E_k(t)\|_{L^\infty(\mathbb{R}^2)} \leq (C_T)^k \frac{t^k}{k!} \|E_1(t) - E_0(t)\|_{L^\infty(\mathbb{R}^2)}$$

which yields that there exists $E \in L^\infty([0,T]\times\mathbb{R}^2)$ such that E_k tends to E in $L^\infty([0,T]\times\mathbb{R}^2)$. Moreover, since $E_k \in \Sigma$ hence we also have $E \in \Sigma$. This allows us to define the action of the map $\mathcal{F}$ on the vector field E as $\mathcal{F}(E) = -\frac{q}{2\pi\epsilon_0}\nabla \ln|\cdot| * \left(n^E - D\right)$ where n^E is the solution of (43) associated with the electric field E. Using again (57) we find

$$\|E_{k+1}(t) - \mathcal{F}(E)(t)\|_{L^\infty(\mathbb{R}^2)} = \|\mathcal{F}(E_k)(t) - \mathcal{F}(E)(t)\|_{L^\infty(\mathbb{R}^2)} \leq C_T \|E_k(t) - E(t)\|_{L^\infty(\mathbb{R}^2)}$$

which leads to $E_{k+1} \to \mathcal{F}(E)$ in $L^\infty([0,T]\times\mathbb{R}^2)$ as $k \to \infty$. Therefore we get $\mathcal{F}(E) = E$ and n^E is the solution of (40), (41), (42). Moreover, by Lemmas 5.1, 5.2 we conclude that $n^E \in L^\infty([0,T]; W^{1,\infty}(\mathbb{R}^2) \cap W^{1,1}(\mathbb{R}^2))$. Hence, from (43), $\partial_t n^E \in L^\infty([0,T]; L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))$. Thanks to Lemma 5.3, we have $\partial_t E \in L^\infty([0,T]\times\mathbb{R}^2)$, thus $E \in W^{1,\infty}([0,T]\times\mathbb{R}^2)$. It remains to verify that the electric field E lies in $L^\infty([0,T]; L^2(\mathbb{R}^2))$. Applying Lemma 2.8, we need to show that $|x|n \in L^\infty([0,T]; L^1(\mathbb{R}^2))$. Indeed, by (45) and the change of variable $x \mapsto X(t;0,x)$ we have

$$\int_{\mathbb{R}^2} |x||n(t,x)|\,\mathrm{d}x = \int_{\mathbb{R}^2} |X(t;0,x)| n_{\mathrm{in}}(x)\,\mathrm{d}x.$$

On the other hand, from (44) we deduce for any $t \in [0,T]$ that

$$|X(t;0,x)| \leq |x| + C(E, B_0, B)T$$

together with $(1+|x|)n_{\mathrm{in}} \in L^1(\mathbb{R}^2)$ yields the desired result.

Uniqueness of smooth solutions

The uniquenness of smooth solution $n(t,x)$ which belongs to $L^\infty([0,T], W^{1,1}(\mathbb{R}^2) \cap W^{1,\infty}(\mathbb{R}^2))$ is immediately derived from the inequality (57) and Gronwall's inequality.

Based on the previous details of the arguments we establish the following result.

Proposition 5.1

⟨main_sol_Lim⟩ *Let $T > 0$. Let $B \in C_b^2(\mathbb{R}^2)$ be a smooth magnetic field, such that $\inf_{x\in\mathbb{R}^2} B(x) = B_0 > 0$ and the fixed background density D verifies $|x|D \in L^1(\mathbb{R}^2)$, $D \in W^{1,1}(\mathbb{R}^2) \cap W^{1,\infty}(\mathbb{R}^2)$. Assume that the initial condition n_{in} satisfies the hypotheses $H4$, $H5$. There is a unique smooth solution $n(t,x)$ on $[0,T]\times\mathbb{R}^2\times\mathbb{R}^2$ of the limit model (40), (41), (42). The solution satisfies*

$$n \geq 0,\ \int_{\mathbb{R}^2} n(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^2} D(x)\,\mathrm{d}x,\ t \in [0,T]$$

$$n \in W^{1,\infty}([0,T]; L^1(\mathbb{R}^2)) \cap W^{1,\infty}([0,T]\times\mathbb{R}^2),\ |x|n \in L^\infty([0,T]; L^1(\mathbb{R}^2))$$

$$E[n] \in W^{1,\infty}([0,T]\times\mathbb{R}^2),\ E[n] \in L^\infty([0,T]; L^2(\mathbb{R}^2)).$$

Remark 5.1

⟨HighOrder⟩ *From the estimates (50), (59), and (52) we realize that there is a relation in the L^∞-norm between the following quantities*

$$\nabla_x n,\ \partial_x X,\ \partial_x E.$$

In the same way, we can extend this relation to the higher order

$$\partial_x^2 n,\ \partial_x^2 X,\ \partial_x^2 E$$

by noting that the inequality (59) can apply to estimate $\partial_x^2 E$ given by

$$\|\partial_x^2 E\|_{L^\infty(\mathbb{R}^2)} \leq C(1 + \|\nabla n\|_{L^\infty(\mathbb{R}^2)}(1 + \ln^+ \|\partial_x^2 n\|_{L^\infty(\mathbb{R}^2)}) + \|\nabla n\|_{L^1(\mathbb{R}^2)}).$$

By similar arguments we can prove further regularity results for the strong solution of the limit model. The proof is standard and is left to the reader.

Proposition 5.2

⟨Regularity⟩ *Let $T > 0$. Let $B \in C_b^3(\mathbb{R}^2)$ be a smooth magnetic field, such that $\inf_{x\in\mathbb{R}^2} B(x) = B_0 > 0$ and the fixed background density D verifies $|x|D \in L^1(\mathbb{R}^2)$, $D \in W^{1,1}(\mathbb{R}^2) \cap W^{2,\infty}(\mathbb{R}^2)$. Assume that the initial condition $n_{\rm in}$ belongs to $W^{2,\infty}(\mathbb{R}^2) \cap W^{2,1}(\mathbb{R}^2)$ and the background density D lies in $W^{2,\infty}(\mathbb{R}^2) \cap W^{2,1}(\mathbb{R}^2)$. Then the global in time strong solution $(n, E[n])$ constructed in Proposition 5.1 satisfies*

$$\partial_x^2 n \in L^\infty([0,T]; L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)),\ E[n] \in W^{2,\infty}([0,T]\times\mathbb{R}^2)$$
$$\partial_t \nabla_x n \in L^\infty([0,T]\times\mathbb{R}^2),\ \partial_t^2 n \in L^\infty([0,T]\times\mathbb{R}^2).$$

In the rest of this section, we provide some estimates on $\|\ln n\|_{L^\infty([0,T];W^{2,\infty}(\mathbb{R}^2))}$ if we assume that $\ln n_{\rm in}$ belongs to $W^{2,\infty}(\mathbb{R}^2)$. Let us start with the estimate of $\|\ln n\|_{L^\infty([0,T]\times\mathbb{R}^2)}$ in the lemma below.

Lemma 5.7

⟨BoundLoga⟩ *Assume that $\ln n_{\rm in} \in L^\infty([0,T]\times\mathbb{R}^2)$ and $B \in C_b(\mathbb{R}^2)$ with $\inf_{x\in\mathbb{R}^2} B(x) = B_0 > 0$. Then, there exists a constant $C > 0$ depends only on $\|\ln n_{\rm in}\|_{L^\infty([0,T]\times\mathbb{R}^2)}, B, B_0$ and $T > 0$ such that*

$$\sup_{t\in[0,T]} \|\ln n\|_{L^\infty(\mathbb{R}^2)} \le C.$$

Proof.

From the equation (43), we deduce that

$$\partial_t \ln\left(\frac{n}{B}\right) + \left(\frac{^\perp E}{B} - \sigma\frac{^\perp\nabla\omega_c}{\omega_c^2}\right)\cdot\nabla\ln\left(\frac{n}{B}\right) = 0. \tag{63}$$ equ:LogaLimMod

Thanks to the formula of the characteristic solution (45), we get

$$\ln\left(\frac{n}{B}\right)(t,x) = \ln\left(\frac{n}{B}\right)(0, X(0,t,x)) \tag{64}$$ SolLogChar

which gives the estimate in the lemma. □

We next provide higher-order estimates on $\ln n$.

Lemma 5.8

⟨BoundLogHigh⟩ *Assume that $\ln n_{\rm in} \in W^{2,\infty}([0,T]\times\mathbb{R}^2)$ and $B \in C_b^3(\mathbb{R}^2)$ with $\inf_{x\in\mathbb{R}^2} B(x) = B_0 > 0$. Then we have*

$$\sup_{[0,T]} \|\partial_t \ln n\|_{L^\infty(\mathbb{R}^2)} + \sup_{[0,T]} \|\nabla \ln n\|_{L^\infty(\mathbb{R}^2)} \le C_1$$

$$\sup_{[0,T]} \|\nabla^2 \ln n\|_{L^\infty(\mathbb{R}^2)} + \sup_{[0,T]} \|\partial_t\nabla_x \ln n\|_{L^\infty(\mathbb{R}^2)} \le C_2$$

where the constants $C_k > 0$, $k = 1, 2$ depend only on $\ln n_0$, B et B_0.

Proof.

From the equation (64) we have

$$\nabla\ln\left(\frac{n}{B}\right) = (^t\partial_x X)(0;t,x)\,(\nabla\ln n_{\rm in})\,(X(0;t,x)) - (^t\partial_x X)(0;t,x)\left(\frac{\nabla B}{B(x)}\right)(X(0;t,x)). \tag{65}$$ GradSolLogChar

By (53), the derivative in x of $X(0;t,x)$ is bounded in $L^\infty([0,T]\times\mathbb{R}^2)$, thus we get the L^∞ bound for the $\nabla\ln n$. Moreover, from (63) we deduce that $\partial_t \ln n \in L^\infty([0,T]\times\mathbb{R}^2)$, together

with the above discussion gives the first assertion in the lemma.
We next estimate $\|\partial_x^2 \ln n\|_{L^\infty(\mathbb{R}^2)}$. We denote by $\partial_i = \partial_{x_i}$ for $i = 1, 2$. Taking the derivative in x_i in the equation (65) we get

$$\begin{aligned}\partial_i \nabla \ln\left(\frac{n}{B}\right) =& {}^t\left[\partial_i \partial_x X\right](0;t,x)\left(\nabla \ln n_{\text{in}}\right)\left(X(0;t,x)\right)\\ &+ {}^t\partial_x X(0;t,x)\left\{\nabla^2(\ln n_{\text{in}})(X(0;t,x))(\partial_i X)(0;t,x)\right\}\\ &- {}^t\left[\partial_i \partial_x X\right](0;t,x)\left(\frac{\nabla B}{B}\right)\left(X(0;t,x)\right)\\ &+ {}^t\partial_x X(0;t,x)\left\{\partial_x\left(\frac{\nabla B}{B}\right)\left(X(0;t,x)\right)(\partial_i X)(0;t,x)\right\}.\end{aligned}$$

By Remark 5.1 it is well known that $\partial_x^2 X(0;t,x) \in L^\infty([0,T]\times\mathbb{R}^2)$. Hence we obtain the L^∞ bound for $\partial_x^2 \ln n$.
Finally we estimate $\|\partial_t \nabla_x \ln n\|_{L^\infty(\mathbb{R}^2)}$. Taking the time derivative in (65) yields

$$\begin{aligned}\partial_t \nabla\left(\ln\frac{n}{B}\right) =& ({}^t\partial_x(\partial_t X))(0;t,x)\left(\nabla \ln n_{\text{in}}\right)\left(X(0;t,x)\right)\\ &+ ({}^t\partial_x X)(0;t,x)\left(\partial_x^2 \ln n_{\text{in}}\right)\left(X(0;t,x)\right)(\partial_t X)(0;t,x)\\ &- ({}^t\partial_x(\partial_t X))(0;t,x)\left(\frac{\nabla B}{B(x)}\right)\left(X(0;t,x)\right)\\ &- ({}^t\partial_x X)(0;t,x)\left(\partial_x\left(\frac{\nabla B}{B(x)}\right)\right)\left(X(0;t,x)\right)(\partial_t X)(0;t,x).\end{aligned}$$

The L^∞ bounds of $\partial_t X(0;t,x)$ and $\partial_x(\partial_t)X(0;t,x)$ is derived from the equation (44) and the regularity of E and B. Combining two of the above discussion yields the second estimate in the lemma. □

6 Convergence results

We now concentrate on the asymptotic behavior as $\varepsilon \searrow 0$ of the family of weak solutions $(f^\varepsilon, E[f^\varepsilon])_{\varepsilon>0}$ of the Vlasov-Poisson-Fokker-Planck system (4), (5), (6) and we establish rigorously the connection to the fluid model (7), (8), (9). We justify the convergence of the solutions $(n[f^\varepsilon], E[f^\varepsilon])_{\varepsilon>0}$ of the system (39) towards the solution $(n, E[n])$ of the limit problem when ε goes to zero by performing the balance of the relative entropy between n^ε and n. The proof requires some regularity properties of the limit solutions as well as the convergence of the initial data.
Let us recall the modulated energy between n^ε and n

$$\mathcal{E}[n^\varepsilon(t)|n(t)] = \int_{\mathbb{R}^2} \sigma n h\left(\frac{n^\varepsilon}{n}\right)\,\mathrm{d}x + \frac{\varepsilon_0}{2m}\int_{\mathbb{R}^2}|\nabla_x\Phi[n^\varepsilon] - \nabla_x\Phi[n]|^2\,\mathrm{d}x.$$

We intend to estimate the modulated energy $\mathcal{E}[n^\varepsilon(t)|n(t)]$, so we will write as

$$\begin{aligned}\mathcal{E}[n^\varepsilon|n] &= \sigma\int_{\mathbb{R}^2} n h\left(\frac{n^\varepsilon}{n}\right)\,\mathrm{d}x + \frac{\varepsilon_0}{2m}\int_{\mathbb{R}^2}|\nabla_x\Phi[n^\varepsilon] - \nabla_x\Phi[n]|^2\,\mathrm{d}x\\ &= \int_{\mathbb{R}^2}(\sigma n^\varepsilon \ln n^\varepsilon + \frac{\varepsilon_0}{2m}|\nabla_x\Phi[n^\varepsilon]|^2)\,\mathrm{d}x - \int_{\mathbb{R}^2}(\sigma n\ln n + \frac{\varepsilon_0}{2m}|\nabla_x\Phi[n]|^2)\,\mathrm{d}x\\ &- \int_{\mathbb{R}^2}\left\{\sigma(1+\ln n) + \frac{q}{m}\Phi[n]\right\}(n^\varepsilon - n)\,\mathrm{d}x\\ &:= \mathcal{E}[n^\varepsilon] - \mathcal{E}[n] - \int_{\mathbb{R}^2} k[n](n^\varepsilon - n)\,\mathrm{d}x \qquad (66)\end{aligned}$$

equ:EntropyDen

where we have been denoted by $k[n] = \sigma(1+\ln n) + \dfrac{q}{m}\Phi[n]$. We introduce as well the modulated energy of f^ε with respect to $n^\varepsilon M$, given by

$$\begin{aligned}
&\sigma\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} n^\varepsilon M h\left(\frac{f^\varepsilon}{n^\varepsilon M}\right)\,\mathrm{d}v\mathrm{d}x + \frac{\varepsilon_0}{2m}\int_{\mathbb{R}^2}\underbrace{|\nabla_x\Phi[f^\varepsilon]-\nabla_x\Phi[n^\varepsilon M]|^2}_{=0}\,\mathrm{d}x\\
&= \sigma\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f^\varepsilon\ln f^\varepsilon - f^\varepsilon\ln n^\varepsilon + f^\varepsilon\ln(2\pi\sigma) + f^\varepsilon\frac{|v|^2}{2\sigma}\,\mathrm{d}v\mathrm{d}x\\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\sigma f^\varepsilon\ln f^\varepsilon + f^\varepsilon\frac{|v|^2}{2}\,\mathrm{d}v\mathrm{d}x + \frac{\varepsilon_0}{2m}\int_{\mathbb{R}^2}|\nabla_x\Phi[f^\varepsilon]|^2\,\mathrm{d}x\\
&- \int_{\mathbb{R}^2}\sigma n^\varepsilon\ln n^\varepsilon\,\mathrm{d}x - \frac{\varepsilon_0}{2m}\int_{\mathbb{R}^2}|\nabla_x\Phi[n^\varepsilon]|^2\,\mathrm{d}x + \sigma\ln(2\pi\sigma)\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f^\varepsilon\,\mathrm{d}v\mathrm{d}x\\
&= \mathcal{E}[f^\varepsilon] - \mathcal{E}[n^\varepsilon] + \sigma\ln(2\pi\sigma)\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f^\varepsilon\,\mathrm{d}v\mathrm{d}x.
\end{aligned}$$

Thanks to the free energy balance and mass conservation of the equation (4) provided by Proposition 3.1 one gets

$$\begin{aligned}
&\mathcal{E}[n^\varepsilon(t)] - \mathcal{E}[n^\varepsilon(0)] + \sigma\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} n^\varepsilon(t) M h\left(\frac{f^\varepsilon(t)}{n^\varepsilon(t)M}\right)\,\mathrm{d}v\mathrm{d}x\\
&\qquad - \sigma\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} n^\varepsilon(0) M h\left(\frac{f^\varepsilon(0)}{n^\varepsilon(0)M}\right)\,\mathrm{d}v\mathrm{d}x\\
&\qquad = -\frac{1}{\varepsilon}\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|\sigma\nabla_v f^\varepsilon + v f^\varepsilon|^2}{f^\varepsilon}\,\mathrm{d}v\mathrm{d}x\mathrm{d}s.
\end{aligned} \tag{67}$$

equ:BalanEnerD

Thanks to Proposition 4.1 and together with (66), (67) leads to

$$\begin{aligned}
&\mathcal{E}[n^\varepsilon(t)|n(t)] + \sigma\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} n^\varepsilon(t) M h\left(\frac{f^\varepsilon(t)}{n^\varepsilon(t)M}\right)\,\mathrm{d}v\mathrm{d}x + \frac{1}{\varepsilon}\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|\sigma\nabla_v f^\varepsilon + v f^\varepsilon|^2}{f^\varepsilon}\,\mathrm{d}v\mathrm{d}x\mathrm{d}s\\
&= \mathcal{E}[n^\varepsilon(0)|n(0)] + \sigma\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} n^\varepsilon(0) M h\left(\frac{f^\varepsilon(0)}{n^\varepsilon(0)M}\right)\,\mathrm{d}v\mathrm{d}x - \int_0^t\frac{\mathrm{d}}{\mathrm{d}s}\int_{\mathbb{R}^2} k[n](n^\varepsilon - n)\,\mathrm{d}x\mathrm{d}s.
\end{aligned} \tag{68}$$

BalModEnerDens

The next task is to evaluate the time derivative of $-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} k[n](n^\varepsilon - n)\,\mathrm{d}x$. To start establishing, let us rewrite the model (39) for the concentration n^ε as following

$$\partial_t n^\varepsilon + \mathrm{div}_x A[n^\varepsilon] = \mathrm{div}_x F^\varepsilon \tag{69}$$

EquivModCon2D

where the flux $A[n^\varepsilon]$ is defined by $A[n^\varepsilon] = n^\varepsilon\left[\frac{{}^\perp E^\varepsilon}{B(x)} - \sigma\frac{{}^\perp\nabla\omega_c(x)}{\omega_c^2(x)}\right]$. Similarly, the limit model (7) for the limit concentration n can be rewritten as

$$\partial_t n + \mathrm{div}_x A[n] = 0 \tag{70}$$

EquivLimMod2D

with the flux $A[n] = n\left[\frac{{}^\perp E}{B(x)} - \sigma\frac{{}^\perp\nabla\omega_c(x)}{\omega_c^2(x)}\right]$. By direct formal computations, we get

$$\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} k[n](n^\varepsilon - n)\,\mathrm{d}x &= -\int_{\mathbb{R}^2}\left(\sigma\frac{\partial_t n}{n} + \frac{q}{m}\partial_t\Phi[n]\right)(n^\varepsilon - n)\,\mathrm{d}x - \int_{\mathbb{R}^2} k[n](\partial_t n^\varepsilon - \partial_t n)\,\mathrm{d}x\\
&= -\int_{\mathbb{R}^2}\partial_t n\left(\sigma\frac{n^\varepsilon - n}{n} + \frac{q}{m}(\Phi[n^\varepsilon] - \Phi[n])\right)\,\mathrm{d}x\\
&\quad - \int_{\mathbb{R}^2}\nabla_x k[n]\left(A[n^\varepsilon] - A[n] + F^\varepsilon\right)\,\mathrm{d}x.
\end{aligned}$$

We shall establish the previous equality for the weak solution of (69) and the strong solution of (70).

Lemma 6.1

With the notations in (69), (70) *we have the equality*

⟨EvoluFirstTerm⟩

$$
\begin{aligned}
&-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\sigma(1+\ln n)(n^\varepsilon-n)\,\mathrm{d}x\\
&=-\sigma\int_{\mathbb{R}^2}(n^\varepsilon-n)\partial_t\ln n\,\mathrm{d}x-\int_{\mathbb{R}^2}(A[n^\varepsilon]-A[n])\cdot[\sigma\nabla(1+\ln n)]\,\mathrm{d}x\\
&+\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\varepsilon\frac{{}^\perp j^\varepsilon}{\omega_c(x)}\cdot\nabla[\sigma(1+\ln n)]\,\mathrm{d}x-\int_{\mathbb{R}^2}\varepsilon\frac{{}^\perp j^\varepsilon}{\omega_c(x)}\partial_t\nabla[\sigma(1+\ln n)]\,\mathrm{d}x\\
&+\int_{\mathbb{R}^2}\frac{{}^\perp j^\varepsilon}{\omega_c(x)}\cdot\nabla[\sigma(1+\ln n)]\,\mathrm{d}x+\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}(v\otimes v-\sigma I_2)f^\varepsilon\,\mathrm{d}v\right):\partial_x\left[\frac{{}^\perp\nabla[\sigma(1+\ln n)]}{\omega_c(x)}\right]\,\mathrm{d}x.
\end{aligned}
$$

Proof.

From (69), (70), we find $n^\varepsilon-n$ satisfying the following equation in the sense of distribution

$$\partial_t(n^\varepsilon-n)+\mathrm{div}_x\left(A[n^\varepsilon]-A[n]\right)=\mathrm{div}_xF^\varepsilon.$$

Then for any test function $\varphi\in C_0^1\left([0,T[\times\mathbb{R}^2\right)$ we have

$$
\begin{aligned}
&\int_0^T\int_{\mathbb{R}^2}(n^\varepsilon-n)\partial_t\varphi\,\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\mathbb{R}^2}(A[n^\varepsilon]-A[n])\cdot\nabla_x\varphi\,\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\mathbb{R}^2}\varepsilon\frac{{}^\perp j^\varepsilon}{\omega_c(x)}\cdot\partial_t\nabla_x\varphi\,\mathrm{d}x\mathrm{d}t\\
&-\int_0^T\int_{\mathbb{R}^2}\frac{{}^\perp j^\varepsilon}{\omega_c(x)}\cdot\nabla_x\varphi\,\mathrm{d}x\mathrm{d}t-\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(v\otimes v-\sigma I_2)\,f^\varepsilon\,\mathrm{d}v:\partial_x\left(\frac{{}^\perp\nabla_x\varphi}{\omega_c(x)}\right)\,\mathrm{d}x\mathrm{d}t\\
&+\int_{\mathbb{R}^2}\varepsilon\frac{{}^\perp j^\varepsilon_{\mathrm{in}}}{\omega_c(x)}\cdot\nabla_x\varphi(0,x)\,\mathrm{d}x+\int_{\mathbb{R}^2}(n^\varepsilon_{\mathrm{in}}-n_{\mathrm{in}})\varphi(0,x)\,\mathrm{d}x=0.
\end{aligned}
\tag{71}
$$

WeakDiffDensi2

We test $\varphi(t,x)=\theta(t)[\sigma(1+\ln n(t,x))]\chi_R(|x|)$ where $\theta\in C_0^1([0,T[)$, χ was defined in Lemma 3.1. Notice that by the Lemmas 5.7, 5.8, and a standard computations, the following sequences are uniformly bounded with respect to R in $L^\infty([0,T]\times\mathbb{R}^2)$

$$
\begin{aligned}
\partial_t\varphi&=\partial_t\theta(1+\ln n)\chi_R(|x|)+\theta\partial_t\ln n\chi_R(|x|)\\
\nabla_x\varphi&=\theta(t)\nabla\ln n\chi_R(|x|)+\theta(t)(1+\ln n)\chi_R'(|x|)\frac{x}{|x|}\\
\partial_t\nabla_x\varphi&=\partial_t\theta\nabla\ln n\chi_R(|x|)+\theta\partial_t\nabla\ln n\chi_R(|x|)\\
&\quad+\partial_t\theta(1+\ln n)\chi_R'(|x|)\frac{x}{|x|}+\theta(t)\partial_t\ln n\chi_R'(|x|)\frac{x}{|x|}\\
\partial_x(\nabla_x\varphi)&=\theta(t)[\partial_x^2\ln n\chi_R(|x|)+\nabla\ln n\otimes\chi_R'(|x|)\frac{x}{|x|}]\\
&\quad+\theta(t)[\nabla\ln n\chi_R'(|x|)\frac{x}{|x|}+(1+\ln n)(\chi''{}_R(|x|)\frac{x}{|x|}\otimes\frac{x}{|x|}+\chi_R'(|x|)(|x|I_2-x\otimes x)/|x|^3)].
\end{aligned}
$$

On the other hand, for each $\varepsilon>0$, using the properties on the solution *i.e.*, taking into account that $(1+|v|^2)f^\varepsilon\in L^\infty([0,T];L^1(\mathbb{R}^2))$, $E^\varepsilon\in L^\infty([0,T]\times\mathbb{R}^2)$, we can easily apply the dominated convergence as $R\to\infty$. Passing to the limit as $R\to\infty$, we get for any test

function $\theta \in C_0^1([0,T[)$ that

$$\int_0^T \int_{\mathbb{R}^2} (n^\varepsilon - n)\partial_t \theta[\sigma(1+\ln n)]\,\mathrm{d}x\mathrm{d}t + \sigma \int_0^T \int_{\mathbb{R}^2} (n^\varepsilon - n)\theta(t)\partial_t \ln n\,\mathrm{d}x\mathrm{d}t$$
$$+ \int_0^T \int_{\mathbb{R}^2} (A[n^\varepsilon] - A[n]) \cdot \theta(t)\nabla[\sigma(1+\ln n)]\,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{R}^2} \varepsilon \frac{{}^\perp j^\varepsilon(t)}{\omega_c(x)} \cdot \dot{\theta}(t)\nabla[\sigma(1+\ln n)]\,\mathrm{d}x\mathrm{d}t$$
$$+ \int_0^T \int_{\mathbb{R}^2} \varepsilon \frac{{}^\perp j^\varepsilon}{\omega_c(x)} \theta(t)\partial_t \nabla[\sigma(1+\ln n)]\,\mathrm{d}x\mathrm{d}t - \int_0^T \int_{\mathbb{R}^2} \frac{{}^\perp j^\varepsilon}{\omega_c(x)} \cdot \theta(t)\nabla[\sigma(1+\ln n)]\,\mathrm{d}x\mathrm{d}t$$
$$- \int_0^T \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} (v \otimes v - \sigma I_2) f^\varepsilon\,\mathrm{d}v \right) : \partial_x \left[\theta(t) \frac{{}^\perp \nabla[\sigma(1+\ln n)]}{\omega_c(x)} \right] \mathrm{d}x\mathrm{d}t$$
$$+ \int_{\mathbb{R}^2} \varepsilon \frac{{}^\perp j_{\mathrm{in}}^\varepsilon}{\omega_c(x)} \cdot \theta(0)\nabla_x(1+\ln n_{\mathrm{in}})\,\mathrm{d}x + \int_{\mathbb{R}^2} (n_{\mathrm{in}}^\varepsilon - n_{\mathrm{in}})\theta(0)(1+\ln n_{\mathrm{in}})\,\mathrm{d}x = 0$$

which implies the desired equality in the Lemma. □

Lemma 6.2

With the notations in (69), (70) *we have the equality*

⟨EvoluSecdTerm⟩

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} \frac{q}{m}\Phi[n](n^\varepsilon - n)\,\mathrm{d}x = -\int_{\mathbb{R}^2} (n^\varepsilon - n)\cdot\frac{q}{m}\partial_t\Phi[n]\,\mathrm{d}x - \int_{\mathbb{R}^2} (A[n^\varepsilon] - A[n])\cdot\frac{q}{m}\nabla_x\Phi[n]\,\mathrm{d}x$$
$$+\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} \varepsilon\frac{{}^\perp j^\varepsilon}{\omega_c(x)}\cdot\frac{q}{m}\nabla_x\Phi[n]\,\mathrm{d}x - \int_{\mathbb{R}^2}\varepsilon\frac{{}^\perp j^\varepsilon}{\omega_c(x)}\cdot\partial_t\frac{q}{m}\nabla_x\Phi[n]\,\mathrm{d}x + \int_{\mathbb{R}^2}\frac{{}^\perp j^\varepsilon}{\omega_c(x)}\cdot\frac{q}{m}\nabla_x\Phi[n]\,\mathrm{d}x$$
$$+\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}(v\otimes v - \sigma I_2)f^\varepsilon\,\mathrm{d}v\right) : \partial_x\left(\frac{\frac{q}{m}{}^\perp\nabla_x\Phi[n]}{\omega_c(x)}\right)\mathrm{d}x.$$

Proof.

We test $\varphi(t,x) = \frac{q}{m}\theta(t)\Phi[n]\chi_R(|x|)$ in (71). Notice that by Proposition 5.2 we have $E[n] \in W^{2,\infty}([0,T]\times\mathbb{R}^2)$, which proves that $E[n]$ is continuously differential with respect to (t,x). So we have $\Phi[n] \in C^2([0,T]\times\mathbb{R}^2)$. Then we use the same argument as Lemma 6.1 which yields the result of the lemma. □

Now we combine the Lemmas 6.1, 6.2 and futher computations, we get

Proposition 6.1

With the notations in (69), (70), *we have the evolution of the following equality*

⟨TimeEvolution2D⟩

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} k[n](n^\varepsilon - n)\,\mathrm{d}x = \int_{\mathbb{R}^2}\frac{{}^\perp\nabla_x k[n]}{B(x)}\cdot(n^\varepsilon - n)(E^\varepsilon - E[n])\,\mathrm{d}x + K(t,x)$$

where we denote by

$$K(t,x) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\varepsilon\frac{{}^\perp j^\varepsilon}{\omega_c(x)}\cdot\nabla_x k[n]\,\mathrm{d}x - \int_{\mathbb{R}^2}\varepsilon\frac{{}^\perp j^\varepsilon}{\omega_c(x)}\partial_t\nabla_x k[n]\,\mathrm{d}x$$
$$+\int_{\mathbb{R}^2}\frac{{}^\perp j^\varepsilon}{\omega_c(x)}\cdot\nabla_x k[n]\,\mathrm{d}x + \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}(v\otimes v - \sigma I_2)f^\varepsilon\,\mathrm{d}v\right) : \partial_x\left[\frac{{}^\perp\nabla_x k[n]}{\omega_c(x)}\right]\mathrm{d}x.$$

Proof.

First thanks to Lemma 2.8 and Poisson's equation, the first term on the right hand side in the equality of Lemma 6.2 can be written as

$$-\int_{\mathbb{R}^2}(n^\varepsilon - n)\cdot\frac{q}{m}\partial_t\Phi[n]\,\mathrm{d}x = -\int_{\mathbb{R}^2}\frac{q}{m}(\Phi[n^\varepsilon] - \Phi[n])\partial_t n\,\mathrm{d}x.$$

Then we together this equality with the first term on the right hand side in the equation of Lemma 6.1 to obtain

$$
\begin{aligned}
&-\int_{\mathbb{R}^2}\partial_t n\left(\sigma\frac{n^\varepsilon-n}{n}+\frac{q}{m}(\Phi[n^\varepsilon]-\Phi[n])\right)\,\mathrm{d}x\\
&=\int_{\mathbb{R}^2}\mathrm{div}_x A[n]\left(\sigma\frac{n^\varepsilon-n}{n}+\frac{q}{m}(\Phi[n^\varepsilon]-\Phi[n])\right)\,\mathrm{d}x\\
&=\int_{\mathbb{R}^2}\sigma\mathrm{div}_x\left(\frac{{}^\perp E[n]}{B(x)}\right)(n^\varepsilon-n)\,\mathrm{d}x\\
&+\sigma\int_{\mathbb{R}^2}\frac{A[n]}{n}\cdot\nabla\ln n(n^\varepsilon-n)\,\mathrm{d}x+\int_{\mathbb{R}^2}A[n]\cdot\frac{q}{m}(E^\varepsilon-E)\,\mathrm{d}x\\
&=\int_{\mathbb{R}^2}\nabla_x k[n]\cdot\frac{A[n]}{n}(n^\varepsilon-n)\,\mathrm{d}x+\int_{\mathbb{R}^2}A[n]\cdot\frac{q}{m}(E^\varepsilon-E)\,\mathrm{d}x
\end{aligned}
$$

Observer that

$$
\begin{aligned}
A[n^\varepsilon]-A[n]-\frac{A[n]}{n}(n^\varepsilon-n)&=n^\varepsilon\left[\frac{{}^\perp E^\varepsilon}{B(x)}-\sigma\frac{{}^\perp\nabla\omega_c(x)}{\omega_c^2(x)}\right]-n\left[\frac{{}^\perp E[n]}{B(x)}-\sigma\frac{{}^\perp\nabla\omega_c(x)}{\omega_c^2(x)}\right]\\
&\quad-\left[\frac{{}^\perp E[n]}{B(x)}-\sigma\frac{{}^\perp\nabla\omega_c(x)}{\omega_c^2(x)}\right](n^\varepsilon-n)\\
&=n^\varepsilon\frac{{}^\perp(E^\varepsilon-E)}{B(x)}
\end{aligned}
$$

and we can write the divergence of the flux $A[n]$ in (70) as

$$
\mathrm{div}_x A[n]=-\mathrm{div}_x\left(\frac{n}{\omega_c(x)}{}^\perp\nabla_x k[n]\right)
$$

Therefore we get

$$
\begin{aligned}
&-\int_{\mathbb{R}^2}\partial_t n\left(\sigma\frac{n^\varepsilon-n}{n}+\frac{q}{m}(\Phi[n^\varepsilon]-\Phi[n])\right)\,\mathrm{d}x\\
&=-\int_{\mathbb{R}^2}\nabla_x k[n]\cdot n^\varepsilon\frac{{}^\perp(E[n^\varepsilon]-E)}{B(x)}\,\mathrm{d}x+\int_{\mathbb{R}^2}A[n]\cdot\frac{q}{m}(E[n^\varepsilon]-E)\,\mathrm{d}x\\
&=\int_{\mathbb{R}^2}{}^\perp\nabla_x k[n]\cdot n^\varepsilon\frac{(E[n^\varepsilon]-E)}{B(x)}\,\mathrm{d}x-\int_{\mathbb{R}^2}{}^\perp\nabla_x k[n]\cdot n\frac{(E[n^\varepsilon]-E)}{B(x)}\,\mathrm{d}x\\
&=\int_{\mathbb{R}^2}\frac{{}^\perp\nabla_x k[n]}{B(x)}\cdot(n^\varepsilon-n)(E[n^\varepsilon]-E[n])\,\mathrm{d}x.
\end{aligned}
$$

So, Proposition 6.1 is proved. □

Coming back to (68), the modulated energy balance becomes

$$
\begin{aligned}
&\mathcal{E}[n^\varepsilon(t)|n(t)]+\sigma\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}n^\varepsilon(t)Mh\left(\frac{f^\varepsilon(t)}{n^\varepsilon(t)M}\right)\,\mathrm{d}v\mathrm{d}x+\frac{1}{\varepsilon}\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|\sigma\nabla_v f^\varepsilon+vf^\varepsilon|^2}{f^\varepsilon}\,\mathrm{d}v\mathrm{d}x\mathrm{d}s\\
&=\mathcal{E}[n^\varepsilon(0)|n(0)]+\sigma\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}n^\varepsilon(0)Mh\left(\frac{f^\varepsilon(0)}{n^\varepsilon(0)M}\right)\,\mathrm{d}v\mathrm{d}x\\
&+\int_0^t\int_{\mathbb{R}^2}\frac{{}^\perp\nabla_x k[n]}{B(x)}\cdot(n^\varepsilon-n)(E[n^\varepsilon]-E[n])\,\mathrm{d}x\mathrm{d}s+\int_0^t K(s,x)\mathrm{d}s
\end{aligned}
\tag{72}
$$

BalModEnerDens

where

$$\begin{aligned}\int_0^t K(s,x)\mathrm{d}s &= \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\int_{\mathbb{R}^2}\varepsilon\frac{{}^\perp j^\varepsilon}{\omega_c(x)}\cdot\nabla_x k[n]\,\mathrm{d}x\mathrm{d}s\\ &\quad-\int_0^t\int_{\mathbb{R}^2}\varepsilon\frac{{}^\perp j^\varepsilon}{\omega_c(x)}\partial_s\nabla_x k[n(s)]\,\mathrm{d}x\mathrm{d}s+\int_0^t\int_{\mathbb{R}^2}\frac{{}^\perp j^\varepsilon}{\omega_c(x)}\cdot\nabla_x k[n]\,\mathrm{d}x\mathrm{d}s\\ &\quad+\int_0^t\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}(v\otimes v-\sigma I_2)f^\varepsilon\,\mathrm{d}v\right):\partial_x\left[\frac{{}^\perp\nabla_x k[n]}{\omega_c(x)}\right]\,\mathrm{d}x\mathrm{d}s\\ &:= K_1+K_2+K_3+K_4.\end{aligned}$$

In order to apply Gronwall's lemma, we will estimate the integrals in the last line of (72). Thanks to the formula

$$\begin{aligned}(n^\varepsilon-n)(E[n^\varepsilon]-E[n]) &= \frac{\varepsilon_0}{q}[\mathrm{div}_x(E[n^\varepsilon]-E[n])](E[n^\varepsilon]-E[n])\\ &= \frac{\varepsilon_0}{q}\mathrm{div}_x\left((E[n^\varepsilon]-E[n])\otimes(E[n^\varepsilon]-E[n])-\frac{|E[n^\varepsilon]-E[n]|^2}{2}I_2\right)\end{aligned}$$

we obtain

$$\begin{aligned}&\int_{\mathbb{R}^2}\frac{{}^\perp\nabla_x k[n]}{B(x)}\cdot(n^\varepsilon-n)(E[n^\varepsilon]-E[n])\,\mathrm{d}x\\ &=\frac{\varepsilon_0}{q}\int_{\mathbb{R}^2}\left((E[n^\varepsilon]-E[n])\otimes(E[n^\varepsilon]-E[n])-\frac{|E[n^\varepsilon]-E[n]|^2}{2}I_2\right):\partial_x\left(\frac{{}^\perp\nabla_x k[n]}{B(x)}\right)\,\mathrm{d}x\\ &\le\frac{\varepsilon_0}{m}\left\|\partial_x\left(\frac{{}^\perp\nabla_x k[n]}{\omega_c(x)}\right)\right\|_{L^\infty(\mathbb{R}^2)}\left(1+\frac{\sqrt{2}}{2}\right)\int_{\mathbb{R}^2}|E[n^\varepsilon]-E[n]|^2\,\mathrm{d}x\end{aligned}$$

where for any matrix $P\in M_{2,2}(\mathbb{R})$, the notation $\|P\|$ stands for $(P:P)^{1/2}$. Next we shall estmate the integrals K_i, for $i=1,...,4$. For K_1, we have

$$\begin{aligned}K_1 &= \varepsilon\int_{\mathbb{R}^2}\frac{{}^\perp j^\varepsilon(t,x)}{\omega_c(x)}\cdot\nabla_x k[n(t)]\,\mathrm{d}x-\varepsilon\int_{\mathbb{R}^2}\frac{{}^\perp j^\varepsilon(0,x)}{\omega_c(x)}\cdot\nabla_x k[n(0)]\,\mathrm{d}x\\ &\le\sqrt{\varepsilon}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(f^\varepsilon(t,x,v)+f^\varepsilon(0,x,v))\left(\varepsilon\frac{|v|^2}{2}+\frac{\|\nabla k[n]\|_{L^\infty}}{2}\right)\,\mathrm{d}v\mathrm{d}x.\end{aligned}$$

For K_2, an elementary estimate yields

$$K_2\le\frac{m}{qB_0}\|\partial_s\nabla k[n]\|_{L^\infty(\mathbb{R}^2)}\varepsilon\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(\frac{|v|^2}{2}+\frac{1}{2}\right)f^\varepsilon(s,x,v)\,\mathrm{d}v\mathrm{d}x\mathrm{d}s.$$

For K_3, since $j^\varepsilon=\int_{\mathbb{R}^2}(\sigma\nabla_v f^\varepsilon+vf^\varepsilon)\,\mathrm{d}v$ we have

$$\begin{aligned}K_3 &= -\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(\sigma\nabla_v f^\varepsilon+vf^\varepsilon)\cdot\frac{{}^\perp\nabla_x k[n]}{\omega_c(x)}\,\mathrm{d}v\mathrm{d}x\mathrm{d}s\\ &\le\frac{1}{4\varepsilon}\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|\sigma\nabla_v f^\varepsilon+vf^\varepsilon|^2}{f^\varepsilon}\,\mathrm{d}v\mathrm{d}x\mathrm{d}s+\frac{m}{qB_0}\|\nabla k[n]\|_{L^\infty}\varepsilon\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}f^\varepsilon\,\mathrm{d}v\mathrm{d}x\mathrm{d}s.\end{aligned}$$

For K_4, since

$$\int_{\mathbb{R}^d}(v\otimes v-\sigma I_2)f^\varepsilon\,\mathrm{d}v=\int_{\mathbb{R}^d}(vf^\varepsilon+\sigma\nabla_v f^\varepsilon)\otimes v\,\mathrm{d}v=\int_{\mathbb{R}^d}\frac{(vf^\varepsilon+\sigma\nabla_v f^\varepsilon)}{\sqrt{\varepsilon f^\varepsilon}}\otimes v\sqrt{\varepsilon f^\varepsilon}\,\mathrm{d}v$$

we have

$$K_4 \leq \frac{1}{4\varepsilon}\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|\sigma\nabla_v f^\varepsilon + v f^\varepsilon|^2}{f^\varepsilon}\,\mathrm{d}v\mathrm{d}x\mathrm{d}s \\ + \left\|\partial_x\left(\frac{{}^\perp\nabla_x k[n]}{\omega_c(x)}\right)\right\|_{L^\infty(\mathbb{R}^2)}\varepsilon\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|v|^2 f^\varepsilon\,\mathrm{d}v\mathrm{d}x\mathrm{d}s.$$

Plugging the above computations in the equality (72), the modulated energy balance becomes for $0 \leq t \leq T$

$$\mathcal{E}[n^\varepsilon(t)|n(t)] + \sigma\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} n^\varepsilon(t) Mh\left(\frac{f^\varepsilon(t)}{n^\varepsilon(t)M}\right)\,\mathrm{d}v\mathrm{d}x + \frac{1}{4\varepsilon}\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|\sigma\nabla_v f^\varepsilon + v f^\varepsilon|^2}{f^\varepsilon}\,\mathrm{d}v\mathrm{d}x\mathrm{d}s$$
$$= \mathcal{E}[n^\varepsilon(0)|n(0)] + \sigma\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} n^\varepsilon(0) Mh\left(\frac{f^\varepsilon(0)}{n^\varepsilon(0)M}\right)\,\mathrm{d}v\mathrm{d}x$$
$$+ \left\|\partial_x\left(\frac{{}^\perp\nabla_x k[n]}{\omega_c(x)}\right)\right\|_{L^\infty(\mathbb{R}^2)}\left(2+\sqrt{2}\right)\frac{\varepsilon_0}{2m}\int_{\mathbb{R}^2}|E[n^\varepsilon]-E[n]|^2\,\mathrm{d}x$$
$$+ \left(\frac{m}{qB_0}\|\partial_s\nabla k[n]\|_{L^\infty(\mathbb{R}^2)} + \left\|\partial_x\left(\frac{{}^\perp\nabla_x k[n]}{\omega_c(x)}\right)\right\|_{L^\infty(\mathbb{R}^2)}\right)\varepsilon\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|v|^2 f^\varepsilon\,\mathrm{d}v\mathrm{d}x\mathrm{d}t$$
$$+ \sqrt{\varepsilon}\sup_{t\in[0,T]}\varepsilon\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|v|^2 f^\varepsilon\,\mathrm{d}v\mathrm{d}x$$
$$+ \sqrt{\varepsilon}\left(\|\nabla k[n]\|_{L^\infty} + \frac{T}{2}\frac{m}{qB_0}\left(\|\partial_s\nabla k[n]\|_{L^\infty(\mathbb{R}^2)} + 2\|\nabla k[n]\|_{L^\infty(\mathbb{R}^2)}\right)\right)\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f^\varepsilon(0,x,v)\,\mathrm{d}v\mathrm{d}x.$$

Thanks to the Lemma 3.4 and (66) for some constant C_T, $0 \leq t \leq T$, $0 < \varepsilon < 1$ we obtain

$$\mathcal{E}[n^\varepsilon(t)|n(t)] + \sigma\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} n^\varepsilon(t) Mh\left(\frac{f^\varepsilon(t)}{n^\varepsilon(t)M}\right)\,\mathrm{d}v\mathrm{d}x + \frac{1}{4\varepsilon}\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|\sigma\nabla_v f^\varepsilon + v f^\varepsilon|^2}{f^\varepsilon}\,\mathrm{d}v\mathrm{d}x\mathrm{d}s$$
$$\leq \mathcal{E}[n^\varepsilon(0)|n(0)] + \sigma\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} n^\varepsilon(0) Mh\left(\frac{f^\varepsilon(0)}{n^\varepsilon(0)M}\right)\,\mathrm{d}v\mathrm{d}x + C_T\int_0^t\mathcal{E}[n^\varepsilon(s)|n(s)]\mathrm{d}s + C_T\sqrt{\varepsilon}.$$

Applying Gronwall's lemma, we deduce that for $0 \leq t \leq T$, $0 < \varepsilon < 1$

$$\mathcal{E}[n^\varepsilon(t)|n(t)] + \sigma\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} n^\varepsilon(t) Mh\left(\frac{f^\varepsilon(t)}{n^\varepsilon(t)M}\right)\,\mathrm{d}v\mathrm{d}x + \frac{1}{4\varepsilon}\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|\sigma\nabla_v f^\varepsilon + v f^\varepsilon|^2}{f^\varepsilon}\,\mathrm{d}v\mathrm{d}x\mathrm{d}s$$
$$\leq \left[\mathcal{E}[n^\varepsilon(0)|n(0)] + \sigma\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} n^\varepsilon(0) Mh\left(\frac{f^\varepsilon(0)}{n^\varepsilon(0)M}\right)\,\mathrm{d}v\mathrm{d}x + C_T\sqrt{\varepsilon}\right]e^{C_T t}.$$

The above inequality says that the particle density f^ε remains close to the Maxwellian with the same concentration, *i.e.*, $n^\varepsilon(t)M$, and $n^\varepsilon(t)$ stays near $n(t)$, provided that analogous behaviour occur for the initial conditions. Therefore, we are ready to prove our main theorem.

Proof. (of Theorem 1.1)
We justify the convergence of f^ε toward nM in $L^\infty(]0,T[;L^1(\mathbb{R}^2\times\mathbb{R}^2))$, the other convergences being obvious. We use the Csisár -Kullback inequality in order to control the L^1 norm by the relative entropy, cf. [13, 23]

$$\int_{\mathbb{R}^n}|g-g_0|\mathrm{d}x \leq 2\max\left\{\left(\int_{\mathbb{R}^n} g_0\mathrm{d}x\right)^{1/2},\left(\int_{\mathbb{R}^n} g\mathrm{d}x\right)^{1/2}\right\}\left(\int_{\mathbb{R}^n} g_0 h\left(\frac{g}{g_0}\right)\mathrm{d}x\right)^{1/2}$$

for any non negative integrable functions $g_0, g : \mathbb{R}^n \to \mathbb{R}$. Applying two times the Csisár -Kullback inequality we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |f^\varepsilon(t,x,v) - n(t,x)M(v)| \,\mathrm{d}v\mathrm{d}x \\
&\leq \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |f^\varepsilon(t,x,v) - n^\varepsilon(t,x)M(v)| \,\mathrm{d}v\mathrm{d}x + \int_{\mathbb{R}^2} |n^\varepsilon(t,x) - n(t,x)| \,\mathrm{d}x \\
&\leq 2\sqrt{M_{\text{in}}} \left(n^\varepsilon(t) M(v) h\left(\frac{f^\varepsilon(t)}{n^\varepsilon(t)M} \right) \right)^{1/2} \\
&+ 2\max\left\{ \sqrt{M_{\text{in}}}, \sqrt{\|n_{\text{in}}\|_{L^1(\mathbb{R}^2)}} \right\} \left(\int_{\mathbb{R}^2} n(t) h\left(\frac{n^\varepsilon(t)}{n(t)} \right) \,\mathrm{d}x \right)^{1/2} \to 0, \text{ as } \varepsilon \searrow 0.
\end{aligned}$$

□

Appendix

A The linear Vlasov-Fokker-Planck equation with external magnetic field

This appendix is devoted to provide a rigorous proof of the Theorem 2.1. The results on the existence and uniqueness of solutions are deeply inspired by those given by Degond in [11]. We recall the linear VFP system in dimension $d = 2$ with the external magnetic field $B(x)$

$$\begin{aligned}
\partial_t f + v\cdot\nabla_x f + E(x)\cdot\nabla_v f + B(x)^\perp v\cdot\nabla_v f = \mathrm{div}_v(\sigma\nabla_v f + vf),\ (t,x,v) \in [0,T]\times\mathbb{R}^2\times\mathbb{R}^2 \\
f(0,x,v) = f_{\text{in}}(x,v). \qquad (73)
\end{aligned}$$

equ:LiVFPAppen

Let us introduce the Hilbert space

$$\mathcal{H} = L^2([0,T]\times\mathbb{R}^2_x, H^1(\mathbb{R}^2_v)) = \left\{ u \in L^2([0,T]\times\mathbb{R}^2\times\mathbb{R}^2) \,|\, \nabla_v u \in L^2([0,T]\times\mathbb{R}^2\times\mathbb{R}^2) \right\}$$

with norm $\|\cdot\|_{\mathcal{H}}$ and scalar product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ defined by

$$\|u\|^2_{\mathcal{H}} = \int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |u|^2 \,\mathrm{d}v\mathrm{d}x\mathrm{d}t + \int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\nabla_v u|^2 \,\mathrm{d}v\mathrm{d}x\mathrm{d}t,\ u\in\mathcal{H},$$

$$\langle u, w\rangle_{\mathcal{H}} = \int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} uw \,\mathrm{d}v\mathrm{d}x\mathrm{d}t + \int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \nabla_v u\cdot\nabla_v w \,\mathrm{d}v\mathrm{d}x\mathrm{d}t,\ u,w\in\mathcal{H}.$$

We also denote $\mathcal{H}'$ is the dual space of $\mathcal{H}$ which is given by $\mathcal{H}' = L^2([0,T]\times\mathbb{R}^2_x, H^{-1}(\mathbb{R}^2_v))$. The symbole $\langle\cdot,\cdot\rangle_{\mathcal{H}',\mathcal{H}}$ represents the dual relation between $\mathcal{H}$ and its dual.
We first state a result on the existence and uniqueness of a weak solution of equation (73) in an L^2 setting, which can be rewritten in the following form

$$\partial_t f + \mathcal{T} f + E(x)\cdot\nabla_v f - 2f - \sigma\Delta_v f = 0.$$

where $\mathcal{T}$ denotes the transport operator given by $\mathcal{T} = v\cdot\nabla_x + (B(x)^\perp v - v)\cdot\nabla_v$. Then we have the following result

Proposition 6.2

Under the hypothesis of Theorem 2.1, there exists a unique weak solution f of equation (73) *in the class of functions* $\mathbb{Y}$ *defined by*

⟨PropExiUniq⟩

$$\mathbb{Y} = \left\{ u\in\mathcal{H} |\ \frac{\partial u}{\partial t} + \mathcal{T}u \in \mathcal{H}' \right\} \qquad (74)$$

?ClassWeakSol?

and satisfying the initial condition in the sense of distribution.

We first recall the theorem of Lions [24], already used in [11].

Theorem 6.1
⟨LionsThm⟩ *Let E be a Hilbert space, provided with a norm $\|\cdot\|_E$ and scalar product $(,)$. Let $\mathcal{V}$ be a subspace of E with a prehilbertian norm $\|\cdot\|_{\mathcal{V}}$ such that the injection $\mathcal{V} \hookrightarrow \mathcal{H}$ is continuous. We consider a bilinear form $\mathcal{E}$*

$$\begin{aligned} \mathcal{E} : E \times \mathcal{V} &\to \mathbb{R} \\ (u, \phi) &\mapsto \mathcal{E}(u, \phi), \end{aligned}$$

such that $\mathcal{E}(\cdot, \phi)$ is continuous on E, for any fixed $\phi \in \mathcal{V}$, and such that

$$|\mathcal{E}(\phi, \phi)| \geq \alpha \|\phi\|_{\mathcal{V}}^2, \ \phi \in \mathcal{V}, \ \alpha > 0.$$

Then given a linear form L in $\mathcal{V}'$, there exists a solution u in E of problem

$$\mathcal{E}(u, \phi) = L(\phi), \ \text{for any } \phi \in \mathcal{V}.$$

Proof. (of Proposition 6.2)
We follow exactly the proof in [11]. First make the change of unknown function $\tilde{f}(t, x, v) = e^{-(\lambda+d)t} f(t, x, e^{-t}v)$, with any $\lambda > 0$ so that $\tilde{f}$ satisfies the equation

$$\frac{\partial \tilde{f}}{\partial t} + e^{-t} v \cdot \nabla_x \tilde{f} + B(x)^{\perp} v \cdot \nabla_v \tilde{f} + e^t E(x) \cdot \nabla_v \tilde{f} + \lambda \tilde{f} - \sigma e^{2t} \Delta_v \tilde{f} = 0,$$
$$\tilde{f}(0, x, v) = \tilde{f}_{\text{in}}(x, v) = f_{\text{in}}(x, v). \qquad (75)$$ equ:NewVFP2D

Now, let E be equal to the space $\mathcal{H}$ and let $\mathcal{V}$ be the space $C_0^{\infty}([0, T) \times \mathbb{R}^2 \times \mathbb{R}^2)$. $\mathcal{V}$ is equipped with a prehilbertian norm defined by

$$\|\phi\|_{\mathcal{V}}^2 = \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |\phi(0, x, v)|^2 \, \mathrm{d}v\mathrm{d}x + \|\phi\|_{\mathcal{H}}^2, \ \phi \in \mathcal{V}.$$

A weak solution of equation (75) in the distribution sense is a function $\tilde{f} \in \mathcal{H}$ such that

$$\int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \tilde{f} \left(-\partial_t \phi - e^{-t} v \cdot \nabla_x \phi - B(x)^{\perp} v \cdot \nabla_v \phi + \lambda \phi \right) \, \mathrm{d}v\mathrm{d}x\mathrm{d}t$$
$$+ \int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \nabla_v \tilde{f} \cdot \left(e^t E(x) \phi + \sigma e^{2t} \nabla_v \phi \right) \, \mathrm{d}v\mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \tilde{f}_{\text{in}}(x, v) \phi(0, x, v) \, \mathrm{d}v\mathrm{d}x \qquad (76)$$ equ:WeakFormNV

for any $\phi \in \mathcal{V}$. We consider the following bilinear form $\mathcal{E}$ as the left-hand side of the variational equation (76) defined by

$$\begin{aligned} \mathcal{E}(\tilde{f}, \phi) = &\int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \tilde{f} \left(-\partial_t \phi - e^{-t} v \cdot \nabla_x \phi - B(x)^{\perp} v \cdot \nabla_v \phi + \lambda \phi \right) \, \mathrm{d}v\mathrm{d}x\mathrm{d}t \\ &+ \int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \nabla_v \tilde{u} \cdot \left(e^t E(x) \phi + \sigma e^{2t} \nabla_v \phi \right) \, \mathrm{d}v\mathrm{d}x\mathrm{d}t \end{aligned}$$

and the linear form

$$L(\phi) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \tilde{f}_{\text{in}}(x, v) \phi(0, x, v) \, \mathrm{d}v\mathrm{d}x.$$

Now, let us check $\mathcal{E}$ satisfies the properties stated in Theorem 6.2. It is easily seen that $\mathcal{E}(\cdot, \phi)$ est continue sur $\mathcal{H}$ since $E \in (L^{\infty}(\mathbb{R}^2))^2$. It remains to show that $\mathcal{E}$ is coercivity on $\mathcal{V} \times \mathcal{V}$.

Indeed, for any $\phi \in \mathcal{V}$ we have

$$\begin{aligned}
\mathcal{E}(\phi,\phi) &= \int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \phi\left(-\partial_t\phi - e^{-t}v\cdot\nabla_x\phi - B(x)^\perp v\cdot\nabla_v\phi + \lambda\phi\right)\,\mathrm{d}v\mathrm{d}x\mathrm{d}t \\
&+ \int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \nabla_v\phi\cdot\left(e^t E(x)\phi + \sigma e^{2t}\nabla_v\phi\right)\,\mathrm{d}v\mathrm{d}x\mathrm{d}t \\
&= \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\phi(0,x,v)|^2\,\mathrm{d}v\mathrm{d}x + \lambda\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|\phi|^2\,\mathrm{d}v\mathrm{d}x\mathrm{d}t \\
&+ \sigma\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{2t}|\nabla_v\phi|^2\,\mathrm{d}v\mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^t\nabla_v|\phi|^2\cdot E(x)\,\mathrm{d}v\mathrm{d}x\mathrm{d}t \\
&= \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\phi(0,x,v|^2\,\mathrm{d}v\mathrm{d}x + \sigma\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{2t}|\nabla_v\phi|^2\,\mathrm{d}v\mathrm{d}x\mathrm{d}t \\
&+ \lambda\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|\phi|^2\,\mathrm{d}v\mathrm{d}x\mathrm{d}t \geq \min(1,\sigma,\lambda)\,\|\phi^2\|_{\mathcal{V}}.
\end{aligned}$$

Then Lion's Theorem 6.1 applies and we get that variational equation $\mathcal{E}(\tilde{f},\phi) = L(\phi)$, for any $\phi\in\mathcal{V}$ admits a solution $\tilde{f}\in\mathcal{H}$. Moreover, $\tilde{f}$ satisfies the equation (76) for any $\phi\in\mathcal{V}$, hence by using the function test $\tilde{\phi} = e^{(\lambda+d)t}\phi(t,x,e^t v)$ we deduce that $f(t,x,v) = e^{(\lambda+2)t}\tilde{f}(t,x,e^t v)$ is a weak solution of (73) in the sense of distribution. This gives that

$$\frac{\partial f}{\partial t} + \mathcal{T}f = -E(x)\cdot\nabla_v f + 2f + \sigma\Delta_v f \in \mathcal{H}'$$

so that f belongs to $\mathbb{Y}$.
We shall call the following Lemma to give a meaning to the initial condition, and also, to show the uniqueness. The proof is very close to the one of Lemma A.1 in [11] and we have been left behind.

Lemma 6.3

⟨GreenFormulas⟩ *1. For $u\in\mathbb{Y}$, u admits continuous trace values $u(0,x,v)$ and $u(T,x,v)$ in $L^2(\mathbb{R}^d\times\mathbb{R}^d)$. This means that the linear map $u\to(u(0,\cdot,\cdot),u(T,\cdot,\cdot))$ is continuous from $\mathbb{Y}$ to $L^2(\mathbb{R}^2\times\mathbb{R}^2)$.*
2. For f and $\tilde{f}$ in $\mathbb{Y}$ we have

$$\begin{aligned}
&\left\langle \partial_t f + \mathcal{T}f, \tilde{f}\right\rangle_{\mathcal{H}'\times\mathcal{H}} + \left\langle \partial_t\tilde{f} + \mathcal{T}\tilde{f}, f\right\rangle_{\mathcal{H}'\times\mathcal{H}} = 2\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f\tilde{f}\,\mathrm{d}v\mathrm{d}x\mathrm{d}t \\
&+ \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(T,x,v)\tilde{f}(T,x,v)\,\mathrm{d}v\mathrm{d}x - \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(0,x,v)\tilde{f}(0,x,v)\,\mathrm{d}v\mathrm{d}x \qquad (77)\ \boxed{\texttt{InteGreen}}
\end{aligned}$$

where $\mathcal{T} = v\cdot\nabla_x + (B(x)^\perp v - v)\cdot\nabla_v$.
3. Similary, for f and $\tilde{f}$ in $\mathbb{Y}$ we have

$$\begin{aligned}
&\left\langle \partial_t f + \mathcal{T}'f, \tilde{f}\right\rangle_{\mathcal{H}'\times\mathcal{H}} + \left\langle \partial_t\tilde{f} + \mathcal{T}'\tilde{f}, f\right\rangle_{\mathcal{H}'\times\mathcal{H}} \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(T,x,v)\tilde{f}(T,x,v)\,\mathrm{d}v\mathrm{d}x - \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(0,x,v)\tilde{f}(0,x,v)\,\mathrm{d}v\mathrm{d}x \qquad (78)\ \boxed{\texttt{InteGreenBis}}
\end{aligned}$$

where $\mathcal{T}' = e^{-t}v\cdot\nabla_x + B(x)^\perp v\cdot\nabla_v$.

Let us now end the proof of Proposition 6.2. Using formula (77) to the solution f of equation (73) and test function ϕ in $\mathcal{V}$ we have

$$\begin{aligned}
&\langle\partial_t f + \mathcal{T}f, \phi\rangle_{\mathcal{H}'\times\mathcal{H}} + \langle\partial_t\phi + \mathcal{T}\phi, f\rangle_{\mathcal{H}'\times\mathcal{H}} \qquad (79)\ \boxed{\texttt{GreenBis1}} \\
&\qquad = 2\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f\phi\,\mathrm{d}v\mathrm{d}x\mathrm{d}t - \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(0,x,v)\phi(0,x,v)\,\mathrm{d}v\mathrm{d}x.
\end{aligned}$$

As f is a solution of (73) in $\mathcal{H}'$ then we get

$$\begin{aligned}\langle \partial_t f + \mathcal{T} f, \phi\rangle_{\mathcal{H}'\times\mathcal{H}} &= \langle -E(x)\cdot\nabla_v f\phi - (\lambda-2)f + \sigma\Delta_v f, \phi\rangle_{\mathcal{H}'\times\mathcal{H}} \\ &= -\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(E(x)\cdot\nabla_v f\phi + (\lambda-2)f\phi + \sigma\nabla_v f\cdot\nabla_v\phi)\,\mathrm{d}v\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Futhermore, f satisfies the variational equality $\mathcal{E}(f,\phi) = L(\phi)$ thus

$$\begin{aligned}\langle \partial_t\phi + \mathcal{T}\phi, f\rangle_{\mathcal{H}'\times\mathcal{H}} = &\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\lambda f\phi + \nabla_v f\cdot(E(x)\phi + \sigma\nabla_v\phi)\,\mathrm{d}v\mathrm{d}x\mathrm{d}t \\ &- \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f_{\mathrm{in}}(x,v)\phi(0,x,v)\,\mathrm{d}v\mathrm{d}x.\end{aligned}$$

Substituting into (79) which yields

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(f(0,x,v) - f_{\mathrm{in}}(x,v))\phi(0,x,v)\,\mathrm{d}v\mathrm{d}x = 0,\ \forall\phi\in\mathcal{V}.$$

Therefore, the initial condition is satisfied in $L^2(\mathbb{R}^2)$. Now for uniqueness, we assume that f is a solution of (73) with $f_{\mathrm{in}} = 0$, which belongs to $\mathbb{Y}$. Proceeding as in Proposition 6.2, we define the function $\tilde{f}$ as $\tilde{f}(t,x,v) = e^{-(\lambda+d)t}f(t,x,e^{-t}v)$ which verifies equation (75) with zero initial data. We apply the formula (78) to the solution $\tilde{f}$ of equation (75) which gives

$$\begin{aligned}0 &= \left\langle \partial_t\tilde{f} + \mathcal{T}'\tilde{f}, \tilde{f}\right\rangle_{\mathcal{H}'\times\mathcal{H}} + \left\langle e^t E(x)\cdot\nabla_v\tilde{f} + \lambda\tilde{f} - \sigma e^{2t}\Delta_v\tilde{f}, \tilde{f}\right\rangle_{\mathcal{H}'\times\mathcal{H}} \\ &= \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|\tilde{f}(T,x,v)|^2\,\mathrm{d}v\mathrm{d}x + \lambda\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|\tilde{f}|^2\,\mathrm{d}v\mathrm{d}x\mathrm{d}t + \sigma\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}e^{2t}|\nabla_v\tilde{f}|^2\,\mathrm{d}v\mathrm{d}x\mathrm{d}t \\ &\geq \lambda\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|\tilde{f}|^2\,\mathrm{d}v\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Therefore we get $\tilde{f} = 0$, which proves uniqueness. □

Proof. (of Lemma 6.3)
Let us consider set Y of C^∞ functions of (x,t) in $[0,T]\times\mathbb{R}^2_x$ with values in $H^1(\mathbb{R}^2_v)$ which are compactly supported in $[0,T]\times\mathbb{R}^2\times\mathbb{R}^2$. Following the arguments in Lemma $A.1$ in [11], we have that the set Y is dense on $\mathbb{Y}$.
Let us take $u\in Y$. Using a partition of unity we can assume, without of loss of generality, that u vanishes on $\{(0,x,v) : (x,v)\in\mathbb{R}^2\times\mathbb{R}^2\}$ or $\{(T,x,v) : (x,v)\in\mathbb{R}^2\times\mathbb{R}^2\}$. Assume that u does not vanish on $\{(0,x,v) : (x,v)\in\mathbb{R}^2\times\mathbb{R}^2\}$. By Green's identity we have

$$\begin{aligned}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|u(0,x,v)|^2\,\mathrm{d}v\mathrm{d}x = &-2\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}u\left[\partial_t + v\cdot\nabla_x + (B(x)^\perp v - v)\cdot\nabla_v\right]u\,\mathrm{d}v\mathrm{d}x\mathrm{d}t \\ &+ 2\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|u|^2\,\mathrm{d}v\mathrm{d}x\mathrm{d}t \\ &\leq 2\left(\left\|\left[\partial_t + v\cdot\nabla_x + (B(x)^\perp v - v)\cdot\nabla_v\right]u\right\|_{\mathcal{H}'} + 2\right)\|u\|_{\mathcal{H}} \leq C\|u\|_{\mathbb{Y}}.\end{aligned}$$

The rest of the lemma follows from straightforward arguments involving the density of Y in $\mathbb{Y}$. □

The following Proposition is devoted to a maximum principle and an L^∞ estimate.

Proposition 6.3

⟨NormInfty2D⟩ *Assume that the initial condition $f_{\rm in}$ is positive and belongs to $L^\infty(\mathbb{R}^2\times\mathbb{R}^2)$. Then the solution f provided by Proposition 6.2 is positive and satisfying*

$$\sup_{[0,T]}\|f(t)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}\leq e^{2T}\|f_{\rm in}\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}.$$

We start by giving the following Lemmas. The proof of these Lemmas are very close to those given by in [11]. We leave it to the reader.

Lemma 6.4

⟨LemNormInftyBis1⟩ *Let $f\in\mathbb{Y}$ then f^+ and f^- defined by $f^+=\max(f,0)\quad f^-=\max(-f,0)$ belong to $\mathcal{H}$ and $\nabla_v f^+=\dfrac{1+\mathrm{sign}(f)}{2}\nabla_v f,\quad \nabla_v f^-=\dfrac{-1+\mathrm{sign}(f)}{2}\nabla_v f$. Futhermore, we have*

$$\begin{aligned}&\left\langle\partial_t f+\mathcal{T}'f,f^-\right\rangle_{\mathcal{H}'\times\mathcal{H}}\\&=\frac12\left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}f(T,x,v)f^-(T,x,v)\,\mathrm{d}v\mathrm{d}x-\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}f(0,x,v)f^-(0,x,v)\,\mathrm{d}v\mathrm{d}x\right)\end{aligned}\tag{80}$$

IntMaxMin1

where $\mathcal{T}'=e^{-t}v\cdot\nabla_x+B(x)^\perp v\cdot\nabla_v$. Similarly, we also have

$$\begin{aligned}&\left\langle\partial_t f+\mathcal{T}f,f^-\right\rangle_{\mathcal{H}'\times\mathcal{H}}=\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}ff^-\,\mathrm{d}v\mathrm{d}x\mathrm{d}t\\&+\frac12\left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}f(T,x,v)f^-(T,x,v)\,\mathrm{d}v\mathrm{d}x-\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}f(0,x,v)f^-(0,x,v)\,\mathrm{d}v\mathrm{d}x\right)\end{aligned}\tag{81}$$

IntMaxMin2

où $\mathcal{T}=v\cdot\nabla_x+(B(x)^\perp v-v)\cdot\nabla_v$.

Lemma 6.5

⟨LemNormInftyBis2⟩ *Let $\mathbf{V}\subset\mathbf{H}\subset\mathbf{V}'$ be a canonical triple of Hilbert spaces. We suppose that the mapping $u\to u^-$ is a contraction on $\mathbf{V}$. Let u belong to $L^2([0,T];\mathbf{V})\cap C^0([0,T];\mathbf{H})$ such that $\frac{du}{dt}\in L^2([0,T];\mathbf{V}')$. Then*

$$\int_0^T\left\langle\frac{du}{dt},u^-\right\rangle_{\mathbf{V}'\times\mathbf{V}}\mathrm{d}t=\frac12\left(|u^-(0)|^2_{\mathbf{H}}-|u^-(T)|^2_{\mathbf{H}}\right).\tag{82}$$

?IntTime?

Proof. (of Proposition 6.3)
We will now show that $f\geq0$ a.e. As above, we define $\tilde f=e^{-(\lambda+2)t}f(t,x,e^{-t}v)$ with any $\lambda>0$ which solves (75) with the initial data $f_{\rm in}$. It is well known that $\tilde f\in\mathbb{Y}$ since $f\in\mathbb{Y}$ and thus $\partial_t\tilde f+\mathcal{T}'\tilde f\in\mathcal{H}'$. Thanks to Lemma 6.4 we have $\tilde f^-\in\mathcal{H}$ which implies from (75) that

$$\left\langle\partial_t\tilde f+\mathcal{T}'\tilde f,\tilde f^-\right\rangle_{\mathcal{H}'\times\mathcal{H}}+\left\langle e^tE(x)\cdot\nabla_v\tilde f+\lambda\tilde f-\sigma e^{2t}\Delta_v\tilde f,\tilde f^-\right\rangle_{\mathcal{H}'\times\mathcal{H}}=0.$$

Then we apply the formula (80) for the function $\tilde f$ to compute $\left\langle\partial_t\tilde f+\mathcal{T}'\tilde f,\tilde f^-\right\rangle_{\mathcal{H}'\times\mathcal{H}}$. Therefore we obtain

$$\begin{aligned}&\left\langle e^tE(x)\cdot\nabla_v\tilde f+\lambda\tilde f-\sigma e^{2t}\Delta_v\tilde f,\tilde f^-\right\rangle_{\mathcal{H}'\times\mathcal{H}}\\&=-\frac12\left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\tilde f(T,x,v)\tilde f^-(T,x,v)\,\mathrm{d}v\mathrm{d}x-\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\tilde f(0,x,v)\tilde f^-(0,x,v)\,\mathrm{d}v\mathrm{d}x\right)\leq0\end{aligned}$$

since $\tilde{f}^-(0,x,v) = f^-(0,x,v) = 0$. Moreover $\tilde{f} = \tilde{f}^+ - \tilde{f}^-$ and $\tilde{f} \in \mathcal{H}$ we have $\left\langle \lambda \tilde{f}, \tilde{f}^- \right\rangle_{\mathcal{H}'\times\mathcal{H}} = -\lambda \left\langle \tilde{f}^-, \tilde{f}^- \right\rangle_{L^2\times L^2}$. Thanks to Lemma 6.5 we deduce that

$$\left\langle -\sigma e^{2t}\Delta_v \tilde{f}, \tilde{f}^- \right\rangle_{\mathcal{H}'\times\mathcal{H}} = -\sigma \left\langle e^{2t}\nabla_v \tilde{f}^-, \nabla_v \tilde{f}^- \right\rangle_{L^2\times L^2} \leq 0,$$

and $\left\langle E(x)\cdot\nabla_v \tilde{f}, \tilde{f}^- \right\rangle_{\mathcal{H}'\times\mathcal{H}} = -\left\langle E(x)\cdot\nabla_v \tilde{f}^-, \tilde{f}^- \right\rangle_{L^2\times L^2} = 0$. Therefore, we get $0 \leq -\lambda \left\langle \tilde{f}^-, \tilde{f}^- \right\rangle_{L^2\times L^2}$ which implies that $\tilde{f}^- = 0$ a.e and $\tilde{f} \geq 0$ a.e so $f \geq 0$ a.e.
Now we estimate the bound of L^∞ norm. First, making the change of unknown function $w(t,x,v) = e^{-2t} f(t,x,v)$ in the equation (73) we get

$$\begin{cases} \frac{\partial w}{\partial t} + \left[v\cdot\nabla_x w + (B(x)^\perp v - v)\cdot\nabla_v w\right] + E(x)\cdot\nabla_v w - \sigma\Delta_v w = 0, \\ w_0(x,v) = f_{\text{in}}(x,v). \end{cases}$$

We will prove that $\|w(t)\|_{L^\infty} \leq \|w_0\|_{L^\infty}$. Putting $w_1(t,x,v) = K(w(t,x,v) - \|w_0\|_{L^\infty})$ where K is a function of class C^2 satisfying

$$K(s) = 0,\ s \leq 0,\ K \text{ is increasing},$$
$$\|K'\|_{L^\infty} \leq C,\ \ K'' \geq 0.$$

We give an example on the function K as $K(y) = \int_0^y g(s)\mathrm{d}s$ with $g(s) = e^{-\frac{1}{s}}$ if $s > 0$ and $f(s) = 0$ if $s \leq 0$. By the construction of K and $w \in \mathbb{Y}$ we deduce that $w_1 \in \mathcal{H}$ and $\partial_t w_1 + \mathcal{T} w_1 = K'(w(t) - \|w_0\|_\infty)(\partial_t w + \mathcal{T} w) \in \mathcal{H}'$. Multiplying the equation for w above by $K'(w(t,x,v) - \|w_0\|_{L^\infty})$ then w_1 belongs to $\mathbb{Y}$ and satisfies the following equation

$$\begin{cases} \partial_t w_1 + \mathcal{T} w_1 + E(x)\cdot\nabla_v w_1 - \sigma\Delta_v w_1 + \sigma|\nabla_v w|^2 K''(w - \|w_0\|_{L^\infty}) = 0, \\ w_1(0) = K(w(0,x,v) - \|w_0\|_{L^\infty}) = 0. \end{cases}$$

We then put $w_2(t,x,v) = e^{-\beta t} w_1(t,x,v)$, with any $\beta > 0$. The function w_2 belongs to $\mathbb{Y}$ and satifies the equation

$$\begin{cases} \partial_t w_2 + \mathcal{T} w_2 + E(x)\cdot\nabla_v w_2 + \beta w_2 - \sigma\Delta_v w_2 + e^{-\beta t}\sigma|\nabla_v w|^2 K''(w - \|w_0\|_{L^\infty}) = 0, \\ w_2(0) = 0. \end{cases}$$

Therefore, w_2 satisfies the variational equation

$$\begin{aligned} &\left\langle \partial_t w_2 + \mathcal{T} w_2, w_2^+ \right\rangle_{\mathcal{H}'\times\mathcal{H}} \\ &+ \left\langle E(x)\cdot\nabla_v w_2 + \beta w_2 - \sigma\Delta_v w_2 + e^{-\beta t}\sigma|\nabla_v w|^2 K''(w - \|w_0\|_{L^\infty}), w_2^+ \right\rangle_{\mathcal{H}'\times\mathcal{H}} = 0. \end{aligned}$$

Using (81) we have

$$\begin{aligned} &\left\langle \partial_t w_2 + \mathcal{T} w_2, w_2^+ \right\rangle_{\mathcal{H}'\times\mathcal{H}} = \int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} w_2 w_2^+ \,\mathrm{d}v\mathrm{d}x\mathrm{d}t \\ &+ \frac{1}{2}\left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} w_2(T,x,v) w_2^+(T,x,v)\,\mathrm{d}v\mathrm{d}x - \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} w_2(0,x,v) w_2^+(0,x,v)\,\mathrm{d}v\mathrm{d}x\right) \\ &= \frac{1}{2}\left(2\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |w_2^+|^2\,\mathrm{d}v\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |w_2^+(T,x,v)|^2\,\mathrm{d}v\mathrm{d}x - \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |w_2^+(0,x,v)|^2\,\mathrm{d}v\mathrm{d}x\right) \\ &\geq \int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |w_2^+|^2\,\mathrm{d}v\mathrm{d}x\mathrm{d}t \end{aligned}$$

car $w_2^+(0,x,v) = w_2(0) = 0$. For the other terms in the privious expression,

$$\left\langle \beta w_2, w_2^+ \right\rangle_{\mathcal{H}'\times\mathcal{H}} = \beta \left\langle w_2^+, w_2^+ \right\rangle_{L^2\times L^2}, \quad \left\langle -\sigma\Delta_v w_2, w_2^+ \right\rangle_{\mathcal{H}'\times\mathcal{H}} = \sigma \left\langle \nabla_v w_2^+, \nabla_v w_2^+ \right\rangle_{L^2\times L^2},$$

and $\left\langle E(x)\cdot\nabla_v w_2, w_2^+ \right\rangle_{\mathcal{H}'\times\mathcal{H}} = \left\langle E(x)\cdot\nabla_v w_2^+, w_2^+ \right\rangle_{L^2\times L^2} = 0$. Therefore we deduce that

$$\int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |w_2^+|^2 \,\mathrm{d}v\mathrm{d}x\mathrm{d}t + \beta \left\langle w_2^+, w_2^+ \right\rangle_{L^2\times L^2} + \sigma \left\langle \nabla_v w_2^+, \nabla_v w_2^+ \right\rangle_{L^2\times L^2} \le 0$$

This implies that $w_2^+ = 0$. Thus $w_2 \le 0$ and $w_1 \le 0$ which yields $\|w(t)\|_{L^\infty} \le \|w_0\|_{L^\infty}$. □

Remark 6.1

⟨RemarkInfty⟩ *If we add the source term $U(t,x,v)$ in the right hand side of* (73)*, that means*

$$\frac{\partial f}{\partial t} + v\cdot\nabla_x f + (B(x)^\perp v - v)\cdot\nabla_v f + E(x)\cdot\nabla_v f - 2f - \sigma\Delta_v f = U, f(0,x,v) = f_{\mathrm{in}}(x,v)$$

and we assume that $U \in L^1([0,T]; L^\infty(\mathbb{R}^2\times\mathbb{R}^2))$. Then we have

$$\|f(t)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \le e^{2T}\|f_{\mathrm{in}}\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} + \int_0^T \|U(s)\|_{L^\infty}\mathrm{d}s.$$

The following estimates relate to the L^p estimate, the kinetic energy and the entropy of equation VFP (73). To establish these estimates, we make the change of unknown function $w(t,x,v) = e^{-2t}f(t,x,e^{-t}v)$. Then w is the solution of the following equation

$$\begin{cases} \dfrac{\partial w}{\partial t} + e^{-t}v\cdot\nabla_x w + B(x)^\perp v\cdot\nabla_v w + e^t E(x)\cdot\nabla_v w - \sigma e^{2t}\Delta_v w = 0 \\ w_0(x,v) = f_{\mathrm{in}}(x,v). \end{cases} \tag{83}$$ equ:NewVFP2DBi

The solution w satisfies $w \in \mathcal{H}$ and $\partial_t w + \mathcal{T}'w \in \mathcal{H}'$ since $f \in \mathbb{Y}$. The estimates of solutions that we will study can be obtained by choosing of an appropriate sequence of functions in the varational equation of w.

Proposition 6.4

Assume that the initial data f_{in} is positive and belongs to $L^p(\mathbb{R}^2\times\mathbb{R}^2)$, with any $p \in [1,\infty[$. Then solution f provided by Proposition 2.1 satisfies

$$\|f\|_{L^\infty(0,T;L^p(\mathbb{R}^2\times\mathbb{R}^2))} \le \mathrm{e}^{\frac{p-1}{p}2T}\|f_{\mathrm{in}}\|_{L^p(\mathbb{R}^2\times\mathbb{R}^2)},\ 1 \le p < \infty, \tag{84}$$ LpNormAppen

$$\|\nabla_v f^{p/2}\|_{L^2(0,T;L^2(\mathbb{R}^2\times\mathbb{R}^2))} \le \sqrt{\frac{p}{4(p-1)\sigma}}\mathrm{e}^{(p-1)T}\|f_{\mathrm{in}}\|_{L^p(\mathbb{R}^2\times\mathbb{R}^2)},\ 1 < p < \infty. \tag{85}$$ LpGradNormAppe

Proof.

First we consider the case $p = 2$. Since w in $\mathcal{H}$ satisfies (83) , we deduce that

$$\left\langle \partial_t w + \mathcal{T}'w, w \right\rangle_{\mathcal{H}'\times\mathcal{H}} = \left\langle -e^t E(x)\cdot\nabla_v w + \sigma e^{2t}\Delta_v w, w \right\rangle_{\mathcal{H}'\times\mathcal{H}}.$$

Since $f \in \mathcal{H}$ the divergence theorem implies that the integral of $-e^t E(x)\cdot\nabla_v w$ vanish on $\mathbb{R}^2\times\mathbb{R}^2$. Then we apply (78) for $\langle\partial_t w + \mathcal{T}'w, w\rangle_{\mathcal{H}'\times\mathcal{H}}$ to obtain

$$2\left\langle \partial_t w + \mathcal{T}'w, w \right\rangle_{\mathcal{H}'\times\mathcal{H}} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |w(T,x,v)|^2|\,\mathrm{d}v\mathrm{d}x - \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |w(0,x,v)|^2\,\mathrm{d}v\mathrm{d}x.$$

Therefore we get for any $T>0$ that

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|w(T,x,v)|^2\,\mathrm{d}v\mathrm{d}x+2\sigma\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}e^{2t}|\nabla_v w|^2\,\mathrm{d}v\mathrm{d}x\mathrm{d}t=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|w(0,x,v)|^2\,\mathrm{d}v\mathrm{d}x$$

which yields the bounds of (84) and (85) when $p=2$.

Next, we consider the case $1\le p<\infty$ and $p\neq 2$. We establish a class of function of approximation C^2 of px^{p-1}, $x\ge 0$ (indeed, the function pw^{p-1} does not belong to $\mathcal{H}$ hence we can not define $\langle\partial_t w+\mathcal{T}'w,pw^{p-1}\rangle_{\mathcal{H}',\mathcal{H}}$ so we need to modify the function px^{p-1}) verifies

(i) $p=1:\psi_\varepsilon(s)=0$ if $s\le 0$, $\psi_\varepsilon(s)=1$ if $\varepsilon\le s$ and $\psi_\varepsilon(s)$ is increasing in $[0,\varepsilon]$.

(ii) $1<p<\infty$, $p\neq 2:\psi_\varepsilon(s)=0$ if $s\le\varepsilon$, $\psi_\varepsilon(s)=ps^{p-1}$ if $\varepsilon\le s\le\dfrac{1}{\varepsilon}$ and $\psi_\varepsilon'(s)=0$ on $[1/\varepsilon,+\infty)$.

It is easily seen that $\psi_\varepsilon\in C^2$ with $\psi_\varepsilon'\in L^\infty(\mathbb{R})$ and $\psi_\varepsilon(0)=0$. Let $\varphi_\varepsilon(s)$ be a primitive of $\psi_\varepsilon(s)$ defined by $\varphi_\varepsilon(t)=\int_{-\infty}^t\psi_\varepsilon(s)\mathrm{d}s$. Since $w\in\mathcal{H}$ we imply that $\psi_\varepsilon(w)$ and $\varphi_\varepsilon(w)$ belong to $\mathcal{H}$ and $\nabla_v\varphi_\varepsilon(w)=\psi_\varepsilon(w)\nabla_v w$. Moreover, the function w in $\mathcal{H}$ satisfies (83), we deduce that

$$\langle\partial_t w+\mathcal{T}'w,\psi_\varepsilon(w)\rangle_{\mathcal{H}'\times\mathcal{H}}+\langle e^tE(x)\cdot\nabla_v w-\sigma e^{2t}\Delta_v w,\psi_\varepsilon(w)\rangle_{\mathcal{H}'\times\mathcal{H}}=0. \tag{86}$$ equ:VarApprox

where $\mathcal{T}'w=e^{-t}v\cdot\nabla_x w+B(x)^\perp v\cdot\nabla_v w$. In the same way of Lemma 6.3 we also have

$$\langle\mathcal{T}'w,\psi_\varepsilon(w)\rangle_{\mathcal{H}'\times\mathcal{H}}=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\varphi_\varepsilon(w(T,x,v))\,\mathrm{d}v\mathrm{d}x-\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\varphi_\varepsilon(w(0,x,v))\,\mathrm{d}v\mathrm{d}x.$$

Since $w\in\mathcal{H}$ the divergence theorem implies the integral of $e^tE(x)\cdot\nabla_v w$ vanish on $\mathbb{R}^2\times\mathbb{R}^2$. If $p=1$, we apply again the divergence theorem to $\langle-\sigma e^{2t}\Delta_v w,\psi_\varepsilon(w)\rangle_{\mathcal{H}'\times\mathcal{H}}$ we have

$$\langle-\sigma e^{2t}\Delta_v w,\psi_\varepsilon(w)\rangle_{\mathcal{H}'\times\mathcal{H}}=\sigma\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}e^{2t}|\nabla_v w|^2\psi_\varepsilon'(w)\cdot 1_{\{0\le w\le\varepsilon\}}\,\mathrm{d}v\mathrm{d}x\mathrm{d}t.$$

Then the equation (86) gives

$$\begin{aligned}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\varphi_\varepsilon(w(T,x,v))\,\mathrm{d}v\mathrm{d}x+\sigma\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}e^{2t}|\nabla_v w|^2\psi_\varepsilon'(w)1_{\{0\le w\le\varepsilon\}}\,\mathrm{d}v\mathrm{d}x\mathrm{d}t\\=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\varphi_\varepsilon(w(0,x,v))\,\mathrm{d}v\mathrm{d}x.\end{aligned}$$

Since $\psi_\varepsilon'\ge 0$ and by using Fatou's Lemma and the dominated convergence theorem we get for any $T>0$ that

$$\|w(T)\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)}=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}w(T,x,v)\,\mathrm{d}v\mathrm{d}x\le\|w_0\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)}$$

which yields (84) with $p=1$.
If $1<p<\infty$ and $p\neq 2$, by the construction of ψ_ε we have

$$\begin{aligned}\langle-\sigma e^{2t}\Delta_v w,\psi_\varepsilon(w)\rangle_{\mathcal{H}'\times\mathcal{H}}&=\sigma\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}e^{2t}|\nabla_v w|^2\psi_\varepsilon'(w)\cdot 1_{\{\varepsilon\le w\le 1/\varepsilon\}}\,\mathrm{d}v\mathrm{d}x\mathrm{d}t\\&=\sigma p(p-1)\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}e^{2t}|\nabla_v w|^2w^{p-2}\cdot 1_{\{\varepsilon\le w\le 1/\varepsilon\}}\,\mathrm{d}v\mathrm{d}x\mathrm{d}t\\&=\frac{4(p-1)}{p}\sigma\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}e^{2t}|\nabla_v w^{p/2}|^2 1_{\{\varepsilon\le w\le 1/\varepsilon\}}\,\mathrm{d}v\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Then the equation (86) becomes

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\varphi_\varepsilon(w(T,x,v))\,\mathrm{d}v\mathrm{d}x+\frac{4(p-1)}{p}\sigma\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}e^{2t}|\nabla_v w^{p/2}|^2 1_{\{\varepsilon\le w\le 1/\varepsilon\}}\,\mathrm{d}v\mathrm{d}x\mathrm{d}t$$
$$=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\varphi_\varepsilon(u(0,x,v))\,\mathrm{d}v\mathrm{d}x.$$

Using Fatou's Lemma and the dominated convergence theorem we get for any $T>0$ that

$$\|w\|^p_{L^\infty([0,T];L^p(\mathbb{R}^2\times\mathbb{R}^2))}+\frac{4(p-1)}{p}\sigma\|\nabla_v w^{p/2}\|^2_{L^2([0,T];L^2(\mathbb{R}^2\times\mathbb{R}^2))}\le\|w\|^p_{L^\infty([0,T];L^p(\mathbb{R}^2\times\mathbb{R}^2))}$$

which yields the estimates of (84) and (85) when $1<p<\infty$ and $p\neq 2$. □

Next we provide the estimates of the kinetic energy and the entropy. First we consider the truncation function $\chi(s)\in C_0^\infty(\mathbb{R})$ such that

$$\chi(s)=1 \text{ if } |s|\le 1,\ \ \chi(s)=0 \text{ if } |s|\ge 2,\ \ \|\chi\|_{W^{1,\infty}(\mathbb{R})}\le 1$$

and we define $\chi_R(z)=\chi\left(\frac{|z|}{R}\right), z\in\mathbb{R}^2, R>0$. We then consider a function of class $C^\infty(\mathbb{R})\cap L^\infty(\mathbb{R})$ satisfying $\psi_\varepsilon(s)=0$ if $s\le 0$, $\psi_\varepsilon(s)=1$ if $s\ge\varepsilon$ and ψ_ε is increasing on $[0,\varepsilon]$. Let φ_ε be a primitive of ψ_ε as $\varphi_\varepsilon(t)=\int_{-\infty}^t\psi_\varepsilon(s)\mathrm{d}s$.

Proposition 6.5

⟨EsKinEnerVFP2D⟩ *Assume that the initial data $f_{\rm in}$ is positive and $(1+|v|^2/2)f_{\rm in}\in L^1(\mathbb{R}^2\times\mathbb{R}^2)$. Then the solution of Proposition 6.2 satisfies*

$$\sup_{[0,T]}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|v|^2}{2}f(t)\,\mathrm{d}v\mathrm{d}x\le C_1+C_2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|v|^2}{2}f_{\rm in}\,\mathrm{d}v\mathrm{d}x$$

for some constants C_1 and C_2, depending only on $\|E\|_{L^\infty}, f_{\rm in}, T, \sigma$.

Proof.
Since $w(t)=e^{-2t}f(t,x,e^{-t}v)$ in $\mathcal{H}$ satisfies the equation (83) we deduce for any function $h\in\mathcal{H}$ that

$$\left\langle\partial_t w+\mathcal{T}'w,h\right\rangle_{\mathcal{H}'\times\mathcal{H}}+\left\langle e^t E(x)\cdot\nabla_v w-\sigma e^{2t}\Delta_v w,h\right\rangle_{\mathcal{H}'\times\mathcal{H}}=0. \tag{87}$$ equ:VariForm

where $\mathcal{T}'=e^{-t}v\cdot\nabla_x w+B(x)^\perp v\cdot\nabla_v$. Taking in (87) the function $h=\chi_R(|v|)\frac{|v|^2}{2}\psi_\varepsilon(w)$. It is easily seen that $h\in\mathcal{H}$ since the function $\chi_R(|v|)\frac{|v|^2}{2}\in L^\infty(\mathbb{R})$ and $\psi(w)\in\mathcal{H}$ by $w\in\mathcal{H}$. In the same way of Lemma 6.3, we have the following formula

$$\begin{aligned}
&\left\langle\partial_t w+\mathcal{T}'w,\chi_R(|v|)\frac{|v|^2}{2}\psi_\varepsilon(w)\right\rangle_{\mathcal{H}'\times\mathcal{H}}\\
&=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left[\varphi_\varepsilon(w(T,x,v))-\varphi_\varepsilon(w(0,x,v))\right]\chi_R(|v|)\frac{|v|^2}{2}\,\mathrm{d}v\mathrm{d}x\\
&-\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}B(x)^\perp v\cdot\left[\left(\frac{v}{|v|}\chi'\left(\frac{v}{R}\right)\frac{|v|^2}{2R}1_{\{|v|\le 2R\}}+\chi_R(|v|)v\right)\varphi_\varepsilon(w)\right]\,\mathrm{d}v\mathrm{d}x\mathrm{d}t\\
&=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left[\varphi_\varepsilon(w(T,x,v))-\varphi_\varepsilon(w(0,x,v))\right]\chi_R(|v|)\frac{|v|^2}{2}\,\mathrm{d}v\mathrm{d}x.
\end{aligned}$$

Before estimating the other terms in (87) we need to observe that $\varphi_\varepsilon(w) = w\Phi_\varepsilon(w)$ with $\Phi_\varepsilon(w) = \int_0^1 \psi_\varepsilon(\theta w)\mathrm{d}\theta$, which implies that

$$\varphi_\varepsilon(w) = |\varphi_\varepsilon(w)| \le w\int_0^1 |\psi_\varepsilon(\theta w)|\mathrm{d}\theta \le w,\ \forall \varepsilon > 0.$$

Moreover, the solution w belongs to $L^1([0,T]\times\mathbb{R}^2\times\mathbb{R}^2)$ beacause $w \in L^\infty([0,T]; L^1(\mathbb{R}^2\times\mathbb{R}^2))$. On the orther hand, since $w \in \mathcal{H}$ the divergence theorem implies that the term $\left\langle e^t E(x)\cdot\nabla_v w, h\right\rangle_{\mathcal{H}\times\mathcal{H}'}$ can be estimated as

$$\begin{aligned}
&\left\langle e^t E(x)\cdot\nabla_v w, h\right\rangle_{\mathcal{H}'\times\mathcal{H}}\\
&= -\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^t E(x)\varphi_\varepsilon(w)\cdot\left(\chi'\left(\frac{|v|}{R}\right)\frac{v}{|v|}\frac{|v|^2}{2R}1_{\{|v|\le 2R\}} + \chi_R(|v|)v\right)\,\mathrm{d}v\mathrm{d}x\mathrm{d}t\\
&\le \|\chi\|_{W^{1,\infty}(\mathbb{R}^2)}\|E\|_{L^\infty}e^T\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} w(t,x,v)|v|\,\mathrm{d}v\mathrm{d}x\mathrm{d}t\\
&\le \frac{1}{2}C(\|E\|_{L^\infty},T)\left(\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} w\,\mathrm{d}v\mathrm{d}x\mathrm{d}t + \int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} w|v|^2\,\mathrm{d}v\mathrm{d}x\mathrm{d}t\right).
\end{aligned}$$

It remains to estimate the contribution of $\left\langle -\sigma e^{2t}\Delta_v w, h\right\rangle_{\mathcal{H}\times\mathcal{H}'}$ in (87). Similarly, applying the divergence theorem and by direct computations we get

$$\begin{aligned}
&\left\langle -\sigma e^{2t}\Delta_v w, h\right\rangle_{\mathcal{H}'\times\mathcal{H}} = \sigma\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{2t}|\nabla_v w|^2\psi_\varepsilon'(w)\chi_R(|v|)\frac{|v|^2}{2}\,\mathrm{d}v\mathrm{d}x\mathrm{d}t\\
&+\sigma\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{2t}\nabla_v w\cdot\left[\left(\chi'\left(\frac{|v|}{R}\right)\frac{v}{|v|}\frac{|v|^2}{2R}1_{\{|v|\le 2R\}} + \chi_R v\right)\psi_\varepsilon(w)\right]\,\mathrm{d}v\mathrm{d}x\mathrm{d}t\\
&\ge \sigma\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{2t}\nabla_v w\cdot\left[\left(\chi'\left(\frac{|v|}{R}\right)v\frac{|v|}{2R}1_{\{|v|\le 2R\}} + \chi_R v\right)\psi_\varepsilon(w)\right]\,\mathrm{d}v\mathrm{d}x\mathrm{d}t\\
&= \sigma\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{2t}\nabla_v\varphi_\varepsilon(w)\cdot\left(\chi'\left(\frac{|v|}{R}\right)v\frac{|v|}{2R}1_{\{|v|\le 2R\}} + \chi_R v\right)\,\mathrm{d}v\mathrm{d}x\mathrm{d}t\\
&= -\sigma\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{2t}\varphi_\varepsilon(w)\left[\left(\chi''\left(\frac{|v|}{R}\right)\frac{|v|^2}{2R^2}1_{\{|v|\le 2R\}} + 2\chi'\left(\frac{|v|}{R}\right)\frac{|v|}{R}1_{\{|v|\le 2R\}} + 2\chi_R\right)\right]\,\mathrm{d}v\mathrm{d}x\mathrm{d}t\\
&\to -2\sigma\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{2t}w\,\mathrm{d}v\mathrm{d}x\mathrm{d}t,\ \text{when } \varepsilon\searrow 0, R\to\infty,
\end{aligned}$$

where we have used the dominated convergence theorem in the last integral. Finally, from the equation (87) we obtain

$$\begin{aligned}
\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\varphi_\varepsilon(w(T,x,v))\chi_R(|v|)\frac{|v|^2}{2}\,\mathrm{d}v\mathrm{d}x &\le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\varphi_\varepsilon(w(0,x,v))\chi_R(|v|)\frac{|v|^2}{2}\,\mathrm{d}v\mathrm{d}x\\
&+\frac{1}{2}C(\|E\|_{L^\infty},T)\left(\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} w\,\mathrm{d}v\mathrm{d}x\mathrm{d}t + \int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} w|v|^2\,\mathrm{d}v\mathrm{d}x\mathrm{d}t\right)\\
&+2\sigma\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{2t}w\,\mathrm{d}v\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Since $\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} w\,\mathrm{d}v\mathrm{d}x\mathrm{d}t \le \int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} w_0\,\mathrm{d}v\mathrm{d}x\mathrm{d}t = T\|f_{\mathrm{in}}\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)}$ we deduce that

$$\begin{aligned}
\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\varphi_\varepsilon(w(T,x,v))\chi_R(|v|)\frac{|v|^2}{2}\,\mathrm{d}v\mathrm{d}x &\le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\varphi_\varepsilon(w(0,x,v))\chi_R(|v|)\frac{|v|^2}{2}\,\mathrm{d}v\mathrm{d}x\\
&+C(\|E\|_{L^\infty},T,\sigma,f_{\mathrm{in}}) + C(\|E\|_{L^\infty},T)\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} w\frac{|v|^2}{2}\,\mathrm{d}v\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Using Fatou's Lemma and then the dominated convergence theorem when $\varepsilon \searrow 0$, $R \to \infty$ we get for any $T > 0$ that

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} w(T,x,v)\frac{|v|^2}{2}\,\mathrm{d}v\mathrm{d}x \le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} w(0,x,v)\frac{|v|^2}{2}\,\mathrm{d}v\mathrm{d}x$$
$$+C(\|E\|_{L^\infty},T,\sigma,f_{\mathrm{in}}) + C(\|E\|_{L^\infty},T)\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} w\frac{|v|^2}{2}\,\mathrm{d}v\mathrm{d}x\mathrm{d}t.$$

By the Growall's inquality we complete the proof. □

In the same way as for the proof of Proposition 6.5, if we take the function h in the equation (87) given by $h(t,x,v) = \chi_R(|x|)|x|\psi_\varepsilon(w)$, we can obtain the following Proposition

Proposition 6.6

?⟨BoundPosit⟩? *Assume that the initial data f_{in} belongs to $L^1(\mathbb{R}^2\times\mathbb{R}^2)$ and satifies $(|x|+|v|^2/2)f_{\mathrm{in}} \in L^1(\mathbb{R}^2\times\mathbb{R}^2)$. Then the solution f is given by Proposition 6.2 satisfies*

$$\sup_{[0,T]}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|x|f(t)\,\mathrm{d}v\mathrm{d}x \le C_1 + C_2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|x|f_{\mathrm{in}}\,\mathrm{d}v\mathrm{d}x$$

for somse constants C_1 and C_2, depending only on f_{in}, T.

Proposition 6.7

?⟨EntropyVFP2D⟩? *Assume that the initial function f_{in} is positve and verifies $(1+|x|+|v|^2/2)f_{\mathrm{in}} \in L^1(\mathbb{R}^2\times\mathbb{R}^2)$. Then the solution f of Proposition 6.2 satisfies*

$$\sup_{[0,T]}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(t)|\ln f(t)|\,\mathrm{d}v\mathrm{d}x \le C + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\sigma f_{\mathrm{in}}|\ln f_{\mathrm{in}}|\,\mathrm{d}v\mathrm{d}x$$

$$\sup_{[0,T]}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|\sigma\nabla_v f(t)|^2}{f(t)}\,\mathrm{d}v\mathrm{d}x \le C + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\sigma f_{\mathrm{in}}|\ln f_{\mathrm{in}}|\,\mathrm{d}v\mathrm{d}x$$

for some constant C, depending only on $\|E\|_{L^\infty}, f_{\mathrm{in}}, T, \sigma$.

Proof.

As before, we will work on $w(t,x,v) = e^{-2t}f(t,x,e^{-t}v)$ which is satisfied by equation (83) and variational equation (87). For any $\varepsilon > 0$, we define the function $g_\varepsilon(w)$ such that

$$1 + 1_{\{\varepsilon\le w\le 1/\varepsilon\}}\ln\varphi_\varepsilon(w) = 1 + 1_{\{\varepsilon\le w\le 1/\varepsilon\}}\ln w = 1 + g_\varepsilon(w)$$

and it is obvious that it belongs to $L^\infty([0,T]\times\mathbb{R}^2\times\mathbb{R}^2)$. Observing that

$$\partial_t w(1 + 1_{\{\varepsilon\le w\le 1/\varepsilon\}}\ln\varphi_\varepsilon(w)) = \partial_t(wg_\varepsilon(w))$$

and

$$\mathcal{T}'w1 + 1_{\{\varepsilon\le w\le 1/\varepsilon\}}\ln\varphi_\varepsilon(w) = \mathcal{T}'(wg_\varepsilon(w))$$

Multiplying the equation (83) by $\sigma(1+1_{\{\varepsilon\le w\le 1/\varepsilon\}}\ln\varphi_\varepsilon(w))$ and then passing to the variational equation with $h = \psi_\varepsilon(w) \in \mathcal{H}$ we get

$$\sigma\left\langle \partial_t(wg_\varepsilon(w)) + \mathcal{T}'(wg_\varepsilon(w)), \psi_\varepsilon\right\rangle_{\mathcal{H}'\times\mathcal{H}}$$
$$+\sigma\left\langle [e^tE(x)\cdot\nabla_v w - \sigma e^{2t}\Delta_v w](1+g_\varepsilon(w)), \psi_\varepsilon(w)\right\rangle_{\mathcal{H}'\times\mathcal{H}} = 0. \qquad (88)$$ equ:VariFormBi

Since $\psi_\varepsilon(w) = 1$ on $\varepsilon \le w \le 1/\varepsilon$ so in the same way of Lemma 6.3, we have the following formula

$$\begin{aligned}
&\sigma \left\langle \partial_t(wg_\varepsilon(w)) + \mathcal{T}'(wg_\varepsilon(w)), \psi_\varepsilon \right\rangle_{\mathcal{H}'\times\mathcal{H}} \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \sigma w(T,x,v)\ln w(T,x,v) 1_{\{\varepsilon\le w\le 1/\varepsilon\}}\,\mathrm{d}v\mathrm{d}x \\
&- \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \sigma w(0,x,v)\ln w(0,x,v) 1_{\{\varepsilon\le w\le 1/\varepsilon\}}\,\mathrm{d}v\mathrm{d}x.
\end{aligned}$$

We estimate now the other terms in (88). Since $w \in \mathcal{H}$ so the divergence theorem implies that

$$\begin{aligned}
\sigma \left\langle e^t E(x)\cdot\nabla_v w(1+g_\varepsilon(w)), \psi_\varepsilon(w)\right\rangle_{\mathcal{H}'\times\mathcal{H}} &= \sigma \left\langle e^t E(x)\cdot\nabla_v (wg_\varepsilon(w)), \psi_\varepsilon(w)\right\rangle_{\mathcal{H}'\times\mathcal{H}} \\
&= \sigma \int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^t E(x)\cdot\nabla_v (wg_\varepsilon(w)) 1_{\{\varepsilon\le w\le 1/\varepsilon\}}\,\mathrm{d}v\mathrm{d}x\mathrm{d}t = 0.
\end{aligned}$$

and

$$\begin{aligned}
\left\langle -\sigma^2 e^{2t}\Delta_v w(1+g_\varepsilon(w)), \psi_\varepsilon(w)\right\rangle_{\mathcal{H}'\times\mathcal{H}} &= \sigma^2 \int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{2t}\nabla_v w\cdot\nabla_v g_\varepsilon(w) 1_{\{\varepsilon\le w\le 1/\varepsilon\}}\,\mathrm{d}v\mathrm{d}x\mathrm{d}t \\
&= \int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{2t}\frac{|\sigma\nabla_v w|^2}{w} 1_{\{\varepsilon\le w\le 1/\varepsilon\}}\,\mathrm{d}v\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Finally, from (88) we obtain for any $T > 0$ that

$$\begin{aligned}
&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \sigma w(T,x,v)\ln w(T,x,v) 1_{\{\varepsilon\le w\le 1/\varepsilon\}}\,\mathrm{d}v\mathrm{d}x + \int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{2t}\frac{|\sigma\nabla_v w|^2}{w} 1_{\{\varepsilon\le w\le 1/\varepsilon\}}\,\mathrm{d}v\mathrm{d}x\mathrm{d}t \\
&\qquad \le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \sigma w(0,x,v)\ln w(0,x,v) 1_{\{\varepsilon\le w\le 1/\varepsilon\}}\,\mathrm{d}v\mathrm{d}x.
\end{aligned}$$

By standard argument, there exists a constant $C > 0$, (see [26], Lemma 2.3) such that

$$|u\ln u| = u\ln u - 2u\ln u_{\{0\le u\le 1\}} \le u\ln u + \frac{1}{4}(|x|+|v|^2)u + Ce^{-\frac{|x|+|v|^2}{2}}$$

therefore

$$\begin{aligned}
&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \sigma w(T,x,v)|\ln w(T,x,v)| 1_{\{\varepsilon\le w\le 1/\varepsilon\}}\,\mathrm{d}v\mathrm{d}x + \int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{2t}\frac{|\sigma\nabla_v w|^2}{w} 1_{\{\varepsilon\le w\le 1/\varepsilon\}}\,\mathrm{d}v\mathrm{d}x\mathrm{d}t \\
&\qquad \le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \sigma w(0,x,v)|\ln w(0,x,v)|\,\mathrm{d}v\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(|x|+|v|^2)w\,\mathrm{d}v\mathrm{d}x + C8\pi
\end{aligned}$$

where we have used that $\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-\frac{|x|+|v|^2}{2}}\,\mathrm{d}v\mathrm{d}x = 8\pi$. Thanks to the hypothesis on the initial data f_{in} we infer that $\frac{1}{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(|x|+|v|^2)w\,\mathrm{d}v\mathrm{d}x \le C(\|E\|_{L^\infty}, f_{\mathrm{in}}, T, \sigma)$. Therefore, Fatou's Lemma implies that

$$\begin{aligned}
&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \sigma w(T,x,v)|\ln w(T,x,v)|\,\mathrm{d}v\mathrm{d}x + \int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{2t}\frac{|\sigma\nabla_v w|^2}{w}\,\mathrm{d}v\mathrm{d}x\mathrm{d}t \\
&\le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \sigma w(0,x,v)|\ln w(0,x,v)|\,\mathrm{d}v\mathrm{d}x.
\end{aligned}$$

Substitutively $w = e^{-2t}f(t,x,e^{-t}v)$ leads to the desired result. □

B Classical solution of the VPFP system with uniform external magnetic field

In this part, we consider the system (10), (11), (12) for uniform magnetic field *i.e.*, $\nabla B(x) = 0$. In order to simplify, we take in the equation (10) with $B = 1$. We focus on the global existence and uniqueness of the smooth solutions. The proof will be based on the approximation scheme $(f^k)_{k\in\mathbb{N}}$ constructed in Theorem 2.2. In order to prove that the system admits a global regular solution, we show that the sequence $(f^k)_{k\in\mathbb{N}}$ is actually bounded, as well as its derivatives, by a function that does not blow up in finite time, if we further assume that the electric field $E^k \in L^\infty_{\rm loc}([0,\infty[; W^{1,\infty}(\mathbb{R}^2))$. We have the following regularity estimate

Lemma 6.6

⟨RegularEsti⟩ *Let $f_{\rm in}$ be a non-negative function such that*

$$f_{\rm in} \in W^{1,1}(\mathbb{R}^2),\ (1+|v|^2)^{\gamma/2}(f_{\rm in} + |\nabla_{x,v} f_{\rm in}|) < +\infty,\ \gamma > 2.$$

Then, there exist two functions $\alpha(t), \beta(t)$ in $L^\infty_{\rm loc}([0,\infty[)$ independent of k, such that for every k and t, we have

$$\|(1+|v|^2)^{\gamma/2} f^k(t,x,v)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \le \alpha(t),\ \|(1+|v|^2)^{\gamma/2} Df^k(t,x,v)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \le \beta(t).$$

Proof.

We define

$$Y^k(t,x,v) = (1+|v|^2)^{\gamma/2} f^k(t,x,v),\quad Z^k(t,x,v) = (1+|v|^2)^{\gamma/2} Df^k(t,x,v).$$

For the L^∞ estimate of the sequence $(Y^k)_{k\in\mathbb{N}}$, we use the same argument as in the Lemma 2.6. We will now focus on estimating Z^k. Taking the derivative with respect to the variables (x,v) in the linear VFP equation (19) for f^{k+1}, we get

$$\begin{aligned}
&\partial_t(Df^{k+1}) + v\cdot\nabla_x(Df^{k+1}) + E^k\cdot\nabla_v(Df^{k+1}) +^\perp v\cdot\nabla_v(Df^{k+1}) \\
&= \mathrm{div}_v\left(\sigma\nabla_v(Df^{k+1}) + v(Df^{k+1})\right) - Dv\cdot\nabla_x f^{k+1} - DE^k\cdot\nabla_v f^{k+1} \\
&\qquad - D^\perp v\cdot\nabla_v f^{k+1} + Dv\cdot\nabla_v f^{k+1}.
\end{aligned}$$

A standard computations, we have the following equalities

$$-Dv\cdot\nabla_x f^{k+1} = -\begin{pmatrix} 0 & 0 \\ I_2 & 0 \end{pmatrix}\begin{pmatrix} \nabla_x f^{k+1} \\ \nabla_v f^{k+1} \end{pmatrix}$$

$$-DE^k\cdot\nabla_v f^{k+1} = -\begin{pmatrix} 0 & \nabla E^k \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \nabla_x f^{k+1} \\ \nabla_v f^{k+1} \end{pmatrix}$$

$$-D^\perp v\cdot\nabla_v f^{k+1} = -\begin{pmatrix} 0 & 0 \\ 0 & \mathcal{R}\left(-\frac{\pi}{2}\right) \end{pmatrix}\begin{pmatrix} \nabla_x f^{k+1} \\ \nabla_v f^{k+1} \end{pmatrix}$$

$$Dv\cdot\nabla_v f^{k+1} = \begin{pmatrix} 0 & 0 \\ 0 & I_2 \end{pmatrix}\begin{pmatrix} \nabla_x f^{k+1} \\ \nabla_v f^{k+1} \end{pmatrix}$$

where $\mathcal{R}\left(-\frac{\pi}{2}\right)$ is a rotation matrix of angle $-\pi/2$. Then, the previous equation can be rewritten as

$$\begin{aligned}
&\partial_t(Df^{k+1}) + v\cdot\nabla_x(Df^{k+1}) + E^k\cdot\nabla_v(Df^{k+1}) + {}^\perp v\cdot\nabla_v(Df^{k+1}) \\
&\qquad = \mathrm{div}_v\left(\sigma\nabla_v(Df^{k+1}) + v(Df^{k+1})\right) + A^k\cdot Df^{k+1}
\end{aligned} \tag{89}$$

equ:Derivative

where $A^k(t,x) \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^2)$ denotes the following matrix

$$A^k(t,x) = \begin{pmatrix} 0 & -\nabla E^k \\ -I_2 & I_2 - \mathcal{R}(-\pi/2) \end{pmatrix}.$$

Now, we multiply equation (89) by $(1+|v|^2)^{\gamma/2}$ and get the following equation for Z^{k+1}

$$\begin{aligned} &\partial_t Z^{k+1} + v \cdot \nabla_x Z^{k+1} + \left(E^k + 2\sigma\gamma \frac{v}{1+|v|^2} \right) \cdot \nabla_v Z^{k+1} \\ &+^\perp v \cdot \nabla_v Z^{k+1} - \sigma \Delta_v Z^{k+1} - \mathrm{div}_v(v Z^{k+1}) = R_1 + R_2 + R_3 \end{aligned} \tag{90}$$

equ:EvoluZk

where R_1 and R_2 are obtained from (25) by replacing f^{k+1} by Df^{k+1} and

$$S_3 = (1+|v|^2)^{\gamma/2} A^k \cdot Df^{k+1}.$$

Thanks to the estimations on R_1 and R_2 in Lemma 2.6, we get

$$\begin{aligned} \|R_1(t)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} &\le \gamma \|E^k(t)\|_{L^\infty(\mathbb{R}^2)} \|(1+|v|^2)^{(\gamma-1)/2} Df^{k+1}\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \\ &\le \gamma \|E^k(t)\|_{L^\infty(\mathbb{R}^2)} \|Z^{k+1}(t)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \\ &\le \gamma C(f_{\mathrm{in}}) \|Y^k(t)\|^{1/\gamma}_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \|Z^{k+1}(t)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \\ &\le C_1(\gamma, f_{\mathrm{in}}) \alpha(t)^{1/\gamma} \|Z^{k+1}(t)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \\ \|R_2(t)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} &\le C_2(\sigma,\gamma) \|Z^{k+1}(t)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \\ \|R_3(t)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} &\le \|A^k(t)\|_{L^\infty(\mathbb{R}^2)} \|Z^{k+1}(t)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}. \end{aligned}$$

Since $\|n^k(t)\|_{L^1(\mathbb{R}^2)} \le \|f_{\mathrm{in}}\|_{L^1(\mathbb{R}^2)}$ and by Lemma 2.2 and $\|Y^k(t)\|_{L^\infty} \le \alpha(t)$ we deduce that $\|n^k(t)\|_{L^\infty(\mathbb{R}^2)} \le C(\gamma, f_{\mathrm{in}}) \alpha(t)^{2/\gamma}$. Moreover,

$$\|\nabla_x n^k(t)\|_{L^\infty(\mathbb{R}^2)} \le \int_{\mathbb{R}^2} \frac{1}{(1+|v|^2)^{\gamma/2}} \,\mathrm{d}x \|Z^k(t)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} = C(\gamma) \|Z^k(t)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}.$$

Combining the above bounds on the density n^k and the inequality (59) in Lemma 5.3 give an estimate for the derivatives of E^k. Therefore we obtain

$$\|A^k(t)\|_{L^\infty(\mathbb{R}^2)} \le C_3(\gamma, f_{\mathrm{in}}) \left(1 + \ln^+ \|Z^k(t)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \right) \|Z^{k+1}(t)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}.$$

So, the maximum principle in Remark 6.1 applied to (90) and the privious estimates of R_1, R_2 and R_3 lead to

$$\begin{aligned} \|Z^{k+1}(t)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} &\le e^{2T} \|Z_0\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} + C_1(\gamma, f_0) \int_0^t \alpha(s)^{1/\gamma} \|Z^{k+1}(s)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \mathrm{d}s \\ &+ C_2(\sigma,\gamma) \int_0^t \|Z^{k+1}(s)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \mathrm{d}s \\ &+ C_3(\gamma, f_{\mathrm{in}}) \int_0^t \alpha(s)^{2/\gamma} \left(1 + \ln^+ \|Z^k(s)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \right) \|Z^{k+1}(s)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \mathrm{d}s. \end{aligned} \tag{91}$$

EstMaxZk

We denote here

$$\psi_1(t) = C_1(\gamma, f_{\mathrm{in}}) \alpha(t)^{1/\gamma} + C_2(\sigma,\gamma), \quad \psi_2(t) = C_3(\gamma, f_{\mathrm{in}}) \alpha(t)^{2/\gamma}.$$

Since $\alpha(t) \in L^\infty_{\rm loc}(\mathbb{R}_+)$ then $\psi_1(t), \psi_2(t) \in L^\infty_{\rm loc}(\mathbb{R}_+)$. Now we introduce the function

$$z^k(t) = \max(1, \|Z^{k+1}(t)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}).$$

We find that there exist a function $\psi_3(t)$ such that

$$1 + \psi_1(t) + \psi_2(t) + \ln^+ \|Z^k(t)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \le \psi_3(t) \ln z^k(t).$$

Indeed,

$$1 + \psi_1(t) + \psi_2(t) + \ln^+ \|Z^k(t)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \le \ln(z^k(t)\psi_3(t)) \le \ln(z^k(t)^{\psi_3(t)}),$$

with $\psi_3(t) = \exp(1 + \psi_1(t) + \psi_2(t))$ since $z^k(t) \ge 1$. Then, from (91) the function $z^{k+1}(t)$ satifies the inequality

$$z^{k+1}(t) \le e^{2T}\|Z_0\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} + \int_0^t \psi_3(s) \ln z^k(s) z^{k+1}(s) \mathrm{d}s.$$

We denote $\beta(t)$ the solution of the differential equation

$$\dot\beta(t) = \psi_3(t) \ln \beta(t)\beta(t), \quad \beta(0) = e^{2T}\|Z_0\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}$$

whose solution is

$$\beta(t) = \exp\left(\ln\beta(0) \exp\int_0^t \psi_3(s)\mathrm{d}s\right).$$

We see that β belongs to $L^\infty_{\rm loc}(\mathbb{R}_+)$ since $\psi_3 \in L^\infty_{\rm loc}(\mathbb{R}_+)$ and the same argument as function $\alpha(t)$ in the Lemma 2.6 show that Z^{k+1} satisfies

$$\|Z^{k+1}(t)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \le \beta(t), \ \forall t \in \mathbb{R}_+.$$

So Lemma 6.6 is proved. □

Corollary 6.1

⟨CoroEstUniformk⟩ *With the same assumptions and notations as in Lemma 6.6, there exist a function η lies in $L^\infty_{loc}(\mathbb{R}_+)$ such that for all $k \in \mathbb{N}$ and $t > 0$*

$$\|n^k(t)\|_\infty + \|\nabla_x n^k(t)\|_\infty + \|E^k(t)\|_\infty + \|\nabla_x E^k(t)\|_\infty \le \eta(t) \tag{92}$$ EstLinftyK

$$\|Df^k(t)\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)} \le \eta(t). \tag{93}$$ EstGradL1K

Proof.
The estimate (92) is a direct consequence of Lemma 6.6. For the estimate (93), going back to equation (89), and applying L^1 estimate, leads to

$$\|Df^{k+1}(t)\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)} \le \|Df_0\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)} + \int_0^t \|A^k(s)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \|Df^{k+1}(s)\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)} \mathrm{d}s.$$

Since $\|A^k(s)\|_{L^\infty}$ is bounded by $\eta(t)$. It implies that $Df^{k+1}(t)\|_{L^1}$ satisfies a linear Gronwall inequality whose coefficients are independent of k and gives (93). □

Global existence of the solution

Let $T > 0$. We then prove the convergence of iterations towards a weak solution. Thanks to Lemma 6.6 and (92) we obtain the following convergences, up to extraction of a subsequence, in the weak star topology of $L^\infty([0,T]\times\mathbb{R}^2\times\mathbb{R}^2)$

$$\begin{aligned} f^k \rightharpoonup f;\ \ (1+|v|^2)f^k \rightharpoonup (1+|v|^2)f,\\ (1+|v|^2)Df^k \rightharpoonup (1+|v|^2)Df, \end{aligned} \tag{94}$$

?ConverWeak1?

and, in $L^\infty([0,T]\times\mathbb{R}^d)$ weak star

$$E^k \rightharpoonup E,\ \ \nabla E^k \rightharpoonup \nabla E. \tag{95}$$

?ConverWeak2?

To take limits in the nonlinear terms of (19), we need strong compactness and convergence of the whole sequence. We will prove that f^k converges to f in the norm of $L^\infty([0,T];L^1(\mathbb{R}^2))$. Indeed, $(f^{k+1}-f^k)$ solves the equation

$$\begin{aligned} &\partial_t(f^{k+1}-f^k) + v\cdot\nabla_x(f^{k+1}-f^k) + E^k\cdot\nabla_v(f^{k+1}-f^k) + {}^\perp v\cdot\nabla_v(f^{k+1}-f^k)\\ &= Q_{FP}\left((f^{k+1}-f^k)\right) - (E^k-E^{k-1})\cdot\nabla_v f^k. \end{aligned}$$

Now, thanks to the L^1 estimate, we obtain

$$\begin{aligned} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|f^{k+1}-f^k|\,\mathrm{d}v\mathrm{d}x &\le \int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|(E^k-E^{k-1})(s,x)||\nabla_v f^k(s,x,v)|\,\mathrm{d}v\mathrm{d}x\mathrm{d}s\\ &\le \frac{1}{2\pi}\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{|x-y|}|\nabla_v f^k(s,x,v)||n^k(s,y)-n^{k-1}(s,y)|\mathrm{d}y\,\mathrm{d}v\mathrm{d}x\mathrm{d}s. \end{aligned} \tag{96}$$

EstDiffL1k

Using the standard interpolation argument and (93) we get

$$\begin{aligned} \sup_y\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{|x-y|}|\nabla_v f^k(s,x,v)|\,\mathrm{d}v\mathrm{d}x &\le \left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|\nabla_v f^k(s,x,v)|\,\mathrm{d}v\mathrm{d}x\right)^{1/2}\\ &\qquad\left(\sup_x\int_{\mathbb{R}^2}|\nabla_v f^k(s,x,v)|\,\mathrm{d}v\right)^{1/2}\\ &\le C(\gamma)\|Df^k(s)\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)}\|Z^k(s)\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} \le C(\gamma,T). \end{aligned}$$

Thus (96) leads to

$$\begin{aligned} \|(f^{k+1}-f^k)(t)\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)} &\le \frac{C(\gamma,T)}{2\pi}\int_0^t\int_{\mathbb{R}^2}|n^k(s,y)-n^{k-1}(s,y)|\mathrm{d}y\mathrm{d}s\\ &\le \frac{C(\gamma,T)}{2\pi}\int_0^t\|(f_{n+1}-f_n)(s)\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)}\mathrm{d}s. \end{aligned}$$

Then $\|(f^{k+1}-f^k)(t)\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)}$ satisfies

$$\|(f^{k+1}-f^k)(t)\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)} \le \left(\frac{C(\gamma,T)}{2\pi}\right)^k\frac{t^k}{k!}\|f_1(t)-f_{\mathrm{in}}\|_{L^\infty(0,T;L^1(\mathbb{R}^2\times\mathbb{R}^2))}.$$

which proves that f^k converges in $L^\infty([0,T];L^1(\mathbb{R}^2\times\mathbb{R}^2))$ to a unique limit which coincides with the function f found previously. It is then easy to prove that f is a weak solution of equation (10).

Uniqueness of the solution

The uniqueness of the solution which belongs to $L^\infty([0,T];W^{1,1}(\mathbb{R}^2\times\mathbb{R}^2)\cap W^{1,\infty}(\mathbb{R}^2\times\mathbb{R}^2))$

can be performed similarly as in the part of existence.

Regularity of the solution

$$f \geq 0,\ f \in L^\infty([0,T]; L^\infty \cap L^1(\mathbb{R}^2 \times \mathbb{R}^2)),$$
$$(1+|v|^2)^{\gamma/2}(f + |Df|) \in L^\infty([0,T] \times \mathbb{R}^2 \times \mathbb{R}^2)).$$

Corollary 6.1 shows that Df^k is bounded in $L^\infty([0,T]; L^1(\mathbb{R}^2 \times \mathbb{R}^2))$. Then, for almost every $t \in [0,T[$, $Df^k(t)$ is a bounded measure, and since it is a function, we obtain $Df \in L^\infty([0,T]; L^1(\mathbb{R}^2 \times \mathbb{R}^2))^2$. Then, thanks to a standard interpolation, we impliy that the density $n(t,x) = \int_{\mathbb{R}^2} f(t,x,v)\, \mathrm{d}v$ belongs to $L^\infty([0,T]; W^{1,1}(\mathbb{R}^2 \times \mathbb{R}^2) \cap W^{1,\infty}(\mathbb{R}^2 \times \mathbb{R}^2))$. So the electric field solves the Poisson equation in a classical sense and we obtain $E \in L^\infty([0,T]; W^{1,\infty}(\mathbb{R}^2))$. We deduce that f is a classical solution of system (10), (11), (12) on $[0,T[$.
We state the following result

Proposition 6.8
Let $T > 0$. Assume that the initial data $f_{\mathrm{in}}(x,v)$ is nonegative and satisfies
ClassicalSolUnif⟩?

$$f_{\mathrm{in}} \in W^{1,1}(\mathbb{R}^2 \times \mathbb{R}^2),\ (1+|v|^2)^{\gamma/2}(f_{\mathrm{in}} + |Df_{\mathrm{in}}|) \in L^\infty(\mathbb{R}^2 \times \mathbb{R}^2) \text{ with } \gamma > 2.$$

Then, there exists a unique smooth solution of the VPFP system on the time interval $[0,T[$. This solution satisfies

$$f \geq 0,\ \ f \in L^\infty([0,T]; W^{1,1}(\mathbb{R}^2 \times \mathbb{R}^2))$$
$$(1+|v|^2)^{\gamma/2}(f + |Df|) \in L^\infty([0,T] \times \mathbb{R}^2 \times \mathbb{R}^2))$$
$$E \in L^\infty([0,T], W^{1,\infty}(\mathbb{R}^2 \times \mathbb{R}^2)).$$

Remark 6.2
When the magnetic field B is non-uniform, observing the derivative in the variable $x_i, i = 1,2$ of the transport in velocity along the magnetic force

$$\partial_{x_i}[B(x)^\perp v \cdot \nabla_v f] = \partial_{x_i} B^\perp v \cdot \nabla_v f + B(x)^\perp v \cdot \nabla_v(\partial_{x_i} f)$$

and then multiplying this identity by $(1+|v|^2)^{\gamma/2}$ we get

$$\begin{aligned}(1+|v|^2)^{\gamma/2}\partial_{x_i}[B(x)^\perp v \cdot \nabla_v f] &= \partial_{x_i} B^\perp v Z_i + B(x)^\perp v \cdot \nabla_v Z_i - B(x)^\perp v \cdot \nabla_v \left(1+|v|^2\right)^{\gamma/2} \partial_{x_i} f \\ &= \partial_{x_i} B^\perp v Z_i + B(x)^\perp v \cdot \nabla_v Z_i\end{aligned}$$

where we denote $Z_i = (1+|v|^2)^{\gamma/2}\partial_{x_i} f$. We cannot apply the maximum principle of Remark 6.1 as in the equation (90), beacause this term $\partial_{x_i} B^\perp v Z_i$ is not bounded in $L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$.

C Preliminary study on the Poisson equation

In this part, we consider the Poisson equation $-\Delta_x \Phi = \rho$ in $\mathbb{R}^d$, where $\rho \in C_0^\infty(\mathbb{R}^d)$ with $d \geq 2$ whose support $\mathrm{supp}\rho \subset \left\{x \in \mathbb{R}^d : |x| \leq R\right\}$. The solution of this equation is given by the convolution with the fundamental solution of the Laplace operator as $\Phi = G_d \star \rho$, where

$$G_d(x) = \begin{cases} -\dfrac{1}{2\pi} \ln|x|, & d = 2 \\ \dfrac{|x|^{2-d}}{(d-2)|B(0,1)|}, & d \geq 3. \end{cases}$$

The purpose of this part is to justify the following identity by using the integral by parts

$$\int_{\mathbb{R}^d} \rho \Phi \mathrm{d}x = \int_{\mathbb{R}^d} |\nabla_x \Phi|^2 \mathrm{d}x.$$

The formula requires that the vector field $\nabla_x \Phi$ decays rapidly at infinity. This is a reasonable condition in the three-dimensional case but not in the two-dimensional one. In three dimensions, the decay of the vector field $\nabla\Phi$ is fast enough so that the integral $\int_{\mathbb{R}^3} |\nabla_x \Phi|^2 \mathrm{d}x$ is finite. In two dimensions, nevertheless, this vector field is not decreasing fast enough to infinity to be in L^2. This is a consequence of the decay property of the kernel $\nabla_x G_d$. You can see that, in two dimensions, $\nabla_x G_2$ decreases to infinity as $1/r$, which is not square integral, while in three dimensions, $\nabla_x G_3$ decreases as $1/r^2$.
Now we consider the case $d = 3$. Since we can write

$$\nabla_x \Phi(x) = \frac{1}{|B(0,1)|} \left(\int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} 1_{\{|x-y|\leq 1\}} \rho(y) \mathrm{d}y + \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} 1_{\{|x-y|\geq 1\}} \rho(y) \mathrm{d}y \right).$$

So, thanks to the Young's inequality for the convolution we get

$$\|\nabla_x \Phi\|_{L^2(\mathbb{R}^3)} \leq \frac{1}{|B(0,1)|} \left(\left\| \frac{1}{|x|^2} 1_{\{|x|\leq 1\}} \right\|_{L^1} \|\rho\|_{L^2} + \left\| \frac{1}{|x|^2} 1_{\{|x|\geq 1\}} \right\|_{L^2} \|\rho\|_{L^1} \right) < +\infty.$$

Another approach to show the vector field $\nabla_x \Phi \in L^2(\mathbb{R}^3)$ is to use the Hardy-Littlewood-Sobolev inequality, see Lemma (6.8) below

$$\|\nabla_x \Phi\|_{L^2(\mathbb{R}^3)} = \left\| \frac{x}{|x|^3} \star \rho \right\|_{L^2(\mathbb{R}^3)} \leq \|\rho\|_{L^{6/5}(\mathbb{R}^3)}.$$

Next, we consider the case $d = 2$. The solution Φ and its gradiant write

$$\Phi(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \ln|x-y| \rho(y) \mathrm{d}y, \ \nabla_x \Phi(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} \rho(y) \mathrm{d}y.$$

Observer for $|x| \neq 0$ that

$$\ln|x-y| = \ln|x| + \ln\left(\left| \frac{x-y}{|x|} \right| \right),$$
$$|x-y|^{-2} = |x|^{-2} \left(1 - 2\frac{x\cdot y}{|x|^2} + \frac{|y|^2}{|x|^2} \right)^{-1}.$$

If $|y| \leq R$ and $|x| \geq 2R$ then for $|x|$ large, we have

$$\ln|x-y| = \ln|x| + \mathcal{O}(|x|), \quad |x-y|^{-2} = |x|^{-2} + \mathcal{O}(|x|^{-3}).$$

Since $w(y)$ has support in $|y| \leq R$ we see that for $|x|$ large

$$\Phi(x) = -\frac{1}{2\pi} \ln|x| \int_{\mathbb{R}^2} \rho(y) \mathrm{d}y + \mathcal{O}(|x|),$$
$$\nabla_x \Phi = -\frac{1}{2\pi} \frac{x}{|x|^2} \int_{\mathbb{R}^2} \rho(y) \mathrm{d}y + \mathcal{O}(|x|^{-2}).$$

Combining with the fact that

$$\int_{\mathbb{R}^d} (1+|x|^2)^{-l/2} \mathrm{d}x < \infty \Leftrightarrow l > d$$

we deduce that $\nabla_x \Phi$ is not square integrable except that $\int_{\mathbb{R}^2} \rho \ \mathrm{d}x = 0$. If $\int_{\mathbb{R}^2} \rho \ \mathrm{d}x = 0$, by adding the decay at infinity of ρ as $(1+|x|)\rho \in L^1(\mathbb{R}^2)$ we can show that $\nabla_x \Phi \in L^2(\mathbb{R}^2)$ and that

$$\int_{\mathbb{R}^2} (-\Delta\Phi)\Phi \ \mathrm{d}x = \int_{\mathbb{R}^2} |\nabla_x \Phi|^2 \ \mathrm{d}x.$$

D Inequalities

Lemma 6.7

⟨Cal-Zyg⟩ *Let $\rho \in L^p(\mathbb{R}^d)$, with $2 \leq p < \infty$, and let $\Phi = G_d \star \rho$. Then*

$$\|D^2\Phi\|_{L^p(\mathbb{R}^d)} \leq Cp\|\rho\|_{L^p(\mathbb{R}^d)},$$

where G_d is fundamental solution of the Laplace equation in $\mathbb{R}^d$, D^2 denotes any second derivative and C is a positive universal constant.

Lemma 6.8

⟨HardySob⟩ *Consider a kernel $K_\alpha(x) = \dfrac{1}{|x|^\alpha}$ and convolution $T_\alpha f = f \star K_\alpha$. If $p > 1$ and $\alpha = d(1-\frac{1}{q}+\frac{1}{p})$, then we have*

$$\|T_\alpha f\|_{L^q(\mathbb{R}^d)} \leq C\|f\|_{L^p(\mathbb{R}^d)}$$

for some positive universal constant C.

Acknowledgement

This work has been carried out within the framework of the EUROfusion Consortium, funded by the European Union via the Euratom Research and Training Programme (Grant Agreement No 101052200 — EUROfusion). Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Commission. Neither the European Union nor the European Commission can be held responsible for them.

References

BerVas2005 [1] F. Berthelin, A. Vasseur, From Kinetic Equations to Multidimensional Isentropic Gas Dynamics Before Shocks, SIAM Journal on Mathematical Analysis. 36(2005) 1807-1835.

BonCarSol [2] L. L. Bonilla, J. A. Carrillo, J. Soler, Asymptotic behaviour of the initial boundary value problem for the three dimensional Vlasov–Poisson–Fokker–Planck system, SIAM J. Appl. Math. 57(1997) 1343–1372.

Bos2019 [3] M. Bostan, Asymptotic behavior for the Vlasov-Poisson equations with strong external magnetic field. Straight magnetic field lines, SIAM J. MATH. ANAL., 51(2019) 2713–2747.

Bos2007 [4] M. Bostan, The Vlasov–Maxwell System with Strong Initial Magnetic Field: Guiding Center Approximation. Multiscale Modeling & Simulation, 6(2007) 1026-1058.

BosGou08 [5] M. Bostan, T. Goudon, High-electric-field limit for the Vlasov-Maxwell-Fokker-Planck system. Annales de l'I.H.P. Analyse non linéaire,25(2008) 1221-1251.

BouDol [6] F. Bouchut, J. Dolbeaut, On long asymptotics of the Vlasov–Fokker–Planck equation and of the Vlasov–Poisson–Fokker–Planck system with Coulombic and Newtonian potentials, Differential Integ. Equations 8(1995) 487–515.

Bre2000 [7] Y. Brenier, Convergence of the Vlasov-Poisson system to the incompressible Euler equations, Comm. Paryial Differential Equations 25(2000) 737-754.

BreMauPue2003 [8] Y. Brenier, N. Mauser, M. Puel, Incompressible Euler and e-MHD scaling limits of the Vlasov-Maxwell system, Commun. Math. Sci. 1(2003) 437-447.

Cha1949 [9] S. Chandrasekhar, Brownian motion, dynamic friction and stellar dynamics, Rev. Mod. Physics 21(1949) 383-388.

VicDwy [10] H. D. Victory and B. P. O'Dwyer, On classical solutions of VPFP systems, Indiana Univ. Math. J. 39(1990) 105–157.

Degond1986 [11] P.Degond, Global existence of smooth solutions for the Vlasov-Fokker-PLanck equation in 1 and 2 space dimensions, Ann Sci. École Norm. Sup., 19(1986), 519-542.

CarSol95 [12] J.A. Carrillo, J. Soler, On the initial value problem for the VPFP system with initial data in L p spaces, Math. Methods Appl. Sci. 18(1995) 825–839.

Csi1967 [13] I. Csiszár , Information-type measures of difference of probability distributons and indirect observation, Studia Sci. Math. Hungar, 2(1967) 299-318.

Victory91 [14] H. D. Victory, On the existence of global weak solutions for VPFP systems, J. Math. Anal. Appl. 160(1991) 515–553.

DegFil16 [15] P. Degond, F. Filbet, On the asymptotic limit of the three dimensional Vlasov-Poisson system for large magnetic field: formal derivation, Journal of Statistical Physics volume 165(2016), 765–784.

DiPeLion [16] R.J. DiPern, P.L. Lions, Ordinary differentail equations, transport theory and Sobolev spaces, Invent. math, 98(1989), 511-547.

GolRay99 [17] F. Golse, L. Saint-Raymond, The Vlasov-Poisson system with strong magnetic field, J. Math. Pures Appl, 78(1999), 791–817.

GolSaintQuas2003 [18] F. Golse, L. Saint-Raymond, The Vlasov-Poisson system with strong magnetic field in quasineutral regime, Math. Models Methods Appl. Sci., 13(2003) 661-714.

GouJabVas2004 [19] T. Goudon, P.-E. Jabin, A. Vasseur, Hydrodynamic limits for the Vlasov-Navier-Stokes equations. Part II: Fine particles regimes, Indiana Univ. Math. J. 53(2004) 1517-1536.

GouNiePouSol [20] T. Goudon, J. Nieto, F. Poupaud, J. Soler, Multidimensional high-field limit of the electrostatic Vlasov–Poisson–Fokker–Planck system, J. Differential Equations 213 (2005) 418-442.

HerRod [21] M. Herda, L.M. Rodrigues, Anisotropic Boltzmann-Gib dynamics of strongly magnetized Vlasov-Fokker-Planck equations, Kinetic and Related Models, 12(2019) 593-636.

KarMelTri [22] T.K. Karper, A. Mellet, K. Trivisa, Existence of weak solution to kinetic flocking models, SIAM Math.Anal., 45(2013), 215-243.

Kul1967 [23] S. Kullback, A lower bound for discrimination in information in terms of variation, IEEE TRans. Information Theory 4(1967) 126-127.

Lions61 [24] J. L. Lions, Equations differentielles operationnelles et problemes aux limites. Springer, Berlin, 1961.

Miot [25] É. Miot, On the gyrokinetic limit for the two-dimensional vlasov-poisson system, arXiv preprint arXiv :1603.04502, 2016.

PouPerSol [26] F. Poupaud, J. Soler, Parabolic limit and stability of the Vlasov-Poisson-Fokker-Planck system, Mathematical Models and Methods in Applied Sciences, 10(2000), 1027–1045.

PerSou [27] B. Perthame, P.E. Souganidis, A limiting case for velocity averaging, Ann. Sci. École Norm Sup, 31(1998) 591-598.

PerLions [28] R.J. Di Perna, P.L. Lions, Global weak solutions of Vlasov-Maxwell Systems, Comm. Pure Appl. Math., Vol. 42(1989) 729-757.

PueSaint2004 [29] M. Puel, L. Saint-Raymond, Quasineutral limit for the relativistic Vlasov-Maxwell system, Asymptot. Anal. 40(2004) 303-352.

Ray2002 [30] L. Saint-Raymond, Control of large velocities in the two-dimensional gyrokinetic approximation, J. Math. Pures Appl, 81(2002) :379–399.

Saint2003 [31] L. Saint-Raymond, Convergence of solutions to the Boltzmann equation in the incompressible Euler lomit, Arch. Ration. Mech. Anal. 166(2003) 47-80.

ReinWeck [32] G. Rein, J. Weckler, Generic global classical solutions of the Vlasov–Fokker–Planck–Poisson system in three dimensions, J. Differential Equations, 99(1992) 59–77.

Yau1991 [33] H.T. Yau, relative entropy and hydrodynamics of Ginzburg-Landau models, Lett. Math. Phys. 22(1991) 63-80.