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LONG TERM SPATIAL HOMOGENEITY FOR A CHEMOTAXIS MODEL WITH LOCAL SENSING AND CONSUMPTION

PHILIPPE LAURENÇOT

ABSTRACT. Global weak solutions to a chemotaxis model with local sensing and consumption are shown to converge to spatially homogeneous steady states in the large time limit, when the motility is assumed to be positive and C^1 -smooth on $[0, \infty)$. The result is valid in arbitrary space dimension $n \geq 1$ and extends a previous result which only deals with space dimensions $n \in \{1, 2, 3\}$.

1. INTRODUCTION

Let Ω be a smooth bounded domain of \mathbb{R}^n , $n \geq 1$, and consider the initial boundary value problem

$$\partial_t u = \Delta(u\gamma(v)) \quad \text{in } (0, \infty) \times \Omega, \quad (1.1a)$$

$$\partial_t v = \Delta v - uv \quad \text{in } (0, \infty) \times \Omega, \quad (1.1b)$$

$$\nabla(u\gamma(v)) \cdot \mathbf{n} = \nabla v \cdot \mathbf{n} = 0 \quad \text{on } (0, \infty) \times \partial\Omega, \quad (1.1c)$$

$$(u, v)(0) = (u^{in}, v^{in}) \quad \text{in } \Omega, \quad (1.1d)$$

which describes the dynamics of a population of bacteria with non-negative density u and of a signal with non-negative concentration v . On the one hand, according to (1.1a), the diffusive motion of the bacteria is not only monitored by the signal through the motility function γ but also biased by a chemotactic effect generated by the signal. On the other hand, the signal is consumed by the bacteria, as reflected by the reaction term on the right hand side of (1.1b). The latter mechanism is in sharp contrast with classical Keller-Segel chemotaxis models [4], in which the sink term $-uv$ in (1.1b) is replaced by $u - v$, so that bacteria produce the signal that alters their motion, see the survey articles [1–3, 8] and the references therein for a more precise account. Therefore, the dynamics of (1.1) is expected to differ significantly. A first hint in that direction is the following property: if (u_s, v_s) is a stationary solution to (1.1) with $u_s \not\equiv 0$, then necessarily $v_s \equiv 0$ by (1.1b). In that case, it readily follows from (1.1a) that $\gamma(0)u_s = \text{const.}$, which reduces to $u_s = \text{const.}$ when $\gamma(0) > 0$. It is thus expected that the positivity of both $\gamma(0)$ and $\|u^{in}\|_1$ implies that any global non-negative

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solution (u, v) to (1.1) satisfies

$$\lim_{t \rightarrow \infty} (u(t), v(t)) = \left(\frac{\|u^{in}\|_1}{|\Omega|}, 0 \right) \quad (1.2)$$

in an appropriate topology. That this convergence holds true in $L^\infty(\Omega, \mathbb{R}^2)$ is shown in [5, Theorem 1.2] when $\gamma \in C^3([0, \infty))$ is positive on $[0, \infty)$ with $\gamma' < 0$ on $(0, \infty)$ and $\|v^{in}\|_\infty$ is sufficiently small and in [6] when $\gamma \in C^3([0, \infty))$ is positive on $[0, \infty)$ and the space dimension n ranges in $\{1, 2, 3\}$. The required regularity of γ is subsequently relaxed in [7, Theorem 1.2], where the validity of (1.2) is established under the sole assumption

$$\gamma \in C^1([0, \infty)), \quad \gamma > 0 \quad \text{on} \quad [0, \infty), \quad (1.3)$$

still for $n \in \{1, 2, 3\}$, though in the weaker topology $H^1(\Omega)' \times L^\infty(\Omega)$. The main purpose of this note is to show that the assumption (1.3) is sufficient to prove that the convergence (1.2) holds true in arbitrary space dimension in $H^1(\Omega)' \times H^1(\Omega)$, see Theorem 1.1 below. When $n \in \{1, 2, 3\}$, we further deduce the convergence of v to zero in $L^\infty(\Omega)$ from Theorem 1.1, the continuous embedding of $H^2(\Omega)$, and the time monotonicity of $\|v\|_\infty$, see Corollary 1.3 below.

The statement of the main result of this note requires to introduce some notation: first, for $z \in H^1(\Omega)'$, we set $\langle z \rangle := \langle z, 1 \rangle_{(H^1)', H^1} / |\Omega|$ and note that

$$\langle z \rangle = \frac{1}{|\Omega|} \int_{\Omega} z(x) \, dx \quad \text{for} \quad z \in H^1(\Omega)' \cap L^1(\Omega).$$

Next, for $z \in H^1(\Omega)'$ with $\langle z \rangle = 0$, let $\mathcal{K}[z] \in H^1(\Omega)$ be the unique (variational) solution to

$$-\Delta \mathcal{K}[z] = z \quad \text{in} \quad \Omega, \quad \nabla \mathcal{K}[z] \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega, \quad (1.4a)$$

satisfying

$$\langle \mathcal{K}[z] \rangle = 0. \quad (1.4b)$$

Also, for $p \in [1, \infty]$, we denote the positive cone of $L^p(\Omega)$ by $L_+^p(\Omega)$.

Theorem 1.1. *Assume that γ satisfies (1.3) and consider $u^{in} \in L_+^1(\Omega) \cap H^1(\Omega)'$ and $v^{in} \in L_+^\infty(\Omega) \cap H^1(\Omega)$ with $M := \langle u^{in} \rangle > 0$. If (u, v) is a global weak solution to (1.1) in the sense of Definition 2.1 below, then*

$$\lim_{t \rightarrow \infty} \|\nabla P(t)\|_2 = \lim_{t \rightarrow \infty} \|v(t)\|_{H^1} = 0, \quad (1.5)$$

$$\lim_{t \rightarrow \infty} \int_t^{t+1} \|u(s) - M\|_2^2 \, ds = \lim_{t \rightarrow \infty} \int_t^{t+1} \|v(s)\|_{H^2}^2 \, ds = 0, \quad (1.6)$$

where $P(t) := \mathcal{K}[u(t) - M]$ for $t \geq 0$.

As already mentioned, Theorem 1.1 supplements previous results in the literature showing the long term convergence of $(u - M, v)$ to zero, either in low space dimension $n \in \{1, 2, 3\}$, see [7, Theorem 1.2], or when $\|v^{in}\|_\infty$ is sufficiently small, see [5, Theorem 1.2]. As in [7], the proof of Theorem 1.1 relies on the so-called duality estimate derived from (1.4a) (Lemma 2.3) and the dissipativity properties of (1.1b) (Lemma 2.4). The building block of the proof is to show that $\|\nabla P\|_2^2 + a\|v\|_2^2$ is a

Liapunov functional for (1.1) for a suitable choice of $a > 0$. This step is the main difference with the approach developed in [7] where a functional of the form $\|\nabla P\|_2^2 + b\|\nabla v\|_2^2$ with $b > 0$ is used.

Remark 1.2. *When $\gamma(0) = 0$, Theorem 1.1 is no longer true and convergence of $u(t)$ as $t \rightarrow \infty$ to a non-constant limit may take place, see [10, Theorem 1.5].*

We do not address here the issue of the existence of global solutions to (1.1) and refer to [5,6,10] for the existence of global bounded classical solutions and to [5–7,9,11] for that of global weak solutions under various assumptions on γ (with either $\gamma(0) = 0$ or $\gamma(0) > 0$) and the space dimension n . In particular, given a global weak solution (u, v) to (1.1) constructed in [6,7] and $t_0 > 0$, $(t, x) \mapsto (u, v)(t + t_0, x)$ is a weak solution to (1.1) in the sense of Definition 2.1, so that the convergence stated in Theorem 1.1 applies to these solutions.

We next combine (1.6) and the time monotonicity of the L^∞ -norm of v to supplement the convergence (1.5) of v in $H^1(\Omega)$ with convergence to zero of v in $L^\infty(\Omega)$ when the space dimension n ranges in $\{1, 2, 3\}$, thereby recovering the outcome of [7, Theorem 1.2].

Corollary 1.3. *Assume that $n \in \{1, 2, 3\}$. Under the assumptions of Theorem 1.1, one has also*

$$\lim_{t \rightarrow \infty} \|v(t)\|_\infty = 0.$$

2. PROOFS

We begin with the definition of a global weak solution to (1.1) and introduce the Hilbert space

$$H_N^2(\Omega) := \{z \in H^2(\Omega) : \nabla z \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\},$$

which is actually the domain of the Laplace operator in $L^2(\Omega)$ supplemented with homogeneous Neumann boundary conditions.

Definition 2.1. *Consider $u^{in} \in L_+^1(\Omega) \cap H^1(\Omega)'$ and $v^{in} \in L_+^\infty(\Omega) \cap H^1(\Omega)$. A global weak solution to (1.1) is a couple of non-negative functions*

$$(u, v) \in C_w([0, \infty), H^1(\Omega)') \times C([0, \infty), L^2(\Omega))$$

satisfying, for any $t > 0$,

$$\begin{aligned} u &\in L^2((0, t) \times \Omega), \\ v &\in L^\infty((0, t) \times \Omega) \cap W^{1,2}((0, t), L^2(\Omega)) \cap L^2((0, t), H_N^2(\Omega)), \end{aligned}$$

along with

$$\int_\Omega u(t)\vartheta(t) \, dx - \int_\Omega u^{in}\vartheta(0) \, dx = \int_0^t \int_\Omega [u\gamma(v)\Delta\vartheta + u\partial_t\vartheta] \, dx ds \quad (2.1a)$$

for all $\vartheta \in W^{1,2}((0, t), L^2(\Omega)) \cap L^2((0, t), H_N^2(\Omega))$ and

$$\begin{aligned} \partial_t v - \Delta v + uv &= 0 \quad \text{a.e. in } (0, t) \times \Omega, \\ \nabla v \cdot \mathbf{n} &= 0 \quad \text{a.e. on } (0, t) \times \partial\Omega. \end{aligned} \quad (2.1b)$$

We recall that, given a Banach space X and $T \in (0, \infty]$, $C_w([0, T], X)$ denotes the space of weakly continuous functions from $[0, T]$ to X .

We next derive several estimates on u and v which are already well-known, see [7]. From now on, $(c_i)_{i \geq 1}$ denote positive constants depending only on Ω , γ in (1.3), u^{in} , and v^{in} .

Lemma 2.2. *For $t \geq 0$,*

$$\langle u(t) \rangle = M = \langle u^{in} \rangle \quad \text{and} \quad \|v(t)\|_\infty \leq V := \|v^{in}\|_\infty. \quad (2.2)$$

Proof. Lemma 2.2 readily follows from (2.1a) (with $\vartheta \equiv 1$), along with (2.1b), the non-negativity of uv , and the comparison principle. \square

We next exploit the specific form of (1.1a) to derive a so-called duality estimate on u .

Lemma 2.3. *Set $P = \mathcal{K}[u - M]$. Then, for $t > 0$,*

$$P \in W^{1,2}((0, t), L^2(\Omega)) \cap L^2((0, t), H_N^2(\Omega))$$

and

$$\frac{d}{dt} \|\nabla P\|_2^2 = -2 \int_\Omega \gamma(v)(u - M)^2 \, dx - 2M \int_\Omega \gamma(v)(u - M) \, dx. \quad (2.3)$$

Proof. As

$$\partial_t P = \langle u\gamma(v) \rangle - u\gamma(v) \quad \text{in} \quad (0, \infty) \times \Omega \quad (2.4)$$

by (1.4a) and (2.1a) (with a suitable choice of test functions), the claimed regularity of P follows from (1.3), (1.4), the square integrability of u , and the boundedness of v . It then follows from (2.4) that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla P\|_2^2 &= - \int_\Omega \partial_t P \Delta P \, dx = \int_\Omega (u - M) [\langle u\gamma(v) \rangle - u\gamma(v)] \, dx \\ &= - \int_\Omega \gamma(v) u (u - M) \, dx \\ &= - \int_\Omega \gamma(v) (u - M)^2 \, dx - M \int_\Omega \gamma(v) (u - M) \, dx, \end{aligned}$$

which completes the proof. \square

We next take advantage of the non-positivity of the right hand side of (1.1b) to obtain a classical energy estimate on v .

Lemma 2.4. *For $t \geq 0$,*

$$\frac{d}{dt} \|v\|_2^2 + 2\|\nabla v\|_2^2 + 2\|v\sqrt{u}\|_2^2 = 0.$$

At this point, we deviate from the proof of [7, Theorem 1.2] and construct a Liapunov functional associated with (1.1), building upon the outcome of Lemma 2.3 and Lemma 2.4. This is clearly

the main building block of the proof. As a preliminary step, we observe that the continuity and positivity (1.3) of γ and the boundedness (2.2) of v imply that

$$\gamma_* := \min_{s \in [0, V]} \{\gamma(s)\} > 0. \quad (2.5)$$

We also recall the Poincaré-Wirtinger inequality: there is $c_1 > 0$ such that

$$\|z - \langle z \rangle\|_2 \leq c_1 \|\nabla z\|_2, \quad z \in H^1(\Omega). \quad (2.6)$$

Proposition 2.5. *There is $c_2 > 0$ such that, for $t \geq 0$,*

$$\frac{d}{dt} \left(\|\nabla P\|_2^2 + c_2 \|v\|_2^2 \right) + \gamma_* \|u - M\|_2^2 + c_2 \|\nabla v\|_2^2 \leq 0.$$

Proof. It follows from (1.3), (2.2), (2.6), and Hölder's inequality that

$$\begin{aligned} 2M \int_{\Omega} \gamma(v)(u - M) \, dx &= 2M \int_{\Omega} [\gamma(v) - \gamma(\langle v \rangle)](u - M) \, dx \\ &\leq 2M \|\gamma'\|_{L^\infty(0, V)} \int_{\Omega} |v - \langle v \rangle| |u - M| \, dx \\ &\leq 2M \|\gamma'\|_{L^\infty(0, V)} \|v - \langle v \rangle\|_2 \|u - M\|_2 \\ &\leq 2M c_1 \|\gamma'\|_{L^\infty(0, V)} \|\nabla v\|_2 \|u - M\|_2. \end{aligned} \quad (2.7)$$

Setting $c_2 := (M c_1 \|\gamma'\|_{L^\infty(0, V)})^2 / \gamma_*$, we infer from Lemma 2.3, Lemma 2.4, (2.5), (2.7), and Young's inequality that

$$\begin{aligned} \frac{d}{dt} \left(\|\nabla P\|_2^2 + c_2 \|v\|_2^2 \right) &\leq -2\gamma_* \|u - M\|_2^2 - 2c_2 \|\nabla v\|_2^2 \\ &\quad + 2M c_1 \|\gamma'\|_{L^\infty(0, V)} \|\nabla v\|_2 \|u - M\|_2 \\ &\leq -2\gamma_* \|u - M\|_2^2 - 2c_2 \|\nabla v\|_2^2 \\ &\quad + \gamma_* \|u - M\|_2^2 + \frac{M^2 c_1^2 \|\gamma'\|_{L^\infty(0, V)}^2}{\gamma_*} \|\nabla v\|_2^2 \\ &= -\gamma_* \|u - M\|_2^2 - c_2 \|\nabla v\|_2^2, \end{aligned}$$

as claimed. □

We next argue as in [7, Lemma 3.2] to obtain additional information on v .

Lemma 2.6. *For $t \geq 0$,*

$$\frac{d}{dt} \|\nabla v\|_2^2 + 2M \|\nabla v\|_2^2 + \|\Delta v\|_2^2 \leq V^2 \|u - M\|_2^2.$$

Proof. We infer from (2.1b), (2.2), and Hölder's and Young's inequalities that

$$\frac{1}{2} \frac{d}{dt} \|\nabla v\|_2^2 + \|\Delta v\|_2^2 = \int_{\Omega} uv \Delta v \, dx$$

$$\begin{aligned}
&= \int_{\Omega} (u - M)v\Delta v \, dx + M \int_{\Omega} v\Delta v \, dx \\
&\leq V\|u - M\|_2\|\Delta v\|_2 - M\|\nabla v\|_2^2 \\
&\leq \frac{\|\Delta v\|_2^2}{2} + \frac{V^2\|u - M\|_2^2}{2} - M\|\nabla v\|_2^2,
\end{aligned}$$

from which Lemma 2.6 follows. \square

Summarizing the outcome of Proposition 2.5 and Lemma 2.6, we have so far obtained the following estimates on u and v .

Proposition 2.7. *There is $c_3 > 0$ such that*

$$\|P(t)\|_{H^1} + \|v(t)\|_{H^1} \leq c_3, \quad t \geq 0, \quad (2.8)$$

$$\int_0^\infty \left[\|u(s) - M\|_2^2 + \|\nabla v(s)\|_2^2 + \|\Delta v(s)\|_2^2 \right] ds \leq c_3, \quad (2.9)$$

and

$$\lim_{t \rightarrow \infty} \|\nabla v(t)\|_2 = 0. \quad (2.10)$$

Proof. The bounds (2.8) and (2.9) being immediate consequences of (1.4b), (2.6), Proposition 2.5 and Lemma 2.6, we are left with proving (2.10). To this end, we deduce from Lemma 2.6 that, for $t \geq 0$,

$$\|\nabla v(t)\|_2^2 \leq \|\nabla v^{in}\|_2^2 e^{-2Mt} + V^2 \int_0^t e^{2M(s-t)} \|u(s) - M\|_2^2 ds. \quad (2.11)$$

At this point, we recall that, if F belongs to $L^1(0, \infty)$, then a straightforward consequence of the Lebesgue dominated convergence theorem is that

$$\lim_{t \rightarrow \infty} \int_0^t e^{\alpha(s-t)} |F(s)| ds = 0 \quad (2.12)$$

for any $\alpha > 0$. Thanks to (2.9), $s \mapsto \|u(s) - M\|_2^2$ belongs to $L^1(0, \infty)$ and we use (2.12) (with $\alpha = 2M$ and $F = \|u - M\|_2^2$) to take the limit $t \rightarrow \infty$ in (2.11) and obtain (2.10), thereby completing the proof. \square

The final step of the proof of Theorem 1.1 deals with the convergence of $\|v(t)\|_1$ and $\|\nabla P(t)\|_2$ as $t \rightarrow \infty$.

Lemma 2.8.

$$\lim_{t \rightarrow \infty} \|v(t)\|_1 = \lim_{t \rightarrow \infty} \|\nabla P(t)\|_2 = 0.$$

Proof. Let us begin with the convergence of $\|v\|_1$ and infer from (2.1b), (2.2), and the non-negativity of v that, for $t \geq 0$,

$$\frac{d}{dt} \|v\|_1 = - \int_{\Omega} uv \, dx = - \int_{\Omega} (u - M)v \, dx - M\|v\|_1$$

$$= - \int_{\Omega} (u - M)(v - \langle v \rangle) \, dx - M \|v\|_1.$$

Hence, by (2.6) and Hölder's inequality,

$$\frac{d}{dt} \|v\|_1 + M \|v\|_1 \leq \|u - M\|_2 \|v - \langle v \rangle\|_2 \leq c_1 \|u - M\|_2 \|\nabla v\|_2.$$

We then integrate with respect to time to find

$$\|v(t)\|_1 \leq \|v^{in}\|_1 e^{-Mt} + c_1 \int_0^t e^{M(s-t)} \|u(s) - M\|_2 \|\nabla v(s)\|_2 \, ds. \quad (2.13)$$

Since $s \mapsto \|u(s) - M\|_2 \|\nabla v(s)\|_2$ belongs to $L^1(0, \infty)$ by (2.9), we deduce from (2.12) (with $\alpha = M$ and $F = \|u - M\|_2 \|\nabla v\|_2$) that the right hand side of (2.13) converges to zero as $t \rightarrow \infty$. Consequently,

$$\lim_{t \rightarrow \infty} \|v(t)\|_1 = 0.$$

Similarly, we infer from (2.3), (2.5), and (2.7) that, for $t \geq 0$,

$$\frac{d}{dt} \|\nabla P\|_2^2 \leq -2\gamma_* \|u - M\|_2^2 + 2Mc_1 \|\gamma'\|_{L^\infty(0,V)} \|\nabla v\|_2 \|u - M\|_2.$$

Moreover, by (1.4), (2.6), and Hölder's inequality,

$$\begin{aligned} \|\nabla P\|_2^2 &= - \int_{\Omega} P \Delta P \, dx = \int_{\Omega} (u - M) P \, dx \\ &\leq \|u - M\|_2 \|P\|_2 \leq c_1 \|u - M\|_2 \|\nabla P\|_2, \end{aligned}$$

so that

$$\|\nabla P\|_2 \leq c_1 \|u - M\|_2.$$

Gathering the above inequalities and setting $c_4 := 2\gamma_*/c_1^2$ and $c_5 := 2Mc_1 \|\gamma'\|_{L^\infty(0,V)}$, we end up with

$$\frac{d}{dt} \|\nabla P\|_2^2 + c_4 \|\nabla P\|_2^2 \leq c_5 \|\nabla v\|_2 \|u - M\|_2.$$

Hence, after integration with respect to time,

$$\|\nabla P(t)\|_2^2 \leq \|\nabla P(0)\|_2^2 e^{-c_4 t} + c_5 \int_0^t e^{c_4(s-t)} \|\nabla v(s)\|_2 \|u(s) - M\|_2 \, ds,$$

and we argue as above to conclude that $\|\nabla P(t)\|_2$ converges to zero as $t \rightarrow \infty$. \square

Theorem 1.1 is now an immediate consequence of Proposition 2.7 and Lemma 2.8.

Proof of Theorem 1.1. The convergences (1.5) follow from (2.6), (2.10), and Lemma 2.8, while the time integrability (2.9) of $\|u - M\|_2$ and $\|\Delta v\|_2$, along with (1.5) and elliptic regularity, gives (1.6). \square

We finally provide the proof of Corollary 1.3.

Proof of Corollary 1.3. Since $n \in \{1, 2, 3\}$, the space $H^2(\Omega)$ is continuously embedded in $L^\infty(\Omega)$ and we deduce from (1.6) that

$$\lim_{t \rightarrow \infty} \int_t^{t+1} \|v(s)\|_\infty^2 \, ds = 0.$$

We then infer from (1.1b), (1.1c), the non-negativity of u and v , and the comparison principle that

$$\|v(t+1)\|_\infty \leq \|v(s)\|_\infty, \quad s \in [t, t+1].$$

Combining the previous two properties readily gives

$$\lim_{t \rightarrow \infty} \|v(t)\|_\infty^2 = \lim_{t \rightarrow \infty} \|v(t+1)\|_\infty^2 \leq \lim_{t \rightarrow \infty} \int_t^{t+1} \|v(s)\|_\infty^2 \, ds = 0,$$

which completes the proof. □

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