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# THE HORIZONTAL CHORD SET TO CIRM AND BACK

DIANA DAVIS AND SERGE TROUBETZKOY

ABSTRACT. We study the set of lengths of the horizontal chords of a continuous function. We show that no matter which function we choose, at least half of the possible lengths occur, and we prove several results about functions for which all the possible lengths occur.

## 1. STORY

On a beautiful summer day, two mathematicians walk from the CIRM research institute down to the sea at the Calanque de Sugiton, and then walk back via the same path, the whole trip taking one hour. They wonder: is there a spot on the route that they passed by two times, exactly 23 minutes apart? How about  $\ell$  hours apart, for any  $\ell \in [0, 1]$ ?

In this paper, we show that the answer to the above question is “yes.” Such a spot exists for every  $\ell$  even if the mathematicians wander back and forth on the trail along the way (left and middle pictures in Figure 1). On the other hand, if the mathematicians pass by CIRM on their return and walk in the other direction, and then go back to part of the way to the Calanque before returning to CIRM (right picture in Figure 1), there is no such guarantee; however, at least half of the possible times must be achieved.

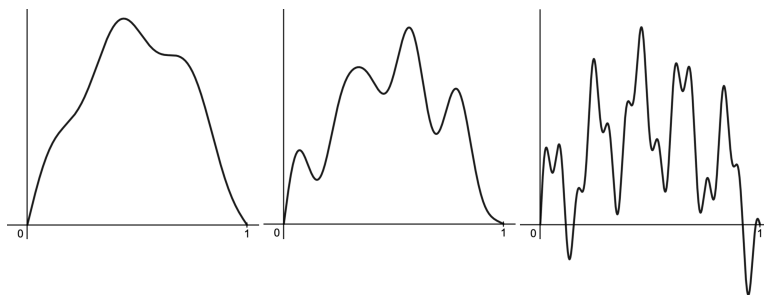


FIGURE 1. A simple hike, a meandering hike, and a wandering hike. Time is on the horizontal axis, and distance from CIRM towards the Calanque is on the vertical axis.

A different application of the issues discussed in this article, to average pace over time, was given in [BDD17].

## 2. DEFINITIONS

By shifting and scaling, we may assume that length (in time) of the hike is 1, and that the common value of  $f(0)$  and  $f(1)$  is 0. Let

$$C := \{f : [0, 1] \rightarrow \mathbb{R}, f \text{ is continuous such that } f(0) = f(1) = 0\}.$$

The *horizontal chord set* is the set of lengths horizontally connecting two points on the graph, i.e.,

$$S(f) := \{\ell \in \mathbb{R} : \text{there exists } s \in [0, 1] \text{ with } f(s) = f(s + \ell)\}.$$

Hopf has characterized which sets  $S \subset \mathbb{R}$  can be a horizontal chord set for some  $f \in C$  ([H36], see [BDD17] for an English version of Hopf's proof). We are interested in the following question: for which functions  $f \in C$  does  $S(f) = [0, 1]$ ? We call this the *full chord property*.

**Definition 1.** Suppose  $f$  is a continuous function,  $s_1 < s_2$ ,  $a > 0$ ,  $f(s_1) = f(s_2) = 0$  and  $f((s_1, s_2)) = (0, a]$ . Then we call  $f|_{[s_1, s_2]}$  a *mountain with endpoints*  $s_1$  and  $s_2$ . Fix  $t \in [s_1, s_2]$  so that  $f(t) = a$ . (Note that while there may be several choices for  $t$ , none of our results depend on the choice.) Then we call  $f|_{[s_1, t]}$  the *ascent*,  $f|_{[t, s_2]}$  the *descent*,  $a$  the *height*, and  $|s_2 - s_1|$  the *width*.

A *valley* is exactly analogous to the above, with  $a < 0$  and “ascent” and “descent” exchanged.

A *mountain range* is a union of contiguous mountains, and possibly also intervals with  $f(x) = 0$  (including intervals degenerate to a point), i.e., essentially the same definition as above but with  $f([s_1, s_2]) = [0, a]$ . The *height* of a mountain range is the height of its tallest mountain, which exists because a continuous function has a maximum on a compact set. The ascent, descent, and width are as above. We define a *valley range* analogously.

Note that mountain ranges and valley ranges can include horizontal intervals satisfying  $f(x) = 0$ . Such a horizontal piece between a mountain range and a valley range can be considered as part of either one; the choice does not affect our results.

When we refer to a mountain or a valley range we will always implicitly assume it is a *maximal* mountain or valley range, modulo the choice of assignment of horizontal pieces at  $f(x) = 0$  as described above.

## 3. THERE AND BACK AGAIN

**Theorem 2.** *If  $f$  is continuous and  $f|_{[s_1, s_2]}$  is a mountain range, then  $[0, s_2 - s_1] \subset S(f)$ .*

**Corollary 3.** *If  $f(x) \geq 0$  for all  $x \in [0, 1]$  then  $f$  has the full chord property.*

In particular the walk to Sugiton has the full chord property. Both of these results also hold for valley ranges.

**Proof.** Fix a mountain range  $f|_{[s_1, s_2]}$ , and consider the shifted mountain range  $f(x - \ell)$  (see Figure 2). For any shift  $\ell \in [0, s_2 - s_1]$ , exactly one endpoint of the mountain range is inside the other copy. Thus it follows from the Intermediate Value Theorem that the two mountain ranges intersect, so  $f(x) = f(x - \ell)$  and there is a horizontal chord of length  $\ell$ .  $\square$

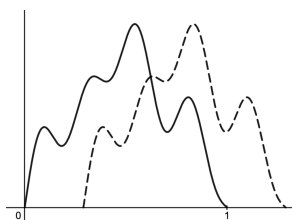


FIGURE 2. A mountain range and its shift always intersect.

#### 4. STRUCTURE OF THE CHORD SET

First, we show that the chord set may not be the entire interval  $[0, 1]$ :

**Theorem 4.** *If  $f$  has a mountain at one endpoint and a valley at the other endpoint, then  $f$  does not have the full chord property.*

**Proof.** Let  $w_1, w_2$  be the width of the mountain and the valley, respectively. Then any chord with horizontal displacement in the range  $[1 - \min\{w_1, w_2\}, 1)$  must have one endpoint in the mountain and one endpoint in the valley. Thus the function values at these two endpoints have different signs, so the chord between them is not horizontal.  $\square$

Most of this paper consists of exploring closed intervals of chord lengths. However, the following figure shows that  $S(f)$  does not necessarily consist of unions of closed intervals; there is a possibility that it can include isolated points:

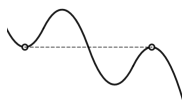


FIGURE 3. A chord that can be slid neither up nor down.

By making lots of bumps, we can make the set of such additional chord lengths into a set with accumulation points.

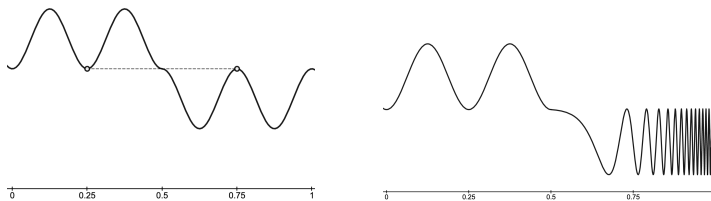


FIGURE 4. A function with  $S(f) = [0, 1/2] \cup \{3/4, 1\}$  and one where  $S(f)$  has many accumulation points.

### 5. AT LEAST HALF THE LENGTHS

In this section, we will use Hopf's classical theorem: *for the horizontal chord set  $S(f)$ , the set  $S^c := \mathbb{R}^+ \setminus S(f)$  is open and additive; i.e., if  $a, b \in S^c$  then  $a + b \in S^c$ .* The other main ingredient in this section is the central symmetry  $1 \rightarrow 1 - \ell$  of the interval  $[0, 1]$ . The image of  $S(f)$  by this symmetry is the set  $T(f) := \{\ell : 1 - \ell \in S(f)\}$ . Denote the Lebesgue measure on  $[0, 1]$  by  $\lambda$ . We have the following result:

**Theorem 5.** *Let  $f : [0, 1] \rightarrow \mathbf{R}$  be a continuous function with  $f(0) = f(1) = 0$ . Then the length of  $S(f)$  satisfies*

$$\lambda(S(f)) \geq \frac{1}{2}.$$

*This bound is sharp.*

**Proof.** Since the map  $\ell \rightarrow 1 - \ell$  preserves the Lebesgue measure we have

$$\lambda(T(f)) = \lambda(S(f)).$$

By Hopf's theorem if  $a + b \notin S^*$  then we must have either  $a$  or  $b$  not in  $S^c$ , i.e., at least one of  $a$  or  $b$  belongs to  $S(f)$ . We apply this to  $a = \ell$  and  $b = 1 - \ell$ , since we know  $1 \in S(f)$  we conclude that either  $\ell$  or  $1 - \ell$  is contained in  $S(f)$ , or equivalently either  $\ell \in S(f)$  or  $\ell \in T(f)$ ; i.e.,

$$S(f) \cup T(f) = [0, 1].$$

If  $\lambda(S(f)) < 1/2$  the two display equations yield an immediate contradiction, thus the first statement of the theorem holds.

To show that the bound is sharp, we construct an example of a function whose chord set has length  $1/2$ , with any desired number  $n$  of mountains and valleys.

For each  $n \in \mathbf{N}$ , consider the complementary sets

$$S_n = \left[ \frac{0}{n}, \frac{1}{n+1} \right], \left[ \frac{1}{n}, \frac{2}{n+1} \right], \dots, \left[ \frac{n-1}{n}, \frac{n}{n+1} \right], \text{ and}$$

$$S_n^* = \left( \frac{1}{n+1}, \frac{1}{n} \right), \left( \frac{2}{n+1}, \frac{2}{n} \right), \dots, \left( \frac{n-1}{n+1}, \frac{n-1}{n} \right), \left( \frac{n}{n+1}, 1 \right).$$

It is clear that  $S_n^*$  is an additive set, because all of the intervals are integer multiples of the first one. Thus since  $S_n^*$  is an open additive set, by Hopf's theorem it is the complement of the chord set of some function on  $[0, 1]$ . That chord set is  $S_n$ .

The total length of each  $S_n$  is  $1/2$ :

$$\begin{aligned} \sum_{k=1}^n \left( \frac{k}{n+1} - \frac{k-1}{n} \right) &= \sum_{k=1}^n \frac{k}{n+1} - \sum_{k=1}^n \frac{k-1}{n} \\ &= \frac{1}{n+1} \frac{n(n+1)}{2} - \frac{1}{n} \frac{(n-1)n}{2} = \frac{1}{2}. \end{aligned}$$

Given any open, additive set  $S^*$ , Hopf gives a construction for a function  $h_S(x)$  whose horizontal chord set is exactly  $S^*$ . Examples of functions  $h_{S_n}(x)$  with  $n$  mountains and valleys, and chord set with length  $1/2$ , given by Hopf's construction are shown in Figure 5. Thus, for each  $n$ , this construction gives the desired function with  $n$  bumps and horizontal chord set of length  $1/2$ .  $\square$

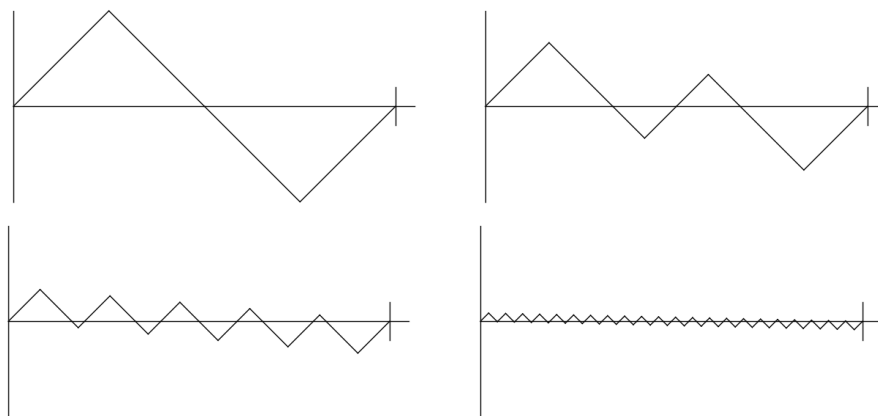


FIGURE 5. Functions having a horizontal chord set of length exactly  $1/2$ , with 1, 2, 5 and 22 mountains, respectively.

Our proof leads to the following natural question, does there exist a example of an  $f$  with countably many mountain ranges and countably many valley ranges such that  $\lambda(S(f)) = 1/2$ ?

## 6. TWO MOUNTAIN RANGES SEPARATED BY A VALLEY RANGE

It would be nice to give a classification of which functions' chord sets have the full chord property, but the general case seems complicated. Thus here we will only analyze the simplest non-trivial case, the piecewise affine case with two mountains and one valley in between.

Let  $w_\ell, w_v, w_r > 0$  be the widths of the left mountain, valley, and right mountain, which are normalized to add up to one, and let  $h_\ell, h_r > 0$  and  $h_v < 0$  be their heights. Since the chord set does not depend on vertical scaling we normalize  $\max(h_\ell, h_r) = 1$ . We also use the widths of the ascent and descent of each mountain, which we call  $a_\ell, d_\ell, a_r$  and  $d_r$ , respectively (see Figure 6).

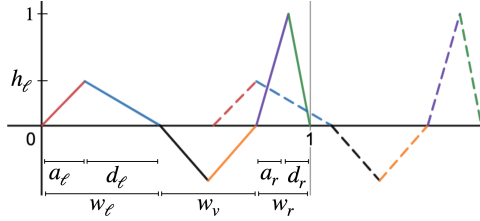


FIGURE 6. A narrow tall mountain always intersects a wide short mountain if their bases intersect. The image also shows the variables defined above.

We will use the notation  $f_{\mathbf{w},\mathbf{a},\mathbf{d},\mathbf{h}}$ . Certain variables are redundant since a width is the sum of its ascent and descent, and one of the ascents or descents or widths is also redundant because of our normalization, thus we have a seven dimensional parameter space along with one discrete parameter. However the constraints in the next theorem live in a 5-dimensional subspace of our 7-dimensional space because of the three parameters needed to describe a valley only its width plays a role.

**Theorem 6.** *The piecewise linear map  $f_{\mathbf{w},\mathbf{d},\mathbf{a},\mathbf{h}}$  does **not** have the full chord property if and only if the parameters satisfy one of the following two conditions*

- (1) (i)  $w_\ell < w_r$ , (ii)  $h_\ell < 1$ , (iii)  $h_\ell \cdot a_r + 1 - w_r - a_\ell < w_v + w_r$  and (iv)  $1 - h_\ell \cdot d_r - a_\ell > \max(w_r; w_\ell + w_v)$ , or
- (2) (i)  $w_r < w_\ell$ , (ii)  $h_r < 1$ , (iii)  $h_r \cdot a_\ell + 1 - w_\ell - a_r < w_v + w_\ell$  and (iv)  $1 - h_r \cdot d_\ell - a_r > \max(w_\ell; w_r + w_v)$ .

**Proof.** There are two different ways to get a chord value. The first possibility is that either the chord connects the two sides of the same mountain or the two side of the same valley, for such chord lengths Theorem 2 implies  $S_{\text{init}} := [0, w_{\text{max}}] \subset S(f)$  where  $w_{\text{max}} := \max(w_\ell, w_v, w_r)$ .

Because of the special form of our graph there is only one other type of chord length possible, namely the lengths of chords that arise between the two mountains, or equivalently when the left mountain shifted by such a length intersects the right mountain.

Clearly if the narrower mountain is at least as high as the wider mountain, then the mountains always intersect when their bases overlap (Figure 6), so such a function has the full chord property. Thus in order not to have the full chord property, one mountain must be narrower and shorter. We can restrict to two cases:

**Case (1)** (Figure 7) Suppose first that the narrower and shorter mountain is on the left ((i)  $w_\ell < w_r$  and (ii)  $h_\ell < h_r = 1$ ).

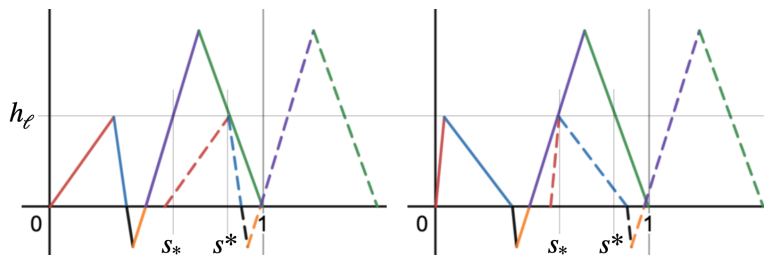


FIGURE 7. The case when a narrower *and shorter mountain is on the left*, along with the definitions of  $s_*$  and  $s^*$ .

No matter which parameters we choose, chord lengths between the two mountains arise if the bases of the shifted left mountain and the right mountain intersect, but neither base is contained in the other base. This set consists of two intervals described below.

When a parameter value belongs to the interval  $S_{\text{mid}} := [w_v, w_\ell + w_v]$ , the shift of the left mountain intersects the ascent of the right mountain, and so  $S_{\text{mid}} \subset S(f)$  (Figure 8).

When a parameter value belongs to the interval  $S_{\text{fin}} := [w_r + w_v, 1]$ , the shift of the left mountain intersects the descent of the right mountain, so  $S_{\text{fin}} \subset S(f)$ .

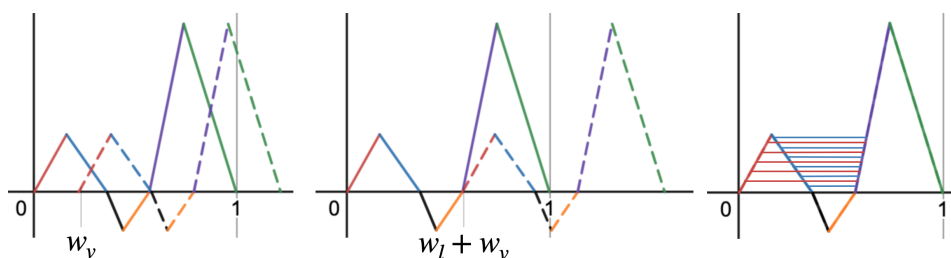


FIGURE 8. (Left) a shift by the parameter  $w_v$  (middle) a shift by the parameter  $w_l + w_v$  (right) the horizontal chord lengths in  $S_{\text{mid}}$

Summarizing, we have shown that  $S_{\text{init}} \cup S_{\text{mid}} \cup S_{\text{fin}} \subset S(f)$  no matter which parameter values we choose, as long as the narrower shorter mountain is on the left.



We will now analyze the complementary set  $T := (S_{\text{init}} \cup S_{\text{mid}} \cup S_{\text{fin}})^c$ . The intervals  $S_{\text{init}}$  and  $S_{\text{mid}}$  abut or overlap since  $w_v \leq w_{\text{max}}$ . The larger of the two right endpoints of these two intervals is

$$\max(w_{\text{max}}, w_\ell + w_v) = \max(w_\ell, w_v, w_r, w_\ell + w_v) = \max(w_r, w_\ell + w_v).$$

Thus we have

$$(1) \quad T = (\max(w_r; w_\ell + w_v), w_v + w_r).$$

Note that the set  $T$  is contained in the set of parameters for which the base of the shifted left mountain is inside the base of the right mountain, and is a strict subset if  $S_{\text{mid}} \subset S_{\text{init}} \setminus [0, w_v)$ , or equivalently if  $w_r > w_\ell + w_v$ .

If the ascent of the smaller mountain is steeper than the ascent of the larger mountain, or if the descent of the smaller mountain is steeper than the descent of the larger mountain, then we get an additional interval of chord values in  $S(f)$  that are in  $T$ , one for each such case.

There are three solutions to the equation  $f(s) = h_\ell$ ,  $a_\ell < s_* < s^*$  (pictured in Figure 7). The zero, one or two additional intervals of chord values in  $T$  are then  $T_1 := [\max(w_r; w_\ell + w_v); s_* - a_\ell]$  and  $T_2 := [s^* - a_\ell; w_v + w_\ell]$ , where here an interval  $[a, b]$  with  $b < a$  is interpreted as the empty interval.

To calculate  $s_*$  and  $s^*$  consider the equations of the ascent and descent of the right mountain, recalling that  $h_r = 1$ :

$$y = \frac{1}{a_r}(s - (1 - w_r)) \quad \text{and} \quad y = \frac{-1}{d_r}(s - 1).$$

Replacing  $y$  by  $h_\ell$  in these two equations yields

$$s_* = h_\ell \cdot a_r + 1 - w_r \quad \text{and} \quad s^* = 1 - h_\ell \cdot d_r.$$

Thus the two additional intervals are

$$(2) \quad T_1 := [\max(w_r; w_\ell + w_v); h_\ell \cdot a_r + 1 - w_r - a_\ell] \quad \text{and}$$

$$(3) \quad T_2 := [1 - h_\ell \cdot d_r - a_\ell; w_v + w_\ell].$$

The intervals  $T_1$  and  $T_2$  never overlap since  $s_* < s^*$ , but either one of them can cover  $T$ . Thus, the full chord property in the case under consideration is equivalent to having either  $T \subset T_1$  or  $T \subset T_2$ .

Comparing equations (1) and (2)–(3), we see that  $T_1$  covers  $T$  if and only if

$$h_\ell \cdot a_r + 1 - w_r - a_\ell \geq w_v + w_r$$

and the right end point of  $T_2$  is larger than the right endpoint of  $T$ , thus  $T_2$  covers  $T$  if and only if

$$1 - h_\ell \cdot d_r - a_\ell \leq \max(w_r; w_\ell + w_v).$$

If neither of these last two equations, which correspond to the negations of (iii) and (iv), hold, then we do not have the full chord property.

**Case (2)** (Figure 9) Now suppose that the narrower and shorter mountain is on the right ((i)  $w_r < w_\ell$  and (ii)  $h_r < h_\ell = 1$ ).

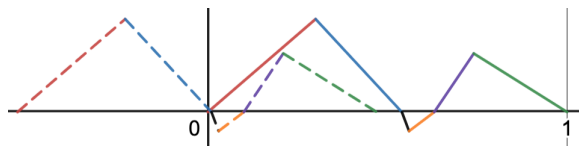


FIGURE 9. The case when the right mountain is narrower and shorter

We take the walk backwards, in other words, we consider the function  $g(s) := f \circ i(s)$  where  $i$  is the isometry  $i : s \mapsto 1 - s$ . The function  $g$  has the same chord set as  $f$ , and furthermore if  $f$  is in case (2) then  $g$  is in case (1). Applying the characterization proven in case (1) to  $g$  and reinterpreting this characterization in terms of the parameters of  $f$  yields: (i)  $w_r < w_\ell$ , (ii)  $h_r < 1$ , (iii)  $h_r \cdot a_\ell + 1 - w_\ell - a_r < w_v + w_\ell$ , and (iv)  $1 - h_r \cdot d_\ell - a_r > \max(w_\ell; w_r + w_v)$ .  $\square$

A Monte Carlo simulation suggests that the percentage of such functions with the full chord property is approximately 70.4%. In principle an exact calculation is possible, but as the constraints are not linear, it is not elementary.

## 7. ACKNOWLEDGMENTS

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## REFERENCES

- [BDD17] Keith Burns, Orit Davidovich, Diana Davis, *Average pace and horizontal chords*, The Mathematical Intelligencer, 39(4) (2017) 41-45.
- [D] *Desmos*, <https://www.desmos.com/calculator/>, accessed 2022-2023.
- [H36] Heinz Hopf, *Über die Sehnen ebener Kontinuen und die Schleifen geschlossener Wege*, Comment. Math. Helv., 9 (1936) 303-319.
- [L34] Paul Lévy, *Sur une Généralisation du Théorème de Rolle*, C.R. Acad. Sci., Paris, 198 (1934), 424-425.

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