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generated by an almost generalized arithmetic sequence,
Frobenius number**

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Marcel Morales. Symmetric and Almost Symmetric semigroups generated by an almost generalized arithmetic sequence, Frobenius number: Symmetric and Almost Symmetric semigroups, Frobenius number. 2021. hal-04043909

HAL Id: hal-04043909

<https://hal.science/hal-04043909>

Preprint submitted on 24 Mar 2023

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SYMMETRIC AND ALMOST SYMMETRIC SEMIGROUPS GENERATED BY AN ALMOST GENERALIZED ARITHMETIC SEQUENCE, FROBENIUS NUMBER

MARCEL MORALES

ABSTRACT. Let a, d, k, h, c be positive integers. Recall that a *numerical almost generalized arithmetic sequence-semigroup* (numerical AAG-semigroup for short) is a semigroup minimally generated by relatively prime integers $a, ha+d, ha+2d, \dots, ha+kd, c$, that is its embedding dimension is $k+2$. In [8] was described a Gröbner basis of the ideal defining S under one technical assumption, the complete case will be published in a forthcoming paper. In this paper we give a complete description of S when is symmetric or almost symmetric and a quadratic formula for its Frobenius number. Note that our results generalizes and extends previous result of [11], [3] and [13]. Given a, d, k, h, c a simple algorithm allows us to determine if S is almost symmetric.

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1. INTRODUCTION

Let a_0, \dots, a_n be natural numbers and $S = \langle a_0, \dots, a_n \rangle = \{k_0a_0 + \dots + k_na_n \mid k_i \in \mathbb{N}\}$ the semigroup generated by $\{a_0, \dots, a_n\}$. Recall that if a_0, \dots, a_n are relatively prime numbers then the *Frobenius number* of S , denoted by $F(S)$, is the biggest integer that does not belong to S . Let $A = K[S] = K[t^k \mid k \in S] = K[t^{a_0}, \dots, t^{a_n}] \subset K[t]$ the *semigroup ring* of S and $R = K[x_0, \dots, x_n]$ the polynomial ring in $n+1$ variables over K graded

2010 *Mathematics Subject Classification*: Primary: 13D40, Secondary 14M25, 13C14, 14M05.

Key words and phrases: Frobenius number, numerical semigroups Pseudo-Frobenius number, Apéry set, Gröbner basis, Semigroup rings, symmetric and almost symmetric semigroups.

by the weights $\deg x_i = a_i$, for all $i = 0, \dots, n$. The *defining ideal* I of $K[S]$ is defined to be the kernel of the K -algebra homomorphism $\Psi : R \rightarrow K[S]$ given by $\Psi(x_i) = t^{a_i}$ for all $i = 0, \dots, n$, we will use often the fact that I is a prime ideal generated by binomials and does not contains monomials. We use the *weighted degree reverse lexicographical order* \prec_w on the monomials of the ring R with $x_0 \prec \dots \prec x_n$, and *the map* $\varphi : [[R]] \rightarrow \mathbb{N}$ defined by $\varphi(M) = k_1 a_1 + \dots + k_n a_n$ for every monomial $M = x_0^{k_0} \dots x_n^{k_n} \in [[R]]$. Let recall the *pseudo-Frobenius set* $PF(S)$ of all integer number a which satisfies $a \notin S$ and $a + s \in S$, for all $0 \neq s \in S$ and the number of elements of $PF(S)$ is called the *type* of S , denoted by $t(S)$. Finally, the *Apéry set* with respect to a_0 plays an important role in our paper which is defined by $\text{Ap}(S, a_0) = \{s \in S \mid s - a_0 \notin S\}$. By defining in [8] a monomial Apéry set $\widetilde{\text{Ap}}(S, a_0)$ of a_0 , that is an algebraic analogous to the Apéry set $\text{Ap}(S, a_0)$ and using the order \prec_w as well as the map φ , we can change from studying the Apéry set to studying the set of monomials of $[[R']]$ which are not in $\text{in}(I)$, where $R' = K[x_0, \dots, x_n]$.

Let a, d, k, h, c be positive integers. Recall that a *numerical almost generalized arithmetic sequence-semigroup* (numerical AAG-semigroup for short) is a semigroup minimally generated by relatively prime integers $a, ha + d, ha + 2d, \dots, ha + kd, c$, that is its embedding dimension is $k + 2$. Our goal is to describe all properties of an AAG-semigroup in terms of a continuous fraction, as an extension of my previous works in [5], [6]. In [8] was described a Gröbner basis of the ideal defining S under one condition, the complete case will be published in a forthcoming paper. In this paper we continue the work of [8] and we can describe the Pseudo Frobenius set (see Theorem 5.2), and so the Frobenius number and its type is at most $2k$. As a consequence we can give a complete description of AAG-semigroups that are symmetric or almost symmetric (see Theorems 6.1, 7.2, 7.3), in particular we prove that if S is almost symmetric its type is at most the embedding dimension minus 1. Another interesting point is that if S is almost symmetric then the Frobenius number is given by a quadratic formula in terms of a, d, k, h, c and $t(S)$. Moreover a simple algorithm using the solutions of some quadratic equations allow us to decide is an AAG-semigroup is almost symmetric. This result extends and generalizes all the results of [13].

The algorithms presented here are the extensions of the previous work by the first author in [5], [6] and can be downloaded in <http://www-fourier.univ-grenoble-alpes.fr/morales/>.

2. FROBENIUS NUMBER AND APÉRY SET

Denote by \mathbb{Z} and \mathbb{N} the set of integers and nonnegative integers respectively. Let S be a semigroup in \mathbb{N} . Given $n \geq 1$ and $a_0, \dots, a_n \in \mathbb{N}$ such that $\gcd(a_0, \dots, a_n) = 1$,

$$S = \langle a_0, \dots, a_n \rangle = \{k_0 a_0 + \dots + k_n a_n \mid k_i \in \mathbb{N}\}.$$

The set $\mathbb{N} \setminus S$ is finite. If S is minimally generated by $\{a_0, \dots, a_n\}$ S is called *numerical semigroup* and $n + 1$ is called the *embedding dimension* of S .

Definition 2.1. Let S be a numerical semigroup generated by a_0, \dots, a_n .

- (i) The number $F(S) = \max\{a \in \mathbb{Z} \mid a \notin S\}$ is called the *Frobenius number* of S .
- (ii) We also define

$$PF(S) = \{a \in \mathbb{Z} \setminus S \mid a + s \in S \text{ if } s \in S \text{ and } s \neq 0\}$$

and an element of $PF(S)$ is called a *pseudo-Frobenius number* of S . Obviously, the Frobenius number is a pseudo-Frobenius number and the number of elements of $PF(S)$ is called the *type* of S , denoted by $t(S)$.

- (iii) The *Apéry set* of a_0 in S is the set

$$\text{Ap}(S, a_0) = \{s \in S \mid s - a_0 \notin S\}.$$

3. FROBENIUS NUMBER AND GRÖBNER BASIS

The definitions and results in this section follow from [8]. Let $R = K[x_0, \dots, x_n]$ be the polynomial ring graded by the weights $\deg x_0 = a_0, \dots, \deg x_n = a_n$, $J \subset R$ a graded ideal and $B = R/J$. We say that R and B are *quasi-homogeneous* rings. Set $R' = K[x_1, \dots, x_n]$ and denote by $[[R']]$ the set of all monomials of R' . Let $\varphi : [[R']] \rightarrow \mathbb{N}$ be the map defined by $\varphi(M) = k_1 a_1 + \dots + k_n a_n$, for every monomial $M = x_1^{k_1} \dots x_n^{k_n} \in [[R']]$.

We consider the weighted degree reverse lexicographical order \prec_w with $x_0 \prec_w \dots \prec_w x_n$ and $\deg x_i = a_i$ for all $0 \leq i \leq n$.

With the notations in the introduction, let $\text{in}(I)$ be the initial ideal of the reduced Gröbner basis $G(S)$ of I for the order \prec_w . Set $R' = K[x_1, \dots, x_n]$ and denote by $[[R']]$ the set of all monomials of R' . Now we consider two sets

$$\widetilde{\text{Ap}}(S, a_0) = \{M \in [[R']] \mid M \notin \text{in}(I)\}$$

and

$$\widetilde{PF}(S) = \{M \in \widetilde{\text{Ap}}(S, a_0) \mid \forall i \neq 0, \exists N_i \in [[R']], \alpha_i > 0 \text{ such that } Mx_i - x_0^{\alpha_i} N_i \in I\}.$$

Corollary 3.1. Assume that $\text{gcd}(a_0, \dots, a_n) = 1$. Then we have

(i) The restriction of φ to $\widetilde{\text{Ap}}(S, a_0)$ is bijective and $\varphi(\widetilde{\text{Ap}}(S, a_0)) = \text{Ap}(S, a_0)$. In particular $\text{card}(\widetilde{\text{Ap}}(S, a_0)) = a_0$ and $F(S) = \max\{\varphi(M) \mid M \notin \text{in}(I)\} - a_0$.

(ii) The restriction of φ to $\widetilde{PF}(S)$ is bijective and $\varphi(\widetilde{PF}(S)) = \widetilde{PF}(S) + a_0$, i.e. each element $\omega \in \widetilde{PF}(S)$ corresponds to exactly one monomial $M_\omega \in \widetilde{PF}(S)$ such that $\varphi(M_\omega) - a_0 = \omega$.

(iii) Let $s \in \text{Ap}(S, a_0)$, $M \in \widetilde{\text{Ap}}(S, a_0)$ and $N \in [[R']]$ such that $s = \varphi(M) = \varphi(N)$. Then $M \prec_w N$.

We denote by $\widetilde{Frob}(S)$ the unique monomial in $\widetilde{PF}(S)$ such that $\varphi(\widetilde{Frob}(S)) = F(S) + a_0$.

The following Lemma is very simple but very useful in order to prove that a set is a Gröbner basis of an ideal in many cases (see [8], [2]).

Lemma 3.2. *Let $R = K[x_0, \dots, x_n]$, $R' = K[x_s, \dots, x_n]$ be the rings with respect to the order \prec_w and $[[R']]$ the set of all monomials of R' . Let $I \subset R$ be an ideal such that the generators of $\text{in}(I)$ belongs to R' and $\text{rad}(\text{in}(I) \cap R') = (x_s, \dots, x_n)R'$. Let $G \subset I$ be a finite set and J the monomial ideal generated by the leading monomials of the elements in G . If $\text{card}([R'] \setminus J) = \text{card}([R'] \setminus (\text{in}(I) \cap R'))$ then G is a Gröbner basis of I .*

4. ALMOST GENERALIZED ARITHMETIC PROGRESSIONS, GROBNER BASIS

Let a, d, k, h, c be positive integers. Recall that a *numerical almost generalized arithmetic sequence-semigroup* (numerical AAG-semigroup for short) is a semigroup minimally generated by relatively prime integers $a, ha + d, ha + 2d, \dots, ha + kd, c$, that is its embedding dimension is $k + 2$. An interesting particular case is a semigroup minimally generated by relatively prime integers $a, a + d, a + 2d, \dots, a + kd, c$, called *numerical almost arithmetic-semigroup* (numerical AA-semigroup for short). AA-semigroups are the case $h = 1$ of AAG-semigroups and were considered by D. P. Patil [9], [10] from the algebraic point of view and by J. L. Ramírez Alfonsín and O. J. Rodseth in [11], [12] from combinatorial point of view.

Let $R = K[x_0, \dots, x_k, x_{k+1}]$ be the polynomial ring in $k + 2$ variables over K graded by the weights $\deg x_i = ha + id$ for $i = 0, \dots, k$, $\deg x_{k+1} = c$ and I the kernel of the homomorphism $\Phi : R \rightarrow K[S]$ of K -algebras defined by $\Phi(x_0) = t^a$, $\Phi(x_i) = t^{ha+id}$ for all $i = 1, \dots, k$ and $\Phi(x_{k+1}) = t^c$. Let $R' = K[x_1, \dots, x_k]$. The following result extends [10][Lemma 1.6.1].

Lemma 4.1. *For $1 \leq i, j < k$, set*

$$\mathcal{A} = \{x_i x_j - x_0^h x_{i+j} \mid \text{if } i + j \leq k, \} \cup \{x_i x_j - x_{i+j-k} x_k \mid \text{if } i + j > k\}.$$

Then every binomial of \mathcal{A} belongs to I and $\text{card}(\mathcal{A}) = \frac{(k-1)k}{2}$.

Proof. Let $1 \leq i, j < k$. The results are implied by the fact that $\varphi(x_i x_j) = ha + id + ha + jd = ha + (i + j)d$ if $i + j \leq k$ or $\varphi(x_i x_j) = (ha + id) + (ha + jd) = (ha + (i + j - k)d) + (ha + kd)$ if $i + j > k$. \square

Corollary 4.2. *Let consider any Gröbner basis with respect to an order \prec_w such that $x_0 \prec_w x_1 \prec_w \dots \prec_w x_k \prec_w x_{k+1}$. The initial ideal $\text{in}(I)$ and $\widetilde{\text{Ap}}(S, a_0)$ can be represented in the plane.*

Proof. By hypothesis and Lemma 4.1 we have only to consider only the monomials in $\text{in}(I)$ which can be written as $L_i x_k^\alpha x_{k+1}^\beta$ where $0 \leq i < k$, $L_0 = 1$ and $L_i = x_i$ for $i > 0$. We associate to $L_i x_k^\alpha x_{k+1}^\beta$ the point $(i + \alpha k, \beta) \in \mathbb{N}^2$. \square

Lemma 4.3. *Let $s \in \mathbb{N}, p, r \in \mathbb{Z}$ such that*

$$ra = sd - pc.$$

Let $s = \sigma k + \rho$, where $0 \leq \rho < k$. For convenience we can write $s = \sigma k + l\rho$, where $l = 0$ if $\rho = 0$ and $l = 1$ if $\rho > 0$. It follows that $ra = \sigma kd + l\rho d - pc$ so that

$$(r + h(\sigma + l))a = \sigma(ha + kd) + l(ha + \rho d) - pc. \quad (*)$$

We set $r' = r + h(\sigma + l)$. We have

- $L_l x_k^\sigma - x_0^{r'} x_{k+1}^p \in I$ if $p, r' \geq 0$,
- $L_l x_k^\sigma x_{k+1}^{-p} - x_0^{r'} \in I$ if $p < 0, r' > 0$,
- $x_{k+1}^p - x_0^{-r'} L_l x_k^\sigma \in I$ if $p \geq 0, r' < 0$.

Our aim is to construct a Gröbner basis and describe the set $\widetilde{\text{Ap}}(S, a_0)$. By applying our algorithm for the case $n = 3$ in Section 4.1 with numbers a, d, c , we get numbers s_i, p_i, q_i, r_i for $0 \leq i \leq m + 1$ such that

$$ar_i = s_i d - p_i c.$$

Let s_0 be the smallest natural number such that $(s_0, 0, r_0)$ is solution of the equation $sd - pc = ra$. Set $p_0 = 0$ and let p_1 be the smallest natural number such that (s_1, p_1, r_1) is solution of the equation $sd - pc = ra$, where $0 \leq s_1 < s_0$. Note that

$$s_0 = \frac{a}{\gcd(a, d)} \text{ and } p_1 = \frac{\gcd(a, d)}{\gcd(a, d, c)}.$$

Now we want to define numbers s_i, p_i, r_i, q_i for $i \geq 2$. We will use the extended Euclid's algorithm for the computation of $\gcd(a, b)$. Namely, let consider the Euclid's algorithm with negative rest:

$$\left\{ \begin{array}{l} s_0 = q_2 s_1 - s_2 \\ s_1 = q_3 s_2 - s_3 \\ \dots = \dots \\ s_{m-1} = q_{m+1} s_m \\ s_{m+1} = 0 \end{array} \right.$$

where $q_i \geq 2$, $s_i \geq 0$ for all $i = 2, \dots, m + 1$. For $i = 1, \dots, m$, let define p_{i+1}, r_{i+1} by

$$p_{i+1} = p_i q_{i+1} - p_{i-1}, r_{i+1} = r_i q_{i+1} - r_{i-1}.$$

It is proved in [6] that for $i = 0, \dots, m$,

$$s_i p_{i+1} - s_{i+1} p_i = s_0 p_1 = \frac{a}{\gcd(a, d, c)},$$

and the sequences s_i, r_i are decreasing, while the sequence p_i is increasing. see [5] and [6].

Let $s_i = \sigma_i k + \rho_i$, where $0 \leq \rho_i < k$. We set $l_i = 0$ if $\rho_i = 0$, $l_i = 1$ if $\rho_i > 0$ and $r'_i = r_i + h(\sigma_i + l_i)$. Since $s_i > s_{i+1}$ we have $\sigma_i \geq \sigma_{i+1}$, if $\sigma_i = \sigma_{i+1}$ then $l_i \geq l_{i+1}$, if $\sigma_i > \sigma_{i+1}$ then $|l_i - l_{i+1}| \leq 1$. In both cases we have $\sigma_i + l_i \geq \sigma_{i+1} + l_{i+1}$, which implies $r'_i > r'_{i+1}$. Let μ be the unique integer such that $r'_\mu > 0 \geq r'_{\mu+1}$. In our next results we suppose that either $r'_\mu \geq h$ or $\rho_\mu = 0$. Note that by the definition of μ $r'_\mu \geq h$ is true when $h = 1$. We give some results from [8] without proofs.

Definition 4.4. *With the above notations. If $r'_\mu \geq h$ or $\rho_\mu = 0$ we set:*

(1) *If $\rho_\mu = 0$ we set*

$$\mathcal{B} = \{x_k^{\sigma_\mu} - x_0^{r'_\mu} x_{k+1}^{p_\mu}\}.$$

If $\rho_\mu \neq 0$ we set

$$\mathcal{B} = \{x_{\rho_\mu} x_k^{\sigma_\mu} - x_0^{r'_\mu} x_{k+1}^{p_\mu}, x_{\rho_\mu+j} x_k^{\sigma_\mu} - x_0^{r'_\mu-h} x_j x_{k+1}^{p_\mu} \mid 1 \leq j \leq k - \rho_\mu\}.$$

Note that since the embedding dimension of the semigroup S is $k+2$ we have $s_\mu > k$.

- (2) Suppose $s_{\mu+1} \neq 0$. Let $s_\mu - s_{\mu+1} = \tilde{\sigma}k + \tilde{\rho}$, with $0 \leq \tilde{\rho} < k$. Set $\tilde{l} = 0$ if $\tilde{\rho} = 0$, $\tilde{l} = 1$ if $\tilde{\rho} > 0$ and $\tilde{r} = r_\mu - r_{\mu+1} + h(\tilde{\sigma} + \tilde{l})$. If $\tilde{\rho} > 0$ then set

$$\mathcal{C} = \{x_{\tilde{\rho}} x_k^{\tilde{\sigma}} x_{k+1}^{p_{\mu+1}-p_\mu} - x_0^{\tilde{r}}, x_{j+\tilde{\rho}} x_k^{\tilde{\sigma}} x_{k+1}^{p_{\mu+1}-p_\mu} - x_0^{\tilde{r}-1} x_j \mid 1 \leq j \leq k - \tilde{\rho}\}$$

and if $\tilde{\rho} = 0$ then set $\mathcal{C} = \{x_k^{\tilde{\sigma}} x_{k+1}^{p_{\mu+1}-p_\mu} - x_0^{\tilde{r}}\}$. Suppose $s_{\mu+1} = 0$. We set $\mathcal{C} = \emptyset$.

- (3) $\mathcal{D} := \{x_{k+1}^{p_{\mu+1}} - x_0^{-r'_{\mu+1}} x_{\rho_{\mu+1}}^{l_{\mu+1}} x_k^{\sigma_{\mu+1}}\}$. By our assumptions the embedding dimension of the semigroup S is $k+2$ so $p_{\mu+1} > 1$, that is $\mu > 0$, and if $r'_{\mu+1} = 0$ we have $s_{\mu+1} > k$.

Now in order to find the set $\widetilde{\text{Ap}}(S, a)$ we need to define two 2 rectangles

$$A = \{(y, z) \in \mathbb{N}^2 \mid 0 \leq y < s_\mu - s_{\mu+1}, 0 \leq z < p_{\mu+1}\}$$

$$B = \{(y, z) \in \mathbb{N}^2 \mid s_\mu - s_{\mu+1} \leq y < s_\mu, 0 \leq z < p_{\mu+1} - p_\mu\}.$$

Note that if $s_{\mu+1} = 0$ then $B = \emptyset$. For $0 \leq i < k$, set $L_i = 1$ if $i = 0$ and $L_i = x_i$ if $i > 0$. To any point $(y, z) \in \mathbb{N}^2$ we associate the monomial $M(y, z) := L_i x_k^\alpha x_{k+1}^z$, where $\alpha = \lfloor \frac{y}{k} \rfloor$ and $i = y - k\alpha$. Conversely, any monomial $L_i x_k^\alpha x_{k+1}^z \in [[R']]$ can be represented by the point $(y, z) \in \mathbb{N}^2$, where $y = \alpha k + i$.

The next theorem allows to compute effectively a system of generators of the ideal semigroup I , it precises and extends the main theorem of [10] where the case $h = 1$ is considered.

Theorem 4.5. *With the above notations, suppose that either $r'_\mu \geq h$ or $\rho_\mu = 0$.*

(i) *We have*

$$\widetilde{\text{Ap}}(S, a) = \left\{ L_i x_k^\alpha x_{k+1}^z \mid (y, z) \in A \cup B, \alpha = \lfloor \frac{y}{k} \rfloor, i = y - k\alpha \right\}.$$

(ii) *If $s_{\mu+1} \neq 0$ then $\mathcal{G} := \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$ is a Gröbner basis of I .*

(iii) *If $s_{\mu+1} = 0$ then $\mathcal{G} := \mathcal{A} \cup \mathcal{B} \cup \mathcal{D}$ is a Gröbner basis of I .*

The following result extends the main result of J. L. Ramírez Alfonsín and O. J. Rodseth in [11], [12], which is the case when $h = 1$.

Corollary 4.6. *With the above notations, suppose that either $r'_\mu \geq h$ or $\rho_\mu = 0$. We have*

$$\text{Ap}(S, a) = \left\{ ha \left\lfloor \frac{y}{k} \right\rfloor + dy + cz \mid (y, z) \in A \cup B \right\}.$$

5. ALMOST GENERALIZED ARITHMETIC PROGRESSIONS, PSEUDO FROBENIUS SET

The case S symmetric was studied in [11] and the case S is pseudo symmetric was studied in [3] with the hypothesis that $h = 1$. Both publications are corollaries of this section. In our work we will describe the Pseudo Frobenius set and characterize when S is almost symmetric in general for $h \geq 1$. In this paper we restrict to the hypothesis $r'_\mu \geq h$ or $\rho_\mu = 0$. Note that $r'_\mu \geq h$ is satisfied if $h = 1$.

Lemma 5.1. *With the above notations, suppose that either $r'_\mu \geq h$ or $\rho_\mu = 0$. Let $\widetilde{PF}(S, a)_1$ be the set of monomials in $\widetilde{PF}(S, a)$ such that the power of x_{k+1} is $p_{\mu+1} - 1$ and let $\widetilde{PF}(S, a)_2$ be the set of monomials in $\widetilde{PF}(S, a)$ such that the power of x_{k+1} is $p_{\mu+1} - p_\mu - 1$. We have*

$$\widetilde{PF}(S, a) = \widetilde{PF}(S, a)_1 \cup \widetilde{PF}(S, a)_2.$$

In particular $1 \leq t(S) \leq 2k$.

Proof. We have

$$\widetilde{\text{Ap}}(S, a) = \{L_i x_k^\alpha x_{k+1}^z \mid i + \alpha k < s_\mu, z < p_{\mu+1} \text{ and } (i + \alpha k < s_\mu - s_{\mu+1} \text{ or } z < p_{\mu+1} - p_\mu)\},$$

Let M be a monomial in $\widetilde{\text{Ap}}(S, a)$, recall that M belongs to $\widetilde{PF}(S, a)$ if and only if for all $i = 1, \dots, k + 1$ we have that $x_i M - x_0^\alpha N_i \in I$ for some monomial N_i and some $\alpha \in \mathbb{N}^*$. We order the monomials in $\widetilde{\text{Ap}}(S, a)$ by saying that $M \ll N$ if $N = x_i M$ for some $i = 1, \dots, k + 1$. So $\widetilde{PF}(S, a)$ is included in the set of maximal elements of $\widetilde{\text{Ap}}(S, a)$ for this order. Note that for a maximal monomial of $\widetilde{\text{Ap}}(S, a)$ we have that the power of x_{k+1} is either $p_{\mu+1} - p_\mu - 1$ or $p_{\mu+1} - 1$. Let $\widetilde{PF}(S, a)_1$ be the set of monomials in $\widetilde{PF}(S, a)$ such that the power of x_{k+1} is $p_{\mu+1} - 1$ and let $\widetilde{PF}(S, a)_2$ be the set of monomials in $\widetilde{PF}(S, a)$ such that the power of x_{k+1} is $p_{\mu+1} - p_\mu - 1$. We have $\text{card}(\widetilde{PF}(S, a)_1), \text{card}(\widetilde{PF}(S, a)_2) \leq k$ so $t(S) \leq 2k$. \square

Note that if M belongs to $\widetilde{PF}(S, a)$ then for all $i = 1, \dots, k + 1$ $x_i M \in \text{in}(I)$, so if $x_i M \in \widetilde{\text{Ap}}(S, a)$ for some $i = 1, \dots, k + 1$ then certainly $M \notin \widetilde{PF}(S, a)$. Note also that $s_\mu - s_{\mu+1} = (\sigma_\mu - \sigma_{\mu+1})k + \rho_\mu - \rho_{\mu+1}$, so if $\rho_{\mu+1} \leq \rho_\mu$ then we have $\tilde{\sigma} = \sigma_\mu - \sigma_{\mu+1}$, $\tilde{\rho} = \rho_\mu - \rho_{\mu+1}$ and if $\rho_{\mu+1} > \rho_\mu$ then we have $\tilde{\sigma} = \sigma_\mu - \sigma_{\mu+1} - 1$, $\tilde{\rho} = k + \rho_\mu - \rho_{\mu+1}$.

Theorem 5.2. *With the above notations, suppose that either $r'_\mu \geq h$ or $\rho_\mu = 0$. We have*

(1) If $r'_{\mu+1} = 0$

(a) If $\rho_{\mu+1} = 0$ then $\widetilde{PF}(S, a)_1 = \emptyset$.

(b) If $\rho_{\mu+1} > 0, \tilde{\rho} = 0$ then

$$\widetilde{PF}(S, a)_1 = \{x_i x_k^{\tilde{\sigma}-1} x_{k+1}^{p_{\mu+1}-1}, i = 1, \dots, k - \rho_{\mu+1}\}.$$

(c) If $\rho_{\mu+1} > 0, \tilde{\rho} = 1, \tilde{\sigma} = 0$ then $\widetilde{PF}(S, a)_1 = \emptyset$.

(d) If $\rho_{\mu+1} > 0, \tilde{\rho} = 1, \tilde{\sigma} > 0$ then

$$\widetilde{PF}(S, a)_1 = \{x_i x_k^{\tilde{\sigma}-1} x_{k+1}^{p_{\mu+1}-1}, i = 1, \dots, k - \rho_{\mu+1}\}.$$

(e) If $\rho_{\mu+1} > 0, \tilde{\rho} > 1$ then

$$\widetilde{PF}(S, a)_1 = \{x_i x_k^{\tilde{\sigma}} x_{k+1}^{p_{\mu+1}-1}, i = 1, \dots, \min\{\tilde{\rho} - 1, k - \rho_{\mu+1}\}\}.$$

(2) If $r'_{\mu+1} < 0$

(a) If $\tilde{\rho} = 0$ then

$$\widetilde{PF}(S, a)_1 = \{x_i x_k^{\tilde{\sigma}-1} x_{k+1}^{p_{\mu+1}-1}, i = 1, \dots, k - 1\}.$$

(b) If $\tilde{\rho} = 1, \tilde{\sigma} = 0$ then

$$\widetilde{PF}(S, a)_1 = \{x_{k+1}^{p_{\mu+1}-1}\}.$$

(c) If $\tilde{\rho} = 1, \tilde{\sigma} > 0$ then

$$\widetilde{PF}(S, a)_1 = \{x_i x_k^{\tilde{\sigma}-1} x_{k+1}^{p_{\mu+1}-1}, i = 1, \dots, k\}.$$

(d) If $\tilde{\rho} > 1$ then

$$\widetilde{PF}(S, a)_1 = \{x_i x_k^{\tilde{\sigma}} x_{k+1}^{p_{\mu+1}-1}, i = 1, \dots, \tilde{\rho} - 1\}.$$

(3) If $s_{\mu+1} = 0$ then $\widetilde{PF}(S, a)_2 = \emptyset$.

(4) $\rho_{\mu} = 0$

(i) If $s_{\mu+1} \geq k - 1$ then

$$\widetilde{PF}(S, a)_2 = \{x_i x_k^{\sigma_{\mu}-1} x_{k+1}^{p_{\mu+1}-p_{\mu}-1}, i = 1, \dots, k - 1\}.$$

(ii) If $s_{\mu+1} < k - 1$ then

$$\widetilde{PF}(S, a)_2 = \{x_i x_k^{\sigma_{\mu}-1} x_{k+1}^{p_{\mu+1}-p_{\mu}-1}, i = \tilde{\rho}, \dots, k - 1\}.$$

(5) $\rho_{\mu} = 1, r'_{\mu} > h$

(i) If $s_{\mu+1} \geq k$ then

$$\widetilde{PF}(S, a)_2 = \{x_i x_k^{\sigma_{\mu}-1} x_{k+1}^{p_{\mu+1}-p_{\mu}-1}, i = 1, \dots, k\}.$$

(ii) If $1 < s_{\mu+1} < k$ then

$$\widetilde{PF}(S, a)_2 = \{x_i x_k^{\sigma_{\mu}-1} x_{k+1}^{p_{\mu+1}-p_{\mu}-1}, i = \tilde{\rho}, \dots, k\}.$$

(iii) If $s_{\mu+1} = 1$ then

$$\widetilde{PF}(S, a)_2 = \{x_k^{\sigma_{\mu}} x_{k+1}^{p_{\mu+1}-p_{\mu}-1}\}.$$

(6) $\rho_{\mu} = 1, r'_{\mu} = h$

(i) If $s_{\mu} - s_{\mu+1} = 1$ then

$$\widetilde{PF}(S, a)_2 = \{x_i x_k^{\sigma_{\mu}-1} x_{k+1}^{p_{\mu+1}-p_{\mu}-1}, i = 1, \dots, k\}.$$

(ii) If $1 < s_\mu - s_{\mu+1} \leq s_\mu - k$ then

$$\widetilde{PF}(S, a)_2 = \{x_1 x_k^{\sigma_\mu - 1} x_{k+1}^{p_{\mu+1} - p_\mu - 1}\}.$$

(iii) If $s_\mu - k < s_\mu - s_{\mu+1}$ then $\widetilde{PF}(S, a)_2 = \emptyset$.

(7) $\rho_\mu > 1$

(i) If $s_{\mu+1} \geq \rho_\mu - 1$ then

$$\widetilde{PF}(S, a)_2 = \{x_i x_k^{\sigma_\mu} x_{k+1}^{p_{\mu+1} - p_\mu - 1}, i = 1, \dots, \rho_\mu - 1\}.$$

(ii) If $s_{\mu+1} < \rho_\mu - 1$ then $\tilde{\rho} = \rho_\mu - s_{\mu+1}$ and

$$\widetilde{PF}(S, a)_2 = \{x_i x_k^{\sigma_\mu} x_{k+1}^{p_{\mu+1} - p_\mu - 1}, i = \tilde{\rho}, \dots, \rho_\mu - 1\}.$$

Before going to the proof let remark:

Remark 5.3. a) Let $i, j = 1, \dots, k$ and M a monomial, if $i + j \leq k$ then we have $x_i x_j - x_0^h x_{i+j} \in I$ so $x_j x_i M - x_0^h x_{i+j} M \in I$.

b) A monomial $x_i M$ for some $i = 1, \dots, k$ belongs to $\widetilde{PF}(S, a)$ if and only if for any $j = 1, \dots, k + 1$ there is a binomial $x_j x_i M - x_0^\alpha N$ for some monomial N and $\alpha \in \mathbb{N}^*$. So in order to check if $x_i M$ belongs to $\widetilde{PF}(S, a)$ we need only to consider j such that $i + j > k$.

c) Let note that the elements in $\widetilde{PF}(S, a)_i$ are ordered by increasing order of evaluation by φ .

Proof. We have to consider all possible cases:

(1) Study of $\widetilde{PF}(S, a)_1$ when $r'_{\mu+1} = 0$,

a) Suppose $\rho_{\mu+1} = 0$. Let $M_i = x_i x_k^\alpha x_{k+1}^{p_{\mu+1} - 1} \in \widetilde{PF}(S, a)_1$ for some $1 \leq i \leq k$, such that $i + k\alpha < s_\mu - s_{\mu+1}$. Since $x_{k+1}^{p_{\mu+1}} - x_k^{\sigma_{\mu+1}} \in I$ we have $x_{k+1} M_i - x_i x_k^\alpha x_k^{\sigma_{\mu+1}} \in I$, but $i + k\alpha + k\sigma_{\mu+1} < s_\mu - s_{\mu+1} + s_{\mu+1} = s_\mu$ so $x_i x_k^\alpha x_k^{\sigma_{\mu+1}} \in \text{Ap}(S, a)$ showing that $M_i \notin \widetilde{PF}(S, a)$ a contradiction, that is $\widetilde{PF}(S, a)_1 = \emptyset$.

b) Suppose $\rho_{\mu+1} > 0, \tilde{\rho} = 0$. We have

$$\widetilde{PF}(S, a)_1 \subset \{x_i x_k^{\tilde{\sigma} - 1} x_{k+1}^{p_{\mu+1} - 1}, i = 1, \dots, k - 1\}$$

and $s_\mu = (\tilde{\sigma} + \sigma_{\mu+1})k + \rho_{\mu+1}$. Let $i = 1, \dots, k - 1, j = 1, \dots, k$ and $M = x_k^{\tilde{\sigma} - 1} x_{k+1}^{p_{\mu+1} - 1}$ such that $i + j > k$. We have $x_i x_j - x_{i+j-k} x_k \in I$ so $x_j x_i M - x_{i+j-k} x_k^{\tilde{\sigma}} x_{k+1}^{p_{\mu+1} - 1} \in I$, since $x_k^{\tilde{\sigma}} x_{k+1}^{p_{\mu+1} - p_\mu} - x_0^{\tilde{r}} \in I$ we have $x_j x_i M - x_0^{\tilde{r}} x_{i+j-k} x_{k+1}^{p_\mu - 1} \in I$ where $\tilde{r} > 0$.

From $x_{k+1}^{p_{\mu+1}} - x_{\rho_{\mu+1}} x_k^{\sigma_{\mu+1}} \in I$, we get $x_{k+1} x_i M - x_i x_{\rho_{\mu+1}} x_k^{\tilde{\sigma} - 1} x_k^{\sigma_{\mu+1}} \in I$. If $i + \rho_{\mu+1} \leq k$ then $x_i x_{\rho_{\mu+1}} - x_0^h x_{1+\rho_{\mu+1}}$, so $x_i M \in \widetilde{PF}(S, a)$. If $i + \rho_{\mu+1} > k$ then $x_{k+1} x_i M - x_{i+\rho_{\mu+1}-k} x_k^{\tilde{\sigma} + \sigma_{\mu+1}} \in I$ but $i + \rho_{\mu+1} - k + (\tilde{\sigma} + \sigma_{\mu+1})k =$

$s_\mu + i - k < s_\mu$ which implies that $x_{i+\rho_{\mu+1}-k}x_k^{\tilde{\sigma}+\sigma_{\mu+1}} \in \widetilde{\text{Ap}}(S, a)$ so $x_iM \notin PF(S, a)$. Hence

$$PF(S, a)_1 = \{x_i x_k^{\tilde{\sigma}-1} x_{k+1}^{p_{\mu+1}-1}, i = 1, \dots, k - \rho_{\mu+1}\}.$$

c) Suppose $\rho_{\mu+1} > 0, \tilde{\rho} = 1, \tilde{\sigma} = 0$. We have

$$PF(S, a)_1 \subset \{x_{k+1}^{p_{\mu+1}-1}\}.$$

The binomial $x_{k+1}(x_{k+1}^{p_{\mu+1}-1}) - x_{\rho_{\mu+1}}x_k^{\sigma_{\mu+1}} \in I$ but $x_{\rho_{\mu+1}}x_k^{\sigma_{\mu+1}} \in \widetilde{\text{Ap}}(S, a)$ so $PF(S, a)_1 = \emptyset$.

d) Suppose $\rho_{\mu+1} > 0, \tilde{\rho} = 1, \tilde{\sigma} > 0$. We have

$$PF(S, a)_1 \subset \{x_i x_k^{\tilde{\sigma}-1} x_{k+1}^{p_{\mu+1}-1}, i = 1, \dots, k\}.$$

Let $i = 1, \dots, k, j = 1, \dots, k, M = x_k^{\tilde{\sigma}-1} x_{k+1}^{p_{\mu+1}-1}$. By the same arguments as in the above item b) we have $x_j x_i M - x_0^\alpha N_i \in I$ for some monomial N_i and $\alpha \in \mathbb{N}^*$. The binomial $x_{k+1}^{p_{\mu+1}} - x_{\rho_{\mu+1}}x_k^{\sigma_{\mu+1}} \in I$ so $x_{k+1}x_iM - x_i x_{\rho_{\mu+1}}x_k^{\tilde{\sigma}-1}x_k^{\sigma_{\mu+1}} \in I$. If $i + \rho_{\mu+1} \leq k$ we have $x_i x_{\rho_{\mu+1}} - x_0^h x_{1+\rho_{\mu+1}}$, so $x_i M \in PF(S, a)$. If $i + \rho_{\mu+1} > k$ we have $x_{k+1}x_iM - x_{i+\rho_{\mu+1}-k}x_k^{\tilde{\sigma}+\sigma_{\mu+1}} \in I$. But $s_\mu = (\tilde{\sigma} + \sigma_{\mu+1})k + \rho_{\mu+1} + 1$ so $i + \rho_{\mu+1} - k + (\tilde{\sigma} + \sigma_{\mu+1})k = s_\mu + i - 1 - k < s_\mu$ and $x_{i+\rho_{\mu+1}-k}x_k^{\tilde{\sigma}+\sigma_{\mu+1}} \in \widetilde{\text{Ap}}(S, a)$ so $x_iM \notin PF(S, a)$. Hence

$$PF(S, a)_1 = \{x_i x_k^{\tilde{\sigma}-1} x_{k+1}^{p_{\mu+1}-1}, i = 1, \dots, k - \rho_{\mu+1}\}.$$

e) Suppose $\rho_{\mu+1} > 0, \tilde{\rho} > 1$. We have

$$PF(S, a)_1 \subset \{x_i x_k^{\tilde{\sigma}} x_{k+1}^{p_{\mu+1}-1}, i = 1, \dots, \tilde{\rho} - 1\}.$$

Let $i = 1, \dots, \tilde{\rho} - 1, j = 1, \dots, k, M = x_k^{\tilde{\sigma}} x_{k+1}^{p_{\mu+1}-1}$. By the same arguments as in the above item b) we have $x_j x_i M - x_0^\alpha N_i \in I$ for some monomial N_i and $\alpha \in \mathbb{N}^*$. Since $x_{k+1}^{p_{\mu+1}} - x_{\rho_{\mu+1}}x_k^{\sigma_{\mu+1}} \in I$, we have $x_{k+1}x_iM - x_i x_{\rho_{\mu+1}}x_k^{\tilde{\sigma}}x_k^{\sigma_{\mu+1}} \in I$. If $i + \rho_{\mu+1} \leq k$ we have $x_{k+1}x_iM - x_0^h x_{i+\rho_{\mu+1}}x_k^{\tilde{\sigma}}x_k^{\sigma_{\mu+1}} \in I$, so $x_iM \in PF(S, a)$. If $i + \rho_{\mu+1} > k$ we have $x_{k+1}x_iM - x_{i+\rho_{\mu+1}-k}x_k^{\tilde{\sigma}+\sigma_{\mu+1}+1} \in I$, $s_\mu = (\tilde{\sigma} + \sigma_{\mu+1})k + \tilde{\rho} + \rho_{\mu+1}$ so $i + \rho_{\mu+1} - k + (\tilde{\sigma} + \sigma_{\mu+1})k + k < s_\mu$, which implies that $x_{i+\rho_{\mu+1}-k}x_k^{\tilde{\sigma}+\sigma_{\mu+1}+1} \in \widetilde{\text{Ap}}(S, a)$ so $x_iM \notin PF(S, a)$. Hence

$$PF(S, a)_1 = \{x_i x_k^{\tilde{\sigma}} x_{k+1}^{p_{\mu+1}-1}, i = 1, \dots, \min\{\tilde{\rho} - 1, k - \rho_{\mu+1}\}\}.$$

(2) Study of $PF(S, a)_1$ when $r'_{\mu+1} < 0$,

a) Suppose $\tilde{\rho} = 0$ and $r'_{\mu+1} < 0$. We have

$$PF(S, a)_1 \subset \{x_i x_k^{\tilde{\sigma}-1} x_{k+1}^{p_{\mu+1}-1}, i = 1, \dots, k - 1\}.$$

Let $i = 1, \dots, k-1, j = 1, \dots, k, M = x_k^{\tilde{\sigma}-1} x_{k+1}^{p_{\mu+1}-1}$. By the same arguments as in the above item 1)b) we have $x_j x_i M - x_0^\alpha N_i \in I$ for some monomial N_i and $\alpha \in \mathbb{N}^*$. Since $x_{k+1}^{p_{\mu+1}} - x_0^{-r'_{\mu+1}} x_{\rho_{\mu+1}} x_k^{\sigma_{\mu+1}} \in I$, we have $x_{k+1} x_i M - x_0^{-r'_{\mu+1}} x_i x_k^{\tilde{\sigma}-1} \in I$ so $x_i M \in \widetilde{PF(S, a)}$. Hence

$$\widetilde{PF(S, a)}_1 = \{x_i x_k^{\tilde{\sigma}-1} x_{k+1}^{p_{\mu+1}-1}, i = 1, \dots, k-1\}.$$

b) Suppose $\tilde{\rho} = 1, \tilde{\sigma} = 0$. We have

$$\widetilde{PF(S, a)}_1 \subset \{x_{k+1}^{p_{\mu+1}-1}\}.$$

c) Suppose $\tilde{\rho} = 1, \tilde{\sigma} > 0$. We have

$$\widetilde{PF(S, a)}_1 \subset \{x_i x_k^{\tilde{\sigma}-1} x_{k+1}^{p_{\mu+1}-1}, i = 1, \dots, k\}.$$

d) Suppose $\tilde{\rho} > 1$. We have

$$\widetilde{PF(S, a)}_1 \subset \{x_i x_k^{\tilde{\sigma}} x_{k+1}^{p_{\mu+1}-1}, i = 1, \dots, \tilde{\rho}-1\}.$$

By the same arguments as in the above item 2a) we have the equality in the items 2b), 2c), 2d).

(3) Study of $\widetilde{PF(S, a)}_2$ when $\rho_\mu = 0$. We have $x_k^{\sigma_\mu} - x_0^{r'_\mu} x_{k+1}^{p_\mu}$,

$$\widetilde{PF(S, a)}_2 \subset \{x_i x_k^{\sigma_\mu-1} x_{k+1}^{p_{\mu+1}-p_\mu-1}, i = 1, \dots, k-1\}.$$

Let $i = 1, \dots, k-1, j = 1, \dots, k, M = x_k^{\sigma_\mu-1} x_{k+1}^{p_{\mu+1}-p_\mu-1}$. If $i+j > k$ we have $x_i x_j - x_{i+j-k} x_k$ then $x_j x_i M - x_0^{r'_\mu} x_{i+j-k} x_{k+1}^{p_{\mu+1}-1} \in I$.

We have $x_{k+1} x_i M = x_i x_k^{\sigma_\mu-1} x_{k+1}^{p_{\mu+1}-p_\mu}$ so by using the set \mathcal{C} we get $x_{k+1} x_i M - x_0^\alpha N \in I$ for some $\alpha \in \mathbb{N}^*$ if and only if $i + (\sigma_\mu - 1) \geq s_\mu - s_{\mu+1}$ that is $i \geq k - s_{\mu+1}$.

(i) If $s_{\mu+1} \geq k-1$ we have $i \geq k - s_{\mu+1}$ for all $i = 1, \dots, k-1$. Hence

$$\widetilde{PF(S, a)}_2 = \{x_i x_k^{\sigma_\mu-1} x_{k+1}^{p_{\mu+1}-p_\mu-1}, i = 1, \dots, k-1\}.$$

(ii) If $s_{\mu+1} < k-1$ We have $s_\mu - s_{\mu+1} = (\sigma_\mu - 1)k + (k - s_{\mu+1})$ so $\tilde{\rho} = k - s_{\mu+1}$
Hence

$$\widetilde{PF(S, a)}_2 = \{x_i x_k^{\sigma_\mu-1} x_{k+1}^{p_{\mu+1}-p_\mu-1}, i = \tilde{\rho}, \dots, k-1\}.$$

(4) If $s_{\mu+1} = 0$ the set $\widetilde{Ap(S, a)}$ is represented by a rectangle, there is no element in $\widetilde{PF(S, a)}$ with power of x_{k+1} equal to $p_{\mu+1} - p_\mu - 1$ so $\widetilde{PF(S, a)}_2 = \emptyset$.

(5) Study of $\widetilde{PF(S, a)}_2$ when $\rho_\mu = 1, r'_\mu > h$. We have $x_1 x_k^{\sigma_\mu} - x_0^{r'_\mu} x_{k+1}^{p_\mu}, x_l x_k^{\sigma_\mu} - x_0^{r'_\mu-h} x_{l-1} x_{k+1}^{p_\mu} \mid l = 2, \dots, k$,

$$\widetilde{PF(S, a)}_2 \subset \{x_i x_k^{\sigma_\mu-1} x_{k+1}^{p_{\mu+1}-p_\mu-1}, i = 1, \dots, k\}.$$

Let $i = 1, \dots, k, j = 1, \dots, k, M = x_k^{\sigma_\mu - 1} x_{k+1}^{p_{\mu+1} - p_\mu - 1}$. If $i + j > k$ then we have $x_i x_j - x_{i+j-k} x_k$. If $i + j = k + 1$ then $x_j x_i M - x_0^{r'_\mu} x_{k+1}^{p_{\mu+1} - 1} \in I$. If $i + j > k + 1$ then $x_j x_i M - x_0^{r'_\mu - h} x_{i+j-k-1} x_{k+1}^{p_{\mu+1} - 1} \in I$. So $x_j x_i M - x_0^{\alpha_{i,j}} N \in I$ for some $\alpha_{i,j} \in \mathbb{N}^*$.

We have $x_{k+1} x_i M = x_i x_k^{\sigma_\mu - 1} x_{k+1}^{p_{\mu+1} - p_\mu}$ so by using the set \mathcal{C} we get $x_{k+1} x_i M - x_0^\alpha N \in I$ for some $\alpha \in \mathbb{N}^*$ if and only if $i + (\sigma_\mu - 1)k \geq s_\mu - s_{\mu+1}$, that is $i + s_\mu - 1 - k \geq s_\mu - s_{\mu+1}$ or $i \geq k + 1 - s_{\mu+1}$. We have to consider several cases.

(i) $s_{\mu+1} \geq k$, the condition $i \geq k + 1 - s_{\mu+1}$ is satisfied for $i = 1, \dots, k$. Hence

$$\widetilde{PF}(S, a)_2 = \{x_i x_k^{\sigma_\mu - 1} x_{k+1}^{p_{\mu+1} - p_\mu - 1}, i = 1, \dots, k\}.$$

(ii) $1 < s_{\mu+1} < k$. We have $s_\mu - s_{\mu+1} = (\sigma_\mu - 1)k + (k + 1 - s_{\mu+1})$ so $k + 1 - s_{\mu+1} = \tilde{\rho}$. Hence

$$\widetilde{PF}(S, a)_2 = \{x_i x_k^{\sigma_\mu - 1} x_{k+1}^{p_{\mu+1} - p_\mu - 1}, i = \tilde{\rho}, \dots, k\}.$$

(iii) $s_{\mu+1} = 1$. We have $i + (\sigma_\mu - 1)k \geq s_\mu - s_{\mu+1} = \sigma_\mu k$ if and only if $i = k$. Hence

$$\widetilde{PF}(S, a)_2 = \{x_k^{\sigma_\mu} x_{k+1}^{p_{\mu+1} - p_\mu - 1}\}.$$

(6) Study of $\widetilde{PF}(S, a)_2$ when $\rho_\mu = 1, r'_\mu = h$. We have $x_1 x_k^{\sigma_\mu} - x_0^h x_{k+1}^{p_\mu}, x_i x_k^{\sigma_\mu} - x_{i-1} x_{k+1}^{p_\mu}$ for $i = 2, \dots, k$ and

$$\widetilde{PF}(S, a)_2 \subset \{x_i x_k^{\sigma_\mu - 1} x_{k+1}^{p_{\mu+1} - p_\mu - 1}, i = 1, \dots, k\}.$$

Let $i = 1, \dots, k, j = 1, \dots, k, M = x_k^{\sigma_\mu - 1} x_{k+1}^{p_{\mu+1} - p_\mu - 1}$. If $i + j > k$ then we have $x_i x_j - x_{i+j-k} x_k$. If $i + j = k + 1$ then $x_j x_i M - x_0^h x_{k+1}^{p_{\mu+1} - 1} \in I$. If $i + j > k + 1$ then $x_j x_i M - x_{i+j-k-1} x_{k+1}^{p_{\mu+1} - 1} \in I$.

(i) If $s_\mu - s_{\mu+1} = 1$ so by using the set \mathcal{C} we get $x_j x_i M - x_0^{\alpha_{i,j}} N \in I$ for some $\alpha_{i,j} \in \mathbb{N}^*$.

On the other side we have $x_{k+1} x_i M = x_i x_k^{\sigma_\mu - 1} x_{k+1}^{p_{\mu+1} - p_\mu}$, so by using the set \mathcal{C} we get $x_i x_k^{\sigma_\mu - 1} x_{k+1}^{p_{\mu+1} - p_\mu} - x_0^\alpha N \in I$ for some $\alpha \in \mathbb{N}^*$. Hence

$$\widetilde{PF}(S, a)_2 = \{x_i x_k^{\sigma_\mu - 1} x_{k+1}^{p_{\mu+1} - p_\mu - 1}, i = 1, \dots, k\}.$$

(ii) If $s_\mu - s_{\mu+1} > 1$. If $i = 1, j = k$ we have $x_k x_1 M - x_0^h x_{k+1}^{p_\mu} \in I$. If $i > 1$ let $j = k + 2 - i$ then $x_j x_i M - x_1 x_{k+1}^{p_{\mu+1} - 1} \in I$ but since $1 < s_\mu - s_{\mu+1}$ we have that $x_1 x_{k+1}^{p_{\mu+1} - 1} \in \widetilde{Ap}(S, a)$. so

$$\widetilde{PF}(S, a)_2 \subset \{x_1 x_k^{\sigma_\mu - 1} x_{k+1}^{p_{\mu+1} - p_\mu - 1}\}.$$

On the other hand we have $x_{k+1} x_1 M = x_1 x_k^{\sigma_\mu - 1} x_{k+1}^{p_{\mu+1} - p_\mu}$. By using the set \mathcal{C} we get we have $x_{k+1} x_1 M = x_0^\alpha N \in I$ for some $\alpha \in \mathbb{N}^*$ if and only if $s_\mu - s_{\mu+1} \leq s_\mu - k$. Hence, if $1 < s_\mu - s_{\mu+1} \leq s_\mu - k$ then $\widetilde{PF}(S, a)_2 = \{x_1 x_k^{\sigma_\mu - 1} x_{k+1}^{p_{\mu+1} - p_\mu - 1}\}$ and if $s_\mu - k < s_\mu - s_{\mu+1}$ then $\widetilde{PF}(S, a)_2 = \emptyset$.

- (7) Study of $\widetilde{PF}(S, a)_2$ when $\rho_\mu > 1$. We have $x_{\rho_\mu} x_k^{\sigma_\mu} - x_0^{r'_\mu} x_{k+1}^{p_\mu}$, $x_{\rho_\mu+l} x_k^{\sigma_\mu} - x_0^{r'_\mu-h} x_l x_{k+1}^{p_\mu}$ for $l = 1, \dots, k - \rho_\mu$ and

$$\widetilde{PF}(S, a)_2 \subset \{x_i x_k^{\sigma_\mu} x_{k+1}^{p_{\mu+1}-p_\mu-1}, i = 1, \dots, \rho_\mu - 1\}.$$

Let $i = 1, \dots, \rho_\mu - 1, j = 1, \dots, k$, $M = x_k^{\sigma_\mu} x_{k+1}^{p_{\mu+1}-p_\mu-1}$. If $i + j > k$ then we have $x_i x_j - x_{i+j-k} x_k$, $x_j x_i M - x_{i+j-k} x_k^{\sigma_\mu+1} x_{k+1}^{p_{\mu+1}-p_\mu-1} \in I$ thus $x_j x_i M - x_{i+j-k} x_0^{r'_\mu-h} x_{k-\rho_\mu} x_{k+1}^{p_{\mu+1}-1} \in I$ but $i + j - k + k - \rho_\mu = i + j - \rho_\mu < j$ so $x_{i+j-k} x_{k-\rho_\mu} - x_0^h x_{i+j-\rho_\mu} \in I$, hence $x_j x_i M - x_0^h N \in I$ for some monomial N . On the other hand we have $x_{k+1} x_i M = x_i x_k^{\sigma_\mu} x_{k+1}^{p_{\mu+1}-p_\mu}$ so by using the set \mathcal{C} we get $x_i x_k^{\sigma_\mu} x_{k+1}^{p_{\mu+1}-p_\mu} - x_0^\alpha N \in I$ for some $\alpha \in \mathbb{N}^*$ if and only if $i + \sigma_\mu k \geq s_\mu - s_{\mu+1}$ that is $i \geq \rho_\mu - s_{\mu+1}$.

- (i) If $\rho_\mu - s_{\mu+1} \leq 1$ we have $i \geq \rho_\mu - s_{\mu+1}$ for any $i \geq 1$. Hence

$$\widetilde{PF}(S, a)_2 = \{x_i x_k^{\sigma_\mu} x_{k+1}^{p_{\mu+1}-p_\mu-1}, i = 1, \dots, \rho_\mu - 1\}.$$

- (ii) If $\rho_\mu - s_{\mu+1} > 1$ we have $\tilde{\rho} = \rho_\mu - s_{\mu+1}$ and

$$\widetilde{PF}(S, a)_2 = \{x_i x_k^{\sigma_\mu} x_{k+1}^{p_{\mu+1}-p_\mu-1}, i = \tilde{\rho}, \dots, \rho_\mu - 1\}.$$

□

6. SYMMETRIC ALMOST GENERALIZED ARITHMETIC PROGRESSIONS

Theorem 6.1. *With the above notations, suppose that $k \geq 3$ and either $r'_\mu \geq h$ or $\rho_\mu = 0$. We have S is symmetric if and only if either*

- (1) $a = (\sigma k + 2)p', d = p'r - pr', c = (\sigma k + 2)r'$ for any $\sigma \geq 1, 1 \leq p < p', h \geq 1$ with $\gcd(p', r') = 1, r + h\sigma > 0, r' < -1$. Moreover I is minimally generated by the Gröbner basis consisting of the set \mathcal{A} and:

$$x_2 x_k^\sigma - x_0^{r+h(\sigma+1)} x_{k+1}^p, \quad x_i x_k^\sigma - x_0^{r+h\sigma} x_{i-2} x_{k+1}^p, \quad i = 3, \dots, k, \quad x_{k+1}^{p'} - x_0^{-r'}$$

The Frobenius number is $(ha + d) + \sigma(ha + kd) + c(p' - 1) - a$.

- (2) $s_{\mu+1} \neq 0, a = (\sigma k + 2)p' - \sigma' k p, d = p'r + ph\sigma', c = \sigma' k r + (\sigma k + 2)\sigma' h$ where $\sigma, \sigma', p', p, r, h$ are integers such that $\sigma > \sigma' \geq 2, p' > p > 0, r + h(\sigma + 1) > 0$ and $(a, d) = 1$. Moreover I is minimally generated by the Gröbner basis consisting of the set \mathcal{A} and:

$$x_2 x_k^{\sigma-\sigma'} x_{k+1}^{p'-p} - x_0^{r+h(\sigma+1)}, \quad x_i x_k^{\sigma-\sigma'} x_{k+1}^{p'-p} - x_0^{r+h\sigma} x_{i-2}, \quad i = 3, \dots, k$$

$$x_2 x_k^\sigma - x_0^{r+h(\sigma+1)} x_{k+1}^p, \quad x_i x_k^\sigma - x_0^{r+h\sigma} x_{i-2} x_{k+1}^p, \quad i = 3, \dots, k, \quad x_{k+1}^{p'} - x_k^{\sigma'}$$

The Frobenius number is $(ha + d) + \sigma(ha + kd) + c(p_{\mu+1} - p_\mu - 1) - a$.

- (3) $k > 2, s_{\mu+1} \neq 0, a = (\sigma k + 2)p' - (\sigma k + 1)p, d = p'r + ph(\sigma + 1), c = (\sigma k + 1)r + (\sigma k + 2)(\sigma + 1)h$ where $\sigma, \sigma', p', p, r, h$ are integers such that $\sigma \geq 1, p' > p >$

$0, r + h(\sigma + 1) > 0$ and $(a, d) = 1$. Moreover I is minimally generated by the Gröbner basis consisting of the set \mathcal{A} and:

$$x_1 x_{k+1}^{p'-p} - x_0^{r+h(\sigma+2)}, \quad x_i x_{k+1}^{p'-p} - x_0^{r+h(\sigma+1)} x_{i-1}, \quad i = 2, \dots, k,$$

$$x_2 x_k^\sigma - x_0^{r+h(\sigma+1)} x_{k+1}^p, \quad x_i x_k^\sigma - x_0^{r+h\sigma} x_{i-2} x_{k+1}^p, \quad i = 3, \dots, k, \quad x_{k+1}^{p'} - x_1 x_k^\sigma$$

The Frobenius number is $(ha + d) + \sigma(ha + kd) + c(p_{\mu+1} - p_\mu - 1) - a$.

- (4) $s_{\mu+1} \neq 0$, $a = (\sigma k + 1)p' - (k - 1)p$, $d = p'r - pr'$, $c = (k - 1)r - (\sigma k + 1)r'$.
 $d > 0$ if and only if $r' < -(p'/p)h\sigma$ with $p < p'$, $r + h(\sigma + 1) = h$, that is
 $r = -\sigma h$, $r' < -\sigma h$.

Moreover I is minimally generated by the Gröbner basis consisting of the set \mathcal{A} and:

$$x_2 x_k^{\sigma-1} x_{k+1}^{p'-p} - x_0^{-r'}, \quad x_i x_k^{\sigma-1} x_{k+1}^{p'-p} - x_0^{-r'-h} x_{i-2}$$

$$x_1 x_k^\sigma - x_0^h x_{k+1}^p, \quad x_i x_k^\sigma - x_{i-1} x_{k+1}^p, \quad i = 2, \dots, k, \quad x_{k+1}^{p'} - x_0^{r'+h} x_{k-1}$$

The Frobenius number is $(ha + d) + (\sigma - 1)(ha + kd) + c(p_{\mu+1} - 1) - a$.

Proof. (1) If $s_{\mu+1} = 0$ then $r'_{\mu+1} \neq 0$ and we have that $\widetilde{PF}(S, a) = \widetilde{PF}(S, a)_1$ with $\tilde{\sigma} = \sigma_\mu$, $\tilde{\rho} = \rho_\mu$. So $\text{card} \widetilde{PF}(S, a) = 1$ if and only if we are in case 2d) with $\rho_\mu = 2$. We have $s_\mu = \sigma k + 2$ for some $\sigma \geq 1$.

That is

$$\begin{array}{ccc|c} s & p & r & r' \\ \dots & & & \\ \sigma k + 2 & p & r & \\ 0 & p' & r' & \end{array}$$

with the condition $p < p'$, $r'_\mu = r + h(\sigma + 1) > 0$, $r' < 0$, $r' < r$. By Lemma 2.2.4 of [6] we get

$$a = (\sigma k + 2)p', d = p'r - pr', c = (\sigma k + 2)r'.$$

The Frobenius number is $\varphi(x_1 x_k^\sigma x_{k-1}^{p_{\mu+1}-1}) - a$.

- (2) If $s_{\mu+1} \neq 0$ and $\text{card} \widetilde{PF}(S, a)_1 = 0$, $\text{card} \widetilde{PF}(S, a)_2 = 1$. We have to consider the cases 1a) and one of the cases 4ii), 5iii), 6ii) 7i), 7ii); or 1c) and case 7i). More precisely

(a) 1a) and 4ii) We have $\rho_\mu = 0$, $\rho_{\mu+1} = 0$, $\tilde{\rho} = 0$, so $\text{card} \widetilde{PF}(S, a)_2 = 1$ if and only if $k = 2$.

(b) 1a) and 5iii) we have $\rho_\mu = 1$, $s_{\mu+1} = 1 = \rho_{\mu+1}$ a contradiction with 1a).

(c) 1a) and 7i) we have $\rho_\mu = 2$, $\rho_{\mu+1} = 0$. So we have $s_\mu = \sigma_\mu k + 2$, $s_{\mu+1} = \sigma_{\mu+1} k$. We set $\sigma = \sigma_\mu$, $\sigma' = \sigma_{\mu+1}$, $p = p_\mu$, $p' = p_{\mu+1}$, $r = r_\mu$, and since $r'_{\mu+1} = 0 = r_{\mu+1} + h\sigma$ we have $r_{\mu+1} = -h\sigma$ and the table

$$\begin{array}{ccc|c} s & p & r & r' \\ \dots & & & \\ \sigma k + 2 & p & r & \\ \sigma' k & p' & -h\sigma' & \end{array}$$

By Lemma 2.2.4 of [6] we get

$$a = (\sigma k + 2)p' - \sigma' k p, \quad d = p'r + ph\sigma', \quad c = \sigma' k r + (\sigma k + 2)\sigma' h$$

for some $\sigma \geq \sigma p' \geq 2, p' > p > 0, r > -h\sigma', r + h(\sigma + 1) > 0$. The Frobenius number is $\varphi(x_1 x_k^\sigma x_{k-1}^{p_{\mu+1}-p_\mu-1}) - a$.

(d) 1a) and 7ii) Since $s_{\mu+1} < \rho_\mu - 1$ we have $s_{\mu+1} = \rho_{\mu+1} = 0$ by 1a), a contradiction.

(e) 1c) and 6i) We have $\text{card}PF(\widetilde{S}, a)_2 = \rho_\mu - 1 = 1$ if and only if $\rho_\mu = 2$. By hypothesis $s_\mu - s_{\mu+1} = 1$. So we have $s_\mu = \sigma_\mu k + 2, s_{\mu+1} = \sigma_\mu k + 1$ for some $\sigma \geq 1$. We set $p = p_\mu, p' = p_{\mu+1}, r = r_\mu$, and since $r'_{\mu+1} = 0 = r_{\mu+1} + h(\sigma + 1)$ we have $r_{\mu+1} = -h(\sigma + 1)$ and the table:

$$\begin{array}{ccc|c} s & p & r & r' \\ \dots & & & \\ \sigma k + 2 & p & r & \\ \sigma k + 1 & p' & -h(\sigma + 1) & \end{array}$$

By Lemma 2.2.4 of [6] we get

$$a = (\sigma k + 2)p' - (\sigma k + 1)p, \quad d = p'r + ph(\sigma + 1), \quad c = (\sigma k + 1)r + (\sigma k + 2)(\sigma + 1)h$$

for some $\sigma \geq 1, p' > p \geq 1, r > -h(\sigma + 1)$. The Frobenius number is $\varphi(x_1 x_k^\sigma x_{k-1}^{p_{\mu+1}-p_\mu-1}) - a$.

(3) Suppose $s_{\mu+1} \neq 0$ and $\text{card}PF(\widetilde{S}, a)_1 = 1, \text{card}PF(\widetilde{S}, a)_2 = 0$. We are in case 6iii), we have $\rho_\mu = 1, r'_\mu = h, 0 < s_{\mu+1} < k$ so $r'_{\mu+1} < 0$ and $s_\mu - s_{\mu+1} = (\sigma_\mu - 1)k + (k - s_{\mu+1} + 1)$. If $s_{\mu+1} = 1$ we have $\tilde{\rho} = 0$, we are in case 2a) and $\text{card}PF(\widetilde{S}, a)_1 = 1$ implies $k = 2$. If $s_{\mu+1} > 1$ we have $0 < \tilde{\rho} = k - s_{\mu+1} + 1 < k$ and $\text{card}PF(\widetilde{S}, a)_1 = 1$ implies $\tilde{\rho} = 2$, that is $s_{\mu+1} = k - 1$. On the other hand $r'_\mu = h = r_\mu + h(\sigma_\mu + 1)$ so $r_\mu = -h\sigma_\mu$. We set $\sigma = \sigma_\mu, p = p_\mu, p' = p_{\mu+1}, r' =$

$$\begin{array}{ccc|c} s & p & r & r' \\ \dots & & & \\ \sigma k + 1 & p & -h\sigma & \\ k - 1 & p' & r' & \end{array} \quad \text{with } \sigma \geq 1, p < p', r' < -(p'/p)h\sigma. \text{ By}$$

Lemma 2.2.4 of [6] we get

$$a = (\sigma k + 1)p' - (k - 1)p, \quad d = p'r - pr', \quad c = (k - 1)r - (\sigma k + 1)r'.$$

The Frobenius number is $\varphi(x_1 x_k^{\sigma-1} x_{k-1}^{p_{\mu+1}-1}) - a$.

□

7. ALMOST SYMMETRIC ALMOST GENERALIZED ARITHMETIC PROGRESSIONS

Lemma 7.1. *With the above notations, suppose that either $r'_\mu \geq h$ or $\rho_\mu = 0$. Suppose that S is almost symmetric of type ≥ 2 then $s_{\mu+1} > 0$ and if $\widetilde{Frob}(S)$ belongs to \widetilde{PSF}_i then $\text{card}(\widetilde{PSF}_i) = 1$ except if $h = 1, s_\mu = k + 1, s_{\mu+1} = k, r'_\mu > 1, r'_{\mu+1} = -1$. Moreover*

- (1) *If $\widetilde{PSF}_1 = \{\widetilde{Frob}(S)\}$ then $\widetilde{Frob}(S) = L_\gamma x_k^{\tilde{\sigma}-\varepsilon} x_{k+1}^{p_{\mu+1}-1}$ where $\gamma, \varepsilon \in \{0, 1\}$.*
- (2) *If $\widetilde{PSF}_2 = \{\widetilde{Frob}(S)\}$ then $\widetilde{Frob}(S) = L_\gamma x_k^{\sigma_\mu-\varepsilon} x_{k+1}^{p_{\mu+1}-p_\mu-1}$ where $\gamma, \varepsilon \in \{0, 1\}$.*
- (3) *If $h = 1, s_\mu = k + 1, s_{\mu+1} = k, r'_\mu > 1, r'_{\mu+1} = 1$ then $a = k + 2, d = 2r_\mu + 2, c = k(r_\mu + 2) + 2$ with k odd, $k \geq 3, r_\mu \geq 0, \text{gcd}(a, d) = 1$. We have $\widetilde{PSF} = \{x_1, \dots, x_k\} \cup \{x_{k+1}\}, t(S) = k + 1, \widetilde{Frob}(S) = x_k, F(S) = kd$.*

Proof. Suppose that $\widetilde{Frob}(S) \in \widetilde{PSF}_{1+\varepsilon}, \text{card}(\widetilde{PSF}_{1+\varepsilon}) \geq 2$ for some $\varepsilon \in \{0, 1\}$. By checking all cases in Theorem 5.2 there exists $2 \leq l \leq k, M$ a monomial such that $\widetilde{Frob}(S) = x_l M$ and $x_{l-1} M \in \widetilde{PSF}_{1+\varepsilon}$. Since S is almost symmetric there exists $M_1 \in \widetilde{PSF}$ such that $M_1 x_{l-1} M - x_0 x_l M \in I$, which implies $x_{l-1} M_1 - x_0 x_l \in I$. We multiply by x_1 and using the Gröbner basis we get $x_0^h x_l M_1 - x_0 x_l x_1 \in I$ that implies $h = 1, M_1 - x_1 \in I$, if $M_1 \neq x_1$ then the embedding dimension of S is less than $k + 2$ contrary to our hypothesis, therefore $M_1 = x_1 \in \widetilde{PSF}$. We have to examine all the possible cases in Theorem 5.2 such that $x_1 \in \widetilde{PSF}$. Since $\sigma_\mu \geq 1, p_{\mu+1} \geq 2$ the possible cases are 4), 5) or 6). In particular we have $s_{\mu+1} > 0$. Case 4) implies $s_\mu = k$ so $x_k - x_0^{r'_\mu} x_{k+1}^{p_\mu} \in I$, that means that the embedding dimension of S is less than $k + 2$ contrary to our hypothesis. Cases 5) and 6) implies $\widetilde{PSF}_2 = \{x_1, \dots, x_k\}$ with $s_\mu = k + 1, s_{\mu+1} = k, \widetilde{Frob}(S) = x_k$ and $\widetilde{PSF}_1 \subset \{x_{k+1}^\beta\}$ for some $\beta \in \mathbb{N}^*$, but this is only possible in case 1a) or 2b). Now we consider the case 1a) so $r'_{\mu+1} = 0$, by the Gröbner basis we have that $x_{k+1}^{p_{\mu+1}} - x_k \in I$ which implies that the embedding dimension of S is less than $k + 2$ contrary to our hypothesis. In case 2b) we have $\widetilde{PSF}_1 = \{x_{k+1}^{p_{\mu+1}-1}\}$, the property almost-symmetry show that $x_{k+1}^{2(p_{\mu+1}-1)} - x_0 x_k \in I$ so that $x_0^{-r'_{\mu+1}} x_k x_{k+1}^{p_{\mu+1}-2} - x_0 x_k \in I$ hence $p_{\mu+1} = 2, r'_{\mu+1} = -1$

and $p_\mu = 1$, so $r_{\mu+1} = r'_{\mu+1} - 1 = -2$ we have the table

\dots	$k + 1$	1	r_μ	r'_μ
	k	2	-2	1

We note that $r'_\mu = h = 1$ if and only if $r_\mu = -1$ which implies $d = 0$, so case 6) is not possible. The case 5) is possible and we have $a = k + 2, d = 2r_\mu + 2, c = k(r_\mu + 2) + 2$ with k odd, $k \geq 3, r_\mu \geq 0, \text{gcd}(a, d) = 1$. We have $\widetilde{PSF} = \{x_1, \dots, x_k\} \cup \{x_{k+1}\}, t(S) = k + 1, F(S) = kd$. \square

Theorem 7.2. *Suppose $\widetilde{PSF}_2 = \{\widetilde{Frob}(S)\}$. Then S is almost symmetric with $k \geq 3, t(S) \geq 2$ if and only if either*

(1) $\widetilde{PSF}_1 = \{x_i x_{k+1}^{p_{\mu+1}-1} \mid i = 1, \dots, k-l\}, \widetilde{PSF}_2 = \{x_1 x_k^\sigma x_{k+1}^{p_{\mu+1}-2}\}$ and

$$\begin{array}{ccc|c} s & p & r & r' \\ \sigma k + 2 & 1 & r_\mu & \\ (\sigma - 1)k + l & p_{\mu+1} & r_{\mu+1} & 0 \end{array}$$

Since $0 = r'_{\mu+1} = r_{\mu+1} + \sigma$ we have $r_{\mu+1} = -\sigma$ with $\sigma \geq 2, l \geq 1$. $t(S) = k-l+1$, $t(S) = 2$ if and only if $l = \rho_{\mu+1} = k-1$.

$$a = (\sigma k + 2)p_{\mu+1} - ((\sigma - 1)k + l)$$

$$d = p_{\mu+1}r_\mu + \sigma$$

$$c = ((\sigma - 1)k + l)r_\mu + (\sigma k + 2)\sigma.$$

where $h = 1, k \geq 3, 1 \leq l \leq k-1$ and $t(S) = k-l+1$, or

(2) $\widetilde{PSF}_1 = \{x_{k+1}^{p_{\mu+1}-1}\}, \widetilde{PSF}_2 = \{x_1 x_k^{\sigma_\mu} x_{k+1}^{p_{\mu+1}-2}\}$ and

$$\begin{array}{ccc|c} s & p & r & r' \\ \sigma k + 2 & 1 & r_\mu & \\ \sigma k + 1 & p_{\mu+1} & r_{\mu+1} & -1 \end{array}$$

but $-1 = r'_{\mu+1} = r_{\mu+1} + h(\sigma + 1)$ so $r_{\mu+1} = -h(\sigma + 1) - 1, t(S) = 2$.

$$a = (\sigma k + 2)p_{\mu+1} - (\sigma k + 1)$$

$$d = p_{\mu+1}r_\mu + h(\sigma + 1) + 1$$

$$c = (\sigma k + 1)r_\mu + (\sigma k + 2)(h(\sigma + 1) + 1).$$

where $h \geq 1, \sigma \geq 1, p_{\mu+1} \geq 2, r_\mu > -h(\sigma + 1) - 1, t(S) = 2$.

Proof. We have $\widetilde{PSF}_2 = \{L_\gamma x_k^{\sigma_\mu - \delta} x_{k+1}^{p_{\mu+1} - p_\mu - 1}\}$ where $\gamma, \delta \in \{0, 1\}$. We set $N_{\sigma_\mu - \delta} := x_k^{\sigma_\mu - \delta} x_{k+1}^{p_{\mu+1} - p_\mu - 1}$. By Theorem 5.2 we have $\widetilde{PSF}_1 = \{x_\Gamma M_{\tilde{\sigma} - \varepsilon}, \dots, x_\Delta M_{\tilde{\sigma} - \varepsilon}\}$ for some $\Gamma \leq \Delta, \varepsilon \in \{0, 1\}$ where $M_{\tilde{\sigma} - \varepsilon} := x_k^{\tilde{\sigma} - \varepsilon} x_{k+1}^{p_{\mu+1} - 1}$. Since S is almost symmetric we have that

$$L_\Gamma L_\Delta M_{\tilde{\sigma} - \varepsilon}^2 - x_0 L_\Gamma N_{\sigma_\mu - \delta} \in I. \quad (1)$$

We have $(x_{k+1}^{p_{\mu+1}-1})^2 = x_{k+1}^{2p_{\mu+1}-2} = x_{k+1}^{p_{\mu+1}} x_{k+1}^{p_{\mu+1}-2}$ since $p_{\mu+1} - 2 \geq 0$, so $(x_{k+1}^{p_{\mu+1}-1})^2 - x_0^{-r'_{\mu+1}} L_{\rho_{\mu+1}} x_k^{\sigma_{\mu+1}} x_{k+1}^{p_{\mu+1}-2} \in I$ so $M_{\tilde{\sigma} - \varepsilon}^2 - x_0^{-r'_{\mu+1}} L_{\rho_{\mu+1}} x_k^{\sigma_{\mu+1} + 2\tilde{\sigma} - 2\varepsilon} x_{k+1}^{p_{\mu+1}-2} \in I$. Note that $p_{\mu+1} - 2 - (p_{\mu+1} - p_\mu - 1) = p_\mu - 1 \geq 0$ so that (1) becomes

$$L_\Gamma L_\Delta x_0^{-r'_{\mu+1}-1} L_{\rho_{\mu+1}} x_k^{\sigma_{\mu+1} + 2\tilde{\sigma} - 2\varepsilon} x_{k+1}^{p_\mu - 1} - L_\Gamma x_k^{\sigma_\mu - \delta} \in I. \quad (2)$$

(1) If $\Gamma + \Delta = 0$ we are in case 2b) so $s_\mu - s_{\mu+1} = 1$. Since $\text{card}(\widetilde{PSF}_2) = 1$ we are in case 7i) hence $\gamma = 1, \delta = 0, s_\mu = \sigma_\mu k + 2, s_{\mu+1} = \sigma_\mu k + 1$. We have

$$x_0^{-r'_{\mu+1}-1} x_1 x_k^{\sigma_\mu} x_{k+1}^{p_\mu - 1} - x_1 x_k^{\sigma_\mu} \in I \quad (3)$$

this is possible only if $r'_{\mu+1} = -1, p_\mu = 1$. We set $\sigma = \sigma_\mu$ so we have

$\widetilde{PSF}_1 = \{x_{k+1}^{p_{\mu+1}-1}\}, \widetilde{PSF}_2 = \{x_1 x_k^{\sigma_\mu} x_{k+1}^{p_{\mu+1}-2}\}$ and

$$\begin{array}{ccc|c} s & p & r & r' \\ \sigma k + 2 & 1 & r_\mu & \\ \sigma k + 1 & p_{\mu+1} & r_{\mu+1} & -1 \end{array}$$
 but $-1 = r'_{\mu+1} = r_{\mu+1} + h(\sigma + 1)$ so $r_{\mu+1} = -h(\sigma + 1) - 1, t(S) = 2$.

$$\begin{aligned} a &= (\sigma k + 2)p_{\mu+1} - (\sigma k + 1) \\ d &= p_{\mu+1}r_\mu + h(\sigma + 1) + 1 \\ c &= (\sigma k + 1)r_\mu + (\sigma k + 2)(h(\sigma + 1) + 1). \end{aligned}$$

(2) If $0 < \Gamma + \Delta \leq k$ we have

$$x_{\Gamma+\Delta}x_0^{h-r'_{\mu+1}-1}L_{\rho_{\mu+1}}x_k^{\sigma_{\mu+1}+2\tilde{\sigma}-2\varepsilon}x_{k+1}^{p_\mu-1} - L_\gamma x_k^{\sigma_\mu-\delta} \in I \quad (4)$$

which implies $h = 1, r'_{\mu+1} = 0$ and after Theorem 5.2 $\rho_{\mu+1} > 0$.

(a) If $\Gamma + \Delta + \rho_{\mu+1} \leq k$ we have

$$x_0x_{\Gamma+\Delta+\rho_{\mu+1}}x_k^{\sigma_{\mu+1}+2\tilde{\sigma}-2\varepsilon}x_{k+1}^{p_\mu-1} - L_\gamma x_k^{\sigma_\mu-\delta} \in I \quad (5)$$

which is impossible since $L_\gamma x_k^{\sigma_\mu-\delta} \in \widetilde{Ap}(S)$. So

(b) $\Gamma + \Delta + \rho_{\mu+1} > k$ we have

$$x_{\Gamma+\Delta+\rho_{\mu+1}-k}x_k^{\sigma_{\mu+1}+2\tilde{\sigma}+1-2\varepsilon}x_{k+1}^{p_\mu-1} - L_\gamma x_k^{\sigma_\mu-\delta} \in I \quad (6)$$

By Theorem 5.2 in all cases with $r'_{\mu+1} = 0, \rho_{\mu+1} > 0$ and $\Gamma + \Delta + \rho_{\mu+1} > k$ we have $\Gamma + \Delta + \rho_{\mu+1} = k + 1$ so

$$x_1x_k^{\sigma_{\mu+1}+2\tilde{\sigma}+1-2\varepsilon}x_{k+1}^{p_\mu-1} - L_\gamma x_k^{\sigma_\mu-\delta} \in I \quad (7)$$

If $\sigma_{\mu+1} + \tilde{\sigma} - \sigma_\mu + \tilde{\sigma} + \delta + 1 - 2\varepsilon > 0$

$$x_1x_k^{\sigma_{\mu+1}+\tilde{\sigma}-\sigma_\mu+\tilde{\sigma}+\delta+1}x_{k+1}^{p_\mu-1} - L_\gamma \in I \quad (8)$$

which leads to a contradiction. So we can assume $\sigma_{\mu+1} + \tilde{\sigma} - \sigma_\mu + \tilde{\sigma} + \delta + 1 - 2\varepsilon \leq 0$. We have

$$x_1x_{k+1}^{p_\mu-1} - L_\gamma x_k^{-(\sigma_{\mu+1}+\tilde{\sigma}-\sigma_\mu+\tilde{\sigma}+\delta+1-2\varepsilon)} \in I. \quad (9)$$

If $x_1x_{k+1}^{p_\mu-1} \neq L_\gamma x_k^{-(\sigma_{\mu+1}+\tilde{\sigma}-\sigma_\mu+\tilde{\sigma}+\delta+1-2\varepsilon)}$, since $L_\gamma x_k^{-(\sigma_{\mu+1}+\tilde{\sigma}-\sigma_\mu+\tilde{\sigma}+\delta+1-2\varepsilon)} \in \widetilde{Ap}(S)$ we have $x_1x_{k+1}^{p_\mu-1} \in \text{in}(I)$ which implies $s_\mu - s_{\mu+1} = 1$ and by 1c) of Theorem 5.2 $\widetilde{PF}_1 = \emptyset$ a contradiction. So $x_1x_{k+1}^{p_\mu-1} = L_\gamma x_k^{-(\sigma_{\mu+1}+\tilde{\sigma}-\sigma_\mu+\tilde{\sigma}+\delta+1-2\varepsilon)}$ which implies $\gamma = 1, p_\mu = 1, \sigma_{\mu+1} + \tilde{\sigma} - \sigma_\mu + \tilde{\sigma} + \delta + 1 - 2\varepsilon = 0$

We have to discuss several cases:

$\varepsilon = 1$: So we are either in case 1b) or 1d) and we have $\tilde{\sigma} > 0$. On the other hand we have either:

1) $\tilde{\sigma} = \sigma_\mu - \sigma_{\mu+1} - 1, \tilde{\rho} > 0, \rho_{\mu+1} + \tilde{\rho} = k + \rho_\mu \geq k$. In case 1b) we have $\tilde{\rho} = 0$ so this case is not possible. Case 1d) implies $\tilde{\rho} = 1$, so

that $\rho_{\mu+1} = k - 1, \rho_\mu = 0$, so we are in case 4), since $\text{card}(\widetilde{PSF_2}) = 1$ we have $k = 2$ a contradiction.

2) $\sigma_\mu = \sigma_{\mu+1} + \tilde{\sigma}$, $\rho_{\mu+1} + \tilde{\rho} = \rho_\mu$, we have $\tilde{\sigma} + \delta - 1 = 0$ so $\tilde{\sigma} = 1, \delta = 0$. $\delta = 0$ implies that we have to consider cases 5iii), 7i) and 7ii). Case 5iii) implies $s_{\mu+1} = 1$, case 7ii) implies $\rho_\mu > s_{\mu+1}$ so $\sigma_{\mu+1} = 0$ in both cases and $x_{k+1}^{p_{\mu+1}} - x_{\rho_{\mu+1}} \in I$, a contradiction. So we have two possible cases 7i)-1b or 7i)-1d. Since $\text{card}(\widetilde{PSF_2}) = 1$ we have $\rho_\mu = 2$ and $\widetilde{PSF_2} = \{x_1 x_k^{\sigma_\mu} x_{k+1}^{p_{\mu+1} - p_\mu - 1}\}$.

◇) 7i) 1b): we have $\tilde{\rho} = 0, \tilde{\sigma} = 1, \rho_{\mu+1} = 2, p_\mu = 1, r'_{\mu+1} = 0$. We set $\sigma = \sigma_\mu$, we have

$$\widetilde{PSF_1} = \{x_i x_{k+1}^{p_{\mu+1} - 1} \mid i = 1, \dots, k - 2\}, \widetilde{PSF_2} = \{x_1 x_k^\sigma x_{k+1}^{p_{\mu+1} - 2}\} \text{ and}$$

$$\begin{array}{ccc|c} s & p & r & r' \\ \sigma k + 2 & 1 & r_\mu & \\ (\sigma - 1)k + 2 & p_{\mu+1} & r_{\mu+1} & 0 \end{array}$$

but $0 = r'_{\mu+1} = r_{\mu+1} + \sigma$ so $r_{\mu+1} = -\sigma, t(S) = k - 1$ and

$$a = (\sigma k + 2)p_{\mu+1} - ((\sigma - 1)k + 2)$$

$$d = p_{\mu+1}r_\mu + \sigma$$

$$c = ((\sigma - 1)k + 2)r_\mu + (\sigma k + 2)\sigma.$$

By using my software we have the following example $k = 4, a = 214, d = 15, c = 236, \sigma = 7, p_{\mu+1} = 8, r_\mu = 9$.

◇◇) 7i) 1d): we have $\tilde{\rho} = 1, \tilde{\sigma} = 1, \rho_{\mu+1} = 1, p_\mu = 1, r'_{\mu+1} = 0$. We set $\sigma = \sigma_\mu$, we have

$$\widetilde{PSF_1} = \{x_i x_{k+1}^{p_{\mu+1} - 1} \mid i = 1, \dots, k - 1\}, \widetilde{PSF_2} = \{x_1 x_k^\sigma x_{k+1}^{p_{\mu+1} - 2}\}. \text{ and}$$

$$\begin{array}{ccc|c} s & p & r & r' \\ \sigma k + 2 & 1 & r_\mu & \\ (\sigma - 1)k + 1 & p_{\mu+1} & r_{\mu+1} & 0 \end{array}$$

but $0 = r'_{\mu+1} = r_{\mu+1} + \sigma$ so $r_{\mu+1} = -\sigma, t(S) = k$ and

$$a = (\sigma k + 2)p_{\mu+1} - ((\sigma - 1)k + 1)$$

$$d = p_{\mu+1}r_\mu + \sigma$$

$$c = ((\sigma - 1)k + 1)r_\mu + (\sigma k + 2)\sigma.$$

By using my software we have the following example $k = 5, a = 487, d = 7, c = 259, \sigma = 7, p_{\mu+1} = 14, r_\mu = 8$.

$\varepsilon = 0$: We are in case 1e). We have $\sigma_{\mu+1} + \tilde{\sigma} - \sigma_\mu + \tilde{\sigma} + \delta + 1 = 0$. If $\tilde{\sigma} = \sigma_\mu - \sigma_{\mu+1}$ we have $\tilde{\sigma} + \delta + 1 = 0$ which is impossible, hence $\sigma_\mu = \sigma_{\mu+1} + \tilde{\sigma} + 1, \tilde{\rho} > 0, \tilde{\rho} + \rho_{\mu+1} = k + \rho_\mu$ and $\tilde{\sigma} = \delta = 0$, so $s_\mu - s_{\mu+1} < k \leq s_\mu - \rho_\mu$ which implies $s_{\mu+1} > \rho_\mu$. Hence we have to consider only case 7i)-1e). We have $\rho_\mu = 2, \tilde{\rho} + \rho_{\mu+1} = k + 2$ which implies $\tilde{\rho} - 1 = k - \rho_{\mu+1} + 1$, and we have seen before that

$\tilde{\sigma} = 0, p_\mu = 1, r'_{\mu+1} = 0$. We set $\sigma = \sigma_\mu, l = \rho_{\mu+1} > 2$, so we have $\widetilde{PSF}_1 = \{x_i x_{k+1}^{p_{\mu+1}-1} \mid i = 1, \dots, k-l\}, \widetilde{PSF}_2 = \{x_1 x_k^\sigma x_{k+1}^{p_{\mu+1}-2}\}$ and

$$\begin{array}{ccc|c} s & p & r & r' \\ \sigma k + 2 & 1 & r_\mu & \\ (\sigma - 1)k + l & p_{\mu+1} & r_{\mu+1} & 0 \end{array}$$

Since $0 = r'_{\mu+1} = r_{\mu+1} + \sigma$ we have $r_{\mu+1} = -\sigma$ with $\sigma \geq 2, l \geq 3$. $t(S) = k - l + 1$. Note that $t(S) = 2$ if and only if $l = \rho_{\mu+1} = k - 1$.

$$a = (\sigma k + 2)p_{\mu+1} - ((\sigma - 1)k + l)$$

$$d = p_{\mu+1}r_\mu + \sigma$$

$$c = ((\sigma - 1)k + l)r_\mu + (\sigma k + 2)\sigma,$$

with $\sigma \geq 1, l \geq 3, p_{\mu+1} \geq 2, r_\mu > -\sigma$. By using my software we have the following example $a = 213, d = 49, c = 209, k = 6, \tilde{\rho} = 5, \rho_{\mu+1} = 3, t(S) = 4$.

- (3) If $\Gamma + \Delta > k$ we are in case 2c), we have $\Gamma = 1, \Delta = k, \varepsilon = 1, r'_{\mu+1} < 0, \tilde{\rho} = 1, \tilde{\sigma} > 0$, and (2) becomes

$$x_1 x_0^{-r'_{\mu+1}} L_{\rho_{\mu+1}} x_k^{\sigma_{\mu+1}+2\tilde{\sigma}-1} x_{k+1}^{p_{\mu}-1} - x_0 L_\gamma x_k^{\sigma_\mu-\delta} \in I. \quad (10)$$

If $\rho_{\mu+1} > 0$ we have

$$x_0^{-r'_{\mu+1}+h-1} x_{\rho_{\mu+1}+1} x_k^{\sigma_{\mu+1}+2\tilde{\sigma}-1} x_{k+1}^{p_{\mu}-1} - L_\gamma x_k^{\sigma_\mu-\delta} \in I. \quad (11)$$

Since $h - r'_{\mu+1} \geq 1$ we will have $L_\gamma x_k^{\sigma_\mu-\delta} x_{k+1}^{p_{\mu+1}-p_{\mu}-1} \notin \widetilde{Ap}(S)$ a contradiction.

If $\rho_{\mu+1} = 0$ then $\rho_\mu = 1, \sigma_\mu = \sigma_{\mu+1} + \tilde{\sigma}, \sigma_{\mu+1} + 2\tilde{\sigma} - 1 = \sigma_\mu + \tilde{\sigma} - 1$ but $x_1 x_k^{\sigma_\mu} - x_0^{r'_\mu} x_{k+1}^{p_\mu} \in I$ so from (10) we have

$$x_0^{r'_\mu - r'_{\mu+1}} x_k^{\tilde{\sigma}-1} x_{k+1}^{2p_\mu-1} - x_0 L_\gamma x_k^{\sigma_\mu-\delta} \in I. \quad (12)$$

Since $r'_\mu - r'_{\mu+1} \geq 2$ we will have $L_\gamma x_k^{\sigma_\mu-\delta} \in \text{in}(I)$ a contradiction. \square

Theorem 7.3. Suppose $\widetilde{PSF}_1 = \{\widetilde{Frob}(S)\}$. Then S is almost symmetric with $k \geq 3, t(S) \geq 2$ if and only if either

- (1) $\widetilde{PSF}_1 = \{x_1 x_k^\sigma x_{k+1}^p\}, \widetilde{PSF}_2 = \{x_2 x_k^\sigma, \dots, x_{\rho_{\mu}-1} x_k^\sigma\}$, we have the table

$$\begin{array}{ccc|c} s & p & r & r' \\ \sigma k + l + 2 & p & -\sigma & 1 \\ l & p + 1 & r_{\mu+1} & < 0 \end{array}$$

$$a = (\sigma k + l + 2)(p + 1) - lp$$

$$d = -(p + 1)\sigma - pr_{\mu+1}$$

$$c = -l\sigma - (\sigma k + l + 2)r_{\mu+1},$$

where $h = 1, k \geq 3, \sigma \geq 1, 1 \leq l \leq k - 3, r_{\mu+1} < -\sigma$ and $t(S) = l + 1$, or

$$(2) \widetilde{PSF}_1 = \{x_1 x_k^{\sigma-1} x_{k+1}^p\}, \widetilde{PSF}_2 = \{x_2 x_k^{\sigma-1}, \dots, x_{k-1} x_k^{\sigma-1}\}.$$

$$\begin{array}{ccc|c} s & p & r & r' \\ \sigma k & p & 1 - \sigma & 1 \\ k - 2 & p + 1 & r_{\mu+1} & < 0 \end{array}$$

$$a = (\sigma k)(p + 1) - (k - 2)p$$

$$d = (1 - h\sigma)(p + 1) - pr_{\mu+1}$$

$$c = (k - 2)(1 - \sigma) - \sigma k r_{\mu+1}.$$

where $h = 1, k \geq 3, p, \sigma \geq 2, r_{\mu+1} < -1$ and $t(S) = k - 1$, or

$$(3) \widetilde{PSF}_1 = \{x_1 x_k^{\sigma-1} x_{k+1}^p\}, \widetilde{PSF}_2 = \{x_2 x_k^{\sigma-1}, \dots, x_k^\sigma\}, t(S) = k.$$

$$\begin{array}{ccc|c} s & p & r & r' \\ \sigma k + 1 & p & 1 - h\sigma & h + 1 \\ k - 1 & p + 1 & r_{\mu+1} & < 0 \end{array}$$

$$a = (\sigma k + 1)(p + 1) - (k - 1)p$$

$$d = (p + 1)(1 - h\sigma) - pr_{\mu+1}$$

$$c = (k - 1)(1 - h\sigma) - (\sigma k + 1)r_{\mu+1}.$$

$h \geq 1, k \geq 3, p, \sigma \geq 1, r_{\mu+1} < \min\{-h, 1 - h\sigma\}$ and $t(S) = k$, or

$$(4) \widetilde{PSF}_1 = \{x_{k+1}^p\}, \widetilde{PSF}_2 = \{x_1, \dots, x_k\}.$$

$$\begin{array}{ccc|c} s & p & r & r' \\ k + 1 & p & -1 & 1 \\ k & p + 1 & r_{\mu+1} & < 0 \end{array}$$

$$a = (k + 1)(p + 1) - kp = k + p + 1$$

$$d = -(p + 1) - pr_{\mu+1}$$

$$c = -k - (k + 1)r_{\mu+1}.$$

with $h = 1, k \geq 3, p \geq 1, r_{\mu+1} < -1$. We have $t(S) = k + 1$, or

$$(5) \widetilde{PSF}_1 = \{x_1 x_k^{\sigma-2} x_{k+1}^p\}, \widetilde{PSF}_2 = \{x_1 x_k^{\sigma-1}\} \text{ and}$$

$$\begin{array}{ccc|c} s & p & r & r' \\ \sigma k + 1 & p & -\sigma & 1 \\ 2k - 1 & p + 1 & r_{\mu+1} & < 0 \end{array}$$

$$a = (\sigma k + 1)(p + 1) - (2k - 1)p$$

$$d = -(p + 1)\sigma - pr_{\mu+1}$$

$$c = -\sigma(2k - 1) - (\sigma k + 1)r_{\mu+1}.$$

with $h = 1, k \geq 3, \sigma \geq 2, p \geq 1, r_{\mu+1} < -\sigma$. We have $t(S) = 2$.

Proof. We have $\widetilde{PSF}_1 = \{L_\gamma x_k^{\tilde{\sigma}-\varepsilon} x_{k+1}^{p_{\mu+1}-1}\}$ where $\gamma, \varepsilon \in \{0, 1\}$. We set $M_{\tilde{\sigma}-\varepsilon} := x_k^{\tilde{\sigma}-\varepsilon} x_{k+1}^{p_{\mu+1}-1}$. By Theorem 5.2 we have $\widetilde{PSF}_2 = \{x_\Gamma N_{\sigma_\mu-\delta}, \dots, x_\Delta N_{\sigma_\mu-\delta}\}$ for some $\Gamma \leq \Delta, \delta \in \{0, 1\}$, where $N_{\sigma_\mu-\delta} := x_k^{\sigma_\mu-\delta} x_{k+1}^{p_{\mu+1}-p_\mu-1}$.

Since S is almost symmetric we have that

$$L_\Gamma L_\Delta N_{\sigma_\mu - \delta}^2 - x_0 L_\gamma M_{\tilde{\sigma} - \varepsilon} \in I, \quad (13)$$

Note that $\gamma = 0$ is possible only in case 2b), where we have $\tilde{\sigma} = 0, \tilde{\rho} = 1, \varepsilon = 0$. In all other cases we have $\gamma = 1$.

- (1) If $\Gamma + \Delta = 0$ then we are in case 5iii), which implies $\rho_\mu = 1, s_{\mu+1} = 1$ so $\tilde{\rho} = 0$, but in all cases of Theorem 5.2 with $\text{card}(\widetilde{PSF}_1) = 1$ satisfying these conditions we have $k = 2$.

- (2) If $0 < \Gamma + \Delta \leq k$ we have

$$x_{\Gamma+\Delta} x_0^{h-1} x_k^{2\sigma_\mu - 2\delta} x_{k+1}^{2(p_{\mu+1} - p_\mu - 1)} - L_\gamma x_k^{\tilde{\sigma} - \varepsilon} x_{k+1}^{p_{\mu+1} - 1} \in I \quad (14)$$

If $\gamma = 0$ or $h > 1$ then $x_{k+1}^{p_{\mu+1} - 1} \in \text{in}(I)$, which is not possible. So we have $\gamma = 1, h = 1$ and

$$x_{\Gamma+\Delta} x_k^{2\sigma_\mu - 2\delta} x_{k+1}^{2(p_{\mu+1} - p_\mu - 1)} - x_1 x_k^{\tilde{\sigma} - \varepsilon} x_{k+1}^{p_{\mu+1} - 1} \in I \quad (15)$$

We consider two cases:

- (a) If $\sigma_\mu = \sigma_{\mu+1} + \tilde{\sigma}$ (so $\rho_\mu = \rho_{\mu+1} + \tilde{\rho}$) then $2\sigma_\mu - 2\delta - (\tilde{\sigma} - \varepsilon) = \sigma_\mu + \sigma_{\mu+1} - 2\delta + \varepsilon$. Note that $\sigma_{\mu+1} - 2\delta + \varepsilon < 0$ if and only if $\delta = 1$ and either $\sigma_{\mu+1} = 0, \varepsilon = 0$, either $\sigma_{\mu+1} = 0, \varepsilon = 1$ or $\sigma_{\mu+1} = 1, \varepsilon = 0$. If $\sigma_{\mu+1} = 0$ we have $\sigma_\mu = \tilde{\sigma}$, but we can check that no case in Theorem 5.2 with $\delta = 1$ satisfy the condition $\sigma_\mu = \tilde{\sigma}$. If $\varepsilon = 0$ then the possible cases are 1e) or 2d), both cases imply that $\tilde{\rho} > 1$ hence $\rho_\mu > 1$. We can check that there is no case in Theorem 5.2 with $\delta = 1, \rho_\mu > 1$.

So we have $\sigma_{\mu+1} - 2\delta + \varepsilon \geq 0$. We have

$$x_{\Gamma+\Delta} x_k^{\sigma_\mu} x_k^{\sigma_{\mu+1} - 2\delta + \varepsilon} x_{k+1}^{2(p_{\mu+1} - p_\mu - 1)} - x_1 x_{k+1}^{p_{\mu+1} - 1} \in I. \quad (16)$$

Since $\sigma_\mu \geq 1$, if $2(p_{\mu+1} - p_\mu - 1) \geq p_{\mu+1} - 1$ then $\varphi(x_1) \geq \varphi(x_k)$ a contradiction, so we have

$$x_{\Gamma+\Delta} x_k^{\sigma_\mu} x_k^{\sigma_{\mu+1} - 2\delta + \varepsilon} - x_1 x_{k+1}^{2p_{\mu+1} - p_{\mu+1}} \in I \quad (17)$$

If $\rho_\mu = 0$ or $\Gamma + \Delta = \rho_\mu$ then $L_{\rho_\mu} x_k^{\sigma_\mu} - x_0^{r'_\mu} x_{k+1}^{p_\mu} \in I$ which leads to a contradiction since $x_1 x_{k+1}^{2p_{\mu+1} - p_{\mu+1}} \in \text{Ap}(S)$. Hence we have either $\Gamma + \Delta > \rho_\mu > 0$ or $\Gamma + \Delta < \rho_\mu, \rho_\mu > 0$.

★ If $\Gamma + \Delta > \rho_\mu > 0$ then

$$x_0^{r'_\mu - 1} x_{\Gamma+\Delta - \rho_\mu} x_k^{\sigma_{\mu+1} - 2\delta + \varepsilon} x_{k+1}^{p_\mu} - x_1 x_{k+1}^{2p_{\mu+1} - p_{\mu+1}} \in I. \quad (18)$$

Since $2p_\mu + 1 - p_{\mu+1} \leq p_\mu$ we get

$$x_0^{r'_\mu - 1} x_{\Gamma+\Delta - \rho_\mu} x_k^{\sigma_{\mu+1} - 2\delta + \varepsilon} x_{k+1}^{p_{\mu+1} - p_\mu - 1} - x_1 \in I \quad (19)$$

This is possible if and only if $r'_\mu = 1, \Gamma + \Delta - \rho_\mu = 1, \sigma_{\mu+1} - 2\delta + \varepsilon = 0$ and $p_{\mu+1} = p_\mu + 1$. We have several cases.

- $\delta = 1$, by Theorem 5.2 we are in case 6) which implies $\rho_\mu = 1$ so $\Gamma + \Delta = 2$ and $\Gamma = \Delta = 1$, the only possible case is 6ii), so $s_\mu - s_{\mu+1} > 1$. Since $1 = \rho_\mu = \rho_{\mu+1} + \tilde{\rho}$ we have either $\tilde{\rho} = 0, \rho_{\mu+1} = 1$ or $\tilde{\rho} = 1, \rho_{\mu+1} = 0$. If $\tilde{\rho} = 0, \rho_{\mu+1} = 1$ by Theorem 5.2 the only possible cases with $\text{card}(PF_1(S)) = 1$, are 1b) and 2a) with $k = 2$ a contradiction. If $\tilde{\rho} = 1, \rho_{\mu+1} = 0$ the only possible case is 2b) with $s_\mu - s_{\mu+1} = 1$, a contradiction, so the case $\delta = 1$ is not possible.
- $\delta = 0$ implies $\sigma_{\mu+1} = \varepsilon = 0$. We have to consider in Theorem 5.2 the cases 1e) or 2d) because $\varepsilon = 0$ and 7i), 7ii) because $\delta = 0$. On the other hand $\Gamma + \Delta = \rho_\mu + 1$ implies that we are in case 7ii) with $\tilde{\rho} = 2$. We set $\sigma = \sigma_\mu, p = p_\mu, l = s_{\mu+1} = \rho_{\mu+1}$ so we have $\widetilde{PSF}_1 = \{x_1 x_k^\sigma x_{k+1}^p\}$, $\widetilde{PSF}_2 = \{x_2 x_k^\sigma, \dots, x_{\rho_{\mu-1}} x_k^\sigma\}$, we have the table

s	p	r	r'
$\sigma k + l + 2$	p	$-\sigma$	1
l	p + 1	$r_{\mu+1}$	< 0

 with $\sigma \geq 1, 1 \leq l < k - 2$. Also $1 = r'_\mu = r_\mu + \sigma + 1$ so $r_\mu = -\sigma, r_{\mu+1} < -\sigma \leq -1$, and $r'_{\mu+1} = r_{\mu+1} + 1 < -\sigma + 1 \leq 0$. So in fact we are in case 2d) 7ii). We have $t(S) = l + 1$. Note that $t(S) = 2$ if and only if $l = 1, \rho_\mu = 3$.

$$\begin{aligned} a &= (\sigma k + l + 2)(p + 1) - lp \\ d &= -(p + 1)\sigma - pr_{\mu+1} \\ c &= -l\sigma - (\sigma k + l + 2)r_{\mu+1}. \end{aligned}$$

★ If $\rho_\mu > 0, \Gamma + \Delta < \rho_\mu$. From (17) we have necessarily $x_{\Gamma+\Delta} x_k^{\sigma_\mu} x_k^{\sigma_{\mu+1}-2\delta+\varepsilon} \in \text{in}(I)$ so $\sigma_{\mu+1} - 2\delta + \varepsilon > 0$. Since $x_{\Gamma+\Delta} x_k^{\sigma_\mu+1} - x_0^{r'_\mu} x_{k+\Gamma+\Delta-\rho_\mu} x_{k+1}^{p_\mu} \in I$ we have

$$x_0^{r'_\mu} x_{k+\Gamma+\Delta-\rho_\mu} x_{k+1}^{p_\mu} x_k^{\sigma_{\mu+1}-1-2\delta+\varepsilon} x_{k+1}^{p_\mu} - x_1 x_{k+1}^{2p_{\mu+1}-p_{\mu+1}} \in I, \quad (20)$$

this is impossible since $x_1 x_{k+1}^{2p_{\mu+1}-p_{\mu+1}} \in \widetilde{Ap}(S)$.

- (b) $\sigma_\mu = \sigma_{\mu+1} + \tilde{\sigma} + 1$ (so $\tilde{\rho} > 0, \rho_\mu = \rho_{\mu+1} + \tilde{\rho} - k > 0$) then $2\sigma_\mu - 2\delta - (\tilde{\sigma} - \varepsilon) = \sigma_\mu + \sigma_{\mu+1} + 1 - 2\delta + \varepsilon$.

Note that $\sigma_{\mu+1} + 1 - 2\delta + \varepsilon < 0$ if and only if $\delta = 1, \sigma_{\mu+1} = 0, \varepsilon = 0$. The possible cases in Theorem 5.2 are 7i), 7ii). In 7ii) we have $\rho_{\mu+1} < \rho_\mu$ so $\sigma_\mu = \sigma_{\mu+1} + \tilde{\sigma}$ a contradiction. In 7i) we have $2\sigma_\mu - 2\delta - (\tilde{\sigma} - \varepsilon) = \sigma_\mu - 1 \geq 0$, $\Gamma + \Delta = \rho_\mu \geq 2$ so (15) becomes

$$x_{\rho_\mu} x_k^{\sigma_\mu-1} x_{k+1}^{2(p_{\mu+1}-p_\mu-1)} - x_1 x_{k+1}^{p_{\mu+1}-1} \in I, \quad (21)$$

if $2(p_{\mu+1} - p_\mu - 1) \geq p_{\mu+1} - 1$ we get

$$x_{\rho_\mu} x_k^{\sigma_\mu-1} x_{k+1}^{(2(p_{\mu+1}-p_\mu-1))-(p_{\mu+1}-1)} - x_1 \in I, \quad (22)$$

which leads to a contradiction; so $2(p_{\mu+1} - p_\mu - 1) < p_{\mu+1} - 1$, we get

$$x_{\rho_\mu} x_k^{\sigma_\mu - 1} - x_1 x_{k+1}^{(p_{\mu+1}-1)-(2(p_{\mu+1}-p_\mu-1))} \in I, \quad (23)$$

but $x_1 x_{k+1}^{(p_{\mu+1}-1)-(2(p_{\mu+1}-p_\mu-1))} \in \widetilde{Ap}(S, a)$ so $x_{\rho_\mu} x_k^{\sigma_\mu - 1} \in \text{in}(I)$ which is not possible.

Hence $\sigma_{\mu+1} + 1 - 2\delta + \varepsilon \geq 0$, and (15) becomes

$$x_{\Gamma+\Delta} x_k^{\sigma_\mu + \sigma_{\mu+1} + 1 - 2\delta + \varepsilon} x_{k+1}^{2(p_{\mu+1}-p_\mu-1)} - x_1 x_{k+1}^{p_{\mu+1}-1} \in I, \quad (24)$$

if $2(p_{\mu+1} - p_\mu - 1) \geq p_{\mu+1} - 1$ we get a contradiction since $\sigma_\mu \geq 1, \varphi(x_k) > \varphi(x_1)$. So

$$x_{\Gamma+\Delta} x_k^{\sigma_\mu + \sigma_{\mu+1} + 1 - 2\delta + \varepsilon} - x_1 x_{k+1}^{2p_{\mu+1}-p_{\mu+1}} \in I. \quad (25)$$

Since $x_1 x_{k+1}^{2p_{\mu+1}-p_{\mu+1}} \in \widetilde{Ap}(S, a)$, we have $x_{\Gamma+\Delta} x_k^{\sigma_\mu + \sigma_{\mu+1} + 1 - 2\delta + \varepsilon} \in \text{in}(I)$. Hence we have either $\rho_\mu = 0$, either $\Gamma + \Delta = \rho_\mu > 0$, either $\Gamma + \Delta > \rho_\mu > 0$ or $0 < \Gamma + \Delta < \rho_\mu, \sigma_{\mu+1} + 1 - 2\delta + \varepsilon > 0$.

- If $\rho_\mu = 0$ or $\Gamma + \Delta = \rho_\mu > 0$ we have $L_{\rho_\mu} x_k^{\sigma_\mu} - x_0^{r'_\mu} x_{k+1}^{p_\mu}$ so from (25) we have

$$x_0^{r'_\mu} x_k^{\sigma_{\mu+1} + 1 - 2\delta + \varepsilon} x_{k+1}^{p_\mu} - x_1 x_{k+1}^{2p_{\mu+1}-p_{\mu+1}} \in I. \quad (26)$$

Since $r'_\mu > 0$ and $x_1 x_{k+1}^{2p_{\mu+1}-p_{\mu+1}} \in \widetilde{Ap}(S, a)$ we get a contradiction.

- If $k \geq \Gamma + \Delta > \rho_\mu > 0$ then we have $x_{\Gamma+\Delta} x_k^{\sigma_\mu} - x_0^{r'_\mu - h} x_{\Gamma+\Delta-\rho_\mu} x_{k+1}^{p_\mu}$, so from (25) we have

$$x_0^{r'_\mu - h} x_{\Gamma+\Delta-\rho_\mu} x_k^{\sigma_{\mu+1} + 1 - 2\delta + \varepsilon} x_{k+1}^{p_\mu} - x_1 x_{k+1}^{2p_{\mu+1}-p_{\mu+1}} \in I. \quad (27)$$

which implies $r'_\mu = h = 1$. We have $p_\mu - (2p_\mu + 1 - p_{\mu+1}) = p_{\mu+1} - p_\mu - 1 \geq 0$ so

$$x_{\Gamma+\Delta-\rho_\mu} x_k^{\sigma_{\mu+1} + 1 - 2\delta + \varepsilon} x_{k+1}^{p_{\mu+1}-p_\mu-1} - x_1 \in I. \quad (28)$$

which is possible only if $\Gamma + \Delta - \rho_\mu = 1, p_{\mu+1} = p_\mu + 1, \sigma_{\mu+1} + 1 - 2\delta + \varepsilon = 0$, which implies $\delta = 1, \sigma_{\mu+1} + \varepsilon = 1$. But we also have $r'_\mu = h = 1, \rho_\mu > 0$ so the only possible case is 6ii) which implies $\rho_\mu = 1, s_{\mu+1} \geq k$ so $\sigma_{\mu+1} = 1, \varepsilon = 0$. We also have $\rho_{\mu+1} + \tilde{\rho} = k + 1$ so $\tilde{\rho} > 1$ which implies either case 1e) or case 2d) with $\tilde{\rho} = 2$ since $\text{card}(\widetilde{PSF}_1) = 1$. Hence $s_\mu = \sigma_\mu k + 1, \sigma_\mu \geq 2, s_{\mu+1} = 2k - 1$, since $r'_\mu = 1 = r_\mu + \sigma_\mu + 1$ we have $r_\mu = -\sigma_\mu$ and $r_{\mu+1} < -\sigma_\mu$ implies $r'_{\mu+1} = r_{\mu+1} + 2 < -\sigma_\mu + 2 \leq 0$, so in fact we are in case 6ii)) and 2d). We set $\sigma = \sigma_\mu, p = p_\mu$ so we have

$$\widetilde{PSF}_1 = \{x_1 x_k^{\sigma-2} x_{k+1}^p\}, \widetilde{PSF}_2 = \{x_1 x_k^{\sigma-1}\} \text{ and}$$

$$\begin{array}{ccc|c} s & p & r & r' \\ \sigma_\mu k + 1 & p & -\sigma_\mu & 1 \\ 2k - 1 & p + 1 & r_{\mu+1} & < 0 \end{array}$$

$$a = (\sigma_\mu k + 1)(p + 1) - (2k - 1)p$$

$$d = -(p + 1)\sigma_\mu - pr_{\mu+1}$$

$$c = -\sigma_\mu(2k - 1) - (\sigma_\mu k + 1)r_{\mu+1}.$$

with $h = 1, k \geq 3, \sigma_\mu \geq 2, p \geq 1, r_{\mu+1} < -\sigma_\mu$ and $t(S) = 2$.

- Suppose that $0 < \Gamma + \Delta < \rho_\mu$ then $\sigma_{\mu+1} + 1 - 2\delta + \varepsilon > 0$. Since $x_{\Gamma+\Delta} x_k^{\sigma_\mu+1} - x_0^{r'_\mu} x_{k+\Gamma+\Delta-\rho_\mu} x_{k+1}^{p_\mu} \in I$ we have from (25)

$$x_0^{r'_\mu} x_{k+\Gamma+\Delta-\rho_\mu} x_k^{\sigma_\mu+1-2\delta+\varepsilon} x_{k+1}^{p_\mu} - x_1 x_{k+1}^{2p_\mu+1-p_\mu+1} \in I. \quad (29)$$

This is not possible since $r'_\mu > 0$ and $x_1 x_{k+1}^{2p_\mu+1-p_\mu+1} \in \widetilde{Ap}(S, a)$.

- (3) If $\Gamma + \Delta > k$, from (13) we have

$$x_{\Gamma+\Delta-k} x_k^{2\sigma_\mu+1-2\delta} x_{k+1}^{2(p_{\mu+1}-p_\mu-1)} - x_0 L_\gamma x_k^{\tilde{\sigma}-\varepsilon} x_{k+1}^{p_{\mu+1}-1} \in I \quad (30)$$

- (a) If $\sigma_\mu = \sigma_{\mu+1} + \tilde{\sigma}$ then we have $\rho_\mu = \rho_{\mu+1} + \tilde{\rho}$, $2\sigma_\mu + 1 - 2\delta - (\tilde{\sigma} - \varepsilon) = \sigma_\mu + \sigma_{\mu+1} + 1 - 2\delta + \varepsilon$. We have several cases.

- (i) $\sigma_{\mu+1} + 1 - 2\delta + \varepsilon < 0$ if and only $\delta = 1, \sigma_{\mu+1} = 0, \varepsilon = 0$. If $\gamma = 0$ then we are in case 2b) so $\tilde{\sigma} = \sigma_{\mu+1} = 0$ which implies $\sigma_\mu = 0$, this is not possible, so $\gamma = 1$. By looking all cases in Theorem 5.2 with $\delta = 1$ we have $\rho_\mu \leq 1$, on the other side $\varepsilon = 0, \gamma = 1$ implies $\tilde{\rho} > 1$ a contradiction since $\tilde{\rho} \leq \rho_\mu$.

- (ii) $\sigma_{\mu+1} + 1 - 2\delta + \varepsilon = 0$ if and only $\delta = 1$ and either $\sigma_{\mu+1} = 0, \varepsilon = 1$ or $\sigma_{\mu+1} = 1, \varepsilon = 0$. We have

$$x_{\Gamma+\Delta-k} x_k^{\sigma_\mu} x_{k+1}^{2(p_{\mu+1}-p_\mu-1)} - x_0 L_\gamma x_{k+1}^{p_{\mu+1}-1} \in I \quad (31)$$

ii-*) Suppose $\gamma = 0$. We are in case 2b), $s_\mu - s_{\mu+1} = 1$, if $\sigma_{\mu+1} = 0$ then $s_\mu = \rho_{\mu+1} + 1$ which is not possible, so $\sigma_{\mu+1} = 1, \tilde{\rho} = 1$ so $\rho_\mu \geq 1$. Since $\delta = 1$ implies $\rho_\mu = 1$, hence $\rho_{\mu+1} = 0$ and since $s_\mu - s_{\mu+1} = 1$ we have $s_\mu = k + 1, s_{\mu+1} = k$. The possible cases are 5i) and 6i), in both cases we have $\Gamma + \Delta - k = 1 = \rho_\mu$. By using the Gröbner basis we have

$$x_0^{r'_\mu-1} x_{k+1}^{2p_{\mu+1}-p_\mu-2} - x_{k+1}^{p_{\mu+1}-1} \in I \quad (32)$$

that is

$$x_0^{r'_\mu-1} x_{k+1}^{p_{\mu+1}-p_\mu-1} - 1 \in I \quad (33)$$

possible only if $r'_\mu = 1, p_{\mu+1} = p_\mu + 1$. Since $r'_\mu \geq h \geq 1$ we have equality, so we are in case 6i) and $1 = r_\mu + 2$ so $r_\mu = -1$. We set $p_\mu = p$, we have

$$\widetilde{PSF}_1 = \{x_{k+1}^p\}, \widetilde{PSF}_2 = \{x_1, \dots, x_k\}.$$

$$\begin{array}{ccc|c} s & p & r & r' \\ k+1 & p & -1 & 1 \\ k & p+1 & r_{\mu+1} & < 0 \end{array}$$

$$a = (k+1)(p+1) - kp = k + p + 1$$

$$d = -(p+1) - pr_{\mu+1}$$

$$c = -k - (k+1)r_{\mu+1}.$$

with $h = 1, k \geq 3, p \geq 1, r_{\mu+1} < -1, t(S) = k + 1$.

ii-**) Suppose $\gamma = 1$. Recall that $\delta = 1$ and either $\sigma_{\mu+1} = 0, \varepsilon = 1$ or $\sigma_{\mu+1} = 1, \varepsilon = 0$. If $\gamma = 1, \varepsilon = 0$ we are in cases 1e) or 2d) so $\tilde{\rho} > 1$. $\delta = 1$ implies $\rho_\mu \leq 1$, but $\rho_\mu = \rho_{\mu+1} + \tilde{\rho}$ so we get a contradiction. Therefore $\delta = 1, \sigma_{\mu+1} = 0, \varepsilon = 1$, again from $\rho_\mu = \rho_{\mu+1} + \tilde{\rho}$ we get $s_{\mu+1} = 1, \rho_\mu = 1, \tilde{\rho} = 0$. The possible cases with $\tilde{\rho} = 0$ are 1b) and 2a) with $k = 2$ a contradiction since we assume $k \geq 3$.

(iii) $\sigma_{\mu+1} + 1 - 2\delta + \varepsilon > 0$. If $\rho_\mu > 0$ we have $x_k^{\sigma_{\mu+1}} - x_0^{r'_\mu - h} x_{k-\rho_\mu} x_{k+1}^{p_\mu}$ so from (30) we have

$$x_0^{r'_\mu - h} x_{k-\rho_\mu} x_{\Gamma+\Delta-k} x_k^{\sigma_{\mu+1}-2\delta+\varepsilon} x_{k+1}^{2p_{\mu+1}-p_\mu-2} - x_0 L_\gamma x_{k+1}^{p_{\mu+1}-1} \in I, \quad (34)$$

hence

$$x_0^{r'_\mu - h} x_{k-\rho_\mu} x_{\Gamma+\Delta-k} x_k^{\sigma_{\mu+1}-2\delta+\varepsilon} x_{k+1}^{p_{\mu+1}-p_\mu-1} - x_0 L_\gamma \in I, \quad (35)$$

recall that $\gamma = 0, 1$, since $k - \rho_\mu \geq 1, \Gamma + \Delta - k \geq 1$ we have $\varphi(x_{k-\rho_\mu} x_{\Gamma+\Delta-k}) > \varphi(x_0 L_\gamma)$ this is not possible.

If $\rho_\mu = 0$ we have $x_k^{\sigma_\mu} - x_0^{r'_\mu} x_{k+1}^{p_\mu}$ so from (30) we have

$$x_0^{r'_\mu} x_{\Gamma+\Delta-k} x_k^{\sigma_{\mu+1}+1-2\delta+\varepsilon} x_{k+1}^{p_{\mu+1}-p_\mu-1} - x_0 L_\gamma \in I, \quad (36)$$

this is not possible since $r'_\mu > 0, \varphi(x_0^{r'_\mu} x_k) > \varphi(x_0 L_\gamma)$.

(b) If $\sigma_\mu = \sigma_{\mu+1} + \tilde{\sigma} + 1, k + \rho_\mu = \rho_{\mu+1} + \tilde{\rho}$ we have $\rho_\mu < \rho_{\mu+1}, \tilde{\rho}$ and $\sigma_\mu + 1 - 2\delta - (\tilde{\sigma} - \varepsilon) = \sigma_{\mu+1} + 2 - 2\delta + \varepsilon \geq 0$.

Suppose $\rho_\mu = 0$, which implies $\sigma_\mu \geq 2$, since $x_k^{\sigma_\mu} - x_0^{r'_\mu} x_{k+1}^{p_\mu} \in I$ from (30) we have

$$x_0^{r'_\mu} x_{\Gamma+\Delta-k} x_k^{\sigma_{\mu+1}+2-2\delta+\varepsilon} x_{k+1}^{p_{\mu+1}-p_\mu-1} - x_0 L_\gamma \in I \quad (37)$$

This is possible if and only if $\gamma = 1, \Gamma + \Delta - k = 1, r'_\mu = 1, p_{\mu+1} = p_\mu + 1$ and $\sigma_{\mu+1} + 2 - 2\delta + \varepsilon = 0$. We note that $\sigma_{\mu+1} + 2 - 2\delta + \varepsilon = 0$. if and only if $\sigma_{\mu+1} = 0, \delta = 1, \varepsilon = 0$, also note that $\sigma_{\mu+1} = 0$ implies $r'_{\mu+1} < 0$. The possible case with $r'_{\mu+1} < 0, \varepsilon = 0$ is 2d) with $\tilde{\rho} = 2$ and $\rho_\mu = 0$ implies the case 4ii). Note that $r'_\mu = 1$ implies $h = 1$. Moreover we have $1 = r'_\mu = r_\mu + \sigma_\mu$ so $r_\mu = 1 - \sigma_\mu$. We set $\sigma := \sigma_\mu, p := p_\mu$ so we have

$$\widetilde{PSF}_1 = \{x_1 x_k^{\sigma-1} x_{k+1}^p\}, \quad \widetilde{PSF}_2 = \{x_2 x_k^{\sigma-1}, \dots, x_{k-1} x_k^{\sigma-1}\}.$$

$$\begin{array}{ccc|c} s & p & r & r' \\ \sigma k & p & 1 - \sigma & 1 \\ k - 2 & p + 1 & r_{\mu+1} & < 0 \end{array}$$

$$\begin{aligned} a &= (\sigma k)(p + 1) - (k - 2)p \\ d &= (1 - \sigma)(p + 1) - pr_{\mu+1} \\ c &= (k - 2)(1 - \sigma) - \sigma kr_{\mu+1}. \end{aligned}$$

$h = 1, \sigma \geq 2, p \geq 1, r_{\mu+1} < -1, t(S) = k - 1$.

Suppose $\rho_\mu > 0$. If $\gamma = 0$ then we are in case 2b), so $\tilde{\rho} = 1, \rho_\mu = 0$, a contradiction. So $\gamma = 1$. We have two cases.

- (i) If $\sigma_{\mu+1} + 2 - 2\delta + \varepsilon = 0$ then we have $\delta = 1, \sigma_{\mu+1} = 0, \varepsilon = 0$. $\delta = 1$ implies $\rho_\mu = 1, \varepsilon = 0$ implies $\tilde{\rho} = 2$ and $\sigma_{\mu+1} = 0$ implies $r'_{\mu+1} < 0, s_\mu - s_{\mu+1} = (\sigma_\mu - 1)k + 2$ so the possible case is 2d) and 5ii) and we have $\Gamma + \Delta - k = 2$. So from (30) we have

$$x_2 x_k^{\sigma_\mu} x_{k+1}^{2(p_{\mu+1} - p_\mu - 1)} - x_0 x_1 x_{k+1}^{p_{\mu+1} - 1} \in I \quad (38)$$

but $x_2 x_k^{\sigma_\mu} - x_0^{r'_\mu - h} x_1 x_{k+1}^{p_\mu} \in I$ so

$$x_0^{r'_\mu - h} x_1 x_{k+1}^{p_{\mu+1} - p_\mu - 1} - x_0 x_1 \in I \quad (39)$$

this is possible only if $r'_\mu - h = 1, p_{\mu+1} = p_\mu + 1$. We have $h + 1 = r'_\mu = r_\mu + h(\sigma + 1)$ so $r_\mu = 1 - h\sigma, r_{\mu+1} \leq -h\sigma$, so $r'_{\mu+1} = r_{\mu+1} + h \leq -h(\sigma - 1)$, so $r'_{\mu+1} < 0$ if and only if $\sigma \geq 2$. We set $\sigma := \sigma_\mu, p := p_\mu$ so we have

$$\widetilde{PSF}_1 = \{x_1 x_k^{\sigma-1} x_{k+1}^p\}, \widetilde{PSF}_2 = \{x_2 x_k^{\sigma-1}, \dots, x_k^\sigma\}, t(S) = k.$$

$$\begin{array}{ccc|c} s & p & r & r' \\ \sigma k + 1 & p & 1 - h\sigma & h + 1 \\ k - 1 & p + 1 & r_{\mu+1} & < 0 \end{array}$$

$$\begin{aligned} a &= (\sigma k + 1)(p + 1) - (k - 1)p \\ d &= (p + 1)(1 - h\sigma) - pr_{\mu+1} \\ c &= (k - 1)(1 - h\sigma) - (\sigma k + 1)r_{\mu+1}, \end{aligned}$$

with $h \geq 1, p \geq 1, \sigma \geq 2, r_{\mu+1} \leq -h\sigma$.

- (ii) Suppose $\sigma_{\mu+1} + 2 - 2\delta + \varepsilon > 0$. We have $x_k^{\sigma_{\mu+1}} - x_0^{r'_\mu - h} x_{k-\rho_\mu} x_{k+1}^{p_\mu} \in I$. From (30) we get

$$x_0^{r'_\mu - h} x_{\Gamma+\Delta-k} x_{k-\rho_\mu} x_k^{\sigma_{\mu+1}+1-2\delta+\varepsilon} x_{k+1}^{p_{\mu+1}-p_\mu-1} - x_0 x_1 \in I, \quad (40)$$

since $\Gamma + \Delta - k \geq 1, k - \rho_\mu \geq 1$ we have $\varphi(x_{\Gamma+\Delta-k} x_{k-\rho_\mu}) > \varphi(x_0 x_1)$ which is impossible.

□

8. FORMULA FOR FROBENIUS NUMBER OF ALMOST SYMMETRIC ALMOST
GENERALIZED ARITHMETIC PROGRESSIONS

This section extends and generalizes all the results of [13].

Theorem 8.1. *Let S be an AAG almost symmetric with $k \geq 3, t(S) \geq 2$. Then there is a quadratic formula for the Frobenius number in terms of a, d, c, k and the type $t(S)$.*

Proof. We have to consider two cases depending on the number i such that $\widetilde{Frob}(S \in (\widetilde{PSF}_i))$.

I) Suppose $\widetilde{Frob}(S \in (\widetilde{PSF}_2))$. Then S is almost symmetric with $k \geq 3, t(S) \geq 2$ if and only if either

- (1) $\widetilde{PSF}_1 = \{x_i x_{k+1}^{p_{\mu+1}-1} \mid i = 1, \dots, k-l\}, \widetilde{PSF}_2 = \{x_1 x_k^\sigma x_{k+1}^{p_{\mu+1}-2}\}$. $t(S) = k-l+1$.
We have

$$a = (\sigma k + 2)p_{\mu+1} - ((\sigma - 1)k + l)$$

$$d = p_{\mu+1}r_\mu + \sigma$$

$$c = ((\sigma - 1)k + l)r_\mu + (\sigma k + 2)\sigma.$$

where $h = 1, k \geq 3, 1 \leq l \leq k-1, \sigma \geq 2$.

Since $x_1 x_{k-l} x_{k+1}^{2(p_{\mu+1}-1)} - x_0 \widetilde{Frob}(S) \in I$ we have $x_{k-l+1} x_{k+1}^{2(p_{\mu+1}-1)} - \widetilde{Frob}(S) \in I$ and $F(S) = 2(p_{\mu+1} - 1)c + a_{k-l+1} - a$ so determine a formula for $F(S)$ consist to determine $p_{\mu+1}$ in terms of a, d, c, k, l .

we set $X = p_{\mu+1}$ we have $x_{k+1}^X - x_l x_k^{\sigma-1}$ so

$$Xc = a_l + \sigma a_k - a_k \tag{41}$$

and

$$\sigma k(X - 1) = a - 2X - k + l \tag{42}$$

We multiply (41) by $k(X - 1)$ and by using (42) we get

$$k(X - 1)Xc = k(X - 1)a_l + a_k(a - 2X - k + l) - k(X - 1)a_k \tag{43}$$

So we get a second order equation in the variable X

$$kcX^2 - (k(c + a_l - a_k) - 2a_k)X - a_k(a + l) + ka_l = 0 \tag{44}$$

so $X = \frac{k(c + a_l - a_k) - 2a_k + \sqrt{(k(c + a_l - a_k) - 2a_k)^2 - 4kc(-a_k(a + l) + ka_l)}}{2kc}$

- (2) $\widetilde{PSF}_1 = \{x_{k+1}^{p_{\mu+1}-1}\}, \widetilde{PSF}_2 = \{x_1 x_k^\sigma x_{k+1}^{p_{\mu+1}-2}\}, t(S) = 2$. We have

$$a = (\sigma k + 2)p_{\mu+1} - (\sigma k + 1)$$

$$d = p_{\mu+1}r_\mu + h(\sigma + 1) + 1$$

$$c = (\sigma k + 1)r_\mu + (\sigma k + 2)(h(\sigma + 1) + 1).$$

where $h \geq 1, \sigma \geq 1, p_{\mu+1} \geq 2, r_{\mu} > -h(\sigma+1) - 1$. Since $x_{k+1}^{2(p_{\mu+1}-1)} - x_0 \widetilde{Frob}(S) \in I$ we have and $F(S) = 2(p_{\mu+1} - 1)c - 2a$ so determine a formula for $F(S)$ consist to determine $p_{\mu+1}$ in terms of a, d, c, k .

We set $X = p_{\mu+1}$ we have $x_{k+1}^X - x_0 x_1 x_k^\sigma$ so

$$Xc = a + a_1 + \sigma a_k \quad (45)$$

and

$$\sigma k(X - 1) = a + 1 - 2X \quad (46)$$

We multiply (45) by $k(X - 1)$ and by using (46) we get

$$k(X - 1)Xc = k(X - 1)(a + a_1) + a_k(a + 1 - 2X) \quad (47)$$

So we get a second order equation in the variable X

$$kcX^2 - (k(c + a + a_1) - 2a_k)X - a_k(a + 1) + k(a + a_1) = 0 \quad (48)$$

$$\text{so } X = \frac{k(c + a + a_1) - 2a_k + \sqrt{(k(c + a + a_1) - 2a_k)^2 - 4kc(-a_k(a + 1) + k(a + a_1))}}{2kc}$$

II) Suppose $\widetilde{Frob}(S) \in (\widetilde{PSF}_1)$. Then S is almost symmetric with $k \geq 3, t(S) \geq 2$ if and only if either

(1) $\widetilde{PSF}_1 = \{x_1 x_k^\sigma x_{k+1}^p\}, \widetilde{PSF}_2 = \{x_2 x_k^\sigma, \dots, x_{p_{\mu}-1} x_k^\sigma\}, t(S) = l + 1$. We have

$$\begin{aligned} a &= (\sigma k + l + 2)(p + 1) - lp \\ d &= -(p + 1)\sigma - pr_{\mu+1} \\ c &= -l\sigma - (\sigma k + l + 2)r_{\mu+1}, \end{aligned}$$

where $h = 1, k \geq 3, \sigma \geq 1, 1 \leq l \leq k - 3, r_{\mu+1} < -\sigma$.

Since $x_2 x_{l+1} x_k^{2\sigma} - x_0 \widetilde{Frob}(S) \in I$ we have $x_{l+3} x_k^{2\sigma} - \widetilde{Frob}(S) \in I$ and $F(S) = a_{l+3} + 2\sigma a_k - a$ but $x_{l+2} x_k^\sigma - x_0 x_{k+1}^{p_{\mu+1}-1} \in I$. We set $X = p_{\mu+1}$ so

$$a_{l+2} + \sigma a_k = a + c(X - 1) \quad (49)$$

and $F(S) = a_{l+3} + 2(a + c(X - 1) - a_{l+2}) - a$. Determine a formula for $F(S)$ consist to determine X in terms of a, d, c, k, l .

By developing the formula for a we have

$$\sigma kX = a - 2X - l \quad (50)$$

We multiply (49) by kX and by using (50) we get

$$ka_{l+2}X + k\sigma a_k X = kaX + kcX(X - 1) \quad (51)$$

So we get a second order equation in the variable X

$$kcX^2 - (k(c - a + a_{l+2}) - 2a_k)X - a_k(a - l) = 0 \quad (52)$$

$$\text{so } X = \frac{k(c - a + a_{l+2}) - 2a_k + \sqrt{(k(c - a + a_{l+2}) - 2a_k)^2 + 4kca_k(a - l)}}{2kc}$$

(2) $\widetilde{PSF}_1 = \{x_1x_k^{\sigma-1}x_{k+1}^p\}$, $\widetilde{PSF}_2 = \{x_2x_k^{\sigma-1}, \dots, x_{k-1}x_k^{\sigma-1}\}$, $t(S) = k - 1$. We have

$$\begin{aligned} a &= (\sigma k)(p + 1) - (k - 2)p \\ d &= (1 - h\sigma)(p + 1) - pr_{\mu+1} \\ c &= (k - 2)(1 - \sigma) - \sigma kr_{\mu+1}. \end{aligned}$$

where $h = 1, k \geq 3, p, \sigma \geq 2, r_{\mu+1} < -1$.

Since our semigroup is almost symmetric we have $x_2x_{k-1}x_k^{2\sigma-2} - x_0\widetilde{Frob}(S) \in I$ that is $x_1x_k^{2\sigma-1} - x_0\widetilde{Frob}(S) \in I$ so $F(S) = a_1 + 2\sigma a_k - a_k - 2a$. But $x_k^\sigma - x_0x_{k+1}^{p\mu+1-1} \in I$, we set $X = p_{\mu+1}$ we have

$$\sigma a_k = a + c(X - 1) \quad (53)$$

and $F(S) = a_1 + 2(a + c(X - 1)) - a_k - 2a$. Determine a formula for $F(S)$ consist to determine X in terms of a, d, c, k .

By developing the formula for a we have

$$\sigma kX = a + kX - 2X - k - 2 \quad (54)$$

We multiply (53) by kX and by using (54) we get

$$a_k(a + kX - 2X - k - 2) = kaX + kcX(X - 1) \quad (55)$$

So we get a second order equation in the variable X

$$kcX^2 - (k(c - a + a_k) - 2a_k)X - a_k(a - k + 2) = 0 \quad (56)$$

and $X = \frac{k(c - a + a_k) - 2a_k + \sqrt{(k(c - a + a_k) - 2a_k)^2 + 4kca_k(a - k + 2)}}{2kc}$.

(3) $\widetilde{PSF}_1 = \{x_1x_k^{\sigma-1}x_{k+1}^p\}$, $\widetilde{PSF}_2 = \{x_2x_k^{\sigma-1}, \dots, x_k^\sigma\}$, $t(S) = k$. We have

$$\begin{aligned} a &= (\sigma k + 1)(p + 1) - (k - 1)p \\ d &= (p + 1)(1 - h\sigma) - pr_{\mu+1} \\ c &= (k - 1)(1 - h\sigma) - (\sigma k + 1)r_{\mu+1}. \end{aligned}$$

$h \geq 1, k \geq 3, p, \sigma \geq 1, r_{\mu+1} < \min\{-h, 1 - h\sigma\}$.

Since $x_2x_k^{2\sigma-1} - x_0\widetilde{Frob}(S) \in I$ we have $F(S) = a_2 + 2\sigma a_k - a_k - 2a$ but $x_1x_k^\sigma - x_0^{h+1}x_{k+1}^{p\mu+1-1} \in I$, we set $X = p_{\mu+1}$, so we have

$$a_1 + \sigma a_k = (h + 1)a + c(X - 1) \quad (57)$$

and $F(S) = a_2 + 2(a + c(X - 1) - a_1) - a_k - 2a$. Determine a formula for $F(S)$ consist to determine X in terms of a, d, c, k .

By developing the formula for a we have

$$\sigma kX = a + X(k - 2) - k + 1 \quad (58)$$

We multiply (57) by kX and by using (58) we get

$$ka_1X + a_k(a + X(k - 2) - k + 1) = k(h + 1)aX + kcX(X - 1) \quad (59)$$

So we get a second order equation in the variable X

$$kcX^2 - (k(a_1 + c - (h + 1)a + a_k) - 2a_k)X - a_k(a - k + 1) = 0 \quad (60)$$

$$X = \frac{k(a_1 + c - (h + 1)a + a_k) - 2a_k + \sqrt{(k(a_1 + c - (h + 1)a + a_k) - 2a_k)^2 + 4kca_k(a - k + 1)}}{2kc}.$$

(4) $\widetilde{PSF}_1 = \{x_{k+1}^p\}$, $\widetilde{PSF}_2 = \{x_1, \dots, x_k\}$, $t(S) = k + 1$. We have

$$\begin{aligned} a &= k + p + 1 \\ d &= -(p + 1) - pr_{\mu+1} \\ c &= -k - (k + 1)r_{\mu+1}. \end{aligned}$$

with $h = 1, k \geq 3, p \geq 1, r_{\mu+1} < -1$. We have $F(S) = pc - a$ and $a = k + p + 1$ so $F(S) = c(a - k - 1) - a$.

(5) $\widetilde{PSF}_1 = \{x_1x_k^{\sigma-2}x_{k+1}^p\}$, $\widetilde{PSF}_2 = \{x_1x_k^{\sigma-1}\}$, $t(S) = 2$. We have

$$\begin{aligned} a &= (\sigma k + 1)(p + 1) - (2k - 1)p \\ d &= -(p + 1)\sigma - pr_{\mu+1} \\ c &= -\sigma(2k - 1) - (\sigma k + 1)r_{\mu+1}. \end{aligned}$$

with $h = 1, k \geq 3, \sigma \geq 2, p \geq 1, r_{\mu+1} < -\sigma$.

Since $x_1^2x_k^{2\sigma-2} - x_0\widetilde{Frob}(S) \in I$ we have $x_2x_k^{2\sigma-2} - \widetilde{Frob}(S) \in I$ so $F(S) = a_2 + 2\sigma a_k - 2a_k - a$ but $x_1x_k^\sigma - x_0x_{k+1}^p \in I$, which gives

$$a_1 + \sigma a_k = a + c(X - 1), \quad (61)$$

where we have set $X = p + 1$. Hence $F(S) = a_2 + 2(a + c(X - 1) - a_1) - 2a_k - a$. Determine a formula for $F(S)$ consist to determine X in terms of a, d, c, k . By developing the formula for a we have

$$\sigma kX = a + X(2k - 2) - 2k + 1 \quad (62)$$

We multiply (61) by kX and by using (62) we get

$$ka_1X + a_k(a + X(2k - 2) - 2k + 1) = kaX + kcX(X - 1) \quad (63)$$

So we get a second order equation in the variable X

$$kcX^2 - (k(c + d + 2a_k) - 2a_k)X - a_k(a - 2k + 1) = 0 \quad (64)$$

$$X = \frac{k(c + d + 2a_k) - 2a_k + \sqrt{(k(c + d + 2a_k) - 2a_k)^2 + 4kca_k(a - 2k + 1)}}{2kc}$$

□

Corollary 8.2. *Given a AAG-semigroup S with data a, d, c, h, k by at most $4k$ tests solving quadratics equations we can determine if S is almost symmetric.*

Proof. The first step is to check if one of the square roots is a natural number, since we don't know $t(S)$ we have to perform k times in the case where the number $1 \leq l \leq k$ appears in this square root. The second step is to check if the solution X as above is a natural number, at this step we know the probably value for $t(S)$ so also the probably case to consider. The third step is to solve a linear system to find the values of $\sigma, p_{\mu+1}, r_{\mu+1}$ from a, d, c, h, k and check if they are natural numbers and satisfy the conditions of the considered case. \square

Example 8.3. *We have implemented the above algorithm and we have for $150 \leq a \leq 160, 1 \leq d \leq 10, 170 \leq c \leq 180, 19 \leq k \leq 20, 2 \leq h \leq 3$ the following values for which the AAG-semigroup is almost symmetric:*

$a = 153, d = 11, c = 177, k = 19, h = 3$, case II.1, $p_{\mu+1} = 7, \sigma = 1, r_{\mu+1} = -3$.

$a = 156, d = 11, c = 174, k = 20, h = 3$, case II.1, $p_{\mu+1} = 7, \sigma = 1, r_{\mu+1} = -3$.

$a = 155, d = 1, c = 177, k = 20, h = 4$, case I.2, $p_{\mu+1} = 8, \sigma = 1, r_{\mu} = -1$.

$a = 152, d = 3, c = 170, k = 21, h = 2$, case I.1, $p_{\mu+1} = 4, \sigma = 2, r_{\mu} = 0$.

$a = 150, d = 4, c = 178, k = 21, h = 3$, case II.1, $p_{\mu+1} = 6, \sigma = 1, r_{\mu+1} = -2$.

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