



**HAL**  
open science

**Symmetric and Almost Symmetric semigroups  
generated by an almost generalized arithmetic sequence,  
Frobenius number**

Marcel Morales

► **To cite this version:**

Marcel Morales. Symmetric and Almost Symmetric semigroups generated by an almost generalized arithmetic sequence, Frobenius number. 2021. hal-04043909

**HAL Id: hal-04043909**

**<https://hal.science/hal-04043909v1>**

Preprint submitted on 24 Mar 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# SYMMETRIC AND ALMOST SYMMETRIC SEMIGROUPS GENERATED BY AN ALMOST GENERALIZED ARITHMETIC SEQUENCE, FROBENIUS NUMBER

MARCEL MORALES

ABSTRACT. Let  $a, d, k, h, c$  be positive integers. Recall that a *numerical almost generalized arithmetic sequence-semigroup* (numerical AAG-semigroup for short) is a semigroup minimally generated by relatively prime integers  $a, ha+d, ha+2d, \dots, ha+kd, c$ , that is its embedding dimension is  $k+2$ . In [8] was described a Gröbner basis of the ideal defining  $S$  under one technical assumption, the complete case will be published in a forthcoming paper. In this paper we give a complete description of  $S$  when is symmetric or almost symmetric and a quadratic formula for its Frobenius number. Note that our results generalizes and extends previous result of [11], [3] and [13]. Given  $a, d, k, h, c$  a simple algorithm allows us to determine if  $S$  is almost symmetric.

## CONTENTS

1.	Introduction	1
2.	Frobenius number and Apéry set	2
3.	Frobenius number and Gröbner basis	3
4.	almost generalized arithmetic progressions, Grobner basis	4
5.	almost generalized arithmetic progressions, Pseudo Frobenius set	7
6.	Symmetric almost generalized arithmetic progressions	13
7.	Almost Symmetric almost generalized arithmetic progressions	16
8.	Formula for Frobenius number of Almost Symmetric almost generalized arithmetic progressions	28
	References	32

## 1. INTRODUCTION

Let  $a_0, \dots, a_n$  be natural numbers and  $S = \langle a_0, \dots, a_n \rangle = \{k_0a_0 + \dots + k_na_n \mid k_i \in \mathbb{N}\}$  the semigroup generated by  $\{a_0, \dots, a_n\}$ . Recall that if  $a_0, \dots, a_n$  are relatively prime numbers then the *Frobenius number* of  $S$ , denoted by  $F(S)$ , is the biggest integer that does not belong to  $S$ . Let  $A = K[S] = K[t^k \mid k \in S] = K[t^{a_0}, \dots, t^{a_n}] \subset K[t]$  the *semigroup ring* of  $S$  and  $R = K[x_0, \dots, x_n]$  the polynomial ring in  $n+1$  variables over  $K$  graded

---

2010 *Mathematics Subject Classification*: Primary: 13D40, Secondary 14M25, 13C14, 14M05.

*Key words and phrases*: Frobenius number, numerical semigroups Pseudo-Frobenius number, Apéry set, Gröbner basis, Semigroup rings, symmetric and almost symmetric semigroups.

by the weights  $\deg x_i = a_i$ , for all  $i = 0, \dots, n$ . The *defining ideal*  $I$  of  $K[S]$  is defined to be the kernel of the  $K$ -algebra homomorphism  $\Psi : R \rightarrow K[S]$  given by  $\Psi(x_i) = t^{a_i}$  for all  $i = 0, \dots, n$ , we will use often the fact that  $I$  is a prime ideal generated by binomials and does not contains monomials. We use the *weighted degree reverse lexicographical order*  $\prec_w$  on the monomials of the ring  $R$  with  $x_0 \prec \dots \prec x_n$ , and *the map*  $\varphi : [[R]] \rightarrow \mathbb{N}$  defined by  $\varphi(M) = k_1 a_1 + \dots + k_n a_n$  for every monomial  $M = x_0^{k_0} \dots x_n^{k_n} \in [[R]]$ . Let recall the *pseudo-Frobenius set*  $PF(S)$  of all integer number  $a$  which satisfies  $a \notin S$  and  $a + s \in S$ , for all  $0 \neq s \in S$  and the number of elements of  $PF(S)$  is called the *type* of  $S$ , denoted by  $t(S)$ . Finally, the *Apéry set* with respect to  $a_0$  plays an important role in our paper which is defined by  $\text{Ap}(S, a_0) = \{s \in S \mid s - a_0 \notin S\}$ . By defining in [8] a monomial Apéry set  $\widetilde{\text{Ap}}(S, a_0)$  of  $a_0$ , that is an algebraic analogous to the Apéry set  $\text{Ap}(S, a_0)$  and using the order  $\prec_w$  as well as the map  $\varphi$ , we can change from studying the Apéry set to studying the set of monomials of  $[[R']]$  which are not in  $\text{in}(I)$ , where  $R' = K[x_0, \dots, x_n]$ .

Let  $a, d, k, h, c$  be positive integers. Recall that a *numerical almost generalized arithmetic sequence-semigroup* (numerical AAG-semigroup for short) is a semigroup minimally generated by relatively prime integers  $a, ha + d, ha + 2d, \dots, ha + kd, c$ , that is its embedding dimension is  $k + 2$ . Our goal is to describe all properties of an AAG-semigroup in terms of a continuous fraction, as an extension of my previous works in [5], [6]. In [8] was described a Gröbner basis of the ideal defining  $S$  under one condition, the complete case will be published in a forthcoming paper. In this paper we continue the work of [8] and we can describe the Pseudo Frobenius set (see Theorem 5.2), and so the Frobenius number and its type is at most  $2k$ . As a consequence we can give a complete description of AAG-semigroups that are symmetric or almost symmetric (see Theorems 6.1, 7.2, 7.3), in particular we prove that if  $S$  is almost symmetric its type is at most the embedding dimension minus 1. Another interesting point is that if  $S$  is almost symmetric then the Frobenius number is given by a quadratic formula in terms of  $a, d, k, h, c$  and  $t(S)$ . Moreover a simple algorithm using the solutions of some quadratic equations allow us to decide is an AAG-semigroup is almost symmetric. This result extends and generalizes all the results of [13].

The algorithms presented here are the extensions of the previous work by the first author in [5], [6] and can be downloaded in <http://www-fourier.univ-grenoble-alpes.fr/morales/>.

## 2. FROBENIUS NUMBER AND APÉRY SET

Denote by  $\mathbb{Z}$  and  $\mathbb{N}$  the set of integers and nonnegative integers respectively. Let  $S$  be a semigroup in  $\mathbb{N}$ . Given  $n \geq 1$  and  $a_0, \dots, a_n \in \mathbb{N}$  such that  $\gcd(a_0, \dots, a_n) = 1$ ,

$$S = \langle a_0, \dots, a_n \rangle = \{k_0 a_0 + \dots + k_n a_n \mid k_i \in \mathbb{N}\}.$$

The set  $\mathbb{N} \setminus S$  is finite. If  $S$  is minimally generated by  $\{a_0, \dots, a_n\}$   $S$  is called *numerical semigroup* and  $n + 1$  is called the *embedding dimension* of  $S$ .

**Definition 2.1.** Let  $S$  be a numerical semigroup generated by  $a_0, \dots, a_n$ .

- (i) The number  $F(S) = \max\{a \in \mathbb{Z} \mid a \notin S\}$  is called the *Frobenius number* of  $S$ .
- (ii) We also define

$$PF(S) = \{a \in \mathbb{Z} \setminus S \mid a + s \in S \text{ if } s \in S \text{ and } s \neq 0\}$$

and an element of  $PF(S)$  is called a *pseudo-Frobenius number* of  $S$ . Obviously, the Frobenius number is a pseudo-Frobenius number and the number of elements of  $PF(S)$  is called the *type* of  $S$ , denoted by  $t(S)$ .

- (iii) The *Apéry set* of  $a_0$  in  $S$  is the set

$$\text{Ap}(S, a_0) = \{s \in S \mid s - a_0 \notin S\}.$$

### 3. FROBENIUS NUMBER AND GRÖBNER BASIS

The definitions and results in this section follow from [8]. Let  $R = K[x_0, \dots, x_n]$  be the polynomial ring graded by the weights  $\deg x_0 = a_0, \dots, \deg x_n = a_n$ ,  $J \subset R$  a graded ideal and  $B = R/J$ . We say that  $R$  and  $B$  are *quasi-homogeneous* rings. Set  $R' = K[x_1, \dots, x_n]$  and denote by  $[[R']]$  the set of all monomials of  $R'$ . Let  $\varphi : [[R']] \rightarrow \mathbb{N}$  be the map defined by  $\varphi(M) = k_1 a_1 + \dots + k_n a_n$ , for every monomial  $M = x_1^{k_1} \dots x_n^{k_n} \in [[R']]$ .

We consider the weighted degree reverse lexicographical order  $\prec_w$  with  $x_0 \prec_w \dots \prec_w x_n$  and  $\deg x_i = a_i$  for all  $0 \leq i \leq n$ .

With the notations in the introduction, let  $\text{in}(I)$  be the initial ideal of the reduced Gröbner basis  $G(S)$  of  $I$  for the order  $\prec_w$ . Set  $R' = K[x_1, \dots, x_n]$  and denote by  $[[R']]$  the set of all monomials of  $R'$ . Now we consider two sets

$$\widetilde{\text{Ap}}(S, a_0) = \{M \in [[R']] \mid M \notin \text{in}(I)\}$$

and

$$\widetilde{PF}(S) = \{M \in \widetilde{\text{Ap}}(S, a_0) \mid \forall i \neq 0, \exists N_i \in [[R']], \alpha_i > 0 \text{ such that } Mx_i - x_0^{\alpha_i} N_i \in I\}.$$

**Corollary 3.1.** Assume that  $\text{gcd}(a_0, \dots, a_n) = 1$ . Then we have

(i) The restriction of  $\varphi$  to  $\widetilde{\text{Ap}}(S, a_0)$  is bijective and  $\varphi(\widetilde{\text{Ap}}(S, a_0)) = \text{Ap}(S, a_0)$ . In particular  $\text{card}(\widetilde{\text{Ap}}(S, a_0)) = a_0$  and  $F(S) = \max\{\varphi(M) \mid M \notin \text{in}(I)\} - a_0$ .

(ii) The restriction of  $\varphi$  to  $\widetilde{PF}(S)$  is bijective and  $\varphi(\widetilde{PF}(S)) = \widetilde{PF}(S) + a_0$ , i.e. each element  $\omega \in \widetilde{PF}(S)$  corresponds to exactly one monomial  $M_\omega \in \widetilde{PF}(S)$  such that  $\varphi(M_\omega) - a_0 = \omega$ .

(iii) Let  $s \in \text{Ap}(S, a_0)$ ,  $M \in \widetilde{\text{Ap}}(S, a_0)$  and  $N \in [[R']]$  such that  $s = \varphi(M) = \varphi(N)$ . Then  $M \prec_w N$ .

We denote by  $\widetilde{Frob}(S)$  the unique monomial in  $\widetilde{PF}(S)$  such that  $\varphi(\widetilde{Frob}(S)) = F(S) + a_0$ .

The following Lemma is very simple but very useful in order to prove that a set is a Gröbner basis of an ideal in many cases (see [8], [2]).

**Lemma 3.2.** *Let  $R = K[x_0, \dots, x_n]$ ,  $R' = K[x_s, \dots, x_n]$  be the rings with respect to the order  $\prec_w$  and  $[[R']]$  the set of all monomials of  $R'$ . Let  $I \subset R$  be an ideal such that the generators of  $\text{in}(I)$  belongs to  $R'$  and  $\text{rad}(\text{in}(I) \cap R') = (x_s, \dots, x_n)R'$ . Let  $G \subset I$  be a finite set and  $J$  the monomial ideal generated by the leading monomials of the elements in  $G$ . If  $\text{card}([R'] \setminus J) = \text{card}([R'] \setminus (\text{in}(I) \cap R'))$  then  $G$  is a Gröbner basis of  $I$ .*

#### 4. ALMOST GENERALIZED ARITHMETIC PROGRESSIONS, GROBNER BASIS

Let  $a, d, k, h, c$  be positive integers. Recall that a *numerical almost generalized arithmetic sequence-semigroup* (numerical AAG-semigroup for short) is a semigroup minimally generated by relatively prime integers  $a, ha + d, ha + 2d, \dots, ha + kd, c$ , that is its embedding dimension is  $k + 2$ . An interesting particular case is a semigroup minimally generated by relatively prime integers  $a, a + d, a + 2d, \dots, a + kd, c$ , called *numerical almost arithmetic-semigroup* (numerical AA-semigroup for short). AA-semigroups are the case  $h = 1$  of AAG-semigroups and where considered by D. P. Patil [9], [10] from the algebraic point of view and by J. L. Ramírez Alfonsín and O. J. Rodseth in [11], [12] from combinatorial point of view.

Let  $R = K[x_0, \dots, x_k, x_{k+1}]$  be the polynomial ring in  $k + 2$  variables over  $K$  graded by the weights  $\deg x_i = ha + id$  for  $i = 0, \dots, k$ ,  $\deg x_{k+1} = c$  and  $I$  the kernel of the homomorphism  $\Phi : R \rightarrow K[S]$  of  $K$ -algebras defined by  $\Phi(x_0) = t^a$ ,  $\Phi(x_i) = t^{ha+id}$  for all  $i = 1, \dots, k$  and  $\Phi(x_{k+1}) = t^c$ . Let  $R' = K[x_1, \dots, x_k]$ . The following result extends [10][Lemma 1.6.1].

**Lemma 4.1.** *For  $1 \leq i, j < k$ , set*

$$\mathcal{A} = \{x_i x_j - x_0^h x_{i+j} \mid \text{if } i + j \leq k, \} \cup \{x_i x_j - x_{i+j-k} x_k \mid \text{if } i + j > k\}.$$

*Then every binomial of  $\mathcal{A}$  belongs to  $I$  and  $\text{card}(\mathcal{A}) = \frac{(k-1)k}{2}$ .*

*Proof.* Let  $1 \leq i, j < k$ . The results are implied by the fact that  $\varphi(x_i x_j) = ha + id + ha + jd = ha + (i + j)d$  if  $i + j \leq k$  or  $\varphi(x_i x_j) = (ha + id) + (ha + jd) = (ha + (i + j - k)d) + (ha + kd)$  if  $i + j > k$ .  $\square$

**Corollary 4.2.** *Let consider any Gröbner basis with respect to an order  $\prec_w$  such that  $x_0 \prec_w x_1 \prec_w \dots \prec_w x_k \prec_w x_{k+1}$ . The initial ideal  $\text{in}(I)$  and  $\widetilde{\text{Ap}}(S, a_0)$  can be represented in the plane.*

*Proof.* By hypothesis and Lemma 4.1 we have only to consider only the monomials in  $\text{in}(I)$  which can be written as  $L_i x_k^\alpha x_{k+1}^\beta$  where  $0 \leq i < k$ ,  $L_0 = 1$  and  $L_i = x_i$  for  $i > 0$ . We associate to  $L_i x_k^\alpha x_{k+1}^\beta$  the point  $(i + \alpha k, \beta) \in \mathbb{N}^2$ .  $\square$

**Lemma 4.3.** *Let  $s \in \mathbb{N}$ ,  $p, r \in \mathbb{Z}$  such that*

$$ra = sd - pc.$$

*Let  $s = \sigma k + \rho$ , where  $0 \leq \rho < k$ . For convenience we can write  $s = \sigma k + l\rho$ , where  $l = 0$  if  $\rho = 0$  and  $l = 1$  if  $\rho > 0$ . It follows that  $ra = \sigma kd + l\rho d - pc$  so that*

$$(r + h(\sigma + l))a = \sigma(ha + kd) + l(ha + \rho d) - pc. \quad (*)$$

We set  $r' = r + h(\sigma + l)$ . We have

- $L_l x_k^\sigma - x_0^{r'} x_{k+1}^p \in I$  if  $p, r' \geq 0$ ,
- $L_l x_k^\sigma x_{k+1}^{-p} - x_0^{r'} \in I$  if  $p < 0, r' > 0$ ,
- $x_{k+1}^p - x_0^{-r'} L_l x_k^\sigma \in I$  if  $p \geq 0, r' < 0$ .

Our aim is to construct a Gröbner basis and describe the set  $\widetilde{\text{Ap}}(S, a_0)$ . By applying our algorithm for the case  $n = 3$  in Section 4.1 with numbers  $a, d, c$ , we get numbers  $s_i, p_i, q_i, r_i$  for  $0 \leq i \leq m + 1$  such that

$$ar_i = s_i d - p_i c.$$

Let  $s_0$  be the smallest natural number such that  $(s_0, 0, r_0)$  is solution of the equation  $sd - pc = ra$ . Set  $p_0 = 0$  and let  $p_1$  be the smallest natural number such that  $(s_1, p_1, r_1)$  is solution of the equation  $sd - pc = ra$ , where  $0 \leq s_1 < s_0$ . Note that

$$s_0 = \frac{a}{\gcd(a, d)} \text{ and } p_1 = \frac{\gcd(a, d)}{\gcd(a, d, c)}.$$

Now we want to define numbers  $s_i, p_i, r_i, q_i$  for  $i \geq 2$ . We will use the extended Euclid's algorithm for the computation of  $\gcd(a, b)$ . Namely, let consider the Euclid's algorithm with negative rest:

$$\left\{ \begin{array}{l} s_0 = q_2 s_1 - s_2 \\ s_1 = q_3 s_2 - s_3 \\ \dots = \dots \\ s_{m-1} = q_{m+1} s_m \\ s_{m+1} = 0 \end{array} \right.$$

where  $q_i \geq 2$ ,  $s_i \geq 0$  for all  $i = 2, \dots, m + 1$ . For  $i = 1, \dots, m$ , let define  $p_{i+1}, r_{i+1}$  by

$$p_{i+1} = p_i q_{i+1} - p_{i-1}, r_{i+1} = r_i q_{i+1} - r_{i-1}.$$

It is proved in [6] that for  $i = 0, \dots, m$ ,

$$s_i p_{i+1} - s_{i+1} p_i = s_0 p_1 = \frac{a}{\gcd(a, d, c)},$$

and the sequences  $s_i, r_i$  are decreasing, while the sequence  $p_i$  is increasing. see [5] and [6].

Let  $s_i = \sigma_i k + \rho_i$ , where  $0 \leq \rho_i < k$ . We set  $l_i = 0$  if  $\rho_i = 0$ ,  $l_i = 1$  if  $\rho_i > 0$  and  $r'_i = r_i + h(\sigma_i + l_i)$ . Since  $s_i > s_{i+1}$  we have  $\sigma_i \geq \sigma_{i+1}$ , if  $\sigma_i = \sigma_{i+1}$  then  $l_i \geq l_{i+1}$ , if  $\sigma_i > \sigma_{i+1}$  then  $|l_i - l_{i+1}| \leq 1$ . In both cases we have  $\sigma_i + l_i \geq \sigma_{i+1} + l_{i+1}$ , which implies  $r'_i > r'_{i+1}$ . Let  $\mu$  be the unique integer such that  $r'_\mu > 0 \geq r'_{\mu+1}$ . In our next results we suppose that either  $r'_\mu \geq h$  or  $\rho_\mu = 0$ . Note that by the definition of  $\mu$   $r'_\mu \geq h$  is true when  $h = 1$ . We give some results from [8] without proofs.

**Definition 4.4.** *With the above notations. If  $r'_\mu \geq h$  or  $\rho_\mu = 0$  we set:*

(1) *If  $\rho_\mu = 0$  we set*

$$\mathcal{B} = \{x_k^{\sigma_\mu} - x_0^{r'_\mu} x_{k+1}^{p_\mu}\}.$$

If  $\rho_\mu \neq 0$  we set

$$\mathcal{B} = \{x_{\rho_\mu} x_k^{\sigma_\mu} - x_0^{r'_\mu} x_{k+1}^{p_\mu}, x_{\rho_\mu+j} x_k^{\sigma_\mu} - x_0^{r'_\mu-h} x_j x_{k+1}^{p_\mu} \mid 1 \leq j \leq k - \rho_\mu\}.$$

Note that since the embedding dimension of the semigroup  $S$  is  $k+2$  we have  $s_\mu > k$ .

- (2) Suppose  $s_{\mu+1} \neq 0$ . Let  $s_\mu - s_{\mu+1} = \tilde{\sigma}k + \tilde{\rho}$ , with  $0 \leq \tilde{\rho} < k$ . Set  $\tilde{l} = 0$  if  $\tilde{\rho} = 0$ ,  $\tilde{l} = 1$  if  $\tilde{\rho} > 0$  and  $\tilde{r} = r_\mu - r_{\mu+1} + h(\tilde{\sigma} + \tilde{l})$ . If  $\tilde{\rho} > 0$  then set

$$\mathcal{C} = \{x_{\tilde{\rho}} x_k^{\tilde{\sigma}} x_{k+1}^{p_{\mu+1}-p_\mu} - x_0^{\tilde{r}}, x_{j+\tilde{\rho}} x_k^{\tilde{\sigma}} x_{k+1}^{p_{\mu+1}-p_\mu} - x_0^{\tilde{r}-1} x_j \mid 1 \leq j \leq k - \tilde{\rho}\}$$

and if  $\tilde{\rho} = 0$  then set  $\mathcal{C} = \{x_k^{\tilde{\sigma}} x_{k+1}^{p_{\mu+1}-p_\mu} - x_0^{\tilde{r}}\}$ . Suppose  $s_{\mu+1} = 0$ . We set  $\mathcal{C} = \emptyset$ .

- (3)  $\mathcal{D} := \{x_{k+1}^{p_{\mu+1}} - x_0^{-r'_{\mu+1}} x_{\rho_{\mu+1}}^{l_{\mu+1}} x_k^{\sigma_{\mu+1}}\}$ . By our assumptions the embedding dimension of the semigroup  $S$  is  $k+2$  so  $p_{\mu+1} > 1$ , that is  $\mu > 0$ , and if  $r'_{\mu+1} = 0$  we have  $s_{\mu+1} > k$ .

Now in order to find the set  $\widetilde{\text{Ap}}(S, a)$  we need to define two 2 rectangles

$$A = \{(y, z) \in \mathbb{N}^2 \mid 0 \leq y < s_\mu - s_{\mu+1}, 0 \leq z < p_{\mu+1}\}$$

$$B = \{(y, z) \in \mathbb{N}^2 \mid s_\mu - s_{\mu+1} \leq y < s_\mu, 0 \leq z < p_{\mu+1} - p_\mu\}.$$

Note that if  $s_{\mu+1} = 0$  then  $B = \emptyset$ . For  $0 \leq i < k$ , set  $L_i = 1$  if  $i = 0$  and  $L_i = x_i$  if  $i > 0$ . To any point  $(y, z) \in \mathbb{N}^2$  we associate the monomial  $M(y, z) := L_i x_k^\alpha x_{k+1}^z$ , where  $\alpha = \lfloor \frac{y}{k} \rfloor$  and  $i = y - k\alpha$ . Conversely, any monomial  $L_i x_k^\alpha x_{k+1}^z \in [[R']]$  can be represented by the point  $(y, z) \in \mathbb{N}^2$ , where  $y = \alpha k + i$ .

The next theorem allows to compute effectively a system of generators of the ideal semigroup  $I$ , it precises and extends the main theorem of [10] where the case  $h = 1$  is considered.

**Theorem 4.5.** *With the above notations, suppose that either  $r'_\mu \geq h$  or  $\rho_\mu = 0$ .*

(i) *We have*

$$\widetilde{\text{Ap}}(S, a) = \left\{ L_i x_k^\alpha x_{k+1}^z \mid (y, z) \in A \cup B, \alpha = \lfloor \frac{y}{k} \rfloor, i = y - k\alpha \right\}.$$

(ii) *If  $s_{\mu+1} \neq 0$  then  $\mathcal{G} := \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$  is a Gröbner basis of  $I$ .*

(iii) *If  $s_{\mu+1} = 0$  then  $\mathcal{G} := \mathcal{A} \cup \mathcal{B} \cup \mathcal{D}$  is a Gröbner basis of  $I$ .*

The following result extends the main result of J. L. Ramírez Alfonsín and O. J. Rodseth in [11], [12], which is the case when  $h = 1$ .

**Corollary 4.6.** *With the above notations, suppose that either  $r'_\mu \geq h$  or  $\rho_\mu = 0$ . We have*

$$\text{Ap}(S, a) = \left\{ ha \left\lfloor \frac{y}{k} \right\rfloor + dy + cz \mid (y, z) \in A \cup B \right\}.$$

## 5. ALMOST GENERALIZED ARITHMETIC PROGRESSIONS, PSEUDO FROBENIUS SET

The case  $S$  symmetric was studied in [11] and the case  $S$  is pseudo symmetric was studied in [3] with the hypothesis that  $h = 1$ . Both publications are corollaries of this section. In our work we will describe the Pseudo Frobenius set and characterize when  $S$  is almost symmetric in general for  $h \geq 1$ . In this paper we restrict to the hypothesis  $r'_\mu \geq h$  or  $\rho_\mu = 0$ . Note that  $r'_\mu \geq h$  is satisfied if  $h = 1$ .

**Lemma 5.1.** *With the above notations, suppose that either  $r'_\mu \geq h$  or  $\rho_\mu = 0$ . Let  $\widetilde{PF}(S, a)_1$  be the set of monomials in  $\widetilde{PF}(S, a)$  such that the power of  $x_{k+1}$  is  $p_{\mu+1} - 1$  and let  $\widetilde{PF}(S, a)_2$  be the set of monomials in  $\widetilde{PF}(S, a)$  such that the power of  $x_{k+1}$  is  $p_{\mu+1} - p_\mu - 1$ . We have*

$$\widetilde{PF}(S, a) = \widetilde{PF}(S, a)_1 \cup \widetilde{PF}(S, a)_2.$$

In particular  $1 \leq t(S) \leq 2k$ .

*Proof.* We have

$$\widetilde{\text{Ap}}(S, a) = \{L_i x_k^\alpha x_{k+1}^z \mid i + \alpha k < s_\mu, z < p_{\mu+1} \text{ and } (i + \alpha k < s_\mu - s_{\mu+1} \text{ or } z < p_{\mu+1} - p_\mu)\},$$

Let  $M$  be a monomial in  $\widetilde{\text{Ap}}(S, a)$ , recall that  $M$  belongs to  $\widetilde{PF}(S, a)$  if and only if for all  $i = 1, \dots, k + 1$  we have that  $x_i M - x_0^\alpha N_i \in I$  for some monomial  $N_i$  and some  $\alpha \in \mathbb{N}^*$ . We order the monomials in  $\widetilde{\text{Ap}}(S, a)$  by saying that  $M \ll N$  if  $N = x_i M$  for some  $i = 1, \dots, k + 1$ . So  $\widetilde{PF}(S, a)$  is included in the set of maximal elements of  $\widetilde{\text{Ap}}(S, a)$  for this order. Note that for a maximal monomial of  $\widetilde{\text{Ap}}(S, a)$  we have that the power of  $x_{k+1}$  is either  $p_{\mu+1} - p_\mu - 1$  or  $p_{\mu+1} - 1$ . Let  $\widetilde{PF}(S, a)_1$  be the set of monomials in  $\widetilde{PF}(S, a)$  such that the power of  $x_{k+1}$  is  $p_{\mu+1} - 1$  and let  $\widetilde{PF}(S, a)_2$  be the set of monomials in  $\widetilde{PF}(S, a)$  such that the power of  $x_{k+1}$  is  $p_{\mu+1} - p_\mu - 1$ . We have  $\text{card}(\widetilde{PF}(S, a)_1), \text{card}(\widetilde{PF}(S, a)_2) \leq k$  so  $t(S) \leq 2k$ .  $\square$

Note that if  $M$  belongs to  $\widetilde{PF}(S, a)$  then for all  $i = 1, \dots, k + 1$   $x_i M \in \text{in}(I)$ , so if  $x_i M \in \widetilde{\text{Ap}}(S, a)$  for some  $i = 1, \dots, k + 1$  then certainly  $M \notin \widetilde{PF}(S, a)$ . Note also that  $s_\mu - s_{\mu+1} = (\sigma_\mu - \sigma_{\mu+1})k + \rho_\mu - \rho_{\mu+1}$ , so if  $\rho_{\mu+1} \leq \rho_\mu$  then we have  $\tilde{\sigma} = \sigma_\mu - \sigma_{\mu+1}$ ,  $\tilde{\rho} = \rho_\mu - \rho_{\mu+1}$  and if  $\rho_{\mu+1} > \rho_\mu$  then we have  $\tilde{\sigma} = \sigma_\mu - \sigma_{\mu+1} - 1$ ,  $\tilde{\rho} = k + \rho_\mu - \rho_{\mu+1}$ .

**Theorem 5.2.** *With the above notations, suppose that either  $r'_\mu \geq h$  or  $\rho_\mu = 0$ . We have*

(1) If  $r'_{\mu+1} = 0$

(a) If  $\rho_{\mu+1} = 0$  then  $\widetilde{PF}(S, a)_1 = \emptyset$ .

(b) If  $\rho_{\mu+1} > 0, \tilde{\rho} = 0$  then

$$\widetilde{PF}(S, a)_1 = \{x_i x_k^{\tilde{\sigma}-1} x_{k+1}^{p_{\mu+1}-1}, i = 1, \dots, k - \rho_{\mu+1}\}.$$



(c) If  $\rho_{\mu+1} > 0, \tilde{\rho} = 1, \tilde{\sigma} = 0$  then  $\widetilde{PF}(S, a)_1 = \emptyset$ .

(d) If  $\rho_{\mu+1} > 0, \tilde{\rho} = 1, \tilde{\sigma} > 0$  then

$$\widetilde{PF}(S, a)_1 = \{x_i x_k^{\tilde{\sigma}-1} x_{k+1}^{p_{\mu+1}-1}, i = 1, \dots, k - \rho_{\mu+1}\}.$$

(e) If  $\rho_{\mu+1} > 0, \tilde{\rho} > 1$  then

$$\widetilde{PF}(S, a)_1 = \{x_i x_k^{\tilde{\sigma}} x_{k+1}^{p_{\mu+1}-1}, i = 1, \dots, \min\{\tilde{\rho} - 1, k - \rho_{\mu+1}\}\}.$$

(2) If  $r'_{\mu+1} < 0$

(a) If  $\tilde{\rho} = 0$  then

$$\widetilde{PF}(S, a)_1 = \{x_i x_k^{\tilde{\sigma}-1} x_{k+1}^{p_{\mu+1}-1}, i = 1, \dots, k - 1\}.$$

(b) If  $\tilde{\rho} = 1, \tilde{\sigma} = 0$  then

$$\widetilde{PF}(S, a)_1 = \{x_{k+1}^{p_{\mu+1}-1}\}.$$

(c) If  $\tilde{\rho} = 1, \tilde{\sigma} > 0$  then

$$\widetilde{PF}(S, a)_1 = \{x_i x_k^{\tilde{\sigma}-1} x_{k+1}^{p_{\mu+1}-1}, i = 1, \dots, k\}.$$

(d) If  $\tilde{\rho} > 1$  then

$$\widetilde{PF}(S, a)_1 = \{x_i x_k^{\tilde{\sigma}} x_{k+1}^{p_{\mu+1}-1}, i = 1, \dots, \tilde{\rho} - 1\}.$$

(3) If  $s_{\mu+1} = 0$  then  $\widetilde{PF}(S, a)_2 = \emptyset$ .

(4)  $\rho_{\mu} = 0$

(i) If  $s_{\mu+1} \geq k - 1$  then

$$\widetilde{PF}(S, a)_2 = \{x_i x_k^{\sigma_{\mu}-1} x_{k+1}^{p_{\mu+1}-p_{\mu}-1}, i = 1, \dots, k - 1\}.$$

(ii) If  $s_{\mu+1} < k - 1$  then

$$\widetilde{PF}(S, a)_2 = \{x_i x_k^{\sigma_{\mu}-1} x_{k+1}^{p_{\mu+1}-p_{\mu}-1}, i = \tilde{\rho}, \dots, k - 1\}.$$

(5)  $\rho_{\mu} = 1, r'_{\mu} > h$

(i) If  $s_{\mu+1} \geq k$  then

$$\widetilde{PF}(S, a)_2 = \{x_i x_k^{\sigma_{\mu}-1} x_{k+1}^{p_{\mu+1}-p_{\mu}-1}, i = 1, \dots, k\}.$$

(ii) If  $1 < s_{\mu+1} < k$  then

$$\widetilde{PF}(S, a)_2 = \{x_i x_k^{\sigma_{\mu}-1} x_{k+1}^{p_{\mu+1}-p_{\mu}-1}, i = \tilde{\rho}, \dots, k\}.$$

(iii) If  $s_{\mu+1} = 1$  then

$$\widetilde{PF}(S, a)_2 = \{x_k^{\sigma_{\mu}} x_{k+1}^{p_{\mu+1}-p_{\mu}-1}\}.$$

(6)  $\rho_{\mu} = 1, r'_{\mu} = h$

(i) If  $s_{\mu} - s_{\mu+1} = 1$  then

$$\widetilde{PF}(S, a)_2 = \{x_i x_k^{\sigma_{\mu}-1} x_{k+1}^{p_{\mu+1}-p_{\mu}-1}, i = 1, \dots, k\}.$$

(ii) If  $1 < s_\mu - s_{\mu+1} \leq s_\mu - k$  then

$$PF(S, a)_2 = \{x_1 x_k^{\sigma_\mu - 1} x_{k+1}^{p_{\mu+1} - p_\mu - 1}\}.$$

(iii) If  $s_\mu - k < s_\mu - s_{\mu+1}$  then  $PF(S, a)_2 = \emptyset$ .

(7)  $\rho_\mu > 1$

(i) If  $s_{\mu+1} \geq \rho_\mu - 1$  then

$$PF(S, a)_2 = \{x_i x_k^{\sigma_\mu} x_{k+1}^{p_{\mu+1} - p_\mu - 1}, i = 1, \dots, \rho_\mu - 1\}.$$

(ii) If  $s_{\mu+1} < \rho_\mu - 1$  then  $\tilde{\rho} = \rho_\mu - s_{\mu+1}$  and

$$PF(S, a)_2 = \{x_i x_k^{\sigma_\mu} x_{k+1}^{p_{\mu+1} - p_\mu - 1}, i = \tilde{\rho}, \dots, \rho_\mu - 1\}.$$

Before going to the proof let remark:

**Remark 5.3.** a) Let  $i, j = 1, \dots, k$  and  $M$  a monomial, if  $i + j \leq k$  then we have  $x_i x_j - x_0^h x_{i+j} \in I$  so  $x_j x_i M - x_0^h x_{i+j} M \in I$ .

b) A monomial  $x_i M$  for some  $i = 1, \dots, k$  belongs to  $PF(S, a)$  if and only if for any  $j = 1, \dots, k + 1$  there is a binomial  $x_j x_i M - x_0^\alpha N$  for some monomial  $N$  and  $\alpha \in \mathbb{N}^*$ . So in order to check if  $x_i M$  belongs to  $PF(S, a)$  we need only to consider  $j$  such that  $i + j > k$ .

c) Let note that the elements in  $PF(S, a)_i$  are ordered by increasing order of evaluation by  $\varphi$ .

*Proof.* We have to consider all possible cases:

(1) Study of  $PF(S, a)_1$  when  $r'_{\mu+1} = 0$ ,

a) Suppose  $\rho_{\mu+1} = 0$ . Let  $M_i = x_i x_k^\alpha x_{k+1}^{p_{\mu+1} - 1} \in PF(S, a)_1$  for some  $1 \leq i \leq k$ , such that  $i + k\alpha < s_\mu - s_{\mu+1}$ . Since  $x_{k+1}^{p_{\mu+1}} - x_k^{\sigma_{\mu+1}} \in I$  we have  $x_{k+1} M_i - x_i x_k^\alpha x_k^{\sigma_{\mu+1}} \in I$ , but  $i + k\alpha + k\sigma_{\mu+1} < s_\mu - s_{\mu+1} + s_{\mu+1} = s_\mu$  so  $x_i x_k^\alpha x_k^{\sigma_{\mu+1}} \in \text{Ap}(S, a)$  showing that  $M_i \notin PF(S, a)$  a contradiction, that is  $PF(S, a)_1 = \emptyset$ .

b) Suppose  $\rho_{\mu+1} > 0, \tilde{\rho} = 0$ . We have

$$PF(S, a)_1 \subset \{x_i x_k^{\tilde{\sigma} - 1} x_{k+1}^{p_{\mu+1} - 1}, i = 1, \dots, k - 1\}$$

and  $s_\mu = (\tilde{\sigma} + \sigma_{\mu+1})k + \rho_{\mu+1}$ . Let  $i = 1, \dots, k - 1, j = 1, \dots, k$  and  $M = x_k^{\tilde{\sigma} - 1} x_{k+1}^{p_{\mu+1} - 1}$  such that  $i + j > k$ . We have  $x_i x_j - x_{i+j-k} x_k \in I$  so  $x_j x_i M - x_{i+j-k} x_k^{\tilde{\sigma}} x_{k+1}^{p_{\mu+1} - 1} \in I$ , since  $x_k^{\tilde{\sigma}} x_{k+1}^{p_{\mu+1} - p_\mu} - x_0^{\tilde{r}} \in I$  we have  $x_j x_i M - x_0^{\tilde{r}} x_{i+j-k} x_{k+1}^{p_\mu - 1} \in I$  where  $\tilde{r} > 0$ .

From  $x_{k+1}^{p_{\mu+1}} - x_{\rho_{\mu+1}} x_k^{\sigma_{\mu+1}} \in I$ , we get  $x_{k+1} x_i M - x_i x_{\rho_{\mu+1}} x_k^{\tilde{\sigma} - 1} x_k^{\sigma_{\mu+1}} \in I$ . If  $i + \rho_{\mu+1} \leq k$  then  $x_i x_{\rho_{\mu+1}} - x_0^h x_{1+\rho_{\mu+1}}$ , so  $x_i M \in PF(S, a)$ . If  $i + \rho_{\mu+1} > k$  then  $x_{k+1} x_i M - x_{i+\rho_{\mu+1}-k} x_k^{\tilde{\sigma} + \sigma_{\mu+1}} \in I$  but  $i + \rho_{\mu+1} - k + (\tilde{\sigma} + \sigma_{\mu+1})k =$

$s_\mu + i - k < s_\mu$  which implies that  $x_{i+\rho_{\mu+1}-k}x_k^{\tilde{\sigma}+\sigma_{\mu+1}} \in \widetilde{\text{Ap}}(S, a)$  so  $x_iM \notin PF(S, a)$ . Hence

$$\widetilde{PF}(S, a)_1 = \{x_i x_k^{\tilde{\sigma}-1} x_{k+1}^{p_{\mu+1}-1}, i = 1, \dots, k - \rho_{\mu+1}\}.$$

c) Suppose  $\rho_{\mu+1} > 0, \tilde{\rho} = 1, \tilde{\sigma} = 0$ . We have

$$\widetilde{PF}(S, a)_1 \subset \{x_{k+1}^{p_{\mu+1}-1}\}.$$

The binomial  $x_{k+1}(x_{k+1}^{p_{\mu+1}-1}) - x_{\rho_{\mu+1}}x_k^{\sigma_{\mu+1}} \in I$  but  $x_{\rho_{\mu+1}}x_k^{\sigma_{\mu+1}} \in \widetilde{\text{Ap}}(S, a)$  so  $\widetilde{PF}(S, a)_1 = \emptyset$ .

d) Suppose  $\rho_{\mu+1} > 0, \tilde{\rho} = 1, \tilde{\sigma} > 0$ . We have

$$\widetilde{PF}(S, a)_1 \subset \{x_i x_k^{\tilde{\sigma}-1} x_{k+1}^{p_{\mu+1}-1}, i = 1, \dots, k\}.$$

Let  $i = 1, \dots, k, j = 1, \dots, k, M = x_k^{\tilde{\sigma}-1} x_{k+1}^{p_{\mu+1}-1}$ . By the same arguments as in the above item b) we have  $x_j x_i M - x_0^\alpha N_i \in I$  for some monomial  $N_i$  and  $\alpha \in \mathbb{N}^*$ . The binomial  $x_{k+1}^{p_{\mu+1}} - x_{\rho_{\mu+1}}x_k^{\sigma_{\mu+1}} \in I$  so  $x_{k+1}x_iM - x_i x_{\rho_{\mu+1}}x_k^{\tilde{\sigma}-1}x_k^{\sigma_{\mu+1}} \in I$ . If  $i + \rho_{\mu+1} \leq k$  we have  $x_i x_{\rho_{\mu+1}} - x_0^h x_{1+\rho_{\mu+1}}$ , so  $x_i M \in PF(S, a)$ . If  $i + \rho_{\mu+1} > k$  we have  $x_{k+1}x_iM - x_{i+\rho_{\mu+1}-k}x_k^{\tilde{\sigma}+\sigma_{\mu+1}} \in I$ . But  $s_\mu = (\tilde{\sigma} + \sigma_{\mu+1})k + \rho_{\mu+1} + 1$  so  $i + \rho_{\mu+1} - k + (\tilde{\sigma} + \sigma_{\mu+1})k = s_\mu + i - 1 - k < s_\mu$  and  $x_{i+\rho_{\mu+1}-k}x_k^{\tilde{\sigma}+\sigma_{\mu+1}} \in \widetilde{\text{Ap}}(S, a)$  so  $x_iM \notin PF(S, a)$ . Hence

$$\widetilde{PF}(S, a)_1 = \{x_i x_k^{\tilde{\sigma}-1} x_{k+1}^{p_{\mu+1}-1}, i = 1, \dots, k - \rho_{\mu+1}\}.$$

e) Suppose  $\rho_{\mu+1} > 0, \tilde{\rho} > 1$ . We have

$$\widetilde{PF}(S, a)_1 \subset \{x_i x_k^{\tilde{\sigma}} x_{k+1}^{p_{\mu+1}-1}, i = 1, \dots, \tilde{\rho} - 1\}.$$

Let  $i = 1, \dots, \tilde{\rho} - 1, j = 1, \dots, k, M = x_k^{\tilde{\sigma}} x_{k+1}^{p_{\mu+1}-1}$ . By the same arguments as in the above item b) we have  $x_j x_i M - x_0^\alpha N_i \in I$  for some monomial  $N_i$  and  $\alpha \in \mathbb{N}^*$ . Since  $x_{k+1}^{p_{\mu+1}} - x_{\rho_{\mu+1}}x_k^{\sigma_{\mu+1}} \in I$ , we have  $x_{k+1}x_iM - x_i x_{\rho_{\mu+1}}x_k^{\tilde{\sigma}}x_k^{\sigma_{\mu+1}} \in I$ . If  $i + \rho_{\mu+1} \leq k$  we have  $x_{k+1}x_iM - x_0^h x_{i+\rho_{\mu+1}}x_k^{\tilde{\sigma}}x_k^{\sigma_{\mu+1}} \in I$ , so  $x_iM \in PF(S, a)$ . If  $i + \rho_{\mu+1} > k$  we have  $x_{k+1}x_iM - x_{i+\rho_{\mu+1}-k}x_k^{\tilde{\sigma}+\sigma_{\mu+1}+1} \in I$ ,  $s_\mu = (\tilde{\sigma} + \sigma_{\mu+1})k + \tilde{\rho} + \rho_{\mu+1}$  so  $i + \rho_{\mu+1} - k + (\tilde{\sigma} + \sigma_{\mu+1})k + k < s_\mu$ , which implies that  $x_{i+\rho_{\mu+1}-k}x_k^{\tilde{\sigma}+\sigma_{\mu+1}+1} \in \widetilde{\text{Ap}}(S, a)$  so  $x_iM \notin PF(S, a)$ . Hence

$$\widetilde{PF}(S, a)_1 = \{x_i x_k^{\tilde{\sigma}} x_{k+1}^{p_{\mu+1}-1}, i = 1, \dots, \min\{\tilde{\rho} - 1, k - \rho_{\mu+1}\}\}.$$

(2) Study of  $\widetilde{PF}(S, a)_1$  when  $r'_{\mu+1} < 0$ ,

a) Suppose  $\tilde{\rho} = 0$  and  $r'_{\mu+1} < 0$ . We have

$$\widetilde{PF}(S, a)_1 \subset \{x_i x_k^{\tilde{\sigma}-1} x_{k+1}^{p_{\mu+1}-1}, i = 1, \dots, k - 1\}.$$

Let  $i = 1, \dots, k-1, j = 1, \dots, k, M = x_k^{\tilde{\sigma}-1} x_{k+1}^{p_{\mu+1}-1}$ . By the same arguments as in the above item 1)b) we have  $x_j x_i M - x_0^\alpha N_i \in I$  for some monomial  $N_i$  and  $\alpha \in \mathbb{N}^*$ . Since  $x_{k+1}^{p_{\mu+1}} - x_0^{-r'_{\mu+1}} x_{\rho_{\mu+1}} x_k^{\sigma_{\mu+1}} \in I$ , we have  $x_{k+1} x_i M - x_0^{-r'_{\mu+1}} x_i x_k^{\tilde{\sigma}-1} \in I$  so  $x_i M \in \widetilde{PF(S, a)}$ . Hence

$$\widetilde{PF(S, a)}_1 = \{x_i x_k^{\tilde{\sigma}-1} x_{k+1}^{p_{\mu+1}-1}, i = 1, \dots, k-1\}.$$

b) Suppose  $\tilde{\rho} = 1, \tilde{\sigma} = 0$ . We have

$$\widetilde{PF(S, a)}_1 \subset \{x_{k+1}^{p_{\mu+1}-1}\}.$$

c) Suppose  $\tilde{\rho} = 1, \tilde{\sigma} > 0$ . We have

$$\widetilde{PF(S, a)}_1 \subset \{x_i x_k^{\tilde{\sigma}-1} x_{k+1}^{p_{\mu+1}-1}, i = 1, \dots, k\}.$$

d) Suppose  $\tilde{\rho} > 1$ . We have

$$\widetilde{PF(S, a)}_1 \subset \{x_i x_k^{\tilde{\sigma}} x_{k+1}^{p_{\mu+1}-1}, i = 1, \dots, \tilde{\rho}-1\}.$$

By the same arguments as in the above item 2a) we have the equality in the items 2b), 2c), 2d).

(3) Study of  $\widetilde{PF(S, a)}_2$  when  $\rho_\mu = 0$ . We have  $x_k^{\sigma_\mu} - x_0^{r'_\mu} x_{k+1}^{p_\mu}$ ,

$$\widetilde{PF(S, a)}_2 \subset \{x_i x_k^{\sigma_\mu-1} x_{k+1}^{p_{\mu+1}-p_\mu-1}, i = 1, \dots, k-1\}.$$

Let  $i = 1, \dots, k-1, j = 1, \dots, k, M = x_k^{\sigma_\mu-1} x_{k+1}^{p_{\mu+1}-p_\mu-1}$ . If  $i+j > k$  we have  $x_i x_j - x_{i+j-k} x_k$  then  $x_j x_i M - x_0^{r'_\mu} x_{i+j-k} x_{k+1}^{p_{\mu+1}-1} \in I$ .

We have  $x_{k+1} x_i M = x_i x_k^{\sigma_\mu-1} x_{k+1}^{p_{\mu+1}-p_\mu}$  so by using the set  $\mathcal{C}$  we get  $x_{k+1} x_i M - x_0^\alpha N \in I$  for some  $\alpha \in \mathbb{N}^*$  if and only if  $i + (\sigma_\mu - 1) \geq s_\mu - s_{\mu+1}$  that is  $i \geq k - s_{\mu+1}$ .

(i) If  $s_{\mu+1} \geq k-1$  we have  $i \geq k - s_{\mu+1}$  for all  $i = 1, \dots, k-1$ . Hence

$$\widetilde{PF(S, a)}_2 = \{x_i x_k^{\sigma_\mu-1} x_{k+1}^{p_{\mu+1}-p_\mu-1}, i = 1, \dots, k-1\}.$$

(ii) If  $s_{\mu+1} < k-1$  We have  $s_\mu - s_{\mu+1} = (\sigma_\mu - 1)k + (k - s_{\mu+1})$  so  $\tilde{\rho} = k - s_{\mu+1}$   
Hence

$$\widetilde{PF(S, a)}_2 = \{x_i x_k^{\sigma_\mu-1} x_{k+1}^{p_{\mu+1}-p_\mu-1}, i = \tilde{\rho}, \dots, k-1\}.$$

(4) If  $s_{\mu+1} = 0$  the set  $\widetilde{Ap(S, a)}$  is represented by a rectangle, there is no element in  $\widetilde{PF(S, a)}$  with power of  $x_{k+1}$  equal to  $p_{\mu+1} - p_\mu - 1$  so  $\widetilde{PF(S, a)}_2 = \emptyset$ .

(5) Study of  $\widetilde{PF(S, a)}_2$  when  $\rho_\mu = 1, r'_\mu > h$ . We have  $x_1 x_k^{\sigma_\mu} - x_0^{r'_\mu} x_{k+1}^{p_\mu}, x_l x_k^{\sigma_\mu} - x_0^{r'_\mu-h} x_{l-1} x_{k+1}^{p_\mu} \mid l = 2, \dots, k$ ,

$$\widetilde{PF(S, a)}_2 \subset \{x_i x_k^{\sigma_\mu-1} x_{k+1}^{p_{\mu+1}-p_\mu-1}, i = 1, \dots, k\}.$$

Let  $i = 1, \dots, k, j = 1, \dots, k, M = x_k^{\sigma_\mu - 1} x_{k+1}^{p_{\mu+1} - p_\mu - 1}$ . If  $i + j > k$  then we have  $x_i x_j - x_{i+j-k} x_k$ . If  $i + j = k + 1$  then  $x_j x_i M - x_0^{r'_\mu} x_{k+1}^{p_{\mu+1} - 1} \in I$ . If  $i + j > k + 1$  then  $x_j x_i M - x_0^{r'_\mu - h} x_{i+j-k-1} x_{k+1}^{p_{\mu+1} - 1} \in I$ . So  $x_j x_i M - x_0^{\alpha_{i,j}} N \in I$  for some  $\alpha_{i,j} \in \mathbb{N}^*$ .

We have  $x_{k+1} x_i M = x_i x_k^{\sigma_\mu - 1} x_{k+1}^{p_{\mu+1} - p_\mu}$  so by using the set  $\mathcal{C}$  we get  $x_{k+1} x_i M - x_0^\alpha N \in I$  for some  $\alpha \in \mathbb{N}^*$  if and only if  $i + (\sigma_\mu - 1)k \geq s_\mu - s_{\mu+1}$ , that is  $i + s_\mu - 1 - k \geq s_\mu - s_{\mu+1}$  or  $i \geq k + 1 - s_{\mu+1}$ . We have to consider several cases.

(i)  $s_{\mu+1} \geq k$ , the condition  $i \geq k + 1 - s_{\mu+1}$  is satisfied for  $i = 1, \dots, k$ . Hence

$$\widetilde{PF}(S, a)_2 = \{x_i x_k^{\sigma_\mu - 1} x_{k+1}^{p_{\mu+1} - p_\mu - 1}, i = 1, \dots, k\}.$$

(ii)  $1 < s_{\mu+1} < k$ . We have  $s_\mu - s_{\mu+1} = (\sigma_\mu - 1)k + (k + 1 - s_{\mu+1})$  so  $k + 1 - s_{\mu+1} = \tilde{\rho}$ . Hence

$$\widetilde{PF}(S, a)_2 = \{x_i x_k^{\sigma_\mu - 1} x_{k+1}^{p_{\mu+1} - p_\mu - 1}, i = \tilde{\rho}, \dots, k\}.$$

(iii)  $s_{\mu+1} = 1$ . We have  $i + (\sigma_\mu - 1)k \geq s_\mu - s_{\mu+1} = \sigma_\mu k$  if and only if  $i = k$ . Hence

$$\widetilde{PF}(S, a)_2 = \{x_k^{\sigma_\mu} x_{k+1}^{p_{\mu+1} - p_\mu - 1}\}.$$

(6) Study of  $\widetilde{PF}(S, a)_2$  when  $\rho_\mu = 1, r'_\mu = h$ . We have  $x_1 x_k^{\sigma_\mu} - x_0^h x_{k+1}^{p_\mu}, x_i x_k^{\sigma_\mu} - x_{i-1} x_{k+1}^{p_\mu}$  for  $i = 2, \dots, k$  and

$$\widetilde{PF}(S, a)_2 \subset \{x_i x_k^{\sigma_\mu - 1} x_{k+1}^{p_{\mu+1} - p_\mu - 1}, i = 1, \dots, k\}.$$

Let  $i = 1, \dots, k, j = 1, \dots, k, M = x_k^{\sigma_\mu - 1} x_{k+1}^{p_{\mu+1} - p_\mu - 1}$ . If  $i + j > k$  then we have  $x_i x_j - x_{i+j-k} x_k$ . If  $i + j = k + 1$  then  $x_j x_i M - x_0^h x_{k+1}^{p_{\mu+1} - 1} \in I$ . If  $i + j > k + 1$  then  $x_j x_i M - x_{i+j-k-1} x_{k+1}^{p_{\mu+1} - 1} \in I$ .

(i) If  $s_\mu - s_{\mu+1} = 1$  so by using the set  $\mathcal{C}$  we get  $x_j x_i M - x_0^{\alpha_{i,j}} N \in I$  for some  $\alpha_{i,j} \in \mathbb{N}^*$ .

On the other side we have  $x_{k+1} x_i M = x_i x_k^{\sigma_\mu - 1} x_{k+1}^{p_{\mu+1} - p_\mu}$ , so by using the set  $\mathcal{C}$  we get  $x_i x_k^{\sigma_\mu - 1} x_{k+1}^{p_{\mu+1} - p_\mu} - x_0^\alpha N \in I$  for some  $\alpha \in \mathbb{N}^*$ . Hence

$$\widetilde{PF}(S, a)_2 = \{x_i x_k^{\sigma_\mu - 1} x_{k+1}^{p_{\mu+1} - p_\mu - 1}, i = 1, \dots, k\}.$$

(ii) If  $s_\mu - s_{\mu+1} > 1$ . If  $i = 1, j = k$  we have  $x_k x_1 M - x_0^h x_{k+1}^{p_\mu} \in I$ . If  $i > 1$  let  $j = k + 2 - i$  then  $x_j x_i M - x_1 x_{k+1}^{p_{\mu+1} - 1} \in I$  but since  $1 < s_\mu - s_{\mu+1}$  we have that  $x_1 x_{k+1}^{p_{\mu+1} - 1} \in \widetilde{Ap}(S, a)$ . so

$$\widetilde{PF}(S, a)_2 \subset \{x_1 x_k^{\sigma_\mu - 1} x_{k+1}^{p_{\mu+1} - p_\mu - 1}\}.$$

On the other hand we have  $x_{k+1} x_1 M = x_1 x_k^{\sigma_\mu - 1} x_{k+1}^{p_{\mu+1} - p_\mu}$ . By using the set  $\mathcal{C}$  we get we have  $x_{k+1} x_1 M = x_0^\alpha N \in I$  for some  $\alpha \in \mathbb{N}^*$  if and only if  $s_\mu - s_{\mu+1} \leq s_\mu - k$ . Hence, if  $1 < s_\mu - s_{\mu+1} \leq s_\mu - k$  then  $\widetilde{PF}(S, a)_2 = \{x_1 x_k^{\sigma_\mu - 1} x_{k+1}^{p_{\mu+1} - p_\mu - 1}\}$  and if  $s_\mu - k < s_\mu - s_{\mu+1}$  then  $\widetilde{PF}(S, a)_2 = \emptyset$ .

- (7) Study of  $\widetilde{PF}(S, a)_2$  when  $\rho_\mu > 1$ . We have  $x_{\rho_\mu} x_k^{\sigma_\mu} - x_0^{r'_\mu} x_{k+1}^{p_\mu}$ ,  $x_{\rho_\mu+l} x_k^{\sigma_\mu} - x_0^{r'_\mu-h} x_l x_{k+1}^{p_\mu}$  for  $l = 1, \dots, k - \rho_\mu$  and

$$\widetilde{PF}(S, a)_2 \subset \{x_i x_k^{\sigma_\mu} x_{k+1}^{p_{\mu+1}-p_\mu-1}, i = 1, \dots, \rho_\mu - 1\}.$$

Let  $i = 1, \dots, \rho_\mu - 1, j = 1, \dots, k$ ,  $M = x_k^{\sigma_\mu} x_{k+1}^{p_{\mu+1}-p_\mu-1}$ . If  $i + j > k$  then we have  $x_i x_j - x_{i+j-k} x_k$ ,  $x_j x_i M - x_{i+j-k} x_k^{\sigma_\mu+1} x_{k+1}^{p_{\mu+1}-p_\mu-1} \in I$  thus  $x_j x_i M - x_{i+j-k} x_0^{r'_\mu-h} x_{k-\rho_\mu} x_{k+1}^{p_{\mu+1}-1} \in I$  but  $i + j - k + k - \rho_\mu = i + j - \rho_\mu < j$  so  $x_{i+j-k} x_{k-\rho_\mu} - x_0^h x_{i+j-\rho_\mu} \in I$ , hence  $x_j x_i M - x_0^h N \in I$  for some monomial  $N$ . On the other hand we have  $x_{k+1} x_i M = x_i x_k^{\sigma_\mu} x_{k+1}^{p_{\mu+1}-p_\mu}$  so by using the set  $\mathcal{C}$  we get  $x_i x_k^{\sigma_\mu} x_{k+1}^{p_{\mu+1}-p_\mu} - x_0^\alpha N \in I$  for some  $\alpha \in \mathbb{N}^*$  if and only if  $i + \sigma_\mu k \geq s_\mu - s_{\mu+1}$  that is  $i \geq \rho_\mu - s_{\mu+1}$ .

- (i) If  $\rho_\mu - s_{\mu+1} \leq 1$  we have  $i \geq \rho_\mu - s_{\mu+1}$  for any  $i \geq 1$ . Hence

$$\widetilde{PF}(S, a)_2 = \{x_i x_k^{\sigma_\mu} x_{k+1}^{p_{\mu+1}-p_\mu-1}, i = 1, \dots, \rho_\mu - 1\}.$$

- (ii) If  $\rho_\mu - s_{\mu+1} > 1$  we have  $\tilde{\rho} = \rho_\mu - s_{\mu+1}$  and

$$\widetilde{PF}(S, a)_2 = \{x_i x_k^{\sigma_\mu} x_{k+1}^{p_{\mu+1}-p_\mu-1}, i = \tilde{\rho}, \dots, \rho_\mu - 1\}.$$

□

## 6. SYMMETRIC ALMOST GENERALIZED ARITHMETIC PROGRESSIONS

**Theorem 6.1.** *With the above notations, suppose that  $k \geq 3$  and either  $r'_\mu \geq h$  or  $\rho_\mu = 0$ . We have  $S$  is symmetric if and only if either*

- (1)  $a = (\sigma k + 2)p', d = p'r - pr', c = (\sigma k + 2)r'$  for any  $\sigma \geq 1, 1 \leq p < p', h \geq 1$  with  $\gcd(p', r') = 1, r + h\sigma > 0, r' < -1$ . Moreover  $I$  is minimally generated by the Gröbner basis consisting of the set  $\mathcal{A}$  and:

$$x_2 x_k^\sigma - x_0^{r+h(\sigma+1)} x_{k+1}^p, \quad x_i x_k^\sigma - x_0^{r+h\sigma} x_{i-2} x_{k+1}^p, \quad i = 3, \dots, k, \quad x_{k+1}^{p'} - x_0^{-r'}$$

The Frobenius number is  $(ha + d) + \sigma(ha + kd) + c(p' - 1) - a$ .

- (2)  $s_{\mu+1} \neq 0, a = (\sigma k + 2)p' - \sigma' k p, d = p'r + ph\sigma', c = \sigma' k r + (\sigma k + 2)\sigma' h$  where  $\sigma, \sigma', p', p, r, h$  are integers such that  $\sigma > \sigma' \geq 2, p' > p > 0, r + h(\sigma + 1) > 0$  and  $(a, d) = 1$ . Moreover  $I$  is minimally generated by the Gröbner basis consisting of the set  $\mathcal{A}$  and:

$$x_2 x_k^{\sigma-\sigma'} x_{k+1}^{p'-p} - x_0^{r+h(\sigma+1)}, \quad x_i x_k^{\sigma-\sigma'} x_{k+1}^{p'-p} - x_0^{r+h\sigma} x_{i-2}, \quad i = 3, \dots, k$$

$$x_2 x_k^\sigma - x_0^{r+h(\sigma+1)} x_{k+1}^p, \quad x_i x_k^\sigma - x_0^{r+h\sigma} x_{i-2} x_{k+1}^p, \quad i = 3, \dots, k, \quad x_{k+1}^{p'} - x_k^{\sigma'}$$

The Frobenius number is  $(ha + d) + \sigma(ha + kd) + c(p_{\mu+1} - p_\mu - 1) - a$ .

- (3)  $k > 2, s_{\mu+1} \neq 0, a = (\sigma k + 2)p' - (\sigma k + 1)p, d = p'r + ph(\sigma + 1), c = (\sigma k + 1)r + (\sigma k + 2)(\sigma + 1)h$  where  $\sigma, \sigma', p', p, r, h$  are integers such that  $\sigma \geq 1, p' > p >$

$0, r + h(\sigma + 1) > 0$  and  $(a, d) = 1$ . Moreover  $I$  is minimally generated by the Gröbner basis consisting of the set  $\mathcal{A}$  and:

$$x_1 x_{k+1}^{p'-p} - x_0^{r+h(\sigma+2)}, \quad x_i x_{k+1}^{p'-p} - x_0^{r+h(\sigma+1)} x_{i-1}, \quad i = 2, \dots, k,$$

$$x_2 x_k^\sigma - x_0^{r+h(\sigma+1)} x_{k+1}^p, \quad x_i x_k^\sigma - x_0^{r+h\sigma} x_{i-2} x_{k+1}^p, \quad i = 3, \dots, k, \quad x_{k+1}^{p'} - x_1 x_k^\sigma$$

The Frobenius number is  $(ha + d) + \sigma(ha + kd) + c(p_{\mu+1} - p_\mu - 1) - a$ .

- (4)  $s_{\mu+1} \neq 0$ ,  $a = (\sigma k + 1)p' - (k - 1)p$ ,  $d = p'r - pr'$ ,  $c = (k - 1)r - (\sigma k + 1)r'$ .  
 $d > 0$  if and only if  $r' < -(p'/p)h\sigma$  with  $p < p'$ ,  $r + h(\sigma + 1) = h$ , that is  
 $r = -\sigma h$ ,  $r' < -\sigma h$ .

Moreover  $I$  is minimally generated by the Gröbner basis consisting of the set  $\mathcal{A}$  and:

$$x_2 x_k^{\sigma-1} x_{k+1}^{p'-p} - x_0^{-r'}, \quad x_i x_k^{\sigma-1} x_{k+1}^{p'-p} - x_0^{-r'-h} x_{i-2}$$

$$x_1 x_k^\sigma - x_0^h x_{k+1}^p, \quad x_i x_k^\sigma - x_{i-1} x_{k+1}^p, \quad i = 2, \dots, k, \quad x_{k+1}^{p'} - x_0^{r'+h} x_{k-1}$$

The Frobenius number is  $(ha + d) + (\sigma - 1)(ha + kd) + c(p_{\mu+1} - 1) - a$ .

*Proof.* (1) If  $s_{\mu+1} = 0$  then  $r'_{\mu+1} \neq 0$  and we have that  $\widetilde{PF}(S, a) = \widetilde{PF}(S, a)_1$  with  $\tilde{\sigma} = \sigma_\mu$ ,  $\tilde{\rho} = \rho_\mu$ . So  $\text{card} \widetilde{PF}(S, a) = 1$  if and only if we are in case 2d) with  $\rho_\mu = 2$ . We have  $s_\mu = \sigma k + 2$  for some  $\sigma \geq 1$ .

That is

$$\begin{array}{ccc|c} s & p & r & r' \\ \dots & & & \\ \sigma k + 2 & p & r & \\ 0 & p' & r' & \end{array}$$

with the condition  $p < p'$ ,  $r'_\mu = r + h(\sigma + 1) > 0$ ,  $r' < 0$ ,  $r' < r$ . By Lemma 2.2.4 of [6] we get

$$a = (\sigma k + 2)p', d = p'r - pr', c = (\sigma k + 2)r'.$$

The Frobenius number is  $\varphi(x_1 x_k^\sigma x_{k-1}^{p_{\mu+1}-1}) - a$ .

- (2) If  $s_{\mu+1} \neq 0$  and  $\text{card} \widetilde{PF}(S, a)_1 = 0$ ,  $\text{card} \widetilde{PF}(S, a)_2 = 1$ . We have to consider the cases 1a) and one of the cases 4ii), 5iii), 6ii) 7i), 7ii); or 1c) and case 7i). More precisely

(a) 1a) and 4ii) We have  $\rho_\mu = 0$ ,  $\rho_{\mu+1} = 0$ ,  $\tilde{\rho} = 0$ , so  $\text{card} \widetilde{PF}(S, a)_2 = 1$  if and only if  $k = 2$ .

(b) 1a) and 5iii) we have  $\rho_\mu = 1$ ,  $s_{\mu+1} = 1 = \rho_{\mu+1}$  a contradiction with 1a).

(c) 1a) and 7i) we have  $\rho_\mu = 2$ ,  $\rho_{\mu+1} = 0$ . So we have  $s_\mu = \sigma_\mu k + 2$ ,  $s_{\mu+1} = \sigma_{\mu+1} k$ . We set  $\sigma = \sigma_\mu$ ,  $\sigma' = \sigma_{\mu+1}$ ,  $p = p_\mu$ ,  $p' = p_{\mu+1}$ ,  $r = r_\mu$ , and since  $r'_{\mu+1} = 0 = r_{\mu+1} + h\sigma$  we have  $r_{\mu+1} = -h\sigma$  and the table

$$\begin{array}{ccc|c} s & p & r & r' \\ \dots & & & \\ \sigma k + 2 & p & r & \\ \sigma' k & p' & -h\sigma' & \end{array}$$

By Lemma 2.2.4 of [6] we get

$$a = (\sigma k + 2)p' - \sigma' k p, \quad d = p'r + ph\sigma', \quad c = \sigma' k r + (\sigma k + 2)\sigma' h$$

for some  $\sigma \geq \sigma p' \geq 2, p' > p > 0, r > -h\sigma', r + h(\sigma + 1) > 0$ . The Frobenius number is  $\varphi(x_1 x_k^\sigma x_{k-1}^{p_{\mu+1} - p_\mu - 1}) - a$ .

(d) 1a) and 7ii) Since  $s_{\mu+1} < \rho_\mu - 1$  we have  $s_{\mu+1} = \rho_{\mu+1} = 0$  by 1a), a contradiction.

(e) 1c) and 6i) We have  $\text{card}PF(\widetilde{S}, a)_2 = \rho_\mu - 1 = 1$  if and only if  $\rho_\mu = 2$ . By hypothesis  $s_\mu - s_{\mu+1} = 1$ . So we have  $s_\mu = \sigma_\mu k + 2, s_{\mu+1} = \sigma_\mu k + 1$  for some  $\sigma \geq 1$ . We set  $p = p_\mu, p' = p_{\mu+1}, r = r_\mu$ , and since  $r'_{\mu+1} = 0 = r_{\mu+1} + h(\sigma + 1)$  we have  $r_{\mu+1} = -h(\sigma + 1)$  and the table:

$$\begin{array}{ccc|c} s & p & r & r' \\ \dots & & & \\ \sigma k + 2 & p & r & \\ \sigma k + 1 & p' & -h(\sigma + 1) & \end{array}$$

By Lemma 2.2.4 of [6] we get

$$a = (\sigma k + 2)p' - (\sigma k + 1)p, \quad d = p'r + ph(\sigma + 1), \quad c = (\sigma k + 1)r + (\sigma k + 2)(\sigma + 1)h$$

for some  $\sigma \geq 1, p' > p \geq 1, r > -h(\sigma + 1)$ . The Frobenius number is  $\varphi(x_1 x_k^\sigma x_{k-1}^{p_{\mu+1} - p_\mu - 1}) - a$ .

(3) Suppose  $s_{\mu+1} \neq 0$  and  $\text{card}PF(\widetilde{S}, a)_1 = 1, \text{card}PF(\widetilde{S}, a)_2 = 0$ . We are in case 6iii), we have  $\rho_\mu = 1, r'_\mu = h, 0 < s_{\mu+1} < k$  so  $r'_{\mu+1} < 0$  and  $s_\mu - s_{\mu+1} = (\sigma_\mu - 1)k + (k - s_{\mu+1} + 1)$ . If  $s_{\mu+1} = 1$  we have  $\tilde{\rho} = 0$ , we are in case 2a) and  $\text{card}PF(\widetilde{S}, a)_1 = 1$  implies  $k = 2$ . If  $s_{\mu+1} > 1$  we have  $0 < \tilde{\rho} = k - s_{\mu+1} + 1 < k$  and  $\text{card}PF(\widetilde{S}, a)_1 = 1$  implies  $\tilde{\rho} = 2$ , that is  $s_{\mu+1} = k - 1$ . On the other hand  $r'_\mu = h = r_\mu + h(\sigma_\mu + 1)$  so  $r_\mu = -h\sigma_\mu$ . We set  $\sigma = \sigma_\mu, p = p_\mu, p' = p_{\mu+1}, r' =$

$$\begin{array}{ccc|c} s & p & r & r' \\ \dots & & & \\ \sigma k + 1 & p & -h\sigma & \\ k - 1 & p' & r' & \end{array} \quad \text{with } \sigma \geq 1, p < p', r' < -(p'/p)h\sigma. \text{ By}$$

Lemma 2.2.4 of [6] we get

$$a = (\sigma k + 1)p' - (k - 1)p, \quad d = p'r - pr', \quad c = (k - 1)r - (\sigma k + 1)r'.$$

The Frobenius number is  $\varphi(x_1 x_k^{\sigma-1} x_{k-1}^{p_{\mu+1} - 1}) - a$ .

□



## 7. ALMOST SYMMETRIC ALMOST GENERALIZED ARITHMETIC PROGRESSIONS

**Lemma 7.1.** *With the above notations, suppose that either  $r'_\mu \geq h$  or  $\rho_\mu = 0$ . Suppose that  $S$  is almost symmetric of type  $\geq 2$  then  $s_{\mu+1} > 0$  and if  $\widetilde{Frob}(S)$  belongs to  $\widetilde{PSF}_i$  then  $\text{card}(\widetilde{PSF}_i) = 1$  except if  $h = 1, s_\mu = k + 1, s_{\mu+1} = k, r'_\mu > 1, r'_{\mu+1} = -1$ . Moreover*

- (1) *If  $\widetilde{PSF}_1 = \{\widetilde{Frob}(S)\}$  then  $\widetilde{Frob}(S) = L_\gamma x_k^{\tilde{\sigma}-\varepsilon} x_{k+1}^{p_{\mu+1}-1}$  where  $\gamma, \varepsilon \in \{0, 1\}$ .*
- (2) *If  $\widetilde{PSF}_2 = \{\widetilde{Frob}(S)\}$  then  $\widetilde{Frob}(S) = L_\gamma x_k^{\sigma_\mu-\varepsilon} x_{k+1}^{p_{\mu+1}-p_\mu-1}$  where  $\gamma, \varepsilon \in \{0, 1\}$ .*
- (3) *If  $h = 1, s_\mu = k + 1, s_{\mu+1} = k, r'_\mu > 1, r'_{\mu+1} = 1$  then  $a = k + 2, d = 2r_\mu + 2, c = k(r_\mu + 2) + 2$  with  $k$  odd,  $k \geq 3, r_\mu \geq 0, \text{gcd}(a, d) = 1$ . We have  $\widetilde{PSF} = \{x_1, \dots, x_k\} \cup \{x_{k+1}\}, t(S) = k + 1, \widetilde{Frob}(S) = x_k, F(S) = kd$ .*

*Proof.* Suppose that  $\widetilde{Frob}(S) \in \widetilde{PSF}_{1+\varepsilon}, \text{card}(\widetilde{PSF}_{1+\varepsilon}) \geq 2$  for some  $\varepsilon \in \{0, 1\}$ . By checking all cases in Theorem 5.2 there exists  $2 \leq l \leq k, M$  a monomial such that  $\widetilde{Frob}(S) = x_l M$  and  $x_{l-1} M \in \widetilde{PSF}_{1+\varepsilon}$ . Since  $S$  is almost symmetric there exists  $M_1 \in \widetilde{PSF}$  such that  $M_1 x_{l-1} M - x_0 x_l M \in I$ , which implies  $x_{l-1} M_1 - x_0 x_l \in I$ . We multiply by  $x_1$  and using the Gröbner basis we get  $x_0^h x_l M_1 - x_0 x_l x_1 \in I$  that implies  $h = 1, M_1 - x_1 \in I$ , if  $M_1 \neq x_1$  then the embedding dimension of  $S$  is less than  $k + 2$  contrary to our hypothesis, therefore  $M_1 = x_1 \in \widetilde{PSF}$ . We have to examine all the possible cases in Theorem 5.2 such that  $x_1 \in \widetilde{PSF}$ . Since  $\sigma_\mu \geq 1, p_{\mu+1} \geq 2$  the possible cases are 4), 5) or 6). In particular we have  $s_{\mu+1} > 0$ . Case 4) implies  $s_\mu = k$  so  $x_k - x_0^{r'_\mu} x_{k+1}^{p_\mu} \in I$ , that means that the embedding dimension of  $S$  is less than  $k + 2$  contrary to our hypothesis. Cases 5) and 6) implies  $\widetilde{PSF}_2 = \{x_1, \dots, x_k\}$  with  $s_\mu = k + 1, s_{\mu+1} = k, \widetilde{Frob}(S) = x_k$  and  $\widetilde{PSF}_1 \subset \{x_{k+1}^\beta\}$  for some  $\beta \in \mathbb{N}^*$ , but this is only possible in case 1a) or 2b). Now we consider the case 1a) so  $r'_{\mu+1} = 0$ , by the Gröbner basis we have that  $x_{k+1}^{p_{\mu+1}} - x_k \in I$  which implies that the embedding dimension of  $S$  is less than  $k + 2$  contrary to our hypothesis. In case 2b) we have  $\widetilde{PSF}_1 = \{x_{k+1}^{p_{\mu+1}-1}\}$ , the property almost-symmetry show that  $x_{k+1}^{2(p_{\mu+1}-1)} - x_0 x_k \in I$  so that  $x_0^{-r'_{\mu+1}} x_k x_{k+1}^{p_{\mu+1}-2} - x_0 x_k \in I$  hence  $p_{\mu+1} = 2, r'_{\mu+1} = -1$

and  $p_\mu = 1$ , so  $r_{\mu+1} = r'_{\mu+1} - 1 = -2$  we have the table

$\dots$	$k + 1$	1	$r_\mu$	$r'_\mu$
$\dots$	$k$	2	-2	1

We note that  $r'_\mu = h = 1$  if and only if  $r_\mu = -1$  which implies  $d = 0$ , so case 6) is not possible. The case 5) is possible and we have  $a = k + 2, d = 2r_\mu + 2, c = k(r_\mu + 2) + 2$  with  $k$  odd,  $k \geq 3, r_\mu \geq 0, \text{gcd}(a, d) = 1$ . We have  $\widetilde{PSF} = \{x_1, \dots, x_k\} \cup \{x_{k+1}\}, t(S) = k + 1, F(S) = kd$ .  $\square$

**Theorem 7.2.** *Suppose  $\widetilde{PSF}_2 = \{\widetilde{Frob}(S)\}$ . Then  $S$  is almost symmetric with  $k \geq 3, t(S) \geq 2$  if and only if either*

(1)  $\widetilde{PSF}_1 = \{x_i x_{k+1}^{p_{\mu+1}-1} \mid i = 1, \dots, k-l\}, \widetilde{PSF}_2 = \{x_1 x_k^\sigma x_{k+1}^{p_{\mu+1}-2}\}$  and

$$\begin{array}{ccc|c} s & p & r & r' \\ \sigma k + 2 & 1 & r_\mu & \\ (\sigma - 1)k + l & p_{\mu+1} & r_{\mu+1} & 0 \end{array}$$

Since  $0 = r'_{\mu+1} = r_{\mu+1} + \sigma$  we have  $r_{\mu+1} = -\sigma$  with  $\sigma \geq 2, l \geq 1$ .  $t(S) = k-l+1$ ,  $t(S) = 2$  if and only if  $l = \rho_{\mu+1} = k-1$ .

$$a = (\sigma k + 2)p_{\mu+1} - ((\sigma - 1)k + l)$$

$$d = p_{\mu+1}r_\mu + \sigma$$

$$c = ((\sigma - 1)k + l)r_\mu + (\sigma k + 2)\sigma.$$

where  $h = 1, k \geq 3, 1 \leq l \leq k-1$  and  $t(S) = k-l+1$ , or

(2)  $\widetilde{PSF}_1 = \{x_{k+1}^{p_{\mu+1}-1}\}, \widetilde{PSF}_2 = \{x_1 x_k^\sigma x_{k+1}^{p_{\mu+1}-2}\}$  and

$$\begin{array}{ccc|c} s & p & r & r' \\ \sigma k + 2 & 1 & r_\mu & \\ \sigma k + 1 & p_{\mu+1} & r_{\mu+1} & -1 \end{array}$$

but  $-1 = r'_{\mu+1} = r_{\mu+1} + h(\sigma + 1)$  so  $r_{\mu+1} = -h(\sigma + 1) - 1, t(S) = 2$ .

$$a = (\sigma k + 2)p_{\mu+1} - (\sigma k + 1)$$

$$d = p_{\mu+1}r_\mu + h(\sigma + 1) + 1$$

$$c = (\sigma k + 1)r_\mu + (\sigma k + 2)(h(\sigma + 1) + 1).$$

where  $h \geq 1, \sigma \geq 1, p_{\mu+1} \geq 2, r_\mu > -h(\sigma + 1) - 1, t(S) = 2$ .

*Proof.* We have  $\widetilde{PSF}_2 = \{L_\gamma x_k^{\sigma_\mu - \delta} x_{k+1}^{p_{\mu+1} - p_\mu - 1}\}$  where  $\gamma, \delta \in \{0, 1\}$ . We set  $N_{\sigma_\mu - \delta} := x_k^{\sigma_\mu - \delta} x_{k+1}^{p_{\mu+1} - p_\mu - 1}$ . By Theorem 5.2 we have  $\widetilde{PSF}_1 = \{x_\Gamma M_{\tilde{\sigma} - \varepsilon}, \dots, x_\Delta M_{\tilde{\sigma} - \varepsilon}\}$  for some  $\Gamma \leq \Delta, \varepsilon \in \{0, 1\}$  where  $M_{\tilde{\sigma} - \varepsilon} := x_k^{\tilde{\sigma} - \varepsilon} x_{k+1}^{p_{\mu+1} - 1}$ . Since  $S$  is almost symmetric we have that

$$L_\Gamma L_\Delta M_{\tilde{\sigma} - \varepsilon}^2 - x_0 L_\Gamma N_{\sigma_\mu - \delta} \in I. \quad (1)$$

We have  $(x_{k+1}^{p_{\mu+1}-1})^2 = x_{k+1}^{2p_{\mu+1}-2} = x_{k+1}^{p_{\mu+1}} x_{k+1}^{p_{\mu+1}-2}$  since  $p_{\mu+1} - 2 \geq 0$ , so  $(x_{k+1}^{p_{\mu+1}-1})^2 - x_0^{-r'_{\mu+1}} L_{\rho_{\mu+1}} x_k^{\sigma_{\mu+1}} x_{k+1}^{p_{\mu+1}-2} \in I$  so  $M_{\tilde{\sigma} - \varepsilon}^2 - x_0^{-r'_{\mu+1}} L_{\rho_{\mu+1}} x_k^{\sigma_{\mu+1} + 2\tilde{\sigma} - 2\varepsilon} x_{k+1}^{p_{\mu+1}-2} \in I$ . Note that  $p_{\mu+1} - 2 - (p_{\mu+1} - p_\mu - 1) = p_\mu - 1 \geq 0$  so that (1) becomes

$$L_\Gamma L_\Delta x_0^{-r'_{\mu+1}-1} L_{\rho_{\mu+1}} x_k^{\sigma_{\mu+1} + 2\tilde{\sigma} - 2\varepsilon} x_{k+1}^{p_\mu - 1} - L_\Gamma x_k^{\sigma_\mu - \delta} \in I. \quad (2)$$

(1) If  $\Gamma + \Delta = 0$  we are in case 2b) so  $s_\mu - s_{\mu+1} = 1$ . Since  $\text{card}(\widetilde{PSF}_2) = 1$  we are in case 7i) hence  $\gamma = 1, \delta = 0, s_\mu = \sigma_\mu k + 2, s_{\mu+1} = \sigma_\mu k + 1$ . We have

$$x_0^{-r'_{\mu+1}-1} x_1 x_k^\sigma x_{k+1}^{p_\mu - 1} - x_1 x_k^\sigma \in I \quad (3)$$

this is possible only if  $r'_{\mu+1} = -1, p_\mu = 1$ . We set  $\sigma = \sigma_\mu$  so we have

$\widetilde{PSF}_1 = \{x_{k+1}^{p_{\mu+1}-1}\}, \widetilde{PSF}_2 = \{x_1 x_k^\sigma x_{k+1}^{p_{\mu+1}-2}\}$  and

$$\begin{array}{ccc|c} s & p & r & r' \\ \sigma k + 2 & 1 & r_\mu & \\ \sigma k + 1 & p_{\mu+1} & r_{\mu+1} & -1 \end{array}$$
 but  $-1 = r'_{\mu+1} = r_{\mu+1} + h(\sigma + 1)$  so  $r_{\mu+1} = -h(\sigma + 1) - 1, t(S) = 2$ .

$$\begin{aligned} a &= (\sigma k + 2)p_{\mu+1} - (\sigma k + 1) \\ d &= p_{\mu+1}r_\mu + h(\sigma + 1) + 1 \\ c &= (\sigma k + 1)r_\mu + (\sigma k + 2)(h(\sigma + 1) + 1). \end{aligned}$$

(2) If  $0 < \Gamma + \Delta \leq k$  we have

$$x_{\Gamma+\Delta}x_0^{h-r'_{\mu+1}-1}L_{\rho_{\mu+1}}x_k^{\sigma_{\mu+1}+2\tilde{\sigma}-2\varepsilon}x_{k+1}^{p_\mu-1} - L_\gamma x_k^{\sigma_\mu-\delta} \in I \quad (4)$$

which implies  $h = 1, r'_{\mu+1} = 0$  and after Theorem 5.2  $\rho_{\mu+1} > 0$ .

(a) If  $\Gamma + \Delta + \rho_{\mu+1} \leq k$  we have

$$x_0x_{\Gamma+\Delta+\rho_{\mu+1}}x_k^{\sigma_{\mu+1}+2\tilde{\sigma}-2\varepsilon}x_{k+1}^{p_\mu-1} - L_\gamma x_k^{\sigma_\mu-\delta} \in I \quad (5)$$

which is impossible since  $L_\gamma x_k^{\sigma_\mu-\delta} \in \widetilde{Ap}(S)$ . So

(b)  $\Gamma + \Delta + \rho_{\mu+1} > k$  we have

$$x_{\Gamma+\Delta+\rho_{\mu+1}-k}x_k^{\sigma_{\mu+1}+2\tilde{\sigma}+1-2\varepsilon}x_{k+1}^{p_\mu-1} - L_\gamma x_k^{\sigma_\mu-\delta} \in I \quad (6)$$

By Theorem 5.2 in all cases with  $r'_{\mu+1} = 0, \rho_{\mu+1} > 0$  and  $\Gamma + \Delta + \rho_{\mu+1} > k$  we have  $\Gamma + \Delta + \rho_{\mu+1} = k + 1$  so

$$x_1x_k^{\sigma_{\mu+1}+2\tilde{\sigma}+1-2\varepsilon}x_{k+1}^{p_\mu-1} - L_\gamma x_k^{\sigma_\mu-\delta} \in I \quad (7)$$

If  $\sigma_{\mu+1} + \tilde{\sigma} - \sigma_\mu + \tilde{\sigma} + \delta + 1 - 2\varepsilon > 0$

$$x_1x_k^{\sigma_{\mu+1}+\tilde{\sigma}-\sigma_\mu+\tilde{\sigma}+\delta+1}x_{k+1}^{p_\mu-1} - L_\gamma \in I \quad (8)$$

which leads to a contradiction. So we can assume  $\sigma_{\mu+1} + \tilde{\sigma} - \sigma_\mu + \tilde{\sigma} + \delta + 1 - 2\varepsilon \leq 0$ . We have

$$x_1x_{k+1}^{p_\mu-1} - L_\gamma x_k^{-(\sigma_{\mu+1}+\tilde{\sigma}-\sigma_\mu+\tilde{\sigma}+\delta+1-2\varepsilon)} \in I. \quad (9)$$

If  $x_1x_{k+1}^{p_\mu-1} \neq L_\gamma x_k^{-(\sigma_{\mu+1}+\tilde{\sigma}-\sigma_\mu+\tilde{\sigma}+\delta+1-2\varepsilon)}$ , since  $L_\gamma x_k^{-(\sigma_{\mu+1}+\tilde{\sigma}-\sigma_\mu+\tilde{\sigma}+\delta+1-2\varepsilon)} \in \widetilde{Ap}(S)$  we have  $x_1x_{k+1}^{p_\mu-1} \in \text{in}(I)$  which implies  $s_\mu - s_{\mu+1} = 1$  and by 1c) of Theorem 5.2  $\widetilde{PF}_1 = \emptyset$  a contradiction. So  $x_1x_{k+1}^{p_\mu-1} = L_\gamma x_k^{-(\sigma_{\mu+1}+\tilde{\sigma}-\sigma_\mu+\tilde{\sigma}+\delta+1-2\varepsilon)}$  which implies  $\gamma = 1, p_\mu = 1, \sigma_{\mu+1} + \tilde{\sigma} - \sigma_\mu + \tilde{\sigma} + \delta + 1 - 2\varepsilon = 0$

We have to discuss several cases:

$\varepsilon = 1$ : So we are either in case 1b) or 1d) and we have  $\tilde{\sigma} > 0$ . On the other hand we have either:

1)  $\tilde{\sigma} = \sigma_\mu - \sigma_{\mu+1} - 1, \tilde{\rho} > 0, \rho_{\mu+1} + \tilde{\rho} = k + \rho_\mu \geq k$ . In case 1b) we have  $\tilde{\rho} = 0$  so this case is not possible. Case 1d) implies  $\tilde{\rho} = 1$ , so

that  $\rho_{\mu+1} = k - 1, \rho_\mu = 0$ , so we are in case 4), since  $\text{card}(\widetilde{PSF_2}) = 1$  we have  $k = 2$  a contradiction.

2)  $\sigma_\mu = \sigma_{\mu+1} + \tilde{\sigma}$ ,  $\rho_{\mu+1} + \tilde{\rho} = \rho_\mu$ , we have  $\tilde{\sigma} + \delta - 1 = 0$  so  $\tilde{\sigma} = 1, \delta = 0$ .  $\delta = 0$  implies that we have to consider cases 5iii), 7i) and 7ii). Case 5iii) implies  $s_{\mu+1} = 1$ , case 7ii) implies  $\rho_\mu > s_{\mu+1}$  so  $\sigma_{\mu+1} = 0$  in both cases and  $x_{k+1}^{p_{\mu+1}} - x_{\rho_{\mu+1}} \in I$ , a contradiction. So we have two possible cases 7i)-1b or 7i)-1d. Since  $\text{card}(\widetilde{PSF_2}) = 1$  we have  $\rho_\mu = 2$  and  $\widetilde{PSF_2} = \{x_1 x_k^{\sigma_\mu} x_{k+1}^{p_{\mu+1} - p_\mu - 1}\}$ .

◇) 7i) 1b): we have  $\tilde{\rho} = 0, \tilde{\sigma} = 1, \rho_{\mu+1} = 2, p_\mu = 1, r'_{\mu+1} = 0$ . We set  $\sigma = \sigma_\mu$ , we have

$\widetilde{PSF_1} = \{x_i x_{k+1}^{p_{\mu+1} - 1} \mid i = 1, \dots, k - 2\}$ ,  $\widetilde{PSF_2} = \{x_1 x_k^\sigma x_{k+1}^{p_{\mu+1} - 2}\}$  and

$$\begin{array}{ccc|c} s & p & r & r' \\ \sigma k + 2 & 1 & r_\mu & \\ (\sigma - 1)k + 2 & p_{\mu+1} & r_{\mu+1} & 0 \end{array}$$

but  $0 = r'_{\mu+1} = r_{\mu+1} + \sigma$  so  $r_{\mu+1} = -\sigma, t(S) = k - 1$  and

$$a = (\sigma k + 2)p_{\mu+1} - ((\sigma - 1)k + 2)$$

$$d = p_{\mu+1}r_\mu + \sigma$$

$$c = ((\sigma - 1)k + 2)r_\mu + (\sigma k + 2)\sigma.$$

By using my software we have the following example  $k = 4, a = 214, d = 15, c = 236, \sigma = 7, p_{\mu+1} = 8, r_\mu = 9$ .

◇◇) 7i) 1d): we have  $\tilde{\rho} = 1, \tilde{\sigma} = 1, \rho_{\mu+1} = 1, p_\mu = 1, r'_{\mu+1} = 0$ . We set  $\sigma = \sigma_\mu$ , we have

$\widetilde{PSF_1} = \{x_i x_{k+1}^{p_{\mu+1} - 1} \mid i = 1, \dots, k - 1\}$ ,  $\widetilde{PSF_2} = \{x_1 x_k^\sigma x_{k+1}^{p_{\mu+1} - 2}\}$ . and

$$\begin{array}{ccc|c} s & p & r & r' \\ \sigma k + 2 & 1 & r_\mu & \\ (\sigma - 1)k + 1 & p_{\mu+1} & r_{\mu+1} & 0 \end{array}$$

but  $0 = r'_{\mu+1} = r_{\mu+1} + \sigma$  so  $r_{\mu+1} = -\sigma, t(S) = k$  and

$$a = (\sigma k + 2)p_{\mu+1} - ((\sigma - 1)k + 1)$$

$$d = p_{\mu+1}r_\mu + \sigma$$

$$c = ((\sigma - 1)k + 1)r_\mu + (\sigma k + 2)\sigma.$$

By using my software we have the following example  $k = 5, a = 487, d = 7, c = 259, \sigma = 7, p_{\mu+1} = 14, r_\mu = 8$ .

$\varepsilon = 0$ : We are in case 1e). We have  $\sigma_{\mu+1} + \tilde{\sigma} - \sigma_\mu + \tilde{\sigma} + \delta + 1 = 0$ . If  $\tilde{\sigma} = \sigma_\mu - \sigma_{\mu+1}$  we have  $\tilde{\sigma} + \delta + 1 = 0$  which is impossible, hence  $\sigma_\mu = \sigma_{\mu+1} + \tilde{\sigma} + 1, \tilde{\rho} > 0, \tilde{\rho} + \rho_{\mu+1} = k + \rho_\mu$  and  $\tilde{\sigma} = \delta = 0$ , so  $s_\mu - s_{\mu+1} < k \leq s_\mu - \rho_\mu$  which implies  $s_{\mu+1} > \rho_\mu$ . Hence we have to consider only case 7i)-1e). We have  $\rho_\mu = 2, \tilde{\rho} + \rho_{\mu+1} = k + 2$  which implies  $\tilde{\rho} - 1 = k - \rho_{\mu+1} + 1$ , and we have seen before that

$\tilde{\sigma} = 0, p_\mu = 1, r'_{\mu+1} = 0$ . We set  $\sigma = \sigma_\mu, l = \rho_{\mu+1} > 2$ , so we have  $\widetilde{PSF}_1 = \{x_i x_{k+1}^{p_{\mu+1}-1} \mid i = 1, \dots, k-l\}, \widetilde{PSF}_2 = \{x_1 x_k^\sigma x_{k+1}^{p_{\mu+1}-2}\}$  and

$$\begin{array}{ccc|c} s & p & r & r' \\ \sigma k + 2 & 1 & r_\mu & \\ (\sigma - 1)k + l & p_{\mu+1} & r_{\mu+1} & 0 \end{array}$$

Since  $0 = r'_{\mu+1} = r_{\mu+1} + \sigma$  we have  $r_{\mu+1} = -\sigma$  with  $\sigma \geq 2, l \geq 3$ .  $t(S) = k - l + 1$ . Note that  $t(S) = 2$  if and only if  $l = \rho_{\mu+1} = k - 1$ .

$$a = (\sigma k + 2)p_{\mu+1} - ((\sigma - 1)k + l)$$

$$d = p_{\mu+1}r_\mu + \sigma$$

$$c = ((\sigma - 1)k + l)r_\mu + (\sigma k + 2)\sigma,$$

with  $\sigma \geq 1, l \geq 3, p_{\mu+1} \geq 2, r_\mu > -\sigma$ . By using my software we have the following example  $a = 213, d = 49, c = 209, k = 6, \tilde{\rho} = 5, \rho_{\mu+1} = 3, t(S) = 4$ .

- (3) If  $\Gamma + \Delta > k$  we are in case 2c), we have  $\Gamma = 1, \Delta = k, \varepsilon = 1, r'_{\mu+1} < 0, \tilde{\rho} = 1, \tilde{\sigma} > 0$ , and (2) becomes

$$x_1 x_0^{-r'_{\mu+1}} L_{\rho_{\mu+1}} x_k^{\sigma_{\mu+1}+2\tilde{\sigma}-1} x_{k+1}^{p_{\mu}-1} - x_0 L_\gamma x_k^{\sigma_{\mu}-\delta} \in I. \quad (10)$$

If  $\rho_{\mu+1} > 0$  we have

$$x_0^{-r'_{\mu+1}+h-1} x_{\rho_{\mu+1}+1} x_k^{\sigma_{\mu+1}+2\tilde{\sigma}-1} x_{k+1}^{p_{\mu}-1} - L_\gamma x_k^{\sigma_{\mu}-\delta} \in I. \quad (11)$$

Since  $h - r'_{\mu+1} \geq 1$  we will have  $L_\gamma x_k^{\sigma_{\mu}-\delta} x_{k+1}^{p_{\mu+1}-p_{\mu}-1} \notin \widetilde{Ap}(S)$  a contradiction.

If  $\rho_{\mu+1} = 0$  then  $\rho_\mu = 1, \sigma_\mu = \sigma_{\mu+1} + \tilde{\sigma}, \sigma_{\mu+1} + 2\tilde{\sigma} - 1 = \sigma_\mu + \tilde{\sigma} - 1$  but  $x_1 x_k^{\sigma_\mu} - x_0^{r'_\mu} x_{k+1}^{p_\mu} \in I$  so from (10) we have

$$x_0^{r'_\mu - r'_{\mu+1}} x_k^{\tilde{\sigma}-1} x_{k+1}^{2p_\mu-1} - x_0 L_\gamma x_k^{\sigma_\mu-\delta} \in I. \quad (12)$$

Since  $r'_\mu - r'_{\mu+1} \geq 2$  we will have  $L_\gamma x_k^{\sigma_\mu-\delta} \in \text{in}(I)$  a contradiction.  $\square$

**Theorem 7.3.** Suppose  $\widetilde{PSF}_1 = \{\widetilde{Frob}(S)\}$ . Then  $S$  is almost symmetric with  $k \geq 3, t(S) \geq 2$  if and only if either

- (1)  $\widetilde{PSF}_1 = \{x_1 x_k^\sigma x_{k+1}^p\}, \widetilde{PSF}_2 = \{x_2 x_k^\sigma, \dots, x_{\rho_{\mu}-1} x_k^\sigma\}$ , we have the table

$$\begin{array}{ccc|c} s & p & r & r' \\ \sigma k + l + 2 & p & -\sigma & 1 \\ l & p + 1 & r_{\mu+1} & < 0 \end{array}$$

$$a = (\sigma k + l + 2)(p + 1) - lp$$

$$d = -(p + 1)\sigma - pr_{\mu+1}$$

$$c = -l\sigma - (\sigma k + l + 2)r_{\mu+1},$$

where  $h = 1, k \geq 3, \sigma \geq 1, 1 \leq l \leq k - 3, r_{\mu+1} < -\sigma$  and  $t(S) = l + 1$ , or

$$(2) \widetilde{PSF}_1 = \{x_1 x_k^{\sigma-1} x_{k+1}^p\}, \widetilde{PSF}_2 = \{x_2 x_k^{\sigma-1}, \dots, x_{k-1} x_k^{\sigma-1}\}.$$

$$\begin{array}{ccc|c} s & p & r & r' \\ \sigma k & p & 1 - \sigma & 1 \\ k - 2 & p + 1 & r_{\mu+1} & < 0 \end{array}$$

$$a = (\sigma k)(p + 1) - (k - 2)p$$

$$d = (1 - h\sigma)(p + 1) - pr_{\mu+1}$$

$$c = (k - 2)(1 - \sigma) - \sigma k r_{\mu+1}.$$

where  $h = 1, k \geq 3, p, \sigma \geq 2, r_{\mu+1} < -1$  and  $t(S) = k - 1$ , or

$$(3) \widetilde{PSF}_1 = \{x_1 x_k^{\sigma-1} x_{k+1}^p\}, \widetilde{PSF}_2 = \{x_2 x_k^{\sigma-1}, \dots, x_k^\sigma\}, t(S) = k.$$

$$\begin{array}{ccc|c} s & p & r & r' \\ \sigma k + 1 & p & 1 - h\sigma & h + 1 \\ k - 1 & p + 1 & r_{\mu+1} & < 0 \end{array}$$

$$a = (\sigma k + 1)(p + 1) - (k - 1)p$$

$$d = (p + 1)(1 - h\sigma) - pr_{\mu+1}$$

$$c = (k - 1)(1 - h\sigma) - (\sigma k + 1)r_{\mu+1}.$$

$h \geq 1, k \geq 3, p, \sigma \geq 1, r_{\mu+1} < \min\{-h, 1 - h\sigma\}$  and  $t(S) = k$ , or

$$(4) \widetilde{PSF}_1 = \{x_{k+1}^p\}, \widetilde{PSF}_2 = \{x_1, \dots, x_k\}.$$

$$\begin{array}{ccc|c} s & p & r & r' \\ k + 1 & p & -1 & 1 \\ k & p + 1 & r_{\mu+1} & < 0 \end{array}$$

$$a = (k + 1)(p + 1) - kp = k + p + 1$$

$$d = -(p + 1) - pr_{\mu+1}$$

$$c = -k - (k + 1)r_{\mu+1}.$$

with  $h = 1, k \geq 3, p \geq 1, r_{\mu+1} < -1$ . We have  $t(S) = k + 1$ , or

$$(5) \widetilde{PSF}_1 = \{x_1 x_k^{\sigma-2} x_{k+1}^p\}, \widetilde{PSF}_2 = \{x_1 x_k^{\sigma-1}\} \text{ and}$$

$$\begin{array}{ccc|c} s & p & r & r' \\ \sigma k + 1 & p & -\sigma & 1 \\ 2k - 1 & p + 1 & r_{\mu+1} & < 0 \end{array}$$

$$a = (\sigma k + 1)(p + 1) - (2k - 1)p$$

$$d = -(p + 1)\sigma - pr_{\mu+1}$$

$$c = -\sigma(2k - 1) - (\sigma k + 1)r_{\mu+1}.$$

with  $h = 1, k \geq 3, \sigma \geq 2, p \geq 1, r_{\mu+1} < -\sigma$ . We have  $t(S) = 2$ .

*Proof.* We have  $\widetilde{PSF}_1 = \{L_\gamma x_k^{\tilde{\sigma}-\varepsilon} x_{k+1}^{p_{\mu+1}-1}\}$  where  $\gamma, \varepsilon \in \{0, 1\}$ . We set  $M_{\tilde{\sigma}-\varepsilon} := x_k^{\tilde{\sigma}-\varepsilon} x_{k+1}^{p_{\mu+1}-1}$ . By Theorem 5.2 we have  $\widetilde{PSF}_2 = \{x_\Gamma N_{\sigma_\mu-\delta}, \dots, x_\Delta N_{\sigma_\mu-\delta}\}$  for some  $\Gamma \leq \Delta, \delta \in \{0, 1\}$ , where  $N_{\sigma_\mu-\delta} := x_k^{\sigma_\mu-\delta} x_{k+1}^{p_{\mu+1}-p_\mu-1}$ .

Since  $S$  is almost symmetric we have that

$$L_\Gamma L_\Delta N_{\sigma_\mu - \delta}^2 - x_0 L_\gamma M_{\tilde{\sigma} - \varepsilon} \in I, \quad (13)$$

Note that  $\gamma = 0$  is possible only in case 2b), where we have  $\tilde{\sigma} = 0, \tilde{\rho} = 1, \varepsilon = 0$ . In all other cases we have  $\gamma = 1$ .

- (1) If  $\Gamma + \Delta = 0$  then we are in case 5iii), which implies  $\rho_\mu = 1, s_{\mu+1} = 1$  so  $\tilde{\rho} = 0$ , but in all cases of Theorem 5.2 with  $\text{card}(\widetilde{PSF}_1) = 1$  satisfying these conditions we have  $k = 2$ .

- (2) If  $0 < \Gamma + \Delta \leq k$  we have

$$x_{\Gamma+\Delta} x_0^{h-1} x_k^{2\sigma_\mu - 2\delta} x_{k+1}^{2(p_{\mu+1} - p_\mu - 1)} - L_\gamma x_k^{\tilde{\sigma} - \varepsilon} x_{k+1}^{p_{\mu+1} - 1} \in I \quad (14)$$

If  $\gamma = 0$  or  $h > 1$  then  $x_{k+1}^{p_{\mu+1} - 1} \in \text{in}(I)$ , which is not possible. So we have  $\gamma = 1, h = 1$  and

$$x_{\Gamma+\Delta} x_k^{2\sigma_\mu - 2\delta} x_{k+1}^{2(p_{\mu+1} - p_\mu - 1)} - x_1 x_k^{\tilde{\sigma} - \varepsilon} x_{k+1}^{p_{\mu+1} - 1} \in I \quad (15)$$

We consider two cases:

- (a) If  $\sigma_\mu = \sigma_{\mu+1} + \tilde{\sigma}$  (so  $\rho_\mu = \rho_{\mu+1} + \tilde{\rho}$ ) then  $2\sigma_\mu - 2\delta - (\tilde{\sigma} - \varepsilon) = \sigma_\mu + \sigma_{\mu+1} - 2\delta + \varepsilon$ . Note that  $\sigma_{\mu+1} - 2\delta + \varepsilon < 0$  if and only if  $\delta = 1$  and either  $\sigma_{\mu+1} = 0, \varepsilon = 0$ , either  $\sigma_{\mu+1} = 0, \varepsilon = 1$  or  $\sigma_{\mu+1} = 1, \varepsilon = 0$ . If  $\sigma_{\mu+1} = 0$  we have  $\sigma_\mu = \tilde{\sigma}$ , but we can check that no case in Theorem 5.2 with  $\delta = 1$  satisfy the condition  $\sigma_\mu = \tilde{\sigma}$ . If  $\varepsilon = 0$  then the possible cases are 1e) or 2d), both cases imply that  $\tilde{\rho} > 1$  hence  $\rho_\mu > 1$ . We can check that there is no case in Theorem 5.2 with  $\delta = 1, \rho_\mu > 1$ .

So we have  $\sigma_{\mu+1} - 2\delta + \varepsilon \geq 0$ . We have

$$x_{\Gamma+\Delta} x_k^{\sigma_\mu} x_k^{\sigma_{\mu+1} - 2\delta + \varepsilon} x_{k+1}^{2(p_{\mu+1} - p_\mu - 1)} - x_1 x_{k+1}^{p_{\mu+1} - 1} \in I. \quad (16)$$

Since  $\sigma_\mu \geq 1$ , if  $2(p_{\mu+1} - p_\mu - 1) \geq p_{\mu+1} - 1$  then  $\varphi(x_1) \geq \varphi(x_k)$  a contradiction, so we have

$$x_{\Gamma+\Delta} x_k^{\sigma_\mu} x_k^{\sigma_{\mu+1} - 2\delta + \varepsilon} - x_1 x_{k+1}^{2p_{\mu+1} - p_{\mu+1}} \in I \quad (17)$$

If  $\rho_\mu = 0$  or  $\Gamma + \Delta = \rho_\mu$  then  $L_{\rho_\mu} x_k^{\sigma_\mu} - x_0^{r'_\mu} x_{k+1}^{p_\mu} \in I$  which leads to a contradiction since  $x_1 x_{k+1}^{2p_{\mu+1} - p_{\mu+1}} \in \text{Ap}(S)$ . Hence we have either  $\Gamma + \Delta > \rho_\mu > 0$  or  $\Gamma + \Delta < \rho_\mu, \rho_\mu > 0$ .

★ If  $\Gamma + \Delta > \rho_\mu > 0$  then

$$x_0^{r'_\mu - 1} x_{\Gamma+\Delta - \rho_\mu} x_k^{\sigma_{\mu+1} - 2\delta + \varepsilon} x_{k+1}^{p_\mu} - x_1 x_{k+1}^{2p_{\mu+1} - p_{\mu+1}} \in I. \quad (18)$$

Since  $2p_\mu + 1 - p_{\mu+1} \leq p_\mu$  we get

$$x_0^{r'_\mu - 1} x_{\Gamma+\Delta - \rho_\mu} x_k^{\sigma_{\mu+1} - 2\delta + \varepsilon} x_{k+1}^{p_{\mu+1} - p_\mu - 1} - x_1 \in I \quad (19)$$

This is possible if and only if  $r'_\mu = 1, \Gamma + \Delta - \rho_\mu = 1, \sigma_{\mu+1} - 2\delta + \varepsilon = 0$  and  $p_{\mu+1} = p_\mu + 1$ . We have several cases.

- $\delta = 1$ , by Theorem 5.2 we are in case 6) which implies  $\rho_\mu = 1$  so  $\Gamma + \Delta = 2$  and  $\Gamma = \Delta = 1$ , the only possible case is 6ii), so  $s_\mu - s_{\mu+1} > 1$ . Since  $1 = \rho_\mu = \rho_{\mu+1} + \tilde{\rho}$  we have either  $\tilde{\rho} = 0, \rho_{\mu+1} = 1$  or  $\tilde{\rho} = 1, \rho_{\mu+1} = 0$ . If  $\tilde{\rho} = 0, \rho_{\mu+1} = 1$  by Theorem 5.2 the only possible cases with  $\text{card}(PF_1(S)) = 1$ , are 1b) and 2a) with  $k = 2$  a contradiction. If  $\tilde{\rho} = 1, \rho_{\mu+1} = 0$  the only possible case is 2b) with  $s_\mu - s_{\mu+1} = 1$ , a contradiction, so the case  $\delta = 1$  is not possible.
- $\delta = 0$  implies  $\sigma_{\mu+1} = \varepsilon = 0$ . We have to consider in Theorem 5.2 the cases 1e) or 2d) because  $\varepsilon = 0$  and 7i), 7ii) because  $\delta = 0$ . On the other hand  $\Gamma + \Delta = \rho_\mu + 1$  implies that we are in case 7ii) with  $\tilde{\rho} = 2$ . We set  $\sigma = \sigma_\mu, p = p_\mu, l = s_{\mu+1} = \rho_{\mu+1}$  so we have  $\widetilde{PSF}_1 = \{x_1 x_k^\sigma x_{k+1}^p\}$ ,  $\widetilde{PSF}_2 = \{x_2 x_k^\sigma, \dots, x_{\rho_{\mu-1}} x_k^\sigma\}$ , we have the table
 

s	p	r	r'
$\sigma k + l + 2$	p	$-\sigma$	1
l	p + 1	$r_{\mu+1}$	$< 0$

 with  $\sigma \geq 1, 1 \leq l < k - 2$ . Also  $1 = r'_\mu = r_\mu + \sigma + 1$  so  $r_\mu = -\sigma, r_{\mu+1} < -\sigma \leq -1$ , and  $r'_{\mu+1} = r_{\mu+1} + 1 < -\sigma + 1 \leq 0$ . So in fact we are in case 2d) 7ii). We have  $t(S) = l + 1$ . Note that  $t(S) = 2$  if and only if  $l = 1, \rho_\mu = 3$ .

$$\begin{aligned} a &= (\sigma k + l + 2)(p + 1) - lp \\ d &= -(p + 1)\sigma - pr_{\mu+1} \\ c &= -l\sigma - (\sigma k + l + 2)r_{\mu+1}. \end{aligned}$$

★ If  $\rho_\mu > 0, \Gamma + \Delta < \rho_\mu$ . From (17) we have necessarily  $x_{\Gamma+\Delta} x_k^{\sigma_\mu} x_k^{\sigma_{\mu+1}-2\delta+\varepsilon} \in \text{in}(I)$  so  $\sigma_{\mu+1} - 2\delta + \varepsilon > 0$ . Since  $x_{\Gamma+\Delta} x_k^{\sigma_\mu+1} - x_0^{r'_\mu} x_{k+\Gamma+\Delta-\rho_\mu} x_{k+1}^{p_\mu} \in I$  we have

$$x_0^{r'_\mu} x_{k+\Gamma+\Delta-\rho_\mu} x_{k+1}^{p_\mu} x_k^{\sigma_{\mu+1}-1-2\delta+\varepsilon} x_{k+1}^{p_\mu} - x_1 x_{k+1}^{2p_{\mu+1}-p_{\mu+1}} \in I, \quad (20)$$

this is impossible since  $x_1 x_{k+1}^{2p_{\mu+1}-p_{\mu+1}} \in \widetilde{Ap}(S)$ .

- (b)  $\sigma_\mu = \sigma_{\mu+1} + \tilde{\sigma} + 1$  (so  $\tilde{\rho} > 0, \rho_\mu = \rho_{\mu+1} + \tilde{\rho} - k > 0$ ) then  $2\sigma_\mu - 2\delta - (\tilde{\sigma} - \varepsilon) = \sigma_\mu + \sigma_{\mu+1} + 1 - 2\delta + \varepsilon$ .

Note that  $\sigma_{\mu+1} + 1 - 2\delta + \varepsilon < 0$  if and only if  $\delta = 1, \sigma_{\mu+1} = 0, \varepsilon = 0$ . The possible cases in Theorem 5.2 are 7i), 7ii). In 7ii) we have  $\rho_{\mu+1} < \rho_\mu$  so  $\sigma_\mu = \sigma_{\mu+1} + \tilde{\sigma}$  a contradiction. In 7i) we have  $2\sigma_\mu - 2\delta - (\tilde{\sigma} - \varepsilon) = \sigma_\mu - 1 \geq 0$ ,  $\Gamma + \Delta = \rho_\mu \geq 2$  so (15) becomes

$$x_{\rho_\mu} x_k^{\sigma_\mu-1} x_{k+1}^{2(p_{\mu+1}-p_\mu-1)} - x_1 x_{k+1}^{p_{\mu+1}-1} \in I, \quad (21)$$

if  $2(p_{\mu+1} - p_\mu - 1) \geq p_{\mu+1} - 1$  we get

$$x_{\rho_\mu} x_k^{\sigma_\mu-1} x_{k+1}^{(2(p_{\mu+1}-p_\mu-1))-(p_{\mu+1}-1)} - x_1 \in I, \quad (22)$$



which leads to a contradiction; so  $2(p_{\mu+1} - p_\mu - 1) < p_{\mu+1} - 1$ , we get

$$x_{\rho_\mu} x_k^{\sigma_\mu - 1} - x_1 x_{k+1}^{(p_{\mu+1}-1)-(2(p_{\mu+1}-p_\mu-1))} \in I, \quad (23)$$

but  $x_1 x_{k+1}^{(p_{\mu+1}-1)-(2(p_{\mu+1}-p_\mu-1))} \in \widetilde{Ap}(S, a)$  so  $x_{\rho_\mu} x_k^{\sigma_\mu - 1} \in \text{in}(I)$  which is not possible.

Hence  $\sigma_{\mu+1} + 1 - 2\delta + \varepsilon \geq 0$ , and (15) becomes

$$x_{\Gamma+\Delta} x_k^{\sigma_\mu + \sigma_{\mu+1} + 1 - 2\delta + \varepsilon} x_{k+1}^{2(p_{\mu+1}-p_\mu-1)} - x_1 x_{k+1}^{p_{\mu+1}-1} \in I, \quad (24)$$

if  $2(p_{\mu+1} - p_\mu - 1) \geq p_{\mu+1} - 1$  we get a contradiction since  $\sigma_\mu \geq 1, \varphi(x_k) > \varphi(x_1)$ . So

$$x_{\Gamma+\Delta} x_k^{\sigma_\mu + \sigma_{\mu+1} + 1 - 2\delta + \varepsilon} - x_1 x_{k+1}^{2p_{\mu+1}-p_{\mu+1}} \in I. \quad (25)$$

Since  $x_1 x_{k+1}^{2p_{\mu+1}-p_{\mu+1}} \in \widetilde{Ap}(S, a)$ , we have  $x_{\Gamma+\Delta} x_k^{\sigma_\mu + \sigma_{\mu+1} + 1 - 2\delta + \varepsilon} \in \text{in}(I)$ . Hence we have either  $\rho_\mu = 0$ , either  $\Gamma + \Delta = \rho_\mu > 0$ , either  $\Gamma + \Delta > \rho_\mu > 0$  or  $0 < \Gamma + \Delta < \rho_\mu, \sigma_{\mu+1} + 1 - 2\delta + \varepsilon > 0$ .

- If  $\rho_\mu = 0$  or  $\Gamma + \Delta = \rho_\mu > 0$  we have  $L_{\rho_\mu} x_k^{\sigma_\mu} - x_0^{r'_\mu} x_{k+1}^{p_\mu}$  so from (25) we have

$$x_0^{r'_\mu} x_k^{\sigma_{\mu+1} + 1 - 2\delta + \varepsilon} x_{k+1}^{p_\mu} - x_1 x_{k+1}^{2p_{\mu+1}-p_{\mu+1}} \in I. \quad (26)$$

Since  $r'_\mu > 0$  and  $x_1 x_{k+1}^{2p_{\mu+1}-p_{\mu+1}} \in \widetilde{Ap}(S, a)$  we get a contradiction.

- If  $k \geq \Gamma + \Delta > \rho_\mu > 0$  then we have  $x_{\Gamma+\Delta} x_k^{\sigma_\mu} - x_0^{r'_\mu - h} x_{\Gamma+\Delta-\rho_\mu} x_{k+1}^{p_\mu}$ , so from (25) we have

$$x_0^{r'_\mu - h} x_{\Gamma+\Delta-\rho_\mu} x_k^{\sigma_{\mu+1} + 1 - 2\delta + \varepsilon} x_{k+1}^{p_\mu} - x_1 x_{k+1}^{2p_{\mu+1}-p_{\mu+1}} \in I. \quad (27)$$

which implies  $r'_\mu = h = 1$ . We have  $p_\mu - (2p_\mu + 1 - p_{\mu+1}) = p_{\mu+1} - p_\mu - 1 \geq 0$  so

$$x_{\Gamma+\Delta-\rho_\mu} x_k^{\sigma_{\mu+1} + 1 - 2\delta + \varepsilon} x_{k+1}^{p_{\mu+1}-p_\mu-1} - x_1 \in I. \quad (28)$$

which is possible only if  $\Gamma + \Delta - \rho_\mu = 1, p_{\mu+1} = p_\mu + 1, \sigma_{\mu+1} + 1 - 2\delta + \varepsilon = 0$ , which implies  $\delta = 1, \sigma_{\mu+1} + \varepsilon = 1$ . But we also have  $r'_\mu = h = 1, \rho_\mu > 0$  so the only possible case is 6ii) which implies  $\rho_\mu = 1, s_{\mu+1} \geq k$  so  $\sigma_{\mu+1} = 1, \varepsilon = 0$ . We also have  $\rho_{\mu+1} + \tilde{\rho} = k + 1$  so  $\tilde{\rho} > 1$  which implies either case 1e) or case 2d) with  $\tilde{\rho} = 2$  since  $\text{card}(\widetilde{PSF}_1) = 1$ . Hence  $s_\mu = \sigma_\mu k + 1, \sigma_\mu \geq 2, s_{\mu+1} = 2k - 1$ , since  $r'_\mu = 1 = r_\mu + \sigma_\mu + 1$  we have  $r_\mu = -\sigma_\mu$  and  $r_{\mu+1} < -\sigma_\mu$  implies  $r'_{\mu+1} = r_{\mu+1} + 2 < -\sigma_\mu + 2 \leq 0$ , so in fact we are in case 6ii)) and 2d). We set  $\sigma = \sigma_\mu, p = p_\mu$  so we have

$$\widetilde{PSF}_1 = \{x_1 x_k^{\sigma-2} x_{k+1}^p\}, \widetilde{PSF}_2 = \{x_1 x_k^{\sigma-1}\} \text{ and}$$

$$\begin{array}{ccc|c} s & p & r & r' \\ \sigma_\mu k + 1 & p & -\sigma_\mu & 1 \\ 2k - 1 & p + 1 & r_{\mu+1} & < 0 \end{array}$$

$$a = (\sigma_\mu k + 1)(p + 1) - (2k - 1)p$$

$$d = -(p + 1)\sigma_\mu - pr_{\mu+1}$$

$$c = -\sigma_\mu(2k - 1) - (\sigma_\mu k + 1)r_{\mu+1}.$$

with  $h = 1, k \geq 3, \sigma_\mu \geq 2, p \geq 1, r_{\mu+1} < -\sigma_\mu$  and  $t(S) = 2$ .

- Suppose that  $0 < \Gamma + \Delta < \rho_\mu$  then  $\sigma_{\mu+1} + 1 - 2\delta + \varepsilon > 0$ . Since  $x_{\Gamma+\Delta} x_k^{\sigma_\mu+1} - x_0^{r'_\mu} x_{k+\Gamma+\Delta-\rho_\mu} x_{k+1}^{p_\mu} \in I$  we have from (25)

$$x_0^{r'_\mu} x_{k+\Gamma+\Delta-\rho_\mu} x_k^{\sigma_\mu+1-2\delta+\varepsilon} x_{k+1}^{p_\mu} - x_1 x_{k+1}^{2p_\mu+1-p_\mu+1} \in I. \quad (29)$$

This is not possible since  $r'_\mu > 0$  and  $x_1 x_{k+1}^{2p_\mu+1-p_\mu+1} \in \widetilde{Ap}(S, a)$ .

- (3) If  $\Gamma + \Delta > k$ , from (13) we have

$$x_{\Gamma+\Delta-k} x_k^{2\sigma_\mu+1-2\delta} x_{k+1}^{2(p_{\mu+1}-p_\mu-1)} - x_0 L_\gamma x_k^{\tilde{\sigma}-\varepsilon} x_{k+1}^{p_{\mu+1}-1} \in I \quad (30)$$

- (a) If  $\sigma_\mu = \sigma_{\mu+1} + \tilde{\sigma}$  then we have  $\rho_\mu = \rho_{\mu+1} + \tilde{\rho}$ ,  $2\sigma_\mu + 1 - 2\delta - (\tilde{\sigma} - \varepsilon) = \sigma_\mu + \sigma_{\mu+1} + 1 - 2\delta + \varepsilon$ . We have several cases.

- (i)  $\sigma_{\mu+1} + 1 - 2\delta + \varepsilon < 0$  if and only  $\delta = 1, \sigma_{\mu+1} = 0, \varepsilon = 0$ . If  $\gamma = 0$  then we are in case 2b) so  $\tilde{\sigma} = \sigma_{\mu+1} = 0$  which implies  $\sigma_\mu = 0$ , this is not possible, so  $\gamma = 1$ . By looking all cases in Theorem 5.2 with  $\delta = 1$  we have  $\rho_\mu \leq 1$ , on the other side  $\varepsilon = 0, \gamma = 1$  implies  $\tilde{\rho} > 1$  a contradiction since  $\tilde{\rho} \leq \rho_\mu$ .

- (ii)  $\sigma_{\mu+1} + 1 - 2\delta + \varepsilon = 0$  if and only  $\delta = 1$  and either  $\sigma_{\mu+1} = 0, \varepsilon = 1$  or  $\sigma_{\mu+1} = 1, \varepsilon = 0$ . We have

$$x_{\Gamma+\Delta-k} x_k^{\sigma_\mu} x_{k+1}^{2(p_{\mu+1}-p_\mu-1)} - x_0 L_\gamma x_{k+1}^{p_{\mu+1}-1} \in I \quad (31)$$

ii-\*) Suppose  $\gamma = 0$ . We are in case 2b),  $s_\mu - s_{\mu+1} = 1$ , if  $\sigma_{\mu+1} = 0$  then  $s_\mu = \rho_{\mu+1} + 1$  which is not possible, so  $\sigma_{\mu+1} = 1, \tilde{\rho} = 1$  so  $\rho_\mu \geq 1$ . Since  $\delta = 1$  implies  $\rho_\mu = 1$ , hence  $\rho_{\mu+1} = 0$  and since  $s_\mu - s_{\mu+1} = 1$  we have  $s_\mu = k + 1, s_{\mu+1} = k$ . The possible cases are 5i) and 6i), in both cases we have  $\Gamma + \Delta - k = 1 = \rho_\mu$ . By using the Gröbner basis we have

$$x_0^{r'_\mu-1} x_{k+1}^{2p_{\mu+1}-p_\mu-2} - x_{k+1}^{p_{\mu+1}-1} \in I \quad (32)$$

that is

$$x_0^{r'_\mu-1} x_{k+1}^{p_{\mu+1}-p_\mu-1} - 1 \in I \quad (33)$$

possible only if  $r'_\mu = 1, p_{\mu+1} = p_\mu + 1$ . Since  $r'_\mu \geq h \geq 1$  we have equality, so we are in case 6i) and  $1 = r_\mu + 2$  so  $r_\mu = -1$ . We set  $p_\mu = p$ , we have

$$\widetilde{PSF}_1 = \{x_{k+1}^p\}, \widetilde{PSF}_2 = \{x_1, \dots, x_k\}.$$

$$\begin{array}{ccc|c} s & p & r & r' \\ k+1 & p & -1 & 1 \\ k & p+1 & r_{\mu+1} & < 0 \end{array}$$

$$a = (k+1)(p+1) - kp = k + p + 1$$

$$d = -(p+1) - pr_{\mu+1}$$

$$c = -k - (k+1)r_{\mu+1}.$$

with  $h = 1, k \geq 3, p \geq 1, r_{\mu+1} < -1, t(S) = k + 1$ .

ii-\*\*) Suppose  $\gamma = 1$ . Recall that  $\delta = 1$  and either  $\sigma_{\mu+1} = 0, \varepsilon = 1$  or  $\sigma_{\mu+1} = 1, \varepsilon = 0$ . If  $\gamma = 1, \varepsilon = 0$  we are in cases 1e) or 2d) so  $\tilde{\rho} > 1$ .  $\delta = 1$  implies  $\rho_\mu \leq 1$ , but  $\rho_\mu = \rho_{\mu+1} + \tilde{\rho}$  so we get a contradiction. Therefore  $\delta = 1, \sigma_{\mu+1} = 0, \varepsilon = 1$ , again from  $\rho_\mu = \rho_{\mu+1} + \tilde{\rho}$  we get  $s_{\mu+1} = 1, \rho_\mu = 1, \tilde{\rho} = 0$ . The possible cases with  $\tilde{\rho} = 0$  are 1b) and 2a) with  $k = 2$  a contradiction since we assume  $k \geq 3$ .

(iii)  $\sigma_{\mu+1} + 1 - 2\delta + \varepsilon > 0$ . If  $\rho_\mu > 0$  we have  $x_k^{\sigma_{\mu+1}} - x_0^{r'_\mu - h} x_{k-\rho_\mu} x_{k+1}^{p_\mu}$  so from (30) we have

$$x_0^{r'_\mu - h} x_{k-\rho_\mu} x_{\Gamma+\Delta-k} x_k^{\sigma_{\mu+1}-2\delta+\varepsilon} x_{k+1}^{2p_{\mu+1}-p_\mu-2} - x_0 L_\gamma x_{k+1}^{p_{\mu+1}-1} \in I, \quad (34)$$

hence

$$x_0^{r'_\mu - h} x_{k-\rho_\mu} x_{\Gamma+\Delta-k} x_k^{\sigma_{\mu+1}-2\delta+\varepsilon} x_{k+1}^{p_{\mu+1}-p_\mu-1} - x_0 L_\gamma \in I, \quad (35)$$

recall that  $\gamma = 0, 1$ , since  $k - \rho_\mu \geq 1, \Gamma + \Delta - k \geq 1$  we have  $\varphi(x_{k-\rho_\mu} x_{\Gamma+\Delta-k}) > \varphi(x_0 L_\gamma)$  this is not possible.

If  $\rho_\mu = 0$  we have  $x_k^{\sigma_\mu} - x_0^{r'_\mu} x_{k+1}^{p_\mu}$  so from (30) we have

$$x_0^{r'_\mu} x_{\Gamma+\Delta-k} x_k^{\sigma_{\mu+1}+1-2\delta+\varepsilon} x_{k+1}^{p_{\mu+1}-p_\mu-1} - x_0 L_\gamma \in I, \quad (36)$$

this is not possible since  $r'_\mu > 0, \varphi(x_0^{r'_\mu} x_k) > \varphi(x_0 L_\gamma)$ .

(b) If  $\sigma_\mu = \sigma_{\mu+1} + \tilde{\sigma} + 1, k + \rho_\mu = \rho_{\mu+1} + \tilde{\rho}$  we have  $\rho_\mu < \rho_{\mu+1}, \tilde{\rho}$  and  $\sigma_\mu + 1 - 2\delta - (\tilde{\sigma} - \varepsilon) = \sigma_{\mu+1} + 2 - 2\delta + \varepsilon \geq 0$ .

Suppose  $\rho_\mu = 0$ , which implies  $\sigma_\mu \geq 2$ , since  $x_k^{\sigma_\mu} - x_0^{r'_\mu} x_{k+1}^{p_\mu} \in I$  from (30) we have

$$x_0^{r'_\mu} x_{\Gamma+\Delta-k} x_k^{\sigma_{\mu+1}+2-2\delta+\varepsilon} x_{k+1}^{p_{\mu+1}-p_\mu-1} - x_0 L_\gamma \in I \quad (37)$$

This is possible if and only if  $\gamma = 1, \Gamma + \Delta - k = 1, r'_\mu = 1, p_{\mu+1} = p_\mu + 1$  and  $\sigma_{\mu+1} + 2 - 2\delta + \varepsilon = 0$ . We note that  $\sigma_{\mu+1} + 2 - 2\delta + \varepsilon = 0$ . if and only if  $\sigma_{\mu+1} = 0, \delta = 1, \varepsilon = 0$ , also note that  $\sigma_{\mu+1} = 0$  implies  $r'_{\mu+1} < 0$ . The possible case with  $r'_{\mu+1} < 0, \varepsilon = 0$  is 2d) with  $\tilde{\rho} = 2$  and  $\rho_\mu = 0$  implies the case 4ii). Note that  $r'_\mu = 1$  implies  $h = 1$ . Moreover we have  $1 = r'_\mu = r_\mu + \sigma_\mu$  so  $r_\mu = 1 - \sigma_\mu$ . We set  $\sigma := \sigma_\mu, p := p_\mu$  so we have

$$\widetilde{PSF}_1 = \{x_1 x_k^{\sigma-1} x_{k+1}^p\}, \widetilde{PSF}_2 = \{x_2 x_k^{\sigma-1}, \dots, x_{k-1} x_k^{\sigma-1}\}.$$

$$\begin{array}{ccc|c} s & p & r & r' \\ \sigma k & p & 1 - \sigma & 1 \\ k - 2 & p + 1 & r_{\mu+1} & < 0 \end{array}$$

$$\begin{aligned} a &= (\sigma k)(p + 1) - (k - 2)p \\ d &= (1 - \sigma)(p + 1) - pr_{\mu+1} \\ c &= (k - 2)(1 - \sigma) - \sigma kr_{\mu+1}. \end{aligned}$$

$h = 1, \sigma \geq 2, p \geq 1, r_{\mu+1} < -1, t(S) = k - 1$ .

Suppose  $\rho_\mu > 0$ . If  $\gamma = 0$  then we are in case 2b), so  $\tilde{\rho} = 1, \rho_\mu = 0$ , a contradiction. So  $\gamma = 1$ . We have two cases.

- (i) If  $\sigma_{\mu+1} + 2 - 2\delta + \varepsilon = 0$  then we have  $\delta = 1, \sigma_{\mu+1} = 0, \varepsilon = 0$ .  $\delta = 1$  implies  $\rho_\mu = 1, \varepsilon = 0$  implies  $\tilde{\rho} = 2$  and  $\sigma_{\mu+1} = 0$  implies  $r'_{\mu+1} < 0, s_\mu - s_{\mu+1} = (\sigma_\mu - 1)k + 2$  so the possible case is 2d) and 5ii) and we have  $\Gamma + \Delta - k = 2$ . So from (30) we have

$$x_2 x_k^{\sigma_\mu} x_{k+1}^{2(p_{\mu+1} - p_\mu - 1)} - x_0 x_1 x_{k+1}^{p_{\mu+1} - 1} \in I \quad (38)$$

but  $x_2 x_k^{\sigma_\mu} - x_0^{r'_\mu - h} x_1 x_{k+1}^{p_\mu} \in I$  so

$$x_0^{r'_\mu - h} x_1 x_{k+1}^{p_{\mu+1} - p_\mu - 1} - x_0 x_1 \in I \quad (39)$$

this is possible only if  $r'_\mu - h = 1, p_{\mu+1} = p_\mu + 1$ . We have  $h + 1 = r'_\mu = r_\mu + h(\sigma + 1)$  so  $r_\mu = 1 - h\sigma, r_{\mu+1} \leq -h\sigma$ , so  $r'_{\mu+1} = r_{\mu+1} + h \leq -h(\sigma - 1)$ , so  $r'_{\mu+1} < 0$  if and only if  $\sigma \geq 2$ . We set  $\sigma := \sigma_\mu, p := p_\mu$  so we have

$$\widetilde{PSF}_1 = \{x_1 x_k^{\sigma-1} x_{k+1}^p\}, \widetilde{PSF}_2 = \{x_2 x_k^{\sigma-1}, \dots, x_k^\sigma\}, t(S) = k.$$

$$\begin{array}{ccc|c} s & p & r & r' \\ \sigma k + 1 & p & 1 - h\sigma & h + 1 \\ k - 1 & p + 1 & r_{\mu+1} & < 0 \end{array}$$

$$\begin{aligned} a &= (\sigma k + 1)(p + 1) - (k - 1)p \\ d &= (p + 1)(1 - h\sigma) - pr_{\mu+1} \\ c &= (k - 1)(1 - h\sigma) - (\sigma k + 1)r_{\mu+1}, \end{aligned}$$

with  $h \geq 1, p \geq 1, \sigma \geq 2, r_{\mu+1} \leq -h\sigma$ .

- (ii) Suppose  $\sigma_{\mu+1} + 2 - 2\delta + \varepsilon > 0$ . We have  $x_k^{\sigma_{\mu+1}} - x_0^{r'_\mu - h} x_{k-\rho_\mu} x_{k+1}^{p_\mu} \in I$ . From (30) we get

$$x_0^{r'_\mu - h} x_{\Gamma+\Delta-k} x_{k-\rho_\mu} x_k^{\sigma_{\mu+1}+1-2\delta+\varepsilon} x_{k+1}^{p_{\mu+1}-p_\mu-1} - x_0 x_1 \in I, \quad (40)$$

since  $\Gamma + \Delta - k \geq 1, k - \rho_\mu \geq 1$  we have  $\varphi(x_{\Gamma+\Delta-k} x_{k-\rho_\mu}) > \varphi(x_0 x_1)$  which is impossible.

□

8. FORMULA FOR FROBENIUS NUMBER OF ALMOST SYMMETRIC ALMOST  
GENERALIZED ARITHMETIC PROGRESSIONS

This section extends and generalizes all the results of [13].

**Theorem 8.1.** *Let  $S$  be an AAG almost symmetric with  $k \geq 3, t(S) \geq 2$ . Then there is a quadratic formula for the Frobenius number in terms of  $a, d, c, k$  and the type  $t(S)$ .*

*Proof.* We have to consider two cases depending on the number  $i$  such that  $\widetilde{Frob}(S \in (\widetilde{PSF}_i))$ .

I) Suppose  $\widetilde{Frob}(S \in (\widetilde{PSF}_2))$ . Then  $S$  is almost symmetric with  $k \geq 3, t(S) \geq 2$  if and only if either

- (1)  $\widetilde{PSF}_1 = \{x_i x_{k+1}^{p_{\mu+1}-1} \mid i = 1, \dots, k-l\}, \widetilde{PSF}_2 = \{x_1 x_k^\sigma x_{k+1}^{p_{\mu+1}-2}\}$ .  $t(S) = k-l+1$ .  
We have

$$a = (\sigma k + 2)p_{\mu+1} - ((\sigma - 1)k + l)$$

$$d = p_{\mu+1}r_\mu + \sigma$$

$$c = ((\sigma - 1)k + l)r_\mu + (\sigma k + 2)\sigma.$$

where  $h = 1, k \geq 3, 1 \leq l \leq k-1, \sigma \geq 2$ .

Since  $x_1 x_{k-l} x_{k+1}^{2(p_{\mu+1}-1)} - x_0 \widetilde{Frob}(S) \in I$  we have  $x_{k-l+1} x_{k+1}^{2(p_{\mu+1}-1)} - \widetilde{Frob}(S) \in I$  and  $F(S) = 2(p_{\mu+1} - 1)c + a_{k-l+1} - a$  so determine a formula for  $F(S)$  consist to determine  $p_{\mu+1}$  in terms of  $a, d, c, k, l$ .

we set  $X = p_{\mu+1}$  we have  $x_{k+1}^X - x_l x_k^{\sigma-1}$  so

$$Xc = a_l + \sigma a_k - a_k \tag{41}$$

and

$$\sigma k(X - 1) = a - 2X - k + l \tag{42}$$

We multiply (41) by  $k(X - 1)$  and by using (42) we get

$$k(X - 1)Xc = k(X - 1)a_l + a_k(a - 2X - k + l) - k(X - 1)a_k \tag{43}$$

So we get a second order equation in the variable  $X$

$$kcX^2 - (k(c + a_l - a_k) - 2a_k)X - a_k(a + l) + ka_l = 0 \tag{44}$$

so  $X = \frac{k(c + a_l - a_k) - 2a_k + \sqrt{(k(c + a_l - a_k) - 2a_k)^2 - 4kc(-a_k(a + l) + ka_l)}}{2kc}$

- (2)  $\widetilde{PSF}_1 = \{x_{k+1}^{p_{\mu+1}-1}\}, \widetilde{PSF}_2 = \{x_1 x_k^\sigma x_{k+1}^{p_{\mu+1}-2}\}, t(S) = 2$ . We have

$$a = (\sigma k + 2)p_{\mu+1} - (\sigma k + 1)$$

$$d = p_{\mu+1}r_\mu + h(\sigma + 1) + 1$$

$$c = (\sigma k + 1)r_\mu + (\sigma k + 2)(h(\sigma + 1) + 1).$$

where  $h \geq 1, \sigma \geq 1, p_{\mu+1} \geq 2, r_{\mu} > -h(\sigma+1) - 1$ . Since  $x_{k+1}^{2(p_{\mu+1}-1)} - x_0 \widetilde{Frob}(S) \in I$  we have and  $F(S) = 2(p_{\mu+1} - 1)c - 2a$  so determine a formula for  $F(S)$  consist to determine  $p_{\mu+1}$  in terms of  $a, d, c, k$ .

We set  $X = p_{\mu+1}$  we have  $x_{k+1}^X - x_0 x_1 x_k^\sigma$  so

$$Xc = a + a_1 + \sigma a_k \quad (45)$$

and

$$\sigma k(X - 1) = a + 1 - 2X \quad (46)$$

We multiply (45) by  $k(X - 1)$  and by using (46) we get

$$k(X - 1)Xc = k(X - 1)(a + a_1) + a_k(a + 1 - 2X) \quad (47)$$

So we get a second order equation in the variable  $X$

$$kcX^2 - (k(c + a + a_1) - 2a_k)X - a_k(a + 1) + k(a + a_1) = 0 \quad (48)$$

$$\text{so } X = \frac{k(c + a + a_1) - 2a_k + \sqrt{(k(c + a + a_1) - 2a_k)^2 - 4kc(-a_k(a + 1) + k(a + a_1))}}{2kc}$$

II) Suppose  $\widetilde{Frob}(S) \in (\widetilde{PSF}_1)$ . Then  $S$  is almost symmetric with  $k \geq 3, t(S) \geq 2$  if and only if either

(1)  $\widetilde{PSF}_1 = \{x_1 x_k^\sigma x_{k+1}^p\}, \widetilde{PSF}_2 = \{x_2 x_k^\sigma, \dots, x_{p_{\mu}-1} x_k^\sigma\}, t(S) = l + 1$ . We have

$$\begin{aligned} a &= (\sigma k + l + 2)(p + 1) - lp \\ d &= -(p + 1)\sigma - pr_{\mu+1} \\ c &= -l\sigma - (\sigma k + l + 2)r_{\mu+1}, \end{aligned}$$

where  $h = 1, k \geq 3, \sigma \geq 1, 1 \leq l \leq k - 3, r_{\mu+1} < -\sigma$ .

Since  $x_2 x_{l+1} x_k^{2\sigma} - x_0 \widetilde{Frob}(S) \in I$  we have  $x_{l+3} x_k^{2\sigma} - \widetilde{Frob}(S) \in I$  and  $F(S) = a_{l+3} + 2\sigma a_k - a$  but  $x_{l+2} x_k^\sigma - x_0 x_{k+1}^{p_{\mu+1}-1} \in I$ . We set  $X = p_{\mu+1}$  so

$$a_{l+2} + \sigma a_k = a + c(X - 1) \quad (49)$$

and  $F(S) = a_{l+3} + 2(a + c(X - 1) - a_{l+2}) - a$ . Determine a formula for  $F(S)$  consist to determine  $X$  in terms of  $a, d, c, k, l$ .

By developing the formula for  $a$  we have

$$\sigma kX = a - 2X - l \quad (50)$$

We multiply (49) by  $kX$  and by using (50) we get

$$ka_{l+2}X + k\sigma a_k X = kaX + kcX(X - 1) \quad (51)$$

So we get a second order equation in the variable  $X$

$$kcX^2 - (k(c - a + a_{l+2}) - 2a_k)X - a_k(a - l) = 0 \quad (52)$$

$$\text{so } X = \frac{k(c - a + a_{l+2}) - 2a_k + \sqrt{(k(c - a + a_{l+2}) - 2a_k)^2 + 4kca_k(a - l)}}{2kc}$$

(2)  $\widetilde{PSF}_1 = \{x_1x_k^{\sigma-1}x_{k+1}^p\}$ ,  $\widetilde{PSF}_2 = \{x_2x_k^{\sigma-1}, \dots, x_{k-1}x_k^{\sigma-1}\}$ ,  $t(S) = k - 1$ . We have

$$\begin{aligned} a &= (\sigma k)(p+1) - (k-2)p \\ d &= (1-h\sigma)(p+1) - pr_{\mu+1} \\ c &= (k-2)(1-\sigma) - \sigma kr_{\mu+1}. \end{aligned}$$

where  $h = 1, k \geq 3, p, \sigma \geq 2, r_{\mu+1} < -1$ .

Since our semigroup is almost symmetric we have  $x_2x_{k-1}x_k^{2\sigma-2} - x_0\widetilde{Frob}(S) \in I$  that is  $x_1x_k^{2\sigma-1} - x_0\widetilde{Frob}(S) \in I$  so  $F(S) = a_1 + 2\sigma a_k - a_k - 2a$ . But  $x_k^\sigma - x_0x_{k+1}^{p\mu+1-1} \in I$ , we set  $X = p_{\mu+1}$  we have

$$\sigma a_k = a + c(X - 1) \quad (53)$$

and  $F(S) = a_1 + 2(a + c(X - 1)) - a_k - 2a$ . Determine a formula for  $F(S)$  consist to determine  $X$  in terms of  $a, d, c, k$ .

By developing the formula for  $a$  we have

$$\sigma kX = a + kX - 2X - k - 2 \quad (54)$$

We multiply (53) by  $kX$  and by using (54) we get

$$a_k(a + kX - 2X - k - 2) = kaX + kcX(X - 1) \quad (55)$$

So we get a second order equation in the variable  $X$

$$kcX^2 - (k(c - a + a_k) - 2a_k)X - a_k(a - k + 2) = 0 \quad (56)$$

and  $X = \frac{k(c - a + a_k) - 2a_k + \sqrt{(k(c - a + a_k) - 2a_k)^2 + 4kca_k(a - k + 2)}}{2kc}$ .

(3)  $\widetilde{PSF}_1 = \{x_1x_k^{\sigma-1}x_{k+1}^p\}$ ,  $\widetilde{PSF}_2 = \{x_2x_k^{\sigma-1}, \dots, x_k^\sigma\}$ ,  $t(S) = k$ . We have

$$\begin{aligned} a &= (\sigma k + 1)(p+1) - (k-1)p \\ d &= (p+1)(1-h\sigma) - pr_{\mu+1} \\ c &= (k-1)(1-h\sigma) - (\sigma k + 1)r_{\mu+1}. \end{aligned}$$

$h \geq 1, k \geq 3, p, \sigma \geq 1, r_{\mu+1} < \min\{-h, 1 - h\sigma\}$ .

Since  $x_2x_k^{2\sigma-1} - x_0\widetilde{Frob}(S) \in I$  we have  $F(S) = a_2 + 2\sigma a_k - a_k - 2a$  but  $x_1x_k^\sigma - x_0^{h+1}x_{k+1}^{p\mu+1-1} \in I$ , we set  $X = p_{\mu+1}$ , so we have

$$a_1 + \sigma a_k = (h+1)a + c(X - 1) \quad (57)$$

and  $F(S) = a_2 + 2(a + c(X - 1) - a_1) - a_k - 2a$ . Determine a formula for  $F(S)$  consist to determine  $X$  in terms of  $a, d, c, k$ .

By developing the formula for  $a$  we have

$$\sigma kX = a + X(k-2) - k + 1 \quad (58)$$

We multiply (57) by  $kX$  and by using (58) we get

$$ka_1X + a_k(a + X(k-2) - k + 1) = k(h+1)aX + kcX(X - 1) \quad (59)$$

So we get a second order equation in the variable  $X$

$$kcX^2 - (k(a_1 + c - (h + 1)a + a_k) - 2a_k)X - a_k(a - k + 1) = 0 \quad (60)$$

$$X = \frac{k(a_1 + c - (h + 1)a + a_k) - 2a_k + \sqrt{(k(a_1 + c - (h + 1)a + a_k) - 2a_k)^2 + 4kca_k(a - k + 1)}}{2kc}.$$

(4)  $\widetilde{PSF}_1 = \{x_{k+1}^p\}$ ,  $\widetilde{PSF}_2 = \{x_1, \dots, x_k\}$ ,  $t(S) = k + 1$ . We have

$$\begin{aligned} a &= k + p + 1 \\ d &= -(p + 1) - pr_{\mu+1} \\ c &= -k - (k + 1)r_{\mu+1}. \end{aligned}$$

with  $h = 1, k \geq 3, p \geq 1, r_{\mu+1} < -1$ . We have  $F(S) = pc - a$  and  $a = k + p + 1$  so  $F(S) = c(a - k - 1) - a$ .

(5)  $\widetilde{PSF}_1 = \{x_1x_k^{\sigma-2}x_{k+1}^p\}$ ,  $\widetilde{PSF}_2 = \{x_1x_k^{\sigma-1}\}$ ,  $t(S) = 2$ . We have

$$\begin{aligned} a &= (\sigma k + 1)(p + 1) - (2k - 1)p \\ d &= -(p + 1)\sigma - pr_{\mu+1} \\ c &= -\sigma(2k - 1) - (\sigma k + 1)r_{\mu+1}. \end{aligned}$$

with  $h = 1, k \geq 3, \sigma \geq 2, p \geq 1, r_{\mu+1} < -\sigma$ .

Since  $x_1^2x_k^{2\sigma-2} - x_0\widetilde{Frob}(S) \in I$  we have  $x_2x_k^{2\sigma-2} - \widetilde{Frob}(S) \in I$  so  $F(S) = a_2 + 2\sigma a_k - 2a_k - a$  but  $x_1x_k^\sigma - x_0x_{k+1}^p \in I$ , which gives

$$a_1 + \sigma a_k = a + c(X - 1), \quad (61)$$

where we have set  $X = p + 1$ . Hence  $F(S) = a_2 + 2(a + c(X - 1) - a_1) - 2a_k - a$ . Determine a formula for  $F(S)$  consist to determine  $X$  in terms of  $a, d, c, k$ . By developing the formula for  $a$  we have

$$\sigma kX = a + X(2k - 2) - 2k + 1 \quad (62)$$

We multiply (61) by  $kX$  and by using (62) we get

$$ka_1X + a_k(a + X(2k - 2) - 2k + 1) = kaX + kcX(X - 1) \quad (63)$$

So we get a second order equation in the variable  $X$

$$kcX^2 - (k(c + d + 2a_k) - 2a_k)X - a_k(a - 2k + 1) = 0 \quad (64)$$

$$X = \frac{k(c + d + 2a_k) - 2a_k + \sqrt{(k(c + d + 2a_k) - 2a_k)^2 + 4kca_k(a - 2k + 1)}}{2kc}$$

□

**Corollary 8.2.** *Given a AAG-semigroup  $S$  with data  $a, d, c, h, k$  by at most  $4k$  tests solving quadratics equations we can determine if  $S$  is almost symmetric.*



*Proof.* The first step is to check if one of the square roots is a natural number, since we don't know  $t(S)$  we have to perform  $k$  times in the case where the number  $1 \leq l \leq k$  appears in this square root. The second step is to check if the solution  $X$  as above is a natural number, at this step we know the probably value for  $t(S)$  so also the probably case to consider. The third step is to solve a linear system to find the values of  $\sigma, p_{\mu+1}, r_{\mu+1}$  from  $a, d, c, h, k$  and check if they are natural numbers and satisfy the conditions of the considered case.  $\square$

**Example 8.3.** *We have implemented the above algorithm and we have for  $150 \leq a \leq 160, 1 \leq d \leq 10, 170 \leq c \leq 180, 19 \leq k \leq 20, 2 \leq h \leq 3$  the following values for which the AAG-semigroup is almost symmetric:*

$a = 153, d = 11, c = 177, k = 19, h = 3$ , case II.1,  $p_{\mu+1} = 7, \sigma = 1, r_{\mu+1} = -3$ .

$a = 156, d = 11, c = 174, k = 20, h = 3$ , case II.1,  $p_{\mu+1} = 7, \sigma = 1, r_{\mu+1} = -3$ .

$a = 155, d = 1, c = 177, k = 20, h = 4$ , case I.2,  $p_{\mu+1} = 8, \sigma = 1, r_{\mu} = -1$ .

$a = 152, d = 3, c = 170, k = 21, h = 2$ , case I.1,  $p_{\mu+1} = 4, \sigma = 2, r_{\mu} = 0$ .

$a = 150, d = 4, c = 178, k = 21, h = 3$ , case II.1,  $p_{\mu+1} = 6, \sigma = 1, r_{\mu+1} = -2$ .

## REFERENCES

- [1] V. BARUCCI, R. FROBERG, *One-Dimensional Almost Gorenstein Rings*, J. of Algebra **188** (1997), 418–442.
- [2] N. T. DUNG, *On the type and generators of monomial curves*, Turkish Journal of Mathematics, **42** (2018), 2112-2124.
- [3] IGNACIO GARCÍA-MARCO, JORGE L. RAMÍREZ ALFONSÍN, ØYSTEIN J. RØDSETH, *Numerical semigroups II: Pseudo-symmetric AA-semigroups*, Journal of Algebra, **470** (2017), 484-498.
- [4] M. MORALES, *Fonctions de Hilbert, genre géométrique d'une singularité quasi-homogène Cohen-Macaulay*. CRAS Paris, t.301, série A n° 14 (1985).
- [5] M. MORALES, *Syzygies of monomial curves and a linear diophantine problem of Frobenius*, Preprint. Max Planck Institut für Mathematik (Bonn-RFA) (1987).
- [6] M. MORALES, *Equations des variétés monomiales en codimension deux*, J. Algebra **175** (1995), 1082-1095.
- [7] M. MORALES, *Software Frobenius number-Groebner basis* download at <https://www-fourier.ujf-grenoble.fr/~morales/>.
- [8] MARCEL MORALES, NGUYEN THI DUNG, *A "pseudo-polynomial" algorithm for the Frobenius number and Gröbner basis*, Preprint.
- [9] DILIP P. PATIL, *Generators for the derivation modules and the defining ideals of certain affine curves*, Thesis, TIFR-Bombay University, (1989).
- [10] DILIP P. PATIL, *Minimal sets of generators for the relation ideals of certain monomial curves*, Manuscripta Math. **80** (1993), 239- 248 .
- [11] J. L. RAMÍREZ ALFONSÍN, AND O. J. RODSETH, *Numerical Semigroups, Apéry set and Hilbert series*, Semigroup Forum, **79(2)** (2009), 323-340.
- [12] J. RODSETH, *On a linear diophantine problem of Frobenius II*, J. Reine Angew. Math. **307/308** (1979), 431-440.
- [13] J.C. ROSALES, AND P.A. GARCÍA-SÁNCHEZ, *Pseudo-symmetric numerical semigroups with three generators* (2005). Published by Elsevier Inc. DOI:10.1016/j.jalgebra.2005.06.005.

UNIVERSITÉ GRENOBLE ALPES, INSTITUT FOURIER, UMR 5582, LABORATOIRE DE MATHÉMATIQUES, B.P.74, 38402 SAINT-MARTIN D'HÈRES CEDEX, FRANCE

*Email address:* marcel.morales@univ-grenoble-alpes.fr