Fluid models for kinetic equations in swarming preserving momentum

Mihaï Bostan, Anh-Tuan Vu

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Mihaï BOSTAN [*], Anh-Tuan VU [†]

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Abstract

We study kinetic models for swarming. The interaction between individuals is given by self-propelling and friction forces, alignment and noise. We consider that each individual relaxes its velocity toward some average velocity, such that the total momentum does not change. We concentrate on fluid models obtained when the time and space scales become very large. We derive first and second order approximations.

Keywords: Swarming, Alignment, Asymptotic analysis.

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1 Introduction

The subject matter of this paper concerns the models for individuals driven by self-propelled forces, with alignment and noise. These models give a mathematical description for self-organizing systems like schools of fish, flocks of birds, swarms of insects, etc. Discrete models where averaging is performed in direction or velocity exist [27, 28]. When the number of agents grows, models based on particle density functions are preferable, in order to avoid prohibitive calculations [20, 25, 26]. The kinetic theory for the description of the collective behavior of large groups of individuals is well developed [41, 17, 32]. The well-posedness theory was established for models including short-range repulsion and long-range attraction, self-propulsion, velocity averaging, that is, each individual adapts its own velocity with respect to that of the neighbors. We refer to [18] for results based on mass transportation distance, [6] for the mean-field limit with locally Lipschitz interactions, [7] for the mean-field limit of Vicsek models. A summary on the mean-field limit for applications in swarming models was presented in [23], see also [24] for a rigorous derivation of the mean-field limit for systems of particles interacting through local sensitivity regions.

In most of the models, the alignment between particles comes by relaxing the individual velocities toward the mean velocity through a pairwise particle interaction kernel depending on the distance between particles [21, 22, 28, 36, 40]. For slowly decaying interaction potentials, it is shown that flocking emerges, that is, the distance between agents remains uniformly

[*]Aix Marseille Université, CNRS, Centrale Marseille, Institut de Mathématiques de Marseille, UMR 7373, Château Gombert 39 rue F. Joliot Curie, 13453 Marseille FRANCE. E-mail : mihai.bostan@univ-amu.fr

[†]Aix Marseille Université, CNRS, Centrale Marseille, Institut de Mathématiques de Marseille, UMR 7373, Château Gombert 39 rue F. Joliot Curie, 13453 Marseille FRANCE. E-mail : anh-tuan.vu@univ-amu.fr

bounded in time and the agent velocities are asymptotically close to the mean velocity. A simple proof of the Cucker-Smale flocking dynamics was obtained in [34] using the Lyapunov functional approach.

The Cucker-Smale model is symmetric: any two agents have mutually the same influence on the alignment. This symmetry is one of the key points for studying the long time behavior, because symmetry implies the conservation of the total momentum and also that the Cucker-Smale model is dissipative. A new framework for analyzing the phenomenon of flocking is presented in [31], which allows for non-symmetric pairwise influence between agents and also the treatment of models with leaders. One way to take into account the interaction between the agents and the environment is to add noise terms [35].

We denote by $f = f(t,x,v) \geq 0$ the particle density in the phase space $(x,v) \in \mathbb{R}^d \times \mathbb{R}^d$, with $d \geq 2$. The self-propulsion and friction mechanism are encoded by the term $\mathrm{div}_v\{f\nabla_v V(|\cdot|)\}$, for a given confining potential $v \to V(|v|)$. We refer to [14] for the potentials $V_{\alpha,\beta}(|v|) = \beta\frac{|v|^4}{4} - \alpha\frac{|v|^2}{2}$, with $\alpha,\beta > 0$. When the coefficients α,β are scaled in such a way that α/β is constant and α and β are large, we obtain measure solutions whose support lies on the sphere of radius $r = \sqrt{\alpha/\beta}$. Macroscopic models resulting from the balance between the reorientation, which tends to align the particle velocities with respect to the mean velocity, and diffusion, which tends to spread the particle velocities isotropically on the sphere $r\mathbb{S}^{d-1}$ were discussed in [15].

The relaxation toward the velocity $u[f]$ is described by the term $\mathrm{div}_v\{f(v-u[f])\}$. Adding the term $\mathrm{div}_v\{\sigma\nabla_v f\}$, which represents the noise with respect to the velocity variable, we obtain the Fokker-Planck type equation

$$\partial_t f + v\cdot\nabla_x f = Q(f) := \mathrm{div}_v\{\sigma\nabla_v f + f(v-u[f]) + f\nabla_v V(|\cdot|)\},\quad (t,x,v)\in\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}^d. \quad (1)$$?Equ1-Swarm?

Many fluid limits, corresponding to various asymptotic regimes are obtained by formal arguments. There are also rigorous derivations, based on the relative entropy method, which provides also a rate of convergence, see [19] for the large friction and high force field limit, [37] for the Cucker-Smale equation with strong noise and strong local alignment. In many models [31], we assume that the velocity $u[f]$ is the standard mean velocity $\frac{\int_{\mathbb{R}^d} f(v)v\,\mathrm{d}v}{\int_{\mathbb{R}^d} f(v)\,\mathrm{d}v}$. In that case, the equilibria of the interaction mechanism Q are von Mises-Fisher distributions, parametrized by the concentration $\rho[f] = \int_{\mathbb{R}^d} f\,\mathrm{d}v$ and the orientation of the mean velocity $\Omega[f] = \frac{u[f]}{|u[f]|}, u[f] \neq 0$. More exactly, for any $u \in \mathbb{R}^d$ we introduce the notations

$$\Phi_u(v) = \frac{|v-u|^2}{2} + V(|v|),\quad Z(\sigma,u) = \int_{\mathbb{R}^d}\exp\left(-\frac{\Phi_u(v)}{\sigma}\right)\,\mathrm{d}v,\quad M_u(v) = \frac{\exp\left(-\frac{\Phi_u(v)}{\sigma}\right)}{Z(\sigma,u)}. \quad (2)$$?Equ2-4-Swarmi

The function Z depends only on σ and $|u|$, see Proposition 2.1, and thus we will write $Z = Z(\sigma, l = |u|)$. For any smooth particle density f and any $u \in \mathbb{R}^d$ we have

$$\sigma\nabla_v f + f(v-u[f]) + f\nabla_v V(|\cdot|) = \sigma M_u(v)\nabla_v\left(\frac{f}{M_u}\right) \quad (3)$$?Equ5-Swarming

and therefore the interaction mechanism writes

$$Q(f) = \sigma\mathrm{div}_v\left(M_{u[f]}\nabla_v\left(\frac{f}{M_{u[f]}}\right)\right).$$

As usual, multiplying by $f/M_{u[f]}$ and integrating by parts with respect to the velocity imply that any equilibrium satisfies

$$f = \rho[f]M_{u[f]},\quad \rho[f] = \int_{\mathbb{R}^d} f(v)\,\mathrm{d}v$$

but we also need to check that $u[\rho[f]M_{u[f]}] = u[f]$. When $u[f]$ is $\frac{\int_{\mathbb{R}^d} f(v)v \, \mathrm{d}v}{f(v)}$, we have to impose

$$\int_{\mathbb{R}^d} f(t,x,v)(v - u[f(t,x,\cdot)]) \, \mathrm{d}v = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$$

and after some computations, see Proposition 2.1, we deduce that the distribution ρM_u is an equilibrium iff $l = |u|$ is a critical point of $Z(\sigma, \cdot)$, that is $\partial_l Z(\sigma, |u|) = 0$. Notice that for any $\sigma > 0$, the distribution

$$\rho M_0(v) = \frac{\rho}{Z(\sigma,0)} \exp\left(-\frac{\Phi_0(v)}{\sigma}\right) = \frac{\rho}{Z(\sigma,0)} \exp\left(-\frac{\frac{|v|^2}{2} + V(|v|)}{\sigma}\right)$$

is an equilibrium of Q, with vanishing mean velocity. But for small enough values $0 < \sigma < \sigma_0$, equilibria with non vanishing mean velocity occur cf. [16, 29, 30, 33], leading to a phase transition at some critical value $\sigma_0 > 0$; several values for $|u|$, or only the trivial one, are admissible, depending on the diffusion coefficient σ being smaller or bigger than the critical value σ_0 . The fluid models corresponding to these values $\sigma \in]0, \sigma_0[$ are derived in [16]. The modulus $l = |u|$ is determined by $\partial_l Z(\sigma, l) = 0$, the concentration $\rho \in \mathbb{R}_+$ and the orientation $\Omega \in \mathbb{S}^{d-1} = \left\{\xi \in \mathbb{R}^d : |\xi| = 1\right\}$ verify a system of d conservation laws (the kernel Q satisfies the mass balance but not the momentum balance).

The fluid models come from the kinetic equations by appealing to the conservations of the mass, momentum etc. We are looking for collision invariants [16, 4, 5, 38, 39], since any collision invariant leads to a conservation law for the macroscopic quantities parametrizing the equilibria of Q. We refer also to [31] for the notion of generalized collision invariants and to [1] Theorem 1.1 for the relation between collision invariants and generalized collision invariants.

When considering the Cucker-Smale model, we combine noise, alignment and self-propulsion force. In [2] the authors investigate first the hydrodynamic limit, when considering strong diffusion and alignment, whereas the self-propulsion force is weaker. Secondly, they study the effect of a large self-propulsion force in the hydrodynamic model : a phase transition appears around some critical temperature. Diffusion corrections were computed as well. Performing the hydrodynamic and large self-propulsion force limit simultaneously, at the level of the kinetic equation is another interesting issue [3].

For the model with mean velocity, $0 < \sigma < \sigma_0$, we obtain a system of d conservation laws (the kernel Q satisfies the mass balance but not the momentum balance) for the concentration $\rho \in \mathbb{R}_+$ and the orientation $\Omega \in \mathbb{S}^{d-1} = \{\xi \in \mathbb{R}^d : |\xi| = 1\}$, whereas the modulus of the mean velocity $l = |u|$ is determined by $\partial_l Z(\sigma, l) = 0$.

In the present work we combine noise, alignment and self-propulsion force, making the assumption that the individuals adapt their velocity by talking into account the neighbors velocities, but also the self-propulsion force of the individuals, such that the momentum conservation hold true $\int_{\mathbb{R}^d} Q(f)(v)v \, \mathrm{d}v = 0$. In that case the velocity u is determined by $\int_{\mathbb{R}^d} f \nabla_v \Phi_u \, \mathrm{d}v = 0$ leading to

$$u^V[f] = \frac{\int_{\mathbb{R}^d} f(v + \nabla_v V(|\cdot|)) \, \mathrm{d}v}{\int_{\mathbb{R}^d} f \, \mathrm{d}v} = \frac{\int_{\mathbb{R}^d} f \nabla_v \Phi_0 \, \mathrm{d}v}{\int_{\mathbb{R}^d} f \, \mathrm{d}v}.$$

It is a generalization of the Fokker-Planck kernel (if $V = 0$ we obtain the usual velocity $u^0[f] = \frac{\int_{\mathbb{R}^d} f \, v \, \mathrm{d}v}{\int_{\mathbb{R}^d} f \, \mathrm{d}v}$, with mass and momentum conservations, as for the Cucker-Smale model. When the alignment velocity is $\frac{\int_{\mathbb{R}^d} f \nabla_v \Phi_0 \, \mathrm{d}v}{\int_{\mathbb{R}^d} f \, \mathrm{d}v}$, any $f = \rho M_u$ ia an equilibrium. Indeed, integrating $\sigma \nabla_v f + f \nabla_v \Phi_u = 0$ with respect to $v \in \mathbb{R}^d$, we obtain $\int_{\mathbb{R}^d} f \nabla_v \Phi_u \, \mathrm{d}v = 0$ leading to $u^V[\rho M_u] =$

u. From now on, the notation $u[f]$ will stand for the expression $\frac{\int_{\mathbb{R}^d} f\nabla_v\Phi_0\ \mathrm{d}v}{\int_{\mathbb{R}^d} f\ \mathrm{d}v}$. When considering large time and space units in (1), we are led to the kinetic equation

$$\partial_t f^\varepsilon + v\cdot\nabla_x f^\varepsilon = \frac{1}{\varepsilon}Q(f^\varepsilon),\ \ (t,x,v)\in\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}^d \tag{4}$$?Equ3-SwarmScal

where, for simplicity, we still denote by (t,x) the new time and space coordinates. At least formally, the asymptotic behavior when $\varepsilon\searrow 0$ comes by the mass and momentum balances, leading to the fluid model for the macroscopic quantities ρ, u which parametrize the equilibrium $\lim_{\varepsilon\searrow 0} f^\varepsilon = f = \rho M_u$. There is no phase transition around some critical value for the diffusion.

Theorem 1.1
Assume that $\lim_{|v|\to+\infty}\frac{\Phi_0(v)}{|v|} = +\infty$, *with* $\Phi_{u=0}$ *defined in* (2) *and consider the function* Z *introduced in* (2), *which depends only on* σ *and* $l = |u|$ *cf. Proposition 2.1. Then, at any* $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$ *the limit* $\lim_{\varepsilon\searrow 0} f^\varepsilon(t,x,\cdot) = f(t,x,\cdot)$ *is a equilibrium of* Q *i.e.,* $f(t,x,v) = \rho(t,x)M_{u(t,x)}(v), (t,x,v)\in\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}^d$ *and*

$$\partial_t\rho + \mathrm{div}_x\left[\rho\left(1+\sigma\frac{\partial_l Z}{|u|Z(\sigma,|u|)}\right)u\right] = 0 \tag{5}$$?EquMassBal?

$$\partial_t\left[\rho\left(1+\sigma\frac{\partial_l Z}{|u|Z(\sigma,|u|)}\right)u\right] + \mathrm{div}_x\left[\rho\left(1+2\sigma\frac{\partial_l Z}{|u|Z}\right)u\otimes u + \rho\sigma^2\frac{\partial^2_{ll}Z - \frac{\partial_l Z}{|u|}}{Z}\Omega[u]\otimes\Omega[u]\right]$$
$$+\sigma\nabla_x\left[\rho\left(1+\sigma\frac{\partial_l Z}{|u|Z(\sigma,|u|)}\right)\right] = 0. \tag{6}$$?EquMomBal?

We are looking for a simplified version of the above fluid model. This can be achieved by writing the above equations with respect to the concentration ρ and the current $j(t,x) = \int_{\mathbb{R}^d} f(t,x,v)v\ \mathrm{d}v$. For any $\sigma > 0$ we introduce the strictly convex function with respect to l given by $\mathcal{E}(\sigma,l) = \frac{l^2}{2} + \sigma\ln Z(\sigma,l), l\in\mathbb{R}_+$, and its convex conjugate function $k\to\mathcal{E}^\star(\sigma,k), k\in\mathbb{R}_+$.

Theorem 1.2
Assume that $\lim_{|v|\to+\infty}\frac{\Phi_0(v)}{|v|} = +\infty$, *with* $\Phi_{u=0}$ *defined in* (2). *Then*

$$\frac{j}{\rho} = \left(1+\sigma\frac{\partial_l Z}{|u|Z(\sigma,|u|)}\right)u$$

and (5), (6) *write*

$$\partial_t\rho + \mathrm{div}_x j = 0$$
$$\partial_t j + \mathrm{div}_x\left[\frac{j\otimes j}{\rho} + \sigma\frac{|j|}{\partial_k\mathcal{E}^\star(\sigma,|j|/\rho)}I_d + \sigma\left(\frac{1}{k^2\partial^2_{kk}\mathcal{E}^\star} - \frac{1}{k\partial_k\mathcal{E}^\star}\right)_{k=|j|/\rho}\frac{j\otimes j}{\rho}\right] = 0.$$

Moreover, we have the entropy inequality

$$\partial_t\{\rho(\sigma\ln\rho + \mathcal{E}^\star(\sigma,|j|/\rho)\} + \mathrm{div}_x\left\{\left[\sigma\ln\rho + \left(\mathcal{E}^\star(\sigma,k) + \sigma\frac{\partial_k\mathcal{E}^\star}{k\partial^2_{kk}\mathcal{E}^\star}\right)_{k=|j|/\rho}\right]j\right\}\le 0.$$

We investigate also the second order aprroximation of (4). For doing that we need to invert the linearization $\mathcal{L}_f$ of the collision operator Q on $(ker\mathcal{L}_f)^\perp$. We need a Poincaré inequality, see Lemma 3.1. Taking into account the first order corrections leads to the following second order approximation.

Theorem 1.3
Assume that $\lim_{|v|\to+\infty}\frac{\Phi_0(v)}{|v|}=+\infty$, *with* $\Phi_{u=0}$ *defined in* (2), *that for any* $u\in\mathbb{R}^d$, *the function* $v\mapsto\frac{1}{4\sigma}|\nabla_v\Phi_u|^2-\frac{1}{2}\Delta_v\Phi_u$ *belongs to* $L^1_{loc}(\mathbb{R}^d)$, *is bounded from below and is coercive i.e.*

$$\lim_{|v|\to+\infty}\left[\frac{1}{4\sigma}|\nabla_v\Phi_u|^2-\frac{1}{2}\Delta_v\Phi_u\right]=+\infty.$$

Consider the family $(f^\varepsilon)_\varepsilon$ *of solutions for* (4). *Then a second order approximation* $(\tilde\rho^\varepsilon,\tilde j^\varepsilon)$ *for* $(\rho^\varepsilon=\int_{\mathbb{R}^3}f^\varepsilon\,\mathrm{d}v, j^\varepsilon=\int_{\mathbb{R}^3}f^\varepsilon v\,\mathrm{d}v)$ *is given by*

$$\partial_t\tilde\rho^\varepsilon+\mathrm{div}_x\tilde j^\varepsilon=0$$

$$\partial_t\tilde j^\varepsilon+\mathrm{div}_x\left(\frac{\tilde j^\varepsilon\otimes\tilde j^\varepsilon}{\tilde\rho^\varepsilon}+\sigma\tilde\rho^\varepsilon\mathcal{M}_{\tilde u^\varepsilon}+\varepsilon\tilde T_2^\varepsilon+\varepsilon\tilde T_3^\varepsilon+\varepsilon\tilde T_4^\varepsilon\right)=0$$

where

$$\mathcal{M}_{\tilde u^\varepsilon}=\frac{\partial_l\mathcal{E}(\sigma,|\tilde u^\varepsilon|)}{|\tilde u^\varepsilon|}(I_d-\tilde\Omega^\varepsilon\otimes\tilde\Omega^\varepsilon)+\partial^2_{ll}\mathcal{E}(\sigma,|\tilde u^\varepsilon|)\tilde\Omega^\varepsilon\otimes\tilde\Omega^\varepsilon$$

$$\tilde T_2^\varepsilon=\frac{\tilde\rho^\varepsilon}{\sigma}c_1(\sigma,\tilde u^\varepsilon)(I_d-\tilde\Omega^\varepsilon\otimes\tilde\Omega^\varepsilon)\{\partial_x\tilde u^\varepsilon+{}^t\partial_x\tilde u^\varepsilon-\frac{2(I_d-\tilde\Omega^\varepsilon\otimes\tilde\Omega^\varepsilon):\partial_x\tilde u^\varepsilon}{d-1}(I_d-\tilde\Omega^\varepsilon\otimes\tilde\Omega^\varepsilon)\}(I_d-\tilde\Omega^\varepsilon\otimes\tilde\Omega^\varepsilon)$$

$$\begin{aligned}\tilde T_3^\varepsilon=\frac{\tilde\rho^\varepsilon}{\sigma}\Big\{&c_2(\sigma,\tilde u^\varepsilon)(I_d-\tilde\Omega^\varepsilon\otimes\tilde\Omega^\varepsilon)\times(I_d-\tilde\Omega^\varepsilon\otimes\tilde\Omega^\varepsilon)+c_3(\sigma,\tilde u^\varepsilon)(\tilde\Omega^\varepsilon\otimes\tilde\Omega^\varepsilon)\times(\tilde\Omega^\varepsilon\otimes\tilde\Omega^\varepsilon)\\&+c_4(\sigma,\tilde u^\varepsilon)[(I_d-\tilde\Omega^\varepsilon\otimes\tilde\Omega^\varepsilon)\times(\tilde\Omega^\varepsilon\otimes\tilde\Omega^\varepsilon)+(\tilde\Omega^\varepsilon\otimes\tilde\Omega^\varepsilon)\times(I_d-\tilde\Omega^\varepsilon\otimes\tilde\Omega^\varepsilon)]\Big\}\partial_x\tilde u^\varepsilon\end{aligned}$$

$$\tilde T_4^\varepsilon=\frac{\tilde\rho^\varepsilon}{\sigma}c_5[(I_d-\tilde\Omega^\varepsilon\otimes\tilde\Omega^\varepsilon)(\partial_x\tilde u^\varepsilon+{}^t\partial_x\tilde u^\varepsilon)(\tilde\Omega^\varepsilon\otimes\tilde\Omega^\varepsilon)+(\tilde\Omega^\varepsilon\otimes\tilde\Omega^\varepsilon)(\partial_x\tilde u^\varepsilon+{}^t\partial_x\tilde u^\varepsilon)(I_d-\tilde\Omega^\varepsilon\otimes\tilde\Omega^\varepsilon)].$$

Here $\tilde\Omega^\varepsilon=\Omega[\tilde j^\varepsilon],\tilde u^\varepsilon=\partial_k\mathcal{E}^\star(\sigma,|\tilde j^\varepsilon|/\tilde\rho^\varepsilon)\tilde\Omega^\varepsilon$. *For any two matrices* $A,B\in\mathcal{M}_d(\mathbb{R})$, *the notation* $A\times B$ *stands for the linear application* $A\times B:\mathcal{M}_d(\mathbb{R})\to\mathcal{M}_d(\mathbb{R})$, *given by* $(A\times B)X=(B:X)A, X\in\mathcal{M}_d(\mathbb{R})$ *and the functions* $(c_i)_{1\le i\le5}$ *are defined in* (38), (39).

Our paper is organized as follows. In Section 2 we establish some preliminaries, concerning the properties of the function Z. It is useful to define the notion of vector and matrix fields which are left invariant by the orthogonal transformations preserving u. We shall continue in this section by studying the structure of such vector and matrix fields. The linearization of the interaction mechanism is studied in Section 3. In Section 4 we concentrate on the first order approximation. The second order approximation is analyzed in Section 5. Some proofs involving technical calculations were postponed to the Appendix A.

2 Preliminaries

For any $u\in\mathbb{R}^d$ we denote by $\mathcal{T}_u$ the family of orthogonal transformations of $\mathbb{R}^d$ preserving u. Clearly $\mathcal{T}_0$ is the family of all orthogonal transformations of $\mathbb{R}^d$. A function $f=f(v)$ is said invariant by the family $\mathcal{T}_u$ iff $f({}^t\mathcal{O}v)=f(v), v\in\mathbb{R}^d,\mathcal{O}\in\mathcal{T}_u$. The structure of the functions which are left invariant by the transformations of $\mathcal{T}_u$ is presented below.

Lemma 2.1
The functions on $\mathbb{R}^d$ *which are left invariant by the family* $\mathcal{T}_0$ *are those depending only on* $|v|$. *The functions on* $\mathbb{R}^d$ *which are left invariant by the family* $\mathcal{T}_u, u\neq0$, *are those depending on* $v\cdot u$ *and* $|v|$.

We recall the following easy result cf. [16].

Lemma 2.2
Let u be a vector in $\mathbb{R}^d$ and $a : \mathbb{R}^d \to \mathbb{R}^d$ be a integrable vector field on $\mathbb{R}^d$, which is left invariant by the family $\mathcal{T}_u$ i.e.,

$$a({}^t\mathcal{O}v) = {}^t\mathcal{O}a(v), \quad v \in \mathbb{R}^d, \quad \mathcal{O} \in \mathcal{T}_u.$$

Then $\int_{\mathbb{R}^d} a(v)\, \mathrm{d}v \in \mathbb{R}u$.

We assume that the following hypothesis holds true

$$\lim_{|v|\to+\infty} \frac{\frac{|v|^2}{2} + V(|v|)}{|v|} = +\infty. \tag{7}$$?Equ11-Swarm?

In that case the function Z is finite for any $\sigma > 0, u \in \mathbb{R}^d$, because we can write

$$\begin{aligned} Z(\sigma, u) &= \exp\left(-\frac{|u|^2}{2\sigma}\right) \int_{\mathbb{R}^d} \exp\left(-\frac{\frac{|v|^2}{2} + V(|v|)}{\sigma} + \frac{v\cdot u}{\sigma}\right) \mathrm{d}v \\ &\leq \exp\left(-\frac{|u|^2}{2\sigma}\right) \int_{\mathbb{R}^d} \exp\left[-\frac{|v|}{\sigma}\left(\frac{\frac{|v|^2}{2} + V(|v|)}{|v|} - |u|\right)\right] \mathrm{d}v. \end{aligned}$$

Similarly, all the moments of M_u are finite

$$\int_{\mathbb{R}^d} |v|^p M_u(v)\, \mathrm{d}v < +\infty, \quad p \in \mathbb{N}.$$

We recall the following formula

$$\int_{\mathbb{R}^d} \chi\left(\frac{v\cdot\Omega}{|v|}, |v|\right) \mathrm{d}v = |\mathbb{S}^{d-2}| \int_{\mathbb{R}_+} r^{d-1} \int_0^\pi \chi(\cos\theta, r) \sin^{d-2}\theta\, \mathrm{d}\theta\mathrm{d}r \tag{8}$$?Equ12-Swarm?

for any non negative measurable function $\chi = \chi(c, r) :]-1, 1[\times\mathbb{R}_+^\star \to \mathbb{R}$, any $\Omega \in \mathbb{S}^{d-1}$ and $d \geq 2$. Here $|\mathbb{S}^{d-2}|$ is the surface of the unit sphere in $\mathbb{R}^{d-1}$, for $d \geq 3$, and $|\mathbb{S}^0| = 2$ for $d = 2$.

Next we concentrate on the properties of the function Z cf. [16].

Proposition 2.1
Assume that the potential $v \mapsto V(|v|)$ satisfies (7).

1. *Then the function $Z(\sigma, u)$ depends only on σ and $|u|$. We will simply write*

$$\int_{\mathbb{R}^d} \exp\left(-\frac{\Phi_u(v)}{\sigma}\right) \mathrm{d}v = Z(\sigma, l = |u|).$$

2. *For any $u \in \mathbb{R}^d$, we have $\int_{\mathbb{R}^d} M_u(v) v\, \mathrm{d}v \in \mathbb{R}_+ u$. In particular we have $\int_{\mathbb{R}^d} M_0(v) v\, \mathrm{d}v = 0$.*

3. $$\partial_l Z(\sigma, |u|) = Z(\sigma, |u|) \int_{\mathbb{R}^d} M_u(v) \frac{v-u}{\sigma}\, \mathrm{d}v \cdot \Omega[u].$$

   In particular $\partial_l Z(\sigma, 0) = 0$ for any $\sigma > 0$.

4. $$\partial_{ll}^2 Z(\sigma, |u|) = Z(\sigma, |u|) \int_{\mathbb{R}^d} M_u(v) \left[\left(\frac{v\cdot\Omega[u] - |u|}{\sigma}\right)^2 - \frac{1}{\sigma}\right] \mathrm{d}v$$

   where the notation $\Omega[u]$ stands for $\frac{u}{|u|}$ if $u \neq 0$ and any vector in $\mathbb{S}^{d-1}$ if $u = 0$.

In the sequel we will appeal to vector and matrix fields, which are left invariant by the family $\mathcal{T}_u$, $u \in \mathbb{R}^d$. We end this section by indicating the structure of these vector and matrix fields. This will be crucial when searching for an antecedent of the pressure tensor through the linearization of the collision mechanism.

Proposition 2.2
Let $a : \mathbb{R}^d \to \mathbb{R}^d$ be a vector field on $\mathbb{R}^d$, which is left invariant by the family $\mathcal{T}_u$, $u \in \mathbb{R}^d$.

1. *If $u = 0$, then there is a function $\beta : \mathbb{R}^d \to \mathbb{R}, \beta(0) = 0$, which is left invariant by the family $\mathcal{T}_0$, such that $a(v) = \beta(v)v, v \in \mathbb{R}^d$.*

2. *If $u \neq 0$, then there are two functions $\alpha : \mathbb{R}^d \to \mathbb{R}$, $\beta : \mathbb{R}^d \to \mathbb{R}$, $\beta|_{\mathbb{R}\Omega[u]} = 0$, which are left invariant by $\mathcal{T}_u$, such that $a(v) = \alpha(v)\Omega[u] + \beta(v)(v - (v \cdot \Omega[u])\Omega[u])$, $v \in \mathbb{R}^d$.*

Proof.
1. Obviously, for any function β which is left invariant by $\mathcal{T}_0$, the vector field $a(v) = \beta(v)v, v \in \mathbb{R}^d$ is left invariant by $\mathcal{T}_0$. Conversely, assume that $a : \mathbb{R}^d \to \mathbb{R}^d$ is left invariant by $\mathcal{T}_0$. For any $\xi \in \mathbb{S}^{d-1} \cap (\mathbb{R}v)^\perp$ we consider $\mathcal{O}_\xi = I_d - 2\xi \otimes \xi \in \mathcal{T}_0$ and therefore we obtain

$$a(v) = a({}^t\mathcal{O}_\xi v) = {}^t\mathcal{O}_\xi a(v) = a(v) - 2(a(v) \cdot \xi)\xi$$

and thus $a(v) \cdot \xi = 0$. We deduce that $a(v) = 0$ if $v = 0$ and $(I_d - v \otimes v/|v|^2)a(v) = 0$ if $v \in \mathbb{R}^d\backslash\{0\}$. The desired result follows by taking the function $\beta(v) = a(v) \cdot v/|v|^2, v \in \mathbb{R}^d\backslash\{0\}$ and $\beta(0) = 0$, which is left invariant by $\mathcal{T}_0$.
2. Clearly, for any functions α, β which are left invariant by $\mathcal{T}_u$, the vector field $a(v) = \alpha(v)\Omega[u] + \beta(v)(v - (v \cdot \Omega[u])\Omega[u])$, $v \in \mathbb{R}^d$ is left invariant by $\mathcal{T}_u$. Conversely, assume that $a : \mathbb{R}^d \to \mathbb{R}^d$ is left invariant by $\mathcal{T}_u$. Then $a_\perp(v) := a(v) - (a(v) \cdot \Omega)\Omega, v \in \mathbb{R}^d$ is left invariant by $\mathcal{T}_u$. For any $\xi \in \mathbb{S}^{d-1} \cap (\mathbb{R}v + \mathbb{R}\Omega)^\perp$, we consider $\mathcal{O}_\xi = I_d - 2\xi \otimes \xi \in \mathcal{T}_u$ and therefore we have

$$a_\perp(v) = a_\perp({}^t\mathcal{O}_\xi v) = {}^t\mathcal{O}_\xi a_\perp(v) = a_\perp(v) - 2(a_\perp(v) \cdot \xi)\xi$$

saying that $a_\perp(v) \cdot \xi = 0$. If $v_\perp := v - (v \cdot \Omega)v = 0$, as we also have $a_\perp(v) \cdot \Omega = 0$, we deduce that $a_\perp(v) = 0$. If $v_\perp \neq 0$ we obtain $(I_d - v_\perp \otimes v_\perp/|v_\perp|^2)a_\perp(v) = 0$. We consider the function $\beta(v) = a_\perp(v) \cdot v_\perp/|v_\perp|^2$, $v \in \mathbb{R}^d\backslash(\mathbb{R}\Omega)$, $\beta|_{\mathbb{R}\Omega} = 0$, which is left invariant by $\mathcal{T}_u$. In that case $a_\perp(v) = \beta(v)v_\perp, v \in \mathbb{R}^d$ and therefore $a(v) = \alpha(v)\Omega + \beta(v)v_\perp$, $v \in \mathbb{R}^d$, where $\alpha = a \cdot \Omega$ is left invariant by $\mathcal{T}_u$. □

A matrix field $A : \mathbb{R}^d \to \mathcal{M}_d(\mathbb{R})$ is said invariant by the family $\mathcal{T}_u$ iff $A({}^t\mathcal{O}v) = {}^t\,\mathcal{O}A(v)\mathcal{O}, v \in \mathbb{R}^d, \mathcal{O} \in \mathcal{T}_u$. Notice that if the matrix field A, B and the vector field a, b are left invariant by $\mathcal{T}_u$, then $\mathrm{tr}A$, Aa, ${}^tA$, $A : B$, $a \cdot b$, $a \otimes b$ are left invariant by $\mathcal{T}_u$.

Proposition 2.3
Let $A : \mathbb{R}^d \to \mathcal{M}_d(\mathbb{R})$ be a matrix field on $\mathbb{R}^d$ which is left invariant by the family $\mathcal{T}_u$, $d \geq 2$.

1. *If $u = 0$, there are two functions $\alpha : \mathbb{R}^d \to \mathbb{R}$, $\beta : \mathbb{R}^d \to \mathbb{R}$ which are left invariant by $\mathcal{T}_0$, $\alpha(0) = 0$, such that*

$$A(v) = \alpha(v)\left(v \otimes v - |v|^2\frac{I_d}{d}\right) + \beta(v)I_d, \ v \in \mathbb{R}^d. \tag{9}$$?Equ9-Swarm?

2. *If $u \neq 0$, there are the functions $\alpha, \beta, \gamma', \gamma'', \delta$ which are left invariant by $\mathcal{T}_u$, $\alpha|_{\mathbb{R}\Omega} = 0$, $\gamma'|_{\mathbb{R}\Omega} = 0$, $\gamma''|_{\mathbb{R}\Omega} = 0$, such that*

$$\begin{aligned} A(v) = {} & \alpha(v)\left(v_\perp \otimes v_\perp - |v_\perp|^2\frac{I_d - \Omega \otimes \Omega}{d-1}\right) + \beta(v)(I_d - \Omega \otimes \Omega) \\ & + \gamma'(v)v_\perp \otimes \Omega + \gamma''\Omega \otimes v_\perp + \delta(v)\Omega \otimes \Omega, \ \ v \in \mathbb{R}^d, \ d \geq 3 \end{aligned} \tag{10}$$?Equ10-Swarm?

and

$$A(v)=\beta(v)(I_2-\Omega\otimes\Omega)+\gamma'(v)v_\perp\otimes\Omega+\gamma''\Omega\otimes v_\perp+\delta(v)\Omega\otimes\Omega,\ \ v\in\mathbb{R}^2.$$

Proof.
1. Clearly, every matrix field in (9) is left invariant by $\mathcal{T}_0$. Conversely, let us consider a matrix field A which is left invariant by $\mathcal{T}_0$. Assume for the moment that $\mathrm{tr}A=0$. For any $\xi\in\mathbb{S}^{d-1}\cap(\mathbb{R}v)^\perp$, we have $\mathcal{O}_\xi=I_d-2\xi\otimes\xi\in\mathcal{T}_0$ and $\mathcal{O}_\xi A(v)=\mathcal{O}_\xi A({}^t\mathcal{O}_\xi v)=A(v)\mathcal{O}_\xi$ implying that

$$\xi\otimes\xi A(v)=A(v)\xi\otimes\xi,\ \text{ and }\ A(v)\xi=(A(v)\xi\cdot\xi)\xi.$$

It is easily seen that there is λ, depending only on v, such that $A(v)\xi=\lambda\xi$, for any $\xi\in(\mathbb{R}v)^\perp$. If $v=0$, as $\mathrm{tr}A=0$, we obtain $A(0)=0$. If $v\in\mathbb{R}^d\backslash\{0\}$, we claim that v is also a proper vector for $A(v)$. Indeed, since the matrix field ${}^tA$ is left invariant by $\mathcal{T}_0$, we have as before, for any $\xi\in\mathbb{S}^{d-1}\cap(\mathbb{R}v)^\perp$

$$A(v)v\cdot\xi=v\cdot{}^tA(v)\xi=v\cdot({}^tA(v)\xi\cdot\xi)\xi=0$$

saying that $A(v)v\in(\mathbb{R}v)^{\perp\perp}=\mathbb{R}v$. Let us consider $\{\xi_1,...,\xi_{d-1}\}$ an orthonormal basis of $(\mathbb{R}v)^\perp$. The matrix $A(v)$ writes

$$A(v)=\lambda(\xi_1\otimes\xi_1+...+\xi_{d-1}\otimes\xi_{d-1})+\mu\frac{v\otimes v}{|v|^2}$$

where $(d-1)\lambda+\mu=\mathrm{tr}A(v)=0$, and therefore

$$A(v)=\lambda\left(I_d-\frac{v\otimes v}{|v|^2}\right)-\lambda(d-1)\frac{v\otimes v}{|v|^2}=\lambda\left(I_d-d\frac{v\otimes v}{|v|^2}\right)=-\frac{\lambda d}{|v|^2}\left(v\otimes v-\frac{|v|^2}{d}I_d\right).$$

By taking $\alpha(v)=\frac{d}{d-1}\frac{A(v):v\otimes v}{|v|^4}$, $v\in\mathbb{R}^d\backslash\{0\}$ and $\alpha(0)=0$, which is left invariant by $\mathcal{T}_0$, we obtain $A(v)=\alpha(v)\left(v\otimes v-\frac{|v|^2}{d}I_d\right),v\in\mathbb{R}^d$. Now, for any matrix field A which is left invariant by $\mathcal{T}_0$, the matrix field $A'=A-\frac{\mathrm{tr}A}{d}I_d$ is left invariant by $\mathcal{T}_0$, and has zero trace. There is $\alpha':\mathbb{R}^d\to\mathbb{R}$, which is left invariant by $\mathcal{T}_0$, such that

$$A'(v)=\alpha'(v)\left(v\otimes v-\frac{|v|^2}{d}I_d\right),\ v\in\mathbb{R}^d$$

and therefore $A(v)=\alpha'(v)\left(v\otimes v-\frac{|v|^2}{d}I_d\right)+\beta(v)I_d,v\in\mathbb{R}^d$, where $\beta=\frac{\mathrm{tr}A}{d}$ is left invariant by $\mathcal{T}_0$.
2. Every matrix field in (10) is left invariant by $\mathcal{T}_u$. Conversely, let us consider a matrix field A, which is left invariant by $\mathcal{T}_u$. For any $\xi\in\mathbb{S}^{d-1}\cap(\mathbb{R}v+\mathbb{R}\Omega)^\perp$ we have $\mathcal{O}_\xi=I_d-2\xi\otimes\xi\in\mathcal{T}_u$, and we deduce as before that there is λ depending only on v such that

$$A(v)\xi=\lambda\xi,\ \xi\in(\mathbb{R}v+\mathbb{R}\Omega)^\perp.$$

If $v\in\mathbb{R}\Omega$, since the matrix field ${}^tA$ is left invariant by $\mathcal{T}_u$, we have for any $\xi\in\mathbb{S}^{d-1}\cap(\mathbb{R}\Omega)^\perp$

$$A(v)\Omega\cdot\xi=\Omega\cdot{}^tA(v)\xi=\Omega\cdot({}^tA(v)\xi\cdot\xi)\xi=0$$

implying that $A(v)\Omega\in(\mathbb{R}\Omega)^{\perp\perp}=\mathbb{R}\Omega$. In that case, $A(v)$ has two proper subspaces and

$$A(v)=\frac{\mathrm{tr}A(v)-A(v)\Omega\cdot\Omega}{d-1}(I_d-\Omega\otimes\Omega)+(A(v)\Omega\cdot\Omega)\Omega\otimes\Omega,\ \ v\in\mathbb{R}\Omega.$$

If $v \in \mathbb{R}^d \backslash (\mathbb{R}\Omega)$, let us consider $\{\xi_1, ..., \xi_{d-2}\}$ an orthonormal basis of $(\mathbb{R}v + \mathbb{R}\Omega)^\perp$. Recall that for any matrix $B \in \mathcal{M}_d(\mathbb{R})$ and orthonormal basis $\{e_1, ..., e_d\}$ of $\mathbb{R}^d$, we have $B = \sum_{i=1}^d e_i \otimes {}^tBe_i$. Therefore we write

$$\begin{aligned} {}^tA(v) &= \frac{v_\perp}{|v_\perp|} \otimes A(v)\frac{v_\perp}{|v_\perp|} + \Omega \otimes A(v)\Omega + \sum_{i=1}^{d-2} \xi_i \otimes A(v)\xi_i \\ &= \frac{v_\perp}{|v_\perp|} \otimes A(v)\frac{v_\perp}{|v_\perp|} + \Omega \otimes A(v)\Omega + \lambda \left(I_d - \frac{v_\perp}{|v_\perp|} \otimes \frac{v_\perp}{|v_\perp|} - \Omega \otimes \Omega \right). \end{aligned}$$

But the vector field $v \to A(v)v_\perp$, $v \to A(v)\Omega$ are left invariant by $\mathcal{T}_u$. We deduce by Proposition 2.2 that

$$A(v)v_\perp = \alpha_1(v)\Omega + \beta_1(v)v_\perp, \ A(v)\Omega = \alpha_2(v)\Omega + \beta_2(v)v_\perp$$

for some functions $\alpha_1, \beta_1, \alpha_2, \beta_2$ which are left invariant by $\mathcal{T}_u$, and therefore

$$\begin{aligned} {}^tA(v) &= \frac{v_\perp}{|v_\perp|} \otimes \frac{\alpha_1\Omega + \beta_1 v_\perp}{|v_\perp|} + \Omega \otimes (\alpha_2\Omega + \beta_2 v_\perp) - \lambda \frac{v_\perp \otimes v_\perp}{|v_\perp|^2} + \lambda(I_d - \Omega \otimes \Omega) \\ &= \frac{\beta_1 - \lambda}{|v_\perp|^2} \left(v_\perp \otimes v_\perp - |v_\perp|^2 \frac{I_d - \Omega \otimes \Omega}{d-1} \right) + [\beta_1 + (d-2)\lambda] \frac{I_d - \Omega \otimes \Omega}{d-1} \\ &+ \frac{\alpha_1}{|v_\perp|^2} v_\perp \otimes \Omega + \beta_2 \Omega \otimes v_\perp + \alpha_2 \Omega \otimes \Omega. \end{aligned}$$

Notice that the matrices $v_\perp \otimes v_\perp - |v_\perp|^2 \frac{I_d - \Omega\otimes\Omega}{d-1}$, $I_d - \Omega \otimes \Omega$, $v_\perp \otimes \Omega$, $\Omega \otimes v_\perp$, $\Omega \otimes \Omega$ are orthogonal with respect to the scalar product $B : C = \mathrm{tr}({}^tBC), B, C \in \mathcal{M}_d(\mathbb{R})$. We obtain easily that

$$\begin{aligned} A(v) &= (d-1) \frac{A(v) : \left(v_\perp \otimes v_\perp - |v_\perp|^2 \frac{I_d - \Omega\otimes\Omega}{d-1} \right)}{(d-2)|v_\perp|^4} \left(v_\perp \otimes v_\perp - |v_\perp|^2 \frac{I_d - \Omega \otimes \Omega}{d-1} \right) \\ &+ A(v) : (I_d - \Omega \otimes \Omega) \frac{I_d - \Omega \otimes \Omega}{d-1} + \frac{A(v) : \Omega \otimes v_\perp}{|v_\perp|^2} \Omega \otimes v_\perp + \frac{A(v) : v_\perp \otimes \Omega}{|v_\perp|^2} v_\perp \otimes \Omega \\ &+ (A(v)\Omega \cdot \Omega)\Omega \otimes \Omega, \ v \in \mathbb{R}^d \backslash (\mathbb{R}\Omega), \ d \geq 3 \end{aligned}$$

and

$$\begin{aligned} A(v) &= A(v) : (I_2 - \Omega \otimes \Omega)(I_d - \Omega \otimes \Omega) + \frac{A(v) : \Omega \otimes v_\perp}{|v_\perp|^2} \Omega \otimes v_\perp + \frac{A(v) : v_\perp \otimes \Omega}{|v_\perp|^2} v_\perp \otimes \Omega \\ &+ (A(v)\Omega \cdot \Omega)\Omega \otimes \Omega, \ v \in \mathbb{R}^d \backslash (\mathbb{R}\Omega), \ d = 2. \end{aligned}$$

The second assertion of the proposition follows by taking

$$\alpha(v) = (d-1) \frac{A(v) : \left(v_\perp \otimes v_\perp - |v_\perp|^2 \frac{I_d - \Omega\otimes\Omega}{d-1} \right)}{(d-2)|v_\perp|^4} \mathbf{1}_{\mathbb{R}^d \backslash (\mathbb{R}\Omega)}, \ v \in \mathbb{R}^d, \ d \geq 3$$

$$\beta(v) = \frac{A(v) : (I_d - \Omega \otimes \Omega)}{d-1}, \ \delta(v) = A(v) : \Omega \otimes \Omega, \ v \in \mathbb{R}^d, \ d \geq 2$$

$$\gamma'(v) = \frac{A(v) : v_\perp \otimes \Omega}{|v_\perp|^2} \mathbf{1}_{\mathbb{R}^d \backslash (\mathbb{R}\Omega)}, \ \gamma''(v) = \frac{A(v) : \Omega \otimes v_\perp}{|v_\perp|^2} \mathbf{1}_{\mathbb{R}^d \backslash (\mathbb{R}\Omega)}, \ v \in \mathbb{R}^d, \ d \geq 2$$

which are left invariant by $\mathcal{T}_u$. □

Corollary 2.1
Let $A : \mathbb{R}^d \to \mathcal{M}_d(\mathbb{R})$ be a field of symmetric matrices on $\mathbb{R}^d$ which is left invariant by the family $\mathcal{T}_u, u \neq 0, d \geq 2$. There are the functions $\alpha, \beta, \gamma, \delta : \mathbb{R}^d \to \mathbb{R}$ which are left invariant by $\mathcal{T}_u$, $\alpha|_{\mathbb{R}\Omega} = 0$, $\gamma|_{\mathbb{R}\Omega} = 0$ such that

$$A(v) = \alpha(v)\left(v_\perp \otimes v_\perp - |v_\perp|^2 \frac{I_d - \Omega \otimes \Omega}{d-1}\right) + \beta(v)(I_d - \Omega \otimes \Omega)$$
$$+ \gamma(v)(v_\perp \otimes \Omega + \Omega \otimes v_\perp) + \delta(v)\Omega \otimes \Omega, \;\; v \in \mathbb{R}^d, \; d \geq 3$$

and

$$A(v) = \beta(v)(I_2 - \Omega \otimes \Omega) + \gamma(v)(v_\perp \otimes \Omega + \Omega \otimes v_\perp) + \delta(v)\Omega \otimes \Omega, \;\; v \in \mathbb{R}^2.$$

3 Linearization of the interaction mechanism

As usual when studying the asymptotic behavior of (4), we introduce the Hilbert development for f^ε

$$f^\varepsilon = f + \varepsilon f_1 + \varepsilon^2 f_2 + ...$$

where $f = f(t,x,v), f_i = f_i(t,x,v), i = 1, 2, ...$ are not depending on ε. We have

$$Q(f^\varepsilon) = Q(f) + \varepsilon \mathcal{L}_f f_1 + \mathcal{O}(\varepsilon^2)$$

where $\mathcal{L}_f := \mathrm{d}_f Q$ is the first derivative of Q at f.

Inserting the expansions for $f^\varepsilon, Q(f^\varepsilon)$ in (4) and identifying terms of equal powers of ε, we obtain

$$\partial_t f + v \cdot \nabla_x f + \varepsilon\left(\partial_t f_1 + v \cdot \nabla_x f_1\right) + ... = \varepsilon^{-1} Q(f) + \mathcal{L}_f(f_1) + ... \qquad (11)$$?developeps?

From the above equation, we infer that $Q(f) = 0$ and as seen before, for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, the individual density $f(t,x,\cdot)$ is a von Mises-Fisher distribution

$$f(t,x,v) = \rho(t,x) M_{u(t,x)}(v), \;\; v \in \mathbb{R}^d.$$

Notice that the equilibria of Q are parametrized by $\rho > 0$ and $u \in \mathbb{R}^d$. Indeed, we have $\sigma \nabla_v M_u + M_u \nabla_v \Phi_u = 0$ and therefore $\int_{\mathbb{R}^d} M_u(v) \nabla_v \Phi_u \, \mathrm{d}v = 0$, saying that $u[M_u] = u$. For any equilibrium $f = \rho M_u$, we investigate the properties of the linear operator $\mathcal{L}_f$. Let us define the Hilbert spaces

$$L^2_{M_u} = \{g : \mathbb{R}^d \to \mathbb{R} \text{ measurable } : \int_{\mathbb{R}^d} \left(\frac{g}{M_u}\right)^2 M_u(v) \, \mathrm{d}v < +\infty\}$$

$$H^1_{M_u} = \{g : \mathbb{R}^d \to \mathbb{R} \text{ measurable } : \int_{\mathbb{R}^d} \left[\left(\frac{g}{M_u}\right)^2 + \left|\nabla_v \left(\frac{g}{M_u}\right)\right|^2\right] M_u(v) \, \mathrm{d}v < +\infty\}$$

endowed with the usual scalar products

$$(g,h)_{M_u} = \int_{\mathbb{R}^d} \frac{g(v)h(v)}{M_u(v)} \, \mathrm{d}v, \;\; g, h \in L^2_{M_u}$$

$$((g,h))_{M_u} = \int_{\mathbb{R}^d} \left[\frac{g(v)}{M_u(v)} \frac{h(v)}{M_u(v)} + \nabla_v \left(\frac{g}{M_u}\right) \cdot \nabla_v \left(\frac{h}{M_u}\right)\right] M_u(v) \, \mathrm{d}v, \;\; g, h \in H^1_{M_u}$$

and we denote by $|\cdot|_{M_u}, \|\cdot\|_{M_u}$ the associated norms.

Proposition 3.1
Let $f = \rho M_u$ be an equilibrium of Q.

1. *The linearization $\mathcal{L}_f = \mathrm{d}Q_f$ is given by*

$$\mathcal{L}_f g = \mathrm{div}_v \left\{ \sigma \nabla_v g + g \nabla_v \Phi_u - M_u(v) \int_{\mathbb{R}^d} g(v') \nabla \Phi_u(v') \, \mathrm{d}v' \right\}.$$

2. *The operator $\mathcal{L}_f$ satisfies the mass and momentum conservations*

$$\int_{\mathbb{R}^d} \mathcal{L}_f g \, \mathrm{d}v = 0, \quad \int_{\mathbb{R}^d} \mathcal{L}_f g \; v \, \mathrm{d}v = 0.$$

3. *The null space of $\mathcal{L}_f$ is given by* $\ker \mathcal{L}_f = \mathrm{span}\{M_u, v_1 M_u, ..., v_d M_u\}$.

Proof.
1. By direct computation we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} u[f+sg] &= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \frac{\int_{\mathbb{R}^d} (v + \nabla_v V(|\cdot|))(f+sg) \, \mathrm{d}v}{\int_{\mathbb{R}^d} (f+sg) \, \mathrm{d}v} \\
&= \frac{\int_{\mathbb{R}^d} (v + \nabla_v V(|\cdot|)) g \, \mathrm{d}v}{\int_{\mathbb{R}^d} f(v) \, \mathrm{d}v} - \frac{\int_{\mathbb{R}^d} (v + \nabla_v V(|\cdot|)) f \, \mathrm{d}v}{(\int_{\mathbb{R}^d} f(v) \, \mathrm{d}v)^2} \int_{\mathbb{R}^d} g(v) \, \mathrm{d}v \\
&= \frac{\int_{\mathbb{R}^d} g(v) \nabla_v \Phi_u \, \mathrm{d}v}{\int_{\mathbb{R}^d} f(v) \, \mathrm{d}v}
\end{aligned}$$

implying that

$$\mathcal{L}_f g = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} Q(f+sg) = \mathrm{div}_v \left\{ \sigma \nabla_v g + g \nabla_v \Phi_u - M_u(v) \int_{\mathbb{R}^d} g(v') \nabla \Phi_u(v') \, \mathrm{d}v' \right\}.$$

2. It is easily seen that $\int_{\mathbb{R}^d} \mathcal{L}_f g \, \mathrm{d}v = 0$ and

$$\begin{aligned}
\int_{\mathbb{R}^d} \mathcal{L}_f g \; v \, \mathrm{d}v &= -\int_{\mathbb{R}^d} \{ \sigma \nabla_v g + g(v) \nabla_v \Phi_u - M_u(v) \int_{\mathbb{R}^d} g(v') \nabla \Phi_u(v') \, \mathrm{d}v' \} \, \mathrm{d}v \\
&= -\int_{\mathbb{R}^d} g(v) \nabla_v \Phi_u \, \mathrm{d}v + \int_{\mathbb{R}^d} g(v') \nabla \Phi_u(v') \, \mathrm{d}v' = 0.
\end{aligned}$$

3. We introduce the notation $W(g) = \sigma \int_{\mathbb{R}^d} M_u(v) \nabla_v \left(\frac{g}{M_u} \right) \, \mathrm{d}v, g \in H^1_{M_u}$. Notice that for any $g \in H^1_{M_u}$, $W(g)$ is well defined and we have

$$\begin{aligned}
|W(g)| &\leq \sigma \int_{\mathbb{R}^d} M_u(v) \left| \nabla_v \left(\frac{g}{M_u} \right) \right| \, \mathrm{d}v \qquad (12) \\
&\leq \sigma \left(\int_{\mathbb{R}^d} M_u(v) \left| \nabla_v \left(\frac{g}{M_u} \right) \right|^2 \, \mathrm{d}v \right)^{1/2} \\
&\leq \sigma \|g\|_{M_u}.
\end{aligned}$$

?EquEstimW?

Moreover, since $\sigma \nabla_v M_u + M_u(v) \nabla_v \Phi_u = 0$, we can write

$$W(g) = -\sigma \int_{\mathbb{R}^d} \nabla_v M_u \frac{g}{M_u(v)} \, \mathrm{d}v = \int_{\mathbb{R}^d} g(v) \nabla_v \Phi_u \, \mathrm{d}v.$$

Notice that, thanks to the momentum balance, we have

$$
\begin{aligned}
-\sigma(\mathcal{L}_f g, h)_{M_u} &= -\sigma \int_{\mathbb{R}^d} \mathcal{L}_f g \frac{h(v)}{M_u(v)}\, \mathrm{d}v \\
&= \int_{\mathbb{R}^d} [\sigma \nabla_v g + g(v)\nabla_v \Phi_u - M_u(v) W(g)] \cdot \sigma \nabla_v \left(\frac{h}{M_u}\right)\, \mathrm{d}v \\
&= \int_{\mathbb{R}^d} [\sigma M_u(v) \nabla_v \left(\frac{g}{M_u}\right) - M_u(v) W(g)] \cdot [\sigma M_u(v) \nabla_v \left(\frac{h}{M_u}\right) - M_u(v) W(h)] M_u^{-1}(v)\, \mathrm{d}v.
\end{aligned}
$$

In particular

$$
-\sigma(\mathcal{L}_f g, g)_{M_u} = \int_{\mathbb{R}^d} \left| \sigma \nabla_v \left(\frac{g}{M_u}\right) - W(g) \right|^2 M_u(v)\, \mathrm{d}v \geq 0 \tag{13}
$$

?EquPositivity

and therefore the operator $-\mathcal{L}_f$ is symmetric and positive with respect to the scalar product of $L^2_{M_u}$. Thanks to the previous computations, we identify the kernel of the operator $\mathcal{L}_f$. From (13) it is clear that if $g \in \ker \mathcal{L}_f$, than $\sigma \nabla_v \left(\frac{g}{M_u}\right) - W(g) = 0$ implying that $g \in \mathrm{span}\{M_u, v_1 M_u, ..., v_d M_u\}$. Therefore we have the inclusion $\ker \mathcal{L}_f \subset \mathrm{span}\{M_u, v_1 M_u, ..., v_d M_u\}$. For the converse inclusion, notice that

$$
W(M_u) = \sigma \int_{\mathbb{R}^d} M_u \nabla_v \left(\frac{M_u}{M_u}\right)\, \mathrm{d}v = 0, \;\; W(v_i M_u) = \sigma \int_{\mathbb{R}^d} M_u \nabla_v \left(\frac{v_i M_u}{M_u}\right)\, \mathrm{d}v = \sigma e_i, 1 \leq i \leq d.
$$

We obtain

$$
\mathcal{L}_f M_u = \mathrm{div}_v \{\sigma M_u \nabla_v \left(\frac{M_u}{M_u}\right) - M_u W(M_u)\} = 0
$$

and

$$
\begin{aligned}
\mathcal{L}_f(v_i M_u) &= \mathrm{div}_v \{\sigma M_u \nabla_v \left(\tfrac{v_i M_u}{M_u}\right) - M_u W(v_i M_u)\} \\
&= \mathrm{div}_v \{\sigma M_u e_i - M_u \sigma e_i\} = 0, \;\; 1 \leq i \leq d
\end{aligned}
$$

implying that $\mathrm{span}\{M_u, v_1 M_u, ..., v_d M_u\} \subset \ker \mathcal{L}_f$. □

We also need to determine the range of the linearization $\mathcal{L}_f$.

Definition 3.1

We say that $g \in H^1_{M_u}$ is a variational solution of $-\mathcal{L}_f g = p$, with $p \in L^2_{M_u}$ iff

$$
\int_{\mathbb{R}^d} \left[\sigma \nabla_v \left(\frac{g}{M_u}\right) - W(g)\right] \cdot \left[\sigma \nabla_v \left(\frac{h}{M_u}\right) - W(h)\right] M_u(v)\, \mathrm{d}v = \sigma \int_{\mathbb{R}^d} \frac{p(v)h(v)}{M_u(v)}\, \mathrm{d}v \tag{14}
$$

?EquVarFor?

for any $h \in H^1_{M_u}$.

If $g \in H^1_{M_u}$ is a variational solution of $-\mathcal{L}_f g = p, p \in L^2_{M_u}$, then taking $h = M_u$ in (14) we obtain $\int_{\mathbb{R}^d} p(v)\, \mathrm{d}v = 0$. Similarly, by taking $h_i = v_i M_u$, we obtain $\int_{\mathbb{R}^d} p(v) v_i\, \mathrm{d}v = 0, 1 \leq i \leq d$, saying that a necessary condition for solving $-\mathcal{L}_f g = p$ is $p \in (\ker \mathcal{L}_f)^\perp$. We claim that this condition is also sufficient. We need a Poincaré inequality, which comes from the equivalence between the Fokker-Planck and Schrödinger operators [8]. Under suitable confining assumptions (cf. Theorem XIII.67 in [42]) we deduce.

Lemma 3.1

Let u be a vector in $\mathbb{R}^d$. Assume that the function $v \mapsto \frac{1}{4\sigma}|\nabla_v \Phi_u|^2 - \frac{1}{2}\Delta_v \Phi_u$ belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^d)$, is bounded from below and is coercive i.e.

$$
\lim_{|v| \to +\infty} \left[\frac{1}{4\sigma}|\nabla_v \Phi_u|^2 - \frac{1}{2}\Delta_v \Phi_u\right] = +\infty.
$$

Therefore there is $\lambda_u > 0$ such that for any $g \in H^1_{M_u}$ we have

$$\sigma \int_{\mathbb{R}^d} \left| \nabla_v \left(\frac{g}{M_u} \right) \right|^2 M_u(v)\, \mathrm{d}v \geq \lambda_u \int_{\mathbb{R}^d} \left| \frac{g(v)}{M_u(v)} - \int_{\mathbb{R}^d} g(v')\, \mathrm{d}v' \right|^2 M_u(v)\, \mathrm{d}v. \qquad (15)$$?PoiIne?

Proposition 3.2
Assume that (7) and the hypotheses of Lemma 3.1 hold true. Then for any $p \in L^2_{M_u}$ such that $\int_{\mathbb{R}^d} p(v)\, \mathrm{d}v = 0, \int_{\mathbb{R}^d} p(v) v\, \mathrm{d}v = 0$, there is a unique variational solution $g \in H^1_{M_u}$ for $-\mathcal{L}_f g = p$, satisfying $\int_{\mathbb{R}^d} g(v)\, \mathrm{d}v = 0, \int_{\mathbb{R}^d} g(v) v\, \mathrm{d}v = 0$. Moreover we have

$$\sup_{p \in (\ker \mathcal{L}_f)^\perp} \frac{\|g\|_{M_u}}{|p|_{M_u}} < +\infty.$$

Proof.
It is a direct consequence of Lax-Milgram lemma, applied to the bilinear symmetric form

$$a : H \times H \to \mathbb{R}, \;\; a(g,h) = \int_{\mathbb{R}^d} \left[\sigma \nabla_v \left(\frac{g}{M_u} \right) - W(g) \right] \cdot \left[\sigma \nabla_v \left(\frac{h}{M_u} \right) - W(h) \right] M_u(v)\, \mathrm{d}v, \;\; g, h \in H$$

and the linear form $l : H \to \mathbb{R}, l(h) = \sigma \int_{\mathbb{R}^d} \frac{p(v)h(v)}{M_u(v)}\, \mathrm{d}v, h \in H$ where $H = \{h \in H^1_{M_u} : \int_{\mathbb{R}^d} h(v)\, \mathrm{d}v = 0, \int_{\mathbb{R}^d} h(v) v\, \mathrm{d}v = 0\}$. Since the linear applications $h \to \int_{\mathbb{R}^d} h(v)\, \mathrm{d}v, h \to \int_{\mathbb{R}^d} h(v) v\, \mathrm{d}v$ are bounded on $L^2_{M_u}$ and thus on $H^1_{M_u}$, we deduce that H is a close subspace in $H^1_{M_u}$ and therefore a Hilbert space with respect to the scalar product of $H^1_{M_u}$. Thanks to (12), we have for any $g, h \in H^1_{M_u}$

$$\begin{aligned}
|a(g,h)| &\leq \left| \left[\sigma \left| \nabla_v \left(\frac{g}{M_u} \right) \right| + |W(g)| \right] M_u \right|_{M_u} \left| \left[\sigma \left| \nabla_v \left(\frac{h}{M_u} \right) \right| + |W(h)| \right] M_u \right|_{M_u} \\
&\leq 4\sigma^2 \left(\int_{\mathbb{R}^d} \left| \nabla_v \left(\frac{g}{M_u} \right) \right|^2 M_u(v)\, \mathrm{d}v \right)^{1/2} \left(\int_{\mathbb{R}^d} \left| \nabla_v \left(\frac{h}{M_u} \right) \right|^2 M_u(v)\, \mathrm{d}v \right)^{1/2} \\
&\leq 4\sigma^2 \|g\|_{M_u} \|h\|_{M_u}
\end{aligned}$$

saying that a is bounded on $H^1_{M_u} \times H^1_{M_u}$, and also on $H \times H$. We concentrate now on the coercivity of a. Notice that for any $g, h \in H^1_{M_u}$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} M_u(v) \sigma \nabla_v \left(\frac{g}{M_u} \right) \cdot \sigma \nabla_v \left(\frac{h}{M_u} \right)\, \mathrm{d}v \\
&= \int_{\mathbb{R}^d} M_u(v) \left[\sigma \nabla_v \left(\frac{g}{M_u} \right) - W(g) + W(g) \right] \cdot \left[\sigma \nabla_v \left(\frac{h}{M_u} \right) - W(h) + W(h) \right]\, \mathrm{d}v \\
&= a(g,h) + W(g) \cdot W(h) + \int_{\mathbb{R}^d} \left[\sigma M_u(v) \nabla_v \left(\frac{g}{M_u} \right) - M_u(v) W(g) \right]\, \mathrm{d}v \cdot W(h) \\
&+ W(g) \cdot \int_{\mathbb{R}^d} \left[\sigma M_u(v) \nabla_v \left(\frac{h}{M_u} \right) - M_u(v) W(h) \right]\, \mathrm{d}v \\
&= a(g,h) + W(g) \cdot W(h).
\end{aligned}$$

We claim that $\inf_{h \in H, |W(h)|=1} a(h,h) > 0$. Indeed, if it is not the case, there is a sequence $(h_n)_n \subset H$, $|W(h_n)| = 1, n \geq 1$ such that $a(h_n, h_n) < \frac{1}{n}, n \geq 1$. This sequence is bounded in $H^1_{M_u}$ because

$$\sigma^2 \int_{\mathbb{R}^d} M_u(v) \left| \nabla_v \left(\frac{h_n}{M_u} \right) \right|^2\, \mathrm{d}v = a(h_n, h_n) + |W(h_n)|^2 < \frac{1}{n} + 1, \;\; n \geq 1$$

and by the Poincaré inequality (15)

$$\lambda_u \int_{\mathbb{R}^d} \left(\frac{h_n}{M_u}\right)^2 M_u(v)\,\mathrm{d}v \le \sigma \int_{\mathbb{R}^d} \left|\nabla_v \left(\frac{h_n}{M_u}\right)\right|^2 M_u(v)\,\mathrm{d}v \le \frac{1}{\sigma}\left(1+\frac{1}{n}\right), \quad n \ge 1.$$

There is a sequence $(h_{n_k})_k$ which converges weakly toward some function h is H. Since the bilinear form a is bounded and non negative on $H \times H$, we have

$$0 \le a(h,h) \le \liminf_{k\to+\infty} a(h_{n_k}, h_{n_k}) \le \liminf_{k\to+\infty} \frac{1}{n_k} = 0.$$

We deduce that $a(h,h) = 0$, saying that $\sigma \nabla_v \left(\frac{h}{M_u}\right) - W(h) = 0$. It comes easily that $h \in \mathrm{span}\{M_u, v_1 M_u, ..., v_d M_u\}$ implying that $h = 0$. Since the linear application W is bounded on $H^1_{M_u}$, we obtain a contradiction

$$0 = |W(h)| = \lim_{k\to+\infty} |W(h_{n_k})| = \lim_{k\to+\infty} 1 = 1.$$

Therefore there is a constant μ_u such that

$$a(h,h) \ge \mu_u |W(h)|^2 \ \text{ for any } h \in H.$$

Finally we obtain for any $h \in H$

$$\begin{aligned} 2a(h,h) &\ge a(h,h) + \mu_u |W(h)|^2 \\ &\ge \min\{1,\mu_u\}\sigma^2 \int_{\mathbb{R}^d} M_u(v) \left|\nabla_v \left(\frac{h}{M_u}\right)\right|^2 \mathrm{d}v \\ &\ge \min\{1,\mu_u\}\sigma\lambda_u \int_{\mathbb{R}^d} \left(\frac{h}{M_u}\right)^2 M_u(v)\,\mathrm{d}v \end{aligned}$$

saying that the bilinear form a is coercive

$$\begin{aligned} 4a(h,h) &\ge \min\{1,\mu_u\}\sigma^2 \int_{\mathbb{R}^d} M_u(v) \left|\nabla_v \left(\frac{h}{M_u}\right)\right|^2 \mathrm{d}v \\ &\quad + \min\{1,\mu_u\}\lambda_u\sigma \int_{\mathbb{R}^d} M_u(v) \left(\frac{h}{M_u}\right)^2 \mathrm{d}v \\ &\ge \sigma \min\{1,\mu_u\} \min\{\sigma,\lambda_u\} \|h\|^2_{M_u}, \quad h \in H. \end{aligned}$$

Clearly, the linear form $l(h) = \sigma \int_{\mathbb{R}^d} p(v)h(v)/M_u(v)\,\mathrm{d}v$ is bounded on H and by Lax-Milgram lemma, there is a unique function $g \in H$ such that

$$a(g,h) = l(h), \quad h \in H.$$

Actually the previous formulation holds true for any $h \in H^1_{M_u}$, because for any $h \in \ker \mathcal{L}_f$ we have $a(g,h) = 0 = l(h)$. With the notation $\alpha = \min\{1,\mu_u\}\min\{\sigma,\lambda_u\}/4$ we have

$$\sigma\alpha \|g\|^2_{M_u} \le a(g,g) = l(g) \le \sigma |p|_{M_u} |g|_{M_u} \le \sigma |p|_{M_u} \|g\|_{M_u}$$

implying that $\|g\|_{M_u} \le \frac{|p|_{M_u}}{\alpha}$. □

4 The first order approximation

The goal of this section is to investigate the formal limit of (4) when $\varepsilon \searrow 0$. The terms of order 1 in (11) lead to

$$\partial_t f + v \cdot \nabla_x f = \mathcal{L}_f f_1, \ \ (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \tag{16}$$?Equ21?

with

$$f(t,x,v) = \rho(t,x) M_{u(t,x)}(v) = \rho(t,x) \frac{\exp\left(-\frac{\Phi_{u(t,x)}(v)}{\sigma}\right)}{Z(\sigma, |u(t,x)|)}.$$

Multiplying (16) by 1 and v, integrating with respect to v and by using the fact that the operator $\mathcal{L}_f$ preserves the mass and momentum, we get

$$\partial_t \int_{\mathbb{R}^d} f \, \mathrm{d}v + \mathrm{div}_x \int_{\mathbb{R}^d} f v \, \mathrm{d}v = \int_{\mathbb{R}^d} \mathcal{L}_f(f_1) \, \mathrm{d}v = 0 \tag{17}$$?EquMass?

$$\partial_t \int_{\mathbb{R}^d} f v \, \mathrm{d}v + \mathrm{div}_x \int_{\mathbb{R}^d} f v \otimes v \, \mathrm{d}v = \int_{\mathbb{R}^d} \mathcal{L}_f(f_1) v \, \mathrm{d}v = 0. \tag{18}$$?EquMom?

We determine the fluid equations satisfied by the macroscopic quantities ρ, u. By Proposition 2.1 we have

$$\int_{\mathbb{R}^d} f(t,x,v) v \, \mathrm{d}v = \rho(t,x) \int_{\mathbb{R}^d} M_u(v)(v \cdot \Omega[u]) \, \mathrm{d}v \, \Omega[u]$$

and

$$\int_{\mathbb{R}^d} M_u(v)(v \cdot \Omega[u]) \, \mathrm{d}v = \sigma \frac{\partial_l Z(\sigma, |u|)}{Z(\sigma, |u|)} + |u| = |u| \left(1 + \sigma \frac{\partial_l Z(\sigma, |u|)}{|u| Z(\sigma, |u|)}\right). \tag{19}$$?Equ22-Swarm?

Therefore the equation (17) becomes

$$\partial_t \rho + \mathrm{div}_x \left[\rho \left(1 + \sigma \frac{\partial_l Z}{|u| Z}\right) u\right] = 0.$$

We analyze now (18). We need to express $\int_{\mathbb{R}^d} M_u(v) v \otimes v \, \mathrm{d}v$ in terms of u. It happens that the elements of the tensor $\int_{\mathbb{R}^d} M_u(v) v \otimes v \, \mathrm{d}v$ depend only on two moments: $\int_{\mathbb{R}^d} M_u(v)(v \cdot \Omega)^2 \, \mathrm{d}v$ and $\int_{\mathbb{R}^d} M_u(v) |v_\perp|^2 \, \mathrm{d}v$, $v_\perp = v - (v \cdot \Omega)\Omega$. The expressions for these two moments, in terms of the function Z are detailed in the following lemma.

Lemma 4.1

1. For any $u \in \mathbb{R}^d \setminus \{0\}$ we have

$$\int_{\mathbb{R}^d} M_u(v) \frac{|v|^2 - (v \cdot \Omega[u])^2}{d-1} \, \mathrm{d}v = \sigma \frac{\int_{\mathbb{R}^d} M_u(v)(v \cdot \Omega[u]) \, \mathrm{d}v}{|u|} = \sigma + \sigma^2 \frac{\partial_l Z(\sigma, |u|)}{|u| Z(\sigma, |u|)}.$$

2. For any $\Omega \in \mathbb{S}^{d-1}$ we have

$$\int_{\mathbb{R}^d} M_0(v)(v \cdot \Omega)^2 \, \mathrm{d}v = \sigma + \sigma^2 \frac{\partial_{ll}^2 Z(\sigma, 0)}{Z(\sigma, 0)}.$$

Proof.

1. Multiplying $\sigma \nabla_v M_u + M_u(v) \nabla_v \Phi_u = 0$ by $(|v|^2 I_d - v \otimes v)\Omega$ with $\Omega = \Omega[u] = u/|u|$, we deduce after integration with respect to $v \in \mathbb{R}^d$

$$\int_{\mathbb{R}^d} \sigma \nabla_v M_u \cdot (|v|^2 I_d - v \otimes v)\Omega \, \mathrm{d}v + \int_{\mathbb{R}^d} M_u(v) \nabla_v \Phi_u \cdot (|v|^2 I_d - v \otimes v)\Omega \, \mathrm{d}v = 0. \tag{20}$$?Equ23-Swarm?

But $\mathrm{div}_v\{(|v|^2 I_d - v\otimes v)\Omega\} = -(d-1)(v\cdot\Omega)$ and

$$\nabla_v \Phi_u \cdot (|v|^2 I_d - v\otimes v)\Omega = [v - u + \nabla_v V(|\cdot|)]\cdot(|v|^2 I_d - v\otimes v)\Omega = -|v|^2|u| + (v\cdot\Omega)^2|u|.$$

Therefore (20) becomes

$$\sigma(d-1)\int_{\mathbb{R}^d} M_u(v)(v\cdot\Omega)\,\mathrm{d}v - \int_{\mathbb{R}^d} M_u(v)[|v|^2 - (v\cdot\Omega)^2]\,\mathrm{d}v\,|u| = 0$$

implying, thanks to (19) that

$$\int_{\mathbb{R}^d} M_u(v)\frac{|v|^2 - (v\cdot\Omega[u])^2}{d-1}\,\mathrm{d}v = \sigma\frac{\int_{\mathbb{R}^d} M_u(v)(v\cdot\Omega[u])\,\mathrm{d}v}{|u|} = \sigma + \sigma^2\frac{\partial_l Z(\sigma,|u|)}{|u|Z(\sigma,|u|)}.$$

2. By the last point in Proposition 2.1, with $u=0$, we have for any $\Omega\in\mathbb{S}^{d-1}$

$$\frac{\partial^2_{ll} Z(\sigma,0)}{Z(\sigma,0)} = \int_{\mathbb{R}^d} M_0(v)\frac{(v\cdot\Omega)^2}{\sigma^2}\,\mathrm{d}v - \frac{1}{\sigma}$$

which is exactly our conclusion. □

The reduction of the tensor $\int_{\mathbb{R}^d} M_u(v)v\otimes v\,\mathrm{d}v$ to the moments $\int_{\mathbb{R}^d} M_u(v)(v\cdot\Omega)^2\,\mathrm{d}v$, $\int_{\mathbb{R}^d} M_u(v)|v_\perp|^2\,\mathrm{d}v$ comes by the invariance of M_u with respect to the transformations of $\mathcal{T}_u$. The details of these computations are postponed to Appendix A, see the proof of Lemma 4.2.

Lemma 4.2
1. For any $u\in\mathbb{R}^d\setminus\{0\}$ we have

$$\int_{\mathbb{R}^d} M_u(v)v\otimes v\,\mathrm{d}v = \left(1 + 2\sigma\frac{\partial_l Z}{|u|Z}\right)u\otimes u + \sigma^2\frac{\partial^2_{ll}Z - \frac{\partial_l Z}{|u|}}{Z}\Omega[u]\otimes\Omega[u] + \left[\sigma + \sigma^2\frac{\partial_l Z}{|u|Z}(\sigma,|u|)\right]I_d$$

2. We have

$$\int_{\mathbb{R}^d} M_0(v)v\otimes v\,\mathrm{d}v = \left[\sigma + \sigma^2\frac{\partial^2_{ll}Z(\sigma,0)}{Z(\sigma,0)}\right]I_d.$$

Remark 4.1
As $\partial_l Z(\sigma,0) = 0, \sigma > 0$, we have

$$\lim_{u\to 0}\left(1 + 2\sigma\frac{\partial_l Z}{|u|Z}(\sigma,|u|)\right)u\otimes u = 0,\quad \lim_{u\to 0}\sigma^2\frac{\partial^2_{ll}Z - \frac{\partial_l Z}{|u|}}{Z} = 0$$

$$\lim_{u\to 0}\left(\sigma + \sigma^2\frac{\partial_l Z(\sigma,|u|)}{|u|Z(\sigma,|u|)}\right) = \sigma + \sigma^2\frac{\partial^2_{ll}Z(\sigma,0)}{Z(\sigma,0)}.$$

Therefore the formulae in Lemma 4.2 are coherent as $u\to 0$.

Based on the previous computations, we have

$$\int_{\mathbb{R}^d}\rho M_u v\otimes v\,\mathrm{d}v = \rho\left(1 + 2\sigma\frac{\partial_l Z}{|u|Z}\right)u\otimes u + \rho\sigma^2\frac{\partial^2_{ll}Z - \frac{\partial_l Z}{|u|}}{Z}\Omega[u]\otimes\Omega[u] + \rho\left(\sigma + \sigma^2\frac{\partial_l Z}{|u|Z}\right)I_d$$

reducing to $\rho\left(\sigma + \sigma^2\frac{\partial^2_{ll}Z(\sigma,0)}{Z(\sigma,0)}\right)I_d$ when $u=0$, and we obtain Theorem 1.1. A more convenient macroscopic quantity when studying fluid models is the current $j(t,x) = \int_{\mathbb{R}^d} f(t,x,v)v\,\mathrm{d}v$, $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$. The mass balance simply writes $\partial_t\rho + \mathrm{div}_x j = 0$. In order to write the momentum balance with respect to ρ, j we need to express u in terms of ρ, j. This can be done by using the function $\mathcal{E}(\sigma,l) = \frac{l^2}{2} + \sigma\ln Z(\sigma,l), (\sigma,l)\in\mathbb{R}_+^\star\times\mathbb{R}_+$. The properties of the function $\mathcal{E}$ are discussed in the next proposition.

Proposition 4.1
The function $\mathcal{E}(\sigma,\cdot)$ is strictly convex for any $\sigma>0$.

Proof.
The first derivative of $\mathcal{E}$ with respect to l is $\partial_l\mathcal{E}=l+\sigma\frac{\partial_l Z}{Z}$ and the second one is $\partial^2_{ll}\mathcal{E}=1+\sigma\frac{\partial^2_{ll}Z}{Z}-\sigma\frac{(\partial_l Z)^2}{Z^2}$. We are done if we prove that $\partial^2_{ll}\mathcal{E}(\sigma,l)>0$ for any $l>0$. For any $l>0,\Omega\in\mathbb{S}^{d-1}$ we know by Proposition 2.1 that

$$\frac{\partial_l Z(\sigma,l)}{Z(\sigma,l)}=\int_{\mathbb{R}^d}M_{l\Omega}(v)\frac{(v-l\Omega)\cdot\Omega}{\sigma}\,\mathrm{d}v$$

and

$$1+\sigma\frac{\partial^2_{ll}Z(\sigma,l)}{Z(\sigma,l)}=\int_{\mathbb{R}^d}M_{l\Omega}(v)\frac{((v-l\Omega)\cdot\Omega)^2}{\sigma}\,\mathrm{d}v.$$

We deduce that

$$\begin{aligned}\sigma\left(\frac{\partial_l Z}{Z}\right)^2&=\sigma\left(\int_{\mathbb{R}^d}M_{l\Omega}(v)\frac{(v-l\Omega)\cdot\Omega}{\sigma}\,\mathrm{d}v\right)^2\\&\le\frac{1}{\sigma}\left(\int_{\mathbb{R}^d}\sqrt{M_{l\Omega}(v)}\sqrt{M_{l\Omega}(v)}|(v-l\Omega)\cdot\Omega|\,\mathrm{d}v\right)^2\\&<\frac{1}{\sigma}\int_{\mathbb{R}^d}M_{l\Omega}(v)((v-l\Omega)\cdot\Omega)^2\,\mathrm{d}v=1+\sigma\frac{\partial^2_{ll}Z(\sigma,l)}{Z(\sigma,l)}\end{aligned}$$

saying that $\partial^2_{ll}\mathcal{E}(\sigma,l)>0$ for any $\sigma>0,l>0$. □

We introduce the convex conjugate function

$$\mathcal{E}^\star(\sigma,k)=\sup_{l\in\mathbb{R}_+}\{kl-\mathcal{E}(\sigma,l)\},\ k\in\mathbb{R}_+.$$

We are looking for a relation between the current j and the velocity u. From Proposition 2.1, second and third statements, we infer that

$$\frac{j}{\rho}=\int_{\mathbb{R}^d}M_u(v)v\,\mathrm{d}v=\underbrace{\int_{\mathbb{R}^d}M_u(v)(v\cdot\Omega[u])\,\mathrm{d}v}_{\ge0}\,\Omega[u]=\left(|u|+\sigma\frac{\partial_l Z}{Z(\sigma,|u|)}\right)\Omega[u].$$

The current j and the velocity u being oriented in the same direction, we get

$$\frac{|j|}{\rho}=\int_{\mathbb{R}^d}M_u(v)(v\cdot\Omega[u])\,\mathrm{d}v=|u|+\sigma\frac{\partial_l Z}{Z(\sigma,|u|)}=\partial_l\mathcal{E}(\sigma,l=|u|)\tag{21}$$?EquModj?

implying that

$$|u|=\partial_k\mathcal{E}^\star\left(\sigma,k=\frac{|j|}{\rho}\right).\tag{22}$$?EquModu?

Moreover taking the derivative with respect to l in $\partial_k\mathcal{E}^\star(\sigma,k=\partial_l\mathcal{E}(\sigma,l))=l$, we obtain $\partial^2_{kk}\mathcal{E}^\star(\sigma,k=\partial_l\mathcal{E}(\sigma,l))\partial^2_{ll}\mathcal{E}(\sigma,l)=1$. We introduce the notation

$$\mathcal{M}_u=\int_{\mathbb{R}^d}M_u(v)\frac{(v-\frac{j}{\rho})\otimes(v-\frac{j}{\rho})}{\sigma}\,\mathrm{d}v.\tag{23}$$?EqucalMu?

The tensor $\int_{\mathbb{R}^d}M_u(v)v\otimes v\,\mathrm{d}v$, computed in Lemma 4.2, in terms of the function Z, can be expressed as well thanks to the convex function $\mathcal{E}$. These calculations are technical, the details are presented in Appendix A, see proof of Lemma 4.3.

Lemma 4.3
1. For any $u \in \mathbb{R}^d \setminus \{0\}$ we have

$$\int_{\mathbb{R}^d} M_u(v) v \otimes v \, \mathrm{d}v = \sigma \frac{\partial_l \mathcal{E}(\sigma,|u|)}{|u|} I_d + \frac{j \otimes j}{\rho^2} + \sigma \left(\partial_{ll}^2 \mathcal{E}(\sigma,|u|) - \frac{\partial_l \mathcal{E}(\sigma,|u|)}{|u|} \right) \Omega[u] \otimes \Omega[u]$$

and

$$\mathcal{M}_u = \frac{\partial_l \mathcal{E}(\sigma,|u|)}{|u|} (I_d - \Omega[u] \otimes \Omega[u]) + \partial_{ll}^2 \mathcal{E}(\sigma,|u|) \Omega[u] \otimes \Omega[u].$$

2. We have $\int_{\mathbb{R}^d} M_0(v) v \otimes v \, \mathrm{d}v = \sigma \partial_{ll}^2 \mathcal{E}(\sigma,0) I_d$ and $\mathcal{M}_0 = \partial_{ll}^2 \mathcal{E}(\sigma,0) I_d$.

Remark 4.2
We have $\partial_l \mathcal{E}(\sigma,0) = \sigma \frac{\partial_l Z(\sigma,0)}{Z(\sigma,0)} = 0$ and therefore $\lim_{u \to 0} \frac{\partial_l \mathcal{E}(\sigma,|u|)}{|u|} = \partial_{ll}^2 \mathcal{E}(\sigma,0)$ saying that the formulae in Lemma 4.3 are coherent as $u \to 0$.

Remark 4.3
1. For any $u \in \mathbb{R}^d \setminus \{0\}$, as in the proof of Lemma 4.2 we obtain

$$\sigma \mathcal{M}_u = \int_{\mathbb{R}^d} M_u(v) ((v - \frac{j}{\rho}) \cdot \Omega[u])^2 \, \mathrm{d}v \, \Omega[u] \otimes \Omega[u] + \int_{\mathbb{R}^d} M_u(v) \frac{|v_\perp|^2}{d-1} \, \mathrm{d}v \, (I_d - \Omega[u] \otimes \Omega[u])$$

where $v_\perp = v - (v \cdot \Omega[u]) \, \Omega[u]$, implying that

$$\int_{\mathbb{R}^d} M_u(v) ((v - \frac{j}{\rho}) \cdot \Omega[u])^2 \, \mathrm{d}v = \sigma \partial_{ll}^2 \mathcal{E}(\sigma,|u|), \quad \int_{\mathbb{R}^d} M_u(v) \frac{|v_\perp|^2}{d-1} \, \mathrm{d}v = \sigma \frac{\partial_l \mathcal{E}(\sigma,|u|)}{|u|}.$$

2. For any $\Omega \in \mathbb{S}^{d-1}$ we have $\int_{\mathbb{R}^d} M_0(v) (v \cdot \Omega)^2 \, \mathrm{d}v = \sigma \partial_{ll}^2 \mathcal{E}(\sigma,0)$.

Based on the results obtained in Lemmas 4.2, 4.3, we are ready to determine the fluid model with respect to ρ, j. We complete this limit model by the entropy inequality satisfied by ρ and j.

Proof. (of Theorem 1.2)
The mass balance gives the continuity equation

$$\partial_t \rho + \mathrm{div}_x j = 0. \tag{24}$$?Equ31-Swarm?

Thanks to the equalities

$$\frac{|j|}{\rho} = \partial_l \mathcal{E}(\sigma,|u|), \quad |u| = \partial_k \mathcal{E}^\star \left(\sigma, \frac{|j|}{\rho} \right), \quad \partial_{ll}^2 \mathcal{E}(\sigma,|u|) = \frac{1}{\partial_{kk}^2 \mathcal{E}^\star(\sigma, \frac{|j|}{\rho})}$$

we obtain

$$\int_{\mathbb{R}^d} \rho M_u(v) v \otimes v \, \mathrm{d}v = \frac{j \otimes j}{\rho} + \sigma \frac{|j|}{|u|} I_d + \sigma \left(\frac{1}{k^2 \partial_{kk}^2 \mathcal{E}^\star} - \frac{1}{k \partial_k \mathcal{E}^\star} \right)_{k=|j|/\rho} \frac{j \otimes j}{\rho}.$$

Therefore the momentum balance becomes

$$\partial_t j + \mathrm{div}_x \left[\frac{j \otimes j}{\rho} + \sigma \frac{|j|}{|u|} I_d + \sigma \left(\frac{1}{k^2 \partial_{kk}^2 \mathcal{E}^\star} - \frac{1}{k \partial_k \mathcal{E}^\star} \right)_{k=|j|/\rho} \frac{j \otimes j}{\rho} \right] = 0. \tag{25}$$?Equ32-Swarm?

Observe that

$$\sigma \frac{|j|}{|u|} I_d + \sigma \left(\frac{1}{k^2 \partial_{kk}^2 \mathcal{E}^\star} - \frac{1}{k \partial_k \mathcal{E}^\star} \right)_{k=|j|/\rho} \frac{j \otimes j}{\rho} = \sigma \rho \frac{\partial_l \mathcal{E}(\sigma,|u|)}{|u|} (I_d - \Omega[u] \otimes \Omega[u])$$
$$+ \sigma \rho \partial_{ll}^2 \mathcal{E}(\sigma,|u|) \Omega[u] \otimes \Omega[u] = \sigma \rho \mathcal{M}_u$$

and therefore (25) becomes

$$\partial_t j + \mathrm{div}_x \left(\frac{j \otimes j}{\rho} + \sigma \rho \mathcal{M}_u \right) = 0. \tag{26}$$?Equ32bis?

Using the continuity equation we observe that

$$\begin{aligned} \partial_t j + \mathrm{div}_x \frac{j \otimes j}{\rho} &= (\partial_t \rho) \frac{j}{\rho} + \rho \partial_t \left(\frac{j}{\rho} \right) + \mathrm{div}_x j \, \frac{j}{\rho} + (j \cdot \nabla_x) \left(\frac{j}{\rho} \right) \\ &= \rho \left[\partial_t \left(\frac{j}{\rho} \right) + \left(\frac{j}{\rho} \cdot \nabla_x \right) \left(\frac{j}{\rho} \right) \right] \end{aligned}$$

and the above momentum balance also writes

$$\begin{aligned} \partial_t \left(\frac{j}{\rho} \right) + \left(\frac{j}{\rho} \cdot \nabla_x \right) \left(\frac{j}{\rho} \right) + \frac{\sigma}{\rho} \nabla_x \left(\rho \frac{k}{\partial_k \mathcal{E}^\star(\sigma, k)} \right)_{k=|j|/\rho} \\ + \frac{\sigma}{\rho} \mathrm{div}_x \left\{ \rho \left(\frac{1}{k^2 \partial^2_{kk} \mathcal{E}^\star} - \frac{1}{k \partial_k \mathcal{E}^\star} \right)_{k=|j|/\rho} \frac{j}{\rho} \otimes \frac{j}{\rho} \right\} = 0. \end{aligned}$$

We concentrate now on the entropy inequality. We have

$$\begin{aligned} &\int_{\mathbb{R}^d} Q(f^\varepsilon) \left[\sigma(1 + \ln f^\varepsilon) + \frac{|v|^2}{2} + V(|v|) \right] \, \mathrm{d}v \\ &\quad = - \int_{\mathbb{R}^d} [\sigma \nabla_v f^\varepsilon + f^\varepsilon (v - u[f^\varepsilon] + \nabla_v V(|\cdot|)] \cdot \left[\sigma \frac{\nabla_v f^\varepsilon}{f^\varepsilon} + v + \nabla_v V(|\cdot|) \right] \, \mathrm{d}v \\ &\quad = - \int_{\mathbb{R}^d} \frac{|\sigma \nabla_v f^\varepsilon + f^\varepsilon \nabla_v \Phi_{u[f^\varepsilon]}|^2}{f^\varepsilon} \, \mathrm{d}v \le 0, \ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d. \end{aligned}$$

Therefore, multiplying (4) by $\sigma(1 + \ln f^\varepsilon) + \frac{|v|^2}{2} + V(|v|)$, one gets after integration with respect to $v \in \mathbb{R}^d$

$$\partial_t \int_{\mathbb{R}^d} f^\varepsilon \left(\sigma \ln f^\varepsilon + \frac{|v|^2}{2} + V(|v|) \right) \, \mathrm{d}v + \mathrm{div}_x \int_{\mathbb{R}^d} v f^\varepsilon \left(\sigma \ln f^\varepsilon + \frac{|v|^2}{2} + V(|v|) \right) \, \mathrm{d}v \le 0.$$

When $\varepsilon \searrow 0$ we expect that

$$\partial_t \int_{\mathbb{R}^d} f \left(\sigma \ln f + \frac{|v|^2}{2} + V(|v|) \right) \, \mathrm{d}v + \mathrm{div}_x \int_{\mathbb{R}^d} v f \left(\sigma \ln f + \frac{|v|^2}{2} + V(|v|) \right) \, \mathrm{d}v \le 0.$$

We have

$$\sigma \ln f + \frac{|v|^2}{2} + V(|v|) = \sigma \ln \frac{\rho}{Z} - \Phi_u(v) + \Phi_0(v) = \sigma \ln \frac{\rho}{Z} - \frac{|u|^2}{2} + v \cdot u$$

implying that

$$\begin{aligned} \int_{\mathbb{R}^d} f(\sigma \ln f + \Phi_0(v)) \, \mathrm{d}v &= \int_{\mathbb{R}^d} f(\sigma \ln \rho - \sigma \ln Z - \frac{|u|^2}{2} + v \cdot u) \, \mathrm{d}v \\ &= \sigma \rho \ln \rho - \rho \sigma \ln Z - \rho \frac{|u|^2}{2} + j \cdot u \\ &= \sigma \rho \ln \rho - \rho \mathcal{E}(\sigma, |u|) + |j| \, |u| \\ &= \sigma \rho \ln \rho + \rho \left(\frac{|j|}{\rho} |u| - \mathcal{E}(\sigma, |u|) \right) \\ &= \sigma \rho \ln \rho + \rho \mathcal{E}^\star \left(\sigma, \frac{|j|}{\rho} \right). \end{aligned}$$

Similarly, thanks to Lemma 4.3, we compute

$$\begin{aligned}\int_{\mathbb{R}^d} f(\sigma \ln f + \Phi_0(v))v \, \mathrm{d}v &= \int_{\mathbb{R}^d} f(\sigma \ln \rho - \sigma \ln Z - \frac{|u|^2}{2} + v \cdot u)v \, \mathrm{d}v \\ &= \sigma \ln \rho \, j - \sigma \ln Z \, j - \frac{|u|^2}{2} j + \rho \int_{\mathbb{R}^d} M_u(v) v \otimes v \, \mathrm{d}v \, u \\ &= \sigma \ln \rho \, j - \mathcal{E}(\sigma, |u|) j + \frac{j \otimes j}{\rho} u + \sigma \rho \partial^2_{ll} \mathcal{E} u \\ &= \sigma \ln \rho \, j + \left[\frac{|j|}{\rho} |u| - \mathcal{E}(\sigma, |u|) \right] j + \sigma \rho \partial^2_{ll} \mathcal{E} u \\ &= \sigma \ln \rho \, j + \mathcal{E}^\star \left(\sigma, \frac{|j|}{\rho} \right) j + \sigma \frac{\partial_k \mathcal{E}^\star(\sigma, |j|/\rho)}{\partial^2_{kk} \mathcal{E}^\star(\sigma, |j|/\rho) |j|/\rho} j.\end{aligned}$$

Therefore ρ and j satisfy the entropy inequality

$$\partial_t \{\rho(\sigma \ln \rho + \mathcal{E}^\star(\sigma, |j|/\rho))\} + \mathrm{div}_x \left\{ \left[\sigma \ln \rho + \left(\mathcal{E}^\star(\sigma, k) + \sigma \frac{\partial_k \mathcal{E}^\star}{k \partial^2_{kk} \mathcal{E}^\star} \right)_{k=|j|/\rho} \right] j \right\} \leq 0. \quad (27)$$?Equ35?

□

Remark 4.4
When $V = 0$, the equilibria are the standard Maxwellians on $\mathbb{R}^d$

$$M_u(v) = \frac{1}{(2\pi\sigma)^{d/2}} \exp\left(-\frac{|v-u|^2}{2\sigma} \right), \quad v \in \mathbb{R}^d.$$

In that case the function $l \to Z(\sigma, l)$ is constant

$$Z(\sigma, l) = \int_{\mathbb{R}^d} \exp\left(-\frac{|v-u|^2}{2\sigma} \right) \mathrm{d}v = (2\pi\sigma)^{d/2}$$

and we obtain $j = \rho u$, that is, in that case u is the usual mean velocity. Clearly we have

$$\mathcal{E}(\sigma, l) = \frac{l^2}{2} + \sigma \ln(2\pi\sigma)^{d/2}, \quad \mathcal{E}^\star(\sigma, k) = \frac{k^2}{2} - \sigma \ln(2\pi\sigma)^{d/2}$$

and therefore

$$\partial_l \mathcal{E}(\sigma, l) = l, \quad \partial^2_{ll} \mathcal{E}(\sigma, l) = 1, \quad \partial_k \mathcal{E}^\star(\sigma, k) = k, \quad \partial^2_{kk} \mathcal{E}^\star(\sigma, k) = 1.$$

The equations (24), (25) become the compressible Euler equations for the gas dynamics

$$\partial_t \rho + \mathrm{div}_x(\rho u) = 0, \quad \partial_t(\rho u) + \mathrm{div}_x(\rho u \otimes u + \sigma \rho I_d) = 0.$$

The entropy inequality (27) writes

$$\partial_t \{\rho(\sigma \ln \rho + \frac{|u|^2}{2} - \sigma \ln(2\pi\sigma)^{d/2})\} + \mathrm{div}_x\{(\sigma \ln \rho + \frac{|u|^2}{2} - \sigma \ln(2\pi\sigma)^{d/2} + \sigma) j\} \leq 0$$

which is the entropy inequality for the Euler equations

$$\partial_t(\sigma\rho \ln \rho + \rho \frac{|u|^2}{2}) + \mathrm{div}_x\{(\sigma\rho \ln \rho + \rho \frac{|u|^2}{2} + \sigma\rho) u\} \leq 0.$$

5 The second order approximation

Motivated by the derivation of fluid models including diffusion terms, we have to consider second order approximations. That is, we need to take into account the first order corrections when approximating the particle density *i.e.,* $f^\varepsilon \approx f + \varepsilon f_1$. In that case we search for fluid equations involving $\rho[f+\varepsilon f_1]$ and $j[f+\varepsilon f_1]$. We are led to a model similar to the compressible Navier-Stokes equations. We appeal to the mass and momentum balances of (4)

$$\partial_t \int_{\mathbb{R}^d} f^\varepsilon \,\mathrm{d}v + \mathrm{div}_x \int_{\mathbb{R}^d} f^\varepsilon v \,\mathrm{d}v = 0 \tag{28}$$?Equ36-Swarm?

$$\partial_t \int_{\mathbb{R}^d} f^\varepsilon \, v \,\mathrm{d}v + \mathrm{div}_x \int_{\mathbb{R}^d} f^\varepsilon \, v \otimes v \,\mathrm{d}v = 0. \tag{29}$$?Equ37-Swarm?

We introduce the notations

$$\tilde{f}^\varepsilon = f + \varepsilon f_1, \;\; \tilde{\rho}^\varepsilon = \rho[\tilde{f}^\varepsilon] = \int_{\mathbb{R}^d} \tilde{f}^\varepsilon \,\mathrm{d}v, \;\; \tilde{j}^\varepsilon = j[\tilde{f}^\varepsilon] = \int_{\mathbb{R}^d} \tilde{f}^\varepsilon v \,\mathrm{d}v$$

and therefore we expect from the balances (28), (29) that

$$\partial_t \tilde{\rho}^\varepsilon + \mathrm{div}_x \tilde{j}^\varepsilon = 0 \tag{30}$$?Equ38-Swarm?

$$\partial_t \tilde{j}^\varepsilon + \mathrm{div}_x \int_{\mathbb{R}^d} \tilde{f}^\varepsilon \, v \otimes v \,\mathrm{d}v = 0. \tag{31}$$?Equ39-Swarm?

We need to compute $\int_{\mathbb{R}^d} \tilde{f}^\varepsilon v \otimes v \,\mathrm{d}v = \int_{\mathbb{R}^d} (f + \varepsilon f_1) v \otimes v \,\mathrm{d}v$. The notation P_u stands for the orthogonal projection on the subspace $\ker \mathcal{L}_f \subset L^2_{M_u}$, for any equilibrium $f = \rho M_u$. We can write

$$\int_{\mathbb{R}^d} \tilde{f}^\varepsilon v \otimes v \,\mathrm{d}v = \int_{\mathbb{R}^d} (f + \varepsilon P_u f_1) v \otimes v \,\mathrm{d}v + \varepsilon \int_{\mathbb{R}^d} (f_1 - P_u f_1) v \otimes v \,\mathrm{d}v.$$

The tensor $\int_{\mathbb{R}^d} M_u(v) v \otimes v \,\mathrm{d}v$ has been computed in Lemma 4.3 and therefore, we have a formula for $\int_{\mathbb{R}^d} f v \otimes v \,\mathrm{d}v$ in terms of $\rho[f], j[f]$. It happens that, up to second order terms, the same formula holds true for $\int_{\mathbb{R}^d} (f + \varepsilon P_u f_1) v \otimes v \,\mathrm{d}v$.

Proposition 5.1
For any equilibrium $f = \rho M_u$ and $g \in \ker \mathcal{L}_f$ we have, as $\varepsilon \searrow 0$

$$\int_{\mathbb{R}^d} (f + \varepsilon g) v \otimes v \,\mathrm{d}v = \frac{j[f+\varepsilon g] \otimes j[f+\varepsilon g]}{\rho[f+\varepsilon g]} + \sigma \frac{|j[f+\varepsilon g]|}{\partial_k \mathcal{E}^\star(\sigma, |j[f+\varepsilon g]|/\rho[f+\varepsilon g])} I_d$$
$$+ \sigma \left(\frac{1}{k^2 \partial^2_{kk} \mathcal{E}^\star} - \frac{1}{k \partial_k \mathcal{E}^\star} \right)_{k = |j[f+\varepsilon g]|/\rho[f+\varepsilon g]} \frac{j[f+\varepsilon g] \otimes j[f+\varepsilon g]}{\rho[f+\varepsilon g]} + \mathcal{O}(\varepsilon^2).$$

Proof.
We introduce the notations $R^\varepsilon = \rho[f+\varepsilon g], J^\varepsilon = j[f+\varepsilon g], U^\varepsilon = \partial_k \mathcal{E}^\star(\sigma, |J^\varepsilon|/R^\varepsilon) \Omega[J^\varepsilon]$ for any $\varepsilon \geq 0$. The equilibria $F^\varepsilon = R^\varepsilon M_{U^\varepsilon}, \varepsilon \geq 0$ verify

$$\int_{\mathbb{R}^d} F^\varepsilon \,\mathrm{d}v = R^\varepsilon = \int_{\mathbb{R}^d} (f + \varepsilon g) \,\mathrm{d}v$$

$$\begin{aligned}\int_{\mathbb{R}^d} F^\varepsilon v\,\mathrm{d}v &= R^\varepsilon \int_{\mathbb{R}^d} M_{U^\varepsilon}(v)\, v\,\mathrm{d}v = R^\varepsilon \int_{\mathbb{R}^d} M_{U^\varepsilon}(v)(v\cdot\Omega[U^\varepsilon])\,\mathrm{d}v\,\Omega[U^\varepsilon]\\ &= R^\varepsilon\left[|U^\varepsilon| + \sigma\frac{\partial_l Z}{Z}(\sigma,|U^\varepsilon|)\right]\Omega[U^\varepsilon]\\ &= R^\varepsilon \partial_l\mathcal{E}(\sigma,|U^\varepsilon|)\,\Omega[U^\varepsilon]\\ &= R^\varepsilon\frac{|J^\varepsilon|}{R^\varepsilon}\Omega[U^\varepsilon]\\ &= J^\varepsilon = \int_{\mathbb{R}^d}(f+\varepsilon g)\, v\,\mathrm{d}v.\end{aligned}$$

Therefore, we have for any $\psi \in \mathrm{span}\{1, v_1, ..., v_d\}$

$$\int_{\mathbb{R}^d}(f+\varepsilon g - F^\varepsilon)\psi(v)\,\mathrm{d}v = 0,\ \ \varepsilon \geq 0.$$

Taking the derivative with respect to ε at $\varepsilon = 0$, one gets for any $\psi \in \mathrm{span}\{1, v_1, ..., v_d\}$

$$\int_{\mathbb{R}^d}\left(g - \frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0}F^\varepsilon\right)\psi(v)\,\mathrm{d}v = 0.$$

It is easily seen, by direct computation, that $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0}F^\varepsilon \in \ker\mathcal{L}_f$ and thus we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0}F^\varepsilon = P_u g = g.$$

Finally we have

$$\begin{aligned}f+\varepsilon g &= F^\varepsilon + f + \varepsilon g - F^\varepsilon\\ &= R^\varepsilon M_{U^\varepsilon} + \underbrace{f - F^0}_{=0} + \varepsilon\underbrace{\left(g - \frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0}F^\varepsilon\right)}_{=0} + \mathcal{O}(\varepsilon^2)\\ &= R^\varepsilon M_{U^\varepsilon} + \mathcal{O}(\varepsilon^2)\end{aligned}$$

and thanks to Lemma 4.3 we deduce

$$\begin{aligned}\int_{\mathbb{R}^d}(f+\varepsilon g)v\otimes v\,\mathrm{d}v &= \int_{\mathbb{R}^d} R^\varepsilon M_{U^\varepsilon} v\otimes v\,\mathrm{d}v + \mathcal{O}(\varepsilon^2)\\ &= \frac{J^\varepsilon\otimes J^\varepsilon}{R^\varepsilon} + \sigma\frac{|J^\varepsilon|}{|U^\varepsilon|}I_d + \sigma\left(\frac{1}{k^2\partial^2_{kk}\mathcal{E}^\star} - \frac{1}{k\partial_k\mathcal{E}^\star}\right)_{k=|J^\varepsilon|/R^\varepsilon}\frac{J^\varepsilon\otimes J^\varepsilon}{R^\varepsilon} + \mathcal{O}(\varepsilon^2)\\ &= \frac{J^\varepsilon\otimes J^\varepsilon}{R^\varepsilon} + \sigma R^\varepsilon\mathcal{M}_{U^\varepsilon} + \mathcal{O}(\varepsilon^2).\end{aligned}$$

□

Recall that we need to compute $\int_{\mathbb{R}^d}(f+\varepsilon f_1)v\otimes v\,\mathrm{d}v$. By the previous proposition we are done if we determine $\varepsilon\int_{\mathbb{R}^d}(f_1 - P_u f_1)v\otimes v\,\mathrm{d}v$. The quantity $f_1 - P_u f_1$ comes by (16)

$$\partial_t f + v\cdot\nabla_x f = \mathcal{L}_f f_1 = \mathcal{L}_f(f_1 - P_u f_1)$$

which is solvable, cf. Proposition 3.2, because the macroscopic quantities defining the equilibria $f(t,x,\cdot)$ were determined by imposing the mass and momentum conservations. We need to compute the antecedent by $\mathcal{L}_f$ of $\partial_t f + v\cdot\nabla_x f$. It is instructive to determine first a fluid second order approximation in a simplified case, when replacing the interaction mechanism Q by a BGK kernel, whose equilibria coincide with that of Q.

5.1 The BGK interaction mechanism

As we will see, the linearization of this BGK operator reduces to the opposite of the identity on the orthogonal of its kernel, and therefore the antecedent of $\partial_t f + v \cdot \nabla_x f$ comes immediately, without any computation.

Proposition 5.2
We consider the kernel $Q_{\mathrm{BGK}}(f) = \rho[f]M_{U[f]} - f$ *where* $\rho[f] = \int_{\mathbb{R}^d} f(v)\,\mathrm{d}v$ *and the velocity* $U[f]$ *is given by*

$$U[f] = \begin{cases} 0, & \text{if } j[f] = 0 \\ l\frac{j[f]}{|j[f]|},\ \frac{|j[f]|}{\rho[f]} = \partial_l \mathcal{E}(\sigma, l), & \text{if } j[f] \neq 0. \end{cases}$$

The interaction mechanisms Q_{BGK}, Q *have the same equilibria. The kernel* Q_{BGK} *conserves the mass and momentum.*

Proof.
Any equilibrium of Q has the form $f = \rho M_u$ where $\rho = \rho[f] \in \mathbb{R}_+$ and $u \in \mathbb{R}^d$. We claim that $U[\rho M_u] = u$. Indeed, if $u = 0$, then $j[f] = \int_{\mathbb{R}^d} \rho M_0(v)\,\mathrm{d}v = 0$ and therefore $U[\rho M_u] = 0$. If $u \in \mathbb{R}^d \setminus \{0\}$ we have by Proposition 2.1

$$\partial_l \mathcal{E}(\sigma, |u|) = |u| + \sigma \frac{\partial_l Z(\sigma, |u|)}{Z(\sigma, |u|)} = \int_{\mathbb{R}^d} M_u(v)(v \cdot \Omega[u])\,\mathrm{d}v = \frac{|j[f]|}{\rho[f]} = \partial_l \mathcal{E}(\sigma, |U[f]|)$$

saying that $|U[f]| = |u|$ and

$$U[f] = |U[f]|\frac{j[f]}{|j[f]|} = |u|\frac{j[f]}{|j[f]|} = u.$$

Conversely, any equilibrium of Q_{BGK} is a equilibrium of Q. We investigate now the mass and momentum balances. For any particle density $f = f(v)$ we have

$$\int_{\mathbb{R}^d} Q_{\mathrm{BGK}}(f)(v)\,\mathrm{d}v = \int_{\mathbb{R}^d} (\rho[f]M_{U[f]}(v) - f(v))\,\mathrm{d}v = 0$$

and

$$\begin{aligned} \int_{\mathbb{R}^d} Q_{\mathrm{BGK}}(f)(v)\,\mathrm{d}v &= \int_{\mathbb{R}^d} \rho[f]M_{U[f]}(v)v\,\mathrm{d}v - j[f] \\ &= \rho[f]\int_{\mathbb{R}^d} M_{U[f]}(v)(v \cdot \Omega[j])\,\mathrm{d}v\,\Omega[j] - j[f] \\ &= \rho[f]\left[|U[f]| + \sigma\frac{\partial_l Z}{Z}(\sigma, |U[f]|)\right]\Omega[j] - j[f] \\ &= |j[f]|\Omega[j] - j[f] = 0. \end{aligned}$$

□

Since the kernels Q, Q_{BGK} have the same equilibria and conservations, the first order approximations for

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon} C(f^\varepsilon)$$

with $C = Q$ or $C = Q_{\mathrm{BGK}}$ coincide, being given by (17), (18). The linearization of Q_{BGK} is given by.

Proposition 5.3
We denote by $\mathcal{L}_{\mathrm{BGK}f}$ the linearization of Q_{BGK} around the equilibrium $f = \rho M_u$. Then we have

$$\mathcal{L}_{\mathrm{BGK}f}\, g = -g + P_u g, \;\; g \in L^2_{M_u}.$$

In particular, the kernels of the linearizations of Q, Q_{BGK} around any equilibrium, coincide.

Proof.
For any $g \in L^2_{M_u}$ we have

$$\begin{aligned}\mathcal{L}_{\mathrm{BGK}f} g &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0} Q_{\mathrm{BGK}}(f+\varepsilon g) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0}\{\rho[f+\varepsilon g]M_{U[f+\varepsilon g]} - f - \varepsilon g\} \\ &= \rho[g]M_{U[f]} + \rho\frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0} M_{U[f+\varepsilon g]} - g \\ &= \rho[g]M_u + \rho\frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0} M_{U[f+\varepsilon g]} - g.\end{aligned}$$

It is easily seen that $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0} M_{U[f+\varepsilon g]} \in \ker \mathcal{L}_f$ and that for any $\psi \in \mathrm{span}\{1, v_1, ..., v_d\}$

$$\int_{\mathbb{R}^d} \mathcal{L}_{\mathrm{BGK}f} g\; \psi(v)\; \mathrm{d}v = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0} \int_{\mathbb{R}^d} Q_{\mathrm{BGK}}(f+\varepsilon g)\psi(v)\; \mathrm{d}v = 0.$$

We deduce that

$$\rho[g]M_u + \rho\frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0} M_{U[f+\varepsilon g]} - P_u g = P_u\left(\rho[g]M_u + \rho\frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0} M_{U[f+\varepsilon g]} - g\right) = 0$$

implying that $\mathcal{L}_{\mathrm{BGK}f} g = P_u g - g, g \in L^2_{M_u}$. Clearly we have

$$\ker \mathcal{L}_{\mathrm{BGK}f} = \mathrm{span}\{M_u, v_1 M_u, ..., v_d M_u\} = \ker \mathcal{L}_f.$$

□

In the sequel we concentrate on the computation of

$$\int_{\mathbb{R}^d} (f_1 - P_u f_1) v \otimes v\; \mathrm{d}v = -\int_{\mathbb{R}^d} \mathcal{L}_{\mathrm{BGK}f}(f_1) v \otimes v\; \mathrm{d}v = -\int_{\mathbb{R}^d} (\partial_t + v\cdot\nabla_x f) v \otimes v\; \mathrm{d}v. \qquad (32)$$?EquBGKTensor?

Let us determine the formula for the orthogonal projection on $\ker \mathcal{L}_f \subset L^2_{M_u}$, for any equilibrium $f = \rho M_u$.

Proposition 5.4
For any function $g \in L^2_{M_u}$, we have

$$P_u g = \rho[g]M_u + \mathcal{M}_u^{-1}\left(j[g] - \frac{j}{\rho}\,\rho[g]\right)\cdot\frac{v - \frac{j}{\rho}}{\sigma} M_u.$$

Proof.
The orthogonal projection of g on $\ker \mathcal{L}_f$ writes

$$P_u g = \alpha M_u + \beta \cdot \frac{v - \frac{j}{\rho}}{\sigma} M_u$$

with $\alpha \in \mathbb{R}, \beta \in \mathbb{R}^d$. The coefficients α, β are determined by imposing

$$\int_{\mathbb{R}^d} (g - P_u g)\; \mathrm{d}v = 0, \;\; \int_{\mathbb{R}^d} (g - P_u g) v\; \mathrm{d}v = 0.$$

We obtain $\alpha = \int_{\mathbb{R}^d} g(v)\,\mathrm{d}v = \rho[g]$ and

$$\int_{\mathbb{R}^d} g(v)\left(v - \frac{j}{\rho}\right)\,\mathrm{d}v = \int_{\mathbb{R}^d} M_u(v)\frac{(v-\frac{j}{\rho})\otimes(v-\frac{j}{\rho})}{\sigma}\,\mathrm{d}v\;\beta = \mathcal{M}_u\beta$$

implying that

$$P_u g = \rho[g]M_u + \mathcal{M}_u^{-1}\left(j[g] - \frac{j}{\rho}\,\rho[g]\right)\cdot\frac{v-\frac{j}{\rho}}{\sigma}M_u.$$

□

In particular we obtain for any $1 \le i \le d$

$$P_u(\partial_{v_i} f) = -\mathcal{M}_u^{-1}e_i\cdot\frac{v-\frac{j}{\rho}}{\sigma}f = -f\mathcal{M}_u^{-1}\frac{v-\frac{j}{\rho}}{\sigma}\cdot e_i$$

and thus $P_u(\nabla_v f) = -f\mathcal{M}_u^{-1}\frac{v-\frac{j}{\rho}}{\sigma}$. We need the following representation formula for $(\partial_t + v\cdot\nabla_x)f$.

Proposition 5.5
For any $f = \rho M_u$ we have

$$P_u(v\cdot\nabla_x f) = \mathrm{div}_x(f\frac{j}{\rho}) + \mathcal{M}_u^{-1}\mathrm{div}_x(\rho\mathcal{M}_u)\cdot(v-\frac{j}{\rho})M_u.$$

Proof.
It is easily seen that $\mathrm{div}_x(f\frac{j}{\rho}) + \mathcal{M}_u^{-1}\mathrm{div}_x(\rho\mathcal{M}_u)\cdot(v-\frac{j}{\rho})M_u$ belongs to $\ker\mathcal{L}_f$. We also have

$$\int_{\mathbb{R}^d}\mathrm{div}_x(f\frac{j}{\rho}) + \mathcal{M}_u^{-1}\mathrm{div}_x(\rho\mathcal{M}_u)\cdot(v-\frac{j}{\rho})M_u\,\mathrm{d}v = \mathrm{div}_x j = \int_{\mathbb{R}^d} v\cdot\nabla_x f\,\mathrm{d}v$$

and thanks to Lemma 4.3

$$\begin{aligned}
\int_{\mathbb{R}^d}&[\mathrm{div}_x(f\frac{j}{\rho}) + \mathcal{M}_u^{-1}\mathrm{div}_x(\rho\mathcal{M}_u)\cdot(v-\frac{j}{\rho})M_u(v)]v\,\mathrm{d}v\\
&= \mathrm{div}_x\int_{\mathbb{R}^d} fv\otimes\frac{j}{\rho}\,\mathrm{d}v + \left(\int_{\mathbb{R}^d} M_u(v)v\otimes(v-\frac{j}{\rho})\,\mathrm{d}v\right)\mathcal{M}_u^{-1}\mathrm{div}_x(\rho\mathcal{M}_u)\\
&= \mathrm{div}_x\left(\frac{j\otimes j}{\rho}\right) + \left(\int_{\mathbb{R}^d} M_u(v)(v-\frac{j}{\rho})\otimes(v-\frac{j}{\rho})\,\mathrm{d}v\right)\mathcal{M}_u^{-1}\mathrm{div}_x(\rho\mathcal{M}_u)\\
&= \mathrm{div}_x\left(\frac{j\otimes j}{\rho}\right) + \sigma\mathrm{div}_x(\rho\mathcal{M}_u) = \mathrm{div}_x\int_{\mathbb{R}^d} fv\otimes v\,\mathrm{d}v = \int_{\mathbb{R}^d}(v\cdot\nabla_x f)v\,\mathrm{d}v.
\end{aligned}$$

Therefore we obtain that

$$P_u(v\cdot\nabla_x f) = \mathrm{div}_x(f\frac{j}{\rho}) + \mathcal{M}_u^{-1}\mathrm{div}_x(\rho\mathcal{M}_u)\cdot(v-\frac{j}{\rho})M_u.$$

□

In order to compute the right hand side in (32), it is very convenient to average with respect to characteristic flows preserving M_u, cf. [10, 9, 11, 13, 12]. For simplicity, when dealing with the BGK kernel, we work in the three dimensional setting $d = 3$, but we will come back to the interaction mechanisme Q in any dimension $d \ge 2$. For any $u \in \mathbb{R}^3\setminus\{0\}$ we consider the characteristic flow

$$\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}\theta} = \mathcal{V}(\theta)\wedge\Omega[u],\ \theta\in\mathbb{R},\ \ \mathcal{V}(0;v) = v. \qquad (33)$$?CharFlow?

Clearly Φ_u together with M_u are left invariant along this flow because $|v|$ and $v \cdot u$ are left invariant

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\theta}|\mathcal{V}(\theta)|^2 = \mathcal{V}(\theta)\cdot\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}\theta} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}\theta}(\mathcal{V}(\theta)\cdot u) = 0, \quad \theta \in \mathbb{R}.$$

Observe also that every trajectory is 2π-periodic

$$\mathcal{V}(\theta; v) = \mathcal{R}(-\theta; \Omega[u])v, \quad \theta \in \mathbb{R}$$

where the notation $\mathcal{R}(\theta;\Omega)$ stands for the rotation

$$\mathcal{R}(\theta;\Omega) = (\Omega\otimes\Omega)v + \cos\theta(I_3 - \Omega\otimes\Omega)v + \sin\theta\,\Omega\wedge v, \quad v\in\mathbb{R}^3.$$

Using the characteristic flow (33) is very useful when computing moments of M_u. For example, if $u \in \mathbb{R}^3\setminus\{0\}$, by performing the change of coordinate $v \to \mathcal{V}(\theta; v), \theta\in\mathbb{R}$, whose jacobian determinant is 1, we can write, using the notation $v_\perp = v - (v\cdot\Omega)\Omega$

$$\begin{aligned}
\mathcal{M}_u &= \int_{\mathbb{R}^3} M_u(v)\frac{(v-\frac{j}{\rho})\otimes(v-\frac{j}{\rho})}{\sigma}\,\mathrm{d}v\\
&= \int_{\mathbb{R}^3} M_u(v)\frac{(\mathcal{V}(\theta;v)-\frac{j}{\rho})\otimes(\mathcal{V}(\theta;v)-\frac{j}{\rho})}{\sigma}\,\mathrm{d}v\\
&= \int_{\mathbb{R}^3} M_u(v)\frac{[(\Omega\otimes\Omega)(v-\frac{j}{\rho}) + \cos\theta\, v_\perp + \sin\theta\, v\wedge\Omega]^{\otimes 2}}{\sigma}\,\mathrm{d}v.
\end{aligned}$$

We average with respect to θ, by taking into account that

$$\frac{1}{2\pi}\int_0^{2\pi}\cos\theta\,\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi}\sin\theta\,\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi}\sin\theta\cos\theta\,\mathrm{d}\theta = 0$$

$$\frac{1}{2\pi}\int_0^{2\pi}\cos^2\theta\,\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi}\sin^2\theta\,\mathrm{d}\theta = \frac{1}{2}.$$

We obtain

$$\begin{aligned}
\sigma\mathcal{M}_u &= \frac{1}{2\pi}\int_0^{2\pi}\sigma\mathcal{M}_u\,\mathrm{d}\theta\\
&= \int_{\mathbb{R}^3} M_u(v)((v-\frac{j}{\rho})\cdot\Omega)^2\,\mathrm{d}v\;\Omega\otimes\Omega + \int_{\mathbb{R}^3} M_u\frac{v_\perp\otimes v_\perp}{2}\,\mathrm{d}v + \int_{\mathbb{R}^3} M_u\frac{(v\wedge\Omega)\otimes(v\wedge\Omega)}{2}\,\mathrm{d}v\\
&= \int_{\mathbb{R}^3} M_u(v)((v-\frac{j}{\rho})\cdot\Omega)^2\,\mathrm{d}v\;\Omega\otimes\Omega + \int_{\mathbb{R}^3} M_u(v)\frac{|v_\perp|^2}{2}\,\mathrm{d}v\;(I_3 - \Omega\otimes\Omega).
\end{aligned}$$

By Remark 4.3 we know that

$$\int_{\mathbb{R}^3} M_u(v)((v-\frac{j}{\rho})\cdot\Omega)^2\,\mathrm{d}v = \sigma\partial_{ll}^2\mathcal{E}(\sigma,|u|), \quad \int_{\mathbb{R}^3} M_u(v)\frac{|v_\perp|^2}{2}\,\mathrm{d}v = \sigma\frac{\partial_l\mathcal{E}(\sigma,|u|)}{|u|}$$

and finally we retrive the formula

$$\mathcal{M}_u = \frac{\partial_l\mathcal{E}(\sigma,|u|)}{|u|}\,(I_3 - \Omega[u]\otimes\Omega[u]) + \partial_{ll}^2\mathcal{E}(\sigma,|u|)\,\Omega[u]\otimes\Omega[u], \quad u\in\mathbb{R}^3\setminus\{0\}.$$

Letting $u\to 0$ while $u/|u|$ is constant, we obtain $\mathcal{M}_0 = \partial_{ll}^2\mathcal{E}(\sigma,0)I_3$. We will average with respect to the characteristic flow (33) in order to compute the moments of M_u in (32). As we

know that $\partial_t f + v\cdot\nabla_x f \perp \mathrm{span}\{M_u, v_1 M_u, v_2 M_u, v_3 M_u\}$, we can write thanks to Proposition 5.5 and by observing that $\partial_t f \in \ker \mathcal{L}_f$

$$
\begin{aligned}
-\int_{\mathbb{R}^3}(\partial_t f + v\cdot\nabla_x f)v\otimes v\,\mathrm{d}v &= -\int_{\mathbb{R}^3}(\partial_t f + v\cdot\nabla_x f)(v-\frac{j}{\rho})\otimes(v-\frac{j}{\rho})\,\mathrm{d}v \qquad (34)\ \texttt{?Equ34-Swarm?}\\
&= -\int_{\mathbb{R}^3}[\partial_t f + v\cdot\nabla_x f - P_u(\partial_t f + v\cdot\nabla_x f)](v-\frac{j}{\rho})\otimes(v-\frac{j}{\rho})\,\mathrm{d}v\\
&= -\int_{\mathbb{R}^3}\mathrm{div}_x[f(v-\frac{j}{\rho})](v-\frac{j}{\rho})\otimes(v-\frac{j}{\rho})\,\mathrm{d}v\\
&+ \int_{\mathbb{R}^3}[\mathcal{M}_u^{-1}\mathrm{div}_x(\rho\mathcal{M}_u)]\cdot(v-\frac{j}{\rho})M_u(v)(v-\frac{j}{\rho})\otimes(v-\frac{j}{\rho})\,\mathrm{d}v\\
&= -\mathcal{A}+\mathcal{B}.
\end{aligned}
$$

Notice that the evaluation of $\mathcal{A},\mathcal{B}$ relies on the computation of third order moments of M_u with respect to $v-j/\rho$. The function $\mathcal{E}$ allowed us to express the second order moments of M_u. We also need to express in term of $\mathcal{E}$ the following third order moments of M_u. These computations are details in Appendix A, cf. proofs of Lemmas 5.1, 5.2.

Lemma 5.1
1. For any $u\in\mathbb{R}^d\backslash\{0\}$ we have

$$\int_{\mathbb{R}^d} M_u((v-\frac{j}{\rho})\cdot\Omega[u])^3\,\mathrm{d}v = \sigma^2\partial^3_{lll}\mathcal{E}(\sigma,|u|),\ \int_{\mathbb{R}^d} M_u((v-\frac{j}{\rho})\cdot\Omega[u])\frac{|v_\perp|^2}{d-1}\,\mathrm{d}v = \sigma^2\partial_l\left(\frac{\partial_l\mathcal{E}}{l}\right)(\sigma,|u|).$$

2. We have for any $\Omega\in\mathbb{S}^{d-1}$

$$\int_{\mathbb{R}^d} M_0(v)(v\cdot\Omega)^3\,\mathrm{d}v = 0,\ \int_{\mathbb{R}^d} M_0(v)(v\cdot\Omega)\frac{|v-(v\cdot\Omega)\Omega|^2}{d-1}\,\mathrm{d}v = 0.$$

Lemma 5.2
Let us consider $\xi\in\mathbb{R}^3$.
1. For any $u\in\mathbb{R}^3\setminus\{0\}$ we have

$$
\begin{aligned}
&\int_{\mathbb{R}^3} M_u(v)\left(\xi\cdot(v-\frac{j}{\rho})\right)(v-\frac{j}{\rho})\otimes(v-\frac{j}{\rho})\,\mathrm{d}v\\
&\qquad = \sigma^2(\xi\cdot\Omega)\mathcal{N}_u + \sigma^2\partial_l\left(\frac{\partial_l\mathcal{E}}{l}\right)(\sigma,|u|)[\Omega\otimes(I_3-\Omega\otimes\Omega)\xi + (I_3-\Omega\otimes\Omega)\xi\otimes\Omega]
\end{aligned}
$$

with

$$\mathcal{N}_u = \partial_l\left(\frac{\partial_l\mathcal{E}}{l}\right)(\sigma,|u|)\ (I_3-\Omega[u]\otimes\Omega[u]) + \partial^3_{lll}\mathcal{E}(\sigma,|u|)\ \Omega[u]\otimes\Omega[u].$$

2. We have $\int_{\mathbb{R}^3} M_0(v)(\xi\cdot v)v\otimes v\,\mathrm{d}v = 0$.

Thanks to Lemma 5.2, we are ready to compute the right hand side in (**??**). Denoting by

$\{e_1, e_2, e_3\}$ the canonical basis of $\mathbb{R}^3$, we have

$$\begin{aligned}\mathcal{A}(\rho, j) &= \int_{\mathbb{R}^3} \partial_{x_i}[f(v-\frac{j}{\rho})_i](v-\frac{j}{\rho}) \otimes (v-\frac{j}{\rho})\,\mathrm{d}v \\ &= \sum_{i=1}^{3} \partial_{x_i} \int_{\mathbb{R}^3} f(v-\frac{j}{\rho})_i (v-\frac{j}{\rho}) \otimes (v-\frac{j}{\rho})\,\mathrm{d}v \\ &+ \sum_{i=1}^{3} \int_{\mathbb{R}^3} f(v-\frac{j}{\rho})_i \partial_{x_i}\left(\frac{j}{\rho}\right) \otimes (v-\frac{j}{\rho})\,\mathrm{d}v \\ &+ \sum_{i=1}^{3} \int_{\mathbb{R}^3} f(v-\frac{j}{\rho})_i (v-\frac{j}{\rho}) \otimes \partial_{x_i}\left(\frac{j}{\rho}\right)\,\mathrm{d}v \\ &= \sum_{i=1}^{3} \partial_{x_i}\left\{\rho\sigma^2\left[\Omega_i \mathcal{N}_u + \partial_l\left(\frac{\partial_l \mathcal{E}}{l}\right)(\sigma, |u|)[\,\Omega \otimes (I_3 - \Omega\otimes\Omega)e_i + (I_3 - \Omega\otimes\Omega)e_i \otimes \Omega\,]\right]\right\} \\ &+ \sigma\rho\left[\partial_x\left(\frac{j}{\rho}\right)\mathcal{M}_u + \mathcal{M}_u\,{}^t\partial_x\left(\frac{j}{\rho}\right)\right].\end{aligned} \tag{35}$$

Similarly we obtain

$$\begin{aligned}\mathcal{B}(\rho, j) &= \sigma^2(\mathcal{M}_u^{-1}\mathrm{div}_x(\rho\mathcal{M}_u)\cdot\Omega)\mathcal{N}_u \\ &+ \sigma^2\partial_l\left(\frac{\partial_l\mathcal{E}}{l}\right)(\sigma,|u|)[\Omega\otimes(I_3-\Omega\otimes\Omega)\mathcal{M}_u^{-1}\mathrm{div}_x(\rho\mathcal{M}_u) + (I_3-\Omega\otimes\Omega)\mathcal{M}_u^{-1}\mathrm{div}_x(\rho\mathcal{M}_u)\otimes\Omega].\end{aligned} \tag{36}$$

The previous computations lead to the second order approximation of (4) corresponding to Q_{BGK}.

Theorem 5.1
Assume that $\lim_{|v|\to+\infty}\frac{\Phi_0(v)}{|v|} = +\infty$ *with* $\Phi_{u=0}$ *defined in* (2) *and consider the family* $(f^\varepsilon)_\varepsilon$ *of solutions for* (4) *with the collision operator* Q_{BGK}. *Then a second order approximation* $(\tilde\rho^\varepsilon, \tilde j^\varepsilon)$ *for* $(\rho^\varepsilon = \int_{\mathbb{R}^d} f^\varepsilon\,\mathrm{d}v, j^\varepsilon = \int_{\mathbb{R}^d} f^\varepsilon v\,\mathrm{d}v)$ *is given by*

$$\partial_t\tilde\rho^\varepsilon + \mathrm{div}_x\tilde j^\varepsilon = 0$$

$$\partial_t\tilde j^\varepsilon + \mathrm{div}_x\left(\frac{\tilde j^\varepsilon\otimes\tilde j^\varepsilon}{\tilde\rho^\varepsilon} + \sigma\tilde\rho^\varepsilon\mathcal{M}_{\tilde u^\varepsilon} - \varepsilon\mathcal{A}(\tilde\rho^\varepsilon,\tilde j^\varepsilon) + \varepsilon\mathcal{B}(\tilde\rho^\varepsilon,\tilde j^\varepsilon)\right) = 0$$

where $\tilde u^\varepsilon = \partial_k\mathcal{E}^\star(\sigma, |\tilde j^\varepsilon|/\tilde\rho^\varepsilon)\Omega[\tilde j^\varepsilon]$ *and* $\mathcal{M}_{\tilde u^\varepsilon}, \mathcal{A}, \mathcal{B}$ *are defined in* (23), (35) *and* (36) *respectively.*

Proof.
Coming back to (31), we can write thanks to Proposition 5.1, Lemma 5.2

$$\begin{aligned}\int_{\mathbb{R}^3}(f+\varepsilon f_1)v\otimes v\,\mathrm{d}v &= \int_{\mathbb{R}^3}(f+\varepsilon P_u f_1)v\otimes v\,\mathrm{d}v - \varepsilon\int_{\mathbb{R}^3}(\partial_t f + v\cdot\nabla_x f)v\otimes v\,\mathrm{d}v \\ &= \frac{j[f+\varepsilon P_u f_1]\otimes j[f+\varepsilon P_u f_1]}{\rho[f+\varepsilon P_u f_1]} + \sigma\rho[f+\varepsilon P_u f_1]\mathcal{M}_{\tilde u^\varepsilon} \\ &- \varepsilon\mathcal{A}(\rho[f], j[f]) + \varepsilon\mathcal{B}(\rho[f], j[f])\end{aligned}$$

where $\tilde u^\varepsilon = \partial_k\mathcal{E}^\star(\sigma, |j[f+\varepsilon P_u f_1]|/\rho[f+\varepsilon P_u f_1])\Omega[j[f+\varepsilon P_u f_1]] = \partial_k\mathcal{E}^\star(\sigma, |j[f+\varepsilon f_1]|/\rho[f+\varepsilon f_1])\Omega[j[f+\varepsilon f_1]]$. Finally, the momentum balance (31) becomes

$$\begin{aligned}\partial_t j[f+\varepsilon f_1] + \mathrm{div}_x\Big(&\frac{j[f+\varepsilon f_1]\otimes j[f+\varepsilon f_1]}{\rho[f+\varepsilon f_1]} + \sigma\rho[f+\varepsilon f_1]\mathcal{M}_{\tilde u^\varepsilon} \\ &-\varepsilon\mathcal{A}(\rho[f+\varepsilon f_1], j[f+\varepsilon f_1]) + \varepsilon\mathcal{B}(\rho[f+\varepsilon f_1], j[f+\varepsilon f_1])\Big) = \mathcal{O}(\varepsilon^2)\end{aligned}$$

justifying the second order approximation

$$\partial_t \tilde{j}^\varepsilon + \mathrm{div}_x \left(\frac{\tilde{j}^\varepsilon \otimes \tilde{j}^\varepsilon}{\tilde{\rho}^\varepsilon} + \sigma \tilde{\rho}^\varepsilon \mathcal{M}_{\tilde{u}^\varepsilon} - \varepsilon \mathcal{A}(\tilde{\rho}^\varepsilon, \tilde{j}^\varepsilon) + \varepsilon \mathcal{B}(\tilde{\rho}^\varepsilon, \tilde{j}^\varepsilon) \right) = 0$$

where $\tilde{u}^\varepsilon = \partial_k \mathcal{E}^\star(\sigma, |\tilde{j}^\varepsilon|/\tilde{\rho}^\varepsilon)\Omega[\tilde{j}^\varepsilon]$. □

Remark 5.1
We have already seen that when $V = 0$, the equilibria are the standard Maxwellians on $\mathbb{R}^d$, $Z = (2\pi\sigma)^{d/2}$, $\frac{j}{\rho} = u$. In that case

$$\partial_l \mathcal{E}(\sigma, l) = l, \;\; \partial^2_{ll} \mathcal{E}(\sigma, l) = 1, \;\; \partial_k \mathcal{E}^\star(\sigma, k) = k, \;\; \partial^2_{kk} \mathcal{E}^\star(\sigma, k) = 1$$

and the first order approximation led to the Euler equations. Moreover, when $V = 0$, $d = 3$, we have

$$\mathcal{M}_u = I_3, \;\; \mathcal{N}_u = O_3, \;\; A(\rho, j) = \sigma\rho \left(\partial_x \left(\frac{j}{\rho} \right) + {}^t\partial_x \left(\frac{j}{\rho} \right) \right), \;\; \mathcal{B}(\rho, j) = O_3$$

and we obtain the Navier-Stokes equations

$$\partial_t \tilde{\rho}^\varepsilon + \mathrm{div}_x \tilde{j}^\varepsilon = 0, \;\; \partial_t \tilde{j}^\varepsilon + \mathrm{div}_x \left[\frac{\tilde{j}^\varepsilon \otimes \tilde{j}^\varepsilon}{\tilde{\rho}^\varepsilon} + \sigma \tilde{\rho}^\varepsilon I_3 - \varepsilon\sigma\tilde{\rho}^\varepsilon \left(\partial_x \left(\frac{\tilde{j}^\varepsilon}{\tilde{\rho}^\varepsilon} \right) + {}^t\partial_x \left(\frac{\tilde{j}^\varepsilon}{\tilde{\rho}^\varepsilon} \right) \right) \right] = 0.$$

5.2 The Fokker-Planck interaction mechanism

We investigate now the fluid second order approximation for the kernel Q. The main difficulty is that we need to determine the antecedent of $\partial_t f + v \cdot \nabla_x f$ by the restriction of $\mathcal{L}_f$ on $(\ker \mathcal{L}_f)^\perp$. We appeal to the structure of matrix fields which are left invariant by $\mathcal{T}_u$, cf. Proposition 2.3. By Proposition 5.5 we have, as before

$$\begin{aligned}
\partial_t f + v \cdot \nabla_x f &= \partial_t f + v \cdot \nabla_x f - P_u(\partial_t f + v \cdot \nabla_x f) \\
&= v \cdot \nabla_x f - P_u(v \cdot \nabla_x f) \\
&= v \cdot \nabla_x f - \mathrm{div}_x \left(f \frac{j}{\rho} \right) - \mathcal{M}_u^{-1} \mathrm{div}_x(\rho \mathcal{M}_u) \cdot \left(v - \frac{j}{\rho} \right) M_u \\
&= (I_d - P_u) \mathrm{div}_x (f(v - j/\rho)) \\
&= \partial_x u : (I_d - P_u) f \frac{(v - j/\rho) \otimes (v - j/\rho)}{\sigma}.
\end{aligned}$$

Therefore we obtain

$$f_1 - P_u f_1 = \mathcal{L}_f^{-1}(\partial_t f + v \cdot \nabla_x f) = \partial_x u : A$$

where A is the unique matrix field satisfying

$$\mathcal{L}_f A = (I_d - P_u) f \frac{(v - j/\rho) \otimes (v - j/\rho)}{\sigma}, \;\; \int_{\mathbb{R}^d} A(v) \, \mathrm{d}v = 0, \;\; \int_{\mathbb{R}^d} v_i A(v) \, \mathrm{d}v = 0, \;\; 1 \le i \le d. \tag{37}$$

?EquMatrixA?

Observe, cf. Proposition 2.3, that the pressure tensor is left invariant by $\mathcal{T}_u$ because if $u = 0$ we have

$$v \otimes v = v \otimes v - |v|^2 \frac{I_d}{d} + |v|^2 \frac{I_d}{d}, \;\; v \in \mathbb{R}^d$$

and if $u \neq 0$ we can write

$$\begin{aligned}(v-j/\rho)\otimes(v-j/\rho) &= [v_\perp + ((v-j/\rho)\cdot\Omega)\Omega]\otimes[v_\perp + ((v-j/\rho)\cdot\Omega)\Omega]\\ &= v_\perp\otimes v_\perp - |v_\perp|^2\frac{I_d-\Omega\otimes\Omega}{d-1} + |v_\perp|^2\frac{I_d-\Omega\otimes\Omega}{d-1}\\ &+ ((v-j/\rho)\cdot\Omega)(v_\perp\otimes\Omega+\Omega\otimes v_\perp) + ((v-j/\rho)\cdot\Omega)^2\Omega\otimes\Omega,\ v\in\mathbb{R}^d.\end{aligned}$$

For any $\mathcal{O}\in\mathcal{T}_u$ and any function $g(v)$, vector field $b(v)$, matrix field $B(v)$, we use the notations

$$g_{\mathcal{O}}(v) = g({}^t\mathcal{O}v),\ b_{\mathcal{O}}(v) = \mathcal{O}b({}^t\mathcal{O}v),\ B_{\mathcal{O}}(v) = \mathcal{O}B({}^t\mathcal{O}v){}^t\mathcal{O},\ v\in\mathbb{R}^d.$$

Notice that the invariance of g, b, B by the family $\mathcal{T}_u$ simply writes $g = g_{\mathcal{O}}$, $b = b_{\mathcal{O}}$, $B = B_{\mathcal{O}}$ for any $\mathcal{O}\in\mathcal{T}_u$. The orthogonal projection P_u commutes with the action of $\mathcal{T}_u$. More exactly the following assertions hold true.

Proposition 5.6
For any function $g\in L^2_{M_u}$ and any $\mathcal{O}\in\mathcal{T}_u$, we have $g_{\mathcal{O}}\in L^2_{M_u}$ and $P_u(g_{\mathcal{O}}) = (P_ug)\mathcal{O}$. In particular if g is left invariant by the family $\mathcal{T}_u$, so is P_ug.

Proof.
We have $|g_{\mathcal{O}}|^2_{M_u} = \int_{\mathbb{R}^d}\left(\frac{g({}^t\mathcal{O}v)}{M_u(v)}\right)^2 M_u(v)\,\mathrm{d}v = |g|^2_{M_u} < +\infty$ and $(g_{\mathcal{O}}, h_{\mathcal{O}})_{M_u} = (g,h)_{M_u}$, for any $h\in L^2_{M_u}$. Observe also that if $h\in\ker\mathcal{L}_f$, then $h_{\mathcal{O}}\in\ker\mathcal{L}_f$. In particular $(P_ug)_{\mathcal{O}}\in\ker\mathcal{L}_f$ and for any $h\in\ker\mathcal{L}_f$ we can write

$$(g_{\mathcal{O}} - (P_ug)_{\mathcal{O}}, h)_{M_u} = (g - P_ug, h_{{}^t\mathcal{O}})_{M_u} = 0$$

saying that $P_u(g_{\mathcal{O}}) = (P_ug)_{\mathcal{O}},\ \mathcal{O}\in\mathcal{T}_u$. □

The previous result extends immediately to vector and matrix fields, where the action of the orthogonal projection P_u on vector fields and matrix fields is understood by

$$P_ub = (P_ub_i)_{1\leq i\leq d},\ P_uB = (P_uB_{ij})_{1\leq i,j\leq d}$$

for any vector field b and matrix field B such that

$$|b|^2_{M_u} = (b,b)_{M_u} = \sum_{i=1}^d (b_i,b_i)_{M_u} = \sum_{i=1}^d\int_{\mathbb{R}^d}\frac{b_i^2(v)}{M_u(v)}\,\mathrm{d}v < +\infty$$

and

$$|B|^2_{M_u} = (B,B)_{M_u} = \sum_{i=1}^d (B_{ij},B_{ij})_{M_u} = \sum_{1\leq i,j\leq d}\int_{\mathbb{R}^d}\frac{B_{i,j}^2(v)}{M_u(v)}\,\mathrm{d}v < +\infty.$$

We have $(b_{\mathcal{O}}, c_{\mathcal{O}})_{M_u} = (b,c)_{M_u}, b,c\in(L^2_{M_u})^d$ and similarly $(B_{\mathcal{O}}, C_{\mathcal{O}})_{M_u} = (B,C)_{M_u}, B,C\in(L^2_{M_u})^{d^2}$. Exactly as for functions, we obtain that $P_u(b_{\mathcal{O}}) = (P_ub)_{\mathcal{O}}$, $P_u(B_{\mathcal{O}}) = (P_uB)_{\mathcal{O}}$, for any $\mathcal{O}\in\mathcal{T}_u$.
The action of $\mathcal{T}_u$ commutes also with the operator $\mathcal{L}_f$. We establish first this result for functions, the analogous statements for vector fields and matrix fields come by similar arguments. Their proofs are left to the reader.

Proposition 5.7
Let us consider a function $p\in(\ker\mathcal{L}_f)^\perp$ and let $g\in H^1_{M_u}\cap(\ker\mathcal{L}_f)^\perp$ be the unique variational solution of $-\mathcal{L}_fg = p$. For any $\mathcal{O}\in\mathcal{T}_u$, we have $p_{\mathcal{O}}\in(\ker\mathcal{L}_f)^\perp$, $g_{\mathcal{O}}\in H^1_{M_u}\cap(\ker\mathcal{L}_f)^\perp$ and $-\mathcal{L}_fg_{\mathcal{O}} = p_{\mathcal{O}}$, in the variational sense. In particular, if p is the left invariant by $\mathcal{T}_u$, so is $g = -\mathcal{L}_f^{-1}p$.

Proof.
Clearly $p_{\mathcal{O}} \in L^2_{M_u}$ and $P_u(p_{\mathcal{O}}) = (P_u p)_{\mathcal{O}} = 0$, saying that $p_{\mathcal{O}} \in (\ker \mathcal{L}_f)^\perp$. Similarly, $g_{\mathcal{O}} \in (\ker \mathcal{L}_f)^\perp$ and

$$\|g_{\mathcal{O}}\|^2_{M_u} = \int_{\mathbb{R}^d} \frac{g^2_{\mathcal{O}}}{M_u}\, \mathrm{d}v + \int_{\mathbb{R}^d} \left| \nabla \left(\frac{g_{\mathcal{O}}}{M_u} \right) \right|^2 M_u(v)\, \mathrm{d}v = \|g\|^2_{M_u}.$$

For any $h \in H^1_{M_u}$ we have

$$\sigma \nabla \left(\frac{h_{\mathcal{O}}}{M_u} \right) - W(h_{\mathcal{O}}) = \mathcal{O}\left[\sigma \nabla \left(\frac{h}{M_u} \right)_{\mathcal{O}} - W(h) \right]$$

and therefore

$$\begin{aligned} a(g_{\mathcal{O}}, h) &= \int_{\mathbb{R}^d} \left[\sigma \nabla \left(\frac{g_{\mathcal{O}}}{M_u} \right) - W(g_{\mathcal{O}}) \right] \cdot \left[\sigma \nabla \left(\frac{(h_{t\mathcal{O}})_{\mathcal{O}}}{M_u} \right) - W((h_{t\mathcal{O}})_{\mathcal{O}}) M_u(v) \right] \mathrm{d}v \\ &= a(g, h_{t\mathcal{O}}) = \sigma \int_{\mathbb{R}^d} \frac{p(v) h_{t\mathcal{O}}(v)}{M_u(v)}\, \mathrm{d}v = \sigma \int_{\mathbb{R}^d} \frac{p_{\mathcal{O}}(v) h(v)}{M_u(v)}\, \mathrm{d}v \end{aligned}$$

saying that $-\mathcal{L}_f g_{\mathcal{O}} = p_{\mathcal{O}}$ in the variational sense. □

Proposition 5.8
Let us consider a vector field $\xi = (\xi_1, ..., \xi_d) \in ((\ker \mathcal{L}_f)^\perp)^d$ and let $b = (b_1, ..., b_d) \in (H^1_{M_u} \cap (\ker \mathcal{L}_f)^\perp)^d$ be the unique variational solution of $-\mathcal{L}_f b = \xi$. For any $\mathcal{O} \in \mathcal{T}_u$, we have $\xi_{\mathcal{O}} \in ((\ker \mathcal{L}_f)^\perp)^d$, $b_{\mathcal{O}} \in (H^1_{M_u} \cap (\ker \mathcal{L}_f)^\perp)^d$ and $-\mathcal{L}_f b_{\mathcal{O}} = \xi_{\mathcal{O}}$, in the variational sense. In particular, if ξ is the left invariant by $\mathcal{T}_u$, so is $b = -\mathcal{L}_f^{-1}\xi$.

Proposition 5.9
Let us consider a matrix field $U \in ((\ker \mathcal{L}_f)^\perp)^{d^2}$ and let $B \in (H^1_{M_u} \cap (\ker \mathcal{L}_f)^\perp)^{d^2}$ be the unique variational solution of $-\mathcal{L}_f B = U$. For any $\mathcal{O} \in \mathcal{T}_u$, we have $U_{\mathcal{O}} \in ((\ker \mathcal{L}_f)^\perp)^{d^2}$, $B_{\mathcal{O}} \in (H^1_{M_u} \cap (\ker \mathcal{L}_f)^\perp)^{d^2}$ and $-\mathcal{L}_f B_{\mathcal{O}} = U_{\mathcal{O}}$, in the variational sense. In particular, if U is the left invariant by $\mathcal{T}_u$, so is $B = -\mathcal{L}_f^{-1} U$.

Based on the above properties of Pu, $\mathcal{L}_f$, we determine the structure of the matrix field A solving (37). Actually only the case with $u \neq 0$ will be used in the sequel, since the pressure tensor is continuous with respect to u, and therefore the formulae for $u = 0$ can be obtained from the formulae with $u_h = h\Omega, \Omega \in \mathbb{S}^{d-1}$, by letting $h \searrow 0$. Nevertheless, we present the structure of these matrix field solutions, for any $u \in \mathbb{R}^d$.

Proposition 5.10

1. *If $u = 0, d \leq 2$, there is a unique function $\alpha : \mathbb{R}^d \to \mathbb{R}$, which is left invariant by $\mathcal{T}_0$, $\alpha(0) = 0$, such that*

$$\mathcal{L}_f \left[\alpha \left(v \otimes v - |v|^2 \frac{I_d}{d} \right) \right] = (I_d - P_0) M_0 \left(v \otimes v - |v|^2 \frac{I_d}{d} \right)$$

and

$$\int_{\mathbb{R}^d} \alpha(v) \left(v \otimes v - |v|^2 \frac{I_d}{d} \right) \mathrm{d}v = 0, \ \int_{\mathbb{R}^d} v_i \alpha(v) \left(v \otimes v - |v|^2 \frac{I_d}{d} \right) \mathrm{d}v = 0, \ 1 \leq i \leq d.$$

2. *If $u \neq 0$, $d \geq 3$, there is a unique function $\alpha : \mathbb{R}^d \to \mathbb{R}$, which is left invariant by $\mathcal{T}_u$, $\alpha|_{\mathbb{R}\Omega} = 0$, such that*

$$\mathcal{L}_f\left[\alpha\left(v_\perp \otimes v_\perp - |v_\perp|^2\frac{I_d - \Omega\otimes\Omega}{d-1}\right)\right] = (I_d - P_u)M_u\left(v_\perp \otimes v_\perp - |v_\perp|^2\frac{I_d - \Omega\otimes\Omega}{d-1}\right)$$

and

$$\int_{\mathbb{R}^d}\alpha\left(v_\perp \otimes v_\perp - |v_\perp|^2\frac{I_d - \Omega\otimes\Omega}{d-1}\right)\,\mathrm{d}v = 0,\ \int_{\mathbb{R}^d} v_i\alpha\left(v_\perp \otimes v_\perp - |v_\perp|^2\frac{I_d - \Omega\otimes\Omega}{d-1}\right)\,\mathrm{d}v = 0$$

for any $1 \leq i \leq d$.

Proof.

1. The matrix field $(I_d - P_0)M_0\left(v\otimes v - |v|^2\frac{I_d}{d}\right)$ is left invariant by $\mathcal{T}_0$. By Proposition 5.9 there is a unique matrix field A, which is left invariant by $\mathcal{T}_0$, such that ${}^tA = A$

$$\mathcal{L}_f A = (I_d - P_0)M_0\left(v\otimes v - |v|^2\frac{I_d}{d}\right),\ \int_{\mathbb{R}^d} A(v)\,\mathrm{d}v = 0, \int_{\mathbb{R}^d} v_i A(v)\,\mathrm{d}v = 0,\ 1 \leq i \leq d.$$

By Proposition 2.3, there are two functions α, β which are left invariant by $\mathcal{T}_0$, $\alpha(0) = 0$, such that

$$A(v) = \alpha(v)\left(v\otimes v - |v|^2\frac{I_d}{d}\right) + \beta(v)I_d,\ d \geq 2.$$

In particular we have $\mathcal{L}_f \mathrm{tr}A = 0$, $\mathrm{tr}A \in (\ker\mathcal{L}_f)^\perp$, implying that $d\beta = \mathrm{tr}A = 0$. Therefore we obtain

$$\mathcal{L}_f\left[\alpha\left(v\otimes v - |v|^2\frac{I_d}{d}\right)\right] = (I_d - P_0)M_0\left(v\otimes v - |v|^2\frac{I_d}{d}\right)$$

where $\int_{\mathbb{R}^d} A(v)\,\mathrm{d}v = 0$, $\int_{\mathbb{R}^d} v_i A(v)\,\mathrm{d}v = 0$, $1 \leq i \leq d$. As $A(v) = \alpha(v)\left(v\otimes v - |v|^2\frac{I_d}{d}\right)$, we deduce that

$$\alpha(v) = \frac{A(v) : v\otimes v}{|v|^4}\frac{d}{d-1},\ v \in \mathbb{R}^d\backslash\{0\}$$

implying the uniqueness of α (recall that $\alpha(0) = 0$).

2. As before, there is unique matrix field, which is left invariant by $\mathcal{T}_u$, such that ${}^tA = A$

$$\mathcal{L}_f A = (I_d - P_u)M_u\left(v_\perp\otimes v_\perp - |v_\perp|^2\frac{I_d - \Omega\otimes\Omega}{d-1}\right)$$

$\int_{\mathbb{R}^d} A(v)\,\mathrm{d}v = 0$, $\int_{\mathbb{R}^d} v_i A(v)\,\mathrm{d}v = 0$, $1 \leq i \leq d$. By Corollary 2.1, there are the functions $\alpha, \beta, \gamma, \delta$, which are left invariant by $\mathcal{T}_u$, $\alpha|_{\mathbb{R}\Omega} = 0$, $\gamma|_{\mathbb{R}\Omega} = 0$ such that

$$\begin{aligned} A(v) = \alpha(v)\left(v_\perp\otimes v_\perp - |v_\perp|^2\frac{I_d - \Omega\otimes\Omega}{d-1}\right) + \beta(v)(I_d - \Omega\otimes\Omega) \\ + \gamma(v)(v_\perp\otimes\Omega + \Omega\otimes v_\perp) + \delta(v)\Omega\otimes\Omega,\ \ v \in \mathbb{R}^d,\ d \geq 3. \end{aligned}$$

We have $\mathcal{L}_f A\Omega\cdot\Omega = 0$, $A\Omega\cdot\Omega \in (\ker\mathcal{L}_f)^\perp$, implying that $\delta = A\Omega\cdot\Omega = 0$. We also have $\mathcal{L}_f A\Omega = 0$, $A\Omega \in ((\ker\mathcal{L}_f)^\perp)^d$ and therefore $A\Omega = \gamma v_\perp = 0$, saying that $\gamma = 0$ (because $\gamma|_{\mathbb{R}\Omega} = 0$). Observe that $\mathcal{L}_f \mathrm{tr}A = 0$, $\mathrm{tr}A \in (\ker\mathcal{L}_f)^\perp$, saying that $\beta(d-1) = \mathrm{tr}A = 0$. Finally we obtain $A(v) = \alpha(v)\left(v_\perp\otimes v_\perp - |v_\perp|^2\frac{I_d - \Omega\otimes\Omega}{d-1}\right)$ where α is given by

$$\alpha(v) = \frac{A(v) : v_\perp\otimes v_\perp}{|v_\perp|^4}\frac{d-1}{d-2},\ v \in \mathbb{R}^d\backslash\mathbb{R}\Omega,\ \alpha|_{\mathbb{R}\Omega} = 0.$$

□

We determine now the antecedent of the pressure tensor by $\mathcal{L}_f$, for $u \neq 0$.

Proposition 5.11
If $u \neq 0$, there are the functions $\alpha, \beta, \gamma, \delta$, which are left invariant by $\mathcal{T}_u$, $\alpha|_{\mathbb{R}\Omega} = \gamma|_{\mathbb{R}\Omega} = 0$, such that

$$\mathcal{L}_f^{-1}(I_d - P_u)M_u(v - j/\rho)\otimes(v - j/\rho) = \beta(v)(I_2 - \Omega\otimes\Omega_2) + \gamma(v)(v_\perp\otimes\Omega + \Omega\otimes v_\perp) + \delta(v)\Omega\otimes\Omega,\ v \in \mathbb{R}^2$$

and

$$\mathcal{L}_f^{-1}(I_d - P_u)M_u(v - j/\rho)\otimes(v - j/\rho) = \alpha(v)\left(v_\perp\otimes v_\perp - |v_\perp|^2\frac{I_d - \Omega\otimes\Omega}{d-1}\right) + \beta(v)(I_d - \Omega\otimes\Omega) + \gamma(v)(v_\perp\otimes\Omega + \Omega\otimes v_\perp) + \delta(v)\Omega\otimes\Omega,\ \ v \in \mathbb{R}^d,\ d \geq 3.$$

Proof.
In the two dimensional case we write

$$\begin{aligned}(v - j/\rho)\otimes(v - j/\rho) &= [v_\perp + ((v - j/\rho)\cdot\Omega)\Omega]\otimes[v_\perp + ((v - j/\rho)\cdot\Omega)\Omega] \\ &= v_\perp\otimes v_\perp + ((v - j/\rho)\cdot\Omega)(v_\perp\otimes\Omega + \Omega\otimes v_\perp) + ((v - j/\rho)\cdot\Omega)^2\Omega\otimes\Omega \\ &= |v_\perp|^2(I_2 - \Omega\otimes\Omega) + ((v - j/\rho)\cdot\Omega)(v_\perp\otimes\Omega + \Omega\otimes v_\perp) \\ &+ ((v - j/\rho)\cdot\Omega)^2\Omega\otimes\Omega\end{aligned}$$

and therefore we obtain

$$\mathcal{L}_f^{-1}(I_d - P_u)M_u(v - j/\rho)\otimes(v - j/\rho) = \beta(v)(I_2 - \Omega\otimes\Omega) + b(v)\otimes\Omega + \Omega\otimes b(v) + \delta(v)\Omega\otimes\Omega$$

whhere $\beta = \mathcal{L}_f^{-1}(I_d - P_u)M_u|v_\perp|^2$, $b = \mathcal{L}_f^{-1}(I_d - P_u)M_u((v - j/\rho)\cdot\Omega)v_\perp$, $\delta = L_f^{-1}(I_d - P_u)M_u((v - j/\rho)\cdot\Omega)^2$. By Proposition 5.7, 5.8, we know that the functions β, γ and the vector field b are left invariant by $\mathcal{T}_u$. Finally we have $\mathcal{L}_f b\cdot\Omega = 0$, $b\cdot\Omega \in (\ker\mathcal{L}_f)^\perp$, saying that $b\cdot\Omega = 0$. Thus, by Proposition 2.2, there is a function γ, $\gamma|_{\mathbb{R}\Omega} = 0$, which is left invariant by $\mathcal{T}_u$, such that $b = \gamma v_\perp$. If $d \geq 3$ we write

$$\begin{aligned}(v - j/\rho)\otimes(v - j/\rho) &= v_\perp\otimes v_\perp - |v_\perp|^2\frac{I_d - \Omega\otimes\Omega}{d-1} + |v_\perp|^2\frac{I_d - \Omega\otimes\Omega}{d-1} \\ &+ ((v - j/\rho)\cdot\Omega)(v_\perp\otimes\Omega + \Omega\otimes v_\perp) + ((v - j/\rho)\cdot\Omega)^2\Omega\otimes\Omega\end{aligned}$$

and our conclusion follows by taking $\alpha, \beta, \gamma, \delta$ such that (cf. Proposition 5.10) $\alpha|_{\mathbb{R}\Omega} = \gamma|_{\mathbb{R}\Omega} = 0$ and

$$\mathcal{L}_f\left[\alpha\left(v\otimes v - |v|^2\frac{I_d}{d}\right)\right] = (I_d - P_0)M_0\left(v\otimes v - |v|^2\frac{I_d}{d}\right)$$

$$\beta = \mathcal{L}_f^{-1}(I_d - P_u)M_u\frac{|v_\perp|^2}{d-1}$$

$$\gamma v_\perp = \mathcal{L}_f^{-1}(I_d - P_u)M_u((v - j/\rho)\cdot\Omega)v_\perp,\ \delta = \mathcal{L}_f^{-1}(I_d - P_u)M_u((v - j/\rho)\cdot\Omega)^2.$$

□

Thanks to the above representation formula for the antecedent by $\mathcal{L}_f$ tensor $(I_d - P_u)M_u(v - j/\rho)\otimes(v - j/\rho)$, we compute

$$\begin{aligned}T_1 &:= \int_{\mathbb{R}^d}(f_1 - P_u f_1)v\otimes v\,\mathrm{d}v = \int_{\mathbb{R}^d}\mathcal{L}_f^{-1}(\partial_t f + v\cdot\nabla_x f)(v - j/\rho)\otimes(v - j/\rho)\,\mathrm{d}v \\ &= \frac{\rho}{\sigma}\int_{\mathbb{R}^d}\mathcal{L}_f^{-1}\left[(I_d - P_u)M_u(v - j/\rho)\otimes(v - j/\rho)\right] : \partial_x u)(v - j/\rho)\otimes(v - j/\rho)\,\mathrm{d}v.\end{aligned}$$

It is useful remark that

Lemma 5.3
Let us consider a function g which is left invariant by $\mathcal{T}_u$, $u \neq 0$, and such that $\int_{\mathbb{R}^d} |v|^2 g(v)\,\mathrm{d}v < +\infty$. Then we have

$$\int_{\mathbb{R}^d} g(v)\left(v_\perp \otimes v_\perp - |v_\perp|^2 \frac{I_d - \Omega\otimes\Omega}{d-1}\right)\,\mathrm{d}v = 0,\ d \geq 3$$

$$\int_{\mathbb{R}^d} g(v) v_\perp\,\mathrm{d}v = 0,\ \int_{\mathbb{R}^d} g(v)(v\cdot\xi)v_\perp \otimes v_\perp\,\mathrm{d}v = 0,\ \xi \in (\mathbb{R}\Omega)^\perp,\ d \geq 2.$$

Proof.
Consider $\{E_1, ..., E_{d-1}\}$ an orthonormal basis of $(\mathbb{R}\Omega)^\perp$. Taking $\mathcal{O}_{ij} = \Omega\otimes\Omega + \sum_{k\notin\{i,j\}} E_k \otimes E_k + E_i \otimes E_j - E_j \otimes E_i \in \mathcal{T}_u$ for any $1 \leq i, j \leq d-1, i \neq j$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} g(v) v_\perp \otimes v_\perp\,\mathrm{d}v &= \sum_{i=1}^{d-1}\int_{\mathbb{R}^d} g(v)(v\cdot E_i)^2 E_i \otimes E_i\,\mathrm{d}v \\ &= \int_{\mathbb{R}^d} g(v)\frac{|v_\perp|^2}{d-1}\,\mathrm{d}v(I_d - \Omega\otimes\Omega),\ d \geq 3.\end{aligned}$$

Using $\mathcal{O}_i = I_d - 2E_i \otimes E_i \in \mathcal{T}_u, 1 \leq i \leq d-1$, we deduce

$$\int_{\mathbb{R}^d} g(v) v_\perp\,\mathrm{d}v = \sum_{i=1}^{d-1}\int_{\mathbb{R}^d} g(v)(v\cdot E_i)E_i\,\mathrm{d}v = 0,\ d \geq 2.$$

Thanks to the decompotion $v\cdot\xi = \sum_{k=1}^{d-1}(v\cdot E_k)(E_k\cdot\xi)$, we are done if we justify for any $1 \leq k \leq d-1$, $d \geq 2$

$$\int_{\mathbb{R}^d} g(v)(v\cdot E_k)v_\perp \otimes v_\perp\,\mathrm{d}v = 0.$$

This comes easily because for any $1 \leq i, j, k \leq d-1$ we have

$$\int_{\mathbb{R}^d} g(v)(v\cdot E_k)(v\cdot E_i)(v\cdot E_j)\,\mathrm{d}v = 0,\ d \geq 2.$$

□

We decompose the contribution T_1 as $T_2 + T_3 + T_4$, where $(T_i)_{2\leq i\leq 4}$ expresse in terms of the Jocobian matrix $\partial_x u$ and some functions given by moments in v of $\alpha, \beta, \gamma, \delta$, with the notations of Proposition 5.11. These calculations are presented in Appendix A, see proof of Proposition 5.12

Proposition 5.12
Assume that $u \neq 0$, $d \geq 3$, then we have

$$T_1 = \int_{\mathbb{R}^d}(f_1 - P_u f_1)v\otimes v\,\mathrm{d}v = T_2 + T_3 + T_4$$

$$T_2 = \frac{\rho}{\sigma}\int_{\mathbb{R}^d}\frac{\alpha|v_\perp|^4}{d^2-1}\,\mathrm{d}v(I_d - \Omega\otimes\Omega)\left(\partial_x u + {}^t\partial_x u - \frac{2(I_d - \Omega\otimes\Omega) : \partial_x u}{d-1}(I_d - \Omega\otimes\Omega)\right)(I_d - \Omega\otimes\Omega)$$

$$\begin{aligned}T_3 = \frac{\rho}{\sigma}\Bigg[&\int_{\mathbb{R}^d}\beta\frac{|v_\perp|^2}{d-1}\,\mathrm{d}v(I_d - \Omega\otimes\Omega)\times(I_d - \Omega\otimes\Omega) + \int_{\mathbb{R}^d}\delta((v - j/\rho)\cdot\Omega)^2\,\mathrm{d}v(\Omega\otimes\Omega)\times(\Omega\otimes\Omega) \\ &+ \int_{\mathbb{R}^d}\delta\frac{|v_\perp|^2}{d-1}\,\mathrm{d}v(I_d - \Omega\otimes\Omega)\times(\Omega\otimes\Omega) + \int_{\mathbb{R}^d}\beta((v - j/\rho)\cdot\Omega)^2\,\mathrm{d}v(\Omega\otimes\Omega)\times(I_d - \Omega\otimes\Omega)\Bigg]\partial_x u\end{aligned}$$

$$T_4 = \frac{\rho}{\sigma}\int_{\mathbb{R}^d}((v-j/\rho)\cdot\Omega)\gamma\frac{|v_\perp|^2}{d-1}\,\mathrm{d}v$$
$$[(I_d-\Omega\otimes\Omega)(\partial_x u+{}^t\partial_x u)(\Omega\otimes\Omega)+(\Omega\otimes\Omega)(\partial_x u+{}^t\partial_x u)(I_d-\Omega\otimes\Omega)]$$

where for any two matrices $A, B \in \mathcal{M}_d(\mathbb{R})$, the notation $A\times B$ stands for the linear application $A\times B : \mathcal{M}_d(\mathbb{R}) \to \mathcal{M}_d(\mathbb{R})$, given by $(A\times B)X = (B:X)A,\ X \in \mathcal{M}_d(\mathbb{R})$.

Remark 5.2
If $u \neq 0$, $d = 2$ we obtain $T_1 = T_3 + T_4$, cf. Proposition 5.11.

In order to phrase out the second order approximation for (4) in a concise fashion, we define the function $(c_i)_{1\le i\le 5}$ given by

$$c_1(\sigma,u) = \int_{\mathbb{R}^d}\frac{\alpha(v)|v_\perp|^4}{d^2-1}\,\mathrm{d}v,\ c_2(\sigma,u) = \int_{\mathbb{R}^d}\frac{\beta(v)|v_\perp|^2}{d-1}\,\mathrm{d}v,\ c_3(\sigma,u) = \int_{\mathbb{R}^d}\delta(v)((v-j/\rho)\cdot\Omega)^2\,\mathrm{d}v \tag{38}$$

?Funcc?

$$c_4(\sigma,u) = \int_{\mathbb{R}^d}\beta(v)((v-j/\rho)\cdot\Omega)^2\,\mathrm{d}v,\ c_5(\sigma,u) = \int_{\mathbb{R}^d}\gamma(v)\frac{|v_\perp|^2}{d-1}((v-j/\rho)\cdot\Omega)\,\mathrm{d}v \tag{39}$$

?FunccBis?

where the functions $\alpha, \beta, \gamma, \delta$ were introduced in Proposition 5.11 (if $d = 2$, then $\alpha = 0$ and $c_1 = 0$). As $\beta = \mathcal{L}_f^{-1}(I_d - P_u)M_u\frac{|v_\perp|^2}{d-1}$, we have by the variational formulation cf. Definition 3.1

$$-a(\beta,\delta) = \sigma\int_{\mathbb{R}^d}(I_d-P_u)M_u\frac{|v_\perp|^2}{d-1}\frac{\delta(v)}{M_u(v)}\,\mathrm{d}v = \sigma\left((I_d-P_u)M_u\frac{|v_\perp|^2}{d-1},\delta\right)_{M_u}$$
$$= \sigma\left(\frac{|v_\perp|^2}{d-1},\delta\right)_{M_u} = \sigma\int_{\mathbb{R}^d}\delta(v)\frac{|v_\perp|^2}{d-1}\,\mathrm{d}v.$$

Similarly, since $\delta = \mathcal{L}_f^{-1}(I_d - P_u)M_u((v-j/\rho)\cdot\Omega)^2$, we obtain

$$-a(\delta,\beta) = \sigma\int_{\mathbb{R}^d}(I_d-P_u)M_u((v-j/\rho)\cdot\Omega)^2\frac{\beta(v)}{M_u(v)}\,\mathrm{d}v = \sigma((I_d-P_u)M_u((v-j/\rho)\cdot\Omega)^2,\beta)_{M_u}$$
$$= \sigma(M_u((v-j/\rho)\cdot\Omega)^2,\beta)_{M_u} = \sigma\int_{\mathbb{R}^d}\beta(v)((v-j/\rho)\cdot\Omega)^2\,\mathrm{d}v.$$

By the symmetry of the bilinear form $a(\cdot,\cdot)$, we deduce that

$$c_4(\sigma,u) = \int_{\mathbb{R}^d}\beta(v)((v-j/\rho)\cdot\Omega)^2\,\mathrm{d}v = \int_{\mathbb{R}^d}\delta(v)\frac{|v_\perp|^2}{d-1}\,\mathrm{d}v.$$

Using the functions $(c_i)_{1\le i\le 5}$, the expression of the terms T_2, T_3, T_4 become

$$T_2 = \frac{\rho}{\sigma}c_1(\sigma,u)(I_d-\Omega\otimes\Omega)\left(\partial_x u+{}^t\partial_x u-\frac{2(I_d-\Omega\otimes\Omega):\partial_x u}{d-1}(I_d-\Omega\otimes\Omega)\right)(I_d-\Omega\otimes\Omega)$$

$$T_3 = \frac{\rho}{\sigma}\{c_2(\sigma,u)(I_d-\Omega\otimes\Omega)\times(I_d-\Omega\otimes\Omega)+c_3(\sigma,u)(\Omega\otimes\Omega)\times(\Omega\otimes\Omega)$$
$$+c_4(\sigma,u)[(I_d-\Omega\otimes\Omega)\times(\Omega\otimes\Omega)+(\Omega\otimes\Omega)\times(I_d-\Omega\otimes\Omega)]\}\,\partial_x u$$

$$T_4 = \frac{\rho}{\sigma}c_5(\sigma,u)[(I_d-\Omega\otimes\Omega)(\partial_x u+{}^t\partial_x u)(\Omega\otimes\Omega)+(\Omega\otimes\Omega)(\partial_x u+{}^t\partial_x u)(I_d-\Omega\otimes\Omega)].$$

We turn to the proof of the second order approximation for (4), with the collision operator $Q(f) = \mathrm{div}_v\{\sigma\nabla_v f + f\nabla_v\Phi_{u[f]}\}$.

Proof. (of Theorem 1.3)
The mass balance is clear. For the momentum balance, we write thanks to Proposition 5.1, 5.12

$$\int_{\mathbb{R}^d}(f+\varepsilon f_1)v\otimes v\,\mathrm{d}v=\int_{\mathbb{R}^d}(f+\varepsilon P_uf_1)v\otimes v\,\mathrm{d}v+\varepsilon\int_{\mathbb{R}^d}(f_1-P_uf_1)v\otimes v\,\mathrm{d}v$$
$$=\frac{j[f+\varepsilon P_uf_1]\otimes j[f+\varepsilon P_uf_1]}{\rho[f+\varepsilon P_uf_1]}+\sigma\rho[f+\varepsilon P_uf_1]\mathcal{M}_{\tilde{u}^\varepsilon}+\varepsilon(T_2+T_3+T_4)(f)+\mathcal{O}(\varepsilon^2)$$
$$=\frac{j[f+\varepsilon P_uf_1]\otimes j[f+\varepsilon P_uf_1]}{\rho[f+\varepsilon P_uf_1]}+\sigma\rho[f+\varepsilon P_uf_1]\mathcal{M}_{\tilde{u}^\varepsilon}+\varepsilon(T_2+T_3+T_4)(f+\varepsilon f_1)+\mathcal{O}(\varepsilon^2)$$

where

$$\tilde{u}^\varepsilon=\partial_k\mathcal{E}\left(\sigma,\frac{|j[f+\varepsilon P_uf_1]|}{\rho[f+\varepsilon P_uf_1]}\right)\Omega[j[f+\varepsilon P_uf_1]]=\partial_k\mathcal{E}^\star\left(\frac{|j[f+\varepsilon P_uf_1]|}{\rho[f+\varepsilon P_uf_1]}\right)\Omega[j[f+\varepsilon f_1]].$$

Therefore we expect that a second order approximation $(\tilde{\rho}^\varepsilon,\tilde{j}^\varepsilon)$ for $(\rho^\varepsilon,j^\varepsilon)$ verifies

$$\partial_t\tilde{j}^\varepsilon+\mathrm{div}_x\left(\frac{\tilde{j}^\varepsilon\otimes\tilde{j}^\varepsilon}{\tilde{\rho}^\varepsilon}+\sigma\tilde{\rho}^\varepsilon\mathcal{M}_{\tilde{u}^\varepsilon}+\varepsilon(\tilde{T}_2+\tilde{T}_3+\tilde{T}_4)\right)=0$$

where $\tilde{u}^\varepsilon=\partial_k\mathcal{E}(\sigma,|\tilde{j}^\varepsilon|/\tilde{\rho}^\varepsilon)\Omega[\tilde{j}^\varepsilon]$. □

A Proofs of Lemmas 4.2, 4.3, 5.1, 5.2, Proposition 5.12

Proof. (of Theorem 4.2)
1. Let us consider $\{E_1,\dots,E_{d-1}\}$ an orthonormal basis of $(\mathbb{R}\Omega)^\perp$, with $\Omega=\Omega[u]=u/|u|$. By using the decomposition

$$v=(\Omega\otimes\Omega)v+\sum_{i=1}^{d-1}(E_i\otimes E_i)v$$

we obtain

$$\int_{\mathbb{R}^d}M_u(v)v\otimes v\,\mathrm{d}v=\int_{\mathbb{R}^d}M_u(v)[(v\cdot\Omega)\Omega+\sum_{i=1}^{d-1}(v\cdot E_i)E_i]\otimes[(v\cdot\Omega)\Omega+\sum_{j=1}^{d-1}(v\cdot E_j)E_j]\,\mathrm{d}v$$
$$=\int_{\mathbb{R}^d}M_u(v)(v\cdot\Omega)^2\,\mathrm{d}v\;\Omega\otimes\Omega+\sum_{i=1}^{d-1}\int_{\mathbb{R}^d}M_u(v)(v\cdot E_i)^2\,\mathrm{d}v\;E_i\otimes E_i\quad(40)$$?Equ40-Swarm?

because

$$\int_{\mathbb{R}^d}M_u(v)(v\cdot\Omega)(v\cdot E_i)\,\mathrm{d}v=0,\;\;1\le i\le d-1\quad(41)$$?Equ41-Swarm?

and

$$\int_{\mathbb{R}^d}M_u(v)(v\cdot E_i)(v\cdot E_j)\,\mathrm{d}v=\int_{\mathbb{R}^d}M_u(v)\frac{|v|^2-(v\cdot\Omega)^2}{d-1}\,\mathrm{d}v,\;\;1\le i,j\le d-1.\quad(42)$$?Equ42-Swarm?

Indeed, (41) comes by performing the change of variable $v=(I_d-2E_i\otimes E_i)v'$ and by noticing that $I_d-2E_i\otimes E_i\in\mathcal{T}_u, 1\le i\le d-1$

$$\int_{\mathbb{R}^d}M_u(v)(v\cdot\Omega)(v\cdot E_i)\,\mathrm{d}v=-\int_{\mathbb{R}^d}M_u(v')(v'\cdot\Omega)(v'\cdot E_i)\,\mathrm{d}v'=0,\;1\le i\le d-1.$$

Similarly, for any $1 \leq i, j \leq d-1, i \neq j$ we have

$$\int_{\mathbb{R}^d} M_u(v)(v \cdot E_i)(v \cdot E_j)\, \mathrm{d}v = -\int_{\mathbb{R}^d} M_u(v')(v' \cdot E_i)(v' \cdot E_j)\, \mathrm{d}v' = 0.$$

For any $1 \leq i, j \leq d-1, i \neq j$, by using the transformation

$$\mathcal{O}_{ij} = \Omega \otimes \Omega + \sum_{k \notin \{i,j\}} E_k \otimes E_k + E_i \otimes E_j - E_j \otimes E_i \in \mathcal{T}_u$$

after performing the change of variable $v = \mathcal{O}_{ij} v'$, one gets

$$\int_{\mathbb{R}^d} M_u(v)(v \cdot E_i)^2\, \mathrm{d}v = \int_{\mathbb{R}^d} M_u(v')(v' \cdot E_j)^2\, \mathrm{d}v' = \int_{\mathbb{R}^d} M_u(v) \frac{|v|^2 - (v \cdot \Omega)^2}{d-1}\, \mathrm{d}v.$$

Finally (40), (41), (42) yield

$$\int_{\mathbb{R}^d} M_u(v) v \otimes v\, \mathrm{d}v = \int_{\mathbb{R}^d} M_u(v)(v \cdot \Omega)^2\, \mathrm{d}v\, \Omega \otimes \Omega + \int_{\mathbb{R}^d} M_u(v) \frac{|v|^2 - (v \cdot \Omega)^2}{d-1}\, \mathrm{d}v\, (I_d - \Omega \otimes \Omega).$$

By Proposition 2.1 we know that

$$\int_{\mathbb{R}^d} M_u(v)(v \cdot \Omega)\, \mathrm{d}v = \sigma \frac{\partial_l Z(\sigma, |u|)}{Z(\sigma, |u|)} + |u|$$

and

$$\int_{\mathbb{R}^d} M_u(v)[(v \cdot \Omega)^2 - 2|u|(v \cdot \Omega) + |u|^2]\, \mathrm{d}v = \sigma + \sigma^2 \frac{\partial_{ll}^2 Z(\sigma, |u|)}{Z(\sigma, |u|)}$$

implying that

$$\begin{aligned}\int_{\mathbb{R}^d} M_u(v)(v \cdot \Omega)^2\, \mathrm{d}v &= 2|u| \left[\sigma \frac{\partial_l Z(\sigma, |u|)}{Z(\sigma, |u|)} + |u|\right] - |u|^2 + \sigma + \sigma^2 \frac{\partial_{ll}^2 Z(\sigma, |u|)}{Z(\sigma, |u|)} \\ &= |u|^2 + 2|u|\sigma \frac{\partial_l Z}{Z} + \sigma + \sigma^2 \frac{\partial_{ll}^2 Z}{Z}.\end{aligned}$$

Thanks to the first statement in Lemma 4.1 we obtain

$$\int_{\mathbb{R}^d} M_u(v) v \otimes v\, \mathrm{d}v = \left(1 + 2\sigma \frac{\partial_l Z}{|u| Z}\right) u \otimes u + \sigma^2 \frac{\partial_{ll}^2 Z - \frac{\partial_l Z}{|u|}}{Z} \Omega \otimes \Omega + \left(\sigma + \sigma^2 \frac{\partial_l Z}{|u| Z}\right) I_d.$$

2. We know that M_0 is left invariant by $\mathcal{T}_0$ and as before we obtain, thanks to the second statement in Lemma 4.1

$$\int_{\mathbb{R}^d} M_0(v) v \otimes v\, \mathrm{d}v = \int_{\mathbb{R}^d} M_0(v) \frac{|v|^2}{d}\, \mathrm{d}v\, I_d = \left[\sigma + \sigma^2 \frac{\partial_{ll}^2 Z(\sigma, 0)}{Z(\sigma, 0)}\right] I_d.$$

□

Proof. (of Theorem 4.3)
1. We have $\frac{|j|}{\rho} = \partial_l \mathcal{E}(\sigma, |u|) = |u| + \sigma \frac{\partial_l Z}{Z(\sigma, |u|)}$ and $\sigma \partial_{ll}^2 \mathcal{E} = \sigma + \sigma^2 \partial_{ll}^2 Z / Z - \sigma^2 (\partial_l Z / Z)^2$. By the first statement in Lemma 4.2 we know that for any $u \in \mathbb{R}^d \setminus \{0\}$ we have

$$\int_{\mathbb{R}^d} M_u(v) v \otimes v\, \mathrm{d}v = \left(|u|^2 + 2|u|\sigma \frac{\partial_l Z}{Z} + \sigma^2 \frac{\partial_{ll}^2 Z - \frac{\partial_l Z}{|u|}}{Z}\right) \Omega[u] \otimes \Omega[u] + \left(\sigma + \sigma^2 \frac{\partial_l Z}{|u| Z}\right) I_d.$$

Notice that

$$\frac{|j|^2}{\rho^2}+\sigma\partial_{ll}^2\mathcal{E}=|u|^2+2|u|\sigma\frac{\partial_l Z}{Z}+\sigma+\sigma^2\frac{\partial_{ll}^2 Z}{Z}$$

and therefore

$$\begin{aligned}\int_{\mathbb{R}^d}M_u(v)v\otimes v\,\mathrm{d}v&=\left(\frac{|j|^2}{\rho^2}+\sigma\partial_{ll}^2\mathcal{E}\right)\Omega[u]\otimes\Omega[u]+\sigma\left(1+\sigma\frac{\partial_l Z}{|u|Z}\right)(I_d-\Omega[u]\otimes\Omega[u])\\&=\left(\frac{|j|^2}{\rho^2}+\sigma\partial_{ll}^2\mathcal{E}\right)\Omega[u]\otimes\Omega[u]+\sigma\frac{|j|}{\rho|u|}(I_d-\Omega[u]\otimes\Omega[u])\\&=\sigma\frac{\partial_l\mathcal{E}(\sigma,|u|)}{|u|}I_d+\frac{j\otimes j}{\rho^2}+\sigma\left(\partial_{ll}^2\mathcal{E}(\sigma,|u|)-\frac{\partial_l\mathcal{E}(\sigma,|u|)}{|u|}\right)\Omega[u]\otimes\Omega[u].\end{aligned}$$

It is easily seen that

$$\int_{\mathbb{R}^d}M_u(v)v\otimes v\,\mathrm{d}v=\int_{\mathbb{R}^d}M_u(v)(v-\frac{j}{\rho})\otimes(v-\frac{j}{\rho})\,\mathrm{d}v+\frac{j\otimes j}{\rho^2}$$

implying that

$$\mathcal{M}_u=\frac{\partial_l\mathcal{E}(\sigma,|u|)}{|u|}\,(I_d-\Omega[u]\otimes\Omega[u])+\partial_{ll}^2\mathcal{E}(\sigma,|u|)\,\Omega[u]\otimes\Omega[u].$$

2. As $\partial_l Z(\sigma,0)=0$, we have $\partial_{ll}^2\mathcal{E}(\sigma,0)=1+\sigma\frac{\partial_{ll}^2 Z}{Z}(\sigma,0)$. By the second statement in Lemma 4.2 we obtain

$$\sigma\mathcal{M}_0=\int_{\mathbb{R}^d}M_0(v)v\otimes v\,\mathrm{d}v=\sigma\left(1+\sigma\frac{\partial_{ll}^2 Z}{Z(\sigma,0)}\right)I_d=\sigma\partial_{ll}^2\mathcal{E}(\sigma,0)\;I_d.$$

□

Proof. (of Lemma 5.1)
1. Thanks to formula (8), we have for any $\Omega\in\mathbb{S}^{d-1}, l>0$

$$\begin{aligned}&\sigma Z(\sigma,l)\partial_{ll}^2\mathcal{E}(\sigma,l)=Z(\sigma,l)\int_{\mathbb{R}^d}M_{l\Omega}(v)((v-\frac{j}{\rho})\cdot\Omega)^2\,\mathrm{d}v\\&=|\mathbb{S}^{d-2}|\int_{\mathbb{R}_+}r^{d-1}\int_0^\pi\exp\left(-\frac{r^2}{2\sigma}-\frac{l^2}{2\sigma}+\frac{lr\cos\theta}{\sigma}-\frac{V(r)}{\sigma}\right)(r\cos\theta-\partial_l\mathcal{E}(\sigma,l))^2\sin^{d-2}\theta\,\mathrm{d}\theta\mathrm{d}r.\end{aligned}$$

Taking the derivative with respect to l, one gets

$$\begin{aligned}&\sigma^2Z(\sigma,l)\partial_{lll}^3\mathcal{E}+\sigma^2\partial_l Z\partial_{ll}^2\mathcal{E}=|\mathbb{S}^{d-2}|\int_{\mathbb{R}_+}r^{d-1}\int_0^\pi\exp\left(-\frac{r^2}{2\sigma}-\frac{l^2}{2\sigma}+\frac{lr\cos\theta}{\sigma}-\frac{V(r)}{\sigma}\right)\\&[(r\cos\theta-l)(r\cos\theta-\partial_l\mathcal{E}(\sigma,l))^2-2(r\cos\theta-\partial_l\mathcal{E}(\sigma,l))\partial_{ll}^2\mathcal{E}(\sigma,l)]\sin^{d-2}\theta\,\mathrm{d}\theta\mathrm{d}r\\&=Z(\sigma,l)\int_{\mathbb{R}^d}M_{l\Omega}(v)(v\cdot\Omega-l)((v-\frac{j}{\rho})\cdot\Omega)^2\,\mathrm{d}v-2Z(\sigma,l)\underbrace{\int_{\mathbb{R}^d}M_{l\Omega}(v)((v-\frac{j}{\rho})\cdot\Omega)\,\mathrm{d}v}_{=0}\partial_{ll}^2\mathcal{E}\\&=Z(\sigma,l)\int_{\mathbb{R}^d}M_{l\Omega}(v)((v-\frac{j}{\rho})\cdot\Omega)^3\,\mathrm{d}v+Z(\sigma,l)(\partial_l\mathcal{E}-l)\int_{\mathbb{R}^d}M_{l\Omega}(v)((v-\frac{j}{\rho})\cdot\Omega)^2\,\mathrm{d}v\\&=Z(\sigma,l)\int_{\mathbb{R}^d}M_{l\Omega}(v)((v-\frac{j}{\rho})\cdot\Omega)^3\,\mathrm{d}v+\sigma^2\partial_l Z\partial_{ll}^2\mathcal{E}\end{aligned}$$

implying that

$$\int_{\mathbb{R}^d} M_{l\Omega}(v)((v-\frac{j}{\rho})\cdot\Omega)^3\,\mathrm{d}v = \sigma^2\partial^3_{lll}\mathcal{E}(\sigma,l).$$

Similarly, by taking the derivative with respect to l in

$$\sigma Z(\sigma,l)\frac{\partial_l\mathcal{E}}{l} = |\mathbb{S}^{d-2}|\int_{\mathbb{R}_+} r^{d-1}\int_0^\pi \exp\left(-\frac{r^2}{2\sigma}-\frac{l^2}{2\sigma}+\frac{lr\cos\theta}{\sigma}-\frac{V(r)}{\sigma}\right)\frac{r^2\sin^2\theta}{d-1}\sin^{d-2}\theta\,\mathrm{d}\theta\mathrm{d}r$$

we obtain

$$\begin{aligned}
&\sigma^2 Z(\sigma,l)\partial_l\left(\frac{\partial_l\mathcal{E}}{l}\right)+\sigma^2\partial_l Z\frac{\partial_l\mathcal{E}}{l} = |\mathbb{S}^{d-2}|\int_{\mathbb{R}_+} r^{d-1}\int_0^\pi \exp\left(-\frac{r^2}{2\sigma}-\frac{l^2}{2\sigma}+\frac{lr\cos\theta}{\sigma}-\frac{V(r)}{\sigma}\right)\\
&(r\cos\theta-l)\frac{r^2\sin^2\theta}{d-1}\sin^{d-2}\theta\,\mathrm{d}\theta\mathrm{d}r\\
&= Z(\sigma,l)\int_{\mathbb{R}^d} M_{l\Omega}(v)((v-\frac{j}{\rho})\cdot\Omega)\frac{|v_\perp|^2}{d-1}\,\mathrm{d}v+\sigma\partial_l Z\int_{\mathbb{R}^d} M_{l\Omega}(v)\frac{|v_\perp|^2}{d-1}\,\mathrm{d}v\\
&= Z(\sigma,l)\int_{\mathbb{R}^d} M_{l\Omega}(v)((v-\frac{j}{\rho})\cdot\Omega)\frac{|v_\perp|^2}{d-1}\,\mathrm{d}v+\sigma^2\partial_l Z\frac{\partial_l\mathcal{E}}{l}.
\end{aligned}$$

Therefore we deduce that

$$\int_{\mathbb{R}^d} M_{l\Omega}(v)((v-\frac{j}{\rho})\cdot\Omega)\frac{|v_\perp|^2}{d-1}\,\mathrm{d}v = \sigma^2\partial_l\left(\frac{\partial_l\mathcal{E}}{l}\right).$$

2. For any $\Omega\in\mathbb{S}^{d-1}$, passing to the limit when $l\searrow 0$ in

$$\int_{\mathbb{R}^d} M_{l\Omega}(v)(v\cdot\Omega-\partial_l\mathcal{E})^3\,\mathrm{d}v = \sigma^2\partial^3_{lll}\mathcal{E},\quad \int_{\mathbb{R}^d} M_{l\Omega}(v)(v\cdot\Omega-\partial_l\mathcal{E})\frac{|v-(v\cdot\Omega)\,\Omega|^2}{d-1}\,\mathrm{d}v = \sigma^2\partial_l\left(\frac{\partial_l\mathcal{E}}{l}\right)$$

yields $\sigma^2\partial^3_{lll}\mathcal{E}(\sigma,0)=\int_{\mathbb{R}^d}M_0(v)(v\cdot\Omega)^3\,\mathrm{d}v=0$ and

$$\lim_{l\searrow 0}\sigma^2\partial_l\left(\frac{\partial_l\mathcal{E}}{l}\right)(\sigma,l) = \int_{\mathbb{R}^d} M_0(v)(v\cdot\Omega)\frac{|v-(v\cdot\Omega)\Omega|^2}{d-1}\,\mathrm{d}v = 0.$$

□

Proof. (of Lemma 5.2)
1. Averaging with respect to the characteristic flow (33) yields

$$\begin{aligned}
&\int_{\mathbb{R}^3} M_u(v)\left((v-\frac{j}{\rho})\cdot\xi\right)(v-\frac{j}{\rho})\otimes(v-\frac{j}{\rho})\,\mathrm{d}v\\
&=\frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{R}^3} M_u(v)\xi\cdot[\Omega\otimes\Omega(v-\frac{j}{\rho})+\cos\theta\,v_\perp+\sin\theta\,v\wedge\Omega]\\
&[\Omega\otimes\Omega(v-\frac{j}{\rho})+\cos\theta\,v_\perp+\sin\theta\,v\wedge\Omega]\otimes[\Omega\otimes\Omega(v-\frac{j}{\rho})+\cos\theta\,v_\perp+\sin\theta\,v\wedge\Omega]\,\mathrm{d}v\mathrm{d}\theta\\
&=\int_{\mathbb{R}^3} M_u(v)((v-\frac{j}{\rho})\cdot\Omega)^3\,\mathrm{d}v(\xi\cdot\Omega)\,\Omega\otimes\Omega\\
&+\int_{\mathbb{R}^3} M_u(v)\left((v-\frac{j}{\rho})\cdot\Omega\right)\frac{|v_\perp|^2}{2}\,\mathrm{d}v(\xi\cdot\Omega)(I_3-\Omega\otimes\Omega)\\
&+\int_{\mathbb{R}^3} M_u(v)(\xi\cdot v_\perp)\left((v-\frac{j}{\rho})\cdot\Omega\right)\frac{\Omega\otimes v_\perp+v_\perp\otimes\Omega}{2}\,\mathrm{d}v\\
&+\int_{\mathbb{R}^3} M_u(v)\left(\xi\cdot(v\wedge\Omega)\right)((v-\frac{j}{\rho})\cdot\Omega)\frac{\Omega\otimes(v\wedge\Omega)+(v\wedge\Omega)\otimes\Omega}{2}\,\mathrm{d}v.
\end{aligned}$$

Notice that we have

$$(\xi\cdot v_\perp)\frac{\Omega\otimes v_\perp+v_\perp\otimes\Omega}{2}+(\xi\cdot(v\wedge\Omega))\frac{\Omega\otimes(v\wedge\Omega)+(v\wedge\Omega)\otimes\Omega}{2}$$
$$=\frac{\Omega\otimes(I_3-\Omega\otimes\Omega)\xi+(I_3-\Omega\otimes\Omega)\xi\otimes\Omega}{2}|v_\perp|^2$$

implying that

$$\int_{\mathbb{R}^3}M_u(v)((v-\frac{j}{\rho})\cdot\xi)\,(v-\frac{j}{\rho})\otimes(v-\frac{j}{\rho})\,\mathrm{d}v$$
$$\left[\int_{\mathbb{R}^3}M_u(v)((v-\frac{j}{\rho})\cdot\Omega)^3\,\mathrm{d}v\,\Omega\otimes\Omega+\int_{\mathbb{R}^3}M_u(v)((v-\frac{j}{\rho})\cdot\Omega)\frac{|v_\perp|^2}{2}\,\mathrm{d}v\,(I_3-\Omega\otimes\Omega)\right](\xi\cdot\Omega)$$
$$+\int_{\mathbb{R}^3}M_u(v)((v-\frac{j}{\rho})\cdot\Omega)\frac{|v_\perp|^2}{2}\,\mathrm{d}v\,[\Omega\otimes(I_3-\Omega\otimes\Omega)\xi+(I_3-\Omega\otimes\Omega)\xi\otimes\Omega].$$

2. When $u=0$, the map $v\to M_0(v)(v\cdot\xi)\,v\otimes v$ is odd and thus its integrals over $\mathbb{R}^3$ vanishes.
□

Proof. (of Proposition 5.12)
If $d\geq 3$, we have by Proposition 5.11 and Lemma 5.3

$$\begin{aligned}T_1&=\frac{\rho}{\sigma}\int_{\mathbb{R}^d}\left[\alpha\left(v_\perp\otimes v_\perp-|v_\perp|^2\frac{I_d-\Omega\otimes\Omega}{d-1}\right)+\beta(I_d-\Omega\otimes\Omega)+\gamma(v_\perp\otimes\Omega+\Omega\otimes v_\perp)+\delta\Omega\otimes\Omega\right]\\&:\partial_x u\left[\left(v_\perp\otimes v_\perp-|v_\perp|^2\frac{I_d-\Omega\otimes\Omega}{d-1}\right)+|v_\perp|^2\frac{I_d-\Omega\otimes\Omega}{d-1}+((v-j/\rho)\cdot\Omega)(v_\perp\otimes\Omega+\Omega\otimes v_\perp)\right.\\&\left.+((v-j/\rho)\cdot\Omega)^2\Omega\otimes\Omega\right]\mathrm{d}v\\&=\frac{\rho}{\sigma}\int_{\mathbb{R}^d}\alpha(v)\left(v_\perp\otimes v_\perp-|v_\perp|^2\frac{I_d-\Omega\otimes\Omega}{d-1}\right):\partial_x u\left(v_\perp\otimes v_\perp-|v_\perp|^2\frac{I_d-\Omega\otimes\Omega}{d-1}\right)\,\mathrm{d}v\\&+\frac{\rho}{\sigma}\int_{\mathbb{R}^d}[\beta(v)(I_d-\Omega\otimes\Omega)+\delta(v)\Omega\otimes\Omega]:\partial_x u\left[\frac{|v_\perp|^2}{d-1}(I_d-\Omega\otimes\Omega)+((v-j/\rho)\cdot\Omega)^2\Omega\otimes\Omega\right]\,\mathrm{d}v\\&+\frac{\rho}{\sigma}\int_{\mathbb{R}^d}((v-j/\rho)\cdot\Omega)\gamma(v)(v_\perp\otimes\Omega+\Omega\otimes v_\perp):\partial_x u(v_\perp\otimes\Omega+\Omega\otimes v_\perp)\,\mathrm{d}v\\&=T_2+T_3+T_4.\end{aligned}$$

The term T_3 writes

$$\begin{aligned}T_3&=\frac{\rho}{\sigma}\int_{\mathbb{R}^d}\frac{\beta(v)|v_\perp|^2}{d-1}\,\mathrm{d}v(I_d-\Omega\otimes\Omega)\times(I_d-\Omega\otimes\Omega)\partial_x u\\&+\frac{\rho}{\sigma}\int_{\mathbb{R}^d}\delta(v)((v-j/\rho)\cdot\Omega)^2\,\mathrm{d}v(\Omega\otimes\Omega)\times(\Omega\otimes\Omega)\partial_x u\\&+\int_{\mathbb{R}^d}\delta(v)\frac{|v_\perp|^2}{d-1}\,\mathrm{d}v(I_d-\Omega\otimes\Omega)\times(\Omega\otimes\Omega)\partial_x u\\&+\frac{\rho}{\sigma}\int_{\mathbb{R}^d}\beta(v)((v-j/\rho)\cdot\Omega)^2\,\mathrm{d}v(\Omega\otimes\Omega)\times(I_d-\Omega\otimes\Omega)\partial_x u.\end{aligned}\tag{43}$$?CalT3?

In order to evaluate T_4 observe that

$$\begin{aligned}
&\int_{\mathbb{R}^d}((v-j/\rho)\cdot\Omega)\gamma(v)(v_\perp\otimes\Omega+\Omega\otimes v_\perp):\partial_x u\, v_\perp\otimes\Omega\,\mathrm{d}v\\
&=\int_{\mathbb{R}^d}(v-j/\rho)\cdot\Omega)\gamma(v)v_\perp\otimes\Omega:(\partial_x u+{}^t\partial_x u)v_\perp\otimes\Omega\,\mathrm{d}v\\
&=\int_{\mathbb{R}^d}(v-j/\rho)\cdot\Omega)\gamma(v)\sum_{k=1}^{d-1}(v\cdot E_k)E_k\otimes\Omega:(\partial_x u+{}^t\partial_x u)\sum_{l=1}^{d-1}(v\cdot E_l)E_l\otimes\Omega\,\mathrm{d}v\\
&=\int_{\mathbb{R}^d}(v-j/\rho)\cdot\Omega)\gamma(v))\sum_{k=1}^{d-1}(v\cdot E_k)^2E_k\otimes\Omega:(\partial_x u+{}^t\partial_x u)E_k\otimes\Omega\,\mathrm{d}v\\
&=\int_{\mathbb{R}^d}(v-j/\rho)\cdot\Omega)\gamma(v))\frac{|v_\perp|^2}{d-1}\sum_{k=1}^{d-1}(\partial_x u+{}^t\partial_x u)\Omega\cdot E_kE_k\otimes\Omega\,\mathrm{d}v\\
&=\int_{\mathbb{R}^d}(v-j/\rho)\cdot\Omega)\gamma(v))\frac{|v_\perp|^2}{d-1}\,\mathrm{d}v(I_d-\Omega\otimes\Omega)(\partial_x u+{}^t\partial_x u)(\Omega\otimes\Omega).
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
T_4=\frac{\rho}{\sigma}\int_{\mathbb{R}^d}(v-j/\rho)\cdot\Omega)\gamma(v))\frac{|v_\perp|^2}{d-1}\,\mathrm{d}v(I_d-\Omega\otimes\Omega)(\partial_x u+{}^t\partial_x u)(\Omega\otimes\Omega)\\
+(\Omega\otimes\Omega)(\partial_x u+{}^t\partial_x u)I_d-\Omega\otimes\Omega).
\end{aligned}\tag{44}$$

?CalT4?

We turn to the evaluation of T_2, which reduces, by Lemma 5.3, to

$$T_2=\frac{\rho}{\sigma}\int_{\mathbb{R}^d}\alpha(v)(v_\perp\otimes v_\perp:\partial_x u)v_\perp\otimes v_\perp\,\mathrm{d}v-\frac{\rho}{\sigma}\int_{\mathbb{R}^d}\alpha(v)\frac{|v_\perp|^4}{(d-1)^2}\,\mathrm{d}v[(I_d-\Omega\otimes\Omega):\partial_x u](I_d-\Omega\otimes\Omega).$$

It is easily seen that the only non vanishing terms $A_{ijkl}=\int_{\mathbb{R}^d}\alpha(v)(v\cdot E_i)(v\cdot E_j)(v\cdot E_k)(v\cdot E_l)\,\mathrm{d}v$, $1\le i,j,k,l\le d-1$, are those such that $1\le\mathrm{card}\{i,j,k,l\}\le 2$. As α is left invariant by $\mathcal{T}_u$, $u\neq 0$, there is a function $\tilde\alpha$ such that $\alpha(v)=\tilde\alpha(|v_\perp|^2,v\cdot\Omega)$. By using polar coordinates, we obtain for any $1\le i\neq j\le d-1$

$$\begin{aligned}
\frac{A_{iiii}}{A_{iijj}}=\frac{A_{1111}}{A_{1122}}&=\frac{\int_{\mathbb{R}^d}\tilde\alpha(z_1^2+...+z_{d-1}^2,z_d)z_1^4\mathrm{d}z}{\int_{\mathbb{R}^d}\tilde\alpha(z_1^2+...+z_{d-1}^2,z_d)z_1^2z_2^2\mathrm{d}z}\\
&=\frac{\int_{\mathbb{R}^{d-2}}\int_{\mathbb{R}_+}\tilde\alpha(r^2+z_3^2+...+z_{d-1}^2,z_d)r^5\int_0^{2\pi}\cos^4\varphi\mathrm{d}\varphi\mathrm{d}r\mathrm{d}z_3...\mathrm{d}z_{d-1}}{\int_{\mathbb{R}^{d-2}}\int_{\mathbb{R}_+}\tilde\alpha(r^2+z_3^2+...+z_{d-1}^2,z_d)r^5\int_0^{2\pi}\cos^2\varphi\sin^2\varphi\mathrm{d}\varphi\mathrm{d}r\mathrm{d}z_3...\mathrm{d}z_{d-1}}=3
\end{aligned}$$

where we have used

$$\int_0^{2\pi}\cos^4\varphi\mathrm{d}\varphi=\frac{3\pi}{4},\ \int_0^{2\pi}\cos^2\varphi\sin^2\varphi\mathrm{d}\varphi=\frac{\pi}{4}.$$

Observe also that

$$\begin{aligned}
\int_{\mathbb{R}^d}\alpha(v)|v_\perp|^4\,\mathrm{d}v&=\int_{\mathbb{R}^d}\alpha(v)[(v\cdot E_1)^2+...+(v\cdot E_{d-1})^2]^2\,\mathrm{d}v\\
&=(d-1)A_{1111}+(d-1)(d-2)A_{1122}=(d-1)(d+1)A_{1122}
\end{aligned}$$

implying that

$$A_{1122}=\frac{\int_{\mathbb{R}^d}\alpha(v)|v_\perp|^4\,\mathrm{d}v}{d^2-1},\ A_{1111}=3\frac{\int_{\mathbb{R}^d}\alpha(v)|v_\perp|^4\,\mathrm{d}v}{d^2-1}.$$

Finally the terms A_{ijkl} write

$$A_{ijkl} = \frac{\int_{\mathbb{R}^d} \alpha(v)|v_\perp|^4 \, \mathrm{d}v}{d^2-1}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \ 1 \le i,j,k,l \le d-1.$$

The previous computations lead to the following expression

$$\begin{aligned}
&\int_{\mathbb{R}^d} \alpha(v)(v_\perp \otimes v_\perp : \partial_x u) v_\perp \otimes v_\perp \, \mathrm{d}v \\
&= \sum_{i,j,k,l<d} \int_{\mathbb{R}^d} \alpha(v)(v\cdot E_i)(v\cdot E_j)(v\cdot E_k)(v\cdot E_l) \, \mathrm{d}v (E_i \otimes E_j : \partial_x u) E_k \otimes E_l \\
&= \int_{\mathbb{R}^d} \alpha(v) \frac{|v_\perp|^4}{d^2-1} \, \mathrm{d}v \sum_{i,j,k,l<d} (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) : \partial_x u)(E_i \otimes E_j : \partial_x u) E_k \otimes E_l \\
&= \int_{\mathbb{R}^d} \alpha(v) \frac{|v_\perp|^4}{d^2-1} \, \mathrm{d}v \sum_{i,j<d} \{[(I_d - \Omega\otimes\Omega) : \partial_x u](I_d - \Omega\otimes\Omega) + (\partial_x u E_j \cdot E_i)[E_i \otimes E_j + E_j \otimes E_i]\} \\
&= \int_{\mathbb{R}^d} \alpha(v) \frac{|v_\perp|^4}{d^2-1} \, \mathrm{d}v \{[(I_d - \Omega\otimes\Omega) : \partial_x u](I_d - \Omega\otimes\Omega) + (I_d - \Omega\otimes\Omega)(\partial_x u + {}^t\partial_x u)(I_d - \Omega\otimes\Omega)\}.
\end{aligned}$$

Consequently the term T_2 is given by

$$\begin{aligned}
T_2 = \frac{\rho}{\sigma} \int_{\mathbb{R}^d} \alpha(v) \frac{|v_\perp|^4}{d^2-1} \, \mathrm{d}v \qquad (45)\ \texttt{?CalT2?} \\
(I_d - \Omega\otimes\Omega)\left(\partial_x u + {}^t\partial_x u - \frac{2(I_d - \Omega\otimes\Omega) : \partial_x u}{d-1}(I_d - \Omega\otimes\Omega)\right)(I_d - \Omega\otimes\Omega).
\end{aligned}$$

The desired result follows from (43), (44), (45). □

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