



**HAL**  
open science

## Dual algebras with a rich set of $w^*$ -continuous characters.

Frédéric Jaeck

► **To cite this version:**

Frédéric Jaeck. Dual algebras with a rich set of  $w^*$ -continuous characters.. Universităţii de vest din Timișoara. Analele. Seria Matematică-Informatică, 2001, 39 (2), pp.87-97. hal-04043703

**HAL Id: hal-04043703**

**<https://hal.science/hal-04043703>**

Submitted on 27 Mar 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

DUAL ALGEBRAS WITH A RICH SET OF  $w^*$ -CONTINUOUS  
CHARACTERS

FRÉDÉRIC JAECK

**Abstract.** Let  $\mathcal{H}$  be an infinite dimensional, separable, complex Hilbert space. We study operator algebras on  $\mathcal{H}$  which possess many  $w^*$ -continuous characters. In this general setting we give a reduction for the invariant subspace problem. We also show that the considered algebras turn out to be uniform algebras. The introduction of certain properties lead to the existence of non trivial invariant subspaces and to a generalisation of a structure theorem for the algebra.

Let  $\mathcal{H}$  be an infinite dimensional, separable, complex Hilbert space. We denote by  $\mathcal{L}(\mathcal{H})$  the algebra of all bounded operators on  $\mathcal{H}$ . As usual  $\mathcal{C}^1(\mathcal{H})$  denotes the ideal of all finite-trace operators, which is a Banach algebra under the trace norm:

$$\mathcal{C}^1(\mathcal{H}) = \left\{ \sum_{i=1}^{\infty} x_i \otimes y_i, (x_i)_i, (y_i)_i \in l^2(\mathcal{H}) \right\}$$

where for  $x, y \in \mathcal{H}$ ,  $x \otimes y(u) = (x|u)y$ , and the trace norm is defined by

$$\|K\|_1 = \sup\{|\text{tr}(T, K)| \mid T \in \mathcal{L}(\mathcal{H}), \|T\| \leq 1\}$$

The duality  $(\mathcal{C}^1(\mathcal{H}))' = \mathcal{L}(\mathcal{H})$  induces a weak- $*$  topology on  $\mathcal{L}(\mathcal{H})$  and a dual algebra is any  $w^*$ -closed sub-algebra of  $\mathcal{L}(\mathcal{H})$ . It is often useful to

---

AMS Subject Classification: 47D27, 47D45.

Keywords and phrases: contraction, Hilbert space, dual algebras, operator algebras, uniform algebras

actually see a dual algebra as the dual of a quotient space of  $\mathcal{C}^1(\mathcal{H})$ . More precisely we denote by  ${}^\perp\mathcal{A} := \{L \in \mathcal{C}^1(\mathcal{H}), \langle L, T \rangle = \text{tr}(T, L) = 0, \forall T \in \mathcal{A}\}$  the preannihilator ideal of  $\mathcal{A}$ . We then have  $\mathcal{A} = (\mathcal{Q}_{\mathcal{A}})'$  where  $\mathcal{Q}_{\mathcal{A}} := \mathcal{C}^\infty(\mathcal{H})/{}^\perp\mathcal{A}$ .

We will denote by  $\mathcal{A}_T$  the dual algebra generated by a single operator  $T$ .

This framework has turned out to be very efficient for the study of absolutely continuous contractions: indeed it is a natural setting to put together many different notions around invariant subspaces, reflexivity and spectral properties (see [2] for initial setting and results up to 1984).

Here we generalize some results for dual algebras which possess enough  $w^*$ -continuous characters. We shall see that in this case the dual algebra turns out to be a uniform algebra.

From now on we will denote by  $\text{Max}\mathcal{A}$  the set of all characters on  $\mathcal{A}$ .

Two well known situations illustrate this setting:

1) The dual algebra generated by a sub-normal operator. In this case S. Brown proved the existence of invariant subspaces (see [5]).

2) Developing considerably Scott Brown's technique it was extended to more general algebras. In 1978, S. Brown, B. Chevreau and C. Pearcy showed that a contraction with a dominating spectrum in the open unit disk had non trivial invariant subspaces (see [3]).

Then the same authors showed in 1986 that a contraction with is spectrum containig  $\mathbf{T}$  had non trivial invariant subspaces (see [4]).

Finally, H. Bercovici and B. Chevreau independantly proved that contractions with an isometric Sz.-Nagy Foiaş functional calculus had property  $\mathbf{A}_1$  (definition reviewed below) (see [1], [10]). S. Brown and B. Chevreau then showed the reflexivity of such algebras (see [6]).

In any case the study of such operator algebras enables to give some classification and results related to the problem of invariant subspaces and reflexivity (see Cassier's articles on singly generated uniform dual algebras [8], [7]).

It is some importance that the following results do not use any dilation theory since collections of three or more contractions do not possess a unitary dilation in general. First of all we recall the definitions of standard properties for general dual algebras. Let  $1 \leq m, n \leq \aleph$ , be two cardinal numbers. A

dual algebra  $\mathcal{A} = (\mathcal{Q}_{\mathcal{A}})^*$  is said to have property  $(\mathbf{A}_{m,n})$  if for any collection  $(L_{i,j})_{i < m, j < n}$  of elements in  $\mathcal{Q}_{\mathcal{A}}$ , there exists vectors  $(x_i)_{i < m}, (y_j)_{j < n}$  such that:

$$L_{ij} = [x_i \otimes y_j]_{\mathcal{Q}_{\mathcal{A}}} \quad \forall i, j$$

where the brackets denote the image in the quotient space  $\mathcal{Q}_{\mathcal{A}}$ . In particular case when  $m = n$  we shall write  $(\mathbf{A}_n)$  for short.

We now introduce new properties which are weaker than  $(\mathbf{A}_n)$  but still lead to the existence of invariant subspaces and structure results: we say that the algebra  $\mathcal{A}$  has property  $(\mathbf{Trig}_n)$  if for any collection  $(L_i)_{0 < i < n}$  of elements in  $\mathcal{Q}_{\mathcal{A}} \cap \text{Max}\mathcal{A}$ , there exists vectors  $(x_i)_{0 \leq i < n}, (y_i)_{0 \leq j < n}$  such that:

$$\begin{cases} [x_i \otimes y_j]_{\mathcal{Q}_{\mathcal{A}}} = 0 & 1 \leq i < j < n \\ [x_i \otimes y_i]_{\mathcal{Q}_{\mathcal{A}}} = L_i & 1 \leq i < n \end{cases}$$

Some intermediate properties were first used by G. Exner and I.B. Jung (see [14]) and more recently by G. Cassier, I. Chalendar and B. Chevreau (see [9]) to give additional information on the classes  $\mathbf{A}_{m,n}$  for one contraction.

Recall that an invariant subspace for  $\mathcal{A}$  is any closed subspace  $E \subset \mathcal{H}$  such that  $TE \subset E$ , for all  $T \in \mathcal{A}$ . A non trivial invariant subspace is an invariant subspace  $E$  such that  $E \neq (0)$  and  $E \neq \mathcal{H}$ . The lattice of all invariant subspaces for  $T$  will be denoted by  $\text{Lat}T$ . If  $N \subset M$  are two invariant subspaces, the space  $K = M \ominus N$  is said to be semi-invariant. In this case, the compression of  $T$  to  $K$  is denoted by  $T_K$  and is defined by  $T_K x = P_K T x$ , for all  $x \in K$ .

## 1. Uniform dual algebras

**Definition 1.1.** *Let  $\mathcal{A}$  be a dual algebra and  $\Lambda = \text{Max}\mathcal{A} \cap \mathcal{Q}_{\mathcal{A}}$  the set of all  $w^*$ -continuous characters on  $\mathcal{A}$ .*

*We say that  $\mathcal{A}$  is of type  $\mathbf{A}$  if  $\Lambda$  is dominating for  $\mathcal{A}$ , i.e.*

$$\|T\| = \sup_{\lambda \in \Lambda} \{|\lambda(T)|\}, \quad \forall T \in \mathcal{A}.$$

**Remark.** This definition has another formulation:  $\Lambda$  is dominating for  $\mathcal{A}$  if and only if  $\overline{\text{aco}}\Lambda = (\mathcal{Q}_{\mathcal{A}})_1$  (closed unit ball). (See proposition 2.2 in [12]).

**Remark.** This is a generalization of the standard class **A** for a single absolutely continuous contraction; such a contraction has a Sz.-Nagy Foias calculus  $\Phi_T: H^\infty(\mathbf{T}) \rightarrow \mathcal{A}_T \subset \mathcal{L}(\mathcal{H})$  which is  $w^*$ -continuous and isometric. In this case for any  $\lambda \in \mathbf{D}$  the point evaluation  $E_\lambda$  at  $\lambda$  is a  $w^*$ -continuous character and we have  $\overline{\text{aco}}(E_\lambda) = (\mathcal{Q}_{H^\infty})_1 = (L^1(\mathbf{T})/H_0^1(\mathbf{T}))_1$ . This property is transmitted through the preadjoint application  $\phi_T$  of  $\Phi_T$  to the predual algebra  $\mathcal{Q}_T$ : we have  $\phi_T: \mathcal{Q}_T \rightarrow \mathcal{Q}_{H^\infty}$  and  $\overline{\text{aco}}\{C_\lambda, \lambda \in \Lambda\} = (\mathcal{Q}_T)_1$  where  $C_\lambda aa\lambda = \phi_T(E_\lambda)$ .

The following proposition says a bit more about the structure algebras of type **A**.

**Proposition 1.1.** *Let  $\mathcal{A}$  be a dual algebra of type **A**. Then*

- 1)  $\mathcal{A}$  is commutative;
- 2)  $\mathcal{A}$  is a uniform algebra.

**Proof.** The commutativity of such algebras is inherited from the dominance of  $\Lambda$ : for any  $V, W \in \mathcal{A}$  and any  $C \in \Lambda$  we have  $C(VW - WV) = 0$  and thus  $\|VW - WV\| = 0$ .

Since  $\mathcal{A}$  is of type **A** we can compute for any  $T \in \mathcal{A}$ :

$$\|T^n\| = \sup_{\lambda \in \Lambda} \{|\lambda(T^n)|\} = \|T\|^n$$

So  $\|\widehat{T}\|_{\text{Max } \mathcal{A}} = \lim \|T^n\|^{1/n} = \|T\|$ , the Gelfand transform is an isometry and  $\mathcal{A}$  can be identified with a uniformly closed sub-algebra of  $C(\text{Max } \mathcal{A})$ , where  $\text{Max } \mathcal{A}$  is endowed with the  $w^*$ -topology. It is easy to notice that  $\widehat{\mathcal{A}}$  always separates the points and contains the constant functions and hence is a uniform algebra.  $\square$

## 2. The property $(\text{Trig}_n)$ and associated structure results

The following definition enlarges in a natural way the one available for algebras possessing a  $H^\infty$   $w^*$ -continuous calculus (see [2]).

**Definition 2.1.** A dual algebra  $\mathcal{A}$  is said to be of type

- 1)  $C_0$ . if  $\forall w \in \mathcal{H}$  and  $\forall (x_n)_n \subset \mathcal{H}$ ,  $x_n \rightarrow 0$  we have  $[w \otimes x_n]_{\mathcal{Q}_\mathcal{A}} \rightarrow 0$ ;
- 2)  $C_{.0}$  if  $\forall w \in \mathcal{H}$  and  $\forall (x_n)_n \subset \mathcal{H}$ ,  $x_n \rightarrow 0$  we have  $[x_n \otimes w]_{\mathcal{Q}_\mathcal{A}} \rightarrow 0$ ;
- 3)  $C_{00}$  if  $\mathcal{A}$  is simultaneously of type  $C_0$ . and of type  $C_{.0}$ .

**Notation.** In the sequel we shall denote by  $\mathcal{L}_n$  ( $1 \leq n \leq \S_l$ ) the set of all elements in  $\mathcal{Q}_\mathcal{A}$  of length  $n$  :  $L \in \mathcal{L}_n$  if there exists  $x_1, \dots, x_n, y_1, \dots, y_n$  in  $\mathcal{H}$  such that  $L = \sum_{i=1}^n [x_i \otimes y_i]_{\mathcal{Q}_\mathcal{A}}$ . Recall that the weak operator (and the strong operator) continuous functionals are the elements of  $\bigcup_{n \geq 1} \mathcal{L}_n$  and the functionals that are continuous with respect to the weak\* operator topology are in  $\mathcal{L}_{\S_l}$ .

**Proposition 2.1.** Let  $\mathcal{A}$  be a dual algebra. If  $C \in \text{Max}(\mathcal{A}) \cap \mathcal{L}_1$  then there exists  $x \in \mathcal{H}$  such that  $C = [x \otimes x]_{\mathcal{Q}_\mathcal{A}}$ .

**Proof.** Since  $C \in \mathcal{L}_1$  then there exists  $u, v \in \mathcal{H}$  such that  $C = [u \otimes v]_{\mathcal{Q}_\mathcal{A}}$ .

We then write  $\mathcal{M} = \overline{\text{Ker}Cu}$ ,  $\mathcal{N} = \overline{Av}$  and  $\mathcal{D} = \mathcal{N} \ominus \mathcal{M}$ .

We can assume that  $P_{\mathcal{N}v} \neq 0$  since  $C \neq 0$  and we can substitute  $P_{\mathcal{N}v}$  to  $v$ .

Since  $\mathcal{A} = \mathcal{C}I + \text{Ker}C$  it is easy to see that  $\mathcal{D} = \mathcal{C}_v$  and we can then write  $C = [u \otimes v]_{\mathcal{Q}_\mathcal{A}} = [v' \otimes v']_{\mathcal{Q}_\mathcal{A}}$  where  $v' = \frac{v}{\|v\|}$ .  $\square$

We now describe the capacity of evaluating characters many times with elements of  $\mathcal{L}_1$  as soon as we have property  $(\text{Trig}_N)$ :

**Proposition 2.2.** Let  $\mathcal{A}$  be a dual algebra over  $\mathcal{H}$  with property  $(\text{Trig}_N)$  for one  $N \in \mathbb{N}$ . If  $C \in \text{Max}(\mathcal{A})$ , then there exists orthonormal collections  $(u_n)_{n=1}^N$  and  $(v_n)_{n=1}^N$  such that  $C = [u_n \otimes v_n]_{\mathcal{Q}_\mathcal{A}}$ .

**Proof.** Let  $C \in \text{Max } \mathcal{A} \cap \mathcal{Q}_\mathcal{A}$ . Thanks to property  $(\text{Trig}_N)$  we can find

vectors  $x_1, \dots, x_N, y_1, \dots, y_N$  in  $\mathcal{H}$  such that:

$$\begin{cases} C = [x_i \otimes y_i] & i = 1, \dots, N \\ 0 = [x_i \otimes y_j] & 1 \leq i < j \leq N \end{cases}$$

We now make a step by step transformation on those equalities.

Step one: If the  $x_i$ s were not independent, there would exist a line, say the  $k^{\text{th}}$ , such that  $x_k \in \text{Vec}\{x_1, \dots, x_{k-1}\}$  and  $[x_k \otimes y_k] = C$  and  $[x_k \otimes y_j] = 0$  for  $j = 1, \dots, k-1$ , which is not possible.

In the same way we can show that the  $y_j$ s are also linearly independent.

Step two: We use the Schmidt orthonormalization method for the  $x_i$ s. First, for  $2 \leq i \leq N$ , we replace  $x_i$  by  $x'_i = x_i + k'_i x_i + k''_i x_2 + \dots + k_i^{(i-1)} x_{i-1}$  so that  $x'_i \perp x_j$  for  $1 \leq j < i$ .

To homogenize the notation, we set  $x'_1 = x_1$ . We then have, for  $2 \leq i \leq N$ ,  $[x'_i \otimes y_i] = \sum_{j=1}^{i-1} k_i^{(j)} [x_i \otimes y_j] + [x_i \otimes y_i] = 0 + C$ .

We also have, for  $2 \leq i < j \leq N$ ,  $[x'_i \otimes y_j] = \sum_{l=1}^{i-1} k_i^{(l)} [x_l \otimes y_j] + [x_i \otimes y_j] = 0$ .

Step three: We go along the same line with the  $y_j$ s: we set  $y'_N = y_N$  and we replace, for  $1 \leq i \leq N-1$ ,  $y_i$  by  $y'_i = y_i + \sum_{k=i+1}^N l_i^{(k)} y_k$  to obtain the desired orthogonal sequence.

Finally we have, for  $1 \leq i \leq N-1$ ,  $[x'_i \otimes y'_i] = \sum_{k=i+1}^N l_i^{(k)} [x'_i \otimes y_k] + [x'_i \otimes y'_i] = 0 + C$  and, for  $1 \leq i < j \leq N$ ,  $[x'_i \otimes y'_j] = \sum_{k=i+1}^N l_i^{(k)} [x'_i \otimes y_k] + [x'_i \otimes y_j] = 0$ , which ends the proof.  $\square$

### 3. Reduction for the invariant subspace problem

In this section we introduce a notion which enables to measure the richness of the set  $\Lambda$  of all  $w^*$ -continuous characters. It is noteworthy that if  $\Lambda$  is a dominating part of the disk  $\mathbf{D}$  for the algebra  $H^\infty(\mathbf{D})$ , then any collection  $\Lambda \setminus \mathcal{F}$ , where  $\mathcal{F}$  is finite, is still dominating. (see [16] for related definitions)

and dominance properties and [11] for applications in the context of dual algebras).

In order to extend these ideas, we set:

**Definition 3.1.** *Let  $\mathcal{A}$  be a dual algebra of type  $\mathbf{A}$ . We say that  $\Lambda = \text{Max}(\mathcal{A}) \cap Q_{\mathcal{A}}$  is rich if it is dominating and if for any finite set  $\mathcal{F} \subset \Lambda$ , the set  $\Lambda \setminus \mathcal{F}$  is still dominating.*

In the next proposition, we show that there is no loss of generality in assuming that  $\Lambda$  is rich for the problem of invariant subspace for algebra  $\mathcal{A}$  of type  $\mathbf{A}$ .

**Theorem 3.1.** *Let  $\mathcal{A}$  be a dual algebra of type  $\mathbf{A}$ . If the set of all  $w^*$ -continuous characters  $\Lambda$  is not rich for  $\mathcal{A}$  then the algebra contains a non trivial algebraic element and has a non trivial hyper-invariant subspace.*

**Proof.** Let us suppose that there exists a finite set  $\mathcal{F} \subset \Lambda$  such that  $\Lambda \setminus \mathcal{F}$  is not longer dominating. Then there exists  $A \in \mathcal{A}$ ,  $\|A\| = 1$  and via the Hahn - Banach theorem there exists  $\lambda_0 \in \Lambda$  such that:

$$\|A\| = 1 = \lambda_0(A) > \sup_{\lambda \in \Lambda} |\lambda(A)|$$

Moreover  $\|A^n\| = \|A\|^n = 1$  and we can extract a  $w^*$ -converging subsequence  $(A^{n_k})_k \rightarrow B$ .

But  $\lambda_0$  is  $w^*$ -continuous and then  $\lambda_0(B) = 1$ , which in turn implies that  $B \neq 0$ .

Notice that for all  $\lambda \in \Lambda \setminus \mathcal{F}$ , we have  $|\lambda(A)| < 1$  and then  $\lambda(B) = 0$ .

We shall denote  $\Lambda_0 = \{\lambda \in \mathcal{F}, |\lambda(A)| = 1\} = \{\lambda_1, \dots, \lambda_N\}$ .

The preceding lines showed that:

$$\begin{cases} \forall \lambda \in \Lambda_0 & |\lambda(B)| = 1 \\ \forall \lambda \notin \Lambda_0 & \lambda(B) = 0 \end{cases}$$

We now set  $P(z) = (z - \lambda_1(B)) \cdots (z - \lambda_N(B))z$ . For every  $\lambda \in \Lambda$  we have  $\lambda(P(B)) = 0$  i.e.  $P(B) = 0$  since  $\Lambda$  is dominating.

Moreover  $\text{Ker} B$  is a non-trivial hyperinvariant subspace for the algebra  $\mathcal{A}$  since  $B \neq 0$  and  $B \neq 1$ .  $\square$



#### 4. Generalisation of one of Exner's results

Before we give a result relative to properties  $(\mathbf{A}_{m,n})$  for algebras of type  $\mathbf{A}$ , we recall a property for general dual algebras which appeared to be one of the most fundamental tools in the study of dual algebras with a  $H^\infty$ -calculus (see [2] for example of application).

If  $\mathcal{A}$  is any dual algebra, and for  $\theta \in [0, 1]$  we have:

$$\begin{aligned} \mathcal{X}_\theta &= \{L \in \mathcal{Q}_\mathcal{A}, \exists (x_n)_n, (y_n)_n \subset (\mathcal{H})_1, \text{ such that} \\ &\quad (1) \overline{\lim}_n \|L - [x_n \otimes y_n]_\mathcal{A}\| \leq \theta \\ &\quad (2) \|[x_n \otimes w]_\mathcal{A}\| \rightarrow 0 \forall w \in \mathcal{H} \\ &\quad (3) \|[w \otimes y_n]_\mathcal{A}\| \rightarrow 0 \forall w \in \mathcal{H}\} \end{aligned}$$

Now for any  $0 \leq \theta < \gamma \leq 1$ , the algebra  $\mathcal{A}$  is said to have property  $X_{\theta,\gamma}$  if  $(\mathcal{Q}_\mathcal{A})_\gamma \subset \mathcal{X}_\theta$ .

It is remarquable that any dual algebra with property  $X_{\theta,\gamma}$  has also property  $A_{X_\theta}$  (see [2]).

Here we give the generalization of one of Exner's well result for the algebra generated by an absolutely continuous contraction (see [13]):

**Theorem 4.1.** *Let  $\mathcal{A}$  be a dual algebra of type  $\mathbf{A}$  and of type  $C_{00}$ . If  $\mathcal{A}$  has property  $(\mathbf{Trig}_n)$  for every  $n \in \mathbb{N}^*$ , then  $\mathcal{A}$  has property  $X_{0,1}$  and thus also property  $(\mathbf{A}_{X_0})$ .*

**Proof.** 1) The algebra is of type  $\mathbf{A}$ , thus, given  $L \in \mathcal{Q}_\mathcal{A}$ ,  $\|L\| \leq 1$ , and  $\epsilon > 0$ , there exist  $\alpha_1, \dots, \alpha_n \in \mathcal{C}$ ,  $\sum_{i=1}^n |\alpha_i| \leq 1$  and  $w^*$ -continuous characters  $C_1, \dots, C_n \in \Lambda$  such that:

$$\|L - \sum_{i=1}^n \alpha_i C_i\| < \epsilon/2$$

Thanks to the hypothesis  $C_{00}$ , it is enough to find vectors  $(x_n)_n$  and  $(y_n)_n$  in  $\mathcal{H}$  such that:

$$\begin{cases} x_n \rightarrow 0, y_n \rightarrow 0 \\ \|L - [x_n \otimes y_n]_{\mathcal{Q}_\mathcal{A}}\| \rightarrow 0 \text{ when } n \rightarrow \infty \end{cases}$$

2) Let us suppose for a while that the algebra  $\mathcal{A}$  has the following property:  $(\mathcal{P}) : \forall \mathcal{C} \in \Lambda$ , there exist sequences  $(e_k)_k \subset \mathcal{H}$  and  $(f_k)_k \subset \mathcal{H}$  such that  $e_k \rightarrow 0$ ,  $f_k \rightarrow 0$  and  $C = [e_k \otimes f_k]_{Q_{\mathcal{A}}}$  for all  $k \in \mathbb{N}$ . Then  $\Lambda \subset \mathfrak{S}_l$  and since  $\mathfrak{S}_l$  is absolutely convex and closed (see [2]) we get  $\overline{\text{acc}\mathcal{O}}(\Lambda) = Q_1 = \mathcal{X}_l$ .

3) We now eventually show that our algebra has property  $(\mathcal{P})$ .

Let  $C$  be a  $w^*$ -continuous character. Since, for every  $N$ ,  $\mathcal{A}$  has property  $(Trig_N)$ , and thanks to proposition 2.2 there exist orthonormal sequences  $u_1, \dots, u_N, v_1, \dots, v_N \in \mathcal{H}$  such that:

$$C = [u_1 \otimes v_1] = \dots = [u_N \otimes v_N]$$

It only remains to pick the  $u_N$ s and the  $v_N$ s that  $u_n \rightarrow 0$  and  $v_n \rightarrow 0$ . In order to achieve this we fix here a sequence  $(x_n)_n \subset \mathcal{H}$  of unit vectors which is dense in the unit ball of  $\mathcal{H}$ . It is then enough via an elementary argument of density to find for each given  $p$ , one  $e_p = u_{n_p}$  and one  $f_p = v_{n_p}$  such that  $|\langle e_p | x_k \rangle| \leq 1/\sqrt{p}$ ,  $1 \leq k \leq p$ ,  $|\langle f_p | x_k \rangle| \leq 1/\sqrt{p}$ ,  $1 \leq k \leq p$  and  $C = [e_p \otimes f_p]_{Q_{\mathcal{A}}}$ .

We start with an elementary remark: given  $x \in \mathcal{H}$ ,  $\|x\| = 1$ , an orthonormal sequence  $t_1, \dots, t_N \in \mathcal{H}$  and  $N \in \mathbb{N}^*$  we have  $\|x\|^2 \geq \sum_{i=1}^N |\langle t_i | x \rangle|^2$ . Thus there exists  $i \in \{1, \dots, N\}$  such that  $|\langle t_i | x \rangle| \leq \frac{1}{\sqrt{N}}$ .

We now consider a fixed  $p$  and we set:

$$\Lambda_1 = \{u_1, \dots, u_{p^{2p}}\} = \bigcup_{k=1}^{p^{2p-2}} S_k^1$$

where  $\text{card}(S_k^1) = p^2$ , for all  $k = 1, \dots, p^{2p-2}$ .

For every  $k = 1, \dots, p^{2p-2}$ , there exists  $w_k^1 \in S_k^1$  such that  $|\langle w_k^1 | x_1 \rangle| \leq \frac{1}{\sqrt{p}}$ , and there exists a corresponding  $y_p^1$  such that  $C = [w_p^1 \otimes y_p^1]$  and  $|\langle x_1 | y_k^1 \rangle| \leq \frac{1}{\sqrt{p}}$  (first pick the  $p$  candidates for  $w_k^1$ , and then chose one corresponding to  $y_k^1$  matching the requirement).

We then set:

$$\Lambda_2 = \{w_k^1, k = 1, \dots, p^{2p-2}\} = \bigcup_{k=1}^{p^{2p-4}} S_k^2$$

where  $\text{card}(S_k^2) = p^2$ , for all  $k$  and we pick  $w_k^2$  and  $y_k^2$ .

We then iterate this process until we get:

$$\Lambda_p = \{w_k^{p-1}, k = 1, \dots, p^2\}$$

and there exists  $k$  such that for  $e_p := w_k^{p-1} \in \Lambda_p$  and for the corresponding  $f_p := y_k^{p-1}$  we have  $|\langle e_p | x_j \rangle| \leq \frac{1}{\sqrt{p}}$ ,  $j = 1, \dots, p$  and  $|\langle f_p | x_j \rangle| \leq \frac{1}{\sqrt{p}}$ ,  $j = 1, \dots, p$ , which is the desired conclusion.  $\square$

We end this paper with natural question associated with what preceded:

**Problem.** Let  $\mathcal{A}$  be an algebra of type **A** with property  $(\mathbf{A}_{\S})$ . Is it always true that  $\mathcal{A}$  has property  $X_{0,1}$ ?

We recall that the answer has been positively established by Chevreau in the case of  $N$ -tuples of commuting contractions with a unitary dilation.

## References

- [1] **H. Bercovici**; *Factorization theorems and the structures of operators on hilbert space*; Ann. of. Math. 128, p. 399 - 413, 1988.
- [2] **H. Bercovici, C. Foiaş, C. Pearcy**; *Dual algebras with applications to invariant subspace and dilation theory*; CBMS n 56, 1985.
- [3] **S. Brown**; *Some invariant subspace for subnormal operators*; Integral Equations Operator Theory , 1 p. 310 - 333, 1978.
- [4] **S. Brown, B. Chevreau**; *Toute contraction à calcul fonctionnel isométrique est réflexive*; C.R. Acad. Sci. Paris Sér. I. Math., 307:185 - 188, 1988.
- [5] **S. Brown, B. Chevreau, C. Pearcy**; *Contraction with rich spectrum have invariant subspaces*; J. Operator Theory, 1: 123 - 136, 1979.

- [6] **S. Brown, B. Chevreau, C. Pearcy**; *On the structure of contraction operators II*; J. Funct. Anal., 76: 30 - 57, 1988.
- [7] **G. Cassier**; *Champs d'algèbres duales uniformes d'opérateurs sur l'espace de Hilbert*; Studia Math., 106.
- [8] **G. Cassier**; *Algèbres duales uniformes d'opérateurs sur l'espace de Hilbert*; Studia Math., 95: 17 - 32, 1989.
- [9] **G. Cassier, I. Chalendar, B. Chevreau**; *New examples of contractions illustrating membership and non-membership in the classes  $G_{m,n}$* ; preprint.
- [10] **B. Chevreau**; *Sur les contractions à calcul fonctionnel isométrique ii*; J. Operator Theory, 20: 269 - 293, 1988.
- [11] **B. Chevreau, C. Pearcy**; *On the structure of contraction operators with applications to invariant subspaces*; J. Funct. Anal., 67: 360 - 378, 1986.
- [12] **B. Chevreau, C. Pearcy, A.L. Shields**; *Finitely connected domains  $G$ , representation of  $H^\infty(G)$ , and invariant subspaces*; J. Operator Theory, 6: 375 - 405, 1981.
- [13] **G. Exner**; *PhD Thesis*; University of Michigan.
- [14] **G.R. Exner, Il B. Jung**; *Compression of contractions in the class  $G_{m,n}$* ; Acta Sci. Math. (Szeged), 59: 545 - 553, 1994.
- [15] **K. Hoffman**; *Banach spaces of analytic functions*; Prentice - Hall Englewood Cliffs, 1965.

Received: 24.02.2000

Accepted: 03.03.2000