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# Optimal interpolation inequalities between generalized means

Petru Mironescu

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## Abstract

We exhibit the optimal inequalities of the form

$$x_1 + \cdots + x_n \leq \alpha(x_1^p + \cdots + x_n^p)^{1/p} + \beta x_1^{1/n} \cdots x_n^{1/n}, \forall x_1, \dots, x_n \geq 0, \\ \text{with } n \geq 2 \text{ and } 1 < p < \infty,$$

respectively

$$(x_1 + \cdots + x_n)^p \leq \alpha(x_1^p + \cdots + x_n^p) + \beta x_1^{p/n} \cdots x_n^{p/n}, \forall x_1, \dots, x_n \geq 0, \\ \text{with } n = 2 \text{ and } 1 < p < \infty \text{ or } n \geq 3 \text{ and } 1 < p \leq n.$$

We prove that, when  $p > n$ , a qualitative change occurs in the second inequality, and we present some partial results in the range  $p > n \geq 3$ .

As a byproduct of our analysis, we obtain a quantitative form of the strict convexity of the Shannon entropy.

## 1 Introduction

The question we address here is the one of the *explicit optimal estimates* interpolating between the standard inequalities involving the generalized means of positive numbers. To give a very simple example of the type of inequalities we have in mind, let  $H$ ,  $G$ , and  $A$  denote the harmonic, geometric, and arithmetic mean of the positive numbers  $x, y$ . Then

$$G \leq \frac{1}{2}A + \frac{1}{2}H, \tag{1}$$

which, granted the inequality

$$H \leq G, \tag{2}$$

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is a refinement of the inequality

$$G \leq A. \quad (3)$$

One may easily prove that (1) is the *optimal* improvement of (3) via (2), in the sense that, if

$$a, b \in (0, 1), \quad a + b = 1 \quad (4)$$

and

$$G \leq aA + bH, \quad \forall x, y > 0, \quad (5)$$

then we necessarily have  $a \geq 1/2$ , and thus all the valid inequalities of the form (5) (under the constraint (4)) are convex combinations of (1), (2), and (3).

Another observation, somewhat marginal in what follows, is that for every  $a \in (0, 1)$  there exists some  $b$ , not necessarily satisfying (4), such that (5) holds. It is actually easy to see that the optimal  $b$  is given by

$$b = b(a) := \begin{cases} 1 - a, & \text{if } 1/2 \leq a < 1 \\ 4/a, & \text{if } 0 < a < 1/2 \end{cases}.$$

More generally, consider the generalized means

$$M_p = M_p(x_1, \dots, x_n) := \begin{cases} \left( \frac{x_1^p + \dots + x_n^p}{n} \right)^{1/p}, & \text{if } p \neq 0 \\ x_1^{1/n} \dots x_n^{1/n}, & \text{if } p = 0 \end{cases},$$

$$n \geq 2, \quad p \in \mathbb{R}, \quad x_1, \dots, x_n > 0.$$

If  $p_0 < p_1 < p_2$ , then

$$M_{p_0}(x_1, \dots, x_n) \leq M_{p_1}(x_1, \dots, x_n) \leq M_{p_2}(x_1, \dots, x_n), \quad \forall x_1, \dots, x_n > 0, \quad (6)$$

and one can ask about the analogues of (1) and (5) in this setting, with focus on the exact value of the optimal  $a$ . I do not know the answer in such a general setting; however, I will give the full analogue of (1) when  $p_0 = 0$ . My original interest for this special case comes from the elementary but noticeable inequality

$$(x_1 + \dots + x_n)^2 \leq (n-1)(x_1^2 + \dots + x_n^2) + nx_1^{2/n} \dots x_n^{2/n}, \quad (7)$$

$$\forall n \geq 2, \quad \forall x_1, \dots, x_n \geq 0,$$

that I learned from Dospinescu [3]. Inequality (7) is quoted in Senderov and Turkevich [4] and attributed to Shleifer. Clearly, (7) can be rewritten as

$$M_1^2 \leq \left(1 - \frac{1}{n}\right) M_2^2 + \frac{1}{n} M_0^2, \quad (8)$$

and thus, indeed, (7) is a refinement of the second inequality in (6) (with  $p_j = j, j = 0, 1, 2$ ).

The not so usual feature of (7) is that equality is achieved not only when  $x_1 = \dots = x_n$  (as one expects), but also when, up to a permutation,  $x_1 = \dots = x_{n-1}$  and  $x_n = 0$ . Motivated by the above considerations, the two inequalities I investigate are

$$M_{p_1}^{p_2} \leq aM_{p_2}^{p_2} + bM_0^{p_2}, \quad (9)$$

respectively

$$M_{p_1} \leq aM_{p_2} + bM_0, \quad (10)$$

where  $0 < p_1 < p_2$ .

Let us note that we may assume, with no loss of generality, that  $p_1 = 1$ . (Perform the substitution  $x_j = y_j^{1/p_1}$ .) From now on, we assume  $p_1 = 1$  and set  $p := p_2 > 1$ ,  $\alpha := an^{p-1}$ ,  $\beta := bn^p$ . Then (4), respectively (9), amount to

$$0 < \alpha < n^{p-1}, 0 < \beta < n^p, n\alpha + \beta = n^p, \quad (11)$$

$$(x_1 + \cdots + x_n)^p \leq \alpha(x_1^p + \cdots + x_n^p) + \beta x_1^{p/n} \cdots x_n^{p/n}, \forall x_1, \dots, x_n \geq 0. \quad (12)$$

Similarly, (4) and (10) amount to

$$0 < \alpha < n^{1-1/p}, 0 < \beta < n, n^{1/p}\alpha + \beta = n, \quad (13)$$

$$x_1 + \cdots + x_n \leq \alpha(x_1^p + \cdots + x_n^p)^{1/p} + \beta x_1^{1/n} \cdots x_n^{1/n}, \forall x_1, \dots, x_n \geq 0. \quad (14)$$

It is easy to see that there exists a minimal  $\alpha_0$  such that (11) and (12) hold; see Proposition 4 below. (The existence of the corresponding  $\alpha_0$  for (14) follows from Theorem 1.) More delicate is the matter of the exact value of  $\alpha_0$  and the characterization of the cases of equality in (12) and (14). As we will see, below, despite their formal resemblance, (12) and (14) exhibit different behaviors when  $p > n$ .

Before proceeding further, let us make a simple observation. Assume that (12) holds. Letting  $x_1 = \cdots = x_{n-1} = 1$  and  $x_n = 0$ , we find that

$$\alpha \geq (n-1)^{p-1}. \quad (15)$$

Similarly, if (14) holds, then

$$\alpha \geq (n-1)^{1-1/p}. \quad (16)$$

We may now state our main result, which, in view of (15) and (16), is optimal.

**Theorem 1.**

1. Let  $n \geq 2$  and  $1 < p \leq n$ . Then

$$(x_1 + \cdots + x_n)^p \leq (n-1)^{p-1}(x_1^p + \cdots + x_n^p) + (n^p - n(n-1)^{p-1})x_1^{p/n} \cdots x_n^{p/n}, \forall x_1, \dots, x_n \geq 0. \quad (17)$$

Moreover, equality holds in (17) if and only if either (i)  $n = p = 2$ , or (ii)  $x_1 = \cdots = x_n$ , or (iii) (up to a permutation)  $x_1 = \cdots = x_{n-1}$  and  $x_n = 0$ .

2. Let  $n \geq 2$  and  $1 < p < \infty$ . Then

$$x_1 + \cdots + x_n \leq (n-1)^{1-1/p}(x_1^p + \cdots + x_n^p)^{1/p} + (n - n^{1/p}(n-1)^{1-1/p})x_1^{1/n} \cdots x_n^{1/n}, \forall x_1, \dots, x_n \geq 0. \quad (18)$$

Moreover, equality holds in (18) if and only if either (i)  $x_1 = \cdots = x_n$  or (ii) (up to a permutation)  $x_1 = \cdots = x_{n-1}$  and  $x_n = 0$ .

**Remark 2.**

1. In some special cases, (17) was known before. Indeed, the case  $p = 2$  corresponds to (7). On the other hand, (17) was already known to hold when  $p = 3$ ,  $p = 4$ , and  $p = n$  (see Cîrtoaje [1, Remarks 3.6 and 3.7, and the references therein]).
2. In the range  $1 < p \leq n$ , (17) and (18) can be embedded into a larger family of inequalities; see Theorems 12 and 13 in Section 4.
3. When  $n = 2$  and  $1 < p < 2$ , an equivalent form of (17) is

$$(1+x)^p + (1-x)^p \geq 2^p - (2^p - 2)(1-x^2)^{p/2}, \quad \forall x \in [0, 1]. \quad (19)$$

(To reduce (17) to (19), assume, without loss of generality, that  $x_1 + x_2 = 2$  and  $x_1 \geq x_2$ , write  $x_1 = 1+x$  and  $x_2 = 1-x$  and plug this into (17).) The interested reader may easily check that (19) is, for small  $x > 0$ , a refinement of the well-known Clarkson's inequality [2, ineq. (7)]

$$(1+x)^p + (1-x)^p \geq 2(1+x^{p/(p-1)})^{p-1}, \quad \forall 1 < p < 2, \quad \forall 0 \leq x \leq 1.$$

We next investigate the case  $p > n$  in (12). Noticeably, (17) does not hold anymore (Proposition 4), so that the restriction  $p \leq n$  is necessary in (17).

A second main result yields the explicit optimal inequality (12) when  $n = 2$  and  $p > 2$ .

**Theorem 3.** Assume that  $n = 2$  and  $p > 2$ . Then the optimal inequality of the form (12) is

$$(x+y)^p \leq \frac{2^{p-1}}{p}(x^p + y^p) + 2^p \frac{p-1}{p} x^{p/2} y^{p/2}, \quad \forall p > 2, \quad \forall x, y \geq 0. \quad (20)$$

Equality occurs in (20) if and only if  $x = y$ .

Note that not only the formula of  $\alpha_0$  changes at the critical value  $p = 2$ , but also the equality case is different.

The exact value of  $\alpha_0$  in (12) when  $n \geq 3$  and  $p > n$  is unknown (at least to me). We warn the reader that, despite what (20) might suggest,  $\alpha_0$  is *not given by*  $\alpha_0 = \frac{n^{p-1}}{p}$ . See Proposition 15 for additional information on  $\alpha_0$  and on the equality case.

The remaining sections are devoted to the proofs. In Section 2, we explain the strategy for obtaining (17), (20), and the equality cases. The main auxiliary estimates, which are at the heart of the proof of Theorems 1 and 3, are obtained in Section 3. In Section 4, we prove the main results and derive some noticeable consequences. The proof of Theorem 1 is not so standard: (i) first, we prove (17); (ii) next, we prove that (17) implies (18) in the range  $1 < p \leq n$ ; (iii) finally, we prove that the validity of (18) for  $p$  close to 1 implies the validity of (18) for every  $p > 1$ . The self-improvement argument in the final step of the proof of Theorem 1 is of independent interest and should be useful in the proof of more general inequalities of the form (10) (but we did not investigate (10) in full generality). Last but not least, we obtain in Section 4 an amusing consequence of Theorem 1: a quantitative strong convexity inequality involving the Shannon entropy (Corollary 14).

## 2 Preliminary remarks

We start by establishing the existence of an optimal  $\alpha_0$  in (12). (The existence of an optimal  $\alpha_0$  in (14) is part of Theorem 1.) It will be clear that the arguments we present here for  $p_0 = 0$  and  $p_1 = 1$  can be easily adapted to all  $p_0 < p_1 < p_2$ . Let us note that, by homogeneity, it suffices to investigate the validity of (12) or (14) when  $x_1 + \dots + x_n = n$ .

**Proposition 4.** Let  $n \geq 2$  and  $1 < p < \infty$ .

(a) Consider the set

$$A := \{0 < \alpha < n^{p-1}; \text{(12) holds with } \beta \text{ as in (11)}\}.$$

Then  $A$  is of the form  $[\alpha_0, n^{p-1})$ , with  $\alpha_0 \geq (n-1)^{p-1}$ .

(b) If  $p > n$  and  $\alpha = (n-1)^{p-1}$ , then there exists no finite  $\beta$  such that (12) holds. In particular, we have  $\alpha_0 > (n-1)^{p-1}$ .

(c) If  $p > n$  and  $(n-1)^{p-1} < \alpha < n^{p-1}$ , then there exists some  $\beta > 0$  (not necessarily satisfying (11)) such that (12) holds.

*Proof. Step 1. Proof of (b).* Argue by contradiction. Plug, into (12),  $x_1 = \dots = x_{n-1} = 1$  and  $x_n = (n-1)\varepsilon > 0$ . Then, for some  $\beta < \infty$ , we have

$$(n-1 + (n-1)\varepsilon)^p = (n-1)^p(1 + \varepsilon)^p \leq (n-1)^p + (n-1)^{2p-1}\varepsilon^p + \beta(n-1)^{p/n}\varepsilon^{p/n},$$

which implies that

$$(n-1)^p + (n-1)^p p \varepsilon + O(\varepsilon^2) \leq (n-1)^p + O(\varepsilon^{p/n}),$$

a contradiction (since  $\frac{p}{n} > 1$ ).

*Step 2. Proof of (c).* Argue by contradiction. Let  $k \geq 1$  be an integer. Consider  $x_{1,k}, \dots, x_{n,k} \geq 0$ , with sum  $n$  and such that (12) with  $\beta = k$  is not satisfied by  $x_{1,k}, \dots, x_{n,k}$ . This implies that  $x_{1,k} \dots x_{n,k} \rightarrow 0$  as  $k \rightarrow \infty$ . Up to a subsequence and a permutation, we may assume that  $x_{j,k} \rightarrow x_j$  as  $k \rightarrow \infty$ , and that  $x_n = 0$ . We find that

$$(x_1 + \dots + x_{n-1})^p \geq \alpha(x_1^p + \dots + x_{n-1}^p) \text{ and } (x_1, \dots, x_{n-1}) \neq (0, \dots, 0),$$

and this contradicts the assumption  $\alpha > (n-1)^{p-1}$ .

*Step 3. Proof of (a).* The only issue is to prove that  $A$  is non empty; if this is the case, then  $A$  clearly has the desired form. Argue by contradiction and assume that, for every  $0 < \alpha < n^{p-1}$ , there exist  $x_{1,\alpha}, \dots, x_{n,\alpha} \geq 0$  with sum  $n$  such that

$$\alpha(x_{1,\alpha}^p + \dots + x_{n,\alpha}^p) + (n^p - n\alpha)x_{1,\alpha}^{p/n} \dots x_{n,\alpha}^{p/n} < n^p. \quad (21)$$

Consider  $x_1, \dots, x_n$  such that, along some subsequence  $\alpha_k \rightarrow n^{p-1}$ , we have  $x_{j,\alpha_k} \rightarrow x_j$  as  $k \rightarrow \infty$ ,  $\forall 1 \leq j \leq n$ . From (21), we find that

$$x_1^p + \dots + x_n^p \leq n = n \left( \frac{x_1 + \dots + x_n}{n} \right)^p,$$

which implies  $x_j = 1, \forall 1 \leq j \leq n$ .

Write  $x_{j,\alpha_k} = 1 + \varepsilon_j = 1 + \varepsilon_{j,k}$ , with  $\varepsilon_{1,k} + \dots + \varepsilon_{n,k} = 0$  and  $\varepsilon_{j,k} \rightarrow 0$  as  $k \rightarrow \infty$ ,  $\forall 1 \leq j \leq n$ . From now on, we omit the subscript  $k$ . Note that, with  $\xi := \varepsilon_1^2 + \dots + \varepsilon_n^2 \rightarrow 0$  as  $k \rightarrow \infty$ , we have  $\xi > 0$  (by (21)) and

$$\begin{aligned} x_1 \dots x_n &= (1 + \varepsilon_1) \dots (1 + \varepsilon_n) = 1 + \varepsilon_1 + \dots + \varepsilon_n + \sum_{i < j} \varepsilon_i \varepsilon_j + o(\xi) \\ &= 1 + \sum_{i < j} \varepsilon_i \varepsilon_j + o(\xi) = 1 - \frac{1}{2} \xi + o(\xi), \end{aligned} \quad (22)$$

where we have used the fact that  $\varepsilon_1 + \dots + \varepsilon_n = 0$  and thus

$$\sum_{i < j} \varepsilon_i \varepsilon_j = -\frac{1}{2} \xi. \quad (23)$$

Using (22) and a second order Taylor expansion, we find that (21) can be rewritten as

$$\begin{aligned} n^p &> \alpha + \alpha \frac{p(p-1)}{2} \xi + o(\xi) + (n^p - n\alpha) - (n^p - n\alpha) \frac{p}{2n} \xi + o(\xi) \\ &= n^p + \underbrace{\left( \alpha - \frac{n^{p-1}}{p} \right)}_{:=\gamma(\alpha)} \frac{p^2}{2} \xi + o(\xi) = n^p + \gamma(\alpha) \xi + o(\xi), \end{aligned} \quad (24)$$

and it is clear that  $o(\xi)$  is locally uniform in  $\alpha$  and  $\xi$ , i.e., given  $\tau > 0$  and an interval  $[\underline{\alpha}, \bar{\alpha}] \subset [0, \infty)$ , there exists some  $\delta > 0$  such that

$$|o(\xi)| \leq \tau \xi, \quad \forall \alpha \in [\underline{\alpha}, \bar{\alpha}], \quad \forall \xi \in [0, \delta].$$

From the above the desired contradiction follows from (24) by letting, in (24),  $\alpha = \alpha_k$  and then  $k \rightarrow \infty$ , provided we have  $\gamma(n^{p-1}) > 0$ , which is clear.  $\square$

We next present the two approaches we use for studying the sharp value of  $\alpha_0$  in (12). While the second one is, in some sense, more general, the first one seems more effective in the proof of Theorems 1 and 3. We will use the second one to give some insight on the optimal inequality (12) in the range  $p > n \geq 3$ .

**First approach.** Let  $\alpha, \beta > 0$ , and consider the constrained minimization problem

$$\mu := \min \left\{ \alpha(x_1^p + \dots + x_n^p) + \beta x_1^{p/n} \dots x_n^{p/n}; x_1, \dots, x_n \geq 0, \right. \\ \left. x_1 + \dots + x_n = n \right\}. \quad (25)$$

Clearly,  $\mu$  is attained, and, at a minimum point  $(x_1, \dots, x_n)$ , one of the following two happens.

*First possibility.* Up to a permutation, we have  $x_n = 0$ . In this case, we have

$$\mu = \alpha \frac{n^p}{(n-1)^{p-1}} \text{ and } x_1 = \dots = x_{n-1} = \frac{n}{n-1}.$$

*Second possibility.* We have  $x_1, \dots, x_n > 0$ . In this case, there exists a Lagrange multiplier  $\lambda > 0$  such that

$$\alpha p x_j^{p-1} + \beta \frac{p}{n} \frac{A}{x_j} = \lambda, \quad \forall 1 \leq j \leq n, \quad (26)$$

where

$$A := x_1^{p/n} \cdots x_n^{p/n} > 0.$$

Since the function

$$(0, \infty) \ni x \mapsto \alpha p x^{p-1} + \beta \frac{p}{n} \frac{A}{x}$$

takes at most twice a given positive value, it follows that, at a minimum point, the  $x_j$ 's can take at most two distinct values. If  $x_1 = \cdots = x_n (= 1)$ , then

$$\mu = n\alpha + \beta.$$

Otherwise, there are integers  $\ell, m \geq 1$  and a number  $x$  such that, up to a permutation,

$$\ell + m = n, 0 < x < \frac{1}{m}, x_1 = \cdots = x_m = 1 + \ell x, x_{m+1} = \cdots = x_n = 1 - mx. \quad (27)$$

(The second approach will tell us that  $\ell = 1$  and  $m = n - 1$ , but we ignore this fact at this stage.) Although this is not the route we will exploit below, we have obtained that

$$\mu = \min \left\{ \alpha \frac{n^p}{(n-1)^{p-1}}, n\alpha + \beta, c_1, \dots, c_{n-1} \right\},$$

where

$$c_m := \min \left\{ f_m(x); 0 \leq x \leq \frac{1}{m} \right\}$$

and

$$f_m(x) := \alpha [m(1 + (n-m)x)^p + (n-m)(1 - mx)^p] \\ + \beta (1 + (n-m)x)^{mp/n} (1 - mx)^{(n-m)p/n}, 1 \leq m \leq n-1, 0 \leq x \leq \frac{1}{m}.$$

Thus the validity of (12) amounts to the estimate of the minimal values of the one variable functions  $f_m$ ,  $1 \leq m \leq n-1$ .

We now turn to the more specific situations considered in Theorems 1 and 3. Let us note that we then have

$$n\alpha + \beta = n^p \text{ and } (n-1)^{p-1} \leq \alpha < n^{p-1}. \quad (28)$$

In particular, since (by (25))  $\mu \leq n\alpha + \beta$ , (28) implies that

$$\mu \leq n^p. \quad (29)$$

In view of the above, we have: either (i)  $\mu = n^p$  (which is the desired inequality (12)) or (ii) (26) and (27) hold. We investigate the latter possibility. Multiplying (26) by  $x_j$  and summing over  $j$ , we find that

$$p\mu = n\lambda. \quad (30)$$

On the other hand, (26) also yields

$$(1 + \ell x)^p - (1 - mx)^p = x_1^p - x_n^p = \frac{\lambda}{p\alpha} ((1 + \ell x) - (1 - mx)) = \frac{\lambda}{p\alpha} n x. \quad (31)$$

Inserting (30) into (31) and using (29), we find that

$$(1 + \ell x)^p - (1 - mx)^p = \frac{\mu}{\alpha} x \leq \frac{n^p}{\alpha} x.$$

We are now at the heart of our first strategy. The above considerations imply the following



**Lemma 5.** Let  $n \geq 2, p > 1, (n-1)^{p-1} \leq \alpha < n^{p-1}$ , and set  $\beta := n^p - n\alpha$ . Assume that:

$$(1 + \ell x)^p - (1 - mx)^p > \frac{n^p}{\alpha} x, \forall \ell, m \geq 1 \text{ integers such that } \ell + m = n, \quad (32)$$

$$\forall 0 < x < \frac{1}{m}.$$

Then (12) holds, with equality if and only if:

- (a) Either  $\alpha > (n-1)^{p-1}$  and  $x_1 = \dots = x_n$ .
- (b) Or  $\alpha = (n-1)^{p-1}$  and either  $x_1 = \dots = x_n$  or, up to a permutation,  $x_1 = \dots = x_{n-1}$  and  $x_n = 0$ .

In order to obtain Theorems 1 and 3, we will prove that, for  $n, p$ , and  $\alpha$  as in these statements, inequality (32) holds. This is the content of the next section.

**Second approach.** This is limited to  $n \geq 3$ , and applies to both (12) and (13). Consider again the minimization problem (25). If  $(x_1, \dots, x_n)$  is a competitor, then  $0 \leq x_1 \dots x_n \leq 1$ , and thus

$$\mu := \min \left\{ \alpha(x_1^p + \dots + x_n^p) + \beta t^p; x_1, \dots, x_n \geq 0, \right. \\ \left. x_1 + \dots + x_n = n, x_1^{1/n} \dots x_n^{1/n} = t \right\}.$$

We next invoke the next result [1, Corollary 1.9]: for fixed  $0 \leq t \leq 1$ , a minimizer of  $x_1^p + \dots + x_n^p$  under the constraints

$$x_1, \dots, x_n \geq 0, x_1 + \dots + x_n = n, x_1 \dots x_n = t^n,$$

is, up to a permutation, an (unique)  $n$  tuple  $(x_1, \dots, x_n)$  such that

$$x_1 = \dots = x_{n-1} \geq x_n.$$

This leads to the following consequence.

**Lemma 6.** Let  $n \geq 3$  and  $1 < p < \infty$ . Then the optimal  $\alpha_0$  in (12) is the smallest  $\alpha$  such that

$$\alpha[(n-1)(1+x)^p + (1-(n-1)x)^p] + (n^p - n\alpha) \times \\ \times (1+x)^{(n-1)p/n} (1-(n-1)x)^{p/n} \geq n^p, \forall 0 \leq x \leq 1/(n-1).$$

Similarly for the optimal  $\alpha_0$  in (14).

A final remark. Going back to (32), the second approach learns us that, in order to use (32) in the proof of Theorem 1, it suffices establish its validity in the case  $\ell = 1$ . However, since we think that the proof Lemma 7 below is of independent interest, we provide the argument in the full range  $1 \leq \ell \leq n-1$ .

### 3 The main auxiliary inequalities

**Lemma 7.** Let  $n \geq 2$  and  $1 < p \leq n$ . Set  $\alpha := (n-1)^{p-1}$ . Assume that  $(n, p) \neq (2, 2)$ . Then (32) holds.

**Lemma 8.** Let  $n = 2$  and  $p > 2$ . Set  $\alpha := \frac{2^{p-1}}{p}$ . Then  $1 < \alpha < 2^{p-1}$  and (32) holds.

*Proof of Lemma 7.* We have to prove that, when  $n \geq 2$ ,  $1 < p \leq n$ ,  $(n, p) \neq (2, 2)$ , and  $\ell, m \geq 1$  are integers with  $\ell + m = n$ , we have

$$f(x) > \frac{n^p}{(n-1)^{p-1}}x, \forall 0 < x < \frac{1}{m}, \quad (33)$$

where

$$f(x) := (1 + \ell x)^p - (1 - mx)^p, \forall 0 \leq x \leq \frac{1}{m}.$$

For further use, let us note that

$$f'(x) = p[\ell(1 + \ell x)^{p-1} + m(1 - mx)^{p-1}], \forall 0 \leq x \leq \frac{1}{m}, \quad (34)$$

$$f''(x) = p(p-1)[\ell^2(1 + \ell x)^{p-2} - m^2(1 - mx)^{p-2}], \forall 0 \leq x < \frac{1}{m}, \quad (35)$$

$$f'''(x) = p(p-1)(p-2)[\ell^3(1 + \ell x)^{p-3} + m^2(1 - mx)^{p-3}], \forall 0 \leq x < \frac{1}{m}, \quad (36)$$

and that (35) still holds for  $x = \frac{1}{m}$  provided  $p > 2$ .

*Step 1. Proof of (33) when  $p = 2$  and  $n \geq 3$ .* After some elementary manipulations, we see that, in this case, (33) amounts to

$$(2m - n)x = (m - \ell)x < 2 - \frac{n}{n-1}, \forall 0 < x < \frac{1}{m}. \quad (37)$$

If  $m \leq \frac{n}{2}$ , then  $m - \ell \leq 0$  and (37) is clear. If  $m > \frac{n}{2}$ , then (37) amounts to

$$(2m - n)\frac{1}{m} = 2 - \frac{n}{m} \leq 2 - \frac{n}{n-1},$$

which holds since  $m \leq n - 1$ .

*Step 2. Proof of (33) when  $2 < p \leq n$ : the case where  $\ell = 1$ .* Set  $b := \frac{1}{m} = \frac{1}{n-1}$ . We note that  $f'''(x) > 0$ ,  $\forall 0 \leq x < \frac{1}{m}$  (since  $p > 2$ ),  $f''(0) < 0$  (since  $\ell = 1 < n - 1 = m$ ), and  $f''(b) > 0$ . Therefore, there exists some  $a \in (0, b)$  such that  $f$  is strictly concave on  $[0, a]$  and strictly convex on  $[a, b]$ .

We next note that

$$f(b) = \frac{n^p}{(n-1)^p} = \frac{n^p}{(n-1)^{p-1}}b \quad (38)$$

and (since  $p \leq n$ )

$$f'(b) = p\frac{n^{p-1}}{(n-1)^{p-1}} \leq \frac{n^p}{(n-1)^{p-1}}. \quad (39)$$

Combining (38) and (39) with Lemma 9 below (whose proof is postponed until the end of this section), applied with  $A = \frac{n^p}{(n-1)^{p-1}}$ , we obtain (32). This completes Step 2.

**Lemma 9.** Let  $0 < a < b$  and let  $f : [0, b] \rightarrow \mathbb{R}$  be a continuous function, which is concave on  $[0, a]$  and strictly convex on  $[a, b]$ .

Let  $A \in \mathbb{R}$ . If  $f(0) = 0$ ,  $f(b) \geq Ab$ , and  $f'(b-) \leq A$ , then  $f(x) > Ax, \forall x \in (0, b)$ .

*Step 3. Proof when  $2 < p \leq n$ : the general case.* The proof is by induction on  $\ell$ . The case  $\ell = 1$  was settled in Step 2. We assume that  $2 \leq \ell \leq n - 1$  and that (33) holds with  $\ell - 1$ , respectively  $m + 1$ , in place of  $\ell$ , respectively  $m$ .

*Step 3.1. Proof of (33) when  $0 < x < \frac{1}{m+1}$ .* Let us note the following easy consequence of Jensen's inequality. If  $g : I \rightarrow \mathbb{R}$  is strictly convex (with  $I$  an interval) and  $x < y < z < t$  are points in  $I$  such that  $x + t = y + z$ , then  $g(x) + g(t) > g(y) + g(z)$ . With the choice  $g(x) := x^p, \forall x \geq 0$ , we obtain that

$$\begin{aligned} (1 + \ell x)^p - (1 - mx)^p &> (1 + (\ell - 1)x)^p - (1 - (m + 1)x)^p \\ &\geq \frac{n^p}{(n - 1)^{p-1}}x, \forall 0 < x \leq \frac{1}{m + 1}, \end{aligned}$$

the latter inequality following from the induction hypothesis. This yields (33) when  $0 < x \leq \frac{1}{m + 1}$ .

It remains to prove (33) when  $\frac{1}{m + 1} < x < \frac{1}{m}$ .

*Step 3.2. Reduction to an inequality at the endpoint  $x = \frac{1}{m}$ .* By (34), we have

$$f'(x) \geq p\ell(1 + \ell x)^{p-1} \geq p\ell \frac{(n + 1)^{p-1}}{(m + 1)^{p-1}}, \forall \frac{1}{m + 1} \leq x \leq \frac{1}{m},$$

and therefore

$$f(x) \geq f\left(\frac{1}{m + 1}\right) + p\ell \frac{(n + 1)^{p-1}}{(m + 1)^{p-1}} \left(x - \frac{1}{m + 1}\right), \forall \frac{1}{m + 1} \leq x \leq \frac{1}{m}.$$

It thus suffices to prove that

$$f\left(\frac{1}{m + 1}\right) + p\ell \frac{(n + 1)^{p-1}}{(m + 1)^{p-1}} \left(x - \frac{1}{m + 1}\right) > \frac{n^p}{(n - 1)^{p-1}}x, \forall \frac{1}{m + 1} \leq x \leq \frac{1}{m}. \quad (40)$$

Since both sides of (40) are affine functions, it suffices to establish (40) when  $x = \frac{1}{m + 1}$  and  $x = \frac{1}{m}$ . When  $x = \frac{1}{m + 1}$ , (40) holds, by Step 3.1. Therefore, it remains to prove (40) when  $x = \frac{1}{m}$ , which amounts to

$$\frac{n^p}{(n - 1)^{p-1}}(m + 1)^p < m[(n + 1)^p - 1] + p(n - m)(n + 1)^{p-1}, \forall 1 \leq m \leq n - 2. \quad (41)$$

*Step 3.3. Proof of (41).* Since the left-hand side of (41) is convex, as a function of  $m > -1$ , and the right-hand side of (41) is affine in  $m$ , it suffices to prove (41) when  $m = 0$ , respectively when  $m = n - 2$ .

When  $m = 0$ , (41) reads

$$\left(\frac{n}{n^2 - 1}\right)^{p-1} < p,$$

which holds since  $p > 1$  and  $n < n^2 - 1$ .

When  $m = n - 2$ , (41) is equivalent to

$$n(n-1) \left( \frac{n}{n+1} \right)^{p-1} + \frac{n-2}{(n+1)^{p-1}} - (n+1)(n-2) - 2p < 0. \quad (42)$$

Since the left-hand side of (42) decreases with  $p$ , it suffices to establish (42) when  $p = 2$ . In this case, (42) is equivalent to  $n^2 + 4 > 0$ . Step 3 is completed.

*Step 4. Proof of (33) when  $1 < p < 2$ : the case where  $\ell \leq m$ .* By (35) and the assumption  $p < 2$ , we have  $f''(x) < 0, \forall 0 < x < \frac{1}{m}$ , and thus  $f$  is strictly concave. The validity of (33) then follows from  $f(0) = 0$  and

$$f\left(\frac{1}{m}\right) = \frac{n^p}{m^p} = \frac{n^p}{m^{p-1}} \frac{1}{m} \geq \frac{n^p}{(n-1)^{p-1}} \frac{1}{m}.$$

*Step 5. Proof of (33) when  $1 < p < 2$ : the general case.* The case  $\ell \leq m$  being settled in Step 4, we assume that  $\ell > m$ . Set  $b := \frac{1}{m}$ . By (35), (36), and the assumption  $\ell > m$ , we have  $f''(0) > 0$ ,  $f''(b-) = -\infty$ , and therefore there exists some  $a \in (0, b)$  such that  $f$  is strictly convex on  $[0, a]$  and strictly concave on  $[a, b]$ .

We next note that

$$f(b) = \frac{n^p}{m^p} \geq \frac{n^p}{(n-1)^{p-1}} b, \quad (43)$$

$$f'(0) = np > \frac{n^p}{(n-1)^{p-1}}. \quad (44)$$

Indeed, inequality (43) is clear. In order to prove (44), which amounts to

$$\left( \frac{n}{n-1} \right)^p - \frac{n}{n-1} p < 0, \quad \forall n \geq 2, \forall 1 < p < 2, \quad (45)$$

we note that the left-hand side of (45) is strictly convex in  $p$ , and therefore it suffices to prove the large inequality in (45) at the endpoints  $p = 1$  and  $p = 2$ ; both these inequalities are obvious.

We obtain (33) from (43), (44), and Lemma 10 below (whose proof is postponed until the end of this section), applied with  $A = \frac{n^p}{(n-1)^{p-1}}$ .

**Lemma 10.** Let  $0 < a < b$  and let  $f : [0, b] \rightarrow \mathbb{R}$  be a continuous function, which is strictly convex on  $[0, a]$  and concave on  $[a, b]$ .

Let  $A \in \mathbb{R}$ . If  $f(0) = 0$ ,  $f(b) \geq Ab$ , and  $f'(0+) \geq A$ , then  $f(x) > Ax, \forall x \in (0, b)$ .

The proofs of Step 5 and of Lemma 5 are complete.  $\square$

*Proof of Lemma 8.* The inequality  $\alpha < 2^{p-1}$  is clear. The inequality  $\alpha > 1$ , or, equivalently,  $2^{p-1} - p > 0, \forall p > 2$ , follows from the fact that

$$(0, \infty) \ni p \mapsto g(p) := 2^{p-1} - p$$

is convex and satisfies  $g(2) = 0$  and  $g'(2) = 2 \ln 2 - 1 > 0$ .

It remains to prove that

$$f(x) > 2px, \quad \forall x \in (0, 1), \quad (46)$$

where

$$f(x) := (1+x)^p - (1-x)^p, \forall x \in [0, 1].$$

Since  $f(0) = 0$ , (46) holds provided we have

$$f'(x) > 2p, \forall x \in (0, 1]. \quad (47)$$

In turn, (47) follows from the strict convexity inequality

$$\begin{aligned} f'(x) &= p(1+x)^{p-1} + p(1-x)^{p-1} = 2p \frac{(1+x)^{p-1} + (1-x)^{p-1}}{2} \\ &> 2p \left( \frac{1}{2}(1+x) + \frac{1}{2}(1-x) \right)^{p-1} = 2p \end{aligned}$$

(recall that  $p > 2$  and thus  $p-1 > 1$ ). □

*Proof of Lemma 9.* It suffices to consider the case where  $A = 0$  (then apply this case to the map  $[0, b] \ni x \mapsto f(x) - Ax$ ). Since  $f$  is strictly convex on  $[a, b]$ , we have

$$f(x) > f(b) + (x-b)f'(b-) \geq 0, \forall x \in [a, b),$$

and, in particular,  $f(a) > 0$ .

Since  $f$  is concave on  $[0, a]$ ,  $f(0) = 0$ , and  $f(a) > 0$ , we have  $f(x) > 0, \forall x \in (0, a)$ . □

*Proof of Lemma 10.* Again, it suffices to consider the case where  $A = 0$ .

Since  $f$  is strictly convex on  $[0, a]$ ,  $f(0) = 0$ , and  $f'(0+) \geq 0$ , we have  $f(x) > 0, \forall x \in (0, a]$ , and in particular  $f(a) > 0$ .

Since  $f$  is concave on  $[a, b]$ ,  $f(a) > 0$  and  $f(b) \geq 0$ , we have  $f(x) > 0, \forall x \in (a, b)$ . □

## 4 Proof of the main results

*Proof of Theorem 1. Step 1. Proof of (17).* The validity of (17) and the equality cases in (17) follow from Lemmas 5 and 7, and the obvious fact that, when  $n = 2$  and  $p = 2$ , the inequality (17) is actually an identity.

*Step 2. Proof of (18) when  $1 < p \leq n$ .* Let  $x_1, \dots, x_n \geq 0$  satisfy  $x_1 + \dots + x_n = n$ . (By homogeneity, this assumption is not restrictive.) Set  $t := x_1^{1/n} \dots x_n^{1/n} \in [0, 1]$ . By (17), we have

$$x_1^p + \dots + x_n^p \geq \frac{n^p}{(n-1)^{p-1}} - \left( \frac{n^p}{(n-1)^{p-1}} - n \right) t^p. \quad (48)$$

In view of (48), we see that the desired inequality (18) and the characterization of the equality cases hold provided

$$\begin{aligned} (n-1)^{1-1/p} \left[ \frac{n^p}{(n-1)^{p-1}} - \left( \frac{n^p}{(n-1)^{p-1}} - n \right) t^p \right]^{1/p} \\ \geq n - [n - n^{1/p}(n-1)^{1-1/p}]t, \forall t \in [0, 1], \end{aligned} \quad (49)$$

with equality only when  $t = 0$  and  $t = 1$ .

We next note that (49) can be rewritten as

$$[n^p - n(n-1)^{p-1}]t^p + (n - [n - n^{1/p}(n-1)^{1-1/p}]t)^p \leq n^p, \quad \forall t \in [0, 1]. \quad (50)$$

Clearly, (i) the left-hand side of (50) is strictly convex in  $t$ , and (ii) (50) holds with equality for  $t = 0$  and  $t = 1$ . Therefore, (50) is valid, with strict inequality when  $t \in (0, 1)$ . This completes the proof of (18) when  $1 < p \leq n$ .

*Step 3. Proof of (18) in the full range  $1 < p < \infty$ .* Set, for  $1 \leq p < \infty$  and  $x = (x_1, \dots, x_n) \in [0, \infty)^n$ ,

$$F(x, p) := (n-1)^{1-1/p} \left( \sum_{j=1}^n x_j^p \right)^{1/p} + (n - n^{1/p}(n-1)^{1-1/p}) \prod_{j=1}^n x_j^{1/n} - \sum_{j=1}^n x_j. \quad (51)$$

Let  $Y$  denote the set of points  $x \in [0, \infty)^n$  such that either (i)  $x_1 = \dots = x_n$  or (ii) up to a permutation,  $x_1 = \dots = x_{n-1}$  and  $x_n = 0$ . Our aim is to prove that

$$\text{for } 1 < p < \infty \text{ and } x \in [0, \infty)^n, F(x, p) > 0 \text{ unless } x \in Y. \quad (52)$$

Clearly, (52) holds if at least one of  $x_1, \dots, x_n$  vanishes. On the other hand, we know, from Step 2, that (52) holds for  $1 < p \leq n$ . Therefore, it suffices to prove that, for  $x \in (0, \infty)^n$ ,  $F(x, \cdot)$  is non-decreasing in  $p$ , i.e., that

$$\frac{\partial F}{\partial p}(x, p) \geq 0, \quad \forall x \in (0, \infty)^n. \quad (53)$$

The key observation is that, since (18) is valid for  $1 \leq p \leq n$ , with equality when  $p = 1$ , we have

$$\frac{\partial F}{\partial p}(y, 1) \geq 0, \quad \forall y \in (0, \infty)^n. \quad (54)$$

We will obtain (53) by combining (54) with the following inequality:

$$\begin{aligned} & \text{for every } x \in (0, \infty)^n \text{ and } 1 < p < \infty, \text{ there exist} \\ & K = K(x, p) > 0 \text{ and } y = y(x, p) \in (0, \infty)^n \text{ such that } \frac{\partial F}{\partial p}(x, p) \geq K \frac{\partial F}{\partial p}(y, 1), \end{aligned} \quad (55)$$

that we next prove.

*Step 4. Proof of (55).* Set

$$S = S(x) := x_1^p + \dots + x_n^p.$$

First, let us note that  $F(tx, p) = tF(x, p)$ ,  $\forall t > 0$ , and thus it suffices to prove (55) for a (positive) multiple of  $x$ . It will be convenient for us to consider points  $x \in (0, \infty)^n$  satisfying the constraint

$$S(x) = 1. \quad (56)$$

If  $x \in (0, \infty)^n$ , then we have

$$\begin{aligned} \frac{\partial F}{\partial p}(x, p) &= \frac{1}{p^2} (n-1)^{1-1/p} [S(x)]^{1/p} \ln(n-1) \\ &\quad + (n-1)^{1-1/p} [S(x)]^{1/p} \left[ -\frac{1}{p^2} \ln(S(x)) + \frac{1}{p} \frac{\sum_{j=1}^n x_j^p \ln x_j}{S(x)} \right] \\ &\quad + \frac{1}{p^2} n^{1/p} (n-1)^{1-1/p} \prod_{j=1}^n x_j^{1/n} \ln \left( \frac{n}{n-1} \right). \end{aligned} \quad (57)$$

Under the constraint (56), (57) reads

$$\begin{aligned} p^2 (n-1)^{1/p-1} \frac{\partial F}{\partial p}(x, p) &= \ln(n-1) + \sum_{j=1}^n x_j^p \ln(x_j^p) \\ &\quad + \left( n^n \prod_{j=1}^n x_j^p \right)^{1/(np)} \ln \left( \frac{n}{n-1} \right). \end{aligned} \quad (58)$$

In particular, (58) becomes, for  $p = 1$  and  $y := (x_1^p, \dots, x_n^p)$  (which satisfies (56) with  $p = 1$ ),

$$\frac{\partial F}{\partial p}(y, 1) = \ln(n-1) + \sum_{j=1}^n x_j^p \ln(x_j^p) + \left( n^n \prod_{j=1}^n x_j^p \right)^{1/n} \ln \left( \frac{n}{n-1} \right). \quad (59)$$

Next, we note that the condition (56) implies that  $n^n x_1^p \cdots x_n^p \leq 1$  and thus

$$\left( n^n \prod_{j=1}^n x_j^p \right)^{1/(np)} \geq \left( n^n \prod_{j=1}^n x_j^p \right)^{1/n}, \quad \forall p \geq 1. \quad (60)$$

Combining (58)–(60), we find that

$$p^2 (n-1)^{1/p-1} \frac{\partial F}{\partial p}(x, p) \geq \frac{\partial F}{\partial p}(y, 1), \quad (61)$$

so that (55) holds, as claimed. This completes Step 3 and the proof of Theorem 1.  $\square$

**Remark 11.** If  $x \notin S$ , then the inequalities (60) and (61) are strict. It follows that, for such  $x$ ,  $F(x, p)$  is increasing in  $p$ .

We next present several noticeable consequences of Theorem 1 and its proof. First, let us note that, in Step 2 in the proof of Theorem 1, the strategy consists of obtaining (18) from (17) and a convexity argument. Following the same strategy, one obtains the family of inequalities (62) below, containing (17) and (18) as special cases.

**Theorem 12.** Let  $n \geq 2$ ,  $1 < p \leq n$ , and  $0 < r \leq p$ . Then

$$\begin{aligned} (x_1 + \cdots + x_n)^r &\leq (n-1)^{r(1-1/p)} (x_1^p + \cdots + x_n^p)^r \\ &\quad + (n^r - n^{r/p} (n-1)^{r(1-1/p)}) x_1^{r/n} \cdots x_n^{r/n}, \quad \forall x_1, \dots, x_n \geq 0, \end{aligned} \quad (62)$$

with equality of and only if either (i)  $n = p = r = 2$ , or (ii)  $x_1 = \cdots = x_n$ , or (iii) (up to a permutation)  $x_1 = \cdots = x_{n-1}$  and  $x_n = 0$ .

Even more generally, we have the following result.

**Theorem 13.** Let  $n \geq 2$  and  $1 < p \leq n$ . Let  $\Phi_1, \Phi_2 : [0, \infty) \rightarrow [0, \infty)$  be concave homeomorphisms. Let

$$\gamma := \inf \left\{ \frac{\Phi_1 \left( \frac{n^p}{(n-1)^{p-1}} x \right) - \Phi_1(nx)}{\Phi_2(x)}; x > 0 \right\}.$$

Then

$$\begin{aligned} & \Phi_1(n^{1-p}(x_1 + \dots + x_n)^p) + \gamma \Phi_2(n^{-p}(x_1 + \dots + x_n)^p) \\ & \leq \Phi_1(x_1^p + \dots + x_n^p) + \gamma \Phi_2(x_1^{p/n} \dots x_n^{p/n}), \forall x_1, \dots, x_n \geq 0. \end{aligned} \quad (63)$$

One recovers, up to a multiplicative constant, (62) from (63) by letting  $\Phi_1(t) = \Phi_2(t) := t^{r/p}, \forall t \geq 0$ . The details of the proofs are left to the reader; one follows exactly the strategy in Step 2 above.

Here is another interesting consequence of the proof of Theorem 1.

**Corollary 14.** Let  $n \geq 2$ . Then

$$\begin{aligned} \sum_{j=1}^n x_j \ln x_j & \geq \sum_{j=1}^n x_j \left( \ln \left( \sum_{k=1}^n x_k \right) - \ln n \right) \\ & + \left( \sum_{j=1}^n x_j - n \prod_{j=1}^n x_j^{1/n} \right) \ln \left( \frac{n}{n-1} \right), \forall x_1, \dots, x_n > 0. \end{aligned} \quad (64)$$

In particular, we have

$$(1+t) \ln(1+t) + (1-t) \ln(1-t) \geq 2 \left( 1 - (1-t^2)^{1/2} \right) \ln 2, \forall 0 < t < 1. \quad (65)$$

Note that (64) is stronger than the convexity of the Shannon entropy, defined (up to a sign convention) by  $f(t) := t \ln t, \forall t > 0$ . Indeed, Jensen's inequality applied to  $f$  yields

$$\sum_{j=1}^n x_j \ln x_j \geq \sum_{j=1}^n x_j \left( \ln \left( \sum_{k=1}^n x_k \right) - \ln n \right). \quad (66)$$

Thus, (64) improves indeed (66), by adding on the right-hand side of (66) a term that is  $> 0$  (except when  $x_1 = \dots = x_n$ ).

It turns out that inequality (65) is quite sharp near the origin, as shows Figure 1 below.

*Proof of Corollary 14.* We note that (64) is (57) with  $p = 1$ . It then suffices to invoke (54).

Finally, one obtains (65) from (64) by letting  $n = 2, x_1 = 1 + t$ , and  $x_2 = 1 - t$ .  $\square$



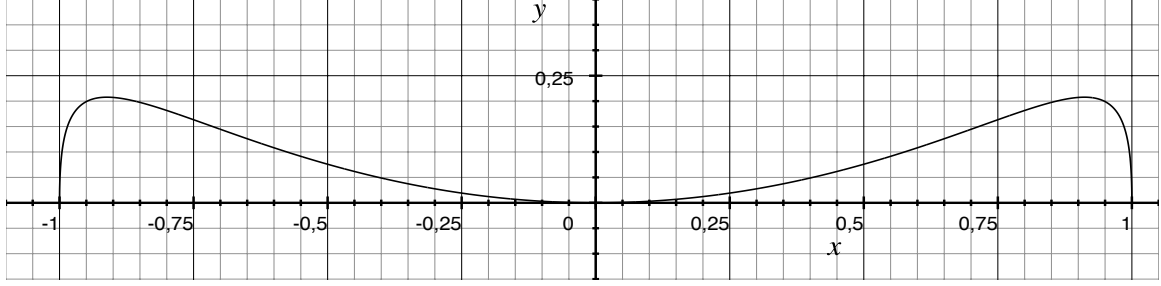


Figure 1: The graph of  $t \mapsto (1+t) \ln(1+t) + (1-t) \ln(1-t) - 2(1 - (1-t^2)^{1/2}) \ln 2$ .

A final consequence of the proof of Theorem 1 and of Remark 11. Given  $x_1, x_2 > 0$  with  $x_1 \neq x_2$ , we know that

$$(0, \infty) \ni p \mapsto (x_1^p + x_2^p)^{1/p} = \|(x_1, x_2)\|_p \text{ is decreasing.} \quad (67)$$

Step 3 in the proof of Theorem 1 (combined with Remark 11) shows that  $F$  given by (51) is increasing with  $p \geq 1$ , and this implies that

$$[1, \infty) \ni p \mapsto (x_1^p + x_2^p)^{1/p} - (2^{1/p} - 1)x_1^{1/2}x_2^{1/2} \text{ is increasing.} \quad (68)$$

Note that the extra term in (68) (vs (67)) is exponentially small, and that (67)–(68) encode the exponential rate of convergence of  $(x_1^p + x_2^p)^{1/p}$  to  $\max\{x_1, x_2\}$  as  $p \rightarrow \infty$ , and lead to the estimate

$$0 \leq (x_1^p + x_2^p)^{1/p} - \max\{x_1, x_2\} \leq (2^{1/p} - 1)x_1^{1/2}x_2^{1/2}, \quad \forall p \geq 1, \forall x_1, x_2 > 0, \quad (69)$$

with strict inequalities if  $x_1 \neq x_2$ .

We now turn to the

*Proof of Theorem 3.* The validity of (20) and the equality case in (20) follow from Lemmas 5 and 8.

In order to prove that  $\alpha_0 = \frac{2^{p-1}}{p}$ , it suffices to prove that if, for some  $\alpha$ , the inequality (12) holds with  $\beta = 2^p - 2\alpha$ , then we necessarily have  $\alpha \geq \frac{2^{p-1}}{p}$ . To see this, we test (12) with  $x_1 = 1+x, x_2 = 1-x, 0 < x < 1$ , and find that

$$2^p \leq \alpha((1+x)^p + (1-x)^p) + (2^p - 2\alpha)(1-x^2)^{p/2}, \quad \forall 0 < x < 1. \quad (70)$$

A second order Taylor expansion in (70) leads to

$$0 \leq \alpha p(p-1)x^2 + O(x^3) - (2^p - 2\alpha)\frac{p}{2}x^2 + O(x^4),$$

which implies

$$\alpha p(p-1) - (2^p - 2\alpha)\frac{p}{2} \geq 0,$$

and therefore  $\alpha \geq \frac{2^{p-1}}{p}$ . □

We end with a qualitative result concerning the optimal minimization problem (12) with  $n \geq 3$  and  $p > n$ .

**Proposition 15.** Let  $n \geq 3$ ,  $p > n$ , and let  $\alpha_0$  be the optimal  $\alpha$  such that (12) holds with  $\beta$  as in (11). Then:

- (a) We have  $\alpha_0 > \max \left\{ (n-1)^{p-1}, \frac{n^{p-1}}{p} \right\}$ .
- (b) There exists some  $0 < x < 1/(n-1)$  such that, in (12) with  $\alpha = \alpha_0$  and  $\beta = n^p - n\alpha_0$ , equality holds when  $x_1 = \dots = x_{n-1} = 1+x$  and  $x_n = 1 - (n-1)x$ .

We do not claim that the above  $x$  is unique (though it is likely to be). Part (b) merely states that, when  $n \geq 3$  and  $p > n$ , there always exists a non symmetric minimizer in (12). This is in sharp contrast with the case  $n = 2$  settled in Theorem 3.

*Proof.* We already know, from Proposition 4, that  $(n-1)^{p-1} < \alpha_0 < \frac{n^{p-1}}{p}$ .

*Step 1.* For any  $n$  and  $p$ , the condition  $\alpha \geq \frac{n^{p-1}}{p}$  is necessary for the validity of (17). Consider the set

$$S := \{x = (x_1, \dots, x_n); x_j \geq 0, \forall j, x_1 + \dots + x_n = n\}.$$

Denote by  $F(x) = F(x_1, \dots, x_n)$  the right-hand side of (17). Let  $x \in S$ . As in the proof of Proposition 4 (a), write  $x_j = 1 + \varepsilon_j$ , with  $\varepsilon_1 + \dots + \varepsilon_n = 0$ , and set  $\xi := \varepsilon_1^2 + \dots + \varepsilon_n^2$ . A second order Taylor expansion of the quantities on the right-hand side of (12) yields, thanks to the identity (23),

$$\begin{aligned} x_1^p + \dots + x_n^p &= n + \frac{p(p-1)}{2}\xi + o(\xi), \\ x_1^{p/n} \dots x_n^{p/n} &= 1 - \frac{p}{2n}\xi + o(\xi), \end{aligned}$$

and thus, assuming (12), we have

$$n^p \leq F(x) = n^p + \frac{p^2}{2} \left[ \alpha_0 - \frac{n^{p-1}}{p} \right] \xi + o(\xi). \quad (71)$$

Letting  $\varepsilon_1 = \dots = \varepsilon_{n-1} = \varepsilon > 0$  and  $\varepsilon_n = -(n-1)\varepsilon$  in (71) and then  $\varepsilon \rightarrow 0$ , we find that  $\alpha \geq \frac{n^{p-1}}{p}$ .

*Step 2.* If  $p \geq 3$  and  $n \geq 3$ , then (17) does not hold when  $\alpha = \frac{n^{p-1}}{p}$ . With  $\varepsilon_j$  as above and  $\alpha = \frac{n^{p-1}}{p}$ , a tedious calculation shows that

$$\begin{aligned} &F(1 + \varepsilon, \dots, 1 + \varepsilon, 1 - (n-1)\varepsilon) \\ &= n^p - \frac{p(n-1)(n-2)}{6} [n\alpha(p-3) + 2n^p] \varepsilon^3 + o(\varepsilon^3) \\ &= n^p - C\varepsilon^3 + o(\varepsilon^3), \end{aligned}$$

with  $C > 0$ , and thus, as claimed, (17) does not hold. (Note that here we use the assumption  $n > 2$ .)

*Step 3. Existence of non symmetric minimizers.* We study the minimizers of the right-hand side of (17) with  $\alpha = \alpha_0$  under the constraint  $x \in S$ . Note that  $x_j > 0, \forall j$  (since  $\alpha_0 > (n-1)^{p-1}$ ). Also note that, as explained in the second strategy (end of Section 2), if  $x$  achieves the equality, then either  $x_1 = \dots = x_n$ , or, up to a permutation, the conclusion of Proposition 15 (b) holds. It thus suffices to prove that there exists some  $x \in S, x \neq \bar{x} := (1, \dots, 1)$ , achieving equality in (17). Argue by contradiction, and suppose that  $F(x) > n^p$  for every  $x \in S \setminus \{\bar{x}\}$ . By (71) and the fact that  $\alpha_0 > \frac{n^{p-1}}{p}$ , there exist some  $C_1 > 0$  and  $\delta$  such that

$$[x \in S, \|x - \bar{x}\|_1 \leq \delta] \implies F(x) - n^p = F(x) - F(\bar{x}) \geq C_1 \|x - \bar{x}\|_1^2. \quad (72)$$

On the other hand, we have assumed, by contradiction, that  $F(x) > n^p$  if  $x \in S \setminus \{\bar{x}\}$ , and therefore, by compactness, there exists some  $C_2 > 0$  such that

$$[x \in S, \|x - \bar{x}\|_1 \geq \delta] \implies F(x) - n^p \geq C_2. \quad (73)$$

Consider now

$$G(x) := x_1^p + \dots + x_n^p - nx_1^{p/n} \dots x_n^{p/n}.$$

Then  $G$  is smooth and bounded in  $S$ , and has a unique global minimum at  $\bar{x}$ , where  $G$  vanishes. Therefore, there exists some  $K > 0$  such that

$$G(x) \leq K_1 \|x - \bar{x}\|_1^2, \forall x \in S. \quad (74)$$

Combining (72)–(74) with the fact that

$$\|x - \bar{x}\|_1 \leq 2(n-1), \forall x \in S,$$

we find that (17) still holds for  $\alpha := \alpha_0 - \varepsilon$  and  $\beta := n^p - n\alpha$ , where  $\varepsilon > 0$  is such that

$$\varepsilon \leq \min\{C_1/K, C_2/(4(n-1)^2 K)\}.$$

This contradicts the minimality of  $\alpha_0$  and completes the proof of Proposition 15.  $\square$

## References

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