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► **To cite this version:**

Cédric Arhancet, Christoph Kriegler, Christian Le Merdy, Safoura Zadeh. Separating Fourier and Schur multipliers. 2023. hal-04043378

**HAL Id: hal-04043378**

**<https://hal.science/hal-04043378>**

Preprint submitted on 23 Mar 2023

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# SEPARATING FOURIER AND SCHUR MULTIPLIERS

CÉDRIC ARHANCET, CHRISTOPH KRIEGLER, CHRISTIAN LE MERDY, AND SAFOURA ZADEH

ABSTRACT. Let  $G$  be a locally compact unimodular group, let  $1 \leq p < \infty$ , let  $\phi \in L^\infty(G)$  and assume that the Fourier multiplier  $M_\phi$  associated with  $\phi$  is bounded on the noncommutative  $L^p$ -space  $L^p(VN(G))$ . Then  $M_\phi: L^p(VN(G)) \rightarrow L^p(VN(G))$  is separating (that is,  $\{a^*b = ab^* = 0\} \Rightarrow \{M_\phi(a)^*M_\phi(b) = M_\phi(a)M_\phi(b)^* = 0\}$  for any  $a, b \in L^p(VN(G))$ ) if and only if there exists  $c \in \mathbb{C}$  and a continuous character  $\psi: G \rightarrow \mathbb{C}$  such that  $\phi = c\psi$  locally almost everywhere. This provides a characterization of isometric Fourier multipliers on  $L^p(VN(G))$ , when  $p \neq 2$ . Next, let  $\Omega$  be a  $\sigma$ -finite measure space, let  $\phi \in L^\infty(\Omega^2)$  and assume that the Schur multiplier associated with  $\phi$  is bounded on the Schatten space  $S^p(L^2(\Omega))$ . We prove that this multiplier is separating if and only if there exist a constant  $c \in \mathbb{C}$  and two unitaries  $\alpha, \beta \in L^\infty(\Omega)$  such that  $\phi(s, t) = c\alpha(s)\beta(t)$  a.e. on  $\Omega^2$ . This provides a characterization of isometric Schur multipliers on  $S^p(L^2(\Omega))$ , when  $p \neq 2$ .

*2020 Mathematics Subject Classification:* Primary 46L51, secondary 43A15, 46B04.

*Key words:* Fourier multipliers, Schur multipliers, noncommutative  $L^p$ -spaces, isometries.

## 1. INTRODUCTION

Let  $\Gamma$  be a locally compact abelian group, let  $1 \leq p \neq 2 < \infty$  and let  $T: L^p(\Gamma) \rightarrow L^p(\Gamma)$  be a bounded Fourier multiplier. A classical theorem going back to Parrott [27] and Strichartz [33] asserts that  $T$  is an isometry if and only if there exists  $c \in \mathbb{C}$ , with  $|c| = 1$ , and  $u \in \Gamma$  such that  $T = c\tau_u$ . Here  $\tau_u: L^p(\Gamma) \rightarrow L^p(\Gamma)$  is the translation operator defined by  $\tau_u(f) = f(\cdot - u)$ .

In the last decade, Fourier multipliers on noncommutative  $L^p$ -spaces associated with group von Neumann algebras emerged as a major topic in noncommutative analysis, with applications to approximation properties of operator algebras, to singular integrals and Calderon-Zygmund operators, as well as to noncommutative probability and quantum information. See in particular [2, 9, 16, 17, 20, 25, 26]. It therefore became a natural issue to understand the structure of isometric Fourier multipliers in the noncommutative framework. Indeed, the original motivation for this work was to extend the Parrott-Strichartz theorem to this setting.

To be more specific, let  $G$  be a locally compact group, let  $VN(G)$  denote its group von Neumann algebra and let  $\lambda: L^1(G) \rightarrow VN(G)$  be the contractive representation associated with the left regular representation of  $G$ . Assume that  $G$  is unimodular. This ensures that the Plancherel weight  $\tau_G$  on  $VN(G)$  is actually a normal semifinite faithful trace. For any  $1 \leq p < \infty$ , let  $L^p(VN(G))$  be the noncommutative  $L^p$ -space associated with  $(VN(G), \tau_G)$ . A Fourier multiplier  $T: L^p(VN(G)) \rightarrow L^p(VN(G))$  is an operator of the form

$$T(\lambda(f)) = \lambda(\phi f),$$

where  $\phi$  is a fixed element of  $L^\infty(G)$  and  $f$  lies in a suitable dense subspace of  $L^1(G)$ . We set  $T = T_\phi$  in this case. See the beginning of Section 3 for more details.

We generalize the Parrott-Strichartz theorem by showing the following result, in which  $VN(G)$  plays the role of  $\Gamma$  and  $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ : If  $p \neq 2$ , a Fourier multiplier  $T_\phi: L^p(VN(G)) \rightarrow L^p(VN(G))$  is an isometry if and only if there exists  $c \in \mathbb{T}$  and a continuous character  $\psi: G \rightarrow \mathbb{T}$  such that  $\phi = c\psi$  locally almost everywhere.

We actually consider the more general class of separating Fourier multipliers. Following [22], we say that an operator  $T: L^p(M) \rightarrow L^p(M)$  acting on some noncommutative  $L^p$ -space  $L^p(M)$  is separating if for any disjoint  $a, b \in L^p(M)$  (that is,  $a^*b = ab^* = 0$ ), the images  $T(a), T(b)$  are disjoint as well. It is well-known that if  $p \neq 2$ , any isometry on  $L^p(M) \rightarrow L^p(M)$  is separating. This follows from Yeadon's characterization of isometries on noncommutative  $L^p$  spaces [36] (see also [22]). We prove that for any  $1 \leq p < \infty$  (including the case  $p = 2$ ), a Fourier multiplier  $T_\phi$  on  $L^p(VN(G))$  is separating if and only if there exists  $c \in \mathbb{C}$  and a continuous character  $\psi: G \rightarrow \mathbb{T}$  such that  $\phi = c\psi$  locally almost everywhere.

The above two characterizations theorems are established in Section 3. Section 4 provides complements on Fourier multipliers.

Section 5 is devoted to Schur multipliers acting on Schatten classes. Let  $(\Omega, \mu)$  be a  $\sigma$ -finite measure space, let  $\phi \in L^\infty(\Omega^2)$  and let  $T_\phi$  denote the associated Schur multiplier acting on the Hilbert-Schmidt space  $S^2(L^2(\Omega))$  (see below for details). Let  $1 \leq p < \infty$  and assume that  $T_\phi$  is bounded on the Schatten space  $S^p(L^2(\Omega))$ . We show that  $T_\phi: S^p(L^2(\Omega)) \rightarrow S^p(L^2(\Omega))$  is separating if and only if there exist a constant  $c \in \mathbb{C}$  and two unitaries  $\alpha, \beta \in L^\infty(\Omega)$  such that  $\phi(s, t) = c\alpha(s)\beta(t)$  a.e. on  $\Omega^2$ . In the case when  $p \neq 2$ , this provides a characterization of isometric Schur multipliers on  $S^p(L^2(\Omega))$ .

## 2. PRELIMINARIES ON SEPARATING MAPS

Let  $\mathcal{M}$  be a semifinite von Neumann algebra equipped with a normal semifinite faithful trace  $\tau$ . Assume that  $\mathcal{M} \subset B(H)$  acts on some Hilbert space  $H$ . Let  $L^0(\mathcal{M})$  denote the  $*$ -algebra of all closed, densely defined (possibly unbounded) operators on  $H$ , which are  $\tau$ -measurable. For any  $1 \leq p < \infty$ , the noncommutative  $L^p$ -space  $L^p(\mathcal{M})$ , associated with  $(\mathcal{M}, \tau)$ , can be defined as

$$L^p(\mathcal{M}) := \{x \in L^0(\mathcal{M}) : \tau(|x|^p) < \infty\}.$$

Let  $\|x\|_p := \tau(|x|^p)^{\frac{1}{p}}$  for any  $x \in L^p(\mathcal{M})$ . Then  $L^p(\mathcal{M})$  equipped with  $\|\cdot\|_p$  is a Banach space. We let  $L^\infty(\mathcal{M}) := \mathcal{M}$  for convenience and we let  $\|\cdot\|_\infty$  denote the operator norm. For any  $1 \leq p < \infty$ , let  $p' = \frac{p}{p-1}$  be the conjugate index of  $p$ . For any  $x \in L^p(\mathcal{M})$  and  $y \in L^{p'}(\mathcal{M})$ , the product  $xy$  belongs to  $L^1(\mathcal{M})$  and  $|\tau(xy)| \leq \|x\|_p \|y\|_{p'}$ . We further have an isometric identification  $L^p(\mathcal{M})^* \simeq L^{p'}(\mathcal{M})$  for the duality pairing given by

$$\langle y, x \rangle = \tau(xy), \quad x \in L^p(\mathcal{M}), \quad y \in L^{p'}(\mathcal{M}).$$

We let  $L^p(\mathcal{M})^+$  denote the cone of positive elements of  $L^p(\mathcal{M})$ . The reader is referred to [29] and the references therein for details on the algebraic operations on  $L^0(\mathcal{M})$ , the construction of  $L^p(\mathcal{M})$ , and for further properties.

We mention that if  $\mathcal{M} = B(H)$  for some Hilbert space  $H$ , then the usual trace  $\text{tr}: B(H)^+ \rightarrow [0, \infty]$  is a normal semifinite faithful one and the resulting noncommutative  $L^p$ -spaces associated with  $(B(H), \text{tr})$  are the Schatten classes  $S^p(H)$ .

We say that  $a, b \in L^0(\mathcal{M})$  are disjoint if  $a^*b = ab^* = 0$ . We say that a bounded operator  $T: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{M})$ ,  $1 \leq p \leq \infty$ , is separating if whenever  $a, b \in L^p(\mathcal{M})$  are disjoint then  $T(a)$  and  $T(b)$  are disjoint as well.

A Jordan  $*$ -homomorphism on a von Neumann algebra  $\mathcal{M}$  is a linear map  $J: \mathcal{M} \rightarrow \mathcal{M}$  that satisfies  $J(a^2) = J(a)^2$  and  $J(a^*) = J(a)^*$ , for every  $a \in \mathcal{M}$ . It is clear that a Jordan  $*$ -homomorphism is positive, i.e. if  $a \in \mathcal{M}^+$  then  $J(a) \in \mathcal{M}^+$ . We warn the reader that Jordan  $*$ -homomorphisms are not always  $*$ -homomorphisms. For example, the transposition map on matrices is a Jordan  $*$ -homomorphism.

However we have the following lemma, in which part (1) follows from the identity  $(a+b)^2 = a^2 + b^2 + (ab+ba)$  and part (2) is given by [18, 10.5.22(iii)].

**Lemma 2.1.** *Let  $J: \mathcal{M} \rightarrow \mathcal{M}$  be a Jordan  $*$ -homomorphism.*

- (1) *For all  $a, b \in \mathcal{M}$ , we have  $J(ab+ba) = J(a)J(b) + J(b)J(a)$ .*
- (2) *If  $a, b \in \mathcal{M}$  satisfy  $ab = ba$ , then we have  $J(ab) = J(a)J(b)$ .*

We also record the following properties for further use. Here a map  $J: \mathcal{M} \rightarrow \mathcal{M}$  is called normal if it is weak\*-continuous.

**Lemma 2.2.** *Let  $J: \mathcal{M} \rightarrow \mathcal{M}$  be a normal Jordan  $*$ -homomorphism.*

- (1) *The kernel  $\ker(J)$  is a  $w^*$ -closed ideal of  $\mathcal{M}$ .*
- (2) *If  $(e_i)_i$  is a bounded net of  $\mathcal{M}$  such that  $e_i \rightarrow 0$  and  $e_i^* \rightarrow 0$  in the strong operator topology, then  $J(e_i) \rightarrow 0$  in the strong operator topology.*

*Proof.* (1) Let  $J: \mathcal{M} \rightarrow \mathcal{M}$  be a normal Jordan  $*$ -homomorphism. A well-known theorem asserts that there exist two von Neumann algebras  $\mathcal{M}_1, \mathcal{M}_2$ , a von Neumann algebra embedding  $\mathcal{M}_1 \oplus_{\infty} \mathcal{M}_2 \subseteq \mathcal{M}$ , a normal  $*$ -homomorphism  $\pi: \mathcal{M} \rightarrow \mathcal{M}_1$  and a normal anti  $*$ -homomorphism  $\sigma: \mathcal{M} \rightarrow \mathcal{M}_2$ , such that  $J(a) = \pi(a) \oplus \sigma(a)$  for all  $a \in \mathcal{M}$ . (See [32, Theorem 3.3] or [11, Corollary 7.4.9.] for this result.) Then  $\ker(J) = \ker(\pi) \cap \ker(\sigma)$ , hence  $\ker(J)$  is a weak\*-closed ideal.

(2) Let  $(e_i)_i$  be a bounded net of  $\mathcal{M}$  such that  $e_i \rightarrow 0$  and  $e_i^* \rightarrow 0$  strongly. Writing  $(e_i^*e_i\zeta|\eta) = (e_i\zeta|e_i\eta)$ , we see that  $(e_i^*e_i\zeta|\eta) \rightarrow 0$  for all  $\zeta, \eta \in \mathcal{H}$ . Since  $(e_i^*e_i)_i$  is bounded, this implies that  $e_i^*e_i \rightarrow 0$  in the weak\*-topology of  $\mathcal{M}$ . Consequently,  $\pi(e_i^*e_i) \rightarrow 0$  weakly. Writing  $\|\pi(e_i)\zeta\|^2 = (\pi(e_i^*e_i)\zeta|\zeta)$ , we deduce that  $\pi(e_i) \rightarrow 0$  strongly. Likewise, using  $e_i e_i^*$  instead of  $e_i^*e_i$ , we have that  $\sigma(e_i) \rightarrow 0$  strongly. Thus,  $J(e_i) \rightarrow 0$  strongly.  $\square$

The next statement plays a fundamental role in the study of separating maps. It was established independently in [22] and [14].

**Proposition 2.3.** [22, Remark 3.3 and Proposition 3.11] *Let  $1 \leq p < \infty$ . A bounded operator  $T: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{M})$  is separating if and only if there exist a normal Jordan  $*$ -homomorphism  $J: \mathcal{M} \rightarrow \mathcal{M}$ , a partial isometry  $w \in \mathcal{M}$ , and a positive operator  $B$  affiliated with  $\mathcal{M}$ , which verify the following conditions:*

- (a)  $T(a) = wBJ(a)$ , for all  $a \in \mathcal{M} \cap L^p(\mathcal{M})$ ;

- (b)  $w^*w = J(1) = s(B)$ ;
- (c) every spectral projection of  $B$  commutes with  $J(a)$ , for all  $a \in \mathcal{M}$ .

Here  $s(B)$  denotes the support of  $B$ . Furthermore, the triple  $(w, B, J)$  is unique.

It was shown by Yeadon [36] that all isometries on  $L^p(\mathcal{M})$ ,  $p \neq 2$ , are separating and have the above mentioned factorisation. For this reason, for  $T$  as above, we refer to  $(w, B, J)$  as the Yeadon triple of  $T$ .

**Lemma 2.4.** *Let  $T: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{M})$  be a separating map and let  $(w, B, J)$  denote its Yeadon triple. If  $T$  has dense range, then  $J(1) = 1$  and  $w$  is a unitary.*

*Proof.* The proof is an easy modification of [24, Remark 3.2].  $\square$

**Lemma 2.5.** *Let  $1 \leq p, q < \infty$ . Let  $T: L^p(\mathcal{M}) + L^q(\mathcal{M}) \rightarrow L^p(\mathcal{M}) + L^q(\mathcal{M})$  and assume that  $T: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{M})$  and  $T: L^q(\mathcal{M}) \rightarrow L^q(\mathcal{M})$  are bounded. If  $T: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{M})$  is separating, then  $T: L^q(\mathcal{M}) \rightarrow L^q(\mathcal{M})$  is separating as well.*

*Proof.* Let  $\mathcal{E} := \{e \in \mathcal{M} : e \text{ is a projection with } \tau(e) < \infty\}$ . Suppose that  $T: L^p(\mathcal{M}) \rightarrow L^p(\mathcal{M})$  is separating. Since,  $\mathcal{E} \subseteq L^p(\mathcal{M}) \cap L^q(\mathcal{M})$ , the operator  $T: L^q(\mathcal{M}) \rightarrow L^q(\mathcal{M})$  also preserves disjointness on  $\mathcal{E}$ . By [22, Remark 3.12 (i)],  $T: L^q(\mathcal{M}) \rightarrow L^q(\mathcal{M})$  is separating.  $\square$

### 3. A CHARACTERIZATION OF SEPARATING FOURIER MULTIPLIERS

Let  $G$  be a locally compact group with left Haar measure  $\mu$  defined on the  $\sigma$ -algebra of Borel sets. We will write  $ds$  for  $d\mu(s)$ . Denote by  $\lambda$  the left regular representation of  $G$  defined by

$$\lambda: G \rightarrow B(L^2(G)); \quad [\lambda(s)f](t) = f(s^{-1}t).$$

The left regular representation  $\lambda$  determines a representation of  $L^1(G)$  also denoted by  $\lambda$  and defined by

$$\lambda: L^1(G) \rightarrow B(L^2(G)), \quad \lambda(g)\eta = g * \eta,$$

for all  $g \in L^1(G)$  and  $\eta \in L^2(G)$ . Here, the convolution is  $g * \eta(t) = \int_G g(s)\eta(s^{-1}t)ds$ . We have that  $\lambda(g) = \int_G g(s)\lambda(s) ds$ , where the operator integral is understood in the strong operator sense.

For any function  $g: G \rightarrow \mathbb{C}$ , we let

$$\check{g}(t) = g(t^{-1}) \quad \text{and} \quad g^*(t) = \overline{g(t^{-1})},$$

for all  $t \in G$ .

We denote by  $e$  the unit element of  $G$ . Also for any Borel set  $A \subseteq G$ , we let  $\chi_A$  denote the indicator function of  $A$ .

Let  $VN(G) \subseteq B(L^2(G))$  be the von Neumann algebra generated by  $\{\lambda(s) : s \in G\}$ . This coincides with the von Neumann algebra generated by  $\{\lambda(g) : g \in L^1(G)\}$ . When  $G$  is abelian, we have  $VN(G) \simeq L^\infty(\widehat{G})$ , where  $\widehat{G}$  is the dual group of  $G$ .

We let  $(\cdot | \cdot)$  denote the inner product on  $L^2(G)$ . The Fourier algebra of the group  $G$  is defined as

$$A(G) = \{(\lambda(\cdot)\zeta | \eta) : \zeta, \eta \in L^2(G)\} \subseteq C_b(G).$$

This is a Banach algebra for the pointwise product and the norm defined, for any  $\psi \in A(G)$ , by

$$\|\psi\|_{A(G)} = \inf\{\|\zeta\|_2\|\eta\|_2\},$$

where the infimum runs over all  $\zeta, \eta \in L^2(G)$  such that  $\psi = (\lambda(\cdot)\zeta|\eta)$ . We note for further use that equivalently, we can write  $A(G) = \{\zeta * \eta : \zeta, \eta \in L^2(G)\}$ .

We recall that  $A(G)^* \simeq VN(G)$  isometrically for the duality pairing given by

$$(3.1) \quad \langle \lambda(s), \psi \rangle = \psi(s), \quad \psi \in A(G), \quad s \in G.$$

Assume that  $G$  is unimodular. We will use the so-called Plancherel trace

$$\tau_G : VN(G)^+ \longrightarrow [0, +\infty],$$

for which we refer to [35, Section VII.3] (see also [3]). We note that  $\tau_G$  is a normal semifinite faithful trace. This allows to consider the noncommutative  $L^p$ -spaces  $L^p(VN(G))$  associated with  $\tau_G$ . We recall that if  $G$  is discrete then  $G$  is unimodular and  $\tau_G$  is normalised. Also, if  $G$  is abelian then  $G$  is unimodular and  $L^p(VN(G)) = L^p(\widehat{G})$ , where  $\widehat{G}$  denotes the dual group of  $G$ .

It is well-known that for any  $g \in L^1(G) \cap L^2(G)$ ,  $\lambda(g) \in L^2(VN(G))$  with  $\|\lambda(g)\|_2 = \|g\|_2$  (see e.g. [3, Section 6.1]). Consequently, the restriction of  $\lambda$  to  $L^1(G) \cap L^2(G)$  extends to an isometry from  $L^2(G)$  into  $L^2(VN(G))$ . It turns out that the latter is onto, which yields a unitary identification

$$(3.2) \quad L^2(VN(G)) \simeq L^2(G).$$

Using the notation  $U_\lambda : L^2(G) \rightarrow L^2(VN(G))$  for the above unitary mapping, we have

$$(3.3) \quad \tau_G(U_\lambda(\zeta)U_\lambda(\eta)) = \int_G \zeta(t)\check{\eta}(t) dt, \quad \zeta, \eta \in L^2(G).$$

Since  $L^1(VN(G))^* \simeq VN(G)$ , we have an isometric identification

$$A(G) \simeq L^1(VN(G)).$$

It is not hard to deduce from (3.1) and (3.3) that this identification is given by the mapping  $A(G) \rightarrow L^1(VN(G))$  taking  $\zeta * \eta$  to  $U_\lambda(\check{\eta})U_\lambda(\zeta)$ , for all  $\zeta, \eta \in L^2(G)$ . Details are left to the reader.

Let  $C_c(G)$  denote the space of continuous and compactly supported functions on  $G$ . We let  $C_c(G) * C_c(G)$  denote the linear span of  $f_1 * f_2$ , where  $f_1, f_2 \in C_c(G)$ . It is well-known that

$$\lambda(C_c(G) * C_c(G)) \subseteq L^1(VN(G)) \cap VN(G),$$

and that  $\lambda(C_c(G) * C_c(G))$  is dense in  $L^p(VN(G))$ , for all  $1 \leq p < \infty$ . For a proof, we refer to [7, Proposition 3.4] for the case  $p = 1$ , and to [5, Proposition 4.7] for the other cases. We also note that since  $C_c(G) * C_c(G)$  is dense in  $L^1(G)$ ,  $\lambda(C_c(G) * C_c(G))$  is weak\*-dense in  $VN(G)$ .

**Lemma 3.1.** *Suppose that  $K \subseteq G$  is a compact set. There is a function  $\psi \in A(G)$  such that  $\psi(s) = \mu(K)$ , for all  $s \in K$ .*

*Proof.* For a proof, we refer to [19, Proposition 2.3.2].

□

Some of the formulations of the main results in this article are easier when the group  $G$  is  $\sigma$ -compact, meaning that  $G$  is the countable union of compact subsets. The following well-known lemma relates this property with other countability properties that the group  $G$  can have. We provide a proof for the sake of completeness.

**Lemma 3.2.** *Let  $G$  be a locally compact group. Then the following implications hold:*

$$G \text{ is second countable} \implies G \text{ is } \sigma\text{-compact} \iff \text{the Haar measure of } G \text{ is } \sigma\text{-finite.}$$

*Moreover, the remaining implication is false. That is, there exists a ( $\sigma$ -)compact group which is not second countable.*

*Proof.* If  $G$  is second countable, then by definition, its topology admits a countable basis. Since  $G$  is locally compact, this basis can be chosen to consist of relatively compact sets  $O_k$ ,  $k \geq 1$ . Thus,

$$G = \bigcup_{k \in \mathbb{N}} O_k = \bigcup_{k \in \mathbb{N}} \overline{O_k},$$

so  $G$  is  $\sigma$ -compact.

Next, we show that for the Haar measure  $\mu$ ,  $\sigma$ -compactness and  $\sigma$ -finiteness are equivalent. Recall [8, Proposition 2.4] that  $G$  admits an open, closed,  $\sigma$ -compact subgroup  $G_0$ . Thus,  $G = \bigcup_{y \in Y} yG_0$  for some subset  $Y \subseteq G$  representing the left cosets, where the union is disjoint. We claim that  $Y$  can be chosen at most countable if and only if  $G$  is  $\sigma$ -compact, if and only if the Haar measure is  $\sigma$ -finite. Indeed, if  $Y$  is at most countable, as  $yG_0$  is  $\sigma$ -compact for all  $y \in Y$ ,  $G$  is  $\sigma$ -compact. If  $G$  is  $\sigma$ -compact, say  $G = \bigcup_{n \in \mathbb{N}} K_n$  with compact  $K_n$ , then  $\mu(K_n) < \infty$  for any  $n \in \mathbb{N}$ , hence  $G$  is  $\sigma$ -finite. Finally, suppose that  $G$  is  $\sigma$ -finite, say  $G = \bigcup_{n \in \mathbb{N}} H_n$  with  $\mu(H_n) < \infty$  for any  $n \in \mathbb{N}$ . According to [8, Proposition 2.22], the fact that  $\mu(H_n)$  is finite implies that  $Y_n := \{y \in Y : H_n \cap yG_0 \neq \emptyset\}$  is at most countable. Thus,  $Y' := \{y \in Y : G \cap yG_0 = \bigcup_{n \in \mathbb{N}} (H_n \cap yG_0) \neq \emptyset\} = \bigcup_{n \in \mathbb{N}} Y_n$  is also at most countable. But clearly,  $Y' = Y$ .

Finally, for the last statement, it suffices to take the compact non first countable group  $G = \mathbb{T}^{\mathbb{R}}$ .  $\square$

In the sequel we use the space  $L^\infty(G)$ . Its definition requires some care. When  $G$  is  $\sigma$ -compact,  $L^\infty(G)$  is defined in the usual way. But when  $G$  is not  $\sigma$ -compact, by the above Lemma 3.2, the left Haar measure  $\mu$  is not  $\sigma$ -finite. In this case, if we define  $L^\infty(G)$  in the usual way, the duality of  $L^1(G)$  and  $L^\infty(G)$  may break down. As it is explained in [8, Section 2.3], it is possible to salvage this duality by modifying the definition of  $L^\infty(G)$  as follows. A set  $E \subseteq G$  is called locally Borel if  $E \cap F$  is Borel whenever  $F$  is Borel and  $\mu(F) < \infty$ . A locally Borel set is locally null if  $\mu(E \cap F) = 0$  whenever  $F$  is Borel and  $\mu(F) < \infty$ . A function  $f: G \rightarrow \mathbb{C}$  is locally measurable if  $f^{-1}(A)$  is locally Borel for every Borel set  $A \subseteq \mathbb{C}$ . A property is true locally almost everywhere if it is true except on a locally null set.

With these definitions in hand, let  $L^\infty(G)$  be the space of all locally measurable functions  $\phi: G \rightarrow \mathbb{C}$  that are bounded except on a locally null set, modulo the functions that are zero locally almost everywhere. Then  $L^\infty(G)$  is a Banach space with the norm

$$\|\phi\|_\infty = \inf \{c : |\phi(t)| \leq c \text{ locally almost everywhere}\}.$$

We note that for any  $1 \leq p < \infty$ , any  $f \in L^p(G)$  has a  $\sigma$ -finite support hence for all  $\phi \in L^\infty(G)$ ,  $\phi f$  is a well-defined element of  $L^p(G)$ .

One may therefore define  $\int_G \phi f$  for all  $\phi \in L^\infty(G)$  and all  $f \in L^1(G)$  and this duality pairing yields an isometric identification  $L^\infty(G) \simeq L^1(G)^*$ . When  $G$  is  $\sigma$ -compact,  $L^\infty(G)$  defined as above coincides with the usual one.

**Definition 3.3.** For any  $\phi \in L^\infty(G)$ , let  $M_\phi: \lambda(C_c(G) * C_c(G)) \rightarrow VN(G)$  be defined by

$$M_\phi(\lambda(f)) := \lambda(\phi f), \quad f \in C_c(G) * C_c(G).$$

For any  $1 \leq p < \infty$ , we say that  $M_\phi$  is a bounded Fourier multiplier on  $L^p(VN(G))$  if the above map extends to a bounded operator (still denoted by)  $M_\phi: L^p(VN(G)) \rightarrow L^p(VN(G))$ . In the sequel we abbreviate this by saying that “ $M_\phi: L^p(VN(G)) \rightarrow L^p(VN(G))$  is a bounded Fourier multiplier”.

Likewise, if  $p = \infty$ , we say that  $M_\phi$  is a bounded Fourier multiplier on  $VN(G)$  if the above map extends to a bounded weak\*-continuous operator  $M_\phi: VN(G) \rightarrow VN(G)$ .

Let  $1 \leq p < \infty$ . We recall from [3, Section 6.1] that if  $M_\phi: L^p(VN(G)) \rightarrow L^p(VN(G))$  is a bounded Fourier multiplier, then  $M_\phi: L^{p'}(VN(G)) \rightarrow L^{p'}(VN(G))$  is a bounded Fourier multiplier as well, where  $p'$  is the conjugate index of  $p$ . Moreover,  $M_{\bar{\phi}}: L^{p'}(VN(G)) \rightarrow L^{p'}(VN(G))$  is a bounded Fourier multiplier, and this operator is actually the adjoint of  $M_\phi: L^p(VN(G)) \rightarrow L^p(VN(G))$ .

Thanks to (3.2), we have that for any  $\phi \in L^\infty(G)$ ,  $M_\phi: L^2(VN(G)) \rightarrow L^2(VN(G))$  is a bounded Fourier multiplier, with

$$(3.4) \quad \|M_\phi: L^2(VN(G)) \rightarrow L^2(VN(G))\| = \|\phi\|_{L^\infty(G)}.$$

Indeed, let us denote as before by  $U_\lambda: L^2(G) \rightarrow L^2(VN(G))$  the unitary mapping taking any  $f \in L^1(G) \cap L^2(G)$  to  $\lambda(f)$ , see (3.2). Let  $\pi: L^\infty(G) \rightarrow B(L^2(G))$  be defined by  $[\pi(\phi)](f) = \phi f$ , for all  $\phi \in L^\infty(G)$  and all  $f \in L^2(G)$ . Then  $\pi$  is an isometry. Since for any  $\phi \in L^\infty(G)$ ,  $U_\lambda \pi(\phi) U_\lambda^*$  coincides with  $M_\phi$  on  $C_c(G) * C_c(G)$ , we obtain that  $M_\phi$  is a bounded Fourier multiplier on  $L^2(VN(G))$  and that  $M_\phi: L^2(VN(G)) \rightarrow L^2(VN(G))$  coincides with  $U_\lambda \pi(\phi) U_\lambda^*$ . Since  $U_\lambda \pi(\cdot) U_\lambda^*$  is an isometry, the equality (3.4) follows. (See also [3, Lemma 6.5].) It follows from above that a Fourier multiplier  $M_\phi: L^2(VN(G)) \rightarrow L^2(VN(G))$  satisfies

$$M_\phi(U_\lambda(f)) = U_\lambda(\phi f), \quad f \in L^2(G).$$

**Remark 3.4.** If  $\phi_1, \phi_2 \in L^\infty(G)$  are such that  $M_{\phi_1}$  and  $M_{\phi_2}$  coincide on  $\lambda(C_c(G) * C_c(G))$ , then by (3.4), the operator  $M_{\phi_1 - \phi_2} = M_{\phi_1} - M_{\phi_2}: L^2(VN(G)) \rightarrow L^2(VN(G))$  is equal to 0, hence  $\phi_1 = \phi_2$  locally almost everywhere.

For the rest of this section, we fix a net  $(f_i)_{i \in I}$  in  $C_c(G) * C_c(G)$  such that  $f_i \geq 0$  and  $\int_G f_i(s) ds = 1$  for all  $i \in I$ , the supports of  $f_i$ 's are contained in some compact neighborhood  $V_i$  of  $e$  where the net  $(V_i)_{i \in I}$ , is decreasing, and  $\bigcap_{i \in I} V_i = \{e\}$ . We set  $e_i := \lambda(f_i)$ . Then for all  $i \in I$ , we have

$$e_i \in L^1(VN(G)) \cap VN(G) \quad \text{and} \quad \|e_i\|_{VN(G)} \leq 1.$$

**Lemma 3.5.** We both have  $e_i \rightarrow 1$  and  $e_i^* \rightarrow 1$  in the strong operator topology.



*Proof.* Let  $\zeta \in L^2(G)$ . Since each  $f_i$  is non-negative and  $L^1$ -normalized, we have

$$\|e_i(\zeta) - \zeta\|_{L^2(G)} = \left\| \int_G f_i(t) (\lambda(t)\zeta - \zeta) dt \right\|_{L^2(G)} \leq \int_G f_i(t) \|\lambda(t)\zeta - \zeta\|_{L^2(G)} dt.$$

Since  $\lambda(t) \rightarrow 1$  as  $t \rightarrow e$  in the strong operator topology, the assumptions on  $(f_i)_{i \in I}$  ensure that the right hand-side tends to 0, when  $i \rightarrow \infty$ . Hence  $\|e_i(\zeta) - \zeta\|_{L^2(G)} \rightarrow 0$ .

This shows that  $e_i \rightarrow 1$  strongly. Since  $e_i^* = \lambda(f_i^*)$  and the  $f_i^*$  have the same features as the  $f_i$ , we also have that  $e_i^* \rightarrow 1$  strongly.  $\square$

We let  $C_b(G)$  be the space of bounded and continuous functions on  $G$ .

The following lemma shows that the definition of bounded Fourier multipliers  $VN(G) \rightarrow VN(G)$  considered in this paper coincide with the one in [6].

**Lemma 3.6.** *Let  $\phi \in L^\infty(G)$  and assume that  $M_\phi$  is a bounded Fourier multiplier on  $VN(G)$ . Then there exists  $\psi \in C_b(G)$  such that  $\phi = \psi$  locally almost everywhere, and*

$$M_\psi(\lambda(s)) = \psi(s)\lambda(s), \quad s \in G.$$

*Proof.* Assume that  $M_\phi: VN(G) \rightarrow VN(G)$  is a bounded Fourier multiplier. Let us show that for any  $s \in G$ ,

$$(3.5) \quad M_\phi(\lambda(s)) \in \text{Span}\{\lambda(s)\}.$$

Fix  $s \in G$ . For all  $i \in I$ ,  $\lambda(s)e_i = \int_G f_i(t)\lambda(st)dt$ , hence  $M_\phi(\lambda(s)e_i) = \int_G \phi(st)f_i(t)\lambda(st) dt$ , where these integrals are defined in the strong operator topology. Therefore, for all  $\varphi \in A(G)$ , we have

$$\begin{aligned} \langle M_\phi(\lambda(s)e_i), \varphi \rangle &= \int_G \phi(st)f_i(t)\langle \lambda(st), \varphi \rangle dt \\ &= \int_G \phi(st)f_i(t)\varphi(st) dt. \end{aligned}$$

Let  $\varphi \in A(G)$  such that  $\varphi(s) = 0$ . From the above, we have

$$|\langle M_\phi(\lambda(s)e_i), \varphi \rangle| \leq \|\phi\|_\infty \int_G f_i(t)|\varphi(st) - \varphi(s)| dt.$$

Since  $\varphi$  is continuous, the right hand-side of the above inequality tends to zero when  $i \rightarrow \infty$ . Moreover by Lemma 3.5,  $\lambda(s)e_i \rightarrow \lambda(s)$  in the strong operator topology. Since  $(e_i)_{i \in I}$  is bounded, this implies that  $\lambda(s)e_i \rightarrow \lambda(s)$  in the weak\*-topology. Therefore,  $M_\phi(\lambda(s)e_i) \rightarrow M_\phi(\lambda(s))$  in the weak\*-topology. Hence,  $\langle M_\phi(\lambda(s)e_i), \varphi \rangle \rightarrow \langle M_\phi(\lambda(s)), \varphi \rangle$ . We obtain that  $\langle M_\phi(\lambda(s)), \varphi \rangle = 0$ . Hence,  $M_\phi(\lambda(s)) \in \{\varphi : \varphi(s) = 0\}^\perp$  in the duality  $A(G)^* \simeq VN(G)$ . Since  $\text{Span}\{\lambda(s)\}^\perp = \{\varphi : \varphi(s) = 0\}$ , we deduce (3.5).

Let  $\psi: G \rightarrow \mathbb{C}$  be the unique function such that for all  $s \in G$ ,  $M_\phi(\lambda(s)) = \psi(s)\lambda(s)$ . The pre-adjoint  $A(G) \rightarrow A(G)$  of  $M_\phi$  is the pointwise multiplication by  $\psi$ . According to the comment following [6, 1.1. Definition], the function  $\psi$  is therefore continuous.

Next, we show that  $\phi = \psi$  locally almost everywhere. For any  $f \in C_c(G) * C_c(G)$ , we have  $\lambda(f) = \int_G f(t)\lambda(t) dt$ . This SOT-integral is absolutely convergent in  $VN(G)$ , hence

$$\begin{aligned} M_\phi(\lambda(f)) &= \int_G f(t)M_\phi(\lambda(t)) dt \\ &= \int_G f(t)\psi(t)\lambda(t) dt \\ &= M_\psi(\lambda(f)). \end{aligned}$$

This implies that  $M_\phi = M_\psi$  on  $C_c(G) * C_c(G)$ , and the result follows by Remark 3.4.

For the regularity of  $G$  needed in the proof of [6, 1.1. Definition], we refer to [19, Theorem 2.3.8 p. 53]. See also the discussion following [3, Definition 6.3].  $\square$

Let  $\mathbb{T}$  denote the unit circle of  $\mathbb{C}$ . A homomorphism  $\varphi: G \rightarrow \mathbb{T}$  is called a character. We let  $Hom(G, \mathbb{T})$  denote the collection of all characters on  $G$ . There is a natural isomorphism between  $Hom(G, \mathbb{T})$  and  $Hom(\frac{G}{[G, G]}, \mathbb{T})$ , where  $[G, G]$  denotes the commutator subgroup of  $G$  [13, (23.8) Theorem p. 358-359]. When  $G$  is a perfect group, i.e.  $[G, G] = G$ , the only character on  $G$  is the trivial one, that is,  $Hom(G, \mathbb{T}) = \{1\}$ . Examples of perfect groups include non-abelian simple groups and the special linear groups  $SL_n(\mathbb{K})$ , for a fixed field  $\mathbb{K}$ .

The following is the main result of this section. In view of this theorem and the observation above, we see that there are groups with relatively few separating Fourier multipliers.

**Theorem 3.7.** *Assume that  $G$  is a locally compact unimodular  $\sigma$ -compact group, let  $1 \leq p < \infty$  and let  $\phi \in L^\infty(G)$ . The following are equivalent.*

- (i) *The mapping  $M_\phi$  is a bounded Fourier multiplier on  $L^p(VN(G))$ , and the operator  $M_\phi: L^p(VN(G)) \rightarrow L^p(VN(G))$  is separating.*
- (ii) *There exist a constant  $c \in \mathbb{C}$  and a continuous character  $\psi: G \rightarrow \mathbb{T}$  such that  $\phi = c\psi$  almost everywhere.*

The proof will be given after a series of intermediate results.

**Lemma 3.8.** *Let  $\mathcal{M}$  be a semifinite von Neumann algebra. Let  $(x_j)_j$  be a net in  $\mathcal{M} \cap L^2(\mathcal{M})$  with  $\sup \|x_j\|_\infty < \infty$ . If  $\|x_j\|_2 \rightarrow 0$ , then  $x_j \rightarrow 0$  in the weak\*-topology of  $\mathcal{M}$ .*

*Proof.* Take  $y \in L^1(\mathcal{M}) \cap L^2(\mathcal{M})$ . We have that

$$|\langle y, x_j \rangle| \leq \|y\|_2 \|x_j\|_2 \rightarrow 0,$$

and therefore  $\langle y, x_j \rangle \rightarrow 0$ . Since  $L^1(\mathcal{M}) \cap L^2(\mathcal{M})$  is dense in  $L^1(\mathcal{M})$  and  $(x_j)_j$  is bounded in  $\mathcal{M}$ , this implies that  $\langle y, x_j \rangle \rightarrow 0$ , for all  $y \in L^1(\mathcal{M})$ . That is,  $x_j \rightarrow 0$  in the weak\*-topology of  $\mathcal{M}$ .  $\square$

In the following lemma,  $G$  is not necessarily  $\sigma$ -compact.

**Lemma 3.9.** *Let  $\phi \in L^\infty(G)$ . Assume that the Fourier multiplier  $M_\phi: L^2(VN(G)) \rightarrow L^2(VN(G))$  is separating and non-zero.*

- (1) *For any compact  $K \subseteq G$ , the restriction  $\phi|_K$  is non-zero almost everywhere on  $K$ .*
- (2) *The operator  $M_\phi: L^2(VN(G)) \rightarrow L^2(VN(G))$  has dense range.*

- (3) For any compact  $K \subseteq G$ , there exists a continuous function  $\Phi: K \rightarrow \mathbb{C}$  such that  $\phi|_K = \Phi$  almost everywhere on  $K$ .

*Proof.* (1) Let  $K$  be a compact subset of  $G$ . Set  $N_K(\phi) := \{s \in K : \phi(s) = 0\}$ . We show that  $N_K(\phi)$  has measure zero. Assume on the contrary that  $N_K(\phi)$  has positive measure. We show that there exists  $a_0 \in K$  such that for any open neighbourhood  $V$  of  $a_0$ ,  $\mu(N_K(\phi) \cap V) > 0$ . Assume on the contrary that for any  $a \in K$  there is an open neighbourhood of  $a$ ,  $V_a$ , such that  $\mu(N_K(\phi) \cap V_a) = 0$ . Since  $K$  is compact and  $\{V_a\}_{a \in K}$  covers  $K$ , there is a finite subcover  $\{V_{a_j}\}_{j=1}^n$  that covers  $K$  as well. Now, note that

$$\mu(N_K(\phi)) = \mu(N_K(\phi) \cap (\cup_{j=1}^n V_{a_j})) = \mu(\cup_{j=1}^n (N_K(\phi) \cap V_{a_j})) \leq \sum_{j=1}^n \mu(N_K(\phi) \cap V_{a_j}) = 0,$$

which contradicts the fact that  $\mu(N_K(\phi)) > 0$ .

Let  $(U_i)_{i \in I}$  be a net of neighbourhoods of  $e$ , the unit element of  $G$ , directed by inclusion, with  $\cap_{i \in I} U_i = \{e\}$ . Then for all  $i \in I$ , we have  $\mu(a_0 U_i \cap N_K(\phi)) > 0$  and we may define

$$h_i := \frac{1}{\mu(a_0 U_i \cap N_K(\phi))} \chi_{U_i} \chi_{N_K(\phi)}(a_0 \cdot).$$

For any  $i \in I$ , we have that  $h_i \geq 0$ ,  $\int h_i = 1$ , and  $h_i \in L^1(G) \cap L^2(G)$ . Moreover,  $\text{supp}(h_i) = U_i \cap a_0^{-1} N_K(\phi)$  and therefore,  $\bigcap_{i \in I} \text{supp}(h_i) = \{e\}$  and  $\phi h_i(a_0^{-1} \cdot) = 0$ , for all  $i \in I$ .

Let  $(w, B, J)$  be the Yeadon triple of  $M_\phi: L^2(VN(G)) \rightarrow L^2(VN(G))$ . For any  $i \in I$ , let  $\varepsilon_i := \lambda(h_i(a_0^{-1} \cdot))$ . We have that  $\varepsilon_i \in L^2(VN(G)) \cap VN(G)$  and  $\|\varepsilon_i\|_{VN(G)} \leq 1$ . By Proposition 2.3(a),  $M_\phi(\varepsilon_i) = wBJ(\varepsilon_i)$ . Also,  $M_\phi(\varepsilon_i) = \lambda(\phi h_i(a_0^{-1} \cdot)) = \lambda(0) = 0$ . Therefore,  $wBJ(\varepsilon_i) = 0$ . We now apply Proposition 2.3(b). Since  $w^*w = s(B)$ , we have  $BJ(\varepsilon_i) = w^*wBJ(\varepsilon_i)$ , hence  $BJ(\varepsilon_i) = 0$ . Further  $0 \leq J(\varepsilon_i) \leq J(1) = s(B)$ , hence  $J(\varepsilon_i)$  is valued in  $\ker(B)^\perp$ . Hence the equality  $BJ(\varepsilon_i) = 0$  implies that  $J(\varepsilon_i) = 0$ , that is  $\varepsilon_i \in \ker(J)$ . By Lemma 2.2(1), this implies that for any  $g \in L^1(G)$  and any  $i \in I$ , we have  $\lambda(g)\varepsilon_i \in \ker(J)$ .

Now, for any  $f \in L^1(G) \cap L^2(G)$ , let  $g = f(\cdot a_0)$ . We have

$$\lambda(g)\varepsilon_i = \lambda(g * h_i(a_0^{-1} \cdot)) = \lambda(g(\cdot a_0^{-1}) * h_i) = \lambda(f * h_i).$$

Since  $\|f * h_i - f\|_2 \rightarrow 0$ , we have  $\|\lambda(f * h_i) - \lambda(f)\|_{L^2(VN(G))} \rightarrow 0$ , by (3.2). Note that,

$$\|\lambda(f * h_i)\|_\infty \leq \|f * h_i\|_1 \leq \|f\|_1 \|h_i\|_1 = \|f\|_1.$$

Hence by Lemma 3.8,  $\lambda(f * h_i) \rightarrow \lambda(f)$ , in the weak\*-topology of  $VN(G)$ . Since  $\ker(J)$  is weak\*-closed, by Lemma 2.2(1), we obtain that  $\lambda(f)$  belongs to  $\ker(J)$ . Finally since the space  $\{\lambda(f) : f \in L^1(G) \cap L^2(G)\}$  is weak\*-dense in  $VN(G)$ , we deduce that  $\ker(J) = VN(G)$ . Hence  $M_\phi = 0$ , which is a contradiction.

(2) Consider any  $F \in C_c(G) * C_c(G)$ . This function has compact support, say  $K \subseteq G$ . Set  $N_\delta = \{s \in K : |\phi(s)| < \delta\}$ , for all  $\delta > 0$ . By part (1),  $\phi \neq 0$  almost everywhere on  $K$ , hence we have  $\lim_{\delta \rightarrow 0} \mu(N_\delta) = 0$ . For any  $\delta > 0$ , set  $g_\delta := \phi^{-1} \chi_{N_\delta} F$ . Since  $\phi^{-1} \chi_{N_\delta} \in L^\infty(G)$ , we have  $g_\delta \in L^1(G) \cap L^2(G)$ . Hence,  $\lambda(g_\delta)$  is well-defined,  $\lambda(g_\delta) \in L^2(VN(G)) \cap VN(G)$  and

$$M_\phi(\lambda(g_\delta)) = \int \phi(t) g_\delta(t) \lambda(t) dt = \lambda(\chi_{N_\delta} F).$$

Therefore,  $\lambda(F) - M_\phi(\lambda(g_\delta)) = \lambda(\chi_{N_\delta}F)$ , and we have that

$$\|\lambda(F) - M_\phi(\lambda(g_\delta))\|_2 = \|\lambda(\chi_{N_\delta}F)\|_2 = \|\chi_{N_\delta}F\|_2,$$

by (3.2). Since  $\overline{\mu(N_\delta)} \rightarrow 0$ , when  $\delta \rightarrow 0$ , this implies that  $\lim_{\delta \rightarrow 0} \|\lambda(F) - M_\phi(\lambda(g_\delta))\|_2 = 0$ . Hence,  $\lambda(F) \in \text{ran}(M_\phi)$ . By density of  $\lambda(C_c(G) * C_c(G))$  in  $L^2(VN(G))$ , this implies that the range of  $M_\phi$  is dense in  $L^2(VN(G))$ .

(3) Let  $w^*M_\phi$  be the operator taking any  $a \in L^2(VN(G))$  to  $w^*M_\phi(a)$ . According to [23, Remark 4.2],  $w^*M_\phi$  is positive, that is, it maps  $L^2(VN(G))^+$  into itself. We use a modification of the argument in [3, Lemma 6.10]. For all  $g \in C_c(G)$ ,  $\lambda(g^* * g) \in L^2(VN(G))^+$ . Hence,  $w^*M_\phi(\lambda(g^* * g)) \in L^2(VN(G))^+$ . Consequently,

$$(M_\phi(\lambda(g^* * g))\zeta|w(\zeta)) \geq 0,$$

for all  $\zeta \in L^2(G)$ . The calculation in the proof of [3, Lemma 6.10] shows that

$$(M_\phi(\lambda(g^* * g))\zeta|w(\zeta)) = \int_G \int_G \overline{g(t)}g(s)\phi(t^{-1}s) (\lambda(t^{-1}s)\zeta|w(\zeta)) dsdt.$$

This implies that for all  $\zeta \in L^2(G)$ , the function  $s \mapsto \phi(s) (\lambda(s)\zeta|w(\zeta))$  is positive definite in the sense of [35, Definition VII.3.20].

By polarization, this implies that for all  $\zeta, \zeta' \in L^2(G)$ ,  $s \mapsto \phi(s) (\lambda(s)\zeta|w(\zeta'))$  is a linear combination of positive definite functions. By part (2) and Lemma 2.4,  $w$  is a unitary. Hence the above actually shows that for all  $\zeta, \eta \in L^2(G)$ ,  $s \mapsto \phi(s) (\lambda(s)\zeta|\eta)$  is a linear combination of positive definite functions. By the definition of  $A(G)$ , this means that for all  $\psi \in A(G)$ ,  $\phi\psi$  is a linear combination of positive definite functions. By [35, Corollary 3.22] and its proof, this implies that for all  $\psi \in A(G)$ ,  $\phi\psi$  is locally almost everywhere equal to a continuous function.

Let  $K \subseteq G$  be a compact set with  $\mu(K) > 0$ . By Lemma 3.1, there is  $\psi \in A(G)$  such that  $\psi|_K > 0$ . Now,  $(\phi\psi)|_K$  is almost everywhere equal to a continuous function. Hence,  $\phi|_K$  is almost everywhere equal to a continuous function.  $\square$

**Remark 3.10.** Note that if  $G$  is  $\sigma$ -compact, the above lemma implies that if  $\phi \in L^\infty(G)$  is such that  $M_\phi: L^2(VN(G)) \rightarrow L^2(VN(G))$  is separating, then  $\phi$  is almost everywhere equal to a continuous function.

In the next statement, we use the net  $(e_i)_{i \in I}$  defined before Lemma 3.5.

**Lemma 3.11.** *Let  $\phi \in C_b(G)$  and consider the Fourier multiplier  $M_\phi: L^2(VN(G)) \rightarrow L^2(VN(G))$ . We have the following convergences in the strong operator topology of  $VN(G)$ .*

- (1) For all  $s \in G$ ,  $M_\phi(\lambda(s)e_i) \xrightarrow{i \rightarrow \infty} \phi(s)\lambda(s)$ .
- (2) For all  $s \in G$ ,  $M_\phi(\lambda(s)e_i^2) \xrightarrow{i \rightarrow \infty} \phi(s)\lambda(s)$ .
- (3) For all  $s \in G$ ,  $M_\phi(e_i\lambda(s)e_i) \xrightarrow{i \rightarrow \infty} \phi(s)\lambda(s)$ .

*Proof.* (1) By Lemma 3.5, it is enough to show that for all  $s \in G$ ,  $M_\phi(\lambda(s)e_i) - \phi(s)\lambda(s)e_i$  converges to 0 in  $VN(G)$ . Using the assumptions on  $(f_i)_{i \in I}$ , and the fact that  $\phi$  is continuous,

we have that

$$\begin{aligned} \|M_\phi(\lambda(s)e_i) - \phi(s)\lambda(s)e_i\|_\infty &= \left\| \int_G (\phi(st)f_i(t)\lambda(st) - \phi(s)f_i(t)\lambda(st)) dt \right\|_\infty \\ &\leq \int_G |\phi(st) - \phi(s)|f_i(t) dt \xrightarrow{i \rightarrow \infty} 0. \end{aligned}$$

This proves the result.

(2) For all  $i \in I$ , we have  $e_i^2 = \lambda(f_i * f_i)$ ,  $f_i * f_i \geq 0$ ,  $\int_G (f_i * f_i)(s) d(s) = 1$ , and  $\text{supp}(f_i * f_i) \subseteq \text{supp}(f_i) \cdot \text{supp}(f_i)$ . Hence using  $f_i * f_i$  instead of  $f_i$ , the convergence in (2) can be shown exactly in the same way as in (1).

(3) We argue as in (1). Let  $s \in G$ . Since  $e_i \rightarrow 1$  in the strong operator topology and  $(e_i)_{i \in I}$  is bounded,  $e_i\lambda(s)e_i \rightarrow \lambda(s)$  in the strong operator topology. Hence we only need to show that  $M_\phi(e_i\lambda(s)e_i) - \phi(s)e_i\lambda(s)e_i$  converges to 0 in  $VN(G)$ . We have that

$$\begin{aligned} &\|M_\phi(e_i\lambda(s)e_i) - \phi(s)e_i\lambda(s)e_i\|_\infty \\ &= \left\| \int_G \int_G \phi(rst)f_i(r)f_i(t)\lambda(rst) drdt - \int_G \int_G \phi(s)f_i(r)f_i(t)\lambda(rst) drdt \right\|_\infty \\ &\leq \int_G \int_G |\phi(rst) - \phi(s)|f_i(r)f_i(t) drdt \xrightarrow{i \rightarrow \infty} 0, \end{aligned}$$

again using the assumptions on  $(f_i)_{i \in I}$ , and the fact that  $\phi$  is continuous.  $\square$

*Proof of Theorem 3.7.*

(ii)  $\Rightarrow$  (i): We may assume that  $c = 1$  and that  $\phi = \psi$ , i.e.  $\phi$  is a continuous character. Any character is positive definite and maps  $e$  to 1. Hence according to [6, Proposition 4.2] and [28, Proposition 3.6],  $M_\phi$  is a bounded Fourier multiplier on  $VN(G)$ , with  $\|M_\phi: VN(G) \rightarrow VN(G)\| = \|M_\phi(1)\|_\infty = |\phi(e)| = 1$ . According to [3, Lemma 6.4 and Lemma 6.6],  $M_\phi$  is therefore a bounded Fourier multiplier on  $L^p(VN(G))$  and

$$\|M_\phi: L^p(VN(G)) \rightarrow L^p(VN(G))\| \leq \|M_\phi: VN(G) \rightarrow VN(G)\|.$$

Thus,

$$(3.6) \quad \|M_\phi: L^p(VN(G)) \rightarrow L^p(VN(G))\| \leq 1.$$

Since  $\phi^{-1}$  is also a continuous character, the above argument shows as well that

$$(3.7) \quad \|M_{\phi^{-1}}: L^p(VN(G)) \rightarrow L^p(VN(G))\| \leq 1.$$

For any  $f \in C_c(G) * C_c(G)$ , we have

$$M_{\phi^{-1}}M_\phi(\lambda(f)) = M_{\phi^{-1}}(\lambda(\phi f)) = \lambda(\phi^{-1}\phi f) = \lambda(f).$$

Similarly,  $M_\phi M_{\phi^{-1}}(\lambda(f)) = \lambda(f)$ . Hence  $M_\phi$  and  $M_{\phi^{-1}}$  are inverse to each other. It therefore follows from (3.6) and (3.7) that  $M_\phi: L^p(VN(G)) \rightarrow L^p(VN(G))$  is an isometry.

If  $p \neq 2$ , it follows from [36] (see also [22]) that  $M_\phi: L^p(VN(G)) \rightarrow L^p(VN(G))$  is separating. If  $p = 2$ , consider any  $1 < q \neq 2 < \infty$ . The above reasoning shows that  $M_\phi: L^q(VN(G)) \rightarrow L^q(VN(G))$  is separating. Applying Lemma 2.5, we deduce that the operator  $M_\phi: L^2(VN(G)) \rightarrow L^2(VN(G))$  is separating.

(i)  $\Rightarrow$  (ii): We assume that  $M_\phi: L^p(VN(G)) \rightarrow L^p(VN(G))$  is separating. By Lemma 2.5,  $\overline{M_\phi}: L^2(VN(G)) \rightarrow L^2(VN(G))$  is separating as well. Let  $(w, B, J)$  be its Yeadon triple. We may assume that  $M_\phi$  is non-zero. Then by Lemma 3.9(2),  $M_\phi: L^2(VN(G)) \rightarrow L^2(VN(G))$  has dense range. It then follows from Lemma 2.4 that  $w$  is a unitary and  $J(1) = 1$ . According to Remark 3.10,  $\phi$  is almost everywhere equal to a continuous function. Replacing  $\phi$  by this function, we may now assume that  $\phi \in C_b(G)$ .

For any  $s \in G$  we have, by Lemma 3.11 and Lemma 2.1(1),

$$\begin{aligned} \phi(s)(\lambda(s) + \lambda(s)) &= \lim_{j \rightarrow \infty} M_\phi(e_j \lambda(s) e_j + \lambda(s) e_j^2) \\ &= \lim_{j \rightarrow \infty} w B J(e_j \lambda(s) e_j + \lambda(s) e_j^2) \\ &= \lim_{j \rightarrow \infty} w B (J(e_j) J(\lambda(s) e_j) + J(\lambda(s) e_j) J(e_j)) \\ &= \lim_{j \rightarrow \infty} (M_\phi(e_j) J(\lambda(s) e_j) + M_\phi(\lambda(s) e_j) J(e_j)), \end{aligned}$$

where the limit is taken in the strong operator topology.

By Lemma 3.5,  $\lambda(s) e_j \rightarrow \lambda(s)$  and  $(\lambda(s) e_j)^* \rightarrow \lambda(s)^*$  strongly. Hence by Lemma 2.2(2),  $J(\lambda(s) e_j) \rightarrow J(\lambda(s))$  strongly. By Lemma 3.11(1),  $M_\phi(e_j) \rightarrow \phi(e)$  strongly. Since  $(M_\phi(e_j))_{j \in I}$  is bounded, we deduce that  $M_\phi(e_j) J(\lambda(s) e_j) \rightarrow \phi(e) J(\lambda(s))$  strongly. Similarly,  $M_\phi(\lambda(s) e_j) J(e_j) \rightarrow \phi(s) \lambda(s) J(1) = \phi(s) \lambda(s)$  strongly. It therefore follows from the previous calculation that  $\phi(s)(\lambda(s) + \lambda(s)) = \phi(e) J(\lambda(s)) + \phi(s) \lambda(s)$ , that is,  $\phi(s) \lambda(s) = \phi(e) J(\lambda(s))$ .

Since  $\phi$  is non-zero, this implies that  $\phi(e) \neq 0$ . Set  $\psi := \phi(e)^{-1} \phi$ . It follows from the above that

$$(3.8) \quad J(\lambda(s)) = \psi(s) \lambda(s), \quad s \in G.$$

We now show that  $\psi$  is a character. Let  $s, t \in G$  and recall that  $\lambda(st) = \lambda(s) \lambda(t)$ . On the one hand, we have that  $J(\lambda(st)) = \psi(st) \lambda(st)$ . On the other hand, we have that  $J(\lambda(s)) J(\lambda(t)) = \psi(s) \psi(t) \lambda(st)$ .

If  $st = ts$ , then  $\lambda(s) \lambda(t) = \lambda(t) \lambda(s)$ , hence by Lemma 2.1(2), we have that  $J(\lambda(st)) = J(\lambda(s) \lambda(t)) = J(\lambda(s)) J(\lambda(t))$ . Hence,  $\psi(st) = \psi(s) \psi(t)$ . Assume now that  $st \neq ts$ . By Lemma 2.1(1), we have

$$J(\lambda(s) \lambda(t) + \lambda(t) \lambda(s)) = J(\lambda(s)) J(\lambda(t)) + J(\lambda(t)) J(\lambda(s)).$$

Therefore,

$$\psi(st) \lambda(st) + \psi(ts) \lambda(ts) = \psi(s) \psi(t) \lambda(st) + \psi(t) \psi(s) \lambda(ts).$$

Since  $\lambda(st)$  and  $\lambda(ts)$  are linearly independent, the above identity implies that  $\psi(st) = \psi(s) \psi(t)$ . This proves that  $\psi$  is a character and therefore,  $\phi = \phi(e) \psi$  is a scalar multiple of a character, as requested.  $\square$

Let us now give a variant of Theorem 3.7 in the general case when  $G$  is not assumed to be  $\sigma$ -compact (see also Remark 3.16). We need the following lemma.

**Lemma 3.12.** *Let  $h_1, h_2: G \rightarrow \mathbb{C}$  be two locally measurable functions. The functions  $h_1$  and  $h_2$  are locally almost everywhere equal if and only if for any compact set  $K \subseteq G$ ,  $h_1|_K = h_2|_K$ , almost everywhere.*

*Proof.* It is enough to show that if  $E \subset G$  is locally Borel, then  $E$  is locally null if (and only if)  $E \cap K$  has measure 0 for any compact set  $K \subseteq G$ . Assume this property. By [8, Proposition 2.4],  $G$  has an open, closed and  $\sigma$ -compact subgroup,  $G_0$ . Let  $Y$  be a subset of  $G$  that contains exactly one element of each of the left cosets of  $G_0$ . Set  $E_y := E \cap yG_0$  for any  $y \in Y$ . Since  $G_0$  is  $\sigma$ -compact,  $yG_0$  is  $\sigma$ -compact as well. It then follows that  $\mu(E_y) = 0$ , for all  $y \in Y$ . Recall from the end of [8, Section 2.3] that  $E$  is locally null if and only if  $\mu(E_y) = 0$ , for every  $y \in Y$ . Hence,  $E$  is locally null.  $\square$

**Corollary 3.13.** *Let  $G$  be locally compact unimodular. Let  $1 \leq p < \infty$  and let  $\phi \in L^\infty(G)$ . The following are equivalent.*

- (i) *The mapping  $M_\phi$  is a bounded Fourier multiplier on  $L^p(VN(G))$ , and the operator  $M_\phi: L^p(VN(G)) \rightarrow L^p(VN(G))$  is separating.*
- (ii) *There exist a constant  $c_0 \in \mathbb{C}$  and a continuous character  $\psi: G \rightarrow \mathbb{T}$  such that  $\phi = c_0\psi$  locally almost everywhere.*

*Proof.* The proof of the implication “(ii)  $\Rightarrow$  (i)” in Theorem 3.7 applies to the non  $\sigma$ -compact case, so we only need to prove that (i) implies (ii).

Assume that  $M_\phi: L^p(VN(G)) \rightarrow L^p(VN(G))$  is separating. As in the proof of Theorem 3.7, we may assume that  $p = 2$  and that  $M_\phi$  is non-zero. We claim that

$$(3.9) \quad \exists \varphi \in C_b(G) \text{ such that } \varphi = \phi, \text{ locally almost everywhere.}$$

To prove this, first note that we may assume that the net  $(f_i)_{i \in I}$  defined prior to Lemma 3.5 has the following property: there exists a compact neighbourhood  $K_0$  of the unit  $e$  such that for all  $i$ ,  $\text{supp}(f_i) \subseteq K_0$ . Let  $L \subseteq G$  be compact. Let

$$K = K_0 L K_0 K_0 = \{strq : (s, t, r, q) \in K_0 \times L \times K_0 \times K_0\}.$$

This is a compact set hence by Lemma 3.9(3) there is a continuous function  $\Phi: K \rightarrow \mathbb{C}$  such that  $\Phi = \phi|_K$  almost everywhere. The proof of Lemma 3.11 shows that for all  $s \in L$ , we have the following convergences in the strong operator topology:

$$(3.10) \quad M_\phi(\lambda(s)e_i) \rightarrow \Phi(s)\lambda(s), \quad M_\phi(\lambda(s)e_i^2) \rightarrow \Phi(s)\lambda(s), \quad \text{and} \quad M_\phi(e_i\lambda(s)e_i) \rightarrow \Phi(s)\lambda(s).$$

In particular, (take  $L = \{e\}$ ) we obtain the existence of  $c_0 \in \mathbb{C}$  such that

$$M_\phi(e_i) \rightarrow c_0,$$

in the strong operator topology.

Let  $(w, B, J)$  be the Yeadon triple of  $M_\phi$ . The argument in the proof of Theorem 3.7 and the convergence properties (3.10) show that for any  $L, K, \Phi$  as above we have

$$(3.11) \quad c_0 J(\lambda(s)) = \Phi(s)\lambda(s), \quad \text{for all } s \in L.$$

By Lemma 3.9(1), this implies that  $c_0 \neq 0$ .

It follows from the above that for all  $s \in G$ ,  $J(\lambda(s))$  is proportional to  $\lambda(s)$ . We therefore have a necessarily unique

$$F: G \rightarrow \mathbb{C}; \quad J(\lambda(s)) = F(s)\lambda(s), \quad \text{for all } s \in G.$$

Moreover, for any  $L, K, \Phi$  as above, we have

$$F|_L = c_0^{-1}\Phi|_L.$$

This implies that  $F$  is continuous. To prove this, fix  $s_0 \in G$  and apply the above with a compact neighbourhood  $L$  of  $s_0$ . Then the continuity of  $\Phi: K \rightarrow \mathbb{C}$  implies the continuity of  $F|_L$ , and hence the continuity of  $F$  at  $s_0$ .

Set  $\varphi := c_0 F$ . Again for  $L, K, \Phi$  as above we obtain that  $\varphi|_K = \phi|_K$ , almost everywhere. By Lemma 3.9, this implies that  $\varphi = \phi$ , locally almost everywhere. Hence (3.9) is proved.

Since  $M_\varphi = M_\phi$ , the argument at the end of the proof of Theorem 3.7 shows that  $\psi := c_0^{-1}\varphi$  is a character.  $\square$

**Remark 3.14.** It follows from the proof of the implication “(ii)  $\Rightarrow$  (i)” in Theorem 3.7 that for any continuous character  $\psi: G \rightarrow \mathbb{T}$  and any  $1 \leq p < \infty$ , the Fourier multiplier  $M_\psi: L^p(VN(G)) \rightarrow L^p(VN(G))$  is an onto isometry. It therefore follows from Corollary 3.13 that if  $\phi \in L^\infty(G) \setminus \{0\}$  is such that  $M_\phi: L^p(VN(G)) \rightarrow L^p(VN(G))$  is bounded and separating, then  $\|M_\phi\|^{-1}M_\phi$  is an onto isometry.

**Corollary 3.15.** *Let  $1 \leq p \neq 2 < \infty$  and let  $\phi \in L^\infty(G)$ . The following are equivalent.*

- (i) *The mapping  $M_\phi$  is a bounded Fourier multiplier on  $L^p(VN(G))$ , and the operator  $M_\phi: L^p(VN(G)) \rightarrow L^p(VN(G))$  is an isometry.*
- (ii) *There exists  $\delta \in \mathbb{T}$  such that  $\delta\phi$  is locally almost everywhere equal to a continuous character.*

*Proof.* It follows from the proof of the implication “(ii)  $\Rightarrow$  (i)” in Theorem 3.7 that for any continuous character  $\psi: G \rightarrow \mathbb{T}$ ,  $M_\psi: L^p(VN(G)) \rightarrow L^p(VN(G))$  is an isometry. Thus, (ii) implies (i). Conversely, assume (i). Since  $p \neq 2$ , any isometry on  $L^p(VN(G))$  is separating, by [36] (see also [22]). Hence by Corollary 3.13, there exist  $c \in \mathbb{C}$  and a continuous character  $\psi: G \rightarrow \mathbb{T}$  such that  $\phi = c\psi$  locally almost everywhere. Then  $M_\phi = cM_\psi$ , hence  $\|M_\phi\| = |c|\|M_\psi\|$ , hence  $|c| = 1$ . This yields (ii), with  $\delta = c^{-1}$ .  $\square$

Note that Corollary 3.15 is not true in the case  $p = 2$ . Indeed, let  $\phi \in L^\infty(G)$ . It follows from the discussion following (3.2) that  $M_\phi: L^2(VN(G)) \rightarrow L^2(VN(G))$  is an isometry if and only if  $|\phi| = 1$  locally almost everywhere. Yet in general, plenty of these isometric Fourier multipliers are not separating. See Section 4 for more on this.

**Remark 3.16.** Corollary 3.13 may be wrong if one replaces “locally almost everywhere” by “almost everywhere” in (ii). Indeed as in [8, Section 2.3], take  $G = \mathbb{R} \times \mathbb{R}_{\text{disc}}$ , where the second factor is equipped with the discrete topology. Consider  $Y = \{0\} \times \mathbb{R}_{\text{disc}}$  which is a closed subset of  $G$ , hence Borel, and set  $\phi = \chi_Y$ . For any compact set  $K \subseteq G$ , we have  $\mu(K \cap Y) = 0$ , by [8, Proposition 2.22]. Hence  $\phi|_K = 0$  almost everywhere. By Lemma 3.12, this implies that  $\phi = 0$  locally almost everywhere. Thus  $\phi$  satisfies the properties of Corollary 3.13, with  $M_\phi = 0$ .

However by [8, Proposition 2.22] again,  $\{s \in G : \phi(s) \neq 0\} = Y$  has infinite Haar measure, hence  $\phi$  is not almost everywhere equal to 0. Consequently,  $\phi$  cannot be almost everywhere equal to a constant times a continuous character.

**Remark 3.17.** In the case when  $G$  is discrete, continuity on  $G$  is automatic and two locally almost everywhere equal functions are equal. Therefore, in the statement of Corollary 3.13, we can replace part (ii) by the following slightly simpler statement: there exist  $c \in \mathbb{C}$  and a character  $\psi: G \rightarrow \mathbb{C}$  such that  $\phi = c\psi$ .



**Remark 3.18.** De Cannière and Haagerup [6] defined Fourier multipliers on  $VN(G)$ , including the case when  $G$  is not unimodular. Let  $\phi \in C_b(G)$  and assume that  $\phi$  induces a Fourier multiplier  $M_\phi : VN(G) \rightarrow VN(G)$  in the sense of [6, Proposition 1.2]. Assume that  $M_\phi$  is separating. If  $G$  is unimodular, then  $M_\phi : L^2(VN(G)) \rightarrow L^2(VN(G))$  is separating by [22, Lemma 3.9]. Hence, by Corollary 3.13,  $\phi$  is a multiple of a character.

However, in the general case of a non-unimodular locally compact group, the description of separating Fourier multipliers on  $VN(G)$  is open.

**Remark 3.19.** Let  $\Gamma$  be a locally compact abelian group. Let  $G = \widehat{\Gamma}$  be its dual group and recall that  $L^\infty(\Gamma) = VN(G)$ . Let  $1 \leq p < \infty$ . For any  $u \in \Gamma$ , let  $\tau_u : L^p(\Gamma) \rightarrow L^p(\Gamma)$  be the translation operator defined by  $\tau_u(f) = f(\cdot - u)$ , for all  $f \in L^p(\Gamma)$ . Note that if we regard  $u \in \Gamma$  as a character  $u : G \rightarrow \mathbb{T}$ , then the associated Fourier multiplier  $M_u : L^p(\Gamma) \rightarrow L^p(\Gamma)$  coincides with  $\tau_u$ .

Let  $T : L^p(\Gamma) \rightarrow L^p(\Gamma)$  be a bounded operator. Then  $T$  commutes with translations, that is,  $T \circ \tau_u = \tau_u \circ T$  for all  $u \in \Gamma$ , if and only if  $T$  is a Fourier multiplier (see e.g. [21, Chapter 4]). Hence Corollary 3.15 implies the following:

(\*) *If  $p \neq 2$ , an isometry  $T : L^p(\Gamma) \rightarrow L^p(\Gamma)$  commutes with translations if and only if there exists  $c \in \mathbb{T}$  and  $u \in \Gamma$  such that  $T = c\tau_u$ .*

This statement is a classical result due to Parrott [27] and Strichartz [33] and Corollary 3.15 should be regarded as a generalization of the latter.

We note that the two papers [27, 33] show (\*) in the case when  $\Gamma$  is not necessarily abelian. If  $\Gamma$  is non-abelian, the statement (\*) is not related to Corollary 3.15.

#### 4. COMPLETELY POSITIVE AND COMPLETELY ISOMETRIC FOURIER MULTIPLIERS

In this section, we complement the characterizations of separating and isometric  $L^p$ -Fourier multipliers from Section 3 with further information. Throughout this section, we assume that  $G$  is a unimodular locally compact group.

Let  $\mathcal{M}$  be a semifinite von Neumann algebra equipped with a normal semifinite faithful trace  $\tau$ . For any  $n \geq 1$ , we equip  $M_n(\mathcal{M})$  with  $\text{tr}_n \otimes \tau$ , where  $\text{tr}_n$  is the usual trace on  $M_n$ . For any  $1 \leq p \leq \infty$ , the resulting noncommutative  $L^p$ -space  $L^p(M_n(\mathcal{M}))$  can be naturally identified (at the algebraic level) with the space of all  $n \times n$  matrices  $[x_{ij}]_{1 \leq i, j \leq n}$  with entries  $x_{ij}$  belonging to  $L^p(\mathcal{M})$ .

Let  $T : L^p(\mathcal{M}) \rightarrow L^p(\mathcal{M})$  be a bounded operator. For any  $n \geq 1$ , let  $T_n : L^p(M_n(\mathcal{M})) \rightarrow L^p(M_n(\mathcal{M}))$  be defined by  $T_n([x_{ij}]) = [T(x_{ij})]$ , for all  $[x_{ij}]_{1 \leq i, j \leq n}$  in  $L^p(M_n(\mathcal{M}))$ . Following usual terminology, we say that  $T$  is completely positive if  $T_n$  is positive for all  $n \geq 1$ . Likewise, we say that  $T$  is a complete contraction if  $\|T_n\| \leq 1$  for all  $n \geq 1$  and that  $T$  is a complete isometry if  $T_n$  is an isometry for all  $n \geq 1$ .

Let  $\psi : G \rightarrow \mathbb{T}$  be a continuous character. Then  $\psi$  is positive definite hence by [6, Proposition 4.2], the Fourier multiplier  $M_\psi : VN(G) \rightarrow VN(G)$  is completely positive. The proof of the implication “(ii)  $\Rightarrow$  (i)” in Theorem 3.7 actually shows that  $M_\psi : L^p(VN(G)) \rightarrow L^p(VN(G))$  is a complete contraction for all  $1 \leq p < \infty$ , and then that  $M_\psi : L^p(VN(G)) \rightarrow L^p(VN(G))$  is a complete isometry for all  $1 \leq p < \infty$ . As a consequence of Corollary 3.13, we therefore obtain the following complement to Remark 3.14.

**Corollary 4.1.** *Let  $1 \leq p < \infty$  and let  $\phi \in L^\infty(G) \setminus \{0\}$ . Assume that  $M_\phi: L^p(VN(G)) \rightarrow L^p(VN(G))$  is bounded and separating. Then  $\|M_\phi\|^{-1}M_\phi$  is a complete isometry.*

**Remark 4.2.** Let  $1 \leq p < \infty$  and let  $\phi \in L^\infty(G) \setminus \{0\}$  such that  $M_\phi: L^p(VN(G)) \rightarrow L^p(VN(G))$  is bounded and separating. Let  $(w, B, J)$  be the Yeadon triple of the latter operator. According to Corollary 4.1 and [15, Theorem 3.2],  $J$  is a  $*$ -homomorphism.

We can make this statement more precise, as follows. Applying Corollary 3.13, let  $c \in \mathbb{C}$  let  $\psi: G \rightarrow \mathbb{T}$  be the continuous character such that  $\phi = c\psi$  locally almost everywhere. Then  $J: VN(G) \rightarrow VN(G)$  is the  $L^\infty$ -Fourier multiplier associated with  $\psi$ ,  $c = \|M_\phi\|$ ,  $B = |c| \cdot 1$  and  $w = |c|^{-1} \cdot 1$ . The easy verification is left to the reader.

**Lemma 4.3.** *Let  $1 \leq p < \infty$  and let  $\phi \in L^\infty(G)$ . If  $M_\phi$  is a bounded Fourier multiplier on  $L^p(VN(G))$  and  $M_\phi: L^p(VN(G)) \rightarrow L^p(VN(G))$  is an isometry, then  $|\phi| = 1$  locally almost everywhere.*

*Proof.* The case  $p = 2$  follows from the paragraph preceding Remark 3.16. Assume that  $1 \leq p \neq 2 < \infty$ . By Corollary 3.13, there exist a constant  $c \in \mathbb{C}$  and a continuous character  $\psi: G \rightarrow \mathbb{T}$  such that  $\phi = c\psi$  locally almost everywhere. We noticed before Corollary 4.1 that  $M_\psi$  is a complete isometry. Since  $cM_\psi = M_\phi$  is also an isometry, we must have that  $|c| = 1$ . Hence,  $|\phi| = |c\psi| = |\psi| = 1$  locally almost everywhere.  $\square$

We have the following partial converse of Lemma 4.3.

**Proposition 4.4.** *Let  $\phi \in L^\infty(G)$  such that  $|\phi| = 1$  locally almost everywhere, let  $1 \leq p < \infty$  and assume that  $M_\phi: L^p(VN(G)) \rightarrow L^p(VN(G))$  is a bounded Fourier multiplier. If  $M_\phi$  is completely positive, then  $\phi$  coincides locally almost everywhere with a continuous character  $\psi: G \rightarrow \mathbb{T}$ .*

*Proof.* Since  $M_\phi$  is completely positive, it follows from [3, Proposition 6.11] that  $\phi$  is locally almost everywhere equal to a continuous positive definite function. Hence, we may assume that  $\phi$  is continuous (and positive definite). By [35, Proposition VII.3.21], there exist a unitary representation  $\pi: G \rightarrow B(H)$  on a Hilbert space  $H$  and a vector  $\xi \in H$  such that

$$(4.1) \quad \phi(s) = \langle \pi(s)\xi, \xi \rangle, \quad s \in G.$$

Since  $|\phi| = 1$ , we have  $\phi(e) = 1$ . Hence it follows from (4.1) that  $1 = \phi(e) = \langle \pi(e)\xi, \xi \rangle = \|\xi\|_H^2$ . Given  $s \in G$ , applying the Cauchy-Schwarz inequality we obtain

$$1 = |\phi(s)| = |\langle \pi(s)\xi, \xi \rangle| \leq \|\pi(s)\xi\|_H \|\xi\|_H = \|\xi\|_H^2 = 1.$$

It follows from the equality condition in the Cauchy-Schwarz inequality that there is  $\psi(s) \in \mathbb{C}$  such that  $\pi(s)\xi = \psi(s)\xi$ .

Now, for any  $s, t \in G$ , on the one hand

$$\pi(st)\xi = \psi(st)\xi,$$

and on the other hand,

$$\pi(s)\pi(t)\xi = \pi(s)\psi(t)\xi = \psi(s)\psi(t)\xi.$$

Hence,  $\psi(st) = \psi(s)\psi(t)$ . Finally,  $\phi(s) = \langle \pi(s)\xi, \xi \rangle = \psi(s)\|\xi\|_H^2 = \psi(s)$  for all  $s \in G$ . Therefore,  $\phi = \psi$  is a character.  $\square$

When  $p = 1$  and  $G$  is assumed to be amenable we can change the assumption of complete positivity in Proposition 4.4 into mere contractivity.

**Proposition 4.5.** *Let  $G$  be an amenable unimodular locally compact group. Let  $\phi \in L^\infty(G)$  and assume that  $M_\phi: L^1(VN(G)) \rightarrow L^1(VN(G))$  is a contractive Fourier multiplier. The following are equivalent.*

- (i)  $M_\phi$  is an isometry.
- (ii)  $|\phi| = 1$  locally almost everywhere.
- (iii) There exist  $c \in \mathbb{T}$  and a continuous character  $\psi: G \rightarrow \mathbb{T}$  such that  $\phi = c\psi$  locally almost everywhere.

*Proof.* The implication “(i)  $\Rightarrow$  (ii)” is established in Lemma 4.3. The implication “(iii)  $\Rightarrow$  (i)” was already discussed several times (see, for example, Remark 3.14). We now show that “(ii)  $\Rightarrow$  (iii)”. Since  $M_\phi$  is a bounded Fourier multiplier on  $L^1(VN(G))$ , we may assume that  $\phi$  is continuous, by Lemma 3.6. Further since  $G$  is amenable, symbols of Fourier multipliers on  $VN(G)$  coincide with the Fourier-Stieltjes algebra of  $G$ . This classical result is mentioned in [6, p. 456], see also [12, Theorem 1]. Hence by [7, Lemma 2.14], there exist a unitary representation  $\pi: G \rightarrow B(H)$  on a Hilbert space  $H$  and vectors  $\xi, \eta$  in  $H$  such that

$$\phi(s) = \langle \pi(s)\xi, \eta \rangle, \quad s \in G, \quad \text{and} \quad \|\xi\|_H = \|\eta\|_H = 1.$$

Assume (ii). Multiplying  $\phi$  by  $\overline{\phi(e)}$ , we may assume that  $\phi(e) = 1$ . This implies that  $1 = \langle \pi(e)\xi, \eta \rangle = \langle \xi, \eta \rangle$ . Since  $\|\xi\|_H = \|\eta\|_H = 1$ , we deduce that  $\eta = \xi$ . Thus,  $\phi$  satisfies (4.1). The proof of Proposition 4.4 therefore shows that  $\phi$  is a character.  $\square$

## 5. A CHARACTERIZATION OF SEPARATING SCHUR MULTIPLIERS

Let  $(\Omega, \Sigma, \mu)$  be a  $\sigma$ -finite measure space. For any  $f \in L^2(\Omega^2)$ , let  $S_f: L^2(\Omega) \rightarrow L^2(\Omega)$  be the bounded operator defined by

$$[S_f(h)](s) = \int_{\Omega} f(s, t)h(t) dt, \quad h \in L^2(\Omega).$$

We recall that  $S_f \in S^2(L^2(\Omega))$  and that the mapping  $f \mapsto S_f$  is a unitary operator from  $L^2(\Omega^2)$  onto  $S^2(L^2(\Omega))$ , see e.g. [30, Theorem VI. 23].

Let  $\phi \in L^\infty(\Omega^2)$ . According to the above identification  $L^2(\Omega^2) \simeq S^2(L^2(\Omega))$ , one may define a bounded operator  $T_\phi: S^2(L^2(\Omega)) \rightarrow S^2(L^2(\Omega))$  by

$$(5.1) \quad T_\phi(S_f) = S_{\phi f}, \quad f \in L^2(\Omega^2).$$

Moreover the norm of this operator is equal to  $\|\phi\|_\infty$ . The operator  $T_\phi$  is called a Schur multiplier.

Let  $1 \leq p < \infty$ . We say that  $T_\phi$  is a bounded Schur multiplier on the Schatten space  $S^p(L^2(\Omega))$  if the restriction of  $T_\phi$  to  $S^p(L^2(\Omega)) \cap S^2(L^2(\Omega))$  extends to a bounded operator from  $S^p(L^2(\Omega))$  into itself. Schur multipliers as defined in this section go back at least to Haagerup [10] and Spronk [31].

For any  $\alpha \in L^\infty(\Omega)$ , we let  $\text{Mult}_\alpha \in B(L^2(\Omega))$  be the multiplication operator taking  $h$  to  $\alpha h$  for all  $h \in L^2(\Omega)$ . Then we let

$$\mathcal{D}(\Omega) = \{\text{Mult}_\alpha : \alpha \in L^\infty(\Omega)\}.$$

This is von Neumann sub-algebra of  $B(L^2(\Omega))$ , which is isomorphic (as a von Neumann algebra) to  $L^\infty(\Omega)$ . We will use the classical fact that

$$(5.2) \quad \mathcal{D}(\Omega)' = \mathcal{D}(\Omega),$$

where  $\mathcal{D}(\Omega)'$  stands for the commutant of  $\mathcal{D}(\Omega)$ . In other words, a bounded operator  $V: L^2(\Omega) \rightarrow L^2(\Omega)$  belongs to  $\mathcal{D}(\Omega)$  if and only if  $V \circ \text{Mult}_\alpha = \text{Mult}_\alpha \circ V$  for all  $\alpha \in L^\infty(\Omega)$ .

We note that for any  $\alpha \in L^\infty(\Omega)$ , the mapping  $x \mapsto \text{Mult}_\alpha \circ x$  is a Schur multiplier. Indeed it coincides with  $T_\phi$ , where the symbol  $\phi \in L^\infty(\Omega^2)$  is given by  $\phi(s, t) = \alpha(s)$ . Likewise, for any  $\beta \in L^\infty(\Omega)$ , the mapping  $x \mapsto x \circ \text{Mult}_\beta$  is a Schur multiplier, with symbol  $\phi$  given by  $\phi(s, t) = \beta(t)$ .

**Theorem 5.1.** *Let  $\phi \in L^\infty(\Omega^2)$  and let  $1 \leq p < \infty$ . The following are equivalent.*

(i) *The mapping  $T_\phi$  is a bounded Schur multiplier on  $S^p(L^2(\Omega))$ , and the resulting operator  $T_\phi: S^p(L^2(\Omega)) \rightarrow S^p(L^2(\Omega))$  is separating.*

(ii) *There exist a constant  $c \in \mathbb{C}$  and two unitaries  $\alpha, \beta \in L^\infty(\Omega)$  such that*

$$\phi(s, t) = c \alpha(s) \beta(t) \quad \text{for almost every } (s, t) \in \Omega^2.$$

(iii) *There exist a constant  $c \in \mathbb{C}$  and two unitaries  $\alpha, \beta \in L^\infty(\Omega)$  such that*

$$(5.3) \quad T_\phi(x) = c \text{Mult}_\alpha \circ x \circ \text{Mult}_\beta, \quad x \in S^2(L^2(\Omega)).$$

*Proof.*

(ii)  $\Rightarrow$  (iii): Let  $c, \alpha, \beta$  as in (ii) and let  $x \in S^2(L^2(\Omega))$ . Let  $f \in L^2(\Omega^2)$  such that  $x = S_f$ . Then for all  $h \in L^2(\Omega)$ , we have

$$[x \circ \text{Mult}_\beta(h)](s) = \int_\Omega f(s, t) \beta(t) h(t) d\mu(t),$$

hence

$$\begin{aligned} [c \text{Mult}_\alpha \circ x \circ \text{Mult}_\beta(h)](s) &= c \alpha(s) \int_\Omega f(s, t) \beta(t) h(t) d\mu(t) \\ &= \int_\Omega \phi(s, t) f(s, t) h(t) d\mu(t), \end{aligned}$$

for a.e.  $s \in \Omega$ . This shows (5.3).

(iii)  $\Rightarrow$  (i): Assume (5.3) for some unitaries  $\alpha, \beta \in L^\infty(\Omega)$ . It is plain that  $T_\phi$  extends to a bounded operator on  $S^p(L^2(\Omega))$  and that the identity (5.3) holds true on  $S^p(L^2(\Omega))$ .

Let  $x, y \in S^p(L^2(\Omega))$  such that  $x^*y = xy^* = 0$ . Then

$$(\text{Mult}_\alpha \circ x \circ \text{Mult}_\beta)^* (\text{Mult}_\alpha \circ y \circ \text{Mult}_\beta) = \text{Mult}_\beta^* \circ x^* \circ \text{Mult}_\alpha^* \text{Mult}_\alpha \circ y \circ \text{Mult}_\beta.$$

Since  $\alpha$  is a unitary of  $L^\infty(\Omega)$ , the operator  $\text{Mult}_\alpha$  is a unitary of  $B(L^2(\Omega))$ , hence the right hand-side of the above equality is equal to  $\text{Mult}_\beta^* \circ x^*y \circ \text{Mult}_\beta$ , hence to 0. Thus  $T_\phi(x)^*T_\phi(y) = 0$ . Likewise,  $T_\phi(x)T_\phi(y)^* = 0$ . This shows that  $T_\phi: S^p(L^2(\Omega)) \rightarrow S^p(L^2(\Omega))$  is separating.

(i)  $\Rightarrow$  (ii): For convenience we let  $H = L^2(\Omega)$  throughout this proof. Owing to Lemma 2.5, we may suppose that  $p = 2$ . We let  $(w, B, J)$  denote the Yeadon triple of the separating map  $T_\phi: S^2(H) \rightarrow S^2(H)$ .

We may assume that  $T_\phi$  is non-zero. Since  $B(H)$  is a factor, it follows from [24, Lemma 4.3] that  $T_\phi$  is 1-1. Applying the definition of  $T_\phi$ , see (5.1), this implies that  $\phi \neq 0$  almost everywhere. Applying this definition again, we obtain that  $T_\phi$  has dense range. By Lemma 2.4, we deduce that  $w$  is a unitary and that  $J(1) = 1$ .

Let  $w^*T_\phi$  denote the operator on  $S^2(H)$  taking any  $x \in S^2(H)$  to  $w^*T_\phi(x)$ . According to [32, Theorem 3.3] (see also [11, Corollary 7.4.9.]), there exists a projection  $q \in B(H)$  such that  $x \mapsto qJ(x)$  is a  $*$ -homomorphism and  $x \mapsto (1 - q)J(x)$  is an anti- $*$ -homomorphism. As explained in [23, Remark 4.3], this implies that  $w^*T_\phi$  is valued in

$$L^2(qB(H)q) \overset{2}{\oplus} L^2((1 - q)B(H)(1 - q)) \subset S^2(H).$$

Since  $T_\phi$  has dense range and  $w$  is a unitary,  $w^*T_\phi$  has dense range as well. This forces  $q$  to be equal either to 0 or 1. Thus  $J: B(H) \rightarrow B(H)$  is either a  $*$ -homomorphism or an anti- $*$ -homomorphism.

Assume first that  $J$  is a  $*$ -homomorphism. Recall that  $J$  is normal. According to the description of normal  $*$ -homomorphisms (see e.g. [34, Theorem IV.5.5]), there exist a Hilbert space  $E$  and a unitary  $u: H \rightarrow H \overset{2}{\otimes} E$  such that

$$J(a) = u^*(a \otimes I_E)u, \quad a \in B(H).$$

Here  $H \overset{2}{\otimes} E$  stands for the Hilbertian tensor product of  $H$  and  $E$  and we regard

$$B(H) \otimes B(E) \subset B(H \overset{2}{\otimes} E)$$

in the usual way. For all  $x \in S^2(H)$ , we have  $T_\phi(x) = wBJ(x)$  hence  $w^*T_\phi(x) = BJ(x)$ . This implies that

$$(5.4) \quad u(w^*T_\phi(x))u^* = uBu^*(x \otimes I_E), \quad x \in S^2(H).$$

Since  $B$  commutes with the range of  $J$ , the operator  $uBu^*$  commutes with  $x \otimes I_E$  for all  $x \in S^2(H)$ . Consequently,  $uBu^* = I_H \otimes c$  for some positive operator  $c$  acting on  $E$ . Then it follows from (5.4) that  $c \in S^2(E)$  and that

$$w^*T_\phi(x) = u^*(x \otimes c)u, \quad x \in S^2(H).$$

Now recall that  $w^*T_\phi$  has dense range. The above identity therefore implies that  $E = \mathbb{C}$ . Thus  $c \in \mathbb{C} \setminus \{0\}$ ,  $u$  is a unitary of  $B(H)$  and  $w^*T_\phi(x) = cu^*xu$  for all  $x \in S^2(H)$ . Let  $v = wu^*$ . This is a unitary of  $B(H)$  and we obtain that

$$T_\phi(x) = cvxu, \quad x \in S^2(H).$$

Let  $(\cdot | \cdot)$  denote the inner product on  $H$ . For any  $g, h \in H$ , let  $g \otimes \bar{h} \in B(H)$  denote the rank one operator taking any  $\xi \in H$  to  $(\xi | h)g$ . Then  $v(g \otimes \bar{h})u = v(g) \otimes \overline{u^*(h)}$ .

Schur multipliers commute with each other, hence for any  $\delta \in \mathcal{D}(\Omega)$ , we have

$$T_\phi(\delta x) = \delta T_\phi(x), \quad x \in S^2(H).$$

Thus  $v\delta xu = \delta vxu$  for all  $\delta \in \mathcal{D}(\Omega)$  and all  $x \in S^2(H)$ . Applying this with  $x = g \otimes \bar{h}$  and using the identities  $\delta(g \otimes \bar{h}) = \delta(g) \otimes \bar{h}$  and  $v(g \otimes \bar{h}) = v(g) \otimes \bar{h}$ , we deduce that

$v(\delta(g) \otimes \bar{h})u = \delta(v(g) \otimes \bar{h})u$  and hence

$$v\delta(g) \otimes \overline{u^*(h)} = \delta v(g) \otimes \overline{u^*(h)}, \quad g, h \in H, \delta \in \mathcal{D}(\Omega).$$

Since  $u^* \neq 0$ , this implies that  $v\delta = \delta v$  for all  $\delta \in \mathcal{D}(\Omega)$ . Thus  $v$  commutes with  $\mathcal{D}(\Omega)$ . According to (5.2), this implies that  $v \in \mathcal{D}(\Omega)$ . Thus there exists a unitary  $\alpha \in L^\infty(\Omega)$  such that  $v = \text{Mult}_\alpha$ . Likewise there exists a unitary  $\beta \in L^\infty(\Omega)$  such that  $u = \text{Mult}_\beta$ . We therefore obtain the identity (5.3), from which (ii) follows at once.

Assume now that  $J$  is an anti  $*$ -homomorphism. For any  $f \in L^2(\Omega^2)$ , let  $\tilde{f} \in L^2(\Omega^2)$  be defined by  $\tilde{f}(s, t) = f(t, s)$ , for a.e.  $(s, t) \in \Omega^2$ . Next, if  $x = S_f$ , set  ${}^t x = S_{\tilde{f}}$ . It is clear that the mapping  $x \mapsto {}^t x$  on  $S^2(L^2(\Omega))$  extends to a normal anti  $*$ -homomorphism

$$\rho: B(H) \longrightarrow B(H).$$

This mapping is an analog of the transposition map on matrices. Obviously, the composition map  $J \circ \rho: B(H) \rightarrow B(H)$  is a normal  $*$ -homomorphism. Now arguing as in the  $*$ -homomorphism case, we obtain the existence of a constant  $c \in \mathbb{C} \setminus \{0\}$  and of two unitaries  $\alpha, \beta \in L^\infty(\Omega)$  such that

$$T_\phi(x) = c \text{Mult}_\alpha \circ {}^t x \circ \text{Mult}_\beta, \quad x \in S^2(L^2(\Omega)).$$

Since  $\alpha, \beta$  are unitaries,  $\text{Mult}_\alpha$  and  $\text{Mult}_\beta$  are unitaries as well and we have  $\text{Mult}_\alpha^{-1} = \text{Mult}_{\bar{\alpha}}$  and  $\text{Mult}_\beta^{-1} = \text{Mult}_{\bar{\beta}}$ . Writing  ${}^t x = c^{-1} \text{Mult}_{\bar{\alpha}} \circ T_\phi(x) \circ \text{Mult}_{\bar{\beta}}$ , we therefore deduce that  $x \mapsto {}^t x$  is a Schur multiplier.

Let us show that this is impossible, except if  $L^2(\Omega)$  has dimension 1. If  $x \mapsto {}^t x$  is a Schur multiplier, then there exists  $\phi_0 \in L^\infty(\Omega^2)$  such that

$$(5.5) \quad \phi_0(s, t)g(s)h(t) = h(s)g(t) \quad a.e.-(s, t) \in \Omega^2,$$

for all  $g, h \in L^2(\Omega)$ . If  $L^2(\Omega)$  has dimension  $\geq 2$ , then there exist  $F_1, F_2 \in \Sigma$  such that  $0 < \mu(F_1) < \infty$ ,  $0 < \mu(F_2) < \infty$  and  $F_1 \cap F_2 = \emptyset$ . The indicator functions  $g = \chi_{F_1}$  and  $h = \chi_{F_2}$  belong to  $L^2(\Omega)$ . Applying (5.5) to these functions, we obtain that  $h(s)g(t) = 0$  for almost every  $(s, t) \in F_2 \times F_1$ . Since  $h(s)g(t) = 1$  for  $(s, t) \in F_2 \times F_1$  and

$$(\mu \otimes \mu)(F_2 \times F_1) = \mu(F_2)\mu(F_1) > 0,$$

we get a contradiction.

Now if we are in the trivial case when  $L^2(\Omega)$  has dimension 1, then (ii) holds true.  $\square$

**Remark 5.2.** Let  $\phi \in L^\infty(\Omega^2)$ , let  $1 \leq p \neq 2 < \infty$  and assume that  $T_\phi$  is a bounded Schur multiplier on  $S^p(L^2(\Omega))$ . It follows from Theorem 5.1 and [36] that if  $T$  is an isometry, then there exist two unitaries  $\alpha, \beta \in L^\infty(\Omega)$  such that  $\phi(s, t) = \alpha(s)\beta(t)$  for a.e.  $(s, t) \in \Omega^2$ . It is clear that the converse is true. For the discrete case (see the following remark), this has been proved in [1].

**Remark 5.3.** Let  $I$  be an index set and let  $(e_i)_{i \in I}$  be the standard basis of  $\ell_I^2$ . Any  $x \in B(\ell_I^2)$  can be represented by a matrix  $[x_{ij}]_{i, j \in I}$  defined by  $x_{ij} = (x(e_j)|e_i)$  for all  $i, j \in I$ . Of course any finitely supported matrix  $[x_{ij}]_{i, j \in I}$  represents an element of  $B(H)$  (actually a finite rank one), and we let  $\|[x_{ij}]\|_p$  denote the  $S^p(\ell_I^2)$ -norm of this element.

Let  $\mathbf{m} = \{m_{ij}\}_{(i,j) \in I^2}$  be a bounded family of complex numbers. If we apply the definitions of this section to  $\Omega = I$  equipped with the counting measure, the Schur multiplier  $T_{\mathbf{m}}$  is defined on finitely supported matrices by

$$T_{\mathbf{m}}([x_{ij}]) = [m_{ij}x_{ij}].$$

It therefore follows from Remark 5.2 that the following are equivalent:

- (i) There exists  $1 \leq p \neq 2 < \infty$  such that

$$\|[m_{ij}x_{ij}]\|_p = \|[x_{ij}]\|_p$$

for all finitely supported matrices  $[x_{ij}]_{i,j \in I}$ .

- (ii) There exist two families  $(\alpha_i)_{i \in I}$  and  $(\beta_j)_{j \in I}$  in  $\mathbb{T}$  such that

$$m_{ij} = \alpha_i \beta_j, \quad \text{for all } (i, j) \in I^2.$$

We conclude with a characterisation of a particular class of Schur multipliers, the separating Herz-Schur multipliers. Let  $G$  be a locally compact  $\sigma$ -compact group (see Lemma 3.2). Suppose  $1 \leq p < \infty$ . Let  $\varphi \in L^\infty(G)$  and define  $\phi \in L^\infty(G^2)$  by  $\phi(s, t) = \varphi(s^{-1}t)$ . The Schur multiplier  $T_\varphi^{HS} := T_\phi$  is called a Herz-Schur multiplier (with symbol  $\varphi$ ).

In [4, Proposition 4.5], it is shown that a Herz-Schur multiplier  $T_\varphi^{HS}: B(L^2(G)) \rightarrow B(L^2(G))$  with positive definite  $\varphi$  such that  $\varphi(e) = 1$ , is a conjugation with a unitary if and only if  $\varphi$  is a character.

**Corollary 5.4.** *Let  $1 \leq p < \infty$ . Let  $G$  be a locally compact  $\sigma$ -compact group. Let  $T_\varphi^{HS}$  be a bounded Herz-Schur multiplier on  $S^p(L^2(G))$ . Then  $T_\varphi^{HS}$  is separating if and only if there exists a continuous character  $\psi: G \rightarrow \mathbb{T}$  and  $c \in \mathbb{C}$  such that  $\varphi = c\psi$  almost everywhere.*

*Proof.* If  $\varphi(s) = c\psi(s)$  a.e.  $s \in G$  for some continuous character  $\psi$ , then the symbol  $\phi$  of the Schur multiplier satisfies  $\phi(s, t) = \varphi(s^{-1}t) = c\psi(s^{-1}t) = c\psi^{-1}(s)\psi(t)$  for a.e.  $(s, t) \in G^2$ . Since  $\psi^{-1}$  and  $\psi$  are clearly unitaries of  $L^\infty(G)$ , by Theorem 5.1,  $T_\varphi^{HS} = T_\phi$  is separating.

Conversely, assume  $T_\varphi^{HS}$  separating. Then by Theorem 5.1, there are unitaries  $\alpha, \beta \in L^\infty(G)$  and some  $c \in \mathbb{C}$  such that  $\varphi(s^{-1}t) = c\alpha(s)\beta(t)$  for a.e.  $(s, t) \in G^2$ . Let  $r \in G$ . Then  $c\alpha(s)\beta(t) = \varphi(s^{-1}t) = \varphi((rs)^{-1}(rt)) = c\alpha(rs)\beta(rt)$  for a.e.  $(s, t) \in G^2$ . Leaving the trivial case  $c = 0$  aside, we deduce that

$$(5.6) \quad \frac{\beta(rt)}{\beta(t)} = \frac{\alpha(s)}{\alpha(rs)}$$

for a.e.  $(s, t) \in G^2$ . Thus there is some  $s \in G$  such that (5.6) holds for a.e.  $t \in G$ . Defining  $\psi(r)$  as the right hand side of (5.6), we then obtain  $\frac{\beta(rt)}{\beta(t)} = \psi(r)$  for a.e.  $t \in G$ . The function  $\psi: G \rightarrow \mathbb{C}$  with this property is necessarily unique. From  $\psi(e) = 1$  and

$$\psi(r_1 r_2) = \frac{\beta(r_1 r_2 t)}{\beta(t)} = \frac{\beta(r_1(r_2 t))}{\beta(r_2 t)} \frac{\beta(r_2 t)}{\beta(t)} = \psi(r_1)\psi(r_2) \quad (\text{a.e. } t \in G)$$

we infer that  $\psi$  is a character. Since  $\psi$  is measurable, by [13, Corollary 22.19 p. 346], it is automatically continuous. From  $\beta(rt) = \beta(t)\psi(r)$  for a.e.  $t \in G$ , we infer by a Fubini argument that there exists some  $t \in G$  such that this equality holds for a.e.  $r \in G$ . Thus,  $\beta$  coincides a.e. with a continuous function. Choosing this continuous representative for  $\beta$ ,

we obtain that for every  $r$  in  $G$ ,  $\beta(rt) = \beta(t)\psi(r)$  for a.e.  $t \in G$ . Since  $\beta$  is continuous, this implies  $\beta(rt) = \beta(t)\psi(r)$  for all  $r, t$  in  $G$ . In particular, we have that  $\beta(r) = \beta(e)\psi(r)$  for all  $r \in G$ . Using (5.6), the same argument as above shows that  $\alpha(s) = \alpha(e)\psi(s^{-1})$  for all  $s$  in  $G$ . Hence we deduce that  $\varphi(s^{-1}t) = c\alpha(e)\beta(e)\psi(s^{-1}t)$  for a.e.  $(s, t) \in G^2$ . Therefore,  $\varphi$  coincides a.e. with a multiple of a continuous character.  $\square$

**Acknowledgements:** Cédric Arhancet and Christoph Kriegler were supported by the grant ANR-18-CE40-0021 of the French National Research Agency ANR (project HAS-CON). Christoph Kriegler was supported by the grant ANR-17-CE40-0021 (project Front). Christian Le Merdy was supported by the ANR project Noncommutative analysis on groups and quantum groups (No./ANR-19-CE40-0002). Safoura Zadeh was supported by I-SITE Emergence project MultiStructure (Harmonic Analysis of Fourier and Schur multipliers) of Clermont Auvergne University.

## REFERENCES

- [1] C. ARHANCET, *Noncommutative Figà-Talamanca–Herz algebras for Schur multipliers*, Integral Equations Operator Theory 70 (2011), no. 4, 485–510.
- [2] C. ARHANCET, *Quantum information theory via Fourier multipliers on quantum groups*, Preprint 2020, arXiv:2008.12019.
- [3] C. ARHANCET AND C. KRIEGLER, *Projections, multipliers and decomposable maps on noncommutative  $L^p$ -spaces*, to appear in Mémoires de la Société Mathématique de France (2023), 177 (new series).
- [4] J. CRANN AND M. NEUFANG, *Quantum channels arising from abstract harmonic analysis*, J. Phys. A 46 (2013), no. 4, 045308, 22 pp.
- [5] M. DAWS, *Representing multipliers of the Fourier algebra on non-commutative  $L^p$  spaces*, Canad. J. Math. 63 (2011), no. 4, 798–825.
- [6] J. DE CANNIÈRE AND U. HAAGERUP, *Multipliers of the Fourier algebras of some simple Lie groups and their discrete subgroups*, Amer. J. Math. 107 (1985), 455–500.
- [7] P. EYMARD, *L’algèbre de Fourier d’un groupe localement compact (French)*, Bull. Soc. Math. France 92 (1964), 181–236.
- [8] G. FOLLAND, *A course in abstract harmonic analysis*, Textbooks in Mathematics. Boca Raton, 2016.
- [9] A. GONZÁLEZ-PÉREZ AND M. JUNGE AND J. PARCET, *Smooth Fourier multipliers in group algebras via Sobolev dimension*, Ann. Sci. Éc. Norm. Supér. (4) 50 (2017), no. 4, 879–925.
- [10] U. HAAGERUP, *Decomposition of completely bounded maps on operator algebras*, Unpublished, Odense University, Denmark, 1980.
- [11] H. HANCHE-OLSEN AND E. STØRMER, *Jordan operator algebras*, Monographs and Studies in Mathematics Series Profile, 21. Boston - London - Melbourne: Pitman Advanced Publishing Program. VIII, 1984.
- [12] C. HERZ, *Une généralisation de la transformée de Fourier-Stieltjes*, Ann. Inst. Fourier 24 (1974), no. 3, 145–157.
- [13] E. HEWITT AND K. A. ROSS, *Abstract harmonic analysis*, Vol. I Second edition, Grundlehren der Mathematischen Wissenschaften, 115. Springer-Verlag, Berlin-New York, 1979.
- [14] G. HONG, S. K. RAY AND S. WANG, *Maximal ergodic inequalities for positive operators on noncommutative  $L^p$ -spaces*, Preprint 2019, arXiv:1907.12967
- [15] M. JUNGE AND Z. RUAN AND D. SHERMAN, *A classification for 2-isometries of noncommutative  $L^p$ -spaces*, Isr. J. Math. 150 (2005), 285–314.
- [16] M. JUNGE AND T. MEI AND J. PARCET, *Smooth Fourier multipliers on group von Neumann algebras*, Geom. Funct. Anal. 24 (2014), no. 6, 1913–1980.



- [17] M. JUNGE AND T. MEI AND J. PARCET, *Noncommutative Riesz transforms—dimension free bounds and Fourier multipliers*, J. Eur. Math. Soc. (JEMS) 20 (2018), no. 3, 529–595.
- [18] R. V. KADISON AND J.R. RINGROSE, *Fundamentals of the theory of operator algebras, Volume II*, Graduate Studies in Mathematics 16, Amer. Math. Soc. 1997.
- [19] E. KANIUTH AND A. T.-M. LAU, *Fourier and Fourier-Stieltjes algebras on locally compact groups*, Mathematical Surveys and Monographs, 231. American Mathematical Society, Providence, RI, 2018. xi+306 pp.
- [20] V. LAFFORGUE AND M. DE LA SALLE, *Noncommutative  $L^p$ -spaces without the completely bounded approximation property*, Duke Math. J. 160 (2011), No. 1, 71–116.
- [21] R. LARSEN, *An introduction to the theory of multipliers*, Springer-Verlag, New York-Heidelberg, 1971. xxi+282 pp.
- [22] C. LE MERDY AND S. ZADEH,  *$\ell^1$ -contractive maps on noncommutative  $L^p$ -spaces*, Journal of Operator Theory, 85:2(2021), 417–442.
- [23] C. LE MERDY AND S. ZADEH, *On factorization of separating maps on noncommutative  $L^p$ -spaces*, Indiana University Mathematics Journal, 71 (2022), no. 5, 1967–2000
- [24] C. LE MERDY AND S. ZADEH, *Surjective separating maps on noncommutative  $L^p$ -spaces*, Mathematische Nachrichten, Math. Nachr. 295 (2022), no. 1, 175–188.
- [25] T. MEI AND É. RICARD, *Free Hilbert transforms*, Duke Math. J. 166 (2017), No. 11, 2153–2182.
- [26] J. PARCET AND É. RICARD AND M. DE LA SALLE, *Fourier multipliers in  $SL_n(\mathbb{R})$* , Duke Math. J. 171 (2022), No. 6, 1235–1297.
- [27] S. PARROTT, *Isometric multipliers*, Pac. J. Math. 25 (1968), 159–166.
- [28] V. PAULSEN, *Completely bounded maps and operator algebras*, Cambridge Univ. Press, 2002.
- [29] G. PISIER AND Q. XU, *Non-commutative  $L^p$ -spaces*, Handbook of the geometry of Banach spaces, Vol. 2, pp. 1459–1517, North-Holland, Amsterdam, 2003.
- [30] M. REED AND B. SIMON, *Methods of modern mathematical physics. I. Functional analysis, Second edition*, Academic Press, New York, 1980, xv+400 pp.
- [31] N. SPRONK, *Measurable Schur multipliers and completely bounded multipliers of the Fourier algebras* Proc. London Math. Soc. (3) 89 (2004), no. 1, 161–192.
- [32] E. STØRMER, *On the Jordan structure of  $C^*$ -algebras*, Trans. Amer. Math. Soc. 120 (1965), 438–447.
- [33] R. STRICHARTZ, *Isomorphisms of group algebras*, Proc. Am. Math. Soc. 17 (1966), 858–862.
- [34] M. TAKESAKI, *Theory of operator algebras. I.*, Springer-Verlag, New York-Heidelberg, 1979.
- [35] M. TAKESAKI, *Theory of operator algebras. II.*, Encyclopaedia of Mathematical Sciences, 125. Operator Algebras and Noncommutative Geometry, 6. Springer-Verlag, Berlin, 2003.
- [36] F. Y. YEADON, *Isometries of noncommutative  $L^p$ -spaces*, Math. Proc. Cambridge Philos. Soc. 90 (1981), 41–50.

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