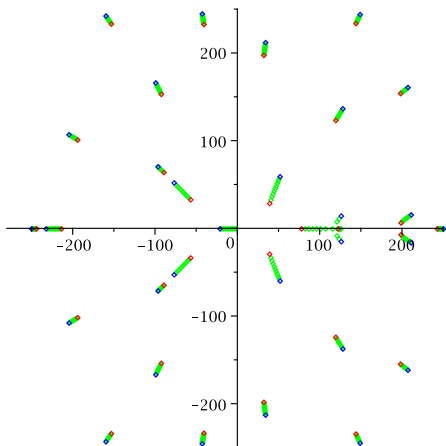


# Asymptotic Limit of Interacting Particle Sets.

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# Patterns of Complex Root sets



The complex root set (blue squares) of a random polynomial and the root set (green squares) of its successive derivatives.

- We study the evolution of root sets of large random polynomials when the polynomial changes in a given way.
- This asymptotic stochastic behaviour is naturally related to combinatorics and probability theory.
- But, since we also consider the evolution of the measure defined by the limit distribution of the root sets, it is also related to PDE theory.
  - Generalising the approach initiated by S. Steinerberger, we derive and study non local, non linear PDEs to describe these 'collective' evolutions.

The two main models we consider are the following:

## Abstract 2

- A random complex polynomial,  $P(z)$  and we look at the zero sets, in  $\mathbb{R}$ ,  $\mathbb{T}^1$  or  $\mathbb{C}$ , under derivations  $\frac{d}{dz}$ . of  $P(z)$ .
- Let  $\beta \in (0, \infty)$  be a real parameter. For each  $n \geq 1$ , we consider  $n$  increasing points on the real line  $(x_j^{n,n})_{1 \leq j \leq n}$ . Then, for  $k$  from  $n - 1$  to  $1$ , we define recursively the  $k$  points set  $x^k := (x_j^{n,k})_{1 \leq j \leq k}$  as the set of zeros of the polynomial

$$\sum_{j=1}^{k+1} w_j^{k+1} \prod_{1 \leq m \leq k+1, m \neq j} (x - x_m^{n,k+1}),$$

where  $(w_j^{k+1})_{1 \leq j \leq k+1}$  follows Dirichlet distribution with  $k + 1$  parameters, all equal to  $\beta/2$ .

- For  $\beta = 1, 2$ , this corresponds to eigenvalues of a random unitary matrix times a diagonal matrix, and the eigenvalues of the successive principal minors of that product.

# Abstract 3

- These 2 models are strongly connected to each other.
- The time steps of the evolution corresponds to degrees;  $\Delta t = \frac{1}{n}$ . When the number of roots tends to infinity their limit set is represented by a density measure, the studied PDE models the "continuous" evolution of this measure.
- The dynamics of root sets of random polynomials under derivations has also been characterised using the so-called finite free probability technics.
- This is a collective motion of systems of particles with long range interactions like in Mathematical Physics.

# Abstract 4

- We focus on convergence of the characteristic polynomials, when  $k$  tends to infinity; it is called "global" behaviour of the limit eigenvalues set. While the study of "local" or short scale behaviour of this limit set should be related to a fine understanding of the  $Sine_\beta$  process.
- The  $Sine_\beta$  process is the thermodynamic limit of the microscopic behaviour in the bulk of a one dimensional log gas system of  $N$  interacting particles, with long range interactions, confined by an external field.
  - With  $\beta = \frac{1}{T}$ , the probability measure on  $\mathbb{R}^N$  is given e.g. by the probability density  $d\mathbb{P}_{N,\beta}(x_1, \dots, x_N) =$

$$\frac{1}{Z_{N,\beta}} \exp(-\beta(\sum_{i<j} -\log|x_i - x_j| + \frac{1}{2} \sum_i x_i^2)).$$

- For  $\beta = 1, 2, 4$ ,  $\mathbb{P}_{N,\beta}$  is the joint eigenvalues law of classical random matrices sets GOE, GUE, GSE.

# Motion

- 1 Plotting roots sets of iterated derivatives, of **large degree random real polynomials**, shows a discretized flow like the motion of a swarm.
- 2 The interactions is described by short range repulsion to prevent collisions between individuals, and long range attraction that keeps the swarm cohesive.
- 3 When all roots are real, considering their density limite, S. Steinerberger derived a non-local PDE to model the asymptotic “motion”

$$\mathbf{u}_t = \frac{1}{\pi} \left[ \arctan \left( \frac{\mathbf{u}}{\mathcal{H}(\mathbf{u})} \right) \right]_{\mathbf{x}}.$$

$\mathcal{H}$  being the Hilbert transformation.

# Intuitive explanation: Mean field equilibrium

- Denote by  $\{x_j\}$  the complex roots of the polynomial  $P$ . By logarithmic derivation, each critical point of  $P$  is an equilibrium of the “electrostatic” interactions

$$P'(z) = 0 \Rightarrow \sum_j \frac{1}{z - x_j} = 0.$$

- Near  $x_0$ ; split the electrostatic field, into a far field and a near field where the roots are **assumed** equi-spaced.
- These fields are balanced at a critical point  $x_0$ .

$$\sum_{|x_j - x_0| \text{ small}} \approx u(x_0) n \pi \cot(u(x_0) n \pi (x - x_0));$$
$$\frac{1}{n} \sum_{|x_j - x_0| \text{ not small}} \approx p v \int_{\mathbb{R}} \frac{u(y) dy}{x - y} = \pi \mathcal{H}(u)(x).$$

- $\mathcal{H}$  being the Hilbert transformation. Equating, we get Steinerberger transport equation.



# Comparison

- In a work in progress, we aim to prove that the same equation governs a more general “motion of eigenvalues”.
- The derivative polynomial  $P'$  is replaced by a polynomial  $\tilde{P}$  depending on weights  $w_j$  following a Dirichlet distribution with equal parameters ( $w_j$  tends to  $\frac{1}{n}$  when  $n$  tends to  $\infty$ ),

$$\tilde{P}(z) = 0 \Rightarrow \sum_{j=1}^{k+1} \frac{w_j}{z - X_j} = 0.$$

- In other words, let  $P_j = \prod_{1 \leq m \leq k+1, m \neq j} (x - x_m)$ , then  $P' = \sum_{j=1}^{k+1} P_j$ , while  $\tilde{P} = \sum_{j=1}^{k+1} w_j P_j$ .
- To compare the asymptotic of  $P$ ,  $P'$  and  $\tilde{P}$ , we view their coefficients as symmetric functions of their roots and we rely on Newton identities to evaluate the moments.

# A first martingale property

- For  $p \leq k \leq n$ , successive root sets  $x^k$  are obtained by iterated application of  $P \rightarrow \tilde{P}$ .

Let us denote  $(q)_r := q(q-1)(q-2)\dots(q-r+1) = r! \binom{q}{r}$ , and by  $e_{k,p}$  the  $p$ -th elementary symmetric function of the points in  $x^k$ ,  $e_{k,p} = \sum_{I \subset \{1,2,\dots,k\}, |I|=p} \prod_{j \in I} x_j^k$ .

- Then, for  $p \leq k \leq n-1$ , we prove the martingale properties, conditioned by the  $x^\ell$  with  $k+1 \leq \ell \leq n$ ,

$$\mathbb{E}\left[\frac{e_{k,p}}{(k)_p}\right] = \frac{e_{k+1,p}}{(k+1)_p},$$

$$\mathbb{E}\left[\sup_{k \leq \ell \leq n} \left(\frac{e_{n,p}}{(n)_p} - \frac{e_{\ell,p}}{(\ell)_p}\right)^2\right] \leq \frac{12 \|x\|_n^{2p}}{\beta(p-1)!^2 (k-p+1)^2}.$$

- Remark : Successive derivations correspond to  $\beta = \infty$ .

- However these bounds on  $e_k$  are not sufficient to prove our objective. So, we consider convergences of sums of powers of roots  $p_k := \sum_{i=1}^n x_i^k$ , and cumulants, as proceeded in free probability.
- The relation between these two symmetric functions of the roots is given by Newton identities
$$p_k = \sum_{i=k-n}^{k-1} (-1)^{k-1+i} e_{k-i} p_i.$$
- Numerical simulations were encouraging and we proved the aimed convergence for few small values of  $k$ , relying on tricky cancellations.
- We are currently working on an intricate recursive argument ...



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