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A SOLUTION TO THE MV-SPECTRUM PROBLEM IN SIZE ALEPH ONE

MIROSLAV PLOŠČICA AND FRIEDRICH WEHRUNG

ABSTRACT. Denote by $\text{Id}_c G$ the lattice of all principal ℓ -ideals of an Abelian ℓ -group G . Our main result is the following.

Theorem. *For every countable Abelian ℓ -group G , every countable completely normal distributive 0-lattice L , and every closed 0-lattice homomorphism $\varphi: \text{Id}_c G \rightarrow L$, there are a countable Abelian ℓ -group H , an ℓ -homomorphism $f: G \rightarrow H$, and a lattice isomorphism $\iota: \text{Id}_c H \rightarrow L$ such that $\varphi = \iota \circ \text{Id}_c f$.*

We record the following consequences of that result:

- (1) A 0-lattice homomorphism $\varphi: K \rightarrow L$, between countable completely normal distributive 0-lattices, can be represented, with respect to the functor Id_c , by an ℓ -homomorphism of Abelian ℓ -groups iff it is closed.
- (2) A distributive 0-lattice D of cardinality at most \aleph_1 is isomorphic to some $\text{Id}_c G$ iff D is completely normal and for all $a, b \in D$ the set $\{x \in D \mid a \leq b \vee x\}$ has a countable coinital subset. This solves Mundici's MV-spectrum Problem for cardinalities up to \aleph_1 . The bound \aleph_1 is sharp.

Item (1) is extended to commutative diagrams indexed by forests in which every node has countable height. All our results are stated in terms of vector lattices over any countable totally ordered division ring.

1. INTRODUCTION

The set $\text{Spec } G$ of all prime ℓ -ideals in an Abelian lattice-ordered group (in short ℓ -group) G with order-unit, endowed with the hull-kernel topology, is a spectral space (as defined in Hochster [8]) called the *spectrum of G* . The topological spaces $\text{Spec } G$ satisfy an additional property called *complete normality*, which states that the specialization order is a root system. The problem of characterizing all spaces $\text{Spec } G$ is stated in the following references:

- In his 1973 paper, Martínez [13, Question II] asks (in an equivalent form) for a characterization of isomorphic copies of ℓ -ideal lattices of *Archimedean* ℓ -groups.
- In his 2011 monograph, Mundici [17, Problem 2] asks “Which topological spaces are homeomorphic to $\text{Spec } A$ for some MV-algebra A ”. Since an MV-algebra and its associated Abelian ℓ -group with unit (through Mundici's equivalence [16]) have homeomorphic spectra, this is equivalent to the corresponding question about ℓ -groups with order-unit.

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Beginning in Martínez' paper [13], most known answers to those problems are stated in terms of the Stone dual of $\text{Spec } G$, which is the lattice $\text{Id}_c G$ of all principal ℓ -ideals of G (cf. Keimel [10]). Isomorphic copies of lattices $\text{Id}_c G$, for Abelian ℓ -groups G , will be called *ℓ -representable* lattices. They are distributive (not necessarily bounded) 0-lattices¹. Owing to Monteiro [15], complete normality of the topological space $\text{Spec } G$ translates to a first-order lattice-theoretical property of $\text{Id}_c G$, also called complete normality, namely

$$(\forall a, b)(\exists x, y)(a \vee b = a \vee y = x \vee b \text{ and } x \wedge y = 0).$$

Delzell and Madden's non- ℓ -representable completely normal lattice from [6] has \aleph_1 elements. On the other hand, a completely normal bounded distributive lattice D is ℓ -representable provided either D is a dual Heyting algebra (cf. Cignoli *et al.* [5], Iberkleid *et al.* [9]) or D is countable² (cf. Wehrung [22]).

There are also negative results, whose spirit is " *ℓ -representability cannot be characterized*" [in a logically simpler way than originally defined]. Lenzi and Di Nola observe in [11] that since the class of all ℓ -representable lattices is not closed under ultraproducts, it is not the class of models of any existential second-order sentence. In Wehrung [24] it is proved, building on Wehrung [23] and an extension of the condensate construction initiated in Gillibert and Wehrung [7], that the class of all ℓ -representable lattices is not the class of all models of any class of sentences in the infinitary language $\mathcal{L}_{\infty\lambda}$ for fixed³ λ . The subsequent paper Wehrung [26] establishes that the class of all ℓ -representable lattices is not the class of models of any sentence ψ of the form $(\forall X)\varphi$ where X is a second-order variable with bounded arity and φ is an $\mathcal{L}_{\infty\infty}$ formula (we say that ψ is *co-projective*). Those non-representability results extend to *Archimedean* ℓ -groups; however, it is still unknown whether they extend to Archimedean ℓ -groups *with order-unit* (cf. Problem 9.1).

Those negative results also apply to the larger class of all *homomorphic images of ℓ -representable lattices*, and in fact to any intermediary class.

A strong necessary condition for ℓ -representability was coined, under different names ("Id ω " and " σ -Conrad", respectively), in Cignoli *et al.* [5] and Iberkleid *et al.* [9]; it got renamed again in Wehrung [22] as follows.

Definition 1.1. A lattice L has *countably based differences* if for all $a, b \in L$ there exists a countable subset C of L such that for every $x \in L$, $a \leq b \vee x$ iff $c \leq x$ for some $c \in C$.

Every ℓ -representable lattice has countably based differences. Delzell and Madden's counterexample from [6] does not have countably based differences. The main counterexample D in Wehrung [23] has size \aleph_2 ; it is completely normal, has countably based differences, but is not what is called there *Cevian*. Being Cevian is an existential second-order property. Homomorphic images of ℓ -representable lattices are always Cevian, thus D is not such a lattice. Ploščica's non ℓ -representable counterexample from [18] is completely normal, has countably based differences, and is Cevian; it has cardinality $(2^{\aleph_0})^+$. Those results show that in some sense, *characterizing ℓ -representability beyond the size \aleph_2 is hopeless*.

¹Where a 0-lattice is just a lattice with a least element, then often denoted by 0.

²Throughout the paper "countable" will mean "at most countable".

³One cannot do better than fixing λ , because any class of models closed under isomorphic copy is the class of all models of all $\mathcal{L}_{\infty\infty}$ sentences satisfied by it

In this note we deal with the remaining gap, that is, the cardinality \aleph_1 . The main consequence, pertaining to the MV-spectrum Problem, of our work is the following extension, from countable to \aleph_1 , of the main result of Wehrung [22], contained in Corollary 8.5:

Theorem. *A distributive 0-lattice of cardinality at most \aleph_1 is ℓ -representable iff it is completely normal and has countably based differences.*

Via Stone duality, this result is a full solution of the MV-spectrum Problem for spaces with at most \aleph_1 compact open subsets. Due to the main counterexample from Wehrung [23], the bound \aleph_1 in this result is sharp (e.g., the theorem above does not extend to the size \aleph_2).

Our main result, Theorem 8.3, deals with countable structures only. It also involves the concept of a *closed homomorphism* between lattices; a lattice homomorphism $f: K \rightarrow L$ is *closed* if for all $a, b \in K$ and all $x \in L$, if $f(a) \leq f(b) \vee x$ then there exists $u \in K$ such that $a \leq b \vee u$ whereas $f(u) \leq x$. In the context of Stone duality between distributive 0-lattices with cofinal 0-lattice homomorphisms and generalized spectral spaces with spectral maps (cf. Rump and Yang [20]), it can be proved that a cofinal 0-lattice homomorphism is closed iff its Stone dual is closed in the topological sense (i.e., it maps closed sets to closed sets); see Iberkleid *et al.* [9, Lemma 1.3.5].

For any ℓ -homomorphism $f: G \rightarrow H$ between Abelian ℓ -groups, the lattice homomorphism $\text{Id}_c f: \text{Id}_c G \rightarrow \text{Id}_c H$ is closed (cf. Wehrung [22, Proposition 2.6]). Our main result implies that conversely, *every closed homomorphism between countable completely normal distributive 0-lattices can be represented in this way*. In fact, that result extends to commutative diagrams indexed by forests in which every node has countable height (cf. Theorem 8.4). Our main result, illustrated on Figure 1.1, states as follows:

$$\begin{array}{ccc}
 \text{Id}_c G & & G & & \text{Id}_c G \\
 \varphi \downarrow \text{closed} & \rightsquigarrow & \downarrow f & & \text{Id}_c f \downarrow & \searrow \varphi \\
 L & & H & & \text{Id}_c H & \xrightarrow{\iota} L
 \end{array}$$

FIGURE 1.1. Illustrating Theorem 8.3

Theorem 8.3 (restated for ℓ -groups). *Let G be a countable Abelian ℓ -group, let L be a countable completely normal distributive 0-lattice, and let $\varphi: \text{Id}_c G \rightarrow L$ be a closed 0-lattice homomorphism. Then there are a countable Abelian ℓ -group H , an ℓ -homomorphism $f: G \rightarrow H$, and a lattice isomorphism $\iota: \text{Id}_c H \rightarrow L$ such that $\varphi = \iota \circ \text{Id}_c f$.*

All our results will be formulated in the context of left vector lattices over *countable* totally ordered division rings \mathbb{k} (due to Wehrung [25, § 9], the countability assumption on \mathbb{k} cannot be dropped). The canonical embedding from any Abelian ℓ -group G into its divisible hull $\mathbb{Q} \otimes G$ is, following the terminology of Anderson and Feil [1], an Archimedean extension, so $\text{Id}_c G \cong \text{Id}_c(\mathbb{Q} \otimes G)$. Conversely, any vector lattice E over \mathbb{Q} (or, more generally, over an Archimedean totally ordered field) has the same ℓ -ideals as an ℓ -group as a vector lattice; hence *the vector lattice results entail the ℓ -group results*.

2. STRATEGY OF THE PROOF

Given a countable completely normal distributive 0-lattice L , a countable left vector lattice E over a countable totally ordered division ring \mathbb{k} , we wish to prove that every closed 0-lattice homomorphism $\varphi: \text{Id}_c E \rightarrow L$ is, up to isomorphism (over $\text{Id}_c E$), $\text{Id}_c f$ for some vector lattice homomorphism $f: E \rightarrow F$. The initial step consists of observing that E is a homomorphic image of the free \mathbb{k} -vector lattice $F_\ell(I, \mathbb{k})$ over any countably infinite set I , thus reducing the problem to the case where $E = F_\ell(I, \mathbb{k})$ is a free vector lattice. Then, owing to the Baker-Bernau-Madden duality for free vector lattices (cf. Lemma 5.5), $\text{Id}_c F_\ell(I, \mathbb{k})$ is isomorphic to the sublattice $\text{Op}^- \mathbb{k}^{(I)}$ of the powerset algebra of the left \mathbb{k} -vector space $\mathbb{k}^{(I)}$ with basis I generated by all open half-spaces defined by elements of $\mathbb{k}^{(I)}$.

The original closed 0-lattice homomorphism $\varphi_0 = \varphi: \text{Op}^- \mathbb{k}^{(I)} \rightarrow L$ is then inductively enlarged to a chain of closed homomorphisms $\varphi_m: \text{Op}^- \mathbb{k}^{(I_m)} \rightarrow L$, with each $I_m \stackrel{\text{def}}{=} I \sqcup \{0, \dots, m-1\}$ (where \sqcup denotes disjoint union). Each map φ_{m+1} is in turn constructed as the common extension of an ascending chain $(\varphi_{m,n} \mid n < \omega)$ of maps $\varphi_{m,n}: \text{Op}^- (\mathbb{k}^{(I_m)} \cup \mathcal{D}_{m,n}) \rightarrow L$ for finite sets $\mathcal{D}_{m,n} \subset \mathbb{k}^{(I_{m+1})}$. Setting $\mathcal{D}_{m,0} \stackrel{\text{def}}{=} \{\delta_m\}$, where δ_m denotes the m th coordinate projection $\mathbb{k}^{(I_{m+1})} \rightarrow \mathbb{k}$, the value of $\varphi_{m,0}$ at the positive open half-space defined by δ_m is initially set at the m th element of L (with respect to a given enumeration). The common extension $\psi: \text{Op}^- \mathbb{k}^{(I \sqcup \omega)} \rightarrow L$ of the φ_m will then be a closed, surjective homomorphism; this easily entails the desired solution.

The construction of the $\varphi_{m,n}$ follows, to some extent, the strategy initiated for the countable case in Wehrung [22]. According to the parity of n , the step from n to $n+1$ may either enlarge $\mathcal{D}_{m,n}$ by one element (“Domain Step”) or solve a “closedness problem” $\varphi_{m,n}[a] \leq \varphi_{m,n}[b] \vee \mathbf{e}$ (“Closure Step”; $[a]$ denotes the positive open half-space associated to a , see Notation 5.1). The Domain Step relies on its original version, stated for finite sets of hyperplanes, stated in Wehrung [22, Lemma 6.6].

However, the original approach to the Closure Step, in the form of Claim 1 from the proof of Wehrung [22, Lemma 7.1], can be proved to fail in this more general context; so the argument of Ploščica and Wehrung [19] is not sufficient to go from “homomorphic image” to “isomorphic copy”. (This should be no surprise, because closed homomorphisms play no role in [19].) It turns out that the Closure Step requires a far more subtle approach.

Our first technical step, handled in Section 6, consists of stating, for a finite subset $\mathcal{D} \cup \{c\}$ of $\mathbb{k}^{(J)}$ where $J = I \sqcup \{o\}$, a criterion ensuring that a 0-lattice homomorphism $\text{Op}^- (\mathbb{k}^{(I)} \cup \mathcal{D}) \rightarrow L$ can be extended to a lattice homomorphism $\text{Op}^- (\mathbb{k}^{(I)} \cup \mathcal{D} \cup \{c\}) \rightarrow L$ sending the pair $([c], [-c])$ to a given pair $(\mathbf{c}^+, \mathbf{c}^-)$ of elements in L . This criterion (cf. Lemma 6.1) is stated in terms of the system of inequalities (6.1)–(6.7), that should be satisfied by $(\mathbf{c}^+, \mathbf{c}^-)$.

Our second technical step, aimed at handling a given closedness problem $\varphi_{m,n}[a] \leq \varphi_{m,n}[b] \vee \mathbf{e}$, consists of taking advantage of the closedness of the map $\varphi_m: \text{Op}^- \mathbb{k}^{(I_m)} \rightarrow L$, leading to the satisfiability of (6.1)–(6.7) with $\mathcal{D} \stackrel{\text{def}}{=} \mathcal{D}_{m,n}$ and $c \stackrel{\text{def}}{=} a - \lambda b$ for large enough $\lambda \in \mathbb{k}$, together with the additional inequality $\mathbf{c}^+ \leq \varphi_{m,n}[-b] \vee \mathbf{e}$. This yields a solution to our closedness problem at the next level $\varphi_{m,n+1}$, and thus the desired conclusion (cf. Section 7).

3. BASIC CONCEPTS AND TERMINOLOGY

A subset C in a poset P is *coinitial* if every element of P lies above some element of C . We say that P is a *forest* if every principal ideal of P is well-ordered under the induced order.

We will denote ℓ -groups additively (even in the non-commutative case). For any structure G with a partially ordered group reduct (e.g., an ordered vector space, an ordered field, and so on), we set $G^+ \stackrel{\text{def}}{=} \{x \in G \mid 0 \leq x\}$. We set $x^+ \stackrel{\text{def}}{=} x \vee 0$ and $|x| \stackrel{\text{def}}{=} x \vee (-x)$ if those elements exist, whenever $x \in G$.

For a totally ordered division ring \mathbb{k} , a \mathbb{k} -*vector lattice* is a left \mathbb{k} -vector space E endowed with a translation-invariant lattice structure such that $\mathbb{k}^+ E^+ \subseteq E^+$. A map $f: E \rightarrow F$, between vector lattices, is a \mathbb{k}, ℓ -*homomorphism* if it is both a vector space homomorphism and a lattice homomorphism. An ℓ -*ideal* of E is a vector subspace of E closed under the lattice operations. Principal (equivalently, finitely generated) ℓ -ideals are the subsets of the form $\langle a \rangle \stackrel{\text{def}}{=} \{x \in E \mid (\exists \lambda \in \mathbb{k}^+)(|x| \leq \lambda|a|)\}$ for $a \in E$. For any $a, b \in E^+$, $\langle a \rangle \vee \langle b \rangle = \langle a \vee b \rangle = \langle a + b \rangle$ whereas $\langle a \rangle \wedge \langle b \rangle = \langle a \wedge b \rangle$. The lattice $\text{Id } E$ of all ℓ -ideals of E is a distributive algebraic lattice, of which its semilattice of compact elements, $\text{Id}_c E \stackrel{\text{def}}{=} \{\langle x \rangle \mid x \in E^+\}$, is a 0-sublattice. Any \mathbb{k}, ℓ -homomorphism $f: E \rightarrow F$ gives rise to a 0-lattice homomorphism $\text{Id}_c f: \text{Id}_c E \rightarrow \text{Id}_c F$, $\langle x \rangle_E \mapsto \langle f(x) \rangle_F$. The assignment $E \mapsto \text{Id}_c E$, $f \mapsto \text{Id}_c f$ is a functor. This functor preserves all directed colimits.

4. CONSONANCE AND CLOSED HOMOMORPHISMS

The following two definitions originate in Wehrung [22] and subsequent papers, where they are widely used. We recall them for convenience.

Definition 4.1. Let K and L be lattices and let $f: K \rightarrow L$ be a lattice homomorphism. We say that f is *closed at* a pair $(a, b) \in K \times K$ if for all $x \in L$ such that $f(a) \leq f(b) \vee x$ there exists $u \in K$ such that $a \leq b \vee u$ and $f(u) \leq x$.

In particular, f is closed iff it is closed at every pair in $K \times K$.

Definition 4.2. A pair (a, b) of elements in a distributive 0-lattice L is *consonant* in L if there is a pair $(u, v) \in L \times L$ (then called a *splitting pair* for (a, b)) such that $a \vee b = a \vee v = u \vee b$ whereas $u \wedge v = 0$; or, equivalently, $a = (a \wedge b) \vee u$, $b = (a \wedge b) \vee v$, and $u \wedge v = 0$. A subset X of L is *consonant in L* if every pair of elements in X is consonant in L .

In particular, L is completely normal iff every pair in $L \times L$ is consonant. The following lemma is a more user-friendly version of Wehrung [22, Lemma 3.7], with a similar proof.

Lemma 4.3. *Let K and L be distributive lattices, such that L has a zero, let $f: K \rightarrow L$ be a lattice homomorphism, and let a, a_1, a_2, b, b_1, b_2 be elements of K . The following statements hold:*

- (1) *If f is closed at both (a_1, b) and (a_2, b) , then it is closed at $(a_1 \vee a_2, b)$.*
- (2) *If f is closed at both (a, b_1) and (a, b_2) , then it is closed at $(a, b_1 \wedge b_2)$.*
- (3) *If $(f(a_1), f(a_2))$ is consonant in L and f is closed at both (a_1, b) and (a_2, b) , then f is closed at $(a_1 \wedge a_2, b)$.*
- (4) *If $(f(b_1), f(b_2))$ is consonant in L and f is closed at both (a, b_1) and (a, b_2) , then f is closed at $(a, b_1 \vee b_2)$.*

Proof. Ad (1) and (2) are both trivial.

Ad (3). Fix a splitting pair (y_1, y_2) for $(f(a_1), f(a_2))$ in L and let $x \in L$ such that $f(a_1 \wedge a_2) \leq f(b) \vee x$. Then each $f(a_i) = f(a_1 \wedge a_2) \vee y_i \leq f(b) \vee x \vee y_i$, so, since f is closed at (a_i, b) , there exists $c_i \in K$ such that $a_i \leq b \vee c_i$ and $f(c_i) \leq x \vee y_i$. Setting $c \stackrel{\text{def}}{=} c_1 \wedge c_2$, we get $a_1 \wedge a_2 \leq b \vee c$ whereas $f(c)$ lies below each $x \vee y_i$, thus below $(x \vee y_1) \wedge (x \vee y_2) = x$.

The proof of (4) is similar to the one of (3). \square

The following lemma stems from Wehrung [22, Lemma 3.9], with a similar proof.

Lemma 4.4. *Let K and L be distributive lattices such that L has a zero, let Σ be a generating subset of K , and let $f: K \rightarrow L$ be a lattice homomorphism such that $f[K]$ is consonant in L . If f is closed at every pair in $\Sigma \times \Sigma$, then it is closed.*

Proof. Owing to Lemma 4.3, for every $a \in \Sigma$, the set

$$X_a \stackrel{\text{def}}{=} \{b \in K \mid f \text{ is closed at } (a, b)\}$$

is a sublattice of K containing Σ ; thus $X_a = K$, that is, f is closed at every pair in $\Sigma \times K$. By the same token, it follows that for every $b \in K$, the set

$$Y_b \stackrel{\text{def}}{=} \{a \in K \mid f \text{ is closed at } (a, b)\}$$

is a sublattice of K containing Σ , thus equal to K . \square

5. THE LATTICES $\text{Op}^- \mathcal{D}$

In this section we state some required facts about the $\text{Op}^- \mathcal{D}$ construction introduced in Wehrung [22] for the case of linear functionals and $\mathbb{k} = \mathbb{Q}$, and extended to arbitrary \mathbb{k} and affine functionals in Wehrung [25]. Throughout this section we fix a totally ordered division ring \mathbb{k} . Further, for any set I we denote by $(\delta_i \mid i \in I)$ the canonical basis of the left \mathbb{k} -vector space $\mathbb{k}^{(I)}$, and for $I \subseteq J$ we identify $\mathbb{k}^{(I)}$ with its canonical copy in $\mathbb{k}^{(J)}$.

Notation 5.1. Let I be a set and let $a \in \mathbb{k}^{(I)}$. We set

$$[a] \stackrel{\text{def}}{=} \left\{ x \in \mathbb{k}^{(I)} \mid (a \mid x) > 0 \right\},$$

whenever $a \in \mathbb{k}^{(I)}$ (where $(a \mid x) \stackrel{\text{def}}{=} \sum_{i \in I} a_i x_i$). Following Wehrung [22, 25], for any $\mathcal{D} \subseteq \mathbb{k}^{(I)}$, we shall denote by $\text{Op}^- \mathcal{D}$ the 0-sublattice of the powerset of $\mathbb{k}^{(I)}$ generated by $\{[x] \mid x \in \mathcal{D}\}$.

For a subset I of a set J , we will occasionally identify $\text{Op}^- \mathbb{k}^{(I)}$ with its canonical image in $\text{Op}^- \mathbb{k}^{(J)}$ (denoted $\text{Op}^- (\mathbb{k}^{(I)}, \mathbb{k}^{(J)})$ in Ploščica and Wehrung [19]).

Notation 5.1 relates to the one from earlier papers such as [19, 22, 25] via the rule $[a] = \llbracket a > 0 \rrbracket$. Our choice of the simpler notation $[a]$ is motivated by both our focus on open sets and the complexity of the equations and inequalities intervening in Sections 6 and 7.

We will be constantly using the obvious properties of the assignment $x \mapsto [x]$ stated in the following lemma.

Lemma 5.2. *The following statements hold, for all $x, y \in \mathbb{k}^{(I)}$ and all $\lambda \in \mathbb{k}^+$:*

- (1) $[x] \cap [y] \subseteq [x + y] \subseteq [x] \cup [y]$.
- (2) $[x] \cap [-x] = \emptyset$.
- (3) $[\lambda x] \subseteq [x]$.

For any subset C of $\mathbb{k}^{(I)}$, we denote by $\text{conv}(C)$ the convex hull of C and by $\text{cone}(C)$ the additive submonoid of $\mathbb{k}^{(I)}$ generated by \mathbb{k}^+C (so $0 \in \text{cone}(C)$). The following lemma is implicit in Lemma 5.5. Although it is certainly well known we could not find any reference stating it explicitly. It can also be used to verify that (1)–(3) from Lemma 5.2 are, actually, defining relations of $\text{Op}^- \mathbb{k}^{(I)}$.

Lemma 5.3. *Let A and B be finite subsets of $\mathbb{k}^{(I)}$. The following are equivalent:*

- (i) $\bigcap_{a \in A} [a] \subseteq \bigcup_{b \in B} [b]$;
- (ii) $\text{conv}(A) \cap \text{cone}(B) \neq \emptyset$;
- (iii) *there are elements $\xi_a, \eta_b \in \mathbb{k}^+$, for $a \in A$ and $b \in B$, such that $\sum_{a \in A} \xi_a a = \sum_{b \in B} \eta_b b$ whereas at least one ξ_a is nonzero.*

Proof. We may assume that I is finite. Also, the result is trivial in case $A = \emptyset$ (for $\text{conv}(\emptyset) = \emptyset$ and, thanks to the zero element, the empty intersection is never contained in $\bigcup_{b \in B} [b]$), so we may assume that $A \neq \emptyset$.

(iii) \Rightarrow (i). Owing to $\sum_{a \in A} \xi_a > 0$, for any $x \in \bigcap_{a \in A} [a]$,

$$0 < \left(\sum_{a \in A} \xi_a \right) \min_{a \in A} (a \mid x) \leq \sum_{a \in A} \xi_a (a \mid x) = \sum_{b \in B} \eta_b (b \mid x),$$

which, since each $\eta_b \geq 0$, entails that $x \in \bigcup_{b \in B} [b]$.

(i) \Rightarrow (ii). Suppose that (ii) fails, that is, $0 \notin \text{conv}(A) + (-\text{cone}(B))$. By the Decomposition Theorem for convex polyhedra (see for example Schrijver [21, Corollary 7.1.b]⁴), $\text{conv}(A) + (-\text{cone}(B))$ is a convex polyhedron, that is, a finite intersection of closed affine half-spaces, of $\mathbb{k}^{(I)}$. Since $0 \notin \text{conv}(A) + (-\text{cone}(B))$, there must exist $x \in \mathbb{k}^{(I)}$ such that $(c \mid x) \geq 1$ whenever $c \in \text{conv}(A) + (-\text{cone}(B))$. Since $0 \in \text{cone}(B)$, it follows that $(a \mid x) > 0$ whenever $a \in A$; so $x \in \bigcap_{a \in A} [a]$. Moreover, for all $a \in A$, $b \in B$, and $\lambda \in \mathbb{k}^+$, $(a - \lambda b \mid x) > 0$, that is, $(a \mid x) > \lambda(b \mid x)$. Taking λ large enough yields $(b \mid x) \leq 0$, whence $x \in \bigcap_{a \in A} [a] \setminus \bigcup_{b \in B} [b]$; that is, (i) fails.

(ii) \Rightarrow (iii) is trivial. \square

Remark 5.4. The case $B = \emptyset$ in Lemma 5.3 states that $\bigcap_{a \in A} [a] = \emptyset$ iff $0 \in \text{conv} A$.

Denote by $F_\ell(I, \mathbb{k})$ the free \mathbb{k} -vector lattice on a set I . As observed in Baker [2], Bernau [4], Madden [12, Ch. III] (see also Wehrung [25, page 13] for a summary), $F_\ell(I, \mathbb{k})$ canonically embeds into $\mathbb{k}^{\mathbb{k}^{(I)}}$. We sum up the corresponding facts that are relevant to our discussion.

Lemma 5.5 (Folklore).

- (1) $F_\ell(I, \mathbb{k})$ is isomorphic to the sublattice of $\mathbb{k}^{\mathbb{k}^{(I)}}$ generated by all linear functionals $\tilde{a}: x \mapsto (a \mid x)$ for $a \in \mathbb{k}^{(I)}$, with the i th generator given by $\tilde{\delta}_i$ whenever $i \in I$;
- (2) The assignment $\langle x \rangle \mapsto \{t \in \mathbb{k}^{(I)} \mid x(t) \neq 0\}$ defines a lattice isomorphism from $\text{Id}_c F_\ell(I, \mathbb{k})$ to $\text{Op}^- \mathbb{k}^{(I)}$, mapping $\langle \tilde{a}^+ \rangle$ to $[a]$ whenever $a \in \mathbb{k}^{(I)}$.

We shall thereon often identify each $a \in \mathbb{k}^{(I)}$ with its associated $\tilde{a} \in F_\ell(I, \mathbb{k})$.

⁴The proof stated there for vector spaces over the reals is valid over any totally ordered division ring (in particular, it involves only [semi]linear algebra).

6. HOMOMORPHISMS ON THE LATTICES $\text{Op}^- \mathcal{D}$

Standing hypothesis: \mathbb{k} is a totally ordered division ring, $J = I \sqcup \{o\}$ is a set, and L is a distributive 0-lattice.

A set \mathcal{D} of vectors is *symmetric* if $-u \in \mathcal{D}$ whenever $u \in \mathcal{D}$. An element $a \in \mathbb{k}^{(J)}$ is *normalized* if $a_o \in \{-1, 0, 1\}$, and a subset \mathcal{D} of $\mathbb{k}^{(J)}$ is normalized if each of its elements is normalized.

A map $\varphi: \text{Op}^- \mathcal{D} \rightarrow L$ *extends* a map $f: \mathcal{D} \rightarrow L$ if $\varphi[u] = f(u)$ whenever $u \in \mathcal{D}$ (in order to avoid cluttering we are writing $\varphi[u]$ instead of $\varphi([u])$).

Lemma 6.1. *Let \mathcal{D} be a normalized symmetric subset of $\mathbb{k}^{(J)} \setminus \mathbb{k}^{(I)}$, let $c \in \mathbb{k}^{(J)}$ be normalized, and let $\mathbf{c}^+, \mathbf{c}^- \in L$. A 0-lattice homomorphism $\varphi: \text{Op}^- (\mathbb{k}^{(I)} \cup \mathcal{D}) \rightarrow L$ extends to some 0-lattice homomorphism $\psi: \text{Op}^- (\mathbb{k}^{(I)} \cup \mathcal{D} \cup \{c, -c\}) \rightarrow L$ such that $(\psi[c], \psi[-c]) = (\mathbf{c}^+, \mathbf{c}^-)$ iff the following inequalities hold for any $u \in \mathcal{D} \cap (c + \mathbb{k}^{(I)})$:*

$$\mathbf{c}^+ \wedge \mathbf{c}^- = 0; \quad (6.1)$$

$$\mathbf{c}^+ \leq \varphi[c - u] \vee \varphi[u]; \quad (6.2)$$

$$\varphi[u] \leq \varphi[u - c] \vee \mathbf{c}^+; \quad (6.3)$$

$$\varphi[c - u] \leq \varphi[-u] \vee \mathbf{c}^+; \quad (6.4)$$

$$\mathbf{c}^- \leq \varphi[u - c] \vee \varphi[-u]; \quad (6.5)$$

$$\varphi[-u] \leq \varphi[c - u] \vee \mathbf{c}^-; \quad (6.6)$$

$$\varphi[u - c] \leq \varphi[u] \vee \mathbf{c}^-. \quad (6.7)$$

Furthermore, if \mathcal{D} is finite and the range of φ is consonant in L , then such a pair $(\mathbf{c}^+, \mathbf{c}^-)$ always exists.

Proof. The given conditions are obviously necessary (cf. Lemma 5.2). This also yields the desired equivalence in case $c \in \mathbb{k}^{(I)} \cup \mathcal{D}$ (in which case $\psi = \varphi$).

Thus suppose, from now on, that $c \notin \mathbb{k}^{(I)} \cup \mathcal{D}$ and that (6.1)–(6.7) all hold. We need to verify that there exists an extension ψ of φ as stated. Set $\mathcal{D}' \stackrel{\text{def}}{=} \mathcal{D} \cup \{c, -c\}$ and define a map $g: \mathbb{k}^{(I)} \cup \mathcal{D}' \rightarrow L$ by setting $g(x) \stackrel{\text{def}}{=} \varphi[x]$ whenever $x \in \mathbb{k}^{(I)} \cup \mathcal{D}$, whereas $g(c) \stackrel{\text{def}}{=} \mathbf{c}^+$ and $g(-c) \stackrel{\text{def}}{=} \mathbf{c}^-$. It suffices to verify that g extends to a 0-lattice homomorphism $\text{Op}^- (\mathbb{k}^{(I)} \cup \mathcal{D}') \rightarrow L$ (for any such homomorphism would necessarily extend φ). By Ploščica and Wehrung [19, Lemma 9]⁵, it suffices to verify that for any integers $k > 0$ and $l \geq 0$ (the case $l = 0$ accounting there for the preservation of the zero), together with elements $u_p, v_q \in \mathbb{k}^{(I)} \cup \mathcal{D}'$ for $1 \leq p \leq k$ and $1 \leq q \leq l$, the containment

$$\bigcap_{1 \leq p \leq k} [u_p] \subseteq \bigcup_{1 \leq q \leq l} [v_q] \quad (6.8)$$

entails the inequality

$$\bigwedge_{1 \leq p \leq k} g(u_p) \leq \bigvee_{1 \leq q \leq l} g(v_q). \quad (6.9)$$

Set $U \stackrel{\text{def}}{=} \{u_1, \dots, u_k\}$ and $V \stackrel{\text{def}}{=} \{v_1, \dots, v_l\}$. We argue by induction on the lexicographically ordered pair $(k+l, m)$ where m is the cardinality of $(U \cup V) \setminus \mathbb{k}^{(I)}$. We may thus assume that no proper subfamily of $(u_1, \dots, u_k, v_1, \dots, v_l)$ satisfies (6.8);

⁵One of the referees informed us that that lemma is also contained in part (i) of the Theorem on page 86 in Balbes and Dwinger's monograph [3], which is not available to us.

we will express this by stating that “ $k + l$ is minimal subjected to (6.9)”. Because of this and since (6.9) trivially holds in case some $u_p = v_q$, we may also assume that $u_1, \dots, u_k, v_1, \dots, v_l$ are pairwise distinct.

Claim. *For all $x, y \in \mathcal{D}'$, $x + y \in \mathbb{k}^{(I)}$ implies that $g(x) \wedge g(y) \leq g(x + y) \leq g(x) \vee g(y)$ and $g(x) \leq g(x + y) \vee g(-y)$.*

Proof of Claim. The inequalities $g(x + y) \leq g(x) \vee g(y)$ and $g(x) \leq g(x + y) \vee g(-y)$ follow from φ being a join-homomorphism, if $\{x, y\} \subseteq \mathcal{D}$, and from (6.1)–(6.7), if $\{x, y\} \cap \{c, -c\} \neq \emptyset$. Since $g(y) \wedge g(-y) = 0$ (use our assumption (6.1)), the second inequality implies that $g(x) \wedge g(y) \leq g(x + y)$. \square Claim.

It follows from Lemma 5.3 that there are $\alpha_p, \beta_q \in \mathbb{k}^+$, for $1 \leq p \leq k$ and $1 \leq q \leq l$, such that some α_p is nonzero and

$$\sum_{1 \leq p \leq k} \alpha_p u_p = \sum_{1 \leq q \leq l} \beta_q v_q. \quad (6.10)$$

Since $k + l$ is minimal subjected to (6.9), all α_p and all β_q are nonzero. If $m = 0$ then $U \cup V \subseteq \mathbb{k}^{(I)}$ and the conclusion (6.9) follows from φ being a lattice homomorphism. If m is nonzero, we separate cases.

Case 1. *There are distinct p, q such that $\{v_p, v_q\} \subseteq \mathcal{D}'$ and $v_p + v_q \in \mathbb{k}^{(I)}$.*

Up to permutation of the v_r , we may assume that $\{v_1, v_2\} \subseteq \mathcal{D}'$, $v_1 + v_2 \in \mathbb{k}^{(I)}$, and $\beta_1 \leq \beta_2$. Now (6.10) can be rewritten as

$$\sum_{1 \leq p \leq k} \alpha_p u_p = \beta_1(v_1 + v_2) + (\beta_2 - \beta_1)v_2 + \sum_{3 \leq q \leq l} \beta_q v_q.$$

It follows that $\bigcap_{1 \leq p \leq k} [u_p] \subseteq [v_1 + v_2] \cup \bigcup_{2 \leq q \leq l} [v_q]$. By our induction hypothesis (which applies since $v_1 \in \mathcal{D}'$ whereas $v_1 + v_2 \in \mathbb{k}^{(I)}$), we get $\bigwedge_{1 \leq p \leq k} g(u_p) \leq g(v_1 + v_2) \vee \bigvee_{2 \leq q \leq l} g(v_q)$, which, by the Claim above applied to the inequality $g(v_1 + v_2) \leq g(v_1) \vee g(v_2)$, entails (6.9).

Case 2. *There are distinct p, q such that $\{u_p, u_q\} \subseteq \mathcal{D}'$ and $u_p + u_q \in \mathbb{k}^{(I)}$.*

Up to permutation of the u_r , we may assume that $\{u_1, u_2\} \subseteq \mathcal{D}'$, $u_1 + u_2 \in \mathbb{k}^{(I)}$, and $\alpha_1 \leq \alpha_2$. Now (6.10) can be rewritten as

$$\alpha_1(u_1 + u_2) + (\alpha_2 - \alpha_1)u_2 + \sum_{3 \leq p \leq k} \alpha_p u_p = \sum_{1 \leq q \leq l} \beta_q v_q.$$

Since at least one element of $\{\alpha_1, \alpha_2 - \alpha_1, \alpha_3, \dots, \alpha_k\}$ is positive, it follows from our induction hypothesis that

$$g(u_1 + u_2) \wedge \bigwedge_{2 \leq p \leq k} g(u_p) \leq \bigvee_{1 \leq q \leq l} g(v_q).$$

Since, by our Claim, $g(u_1) \wedge g(u_2) \leq g(u_1 + u_2)$, (6.9) follows.

Now suppose that neither Case 1 nor Case 2 occurs. Since \mathcal{D}' is normalized, all u_p from \mathcal{D}' have the same σ -coordinate ε and all v_q from \mathcal{D}' have the same σ -coordinate η (so $\{\varepsilon, \eta\} \subseteq \{-1, 1\}$). Since $m > 0$, there must be at least one such vector on each side, and further, due to (6.10), we get $\varepsilon = \eta$. We may thus assume that $\{u_1, v_1\} \subseteq \mathcal{D}'$ and $u_1 - v_1 \in \mathbb{k}^{(I)}$. Two cases may then occur.

Case 3. $\alpha_1 > \beta_1$.

Rewriting (6.10) as

$$(\alpha_1 - \beta_1)u_1 + \sum_{2 \leq p \leq k} \alpha_p u_p = \beta_1(v_1 - u_1) + \sum_{2 \leq q \leq l} \beta_q v_q,$$

our induction hypothesis entails

$$\bigwedge_{1 \leq p \leq k} g(u_p) \leq g(v_1 - u_1) \vee \bigvee_{2 \leq q \leq l} g(v_q). \quad (6.11)$$

Now our Claim entails $g(v_1 - u_1) \leq g(v_1) \vee g(-u_1)$, thus, since the left hand side of (6.11) lies below $g(u_1)$ and $g(u_1) \wedge g(-u_1) = 0$, (6.9) holds.

Case 4. $\alpha_1 \leq \beta_1$.

Rewriting (6.10) as

$$\alpha_1(u_1 - v_1) + \sum_{2 \leq p \leq k} \alpha_p u_p = (\beta_1 - \alpha_1)v_1 + \sum_{2 \leq q \leq l} \beta_q v_q,$$

our induction hypothesis entails

$$g(u_1 - v_1) \wedge \bigwedge_{2 \leq p \leq k} g(u_p) \leq \bigvee_{1 \leq q \leq l} g(v_q).$$

Since, by our Claim, $g(u_1) \leq g(u_1 - v_1) \vee g(v_1)$, (6.9) follows. This concludes the proof of the equivalence between the existence of ψ and the system of inequalities (6.1)–(6.7).

Now suppose that \mathcal{D} is finite and the range of φ is consonant in L . Since the set $\mathcal{K} \stackrel{\text{def}}{=} \mathcal{D} \cup (\mathbb{k}^{(I)} \cap ((c + \mathcal{D}) \cup (-c + \mathcal{D})))$ is finite, it follows from Wehrung [22, Lemma 6.6] (stated there on real vector spaces; the \mathbb{k} -vector space version, stated in Wehrung [25, Lemma 4.7], is, *mutatis mutandis*, identical) that the restriction of φ to $\text{Op}^- \mathcal{K}$ extends to a 0-lattice homomorphism $\psi_0: \text{Op}^- (\mathcal{K} \cup \{c, -c\}) \rightarrow L$. Since $\{u, -u, u - c, c - u\}$ is contained in \mathcal{K} whenever $u \in \mathcal{D} \cap (c + \mathbb{k}^{(I)})$, all conditions (6.1)–(6.7), with $\mathbf{c}^+ \stackrel{\text{def}}{=} \psi_0[c]$ and $\mathbf{c}^- \stackrel{\text{def}}{=} \psi_0[-c]$, are satisfied. \square

7. THE CLOSURE STEP FOR OPEN HALF-SPACES

Standing hypothesis: \mathbb{k} is a totally ordered division ring and L is a distributive 0-lattice.

The following preparatory lemma aims at providing a better understanding of what it means for a map to be closed at a pair of open half-spaces.

Lemma 7.1. *The following statements hold, for every 0-lattice homomorphism $\varphi: \text{Op}^- \mathbb{k}^{(I)} \rightarrow L$ and all $a, b \in \mathbb{k}^{(I)}$:*

- (1) *For every $\lambda \in \mathbb{k}^+$ and every $\mathbf{e} \in L$, the inequality $\varphi[a] \wedge \varphi[a - \lambda b] \leq \mathbf{e}$ is equivalent to the conjunction of $\varphi[a] \leq \varphi[b] \vee \mathbf{e}$ and $\varphi[a - \lambda b] \leq \varphi[-b] \vee \mathbf{e}$.*
- (2) *Suppose that $\varphi[a] \leq \varphi[b] \vee \mathbf{e}$. Then for all $\lambda, \lambda' \in \mathbb{k}$ such that $0 \leq \lambda \leq \lambda'$, $\varphi[a] \wedge \varphi[a - \lambda b] \leq \mathbf{e}$ implies that $\varphi[a] \wedge \varphi[a - \lambda' b] \leq \mathbf{e}$.*
- (3) *The map φ is closed at $([a], [b])$ iff for every $\mathbf{e} \in L$ such that $\varphi[a] \leq \varphi[b] \vee \mathbf{e}$ there exists $\lambda \in \mathbb{k}^+$ such that $\varphi[a - \lambda b] \leq \varphi[-b] \vee \mathbf{e}$.*

Proof. Ad (1). Suppose first that $\varphi[a] \wedge \varphi[a - \lambda b] \leq \mathbf{e}$. From $\lambda \geq 0$ it follows that $[a] \subseteq [b] \cup ([a] \cap [a - \lambda b])$, thus

$$\varphi[a] \leq \varphi[b] \vee (\varphi[a] \wedge \varphi[a - \lambda b]) \leq \varphi[b] \vee \mathbf{e}.$$

Moreover, from $\lambda \geq 0$ it follows that $[a - \lambda b] \subseteq [a] \cup [-b]$, so we get

$$\varphi[a - \lambda b] \leq (\varphi[a] \wedge \varphi[a - \lambda b]) \vee \varphi[-b] \leq \varphi[-b] \vee \mathbf{e}.$$

Suppose, conversely, that $\varphi[a] \leq \varphi[b] \vee \mathbf{e}$ and $\varphi[a - \lambda b] \leq \varphi[-b] \vee \mathbf{e}$. From the former inequality together with $\varphi[b] \wedge \varphi[-b] = 0$ it follows that $\varphi[a] \wedge \varphi[-b] \leq \mathbf{e}$, whence

$$\varphi[a] \wedge \varphi[a - \lambda b] \leq \varphi[a] \wedge (\varphi[-b] \vee \mathbf{e}) \leq \mathbf{e}.$$

Ad (2). Suppose that $\varphi[a] \wedge \varphi[a - \lambda b] \leq \mathbf{e}$. By (1) above, this means that $\varphi[a - \lambda b] \leq \varphi[-b] \vee \mathbf{e}$. Hence $\varphi[a - \lambda' b] \leq \varphi[a - \lambda b] \vee \varphi[(\lambda - \lambda')b] \leq \varphi[-b] \vee \mathbf{e}$, so, by (1) above, $\varphi[a] \wedge \varphi[a - \lambda' b] \leq \mathbf{e}$.

Ad (3). Suppose first that φ is closed at $([a], [b])$. Identifying the lattices $\text{Op}^- \mathbb{k}^{(I)}$ and $\text{Id}_c \text{F}_\ell(I, \mathbb{k})$ via the isomorphism $[x] \mapsto \langle x^+ \rangle$ given by Lemma 5.5, any inequality of the form $\varphi[a] \leq \varphi[b] \vee \mathbf{e}$, where $\mathbf{e} \in L$, translates to $\varphi\langle a^+ \rangle \leq \varphi\langle b^+ \rangle \vee \mathbf{e}$, thus, since φ is closed at $([a], [b])$, there exists $e \in \text{F}_\ell(I, \mathbb{k})^+$ such that $\langle a^+ \rangle \leq \langle b^+ \rangle \vee \langle e \rangle$ whereas $\varphi\langle e \rangle \leq \mathbf{e}$. The former inequality means that $a^+ \leq \lambda(b^+ + e)$ for some $\lambda \in \mathbb{k}^+$, so $(a^+ - \lambda b^+)^+ \leq \lambda e$. By virtue of the identity $(a^+ - \lambda b^+)^+ = a^+ \wedge (a - \lambda b)^+$ we thus get

$$\varphi\langle a^+ \rangle \wedge \varphi\langle (a - \lambda b)^+ \rangle \leq \varphi\langle \lambda e \rangle \leq \varphi\langle e \rangle \leq \mathbf{e},$$

which, owing to the identification given by $\text{Op}^- \mathbb{k}^{(I)} \cong \text{Id}_c \text{F}_\ell(I, \mathbb{k})$, can be written

$$\varphi[a] \wedge \varphi[a - \lambda b] \leq \mathbf{e}.$$

Conversely, if for every $\mathbf{e} \in L$ such that $\varphi[a] \leq \varphi[b] \vee \mathbf{e}$ there exists $\lambda \in \mathbb{k}^+$ such that $\varphi[a - \lambda b] \leq \varphi[-b] \vee \mathbf{e}$, that is, due to (1) above, $\varphi([a] \cap [a - \lambda b]) \leq \mathbf{e}$, then, due to the containment $[a] \subseteq [b] \cup ([a] \cap [a - \lambda b])$, φ is closed at $([a], [b])$. \square

For the remainder of this section let $J = I \cup \{o\}$ with $o \notin I$, let \mathcal{D} be a symmetric, normalized subset of $\mathbb{k}^{(J)} \setminus \mathbb{k}^{(I)}$, let $\{a, b\} \subseteq \mathbb{k}^{(I)} \cup \mathcal{D}$, let $\mathbf{e} \in L$, and let $\varphi: \text{Op}^-(\mathbb{k}^{(I)} \cup \mathcal{D}) \rightarrow L$ be a 0-lattice homomorphism. We also assume that the restriction of φ to $\text{Op}^- \mathbb{k}^{(I)}$ is closed and that $\varphi[a] \leq \varphi[b] \vee \mathbf{e}$.

Notation 7.2. For all $x \in \mathbb{k}^{(J)}$ and $u \in \mathbb{k}^{(J)} \setminus \mathbb{k}^{(I)}$, we denote by x^u the unique element in $\mathbb{k}^{(I)} \cap (x + \mathbb{k}u)$. That is, $x^u = x - x_o u_o^{-1} u$.

Observe that $x^u = x^{-u}$. Also, in all the cases that we will consider, u will be normalized (i.e., $u_o \in \{-1, 1\}$); so $u_o^{-1} = u_o$.

Lemma 7.3. *Assume $\{a, b\} \not\subseteq \mathbb{k}^{(I)}$ and let $u \in \mathcal{D}$ such that either $b_o u_o = 1$ or ($b_o = 0$ and $a_o u_o = -1$). The following statements hold:*

- (1) $\varphi[a^u] \leq \varphi[b^u] \vee \varphi[u] \vee \varphi[-b] \vee \mathbf{e}$.
- (2) For all large enough $\lambda \in \mathbb{k}^+$, $\varphi[a^u - \lambda b^u] \leq \varphi[u] \vee \varphi[-b] \vee \mathbf{e}$.

Proof. *Ad (1).* From $a^u = a - a_o u_o u$ and $\varphi[a] \leq \varphi[b] \vee \mathbf{e}$ it follows that

$$\varphi[a^u] \leq \varphi[a] \vee \varphi[-a_o u_o u] \leq \varphi[b] \vee \varphi[-a_o u_o u] \vee \mathbf{e}. \quad (7.1)$$

If $b_o = 0$ and $a_o u_o = -1$, then $b^u = b$ and (1) follows. Let $b_o u_o = 1$; so $b^u = b - u$. Then $\varphi[b] \leq \varphi[b^u] \vee \varphi[u]$ thus, by (7.1),

$$\varphi[a^u] \leq \varphi[b^u] \vee \varphi[u] \vee \varphi[-a_o u_o u] \vee \mathbf{e}. \quad (7.2)$$

Hence, if $a_o u_o \in \{-1, 0\}$ then we get (1) right away. If $a_o u_o = 1$, then

$$\varphi[-a_o u_o u] = \varphi[-u] \leq \varphi[b^u] \vee \varphi[-b],$$

which, combined with (7.2), yields again (1).

Ad (2). Since φ is closed at (a^u, b^u) and by (1) together with Lemma 7.1, there exists $\lambda_0 \in \mathbb{k}^+$ such that

$$\varphi[a^u - \lambda b^u] \leq \varphi[-b^u] \vee \varphi[u] \vee \varphi[-b] \vee \mathbf{e} \quad \text{whenever } \lambda > \lambda_0. \quad (7.3)$$

If $b_o = 0$, then $b^u = b$ and we get (2) right away. If $b_o u_o = 1$ then $b^u = b - u$, thus $\varphi[-b^u] \leq \varphi[u] \vee \varphi[-b]$, which, together with (7.3), yields (2) again. \square

Lemma 7.4. *Assume $\{a, b\} \not\subseteq \mathbb{k}^{(I)}$ and suppose that either $b_o u_o = -1$ or ($b_o = 0$ and $a_o u_o = 1$). Then*

$$\varphi[u] \leq \varphi[\lambda b^u - a^u] \vee \varphi[-b] \vee \mathbf{e} \quad \text{for all large enough } \lambda \in \mathbb{k}^+. \quad (7.4)$$

Proof. Since $-u$ satisfies the assumptions of Lemma 7.3, there is $\lambda_0 \in \mathbb{k}^+$ such that

$$\varphi[a^u - \lambda b^u] \leq \varphi[-u] \vee \varphi[-b] \vee \mathbf{e} \quad \text{whenever } \lambda > \lambda_0. \quad (7.5)$$

Let $\lambda > \lambda_0$; set $\mathbf{e}' \stackrel{\text{def}}{=} \varphi[\lambda b^u - a^u] \vee \varphi[-b] \vee \mathbf{e}$ and $\mathbf{e}'' \stackrel{\text{def}}{=} \mathbf{e}' \vee \varphi[-u]$.

We claim that $\varphi[b^u] \leq \mathbf{e}''$. Pick $\lambda' \in \mathbb{k}$ with $\lambda_0 < \lambda' < \lambda$. From the containment $[b^u] \subseteq [\lambda b^u - a^u] \cup [a^u - \lambda' b^u]$ it follows that $\varphi[b^u] \leq \varphi[\lambda b^u - a^u] \vee \varphi[a^u - \lambda' b^u]$, which, by (7.5) together with the definition of \mathbf{e}'' , entails $\varphi[b^u] \leq \mathbf{e}''$, as claimed.

We next claim that $\varphi[u] \leq \mathbf{e}''$. If $b_o u_o = -1$ then $b^u = b + u$, thus

$$\varphi[u] \leq \varphi[b^u] \vee \varphi[-b] \leq \mathbf{e}'',$$

as desired. Suppose now that $b_o = 0$; so $b^u = b$ and $a^u = a - u$. From the containment $[-a^u] \subseteq [\lambda b^u - a^u] \cup [-b]$ we get $\varphi[-a^u] \leq \varphi[\lambda b^u - a^u] \vee \varphi[-b] \leq \mathbf{e}'$, thus, since $[u] \subseteq [a] \cup [-a^u]$, we get

$$\begin{aligned} \varphi[u] &\leq \varphi[a] \vee \varphi[-a^u] \\ &\leq \varphi[a] \vee \mathbf{e}' \\ &\leq \varphi[b] \vee \mathbf{e}' && \text{(because } \varphi[a] \leq \varphi[b] \vee \mathbf{e} \text{)} \\ &= \varphi[b^u] \vee \mathbf{e}' && \text{(because } b^u = b \text{)} \\ &\leq \mathbf{e}'', \end{aligned}$$

thus completing the proof of our second claim. Since $\varphi[u] \leq \mathbf{e}'' = \mathbf{e}' \vee \varphi[-u]$ and $\varphi[u] \wedge \varphi[-u] = 0$, it follows that $\varphi[u] \leq \mathbf{e}'$. \square

Lemma 7.5. *Suppose that \mathcal{D} is finite and the range of φ is consonant in L . Then for all large enough $\lambda \in \mathbb{k}^+$, the map φ extends to a lattice homomorphism $\psi: \text{Op}^-(\mathbb{k}^{(I)} \cup \mathcal{D} \cup \{a - \lambda b, \lambda b - a\}) \rightarrow L$ such that $\psi[a - \lambda b] \leq \varphi[-b] \vee \mathbf{e}$.*

Proof. If $\{a, b\} \subseteq \mathbb{k}^{(I)}$ then the desired conclusion follows from the restriction of φ to $\text{Op}^-\mathbb{k}^{(I)}$ being closed (cf. Lemma 7.1).

Suppose from now on that $\{a, b\} \not\subseteq \mathbb{k}^{(I)}$. Let $\lambda_0 \in \mathbb{k}^+$ such that all inequalities in Lemmas 7.3 and 7.4 are satisfied whenever $u \in \mathcal{D}$. We may also assume that $a_o - \lambda b_o$ has constant sign, necessarily nonzero, over $\lambda > \lambda_0$. Fixing such a λ , the scalar $\xi \stackrel{\text{def}}{=} |a_o - \lambda b_o|^{-1}$ is positive, and further, $c \stackrel{\text{def}}{=} \xi \cdot (a - \lambda b)$ is a normalized element of $\mathbb{k}^{(J)} \setminus \mathbb{k}^{(I)}$. For any $u \in \mathcal{D}$, two cases may occur.

Case 1. $u_o + c_o = 0$ (i.e., $u + c \in \mathbb{k}^{(I)}$).

It follows that either $b_o u_o = 1$ or ($b_o = 0$ and $a_o u_o = -1$). Then $u + c = c^u = \xi \cdot (a^u - \lambda b^u)$, whence, applying Lemma 7.3,

$$\varphi[u + c] = \varphi[a^u - \lambda b^u] \leq \varphi[u] \vee \varphi[-b] \vee \mathbf{e}. \quad (7.6)$$

Case 2. $u_o = c_o$ (i.e., $u - c \in \mathbb{k}^{(I)}$).

It follows that either $b_o u_o = -1$ or ($b_o = 0$ and $a_o u_o = 1$). Then $u - c = -c^u = \xi \cdot (\lambda b^u - a^u)$, whence $\varphi[u - c] = \varphi[\lambda b^u - a^u]$, and so, by Lemma 7.4,

$$\varphi[u] \leq \varphi[\lambda b^u - a^u] \vee \varphi[-b] \vee \mathbf{e} = \varphi[u - c] \vee \varphi[-b] \vee \mathbf{e}. \quad (7.7)$$

Since \mathcal{D} is finite and the range of φ is consonant in L , it follows from the last statement of Lemma 6.1 that the system of inequalities (6.1)–(6.7) has a solution $(\mathbf{c}^+, \mathbf{c}^-)$. Set $\mathbf{c}^* \stackrel{\text{def}}{=} \mathbf{c}^+ \wedge (\varphi[-b] \vee \mathbf{e})$. We claim that the pair $(\mathbf{c}^*, \mathbf{c}^-)$ also satisfies the system of inequalities (6.1)–(6.7): indeed, since $(\mathbf{c}^+, \mathbf{c}^-)$ already satisfies that system and $\mathbf{c}^* \leq \mathbf{c}^+$, the only inequalities that need to be taken care of are (6.3) and (6.4), which then follow from (7.6) and (7.7) above.

By Lemma 6.1, the homomorphism φ extends to a lattice homomorphism $\psi: \text{Op}^-(\mathbb{k}^{(I)} \cup \mathcal{D} \cup \{c, -c\}) \rightarrow L$ such that $\psi[c] = \mathbf{c}^*$ and $\psi[-c] = \mathbf{c}^-$. In particular, $\psi[a - \lambda b] = \psi[c] \leq \varphi[-b] \vee \mathbf{e}$. \square

8. PROOF OF THE MAIN RESULT

Theorem 8.1 (The Closure Step). *Let $J \stackrel{\text{def}}{=} I \sqcup \{o\}$ be a countable set, let \mathbb{k} be a countable totally ordered division ring, let L be a countable completely normal distributive 0-lattice, and let $\mathbf{d}_0, \mathbf{d}_1 \in L$ such that $\mathbf{d}_0 \wedge \mathbf{d}_1 = 0$. Then every closed 0-lattice homomorphism $\varphi: \text{Op}^-\mathbb{k}^{(I)} \rightarrow L$ extends to a closed 0-lattice homomorphism $\psi: \text{Op}^-\mathbb{k}^{(J)} \rightarrow L$ such that $\psi[\delta_o] = \mathbf{d}_0$ and $\psi[-\delta_o] = \mathbf{d}_1$.*

Proof. Denote by \mathcal{N} the set of all normalized elements of $\mathbb{k}^{(J)} \setminus \mathbb{k}^{(I)}$. Since \mathbb{k} , I , J , and L are all countable, we can write $\mathcal{N} = \{c_n \mid n < \omega\}$, $\mathbb{k}^{(I)} = \{f_n \mid n < \omega\}$, and $L = \{e_n \mid n < \omega\}$.

Define inductively finite symmetric subsets \mathcal{D}_n of \mathcal{N} and 0-lattice homomorphisms $\varphi_n: \text{Op}^-(\mathbb{k}^{(I)} \cup \mathcal{D}_n) \rightarrow L$, for $n < \omega$, as follows. Set $\mathcal{D}_0 \stackrel{\text{def}}{=} \{\delta_o, -\delta_o\}$. By the argument of Wehrung [22, Lemma 8.3] (alternatively, apply Lemma 6.1 with $\mathcal{D} \stackrel{\text{def}}{=} \emptyset$), there is a unique 0-lattice homomorphism $\varphi_0: \text{Op}^-(\mathbb{k}^{(I)} \cup \mathcal{D}_0) \rightarrow L$ such that $\varphi_0[\delta_o] = \mathbf{d}_0$ and $\varphi_0[-\delta_o] = \mathbf{d}_1$.

Suppose \mathcal{D}_n and φ_n defined.

If n is even, set $\mathcal{D}_{n+1} \stackrel{\text{def}}{=} \mathcal{D}_n \cup \{c_{n/2}, -c_{n/2}\}$. By the last statement of Lemma 6.1, φ_n extends to a lattice homomorphism $\varphi_{n+1}: \text{Op}^-(\mathbb{k}^{(I)} \cup \mathcal{D}_{n+1}) \rightarrow L$.

If n is odd, then a straightforward finite iteration of Lemma 7.5 yields a finite symmetric subset \mathcal{D}_{n+1} of \mathcal{N} containing \mathcal{D}_n and a 0-lattice homomorphism $\varphi_{n+1}: \text{Op}^-(\mathbb{k}^{(I)} \cup \mathcal{D}_{n+1}) \rightarrow L$ such that for all $a, b \in \mathcal{D}_n \cup \{f_l \mid l \leq n\}$ and all $k \leq n$,

$$\varphi_n[a] \leq \varphi_n[b] \vee \mathbf{e}_k \Rightarrow (\exists \lambda \in \mathbb{k}^+) (\varphi_{n+1}[a - \lambda b] \leq \varphi_n[-b] \vee \mathbf{e}_k) \quad (8.1)$$

(where it is understood, from the notation $\varphi_{n+1}[a - \lambda b]$, that $\xi \cdot (a - \lambda b)$ belongs to $\mathbb{k}^{(I)} \cup \mathcal{D}_{n+1}$ for some $\xi > 0$). By construction, $\mathcal{N} = \bigcup_{n < \omega} \mathcal{D}_n$ and the union $\psi \stackrel{\text{def}}{=} \bigcup_{n < \omega} \varphi_n$ is a 0-lattice homomorphism from $\text{Op}^-(\mathbb{k}^{(I)} \cup \mathcal{N}) = \text{Op}^-\mathbb{k}^{(J)}$ to L . Moreover, owing to (8.1) together with Lemma 7.1, ψ is closed at every pair $([a], [b])$ where $a, b \in \mathbb{k}^{(J)}$. Since $\{[x] \mid x \in \mathbb{k}^{(J)}\}$ generates $\text{Op}^-\mathbb{k}^{(J)}$ as a sublattice, it follows from Lemma 4.4 that ψ is closed. \square

Theorem 8.2. *Let $J = I \sqcup \omega$ be a countable set, let \mathbb{k} be a countable totally ordered division ring, and let L be a countable completely normal distributive 0-lattice. Then every closed 0-lattice homomorphism $\psi_0: \text{Op}^- \mathbb{k}^{(I)} \rightarrow L$ extends to a surjective closed 0-lattice homomorphism $\psi: \text{Op}^- \mathbb{k}^{(J)} \rightarrow L$.*

Proof. Set $I_n \stackrel{\text{def}}{=} I \cup \{0, 1, \dots, n-1\}$, whenever $n < \omega$. Moreover, let $L = \{e_n \mid n < \omega\}$. By a repeated use of Theorem 8.1 we obtain a sequence of closed 0-lattice homomorphisms $\psi_n: \text{Op}^- \mathbb{k}^{(I_n)} \rightarrow L$, each ψ_n extending the previous one, and satisfying $\psi_{n+1}[\delta_n] = e_n$ and $\psi_{n+1}[-\delta_n] = 0$. Define ψ as the common extension of all ψ_n . \square

Theorem 8.3 (Main Theorem). *Let \mathbb{k} be a countable totally ordered division ring, let E be a countable \mathbb{k} -vector lattice, let L be a countable completely normal distributive 0-lattice, and let $\varphi: \text{Id}_c E \rightarrow L$ be a closed 0-lattice homomorphism. Then there are a countable \mathbb{k} -vector lattice F , a \mathbb{k}, ℓ -homomorphism $f: E \rightarrow F$, and a lattice isomorphism $\iota: \text{Id}_c F \rightarrow L$ such that $\varphi = \iota \circ \text{Id}_c f$.*

Proof. Throughout the proof let us fix a countably infinite set I , disjoint from ω , and set $J \stackrel{\text{def}}{=} I \sqcup \omega$.

Since E is countable, there exists a surjective \mathbb{k}, ℓ -homomorphism $p: F_\ell(I, \mathbb{k}) \rightarrow E$. By Wehrung [22, Proposition 2.6] (also valid for \mathbb{k} -vector lattices, with the same proof), $\text{Id}_c p$ is a surjective closed 0-lattice homomorphism from $\text{Id}_c F_\ell(I, \mathbb{k})$ onto $\text{Id}_c E$. The map $\psi_0 \stackrel{\text{def}}{=} \varphi \circ \text{Id}_c p$ is a closed 0-lattice homomorphism from $\text{Id}_c F_\ell(I, \mathbb{k})$ to L . By Theorem 8.2, together with the canonical isomorphism $\text{Id}_c F_\ell(I, \mathbb{k}) \cong \text{Op}^- \mathbb{k}^{(I)}$ given by Lemma 5.5, ψ_0 can be extended to a surjective closed 0-lattice homomorphism from $\text{Id}_c F_\ell(J, \mathbb{k})$ onto L . (Here we identify $\text{Id}_c F_\ell(I, \mathbb{k})$ with its canonical image into $\text{Id}_c F_\ell(J, \mathbb{k})$.)

Then $Q \stackrel{\text{def}}{=} \{x \in F_\ell(J, \mathbb{k}) \mid \psi(\langle x \rangle) = 0\}$ is an ℓ -ideal of $F_\ell(J, \mathbb{k})$, and further, denoting by q the canonical projection from $F_\ell(J, \mathbb{k})$ onto $F \stackrel{\text{def}}{=} F_\ell(J, \mathbb{k})/Q$, there exists a unique isomorphism $\iota: \text{Id}_c F \rightarrow L$ such that $\psi = \iota \circ \text{Id}_c q$ (cf. Wehrung [22, Lemma 2.5]). The argument can be followed on Figure 8.1.

$$\begin{array}{ccccc}
 \text{Id}_c F_\ell(I, \mathbb{k}) & \longleftrightarrow & \text{Id}_c F_\ell(J, \mathbb{k}) & & \\
 \downarrow \text{Id}_c p & \searrow \psi_0 & \downarrow \psi & \searrow \text{Id}_c q & \\
 \text{Id}_c E & \xrightarrow[\text{closed}]{\varphi} & L & \xleftarrow[\cong]{\iota} & \text{Id}_c F
 \end{array}$$

FIGURE 8.1. Illustrating the proof of Theorem 8.3

Now observe the following implications, for any $x \in F_\ell(I, \mathbb{k})$:

$$\begin{aligned}
 p(x) = 0 &\Leftrightarrow (\text{Id}_c p)(\langle x \rangle) = 0 \\
 &\Rightarrow \psi_0(\langle x \rangle) = 0 && \text{(because } \psi_0 = \varphi \circ \text{Id}_c p \text{)} \\
 &\Leftrightarrow \psi(\langle x \rangle) = 0 \\
 &\Leftrightarrow q(x) = 0.
 \end{aligned}$$

Hence, by the First Isomorphism Theorem (for \mathbb{k} -vector lattices), there exists a unique \mathbb{k}, ℓ -homomorphism $f: E \rightarrow F$ such that $f \circ p = q \upharpoonright_{F_\ell(I, \mathbb{k})}$. It follows that

$$\iota \circ \text{Id}_c f \circ \text{Id}_c p = \iota \circ \text{Id}_c (q \upharpoonright_{F_\ell(I, \mathbb{k})}) = \psi \upharpoonright_{\text{Id}_c F_\ell(I, \mathbb{k})} = \varphi \circ \text{Id}_c p.$$

Since the map $\text{Id}_c p$ is surjective, it follows that $\varphi = \iota \circ \text{Id}_c f$. \square

It is then an easy matter to extend Theorem 8.3 to its following diagram version.

Theorem 8.4. *Let \mathbb{k} be a countable totally ordered division ring and let T be a forest in which every element has countable height. Then every T -indexed commutative diagram $\vec{L} \stackrel{\text{def}}{=} (L_s, \varphi_{s,t} \mid s \leq t \text{ in } T)$ of countable completely normal distributive 0-lattices, with closed 0-lattice homomorphisms, is isomorphic to $\text{Id}_c \vec{E}$ for some T -indexed commutative diagram \vec{E} of \mathbb{k} -vector lattices with \mathbb{k}, ℓ -homomorphisms.*

We illustrate Theorem 8.4 on Figure 8.2, with o a minimal element of T and p an atom of T .

$$\begin{array}{ccccccc} \text{Id}_c E_o & \xrightarrow{\text{Id}_c f_{o,p}} & \text{Id}_c E_p & \longrightarrow & \cdots & \longrightarrow & \text{Id}_c E_t \longrightarrow \cdots \\ \cong \downarrow \chi_o & & \cong \downarrow \chi_p & & & & \cong \downarrow \chi_t \\ L_o & \xrightarrow[\text{closed}]{\varphi_{o,p}} & L_p & \xrightarrow{\text{closed}} & \cdots & \xrightarrow{\text{closed}} & L_t \xrightarrow{\text{closed}} \cdots \end{array}$$

FIGURE 8.2. Illustrating Theorem 8.4

Outline of proof. An inductive argument within T , similar, *mutatis mutandis*, to the one of Ploščica and Wehrung [19, Theorem 7.3], with Theorem 8.3 used instead of [19, Theorem 1], E_s instead of $F_\ell(I_s, \mathbb{k})$, $f_{s,t}$ instead of η_{I_s, I_t} , and all maps χ_t now being isomorphisms. At successor stages we apply Theorem 8.3, while at limit stages we apply the preservation of directed colimits by the functor Id_c . \square

The above argument yields in fact that *every partial lifting of \vec{L} , defined on a lower subset of T , can be extended to a full lifting of \vec{L} .*

Corollary 8.5. *The following are equivalent, for any countable totally ordered division ring \mathbb{k} and any distributive 0-lattice L of cardinality at most \aleph_1 :*

- (i) *There exists a \mathbb{k} -vector lattice E such that $L \cong \text{Id}_c E$.*
- (ii) *There exists an Abelian ℓ -group G such that $L \cong \text{Id}_c G$.*
- (iii) *The lattice L is completely normal and it has countably based differences.*

Proof. (ii) \Rightarrow (iii) is well known (and easy), see Cignoli *et al.* [5, Thm. 2.2]; see also Iberkleid *et al.* [9, Prop. 4.1.2], Wehrung [22, Lemma 10.1]. A similar argument also yields the implication (i) \Rightarrow (iii) (the countability assumption on \mathbb{k} can there be weakened by just saying that \mathbb{k} has a countable cofinal sequence).

(iii) \Rightarrow (i). Let L be a completely normal distributive 0-lattice with countably based differences, of cardinality at most \aleph_1 . Let us write $L = \{e_\xi \mid \xi < \omega_1\}$. The assumption that L has countably based differences means that there exists a sequence $(\searrow_n \mid n < \omega)$ of binary operations on L such that for all $a, b, c \in L$, $a \leq b \vee c$ iff there exists $n < \omega$ such that $a \searrow_n b \leq c$. For each $\alpha < \omega_1$, the closure L_α of $\{0\} \cup \{e_\xi \mid \xi < \alpha\}$ under the lattice operations together with all \searrow_n is a countable completely normal 0-sublattice of L , and L is the ascending union of the L_α .

Moreover, due to all L_ξ being closed under all operations \searrow_n , the inclusion map from L_α into L_β is a closed 0-lattice embedding whenever $\alpha < \beta < \omega_1$. Hence, by applying Theorem 8.4 to the well-ordered chain $\vec{L} \stackrel{\text{def}}{=} (L_\alpha \mid \alpha < \omega_1)$ of sublattices

of L , with the inclusion maps as transition maps (and $T \stackrel{\text{def}}{=} \omega_1$), we obtain an ω_1 -indexed commutative diagram $\vec{E} = (E_\alpha, f_{\alpha,\beta} \mid \alpha \leq \beta < \omega_1)$ of \mathbb{k} -vector lattices such that $\text{Id}_c \vec{E} \cong \vec{L}$. Setting $E \stackrel{\text{def}}{=} \varinjlim \vec{E}$, the preservation of all directed colimits by the functor Id_c , together with the universal property of the colimit, yields $\text{Id}_c E \cong L$.

(iii) \Rightarrow (ii) is just (iii) \Rightarrow (i) with $\mathbb{k} \stackrel{\text{def}}{=} \mathbb{Q}$. \square

Recall from Wehrung [25, § 9] that the direction (iii) \Rightarrow (i) from Corollary 8.5 fails at every uncountable \mathbb{k} , even for countable lattices L .

9. OPEN PROBLEMS

Martínez asks in [13, Question II] for a characterization of all lattices $\text{Id}_c G$ for G an Archimedean ℓ -group. Since all members of the class $\mathcal{A}(\theta, \vec{A})$ introduced in Wehrung [24, Notation 12.2] are Archimedean ℓ -groups whenever $\theta > \omega$ (see the comments following Notation 12.2 in [24]), [24, Theorem 12.3] already yields a negative answer to Martínez' question: *the class of all isomorphic copies of $\text{Id}_c G$, for G Archimedean, is not the class of models of any class of $\mathcal{L}_{\infty\lambda}$ sentences for any λ ; nor is it co-projective* (as defined on page 2).

However, *the lattices in $\mathcal{A}(\theta, \vec{A})$ have no top element*, because the arrows in the cube diagram \vec{A} from Wehrung [23] are not unit-preserving. This suggests the following problem, already hinted at in the comments following [24, Corollary 12.9].

Problem 9.1. Is the class of all isomorphic copies of $\text{Id}_c G$, for G an Archimedean ℓ -group *with order-unit*, the class of all models of some $\mathcal{L}_{\infty\infty}$ sentence?

Complete normality is a first-order sentence, while “countably based differences” is an $\mathcal{L}_{\omega_1\omega_1}$ sentence. The following problem asks for an extension of Corollary 8.5 to real spectra of commutative unital rings.

Problem 9.2. Denote by $\Phi(A)$ the Stone dual of the real spectrum $\text{Spec}_r A$ of any commutative unital ring A . Can the isomorphic copies of the $\Phi(A)$, with cardinality at most \aleph_1 , be characterized (within structures of size $\leq \aleph_1$) by an $\mathcal{L}_{\omega_1\omega_1}$ sentence?

In more detail: it is well known that every $\Phi(A)$ is a completely normal bounded distributive lattice. The main result of Wehrung [25] states that every *countable* completely normal bounded distributive lattice can be obtained in this way. As observed in Wehrung [27], this fails in a strong sense for lattices of cardinality \aleph_1 : *there exists an Abelian ℓ -group G of size \aleph_1 such that $\text{Id}_c G$ is not a homomorphic image of any $\Phi(A)$* . It is also observed in [27] that a homomorphic image of some $\Phi(A)$ of cardinality \aleph_1 need not be isomorphic to any $\Phi(A')$.

The cardinality assumption in Problem 9.2 cannot be omitted: by Mellor and Tressl [14], the class \mathcal{R} of isomorphic copies of all $\Phi(A)$ is not the class of models of any class of $\mathcal{L}_{\infty\lambda}$ sentences for fixed λ . This result is extended in Wehrung [26], by proving that \mathcal{R} is not co-projective.

For any chain T , denote by $\mathcal{O}(T)$ the sublattice of the powerset of T generated by all open intervals $\{x \in T \mid x < a\}$ and $\{x \in T \mid x > a\}$ for $a \in T$. It is not hard to verify that $\mathcal{O}(T)$ is a completely normal bounded distributive lattice.

Problem 9.3. Let T be a chain such that $\mathcal{O}(T)$ has countably based differences. Does there exist an Abelian ℓ -group G such that $\mathcal{O}(T) \cong \text{Id}_c G$?

By Corollary 8.5, the answer to Problem 9.3 is positive for $\text{card } T \leq \aleph_1$.

10. DECLARATIONS

Availability of Data and Materials:

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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REFERENCES

- [1] Marlow Anderson and Todd Feil, *Lattice-Ordered Groups*, Reidel Texts in the Mathematical Sciences, D. Reidel Publishing Co., Dordrecht, 1988, An introduction. MR 937703 (90b:06001)
- [2] Kirby A. Baker, *Free vector lattices*, Canad. J. Math. **20** (1968), 58–66. MR 0224524
- [3] Raymond Balbes and Philip Dwinger, *Distributive Lattices*, Abstract Space Publishing, 2011.
- [4] Simon J. Bernau, *Free abelian lattice groups*, Math. Ann. **180** (1969), 48–59. MR 0241340
- [5] Roberto Cignoli, Daniel Gluschankof, and François Lucas, *Prime spectra of lattice-ordered abelian groups*, J. Pure Appl. Algebra **136** (1999), no. 3, 217–229. MR 1675803
- [6] Charles N. Delzell and James J. Madden, *A completely normal spectral space that is not a real spectrum*, J. Algebra **169** (1994), no. 1, 71–77. MR 1296582
- [7] Pierre Gillibert and Friedrich Wehrung, *From Objects to Diagrams for Ranges of Functors*, Lecture Notes in Mathematics, vol. 2029, Springer, Heidelberg, 2011. MR 2828735 (2012i:18001)
- [8] Melvin Hochster, *Prime ideal structure in commutative rings*, Trans. Amer. Math. Soc. **142** (1969), 43–60. MR 0251026
- [9] Wolf Iberkleid, Jorge Martínez, and Warren Wm. McGovern, *Conrad frames*, Topology Appl. **158** (2011), no. 14, 1875–1887. MR 2823701
- [10] Klaus Keimel, *The representation of lattice-ordered groups and rings by sections in sheaves*, (1971), 1–98. Lecture Notes in Math., Vol. 248. MR 0422107
- [11] Giacomo Lenzi and Antonio Di Nola, *The spectrum problem for Abelian ℓ -groups and MV-algebras*, Algebra Universalis **81** (2020), no. 3, Paper No. 39, 43. MR 4120404
- [12] James J. Madden, *Two methods in the study of k -vector lattices*, Ph.D. thesis, Wesleyan University, 1984. MR 2633496
- [13] Jorge Martínez, *Archimedean lattices*, Algebra Universalis **3** (1973), 247–260. MR 0349503
- [14] Timothy Mellor and Marcus Tressl, *Non-axiomatizability of real spectra in $\mathcal{L}_{\infty\lambda}$* , Ann. Fac. Sci. Toulouse Math. (6) **21** (2012), no. 2, 343–358. MR 2978098
- [15] António A. Monteiro, *L'arithmétique des filtres et les espaces topologiques*, Segundo symposium sobre algunos problemas matemáticos que se están estudiando en Latino América, Julio, 1954, Centro de Cooperación Científica de la UNESCO para América Latina, Montevideo, Uruguay, 1954, pp. 129–162. MR 0074805
- [16] Daniele Mundici, *Interpretation of AF C^* -algebras in Łukasiewicz sentential calculus*, J. Funct. Anal. **65** (1986), no. 1, 15–63. MR 819173 (87k:46146)
- [17] ———, *Advanced Łukasiewicz Calculus and MV-Algebras*, Trends in Logic—Studia Logica Library, vol. 35, Springer, Dordrecht, 2011. MR 2815182
- [18] Miroslav Ploščica, *Cevian properties in ideal lattices of Abelian ℓ -groups*, Forum Math. **33** (2021), no. 6, 1651–1658. MR 4333993
- [19] Miroslav Ploščica and Friedrich Wehrung, *Spectral subspaces of spectra of Abelian lattice-ordered groups in size aleph one*, Acta Sci. Math. (Szeged) (2023), DOI 10.1007/s44146-023-00080-z.
- [20] Wolfgang Rump and Yi Chuan Yang, *The essential cover and the absolute cover of a schematic space*, Colloq. Math. **114** (2009), no. 1, 53–75. MR 2457279
- [21] Alexander Schrijver, *Theory of Linear and Integer Programming*, Wiley-Interscience Series in Discrete Mathematics, John Wiley & Sons, Ltd., Chichester, 1986, A Wiley-Interscience Publication. MR 874114
- [22] Friedrich Wehrung, *Spectral spaces of countable Abelian lattice-ordered groups*, Trans. Amer. Math. Soc. **371** (2019), no. 3, 2133–2158. MR 3894048
- [23] ———, *Cevian operations on distributive lattices*, J. Pure Appl. Algebra **224** (2020), no. 4, 106202. MR 4021916

- [24] ———, *From noncommutative diagrams to anti-elementary classes*, J. Math. Log. **21** (2021), no. 2, Paper No. 2150011, 56 p. MR 4290500
- [25] ———, *Real spectra and ℓ -spectra of algebras and vector lattices over countable fields*, J. Pure Appl. Algebra **226** (2022), no. 4, Paper No. 106861, 25 p. MR 4301012
- [26] ———, *Projective classes as images of accessible functors*, J. Logic Comput. **33** (2023), no. 1, 90–135. MR 4535824
- [27] ———, *Real spectrum versus ℓ -spectrum via Brumfiel spectrum*, Algebr. Represent. Theory **26** (2023), no. 1, 137–158. MR 4546136

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