Possibility theory in constraint satisfaction problems: Handling priority, preference and uncertainty

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Abstract

In classical Constraint Satisfaction Problems (CSPs) knowledge is embedded in a set of hard constraints, each one restricting the possible values of a set of variables. However constraints in real world problems are seldom hard, and CSP's are often idealizations that do not account for the preference among feasible solutions. Moreover some constraints may have priority over others. Lastly, constraints may involve uncertain parameters. This paper advocates the use of fuzzy sets and possibility theory as a realistic approach for the representation of these three aspects. Fuzzy constraints encompass both preference relations among possible instanciations and priorities among constraints. In a Fuzzy Constraint Satisfaction Problem (FCSP), a constraint is satisfied to a degree (rather than satisfied or not satisfied) and the acceptability of a potential solution becomes a gradual notion. Even if the FCSP is partially inconsistent, best instanciations are provided owing to the relaxation of some constraints. Fuzzy constraints are thus flexible. CSP notions of consistency and k-consistency can be extended to this framework and the classical algorithms used in CSP resolution (e.g., tree search and filtering) can be adapted without losing much of their efficiency. Most classical theoretical results remain applicable to FCSPs. In the paper, various types of constraints are modelled in the same framework. The handling of uncertain parameters is carried out in the same setting because possibility theory can account for both preference and uncertainty. The presence of uncertain parameters lead to ill-defined CSPs, where the set of constraints which defines the problem is not precisely known.

Keywords: Constraint satisfaction problem; possibility theory; fuzzy restriction; softness; uncertainty; preference; priority.

1. Introduction

Classical Constraint Satisfaction Problems (CSPs) only consider a set of hard constraints that every solution must satisfy. This rigid representation framework has several drawbacks. First, some problems are over-constrained and have no solutions. A relaxation of the less rigid or important constraints must be performed in order to obtain a solution. Discovering that a problem has no solution may be time-consuming and devising an efficient constraint relaxation method is far from easy. Alternatively, other problems lead to a large set of equally possible solutions, although there often exist preferences among them which remain unexpressed. But, a standard CSP procedure will pick a solution at random. As a matter of fact, in practice, constraints are not always strict and it is desirable to extend the CSP framework in order to accommodate flexible constraints. Devising a framework for representing the flexibility of constraints will avoid artificially unfeasible problems (constraints being self-relaxable), and will avoid the random choice of solutions to loosely constrained problems. By flexible constraints, we mean either (i) *soft* constraints, which directly express preferences among solutions (i.e., this is a ranking of instantiations which are more or less acceptable for the satisfaction of a soft constraint), or (ii) *prioritized* constraints, that can be violated if they conflict with more prioritary constraints.

In soft constraints, the flexibility accounts for the possibility of going away from instantiations that satisfy the constraints ideally. Notice that the interest in soft constraints can be traced back to the early CSP litterature; in 1975, Waltz [1] mentioned that he heuristically distinguished between "likely" instanciations of a constraint and "unlikely" ones which are only considered if necessary. Also in computer vision, in 1976 Rosenfeld et al. [2] modelled preference in the detection of convex objects for scene labeling problems, and proposed to use a fuzzy degree of constraint satisfaction. The idea of representing relative preferences by means of weights is also at work in the relaxation labeling process described in 1983 by Hummel and Zucker [3]. More recent works either propose to use fuzzy sets in modeling such constraints [4][5][6][7][8] or to progressively relax the constraints when preferences are in conflict [9].

In prioritized constraints the flexibility lies in the ability to discard constraints involved in inconsistencies, provided that they are not too important. Generally, a weight is associated with each constraint and the request is to minimize the greatest priority levels of the violated constraints [10][11]. More generally, Brewka et al. [12], and Borning et al. [13] identify different forms of constraint relaxation, viewing each constraint as a strict partial order on value assignment and weighting the importance of constraints; in particular, Brewka et al. provide a formal semantics in relation to nonmonotonic reasoning by means of maximal-consistent subsets of constraints.

Freuder [14], Freuder and Wallace [15], and Satoh [16] have devised theoretical foundations for the treatment of flexibility in CSPs. Satoh tries to apply results in nonmonotonic reasoning based on circumscription to the handling of prioritized constraints so as to induce preference relations on the solution set. A similar point of view is adopted by Lang [17] where prioritized constraints are expressed in possibilistic logic (i.e., a logic with weigthed formulas which has a nonmonotonic behaviour in case of partial inconsistency). Taking a dual point of view, Freuder [14] regards a flexible problem as a collection of classical CSPs. A metric can then be defined that evaluates the distance between them. Then, the question is to "find the solutions to the closest solvable problems".

In order to take into account *both* types of flexibility, a generalization of the CSP framework has been proposed [18], based on Zadeh's possibility theory [19]: the Fuzzy Constraint Satisfaction Problem framework (FCSP). The main point is that both types of flexible constraints are regarded as local criteria that rank-order (partial) instantiations and can be represented by means of fuzzy relations. In a FCSP, constraint satisfaction or violation are no longer an all-or-nothing notion: an instanciation is compatible with a flexible constraint to a degree (belonging to some totally ordered scale). The notion of consistency of a FCSP also becomes a matter of degree. The question is then to combine the satisfaction degrees of the fuzzy constraints in order to determine the total ordering induced over the potential solutions and to choose the best ones. Making a step further, we propose to use this framework also to handle more complex constraints, e.g., nested conditional constraints.

Moreover, the framework offered by possibility theory enables us to represent ill-known parameters, whose precise value is neither accessible nor under our control, under the form of so-called possibility distributions (where the possible values are rank-ordered according to their level of plausibility). Ill-known parameters contrast with decision variables on which a decision-maker has control. This paper shows that constraints whose satisfaction depends on these ill-known parameters can be represented in the setting of possibility theory as well. In the presence of ill-known parameters, robust solutions should be searched for, such that the constraints be satisfied whatever the values of these ill-known parameters. Possibility theory implements this idea in a flexible way. Lastly, ill-known parameters lead to the idea of ill-defined CSPs; by ill-defined CSP we mean a CSP for which we are uncertain about the precise set of constraints which defines it, this uncertainty being due to ill-known factors. This aspect can also be accounted for in our framework.

From an algorithmic point of view, the possibility of extending Waltz' algorithm to fuzzy constraints has been pointed out by Dubois and Prade [20] and by Yager [21]. As we will show, all the classical CSP algorithms (e.g., tree search, AC3, PC2) can easily be adapted to FCSPs. More generally, our framework reveals itself powerful enough to accommodate the definitions of local consistency of a problem (arc-consistency, 3-consistency, k-consistency) — interestingly enough, investigations by the second author [22] indicate that the theoretical results relating levels of local consistency of a CSP to its global consistency [23][24] remain valid in FCSPs.

The next section deals with representation issues concerning flexible constraints. Fuzzy subsets on Cartesian products of domains, i.e., fuzzy relations, are used to model soft and/or prioritized constraints. An illustrative example is provided. The agreement of this representation with the preferential semantics of possibility theory is emphasized. Then the extension (resp.: projection) of fuzzy constraints to larger (resp.: smaller) Cartesian products of domains is recalled as well as the conjunctive or disjunctive combinations of fuzzy relations for representing compound constraints. Finally, this section devoted to representation issues discusses the modelling of more sophisticated constraints, namely prioritized constraints with safeguard (in order to guarantee the satisfaction of a weaker constraint in case of violation of the prioritized one) and conditional constraints. Then Section 3 formally defines the FCSP framework and compares it to other approaches to flexibility in CSP. This section then presents the essentials of a Branch and Bound algorithm performing the search for the best solutions. Nonmonotonic aspects of FCSPs are also outlined. Different notions of local consistency (arc-consistency, k-consistency) of a FCSP are defined in Section 4; the complexity of extensions of filtering algorithms (e.g., AC3) to the fuzzy set framework is also discussed. Section 5 explains how to handle ill-known parameters pervaded with uncertainty in FCSPs. Before the general conclusion, Section 6 briefly discusses the modelling of ill-defined CSPs in the possibilistic framework. In this setting, ill-defined hard CSPs, where the belonging to the problem of each constraint is (individually) uncertain, are shown to be formally equivalent to FCSPs made of prioritized hard constraints. The relation between ill-defined hard CSPs and CSPs with ill-known parameters is also briefly discussed.

2. Representing Flexible Constraints

A hard constraint C relating a set of decision variables $\{x_1, \ldots, x_n\}$ ranging on respective domains $D_1, \ldots, D_n$ is classically described by an associated relation R: R is the crisp subset of $D_1 \times \ldots \times D_n$ that specifies the tuples $d = (d_1, \ldots, d_n)$ of values which are compatible with C. The set $\{x_1, \ldots, x_n\}$ of variables related by R will be denoted by V(R).

2.1. Fuzzy Model of a Soft Constraint

A soft constraint C will be described by means of an associated *fuzzy* relation R [25], i.e., the fuzzy subset of $D_1 \times \ldots \times D_n$ of values that more or less satisfy C. R is defined by a membership function μ_R which associates a level of satisfaction $\mu_R(d_1, \ldots, d_n)$ in a totally ordered set L (with top denoted 1 and bottom denoted 0) to each tuple $(d_1, \ldots, d_n) \in D = D_1 \times \ldots \times D_n$. This membership grade indicates to what extent $d = (d_1, \ldots, d_n)$ is compatible with (or satisfies) C. Thus, the notion of constraint satisfaction becomes a matter of degree:

$\mu_R(d_1, \ldots, d_n) = 1$	means	$(d_1, \ldots, d_n)$ totally satisfies C
$\mu_R(d_1, \ldots, d_n) = 0$	means	$(d_1, \ldots, d_n)$ totally violates C
$0 < \mu_R(d_1, \ldots, d_n) < 1$	means	$(d_i, \ldots, d_n)$ partially satisfies C

Hard constraints are particular cases of soft constraints, since they involve levels 0 and 1 only. A soft constraint involving preferences between values is regarded as a local criterion ordering the instantiations of C, preferences levels being represented in the scale L: $\mu_R(d_1, \ldots, d_n) > \mu_R(d'_1, \ldots, d'_n)$ means that the first instantiation is preferred to the second one. Interpreting the preference degrees as membership degrees leads to represent a soft constraint by a fuzzy relation.

The assumption of a totally ordered satisfaction scale underlying the above setting may be questioned. The very use of a satisfaction scale instead of just an ordering relation is crucial when it comes to the aggregation of local satisfaction levels. Indeed due to the famous Arrow theorem (e.g., Moulin [26]), it is very difficult to merge several ordering relations that are not commensurate. The satisfaction scale needs not be totally ordered stricly speaking, since a complete lattice will do as well. In the following we assume that L is a totally ordered set, i.e., a chain. But the scale of membership grades needs not be numerical, as pointed out years ago [27]. A qualitative scale makes sense on finite domains. However on continuous domains, as in the case of temporal constraints with continuous time, it is much more natural and simple to assume that the satisfaction scale is the unit interval; then levels of satisfaction reflect distances to ideal values in the domain.

2.2. Fuzzy Model of a Prioritized Constraint

Fuzzy relations also offer a suitable formalism for the expression of prioritized constraints. When it is possible to a priori exhibit a total preorder over the respective priorities of the constraints, these priorities will be represented by levels in another scale V: a priority degree Pr(C) is attached to each constraint C and indicates to what extent it is imperative that C be satisfied. First consider the case of hard constraints. Pr(C) = 1 means that C is an absolutely imperative constraint while Pr(C) = 0 indicates that it is completely possible to violate C (C has no incidence in the problem). Given two constraints C and C', Pr(C) > Pr(C') means that the satisfaction of C is more necessary than the satisfaction of C'. If C and C' cannot be satisfied simultaneously, solutions compatible with C will be preferred to solutions compatible with C'.

In fact, the scale V can be interpreted as a "violation scale": the greater Pr(C), the less it is possible to violate C. This remark leads us to relate the satisfaction scale L to the violation scale V, considering that there exists an order-reversing bijection from V to L such that $L = c(V) = \{c(v), v \in V\}$: c(0) and c(1) are respectively the top element and the bottom element of L, and $v \geq v'$ in V implies $c(v) \leq c(v')$ in L. This is one of the basic modeling assumptions in this paper: the c-complement of the level of priority of a constraint is interpreted as the extent to which the constraint can be violated, using the reversed priority scale L = c(V) as a satisfaction scale; L is nothing but V put upside down. Since Pr(C) represents to what extent it is necessary to satisfy C, c(Pr(C)) indicates to what extent it is possible to violate C, i.e., to satisfy its negation. In other words, the constraint C is considered as satisfied at least to degree c(Pr(C)) whatever the considered solution, whether it satisfies C or not. More precisely, the prioritized constraint (C, Pr(C)) is considered as totally satisfied by a tuple if C is satisfied, and satisfied to degree c(Pr(C)) if the tuple violates C. Hence c(V) can be identified to a satisfaction scale as in the previous section, and a prioritized constraint C may be represented by the fuzzy relation (see Figure 1):

$$\mu_R(d_1, \ldots, d_n) = c(0) = 1 \quad \text{if} \quad (d_1, \ldots, d_n) \text{ satisfies } C;$$

$$\mu_R(d_1, \ldots, d_n) = c(Pr(C)) \quad \text{if} \quad (d_1, \ldots, d_n) \text{ violates } C.$$

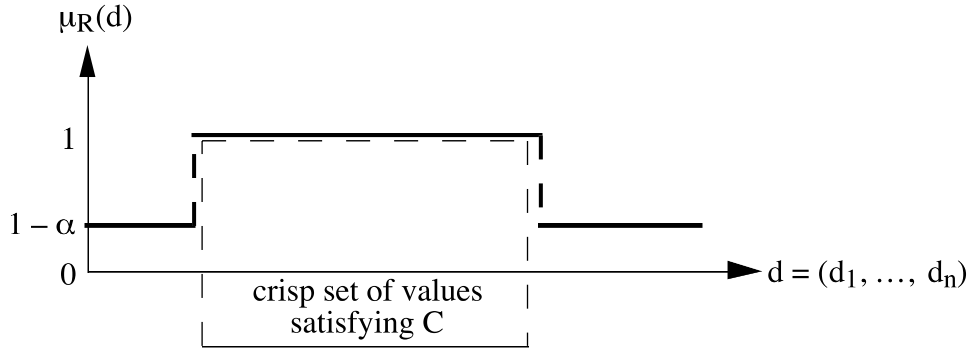


Figure 1. A hard (or crisp) constraint C with priority $Pr(C) = \alpha$ when $c(x) = 1 - x$.

Note that when $Pr(C) = 1$, the characteristic function of C is recovered, while when $Pr(C) = 0$ the constraint C degenerates into the whole domain D.

Conversely, a soft constraint C where preferences are described in terms of a finite number of satisfaction degrees $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_p < 1$ in a scale L, can be represented by a finite set of prioritized constraints $\{C^j, 0 \leq j < p\}$ using the scale L put upside down as a priority scale, via an order-reversing map c:

$$Pr(C^j) = c(\alpha_j) \qquad \text{defining } R^j = \{(d_1, \ldots, d_n), \mu_R(d_1, \ldots, d_n) \geq \alpha_{j+1}\}, j = 0, p - 1.$$

If moreover it is assumed that c is involutive, that is $c(c(\alpha)) = \alpha$ (this hypothesis is made throughout the whole paper), then it is straightforward to reconstruct the soft constraint C by means of the set of prioritized constraints $\{(C^j, Pr(C^j)), 0 \leq j < p\}$ as shown in Figure 2 where:

$$\mu_R(d) = \min_j \max(c(Pr(C^j)), \mu_{R^j}(d)) \text{ for every tuple } d = (d_1, \ldots, d_n) \tag{1}$$

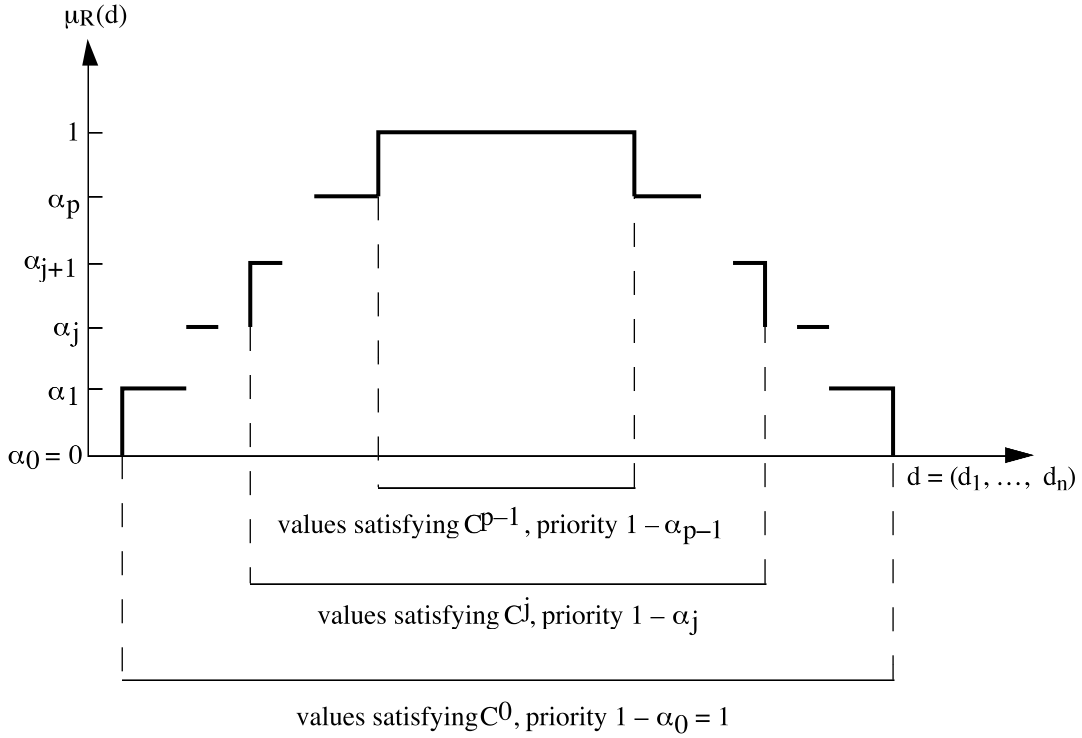


Figure 2. Decomposition of a soft constraint into a family of prioritized constraints when $c(x) = 1 - x$.

Finally, a prioritized soft constraint C corresponds to the following fuzzy relation:

$$\mu_{R'}(d_1, \ldots, d_n) = \max(c(\Pr(C)), \mu_R(d_1, \ldots, d_n)) \tag{2}$$

where R is the fuzzy relation describing the preferences of C only. Viewing the soft constraint expressed by R as a family of nested prioritized constraints, the global priority Pr(C) attached to the soft constraint C means that we forget the priorities higher than Pr(C) in the expression of R since

$$\begin{aligned}\max(c(\Pr(C)), \mu_R(d)) &= \max(c(\Pr(C)), \min_j \max(c(\Pr(C^j)), \mu_{R^j}(d))) \\ &= \min_j \max(c(\min(\Pr(C), \Pr(C^j))), \mu_{R^j}(d)).\end{aligned}$$

To conclude with representation issues, prioritized and soft constraints can be cast in a unique setting that we call "flexible constraints", modelled by fuzzy sets, where flexibility means the capability of self-relaxation. This capability is locally imbedded in the description of the constraint, thus avoiding the necessity of a specific constraint relaxation procedure to be triggered when a set of constraints is found inconsistent. This unification presupposes a strong link between levels of constraint satisfaction, and levels of constraint priority, using a single ordered scale L for both priority and satisfaction and an order-reversing map c that changes one notion into the other. For simplicity, we sometimes use $L = [0,1]$ and $c(x) = 1 - x$ in the following. However all results to be presented remain valid on a qualitative scale.

2.3. Possibility as Preference

The above approach to the joint handling of soft and prioritized constraints is in complete accordance with the basic principles of possibility theory (Zadeh [19], Dubois and Prade [28]). A possibility distribution π is a mapping from a domain D to a linearly ordered scale L ([0,1] in general). Attached to a variable x, a possibility distribution expresses that the value of x is incompletely specified, as soon as $\exists\ d_1 \neq d_2, \pi(d_1) > 0$ and $\pi(d_2) > 0$. $\pi(d) = 0$ means that it is impossible, or ruled out, that $x = d$. A normalized possibility distribution π is such that $\pi(d) = 1$ for some d, expressing that no conflict on the value of x is present. In the context of constraint satisfaction problems, x is a decision variable, i.e., its value is controllable, and the problem is to select a suitable value for x. The fuzzy set of admissible values for x according to the associated constraint (more generally the fuzzy relation in case x is a vector of elementary variables) can be viewed as a possibility distribution prescribing to what extent a value is judged to be suitable for x according to the constraint. Hence the degree of possibility $\pi(d)$ is the degree of preference for choosing $x = d$, with the convention that when $\pi(d) = 0$, d is a forbidden value of x, and when $\pi(d) = 1$, d is among the values which are definitely preferred, or, more specifically, against which no objection exists. Hence flexible constraints are naturally described by means of possibility distributions.

Given a possibility distribution π attached to a variable x, the occurrence of events of the form $x \in A$ can be assessed by means of possibility and necessity degrees, respectively defined as (see, e.g., [28])

$$\Pi(A) = \sup_{d \in A} \pi(d)\ ,\ N(A) = \inf_{d \notin A} c(\pi(d)) \tag{3}$$

where c is the order-reversing map on L. They are such that $\Pi(A) = c(N(\bar{A}))$, where the overbar denotes complementation, i.e., an event necessarily occurs if its contrary is impossible. $\Pi(A) = 1$ only means that A is consistent with the constraint represented by π while $N(A) = 1$ means that the satisfaction, even partial, of the constraint represented by π entails the occurrence of A (i.e., the fuzzy set of solutions which more or less satisfy the constraint represented by π is included in A).

In the case of prioritized constraints, the degree of priority α of a contraint C is viewed as a degree of necessity of the subset R modelling the constraint, i.e., corresponds to the higher level constraint $N(R) \geq \alpha$. The possibility distribution that accounts for the priority level α, as pictured in Figure 1, is the least specific, or equivalently, the largest, the least restrictive possibility distribution π such that $N(R) \geq \alpha$ holds, i.e., $\pi(d)$ is maximal for each $d \in D$. Indeed $N(R) \geq \alpha$ is equivalent to $\inf_{d \notin A} c(\pi(d)) \geq \alpha$, that is $\pi(d) \leq c(\alpha)$ for all $d \notin A$. The least restrictive soft constraint that respects this condition is $\pi^*(d) = 1$ if $d \in A$ and $c(\alpha)$ otherwise. This is a formal justification of the treatment of prioritized constraints in the previous section.

When R is itself a possibility distribution modelling a soft constraint, whose priority is α, the notion in possibility theory that can account for priority is the necessity of the fuzzy event R,

$$N(R) = \inf_{d \in D} \pi(d) \rightarrow \mu_R(d) \geq \alpha \tag{4}$$

where the arrow $\rightarrow$ is a multiple-valued implication, that is, a function that is decreasing in the first argument and increasing in the second one; N(R) is the degree of inclusion in R of the fuzzy set with membership function π. When $\alpha = 1$, the least specific solution of the above inequality should be $\pi = \mu_R$. Indeed "R is fully imperative" is equivalent to the soft constraint R itself. This forces the multiple-valued implication to satisfy $\pi(d) \rightarrow \mu_R(d) = 1$ iff $\pi(d) \leq \mu_R(d)$ (that is, when the constraint R is looser than the one described by π). Moreover, if $\alpha = 0$, then the least specific solution of the above inequality should again be $\pi(d) = 1, \forall d$ (the noninformative possibility distribution, expressing no constraint). This condition enforces $1 \to \mu_R(d) = 0$ if $\mu_R(d) < 1$. Finally, a simplicity requirement, in agreement with the latter, is that $\pi(d) \to \mu_R(d)$ should depend only on (and is decreasing with) $\pi(d)$ when $\pi(d) > \mu_R(d)$. Hence $\pi(d) \to \mu_R(d) = c(\pi(d))$ if $\pi(d) > \mu_R(d)$ (where c is the order-reversing mapping considered in Subsection 2.2; often for simplicity $c(a) = 1 - a$). With this choice we do have the fuzzy model of a soft, prioritized constraint suggested at the end of the previous subsection, i.e.

$$\Pr(C) = \alpha \Leftrightarrow \pi(d) = \max(c(\alpha), \mu_R(d)) \tag{5}$$

when $\Pr(C) = \alpha$ is interpreted as $N(R) \geq \alpha$. The fuzzy set of admissible values with respect to a soft constraint with priority α, is thus included to degree α in the set of values compatible with the same constraint with maximal priority, and thus the satisfaction of this prioritized constraint cannot fall under level $c(\alpha)$.

The present semantics of possibility distributions in terms of preference over the possible values of a variable, among which one must choose, contrasts with the alternative semantics in terms of plausibility that a parameter supposedly uncontrollable, or unknown, takes some value. The latter semantics will be envisaged in Section 5.

2.4. Example

A course must involve 7 sessions, namely x lectures, y exercise sessions and z training sessions (C1). There must be about 2 training sessions (C2), i.e., ideally 2, possibly 1 or 3. Dr. B, which gives the exercise part of the course, wants to manage 3 or 4 sessions (C3). Prof A, which gives the lectures, wants to give about 4 lectures (C4), i.e., ideally 4 lectures, possibly 3 or 5). The request of Dr.B is less important than the one of Prof. A and is itself less important than the imperative constraints C1 and C2. In this example, flexibility is modeled using a five level scale $L = (\alpha_0 = 0 < \alpha_1 = c(\alpha_3) < \alpha_2 = c(\alpha_2) < \alpha_3 = c(\alpha_1) < \alpha_4 = 1)$, where c is the order-reversing operation. The priorities of C3 and C4 are respectively α_2 and

α_3 ($\alpha_2 < \alpha_3$). The domain of variables x, y and z is the set {0,1,2,3,4,5,6,7}. The following model can be used:

C_1: classical hard constraint

$\mu_{R1}(x,y,z) = 1$ if $x + y + z = 7$;

$\mu_{R1}(x, y, z) = 0$ otherwise.

C_2: soft constraint (see Figure 3a)

$\mu_{R2}(z) = 1$ if $z = 2$;

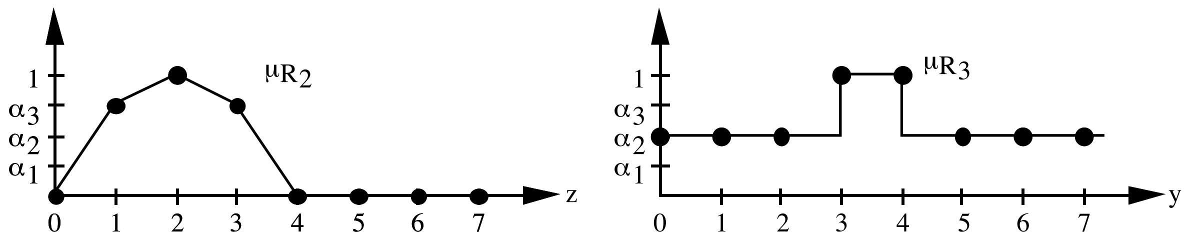
$\mu_{R2}(x) = \alpha_3$ if $z = 1$ or $z = 3$;

$\mu_{R2}(z) = 0$ otherwise.

C_3: prioritized constraint $Pr(C_3) = \alpha_2$ (see Figure 3b)

$\mu_{R3}(y) = 1$ if $y = 3$ or $y = 4$;

$\mu_{R3}(y) = c(\alpha_2) = \alpha_2$ otherwise.



Figures 3a and 3b. Modeling of C_2 (a) and C_3 (b) by means of fuzzy unary restrictions

C_4: soft and prioritized constraint $Pr(C_4) = \alpha_3$ (see Figure 4)

$\mu_{R4}(x) = 1$ if $x = 4$;

$\mu_{R4}(x) = \max(\alpha_3, c(\alpha_3)) = \alpha_3$ if $x = 3$ or $x = 5$;

$\mu_{R4}(x) = c(\alpha_3) = \alpha_1$ otherwise.

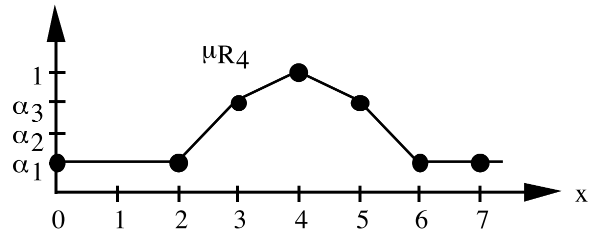


Figure 4. Modeling of C_4 by means of a fuzzy unary restriction

2.5. Operations on Fuzzy Relations

Flexible constraints are modelled by qualitative fuzzy relations. The usual operations on crisp relations can be easily generalized to fuzzy relations (Zadeh [25]). To do so, we exploit the fact that , being totally ordered, the satisfaction scale L is a complete distributive lattice, where the minimum and the maximum of two elements make sense. The following definitions extend classical set-theoretic notions used in constraint-directed problem-solving:

- A fuzzy relation R' is said to be *included* into R if and only if (see Figure 5):

$$\forall\ (d_1, \ldots, d_n) \in D_1 \times \ldots \times D_n\ , \quad \mu_{R'}(d_1, \ldots, d_n) \leq \mu_R(d_1, \ldots, d_n).$$

  This definition is a generalization of the classical set inclusion. In terms of constraints, C' is tighter that C and C is a relaxation (or a weakening) of C'.

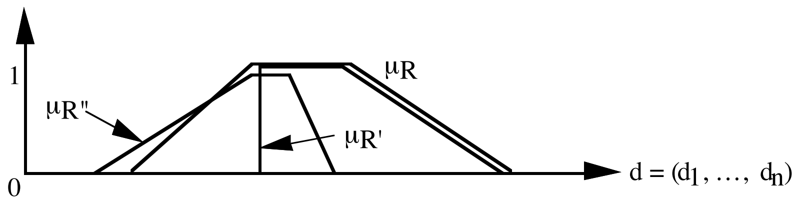


Figure 5. $R' \subseteq R$ and $R'' \not\subseteq R$

- The *projection* of a fuzzy relation R on $\{x_{k1}, \ldots, x_{kn_k}\} \subseteq V(R)$ is a fuzzy relation $R^{\downarrow\{x_{k1},\ldots,x_{kn_k}\}}$ on $\{x_{k1}, \ldots, x_{kn_k}\}$ such that:

$$\mu_{R^{\downarrow\{x_{k1},\ldots,x_{kn_k}\}}}(d_{k1}, \ldots, d_{kn_k}) = \sup_{\{d\ /\ d^{\downarrow\{x_{k1},\ldots,x_{kn_k}\}} = (d_{k1},\ldots,d_{kn_k})\}} \mu_R(d)$$

where $d^{\downarrow\{x_{k1},\dots,x_{kn_k}\}}$ denotes the classical restriction of $d = (d_1, \dots, d_n)$ to $\{x_{k_1}, \dots, x_{kn_k}\}$. This definition is a generalization of the projection of ordinary relations. $\mu_{R^{\downarrow\{x_{k1},\dots,x_{kn_k}\}}}(d_{k1}, \dots, d_{kn_k})$ estimates to what level of satisfaction the instantiation $(d_{k1}, \dots, d_{kn_k})$ can be extended to an instantiation that satisfies C.

- The *cylindrical extension* of a fuzzy relation R to $\{x_{k1}, \dots, x_{kn_k}\} \supseteq V(R)$ is a fuzzy relation $R^{\uparrow\{x_{k1},\dots,x_{kn_k}\}}$ on $\{x_{k1}, \dots, x_{kn_k}\}$ such that:

$$\mu_{R^{\uparrow\{x_{k1},\dots,x_{kn_k}\}}}(d_{k1}, \dots, d_{kn_k}) = \mu_R((d_{k1}, \dots, d_{kn_k})^{\downarrow V(R)})$$

  This definition is a generalization of the cylindrical extension of ordinary relations. $\mu_{R^{\uparrow\{x_{k1},\dots,x_{kn_k}\}}}(d_{k1}, \dots, d_{kn_k})$ estimates to what extent the instantiation $(d_{k1}, \dots, d_{kn_k})$ satisfies C.

- The *conjunctive combination* (or join) of two fuzzy relations R_i and R_j is a fuzzy relation $R_i \otimes R_j$ over $V(R_i) \cup V(R_j) = \{x_1, \dots, x_k\}$ such that (see Figure 6):

$$\mu_{R_i \otimes R_j}(d_1, \dots, d_k) = \min(\mu_{R_i}((d_1, \dots, d_k)^{\downarrow V(R_i)}), \mu_{R_j}((d_1, \dots, d_k)^{\downarrow V(R_j)})).$$

  $\mu_{R_i \otimes R_j}(d_1, \dots, d_k)$ estimates to what extent $(d_1, \dots, d_k)$ satisfies both C_i and C_j. When $V(R_i) = V(R_j)$, $\otimes$ is a generalization of classical set intersection. All properties of the standard intersection (associativity, commutativity, etc.) hold as long as negation is not involved; in particular, there holds $(R_i \otimes R_j)^{\downarrow V(R_i)} \subseteq R_i$ and $(R_i \otimes R_i) = R_i$.

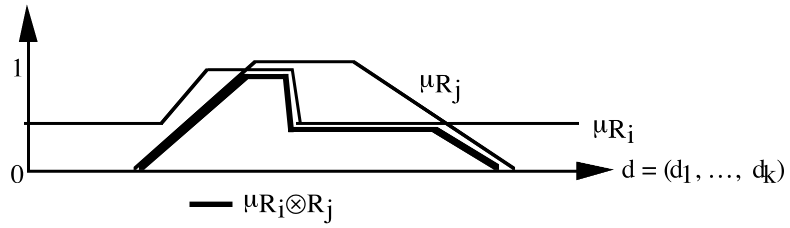


Figure 6. Conjunctive combination of two fuzzy relations R_i and R_j

Note that the use of the combination rule, allowed by the presence of a unique satisfaction scale, underlies an assumption of *commensurability* between satisfaction levels pertaining to different constraints: the user who specifies the constraints must describe them by means of this *unique* scale L (or by means of the dual scale L^T). For instance, in the example of Section 2.4, the satisfaction level α_3 of C4 for $x \in \{3,5\}$ is assumed to be equal to the satisfaction level for $z \in \{1,3\}$ and $\alpha_1 < c(Pr(C3)) < c(Pr(C4))$. Although natural and often implicit, this assumption must be emphasized.

- The *disjunctive combination* of two fuzzy relations R_i and R_j is a fuzzy relation $R_i \oplus R_j$ over $V(R_i) \cup V(R_j) = \{x_1, \ldots, x_k\}$ such that (see Figure 7):

$$\mu_{R_i \oplus R_j}(d_1, \ldots, d_k) = \max(\mu_{R_i}((d_1, \ldots, d_k)^{\downarrow V(R_i)}), \mu_{R_j}((d_1, \ldots, d_k)^{\downarrow V(R_j)})).$$

  $\mu_{R_i \oplus R_j}(d_1, \ldots, d_k)$ estimates to what extent $(d_1, \ldots, d_k)$ satisfies either C_i or C_j. When $V(R_i) = V(R_j)$, $\oplus$ is a generalization of classical set union. All properties of set union (associativity, commutativity, distributivity over intersection, etc.) hold, if negation is not involved.

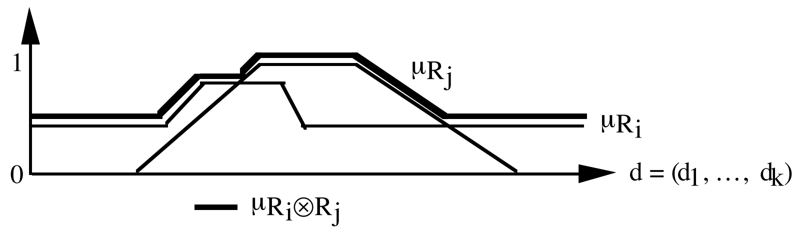


Figure 7. Disjunctive combination of two fuzzy relations R_i and R_j

2.6. Prioritized Constraints with Safeguard

The framework of fuzzy constraints offers a convenient tool for representing more sophisticated constraints than the previously encountered ones, for instance prioritized constraints with safeguard, as well as nested conditional constraints as we are going to see. First one may like to express that a constraint C, even with a rather low priority $Pr(C) = \alpha$, cannot never be completely violated, in the sense that if C is violated, at least a more permissive, minimal, constraint C' is still satisfied. Let R and R' be the fuzzy relations associated with C and C' respectively, with $R \subseteq R'$ (C' is more permissive than C, i.e., C' is a relaxation of C). The whole constraint C* corresponding to the pair (C, C') can be viewed as the conjunction of a prioritized constraint (C) and a weaker but imperative, possibly soft, constraint (C'). This conjunction is represented by the fuzzy relation R*, pictured in Figure 8, and expressed by:

$$\forall\, d \in D_1 \times \ldots \times D_n, \quad \mu_{R^*}(d) = \min(\max(\mu_R(d), c(\alpha)), \mu_{R'}(d)). \qquad (6)$$

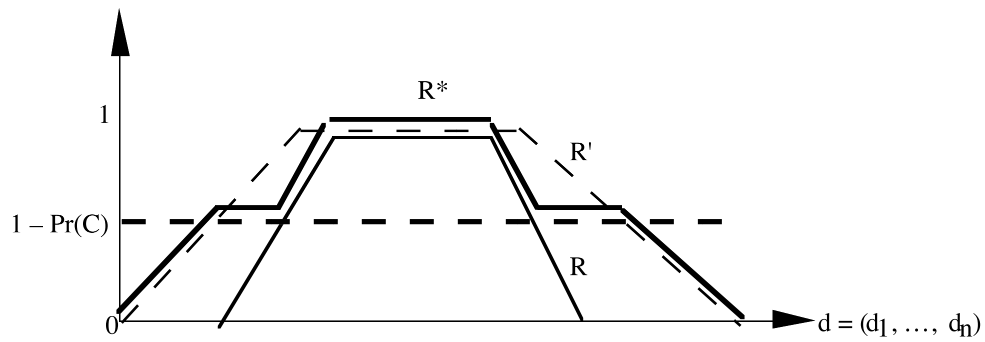


Figure 8. Representation of a prioritized fuzzy constraint with safeguard

This is a particular case of the decomposition of a soft constraint into prioritized ones when C and C' are hard. Indeed, such constraints express both a requirement with priority α less than 1 and a weaker requirement with priority 1 and R* is of the form (1):

$$\mu_{R^*}(d) = \min(\max(\mu_R(d), c(\alpha)), \max(\mu_{R'}(d), c(\Pr(C')))) \text{ with } \Pr(C') = 1.$$

See [29] for the use of such constraints in fuzzy database querying systems. Interestingly enough, R* can be decomposed either as a disjunction or as a conjunction of two fuzzy relations, depending on which fuzzy relation, R or R', the priority weight is combined with. Indeed

$$\begin{aligned}\mu_{R^*}(d) &= \min(\max(\mu_R(d), c(\alpha)), \mu_{R'}(d)) \\ &= \min(\max(\mu_R(d), c(\alpha)), \max(\mu_R(d), \mu_{R'}(d))) \text{ since } R \subseteq R' \\ &= \max(\mu_R(d), \min(c(\alpha), \mu_{R'}(d))).\end{aligned}$$

It expresses that satisfying a constraint with safeguard corresponds to either satisfying its stronger form C, or its weaker form C' the satisfaction degree being upper-bounded in this second case by $c(\alpha)$.

For instance, a flexible C_5 constraint prescribing: "Prof. A wants to give about four lectures; anyway, he will never accept to give no lecture" is represented by the fuzzy relation R_5^* pictured in Figure 9:

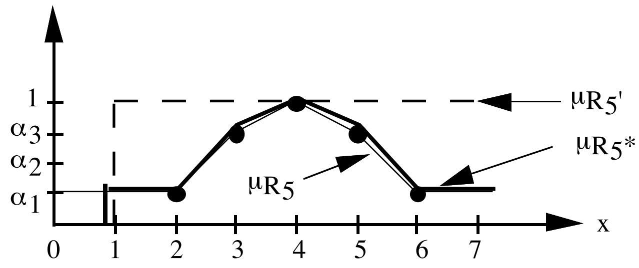


Figure 9. Modeling of C_5 by fuzzy constraint with safeguard constraint

2.7. Conditional and Hierarchically Organized Constraints

A conditional constraint is a constraint which applies only if another one is satisfied. This notion will be interpreted as follows: A constraint C_j conditioned by a hard constraint C_i (associated with fuzzy relations R_j and R_i respectively) is imperative if C_i is satisfied and can be dropped otherwise. More generally, the level of satisfaction $\mu_{R_i}(d)$ of a soft conditioning constraint C_i by an instance d is viewed as the level of priority of the conditioned constraint C_j, i.e., the greater the level of satisfaction of C_i, the greater the priority of C_j is. A conditional constraint is then naturally represented by a fuzzy relation $R_i \Rightarrow R_j$ over $V(R_i) \cup V(R_j) = \{x_1, \ldots, x_k\}$ such that:

$$\mu_{R_i \Rightarrow R_j}(d_1, \ldots, d_k) = \max(\mu_{R_j}((d_1, \ldots, d_k)^{\downarrow V(R_j)}), c(\mu_{R_i}((d_1, \ldots, d_k)^{\downarrow V(R_i)})))$$

$R_i \Rightarrow R_j$ is a prioritized constraint with variable priority: C_j has a priority 1 if $\mu_{R_i}((d_1, \ldots, d_k)^{\downarrow V(R_i)}) = 1$, i.e., if C_i is satisfied, and has a priority 0 (which means that C_j can be forgotten), if C_i is not satisfied. $\mu_{R_i \Rightarrow R_j}(d_1, \ldots, d_k)$ estimates to what extent $d = (d_1, \ldots, d_k)$ satisfies the proposition "if C_i is satisfied, then C_j must be satisfied too"; the function $\max(b, 1 - a)$ is indeed a multiple-valued implication. Note that the conjunction of the two constraints "C_i" and "C_j conditioned by C_i" is not equivalent to the conjunction "C_i and C_j" in general, since min(a, max(1 – a, b)) ≠ min(a,b). The equivalence holds however if C_i is a crisp constraint (a = 1 or 0). This is not equivalent when C_i is a soft constraint since when C_i is not completely satisfied, C_j has a priority less than the one of C_i.

Let us now show how to represent nested requirements with preferences, such as the ones considered by database authors [30][31], by means of conditional prioritized constraints. Lacroix and Lavency [30] deal with requirements of the form "C_1 should be satisfied, and among the solutions to C_1 (if any) the ones satisfying C_2 are preferred, and among those satisfying both C_1 and C_2, those satisfying C_3 are preferred, and so on", where C_1, C_2, C_3..., are hard constraints. It should be understood in the following way: satisfaying C_2 if C_1 is not satisfied is of no interest; satisfying C_3 if C_2 is not satisfied is of no use even if C_1 is satisfied. Thus there is a hierarchy between the constraints. For the sake of simplicity, let us consider the case of a compound constraint C made of three nested constraints. Thus, one would like to express that C_1 should hold (with priority 1), and that if C_1 holds, C_2 holds with priority α_2, and if C_1 and C_2 hold, C_3 holds with priority α_3 (with $\alpha_3 < \alpha_2 < 1$). The constraints C_1,C_2 and C_3 are supposed to restrict the possible values of the same set of variables (the relations are defined on the same referential $D_1 \times \ldots \times D_n$). It is always possible to be in this situation taking the cylindrical extensions of R_1, R_2 and R_3 in $V(R_1) \cup V(R_2) \cup V(R_3)$. Using the representation of conditional constraints presented above, this nested conditional constraint may be represented by means of the fuzzy relation R* defined on $D_1 \times \ldots \times D_n$:

$$
\begin{aligned}
\mu_{R^*}(d) = \min(\ & \mu_{R_1}(d), \\
& \max[c(\mu_{R_1}(d)), \max(\mu_{R_2}(d), c(\alpha_2))], \\
& \max[c[\min(\mu_{R_1}(d), \mu_{R_2}(d))], \max(\mu_{R_3}(d), c(\alpha_3))] \\
= \min(\ & \mu_{R_1}(d), \\
& \max(\mu_{R_2}(d), c[\min(\mu_{R_1}(d), \alpha_2)]), \\
& \max(\mu_{R_3}(d), c[\min(\mu_{R_1}(d), \mu_{R_2}(d), \alpha_3)])
\end{aligned}
$$

In the above expression, it is clear that the priority level of C_2 is min(μR_1(d), α_2), i.e., is α_2 if C_1 is completely satisfied and is zero if C_1 is not at all satisfied. Similarly, the priority level of C_3 is actually min(μR_1(d), μR_2(d), α_3). Note that it is zero if C_1 is not satisfied even if C_2 is satisfied. It is easy to check that:

$\mu R_1(d) = 1$ and $\mu R_2(d) = 1$ and $\mu R_3(d) = 1 \quad \Rightarrow \mu R^*(d) = 1$

$\mu R_1(d) = 1$ and $\mu R_2(d) = 1$ and $\mu R_3(d) = 0 \quad \Rightarrow \mu R^*(d) = c(\alpha_3)$

$\mu R_1(d) = 1$ and $\mu R_2(d) = 0$ and $\mu R_3(d) = 1 \quad \Rightarrow \mu R^*(d) = c(\alpha_2) < c(\alpha_3)$

$\mu R_1(d) = 1$ and $\mu R_2(d) = 0$ and $\mu R_3(d) = 0 \quad \Rightarrow \mu R^*(d) = c(\alpha_2)$

$\mu R_1(d) = 0 \quad \Rightarrow \mu R^*(d) = 0$

Thus, as soon as C_2 is not satisfied, the satisfaction of C_3 or its violation make no difference; in both cases $\mu R^*(d) = c(\alpha_2) < c(\alpha_3)$. R* reflects that we are completely satisfied if C_1,C_2 and C_3 are completely satisfied,we are less satisfied if C_1 and C_2 only are satisfied, and we are even less satisfied if only C_1 is satisfied. This is pictured on Figure 10.

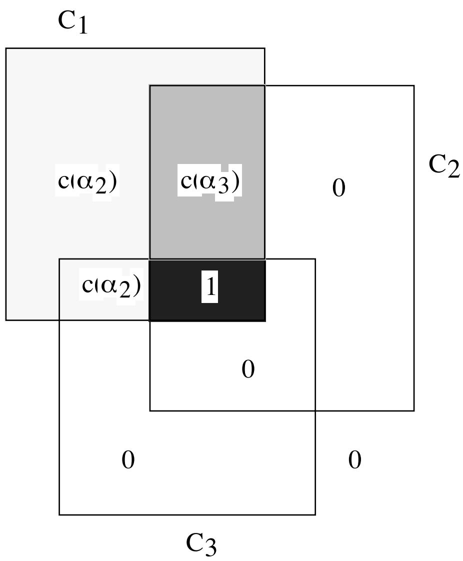


Figure 10. Levels of satisfaction of a hierarchy of constraints

In the preceding example an unconditioned constraint (C_1) was refined by a hierarchy of conditional prioritized constraints (C_2,C_3). A request looking for candidates such that "if they are not graduated they should have professional experience, and if they have professional experience, they should preferably have communication abilities", is an example where only conditional constraints, organized in a hierarchical way, take place. It will be represented by an expression of the form

$$\min[\max(c(\mu_{R_1}(d)), \mu_{R_2}(d)), \max(\mu_{R_3}(d), c(\min(\mu_{R_1}(d), \mu_{R_2}(d), \alpha))]$$

with $\mu_{R_1} = c(\mu_{grad.})$, $\mu_{R_2} = \mu_{prof.exp.}$ and $\mu_{R_3} = \mu_{com.ab.}$, i.e.,

$$\min[\max(\mu_{prof.exp.}(d), \mu_{grad.}(d)), \max(\mu_{com.ab.}(d), c(\min(c(\mu_{grad.}(d)), \mu_{prof.exp.}(d), \alpha))]$$

so that if d has professional experience and communication abilities d completely satisfies the request, as well as if d is graduated; d satisfies the request to the degree $c(\alpha)$ if d is not graduated and has professional experience only. d does not satisfy the request at all if d is neither graduated nor has professional experience (even if d has communication abilities).

3. Stating and Solving Fuzzy Constraint Satisfaction Problems

3.1. Definition

A Fuzzy Constraint Satisfaction Problem (FCSP) P involves a set of n decision variables $X = \{x_1, \ldots, x_n\}$ each ranging on its respective domain $D_1, \ldots, D_n$ and a set of m fuzzy relations $R = \{R_1, \ldots, R_m\}$ representing a set $C = \{C_1, \ldots, C_m\}$ of hard, soft or prioritized constraints (domains are assumed to be discrete in the following). A unary relation R_j of R is supposed to be associated to each variable x_j. It represents the values which are a priori more or less feasible (i.e., here, preferred) for x_j (by default, $R_j = D_j$). If all the constraints are unary or binary, the FCSP is called a fuzzy constraint network.

Classically, an instantiation of $\{x_{k_1}, \ldots, x_{k_{nk}}\} \subseteq X$ is locally consistent if it satisfies all the constraints in the subnetwork restricted to $\{x_{k_1}, \ldots, x_{k_{nk}}\}$. Like constraint satisfaction, the notion of consistency is now a matter of degree. The definition of the conjunctive combination states that $\mu_{[R_{i1} \otimes \ldots \otimes R_{ip}]}(d_{k_1}, \ldots, d_{k_{nk}})$ estimates to what extent $(d_{k_1}, \ldots, d_{k_{nk}})$ satisfies all the constraints $C_{i1}, \ldots, C_{ip}$. Hence, the degree of local consistency of $(d_{k_1}, \ldots, d_{k_{nk}})$ is defined by:

$$
\begin{aligned}
\mathrm{Cons}(d_{k_1}, \ldots, d_{k_{nk}}) &= \mu_{[\otimes_{\{R_i \in R \,/\, V(R_i) \subseteq \{x_{k_1}, \ldots, x_{k_{nk}}\}\}} R_i]}\,(d_{k_1}, \ldots, d_{k_{nk}}) \\
&= \min_{\{R_i \in R \,/\, V(R_i) \subseteq \{x_{k_1}, \ldots, x_{k_{nk}}\}\}} (\mu_{R_i}\,((d_{k_1}, \ldots, d_{k_{nk}})^{\downarrow V(R_i)})).
\end{aligned}
$$

It should be noticed that:

$$\forall\, Y \subseteq \{x_{k_1}, \ldots, x_{k_{nk}}\},\ \mathrm{Cons}\,((d_{k_1}, \ldots, d_{k_{nk}})^{\downarrow Y}) \geq \mathrm{Cons}\,(d_{k_1}, \ldots, d_{k_{nk}}).$$

Considering a complete instantiation of X, $\mu_{[R_1 \otimes \ldots \otimes R_m]}(d_1, \ldots, d_n)$ is the satisfaction degree of all the constraints by $(d_1, \ldots, d_n)$, i.e., the satisfaction degree of problem P by $(d_1, \ldots, d_n)$. It is the membership degree of $(d_1, \ldots, d_n)$ to the fuzzy set $\rho = R_1 \otimes \ldots \otimes R_m$ which is nothing but the (fuzzy) set of solutions of P. As for classical CSPs, solutions are consistent instantiations of X: $\mathrm{Cons}(d_1, \ldots, d_n) = \mu_\rho(d_1, \ldots, d_n) > 0$, i.e., solutions that are not totally unfeasible.

These degrees discriminate among the potential solutions since they induce a total preorder over the instantiations; this preorder does not depend on whether L is a numerical scale or not. In other terms, the FCSP approach to flexibility is more qualitative than quantitative. Actually, solving a classical CSP means separating the set of all instantiations into two classes: the instantiations which are solutions to the problem, and those which are not. Introducing flexibility just refines this order.

It should be noticed that the best instantiations of X may get a satisfaction degree lower than 1 if some constraints are conflicting: the FCSP approach can handle partially inconsistent problems. The consistency degree of the FCSP is the satisfaction degree of the best instantiations:

$$\begin{aligned} \mathrm{Cons}(P) &= \sup_{\{(d_1,\ldots,d_n) \in D_1\times\ldots\times D_n\}} \mu_\rho(d_1, \ldots, d_n) \\ &= \sup_{\{(d_1,\ldots,d_n)\in D_1\times\ldots\times D_n\}} [\min_{\{R_i\in R\}} \mu_{R_i}((d_1, \ldots, d_n)^{\downarrow V(R_i)})] \end{aligned}$$

The best solutions of P are those which satisfy the global problem to the maximal degree $\mu_{R_1\otimes\cdots\otimes R_m}(d_1, \ldots, d_n) = \mu_\rho(d_1, \ldots, d_n)$ (= Cons(P)), i.e., those which maximize the satisfaction level of the least satisfied constraint. If there are some instantiations which perfectly satisfy all the constraints (Cons(P) = 1), they are the best solutions. Otherwise, an implicit relaxation of flexible constraints is performed, achieving a trade-off between antagonistic constraints in the spirit of [10]: a solution will be found as long as the problem is not totally inconsistent.

Our example of Section 2.4 is partially inconsistent. The best solution is (x = 3, y = 3, z = 1) and the consistency degree is $\alpha_3 < 1$: the constraint over the number of training sessions and Prof. A's constraint are slightly relaxed according to their flexibility. The other potential solutions (e.g., (x = 4, y = 1, z = 2) or (x = 2, y = 3, z = 2)) are less consistent (their respective satisfaction degrees are α_2 and α_1).

3.2. Discussion

The FCSP approach is in accordance with Freuder's view of constraint relaxation by partial satisfaction [14]. Indeed, a FCSP involving p different satisfaction levels is equivalent to p CSPs: for each level $\alpha_j > 0$, $\alpha_j \in L$, a CSP P^{α_j} is formed by the set of hard constraints $C_i^{\alpha_j}$ containing the tuples that satisfy C_i to a degree greater than or equal to α_j. Considering that a weight is associated to each possible relaxation of each constraint C_i, the metric associated to this space is defined by the maximum among the weights of the relaxations performed. The set of best solutions to the flexible problem is the set of solutions of the consistent P^{α_j} of highest α_j (the closest solvable problem using Freuder's terminology).

The FCSP approach is different from probabilistic or cost-based approaches, where the best solutions are those satisfying the maximal number of constraints [15] or those for which the sum of satisfaction degrees is maximal [32]. These additive approaches allow for the violation of a constraint to be counterbalanced by the satisfaction of other constraints. The word "constraint" is then hardly justified. In a FCSP, as soon as an instantiation violates a hard constraint, it is totally inconsistent: $\mu_{R_1 \otimes \dots \otimes R_m}(d_1, \dots, d_n) = 0$. Thus, we are in accordance with the principle of constraint satisfaction: no constraint can be violated — except according to its relaxation capacities, which are expressed by the FCSP formalism. Additive satisfaction pooling methods also presuppose that constraints are independent or at least, not redundant. This ideal is difficult to achieve and looks contradictory with the purpose of constraint propagation, which is to produce redundant constraints. Note that the two methods of aggregation of satisfaction levels correspond to the two basic approaches to the definition of social welfare in utility theory (e.g., Moulin [26]): utilitarianism which maximizes the sum of the individual utilities, and egalitarianism which maximizes the minimal individual utility. Only the latter is compatible with the usual treatment of constraints.

Although in accordance with ours, Satoh's approach [16] differs in the way priorities between constraints are expressed. Indeed, Satoh uses second-order logic to describe priorities. Moreover the ordering of solutions depends on how many constraints are satisfied. In our approach, solutions which satisfy a FCSP to the same degree are not discriminated, even if some of them satisfy more constraints. In other terms, the best solutions in the sense of Satoh are among the best according to the FCSP definition. However, a so-called lexicographic ordering may be used in FCSP, if needed, to discriminate solutions sharing the same global satisfaction degree, as for instance in [10][7]. This mode of aggregation is also known in the social welfare literature under the name "leximin aggregation" (see Moulin [26]). The definition of the leximin ordering of two vectors $v_1 = (\mu_1, \dots, \mu_n)$ and $v_2 = (\lambda_1, \dots, \lambda_n)$ in L^n is as follows:

1) rearrange the vectors in increasing order, say $\mu_{i_1} \le \mu_{i_2} \le \ldots \le \mu_{i_n}$ and $\lambda_{j_1} \le \lambda_{j_2} \le \ldots \le \lambda_{j_n}$;

2) perform a lexicographical comparison starting from the first component, i.e.,

$$v_2 > v_1 \Leftrightarrow \exists\, k \le n \text{ such that } \forall\, m < k$$
$$\lambda_{j_m} = \mu_{i_m} \text{ and } \lambda_{j_k} > \mu_{i_k}.$$

In the example, the instantiation (z = 2, y = 1, x = 4) which satisfies C1, C2, C3 and C4 to degrees $(1, 1, \alpha_2, 1)$ is considered in a FCSP as equally good as (z = 1, y = 1, x = 5) which satisfies the constraints to degrees $(1, \alpha_3, \alpha_2, \alpha_3)$. The lexicographic ordering, which is a refinement of the min-induced ordering, will prefer the first instantiation to the second one. Note that if L = {0,1}, i.e., if the FCSP is a classical CSP, the solutions which are the best according to the lexicographic ordering are those satisfying the maximal number of constraints, as in Freuder's view of partial constraint satisfaction [15]. In other terms, the lexicographic ordering in a FCSP, which is more precisely studied in [33][34], is a generalization of Freuder's ordering in a classical CSP.

As a general model based on possibility theory, the FCSP approach generalizes the frameworks that model softness by means of fuzzy sets [4][7][8] as well as those dealing with constraint priorities by searching to minimize the priority of the violated constraints [10][7][13][11]. More precisely, some of them use an inclusion-based refinement of the min-induced ordering [13], or a lexicographic refinement [10][7] — which is itself a refinement of the inclusion-based ordering. See [33][34] for a discussion on the selection of preferred solutions in FCSP by means of these three criteria.

3.3. A Generic Solving Method for FCSPs

Finding a solution to a classical CSP is a NP-complete task. Hence, finding the best solution of FCSP is at least NP-hard. In fact, it reduces to a sup/min optimization formulation:

$$\mathrm{Sup}_{\{(d_1,\ldots,d_n)\in D_1\times\ldots\times D_n\}}[\min_{\{R_i\in\{R_1,\ldots,R_m\}\}}(\mu_{R_i}((d_1, \ldots, d_n)^{\downarrow V(R_i)}))].$$

This kind of problem can be solved using classical Branch and Bound algorithms [14][17][11], such as Depth-First Branch and Bound. It is a natural extension of backtracking, the standard approach to CSPs. Using such a classical tree search algorithm, variables are instanciated in a predetermined sequence, say $(x_1, \ldots, x_n)$. The root of the tree is the empty assignment. Intermediate nodes $(d_1, \ldots, d_k)$ denote partial instantiations and leaves are complete instantiations of $(x_1, \ldots, x_n)$. For each leaf $(d_1, \ldots, d_n)$ in the tree, we may compute $\mu_\rho(d_1, \ldots, d_n)$. The leaves that maximize μ_ρ are searched for via a depth-first exploration of the tree.

The use of fuzzy constraints makes it possible to prune each branch that necessary leads to suboptimal leaves that can be proved worse than the best of the already evaluated solutions. In other terms, it is useless to extend intermediary nodes $(d_1, \ldots, d_k)$ such that $\mu_{[\rho\downarrow\{x1,\ldots,xk\}]}(d_1, \ldots, d_k) \le b_{inf}$, b_{inf} being a lower bound of Cons(P). The calculation of $\mu_{[\rho\downarrow\{x1,\ldots,xk\}]}(d_1, \ldots, d_k)$ requires the extension of $(d_1, \ldots, d_k)$ into a complete instantiation but the definition of local consistency provides an upper bound for it. Indeed:

$$\begin{aligned}\mathrm{Cons}(d_1, \ldots, d_k) &= \mu_{[\otimes_{\{R_i\in R\,/\,V(R_i)\subseteq\{x_1,\ldots,x_k\}\}} R_i]}(d_1, \ldots, d_k)\\ &= \min_{\{R_i\in R\,/\,V(R_i)\subseteq\{x_1,\ldots,x_k\}\}}(\mu_{R_i}((d_1, \ldots, d_k)^{\downarrow V(R_i)}))\\ &\ge \min_{\{R_i\in R\}}(\mu_{R_i}((d_1, \ldots, d_k)^{\downarrow V(R_i)})) = \mu_{[\rho\downarrow\{x1,\ldots,xk\}]}(d_1, \ldots, d_k)\end{aligned}$$

Hence $\mathrm{Cons}\,(d_1, \ldots, d_k) \ge \mu_{[\rho\downarrow\{x1,\ldots,xk\}]}(d_1, \ldots, d_k)$.

This bound decreases when extending the nodes of the search tree and becomes exact for the leaves. Moreover, it may be incrementally computed as the tree is explored downward:

$$\text{Cons}(d_1,\ldots, d_{k+1}) = \min(\text{Cons}(d_1,\ldots,d_k), \min_{\{R_i\in R, x_{k+1}\in V(R_i) \text{ and } V(R_i)\subseteq\{x_1,\ldots,x_{k+1}\}\}}\mu_{R_i}((d_1,\ldots,d_{k+1})^{\downarrow V(R_i)}))).$$

Like the incremental computation of consistency in classical CSPs, the incremental computation of $\text{Cons}(d_1, \ldots, d_{k+1})$ considers each constraint only once.

Hence, the search starts with a lower bound b_{inf} (for pruning) and an upper bound b_{sup} of Cons(P); b_{inf} and b_{sup} may respectively be initialized to 0 and 1, or to better lower and upper bounds of Cons(P) if available. The consistency of the root is taken as b_{sup}. At each step, the current partial instantiation $(d_1, \ldots, d_k)$ is tentatively extended to variable x_{k+1}. If there is a value d_{k+1} such that $\text{Cons}(d_1, \ldots, d_{k+1}) > b_{inf}$, d_{k+1} is assigned to x_{k+1}. If no value consistent enough can be found for x_{k+1}, the algorithm backtracks to the most recent variable assignment. When a solution $(d_1, \ldots, d_n)$ is reached whose consistency is greater than b_{inf}, it is thus the best current solution; b_{inf} is updated to $\text{Cons}(d_1, \ldots, d_n)$ since it is a better lower bound of Cons(P). If $\text{Cons}(d_1, \ldots, d_n) < b_{sup}$, the algorithm backtracks in order to find a solution better than the current one. It should be noticed that partial instantiations $(d_1, \ldots, d_k)$ which have been extended to a solution $(d_1, \ldots, d_n)$ whose consistency is equal to $\text{Cons}(d_1, \ldots, d_k)$ do not have better extensions; hence, these extensions do not have to be explored.

Figure 11 shows a search tree corresponding to the example of Section 2.4.

Circumstances may impose resource bounds. In particular, real time processing may require immediate answers that can be refined later if time allows. The Depth-First Branch and Bound process is well suited to provide resource-bounded solutions. We can simply report the best instantiation available when, for example, a time limit is exceeded. In our example, the discovery of the best solution requests 10 nodes and 37 checks of consistency (i.e., computations of the satisfaction degree of a constraint): the first solution (consistency: α_1) is reached after 3 node extensions and 10 consistency checks and the best solution (consistency:

α_3) is encountered after 7 node extensions and 23 consistency checks, the remaining computational effort being used to prove that there is no better solution. See Figure 11.

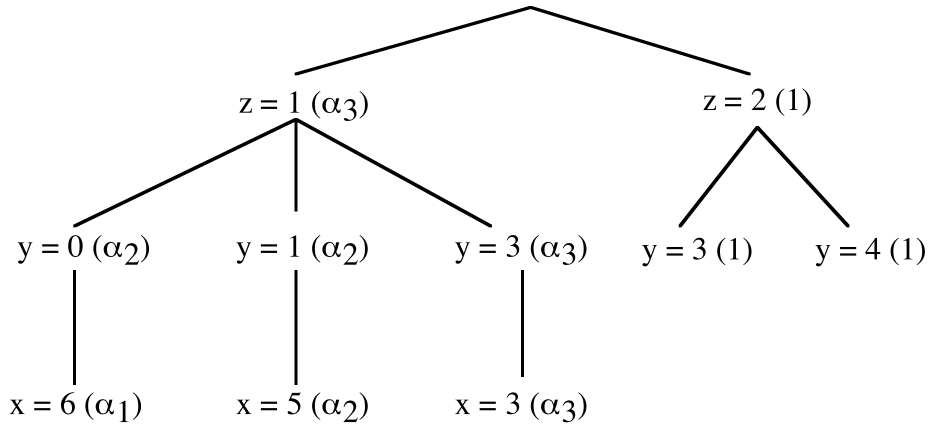


Figure 11. A search tree for the example of Section 2.4.

(The tree is explored from the root to leaves and from left to right)

This kind of algorithm has clearly a worst-case behaviour not worse than classical backtracking: both algorithms, in the worst case, will end up trying all possible combinations of values, and testing all the constraints among them. It may actually save effort as stressed in [14]. Using a pure backtrack search, 44 nodes and 155 consistency checks were needed, the best solution being reached after 9 node extensions and 28 consistency checks.

The improvement of the search depends on the bounds b_{inf} and b_{sup}: the higher b_{inf} the more efficient the pruning of useless branches and the lower b_{sup}, the sooner the search will stop. For instance, there are:

– 6 nodes and 24 consistency checks for $b_{inf} = \alpha_3$ and $b_{sup} = 1$.

– 7 nodes and 23 consistency checks for $b_{inf} = 0$ and $bsup = \alpha_3$.

– 3 nodes and 10 consistency checks for $binf = bsup = \alpha_3$.

It is possible to develop a large class of tree search algorithms (e.g., beam search as in [35]) based on the same principles and integrating different enhancements or variants (see [11]). Heuristics for choosing the instatiation ordering of the variables like those proposed by

Dechter and Meiri [36] may be used, since they only consider structural characteristics. Dynamic search rearrangement may also be applied: when extending the current instantiation, the variable having the least number of values whose degree of satisfaction is greater than b_{inf} should be chosen first. A variant for assessing the priority of a variable may be to consider the set of the values which are consistent with the current instantiation with a degree greater than b_{inf}:

$$\text{Priority}(x_j \,/\, d_1, \ldots, d_k) = \text{Cardinality}(\{d_j, \text{Cons}(d_1,\ldots,d_k,d_j) > b_{inf}\}).$$

For the selection of the value of a variable, the value(s) having the highest degree of satisfaction may be chosen first.

3.4. Nonmonotonicity in FCSPs

Using the classical CSP approach, the set of solutions shrinks when new constraints are added, and eventually becomes empty in case of conflicting constraints. In the FCSP framework, adding a new constraint to a problem P may rule out all the previously best solutions if they satisfy the new constraint to a degree lower than Cons(P). But as long as the new problem (say P') is not totally inconsistent, a new set of best solutions appears that satisfies the new problem to a degree Cons(P') ≤ Cons(P). Indeed, it holds that:

$$R_1 \otimes \ldots \otimes R_m \otimes R_{m+1} \subseteq R_1 \otimes \ldots \otimes R_m$$

where $\subseteq$ stands for the fuzzy set inclusion, but generally:

$$\{(d_1, \ldots, d_n) \,/\, \mu_{R_1\otimes\cdots\otimes R_m\otimes R_{m+1}}(d_1, \ldots, d_n) = \text{Cons}(P')\}$$
$$\not\subset \{(d_1, \ldots, d_n) \,/\, \mu_{R_1\otimes\cdots\otimes R_m}(d_1, \ldots, d_n) = \text{Cons}(P)\}.$$

Hence, the set of best solutions does not decrease monotonically when new constraints are added. The nomonotonic behaviour of soft constraints has been noticed by Satoh [16]. The type of nonmonotononicity at work here is the same as the one captured by possibilistic logic (Dubois and Prade [37]) and appears only in the presence of inconsistency. It has been precisely characterized by Benfehrat et al. [38] as the class of preferential inference relations satisfying the rational monotonicity property of Lehmann [39]. In fact, adding a new constraint may lead to four situations:

- the new constraint is redundant: $R_1 \otimes \ldots \otimes R_m \subseteq R_{m+1}$; the set of best solutions remains unchanged.
- the new constraint is totally compatible with P: Cons(P) = Cons(P'); the set of best solutions is included in the previous one but may remain unchanged.
- the new constraint is partially inconsistent with P: Cons(P') < Cons(P); constraints are implicitly relaxed according to their flexibility and the set of best solutions is not necessarily included in the previous one.
- the new constraint is totally incompatible with P: Cons(P') = 0; the set of best solutions is empty.

In the example of Section 2.4, the consistency of the problem is α_3 and the set of best solutions consists of a single one, namely $\{(x = 3, y = 3, z = 1)\}$:

- adding the redundant hard constraint $z \leq 3$ does not change the consistency of the problem nor the set of best solutions.
- adding the compatible hard constraint $y + z = 4$ neither changes the consistency of the problem nor the set of best solutions; however, the satisfaction degrees of other instanciations decreases (e.g., the satisfaction degrees of $(x = 4, y = 1, z = 2)$ become 0 instead of α_2).
- adding the hard constraint $y + z = 3$, the consistency of the the problem becomes α_2 and the new set of best solutions is $\{(x = 4, y = 1, z = 2), (x = 4, y = 2, z = 1), (x = 4, y = 0, z = 3)\}$.
- adding the hard constraint $x + y = 3$, the problem becomes totally inconsistent.

As a consequence of this nonmonotonic behavior, the problem of solution maintenance in FCSPs appears to be more complex than in classical CSPs: pruned branches in a previous search through the tree have to be developed contrary to the method proposed by Van Hentenryck [40]. The question of relaxing or deleting a constraint is not separately considered in the FCSP model, since the relaxation capacities of the constraints are supposed to be explicitly represented by means of preference among values and priority degrees. In other terms, in the FCSP model, the allowed weakening and deletion of constraints are already captured by the flexibility of the constraints (as far as preferences remain unchanged). On the contrary, when constraints have to be dynamically added or strengthened, e.g., the priority of a constraint (resp. the satisfaction degree of a value) increase (resp. decrease) the nonmonotonicity phenomena described above takes place.

4. Local Consistency in FCSPs

In the classical CSP framework, local consistency techniques can be used to improve the eficiency of the search algorithms. The most common techniques enforce arc-consistency (e.g., the AC-3 algorithm proposed by Mackworth [41] which is a generalization and a simplification of the earlier "filtering" algorithm by Waltz [1]) or path-consistency (e.g., the PC-2 algorithm proposed also by Mackworth [41]). These preliminary notions of local consistency have been generalized by the concept of k-consistency [42]. Actually, the theoretical foundations of the FCSP framework, as presented in the previous sections, allow us to easily extend all these definitions — and the associated algorithms (e.g., AC3) [22].

4.1. Local Consistency for Fuzzy Constraints

- ***k-consistency***

A classical CSP is said to be k-consistent if any consistent instantiation of k – 1 variables can be extended to a consistent instantiation involving any k^{th} variable. A FCSP is said to be k-consistent if any instantiation of k – 1 variables can be extended to a partially consistent instantiation involving any k^{th} variable, and this instanciation must be as consistent as the instantiation of k – 1 variables. Formally, a FCSP is k-consistent if and only if:

$$\forall \{x_{j1}, \ldots, x_{jk-1}\} \subseteq X, \forall x_{jk} \in X \text{ such as } x_{jk} \notin \{x_{j1}, \ldots, x_{jk-1}\},$$
$$\forall (d_{j1}, \ldots, d_{jk-1}) \in D_{j1} \times \ldots \times D_{jk-1},$$
$$\exists d_{jk} \in D_{jk} \text{ such as } Cons(d_{j1}, \ldots, d_{jk-1}, d_{jk}) = Cons(d_{j1}, \ldots, d_{jk-1})$$

A FCSP is strongly k-consistent if it is j-consistent for every j≤k. According to this definition, a necessary and sufficient condition for k-consistency is:

$$\forall \{x_{j1}, \ldots, x_{jk-1}\} \subseteq X, \forall x_{jk} \in X \text{ such as } x_{jk} \notin \{x_{j1}, \ldots, x_{jk-1}\},$$
$$\otimes_{\{R_i, V(R_i) \subseteq \{x_{j1}, \ldots, x_{jk-1}\}\}} R_i = [\otimes_{\{R_i, V(R_i) \subseteq \{x_{j1}, \ldots, x_{jk}\}\}} R_i]^{\downarrow \{x_{j1}, \ldots, x_{jk-1}\}}$$

- ***Arc-consistency***

A network of fuzzy constraints is arc-consistent (or equivalently 2-consistent) if and only if:

$$\forall x_i \in X, \forall x_j \in X \text{ such as } x_i \neq x_j, R_i \subseteq [R_{ij} \otimes R_j]^{\downarrow \{x_i\}}$$

R_i, R_j and R_{ij} respectively denoting the relation associated to the unary constraint on x_i, x_j and the binary constraint relating x_i and x_j. This condition is a generalization to fuzzy relations of the usual condition of arc consistency.

- ***3-consistency***

A network of fuzzy constraints is 3-consistent if and only if:

$$\forall \{x_i, x_j\} \subseteq X, \forall x_k \in X - \{x_i, x_j\}, R_i \otimes R_{ij} \otimes R_j \subseteq [R_{ik} \otimes R_k \otimes R_{kj}]^{\downarrow \{x_i, x_j\}}$$

Combination and projection of fuzzy relations share all properties of combination and projection of crisp relations. Hence basic results in classical CSPs extend right away to the FCSP framework, like Mackworth's theorem relating 3-consistency to path consistency [41] and backtrack-free sufficient conditions of Freuder [23] or Dechter [24]. See Fargier [22] for detailed proofs. In fact, properties which cannot be used in FCSPs are the monotonicity, as outlined previously, and properties related to the negation: if R_i represents a flexible constraint C_i and $R_i{}^c$ the constraint not(C_i), $R_i \otimes R_i{}^c = \emptyset$ does not hold generally. As soon as the description language does not include the negation of constraints, all results in classical static CSPs still hold in FCSPs.

4.2. Network Consistency Algorithms for FCSPs

All the classical filtering algorithms extend to FCSPs as well. The following algorithm is an extension of AC3 which ensures arc-consistency in a network of fuzzy constraints (L = [0,1]). It provides in addition an upper approximation of the overall consistency degree Cons(P).

Procedure FAC-3

Cons-P-sup $\leftarrow$ 1

Q $\leftarrow$ $\{(i,j) / \exists R_h \in R$ s.t. $V(R_h) = \{x_i, x_j\}, i \neq j\}$

While Q not empty, do

select and delete any arc (k,m) from Q

if Revise(k, m, Cons-P-sup) do

Q $\leftarrow$ Q $\cup$ $\{(i,k) / \exists R_h \in R$ s.t. $V(R_h) = \{x_i, x_k\}, i \neq k, i \neq m\}$

return Cons-P-sup.

Procedure Revise (i,j,Cons-P-sup)

Changed $\leftarrow$ false

Height $\leftarrow$ 0

for each d_i in D_i do

for each d_j in D_j do

new-degree $\leftarrow$ min ($\mu_{Ri}(d_i)$, $\mu_{Rij}(di,dj)$, $\mu_{Rj}(d_j)$)

Height $\leftarrow$ max(new-degree, Height)

if new-degree = 0, delete d_i from D_i.

if new-degree $\neq \mu_{Ri}(d_i)$, do

Changed $\leftarrow$ true

$\mu_{Ri}(d_i) \leftarrow$ new-degree.

Cons-P-sup $\leftarrow$ min (Cons-P-sup, Height)

return Changed

Other classical filtering algorithms can be straightforwardly extended by changing the "revise" procedure. For instance, PC2 is to be performed as defined by Mackworth [41], replacing the updating pattern by its fuzzy counterpart:

$$R_{ij} \leftarrow R_{ij} \otimes [R_{ik} \otimes R_k \otimes R_{kj}]^{\downarrow \{x_i,\, x_j\}}$$

$\otimes$ and $\downarrow$ denoting the fuzzy conjunctive combination and projection.

When performed on a FCSP, the complexity of any classical filtering algorithm must at most be augmented by a factor p, if p is the number of different levels of satisfaction used to describe the flexibility of the problem. This result can be intuitively understood since a FCSP is equivalent to p classical CSPs, as outlined in Section 3. As a matter of fact, consider FAC3, d denoting the maximal cardinality of domains, p being the number of different levels of satisfaction, m the number of binary constraints and d the maximal cardinality of domains. The cost of the "revise" procedure remains unchanged. The list Q is increased when a call to revise has succeeded. In classical CSPs, revise(x_i,x_j) is called with success at most d times. In a FCSP, revise(x_i,x_j) is called with success at most p*d times since each possible value for x_i may have its degree diminished at most p times: combination is idempotent, leads to decrease satisfaction levels without generating levels other than the original ones. Successful calls of "revise" concern at least one of the d possible values for x_i. Hence, the theoretical complexity of FAC3 is $O(pd^3m)$. More sophisticated algorithms, like AC4 [43], can also be straighforward adapted to FCSP (see [44]). In [45][46][47] the use of hypergraph structures appears to be a generalization of tree clustering [48] to fuzzy constraints.

5. Integrating Uncertain Parameters in FCSPs

So far, each relevant parameter of a CSP (or a FCSP) was supposed to be controllable, i.e., it is a decision variable whose value must be chosen according to the constraints relating it to other decision variables. Nevertheless, many real world decision problems must take into account non-controllable parameters, i.e., ill-known quantities whose precise value is neither accessible, nor under user's control. Uncertain as these parameters can be, some knowledge is often available about their plausible values. For instance, in scheduling problems, the duration of a task may be uncertain, due to possible perturbations ("The task Oi will have a duration of approximately five time units"); for more details about the handling of flexibility and uncertainty in scheduling problems, see [49]. Or, going back to our tutorial problem, the number of possible training sessions in the tutorial (z) may depend on the number of people that will attend the tutorial: 1 session if there are 16 people or more, 2 sessions if there are from 9 to 15 people, 3 sessions otherwise (the greater the number of attendees, the less the number of training sessions which can be offered, due to a limited amount of resources); all that we know is that the number of actual participants is from 7 to 17, more possibly between 10 and 14. In this case, the set of decision variables of the FCSP is X = {x,y}, since the value of z is no longer under our control. In the following z denotes an uncontrollable variable.

Possibility theory (Zadeh [19], Dubois and Prade [28]) can represent such uncertain quantities under the form of possibility distributions π_Z, where the values are ranked according to their level of plausibility, taken in a totally ordered scale U. In the example of the tutorial, can be defined as follows:

$$\pi_Z: \quad Dz \to U$$

$$\pi_Z(1) = \pi_Z(3) = \alpha \quad (1 > \alpha > 0);$$
$$\pi_Z(2) = 1;$$
$$\pi_Z(a) = 0 \text{ if } a \notin \{1,2,3\}.$$

This interpretation of possibility distributions for uncontrollable parameters is in full contrast with the alternative interpretation in terms of preference for controllable decision variables that was the one in previous sections. Then the possibility measure $\prod(A) = \sup_{d \in A} \pi_Z(d)$ and the necessity measure $N(A) = c(\prod(\bar{A})) = \inf_{d \notin A} c(\pi_Z(d))$, introduced in Section 2.3, are now respectively estimating the extent to which the event A is unsurprizing and the extent to which $\bar{A}$ is surprizing, i.e., A is believed in spite of the uncertainty pervading the value of z.

Let us now study to what extent a classical constraint relating such an uncertain (non controllable) parameter to a decision variable (controllable) will be satisfied. Consider for instance a crisp constraint C_{XZ} relating z to x. In order to satisfy C_{XZ}, we must choose a value of x such as C_{XZ} is satisfied *whatever the value of z turns out to be*. In other terms, the satisfaction degree of the constraint for the value $d \in D_X$ is the necessity degree [28] of the event $z \in (R_{XZ} \cap \{d\})^{\downarrow D_Z}$, given that the the possible values of z are restricted by π_Z:

$$\begin{aligned} N(x = d \text{ satisfies } C_{XZ}) &= N(z \in (R_{XZ} \cap \{d\})^{\downarrow D_Z}) \\ &= \inf_{a \notin (R_{XZ} \cap \{d\})^{\downarrow D_Z},\, a \in D_Z} c(\pi_Z(a)) \\ &= c(\sup_{a \notin (R_{XZ} \cap \{d\})^{\downarrow D_Z},\, a \in D_Z} \pi_Z(a)) \end{aligned}$$

This degree evaluates to what extent it is impossible to have a whatsoever possible value of z violating the constraint: it is equal to 0 if there is a totally plausible value a_1 for z such that (a_1,d) violates the constraint and equal to 1 if all the values a such that (z, x) = (a,d) violates C_{XZ} are impossible. In the crisp case, π_Z is the characteristic function of the set A of possible values of z, and $N(x = d \text{ satisfies } C_{XZ}) = 1$ if and only if $A \subseteq (R_{XZ} \cap \{d\})^{\downarrow D_Z}$ and 0 otherwise. In terms of fuzzy sets, this necessity degree evaluates a form of inclusion of the fuzzy set of possible values for z in the set $(R_{XZ} \cap \{d\})^{\downarrow D_Z}$. For instance, if C_{XZ} requires $x + z \leq 5$, the following satisfaction degrees for the different values of x are obtained:

$N(x = d \text{ satisfies } C_{XZ}) = 1$ for $d \in \{0,1,2\}$;

$N(x = d \text{ satisfies } C_{XZ}) = c(\alpha)$ for $d \in \{3\}$

$N(x = d \text{ satisfies } C_{XZ}) = 0$ for $d \in \{4,5,6,7\}$.

A constraint involving a decision variable x and an uncertain parameter z can thus be interpreted as the unary soft constraint C_X on x defined by the fuzzy relation R_X:

$$\mu_{R_X}(d) = N(x = d \text{ satisfies } C_{XZ}).$$

This assumption corresponds to a very cautious attitude, since the constraint C_X will be considered as violated as soon as a totally plausible value z may lead to an actual violation of $C_{XZ} \cdot \mu_{R_X}(d) = 0$ just means: in the normal course of things C_X is violated by decision d. Note that doing so, a form of commensurability between uncertainty degrees and satisfaction degrees is assumed, since here the degree of feasibility of the constraint, that lies in L is computed from a degree of necessity in U via a mapping that equates L and U . It agrees with commonsense which allows the interpretation of "this constraint is not very satisfied" as "it is not very certain that the constraint should be satisfied".

More generally, consider a (possibly soft) constraint C relating the set of decision variables $Y = \{x_1, \ldots, x_n\} \subseteq X$ to a set of uncertain parameters $Z = \{z_1, \ldots, z_k\}$ respectively defined on $A_1, \ldots, A_k$; let us denote π_Z a possibility distribution defined on $A_Z = A_1 \times \ldots \times A_k$ modeling our knowledge of the uncertain parameters. The constraint is satisfied by the instanciation $d = (d_1, \ldots, d_n) \in D_1 \times \ldots \times D_n$ of the decision variables if, whatever the values of $z = (z_1, \ldots, z_k)$, these values are compatible with d, i.e., if the set of possible values for z is included in $T = (R \otimes \{(d_1, \ldots, d_n)\})^{\downarrow Z}$. It is obvious that $\mu_T(a) = \mu_R(a,d)$ and

$$\begin{aligned} N(d \text{ satisfies } C) &= N(z \in T) \\ &= \inf_{a \in A_Z} \max(\mu_T(a), c(\pi_Z(a))) \\ &= c(\sup_{a \in A_Z} \min(c(\mu_T(a)), \pi_Z(a))). \end{aligned}$$

The above extension of the degree of necessity to a fuzzy event T is such that N(d satisfies C) = 1 if and only if $\forall a, \pi_z(a) > 0 \Rightarrow \mu_T(a) = 1$, i.e., any value of z which is whatsoever plausible leads to a total satisfaction of constraint C. It is strikingly different from the necessity of a fuzzy event that is used for expressing the degree of priority of a fuzzy constraint, as in Section 2.3. In the latter case, the underlying fuzzy set inclusion is Zadeh's one ($F \subseteq G \Leftrightarrow \mu_F \leq \mu_G$) while the inclusion underlying N(d satisfies C) is stricter since it requires here that the support of π_z, (i.e., the support $\{a, \pi_z(a) > 0\}$ of the fuzzy set of more or less plausible values of z) be a subset of the core of T, that is, $\{a, \mu_T(a) = 1\}$. In that situation only, N(d satisfies C) = 1. Similarly, N(d satisfies C) >0 only if the core of of π_z, gathering all the normal values of z, is contained in the support of T, i.e., the constraint C is normally satisfied.

The degree of satisfaction of constraint C by decision d is again taken as equal to N(d satisfies C). If the uncertain parameters are logically independent from each other, our knowledge about each z_j is completely described by a possibility distribution π_{z_j} and:

$$\forall\, a = (a_1, \ldots, a_k) \in A_1 \times \ldots \times A_k \qquad \pi_z(a) = \min_{j=1,\ldots,k} \pi_{z_j}(a_j).$$

Let C' be the constraint on $\{x_1, \ldots, x_n\}$ whose associated fuzzy relation R' is defined by $\mu_{R'}(d)$ = N(d satisfies C). In practice, applying possibility theory, this fuzzy restriction may be computed more generally as:

$$R' = \overline{((\overline{R} \otimes F_z)^{\downarrow Y})}$$

where F_z is the fuzzy set whose membership function is π_z and where $\overline{R}$ denotes the complement of R, i.e., $\mu_{\overline{R}} = c(\mu_R)$. The available information about uncertain parameters can be assimilated by changing each constraint C_i into a companion one C'_i using the above computation. When F_z is not fuzzy and is a set A of possible values for z, the value $\mu_{R'}(d)$ = N(d satisfies C) = $\min_{a \in A} \mu_R(d,a)$, i.e., it corresponds to a cautious Wald criterion for decision under uncertainty whereby the worst resulting situation is used to evaluate the worth of a decision. The index N(d satisfies C) is a qualitative extension of this criterion involving a trade-off between graded uncertainty, described by π_Z, and preference described by μ_R. It can be axiomatically justified as a qualitative counterpart of a utility function [50]. In the general case, $\mu_{R'}(d) \geq \alpha$ means that if it is taken for granted that the actual value of z has plausibility at least $c(\alpha)$, then it is sure that the decision d satisfies C at least at level α. This technique has been used to handle imprecise processing times of operations in scheduling problems [49].

Note that if z becomes controllable, then it is enough to change N(d satisfies C) into Π(d satisfies C) = $\sup_{a \in D_Z} \min(\mu_{T_i}(a), \pi_Z(a))$, in order to recover the standard FCSP framework. π_Z is then regarded as the membership function of just a constraint among other ones. This remark emphasizes the convenience of the possibilistic framework.

6. Possibilistic Modelling of an Ill-Defined CSP

An ill-known CSP is a CSP for which we are unsure about the set of constraints which defines it. More formally, given a set of possible constraints $\mathcal{C} = \{C_1, \ldots, C_n\}$, the uncertainty about the CSP means that there exist several subsets of $\mathcal{C}$ which are possible candidates as defining what is the real CSP under concern. For the sake of simplicity, the C_i's are assumed to represent non-fuzzy constraints. Recently, Fargier and Lang [51] have proposed a probabilistic handling of ill-known CSPs. In their approach, each constraint C_i has a probability p_i to be present in the real CSP (and $1 - p_i$ to be absent, mind that $1 - p_i$ is *not* the probability of the presence of the complementary constraint $\overline{C_i}$!). Then, using an independence hypothesis, the probability that the real problem exactly corresponds to the subset $P \subseteq \mathcal{C}$ of constraints is computed as $\text{prob}(P) = \Pi_{i:C_i \in P}\, p_i \cdot \Pi_{j:C_j \notin P}\, (1 - p_j)$. Clearly $\sum_P \text{prob}(P) = 1$. Thus, a probability is attached to each subset of constraints (including the empty subset) which is a candidate for representing the real problem.

A possibilistic model can be also proposed. The idea is to rank the subsets of constraints which are candidates for describing the real problem, using a scale of possibility levels in order to rely on a purely ordinal structure (rather than the richer additive structure of probabilities).

Thus, a possibility distribution $\pi_{\mathcal{P}}$ is defined on $2^{\mathcal{C}}$, and $\forall$ P $\subseteq \mathcal{C}$, $\pi_{\mathcal{P}}$(P) represents to what extent it is possible that P exactly describes the real problem $\mathcal{P}$. $\pi_{\mathcal{P}}$ is assumed to be normalized. Then, as pointed out by Fargier and Lang [51] in the probabilistic approach, we are not necessarily interested in the CSP with the highest plausibility, but rather in solving a consistent CSP (i.e., which has a solution) such that its solution(s) has/have a maximal certainty to be solution(s) of the real problem (if the real problem has a solution). For instance, if $\mathcal{C} = \{C_1, C_2, C_3\}$ is consistent and $\pi(\mathcal{C}) = 0$, $\pi(\{C_1,C_2\}) = \pi(\{C_1,C_3\}) = \pi(\{C_2,C_3\}) = 1$, a solution to $\mathcal{C}$ is certainly a solution to the real problem although it is impossible that the real problem be made of the three constraints C_1, C_2, C_3 altogether. The certainty N(d) that the instantiation d is a solution of the real problem is computed as the impossibility that d violates some constraint of the real problem $\mathcal{P}$, namely

$$N(d) = c(\max_{P:d\notin sol(P)} \pi_{\mathcal{P}}(P))$$

where sol(P) is the set of solutions of the CSP problem represented by the set P of constraints. Note that any solution of P is also a solution of P' such that P' $\subseteq$ P, and if d $\notin$ sol(P') then d $\notin$ sol(P). The possibility that d is a solution of $\mathcal{P}$ is given by

$$\Pi(d) = \max_{P:d\in sol(P)} \pi_{\mathcal{P}}(P).$$

Due to the normalization of $\pi_{\mathcal{P}}$, $\max(\Pi(d), c(N(d))) = 1$ and thus $\Pi(d) \geq N(d)$. In particular, if d is a solution of the CSP problem corresponding to the whole set $\mathcal{C}$ of constraints then N(d) = 1. When $\mathcal{C}$ does not correspond to a consistent CSP, the solutions d of subsets which maximize(s) N(d) and Π(d) must be searched for (taking into account that N(d) = 0 when Π(d) < 1). Note that N(d) = 0 as soon as $\exists$P, d $\notin$ sol(P), $\pi_{\mathcal{P}}$(P) = 1. Thus, when there exists an inconsistent set of constraints P such that π(P) = 1, it means that we cannot be somewhat certain that the real problem has a solution, since this certainty is expressed by

$$N(\mathcal{P} \text{ has a solution}) = c(\max_{P \text{ inconsistent}} \pi_{\mathcal{P}}(P)).$$

The possibility that $\mathcal{P}$ has a solution is given by

$$\prod(\mathcal{P} \text{ has a solution}) = \max_{P \text{ consistent}} \pi_{\mathcal{P}}(P).$$

This possibility will be zero for instance if $\pi_{\mathcal{P}}(\emptyset) = 0$, $\forall i$, $\pi_{\mathcal{P}}(\{C_i\}) = 0$ and $\forall i$, $\forall j$, $\{C_i,C_j\}$ is inconsistent.

It can be suspected that there is a link between ill-defined problems and CSP's involving uncertain parameters. In fact, the two models are equivalent in the following sense: given a possibility distribution $\pi_{\mathcal{P}}$ over a set of problems $P \subseteq \mathcal{C}$, it is enough to introduce a parameter z ranging on $\{1, 2, \ldots, 2^{|\mathcal{C}|}\}$ such $z = k$ if and only if the corresponding problem is $P_k = \{C_i, i \in I(k)\}$, where I(k) is the set of indices of constraints in problem P_k.

Then define a set $\mathcal{C}'$ of constraints involving the uncertain parameter z. For each constraint $C_i \in \mathcal{C}$ define C'_i as follows:

$$\begin{aligned} \mu_{R'_i}(d,k) &= \mu_{R_i}(d) \text{ if } i \in I(k) \\ &= 1 \text{ otherwise.} \end{aligned}$$

Also let $\pi(k) = \pi_{\mathcal{P}}(\{C_i \mid i \in I(k)\})$. Then according to the results of the previous section

$$\begin{aligned} N(d \text{ sat } \mathcal{C}') &= \min_{i=1,n} \min_z \max(c(\textstyle\prod(z)), \mu_{R'_i}(d,z)) \\ &= \min_z \min_{i=1,n} \max(c(\textstyle\prod(z)), \mu_{R'_i}(d,z)) \\ &= \min_z \max(c(\textstyle\prod(z)), \min_{i=1,n} \mu_{R'_i}(d,z)) \\ &= \min_z \max(c(\textstyle\prod(z)), \min_{i\in I(z)} \mu_{R_i}(d)) \\ &= \min_{P\in\mathcal{C}} \max(c(\pi_{\mathcal{P}}(P)), \min_{C_i\in P} \mu_{R_i}(d)) = N(d). \end{aligned}$$

Conversely a problem with constraints $C'_1, \ldots, C'_m$ involving uncertain parameters $z = (z_1, \ldots, z_p)$ can be understood as an ill-defined problem. Namely fixing the uncertain parameter value to $z = a$, a set $P_a = \{C^a{}_1, \ldots, C^a{}_m\}$ of constraints involving only controllable variables is obtained, with possibility $\pi_{\mathcal{P}}(P_a) = \pi(a)$ the set $\mathcal{C}$ of potential problems is then $\{P_a, \pi(a) > 0\}$. In that case the possibility distribution $\pi_{\mathcal{P}}$ is constructed from the available knowledge of uncontrollable parameter values.

The possibility distribution $\pi_{\mathcal{P}}$ over $2^{\mathcal{C}}$ can also obtained from the levels of possibility $\pi_i(C_i)$ that C_i belongs to $\mathcal{P}$ and $\pi_i(\text{not } C_i)$ that C_i is absent from $\mathcal{P}$ (which again does not mean that $\overline{C_i}$ is present in $\mathcal{P}$!), assuming non-interactivity (see [52][28]) or if we prefer logical independence between the presence or absence of C_i, and the presence or absence of C_j, $\forall i$, $\forall j$. Then $\pi_{\mathcal{P}} = \min_i \pi_i$, i.e.,

$$\forall\, P \subseteq \mathcal{C},\ \pi_{\mathcal{P}}(P) = \min(\min_{i:C_i \in P} \pi_i(C_i), \min_{j:C_j \notin P} \pi_j(\text{not } C_j)).$$

Clearly, the normalization condition of the π_i's ($\max(\pi_i(C_i), \pi_i(\text{not } C_i)) = 1$) entails that $\pi_{\mathcal{P}}$ is normalized. Note that $\pi_{\mathcal{P}}(P) = 0$ as soon as $\exists i, C_i \in P, \pi_i(C_i) = 0$ or $\exists j, C_j \notin P, \pi_j(\text{not } C_j) = 0$, i.e., P cannot be the real problem if it contains constraints that cannot be present or it excludes constraints which should be present. Note also that if we consider a constraint C_i such that we totally ignore if it is present or absent in $\mathcal{P}$ (i.e., $\pi_i(C_i) = 1 = \pi_i(\text{not } C_i)$), it does not make any difference to have C_i or not in P.

In the general case, the degrees of possibility $\pi_i(C_i)$ and $\pi_i(\text{not } C_i)$ should be computed from the possibility distribution $\pi_{\mathcal{P}}$, via projection, namely:

$$\pi_i(C_i) = \max_{P \ni C_i} \pi_{\mathcal{P}}(P)$$
$$\pi_i(\text{not } C_i) = \max_{P \not\ni C_i} \pi_{\mathcal{P}}(P).$$

Although only an upper bound on $\pi_{\mathcal{P}}$ can be recovered from the knowledge of $\{\pi_i(C_i), \pi_i(\neg C_i), i=1,n\}$ it is possible to express N(d) only in terms of $\{\pi_i(C_i), i=1,n\}$. Indeed $\{P: d \notin \text{sol}(P)\} = \{P, \exists\, C_i \in P, d \text{ violates } C_i\} = \bigcup_{i:\, d \text{ violates } C_i} \{P \mid C_i \in P\}$. Hence

$$\begin{aligned} N(d) &= c(\max_{P:\, d \notin \text{sol}(P)} \pi_{\mathcal{P}}(P)) = c(\max_{i:\, d \text{ violates } C_i} \max_{C_i \in P} \pi_{\mathcal{P}}(P)) \\ &= c(\max_{i:\, d \text{ violates } C_i} \pi_i(C_i)) \\ &= c(\max_i \min(c(\mu_{R_i}(d)), \pi_i(C_i))) \\ &= \min_i \max(\mu_{R_i}(d), c(\pi_i(C_i))). \end{aligned}$$

The above result emphasizes that the problem of finding a certainly feasible solution to an ill-defined CSP problem (by maximizing N(d)) comes down to a prioritized classical CSP problem where the priority of constraint C_i reflects exactly the degree of possibility that the real problem contains C_i. Note that this problem reduces to a non-prioritized problem when $\pi_i(C_i) = 1, \forall i$. This is the case when $\pi_i(\text{not } C_i) < 1, \forall i$, namely it is somewhat certain that all constraints appear in the real problem. Then N(d) = 0 or 1, and N(d) = 1 only if d satisfies all constraints in $\mathcal{C}$. When $\mathcal{C}$ is inconsistent $N(d) = 0, \forall d$. The case $N(d) \notin \{0,1\}$ is when it is believed that some constraints are *not* in the real problem ($\pi_i(C_i) < 1, \pi_i(\text{not } C_i) = 1$).

When $\pi_i(C_i) = 1, \forall i$ and $N(d) = 0$, one may be interested in maximizing $\prod(d)$. Let $\mathcal{C}(d) = \{C_i$ such that d satisfies $C_i\}$ then $d \in sol(P)$ if and only if $P \subseteq \mathcal{C}(d)$. Hence

$$\prod(d) = \max_{P \subseteq \mathcal{C}(d)} \pi_{\mathcal{P}}(P).$$

Let us consider the case where all the C_i's are such that $\pi_i(C_i) = 1$ and where the above independence assumption holds between the presence or absence of one constraint with respect to another. In that case, $n_i(C_i) = c(\pi_i(\text{not } C_i))$ estimate to what extent it is certain that C_i is in $\mathcal{P}$. Then, it holds

$$\forall\ P \subseteq \mathcal{C}, \pi_{\mathcal{P}}(P) = \min_{i: C_i \notin P} c(n_i(C_i)).$$

An example of the lattice of possible representations of $\mathcal{P}$ when $\mathcal{C} = \{C_1, C_2, C_3\}$, in the particular case where $\pi_i(C_i) = 1, \forall\ i = 1,3$, is given on Figure 12

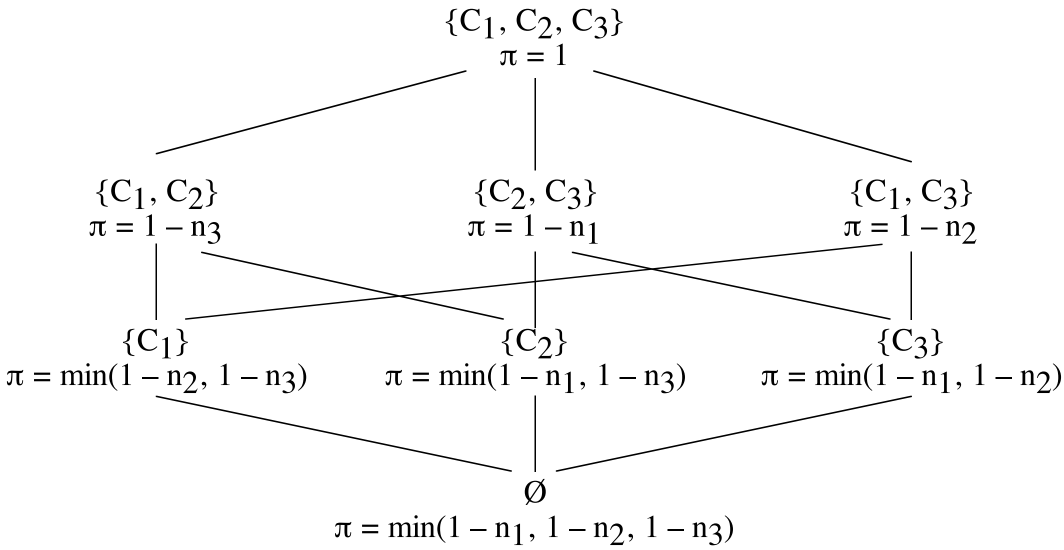


Figure 12.

It can be easily checked that $P' \subseteq P \Rightarrow \pi(P) \geq \pi(P')$ when $\forall i, \pi_i(C_i) = 1$. It can be checked that in that particular case

$$\begin{aligned} \prod(d) &= \pi_{\mathcal{P}}(\mathcal{C}(d)) \\ &= \min_{i:\, C_i \notin \mathcal{C}(d)} c(n_i(C_i)) \\ &= \min_{i=1,n} \max(\mu_{R_i}(d), c(n_i(C_i))). \end{aligned}$$

Thus a problem with prioritized constraints, as presented in Section 2, is recovered again. However note the slight difference of interpretation: certainty of presence of the constraint in the real problem versus level of priority attached to constraint in Section 2, this priority level being a part of the specification of a flexible problem. It has also been observed that maximizing the certainty of feasibility of a solution to an ill-defined problem, it also comes down to attach to each constraint a priority level that expresses the level of possibility of presence of the constraint. When these constraints are not fuzzy the three problems (priority handling, certainty of presence of constraints, possibility of presence of constraints) can be addressed in the same setting, which in logical terms is the one of possibilistic logic. A possibilistic logic program can solve a set of prioritized, logically expressed constraints as well as find the best model of a set of uncertain sentences (cf. Lang [17]).

7. Conclusion

The rich expressive power of possibility theory provides a general and unified framework for the representation and the management of flexible constraints involving preferences on values as well as prioritized constraints. It also allows for the representation of conditional constraints and of constraints whose satisfaction depends on uncertain parameters. The FCSP formalism, which is a generalization of classical CSPs, nevertheless offers a large variety of efficient problem solving tools: most classical CSP algorithms easily extend, as well asmost of the CSP theoretical results and their applications. This is due to the fact that FCSP's are not additive, but solely based on commensurate orderings, so that all useful properties of the Boolean structure underlying classical CSP's remain valid. The FCSP framework is currently applied to constraint-based approaches in jobshop scheduling [49] where flexible constraints and uncertain parameters are usual features.

As it turns out, explicitly taking the flexibility of the problem into account does not drastically increase the worst-case computational cost of the search procedure; the complexity of filtering procedures may be increased by a factor reflecting the number of different levels used to describe flexibility in the application under concern. Moreover the problem of finding a feasible solution is changed into an optimization problem of the bottleneck kind, to which Branch and Bound procedures may apply. Of course, in practice, finding an optimal solution is generally more computationally expensive than finding a feasible solution. Obviously, preferences and priorities can be used for expressing heuristics focusing the search on the more promising instantiations. Since the first solution which is found is feasible and usually good, it is not always necessary to proceed with the search for an optimal solution. Experiments carried out in the area of scheduling indicate that the first feasible solution found in the FCSP framework is often obtained more quickly than when preferences are neglected [22]. Moreover, the FCSP approach bypasses empirical relaxation techniques which are needed when a set of constraint is globally unfeasible. Constraint relaxation often happens to be more expensive, difficult to formulate, and suboptimal. On the contrary, the FCSP approach can handle partially inconsistent problems. A solution (the instantiation with the maximal satisfaction degree) will be provided as long as the problem is not totally inconsistent. Hence fuzzy constraints are also useful to guide the search procedure towards "interesting" solutions. Theoretical extensions of the framework are planned with a view to developing computational tools for handling refinements of the global minimum-based satisfaction ordering used here that may be judged as not enough discriminant [33][34]. Finally, this formalism suggests a nonmonotonic framework for dynamic CSPs, when for instance in computer aided-design, default constraints which are used in a first step analysis, are then dynamically modified by the designer.

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