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# The two-dimensional one-component plasma is hyperuniform

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We prove that at all positive temperatures in the bulk of a classical two-dimensional one-component plasma (also called Coulomb or log-gas, or jellium) the variance of the number of particles in large disks grows (strictly) more slowly than the area. In other words the system is hyperuniform.

## 1. Introduction

Let  $N \geq 1$  be an integer, let  $\Sigma_N$  be the disk of center 0 and radius  $\sqrt{\frac{N}{\pi}}$ , let  $\mathbf{X}_N := (x_1, \dots, x_N)$  denote a  $N$ -tuple of points in  $\Sigma_N$  and let  $\mathbf{X}_N := \sum_{i=1}^N \delta_{x_i}$  be the associated atomic measure. We let  $\mathbf{f}_N$  be the signed “fluctuation” measure on  $\Sigma_N$  defined by  $\mathbf{f}_N := \mathbf{X}_N - \mathbf{m}_0$ , where  $\mathbf{m}_0$  is the Lebesgue measure on  $\Sigma_N$ . We think of  $\mathbf{X}_N$  as a collection of *point particles* in  $\Sigma_N$  and of  $\mathbf{m}_0$  as the *background measure*. We define the *logarithmic interaction energy*  $F_N(\mathbf{X}_N)$  as:

$$F_N(\mathbf{X}_N) := \frac{1}{2} \iint_{(x,y) \in \Sigma_N \times \Sigma_N, x \neq y} -\log|x-y| d\mathbf{f}_N(x) d\mathbf{f}_N(y). \quad (1.1)$$

Let  $\beta$  be a positive real number that will be fixed throughout the paper. We define a probability density  $\mathbb{P}_N^\beta$  on the space of  $N$ -tuples of points in  $\Sigma_N$  by setting:

$$d\mathbb{P}_N^\beta(\mathbf{X}_N) := \frac{1}{K_N^\beta} \exp(-\beta F_N(\mathbf{X}_N)) d\mathbf{X}_N, \quad (1.2)$$

with a normalizing constant  $K_N^\beta := \int_{\Sigma_N \times \dots \times \Sigma_N} \exp(-\beta F_N(\mathbf{X}_N)) d\mathbf{X}_N$  called the *partition function*. Here and below,  $d\mathbf{X}_N$  denotes the Lebesgue measure on the Cartesian product  $\Sigma_N \times \dots \times \Sigma_N \subset (\mathbb{R}^2)^N$ . The probability measure  $\mathbb{P}_N^\beta$  is the *canonical Gibbs measure of the two-dimensional one-component plasma* (2DOCP) at *inverse temperature*  $\beta$ . We denote the expectation under  $\mathbb{P}_N^\beta$  by  $\mathbb{E}_N^\beta$ .

Let  $\Omega$  some (Borel measurable) subset of  $\Sigma_N$ . We denote by  $\text{Pts}(\mathbf{X}_N, \Omega)$  the number of points of  $\mathbf{X}_N$  in  $\Omega$  and by  $\text{Dis}(\mathbf{X}_N, \Omega)$  the *discrepancy* (of  $\mathbf{X}_N$ ) in  $\Omega$ , defined by:

$$\text{Dis}(\mathbf{X}_N, \Omega) := \text{Pts}(\mathbf{X}_N, \Omega) - \mathbf{m}_0(\Omega).$$

By construction we have  $\text{Dis}(\mathbf{X}_N, \Sigma_N) = 0$ , which corresponds to the fact that the system is globally neutral. However it cannot be perfectly locally neutral, and  $\text{Dis}(\mathbf{X}_N, \Omega)$  is meant to measure *charge fluctuations* in  $\Omega$ . The *number variance* in  $\Omega$  is defined as the variance under  $\mathbb{P}_N^\beta$  of  $\text{Pts}(\mathbf{X}_N, \Omega)$ , or equivalently of  $\text{Dis}(\mathbf{X}_N, \Omega)$ .

For all  $x$  in  $\mathbb{R}^2$  and  $R > 0$ , we let  $\mathfrak{D}(x, R)$  be the disk of center  $x$  and radius  $R$ .

## 1.1. Main result

Hyperuniformity of a system is defined by the fact that:

[Tor18, Section 1] (...) *the number variance of particles within a spherical observation window of radius  $R$  grows more slowly than the window volume in the large- $R$  limit.*

Our main conclusion is the following:

**Theorem** (The 2DOCP is hyperuniform). For each  $N \geq 1$ , let  $x = x(N)$  be a point in the bulk of  $\Sigma_N$  and let  $R = R(N)$  be such that  $R(N) \rightarrow \infty$ . Then the number variance in  $\mathfrak{D}(x, R)$  is  $o(R^2)$  as  $N \rightarrow \infty$ .

We give a more precise statement in Section 1.5, explaining what we mean by “the bulk of  $\Sigma_N$ ” and specifying a quantitative upper bound on the number variance of the form  $\mathcal{O}\left(\frac{R^2}{\log^c(R)}\right)$  for some  $c > 0$ .

Hyperuniformity of the 2DOCP *at all positive temperatures* is a forty year old prediction<sup>1</sup> from statistical physics, see [MY80; Leb83; Mar88; JLM93; LWL00] (which use a different terminology), or [Tor18] again: “*OCP fluid phases at [all] temperatures must always be hyperuniform*”. In fact, the full physical prediction says not only that the number variance is negligible with respect to the area of the disk, but that it should even be *comparable to the perimeter* ( $\mathcal{O}(R)$  and not only  $o(R^2)$ ). We give here the first mathematical proof of hyperuniformity at all temperatures, however our upper bound on the number variance remains far from the conjectured sharp estimate.

## 1.2. The 2DOCP

The two-dimensional *one-component plasma* is a well-studied model of statistical physics. Besides the papers cited above, let us refer to [For10, Chapter 15] for a presentation of exact and approximate results. When defining the energy  $F_N$  in (1.1) and the canonical Gibbs measure  $\mathbb{P}_N^\beta$  in (1.2), we think of  $(x_1, \dots, x_N)$  as the positions of point particles in  $\Sigma_N$  which all carry the same electric charge  $+1$ , and which are immersed in a uniform neutralizing background of constant density on  $\Sigma_N$ . The logarithmic potential is then the electrostatic interaction potential in dimension 2. Instead of directly placing a background measure, one sometimes imposes an harmonic confining potential. This would not change our analysis, see Appendix A for a discussion.

### Mathematical aspects of Coulomb gases

As mathematical objects, Coulomb gases (under various forms: one-, two- or three-dimensional, with one or two components...) have attracted much interest. Concerning the 2DOCP itself, topics that have been investigated in the last years alone are very diverse and include: lower bounds on the minimal distance between points [Ame18], concentration inequalities for the empirical measure of the particles [CHM18; GZ19b], upper bounds for the local density of points [LRY19], generalizations to Riemannian manifolds [Ber19], Wegner’s and clustering estimates [Tho22]... to quote only a few among many others. We refer to [Ser18] for an overview of motivations, ranging from constructive approximation to the study of the Quantum Hall Effect, via random matrix theory. The long-range and singular nature of the pairwise potential raises considerable analytic challenges.

In [LS18; BBNY19; Ser20] it was shown that the 2DOCP exhibits strong forms of rigidity at all scales and for all values of the temperature as far as fluctuations of *smooth* linear statistics are concerned. Regarding fluctuations of charges, i.e. the statistics of an indicator function, [AS21] (see also [Leb17; BBNY17] for weaker controls) imply that the discrepancy within a disk of radius  $R$  is  $\mathcal{O}(R)$  with high probability. Our goal here is improve such bounds to  $o(R)$  and to thus prove hyperuniformity.

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<sup>1</sup>Those results are rigorous to the extent that authors make use of so-called “clustering assumptions”, i.e. they assume properties of the two-point correlation function at large distances in order to derive certain identities (called “sum rules”) which, among other things, imply hyperuniformity. As explained in [Mar88]: “*The results obtained in this way are exact (i.e. do not follow from approximations), but not all of them are rigorously proven, in so far as some reasonable properties (e.g. the type of decay of the correlations) are assumed to hold a priori.*”. Unfortunately, obtaining mathematically rigorous statements about the large-distance properties of the 2DOCP’s two-point correlation function (for  $\beta \neq 2$ ) is extremely challenging.

## The Ginibre ensemble

When the inverse temperature parameter  $\beta$  is equal to 2, the probability measure  $\mathbb{P}_N^\beta$  admits an interpretation as the joint law of the complex eigenvalues of a  $N \times N$  non-Hermitian random matrix, known as the (finite) Ginibre ensemble, after [Gin65]. The model belongs to the class of *determinantal processes* (see e.g. [HKPV06, Section 4.3.7]) and is amenable to exact computations. In that specific case, the standard deviation of the number of points in a disk of radius  $R$  is known: it scales like  $R^{\frac{1}{2}}$  (in accordance with predictions from physics), see [Shi06] and [OS08]. Thus for  $\beta = 2$  our result is not new, and our bound is far from the optimal one.

## The hierarchical model

In [Cha19], S. Chatterjee studied Coulomb gases in dimensions  $\{1, 2, 3\}$  (see [GS20] for an extension to higher dimensions), giving sharp estimates on the number variance up to logarithmic corrections. His model is a *hierarchical Coulomb gas*, where the physical space is decomposed into a tree-like structure following an old recipe of Dyson. The hierarchical model has two built-in properties: conditional independence of subsystems and a self-similar nature. A large part of the present analysis is devoted to finding approximate analogues of those features for the “true” (non-hierarchical) model.

### 1.3. Hyperuniformity

The term “hyperuniform(ity)” has been coined in the theoretical chemistry literature by S. Torquato (see [TS03; Tor18] for surveys), an alternative terminology due to J. Lebowitz is “superhomogeneous/ity”. As we already mentioned, a system is hyperuniform when the number variance in a large ball is asymptotically negligible with respect to the volume of said ball. Of course the definition of hyperuniformity needs an adaptation for finite systems, as one should take both the size of the system and of the “spherical window” to infinity. We refer to [Cos21] for a mathematical presentation of hyperuniformity and a survey of related results.

#### Examples of hyperuniform systems

For non-interacting random particles (forming a Bernoulli point process or a Poisson point process with constant intensity) the number variance grows exactly like the volume, hence hyperuniformity can be thought of as the property of systems that are “much more rigid” than independent particles regarding discrepancy at large scales. Examples of two-dimensional objects that are proven to be hyperuniform include: the Ginibre ensemble (as mentioned above) and its generalizations (see [CE19] for a recent result in connection with the Quantum Hall Effect), averaged and perturbed lattices (see [GS75]), the zeroes of random polynomials with i.i.d Gaussian coefficients (see [FH99; SZ08])...

In some cases (see [Tor18, Section 5.3.2]) the number variance grows as the perimeter (in fact that was initially chosen as the definition of hyperuniformity in [TS03] but is now called “class I hyperuniformity”), which is the slowest possible growth (see [Bec87]). The 2DOCP is predicted to be class I hyperuniform at all temperatures, with good tails on the probability of large charge fluctuations, as we explain next.

#### The Jancovici-Lebowitz-Manificat law

In [JLM93], Jancovici, Lebowitz and Manificat made precise predictions concerning the probability of observing large charge fluctuations within the 2DOCP (and its three-dimensional version). Their statement is significantly more precise<sup>2</sup> than hyperuniformity, as they argue that for all  $\alpha > \frac{1}{2}$ :

$$\mathbb{P}[\text{Discrepancy of size } R^\alpha \text{ in a disk of radius } R] \sim \exp(-R^{\varphi(\alpha)}) \quad (\text{“JLM law”}) \quad (1.3)$$

where the rate  $\varphi(\alpha) > 0$  is an explicit piecewise affine function of  $\alpha$ . This was later checked for  $\beta = 2$  through explicit computations, see [Shi06; FL21], while the general case is open. Remarkably, although the original prediction of [JLM93] deals with Coulomb gases, it was first verified in [NSV08] for a different model, namely the zeroes of the Gaussian Entire Function (see [GN18] for a survey of related results).

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<sup>2</sup>Tail estimates as in (1.3) readily imply the strongest type of hyperuniformity (Type I).

## Fekete points and the low temperature regime

A situation of interest that we do not consider here is the case of energy minimizers, which formally corresponds to taking  $\beta = +\infty$  i.e. a zero temperature. A long-standing conjecture posits that minimizers of  $F_N$  form a lattice as  $N \rightarrow \infty$  and in that regard Fekete points should exhibit excellent rigidity properties at finite  $N$ . For studies of such questions we refer to [AOC12; Ame17] and, in a different line of work, to [NS15, Thm. 3]. See also [AR22; MR22] or the  $\beta$ -dependent statements of [AS21] for a more general “low-temperature” regime where  $\beta$  is large but not necessarily infinite.

### 1.4. Open questions

Besides the obvious natural goal of obtaining the sharp number variance estimate as well as proving the aforementioned “JLM laws”, let us mention three other open directions of research.

**The 3DOCP.** The claims of [JLM93] are made in dimension 2 and 3, moreover sharp hyperuniformity estimates were obtained for the *hierarchical* model in dimension 3 in [Cha19], it is thus natural to ask whether one can prove hyperuniformity for the 3DOCP. Unfortunately, in dimension 3 there is no (known) value of  $\beta$  for which the model would be integrable and predictions could be tested. Some understanding of fluctuations of smooth linear statistics was recently obtained in [Ser20] but with considerable new difficulties compared to the two-dimensional case.

**Riesz interactions.** The 2DOCP can be seen as a member of one-parameter family of two-dimensional systems called *Riesz gases* for which the interaction potential is taken as a certain power  $|x - y|^{-s}$  of the distance (the case  $s = 0$  corresponds by convention to the logarithmic interaction). Rigidity properties of those systems depend on the value of  $s$  as it governs both the level of repulsiveness at 0 and, most importantly, the long- or short-range nature of the interaction. Rough bounds on the discrepancies were given in [LS17, Lemma 3.2] in the infinite-volume limit, which say that the number variance in a disk of radius  $R$  grows at most as  $R^{2+s}$ . We believe the sharp estimate to be  $R^{1+s}$  and thus that two-dimensional Riesz gases should remain hyperuniform for  $s < 1$ . For one-dimensional systems, such questions were recently answered in [Bou21].

**High temperature regime.** There has been recent interest in studying the “high-temperature” regime where  $\beta \rightarrow 0$  as  $N \rightarrow \infty$  for Coulomb gases (and other related systems), see e.g. [AB19; Lam21] in the two-dimensional setting. Those results have put forward that the inverse temperature regime where  $\beta \sim \frac{c}{N}$  is the threshold that distinguishes between “Ginibre-like” ( $c \rightarrow +\infty$ ) and “Poisson-like” ( $c \rightarrow 0$ ) properties. It is thus tempting to ask whether the 2DOCP stays hyperuniform as long as  $\beta N \rightarrow \infty$ , and whether one can observe an interesting transition along  $\beta = \frac{c}{N}$ , as  $c$  moves down from  $+\infty$  to 0.

### 1.5. Precise statement of the result, proof strategy and tools

**Theorem 1.** Let  $\delta > 0$  be fixed. For all  $N$  and  $R$  large enough (both depending on  $\beta$  and  $\delta$ ), for all  $x$  in  $\Sigma_N$  such that  $x$  is “in the bulk” in the following sense:

$$\text{dist}(\mathfrak{D}(x, R), \partial\Sigma_N) \geq \delta\sqrt{N}, \quad (1.4)$$

we have:

$$\mathbb{P}_N^\beta \left( \{ |\text{Dis}(\mathbf{X}_N, \mathfrak{D}(x, R))| \geq R(\log R)^{-0.3} \} \right) \leq \exp(-\log^{1.5} R). \quad (1.5)$$

Since good exponential tails for  $\text{Dis}$  at values higher than  $R$  are already known (see e.g. [AS21, Thm 1]), this new sub-algebraic tail valid for values of the discrepancy between  $R \log^{-0.3} R$  and  $R$  imply that:

$$\text{Var}[\text{Dis}(\mathbf{X}_N, \mathfrak{D}(x, R))] \leq \mathbb{E}_N^\beta [\text{Dis}^2(\mathbf{X}_N, \mathfrak{D}(x, R))] = \mathcal{O}\left(\frac{R^2}{\log^{0.6} R}\right) = o(R^2),$$

and the 2DOCP is thus indeed *hyperuniform*. The proof will be given in Section 7.

## General strategy

Our main source of inspiration is [NSV08, Section 4], in which Nazarov-Sodin-Volberg prove JLM-like estimates on the probability of having large “charge” fluctuations in the disk  $\mathfrak{D}(0, R)$  for zeros of the *Gaussian Entire Function*. Their argument can be roughly summarized as follows: fix  $\alpha \in (\frac{1}{2}, 1)$  and assume there is a discrepancy of size  $R^\alpha$  in the disk  $\mathfrak{D}(0, R)$ .

1. Show that the discrepancy can be captured along the boundary  $\partial\mathfrak{D}(0, R)$ .
2. Split that boundary into  $\approx R$  pieces of size  $\approx 1$ . Take  $M$  large and apply a basic pigeonhole argument: there exists a family of  $\approx \frac{R}{M}$  pieces that capture a discrepancy of size at least  $\frac{R^\alpha}{M}$  and which are “well-separated” (distances between pieces are multiples of  $M$ ).
3. Then comes the main probabilistic work:
  - a) Show that these well-separated pieces are *approximately independent*.
  - b) Show that the discrepancy on each piece (of size  $\approx 1$ ) is typically  $\mathcal{O}(1)$ .
  - c) Show that the discrepancy on each piece is centered.
4. Apply Bernstein’s concentration inequality: if  $\{D_i\}_i$  is a family of  $\approx \frac{R}{M}$  independent centered random variables of size  $\mathcal{O}(1)$  then:

$$\mathbb{P}\left(\left\{\sum_i D_i \geq R^\alpha\right\}\right) \lesssim \exp\left(-\frac{R^{2\alpha-1}}{M}\right).$$

This gives them *sharp* bounds on the number variance *and* the correct tail probabilities. Whilst not aiming so high, we follow here a similar strategy, with much effort to translate it to the context of the 2DOCP (besides point (2) and (4), which are easy).

## Electric approach, sub-systems

The technical core is the the “electric approach” to Coulomb gases as developed by S. Serfaty and co-authors. In that regard, our main imports are:

- The general spirit of controlling fluctuations through the electric energy (see Section 2.3).
- Local laws (up to microscopic scale) as in [AS21] (see Proposition 2.4 and Proposition 5.8).
- Optimal bounds for fluctuations of smooth linear statistics as in [LS18; Ser20; BBNY19]. Point (1), for example follows fairly easily from such bounds.
- “Smallness of the anisotropy”, for which we refer to [LS18; Ser20] (see also [BBNY19]).

In addition, we put forward the role of so-called “sub-systems”, which arise as restrictions of the full system to some region and which we view as two-dimensional Coulomb systems in their own right, possibly with a small global non-neutrality, and (most importantly) feeling the effect of some harmonic exterior potential. As such, this object is not new - it has been sometimes called a “conditional” or “local” measure in the literature (see e.g. [BBNY19; BEY14]). We provide here a thorough study of its behavior, through global and local laws, and show that with high probability the sub-systems, although under the influence of an external potential, retain most of the rigidity properties of the full system - as is the case “for free” in the hierarchical model studied in [Cha19]. This requires a precise study of the external potential “felt” by a sub-system and of the perturbation of the equilibrium measure that it induces within the sub-system (for which we rely on the analysis of [BBNY17, Section 2.], with some modifications). The proof of local laws given in [AS21; Leb17] extends to sub-systems. This allows one for example to treat Point (3) (b), which seem unsurprising, but which requires to consider objects of size  $\sim 1$ , hence to control the system *down to the smallest scales*. Such local laws are also crucial for the approximate translation-invariance argument which we present now.

## Approximate translation-invariance

Step (3) (c) is void in the context of [NSV08] because the underlying point process is infinite and translation-invariant (which implies that linear statistics are centered). However, it turns out to be a major roadblock when adapting the proof to a Coulomb gas context. Indeed, in sub-systems (or even in the full system if one does not impose artificial boundary conditions) there is absolutely no obvious translation-invariance. We introduce an “approximate translation-invariance” result, valid for both the full system and sub-systems, which is crucial for Step (3) (c). The point being that shifting a function (or more precisely, averaging over shifts) acts as a mollification and enables us to compare the expectation of a discrepancy, or of any non-smooth linear statistic, to that of a smoother one.

There is a series of results in mathematical statistical mechanics *à la* H.-O. Georgii devoted to prove translation invariance of infinite volume Gibbs measures in contexts where stationarity is not built-in, see e.g. [Geo99; Ric07], following earlier works by Fröhlich-Pfister [FP81; FP86]. The basic idea is to construct suitable “localized translations” in the form of diffeomorphisms acting as a given translation in a large box while leaving the majority of the system unchanged, and to control the effect on the energy of such changes of variables. Following the wisdom found in a remark of [Sim14, Chap. 3, Sec. III.7], we construct a localized translation that varies very slowly (in terms of its  $H^1$  norm), and a careful revisit of a computation done in [Ser20], together with Serfaty’s “smallness of anisotropy” trick, allows us to conclude that this localized translation can be chosen to have an arbitrarily small effect on the energy.

## Approximate independence

Finally, one needs to find some kind of independence between the sub-systems. We introduce a new “approximate independence” argument (which we believe to be of interest on its own) for sub-systems that are well-separated, conditionally on the *number of points* in each of them. The simple idea is that two domains  $\Lambda_i, \Lambda_j$  with  $n_i, n_j$  points contribute an interaction energy given to first order by:

$$-(n_i - |\Lambda_i|)(n_j - |\Lambda_j|) \log \text{dist}(\Lambda_i, \Lambda_j),$$

while the precise arrangements of the points inside each domain should matter only to a lower order.

**Remark 1.1.** In the rest of the paper we often use results found in [AS21] and [Ser20]. It is worth noting that although both papers deal (among other things) with the same 2DOCP model, they do not use the same scaling convention. In [AS21] the authors work with the so-called “blown-up scaling” (as in the present paper), which is more common in the physics literature and for which lengthscales range from  $\sim 1$  (the nearest-neighbor scale) to  $\sim N^{1/2}$  (the diameter of the system), whereas [Ser20] uses the random matrix theory convention where the local scale is  $\sim N^{-1/2}$  and the global scale is  $\sim 1$ . In particular, a length scale  $\ell$  in the present paper translates into  $N^{1/2}\ell$  in [Ser20], and the corresponding modifications must be applied when quoting results.

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## 2. Preliminary results

### 2.1. Some notation.

We denote indicator functions by  $\mathbf{1}$ . We denote by  $|\Omega|$  the Lebesgue measure of a measurable subset  $\Omega$ . For  $x \in \mathbb{R}^2$  and  $r > 0$  we let  $\mathfrak{D}(x, r)$  be the (closed) disk of center  $x$  and radius  $r$ . We let  $\square(x, r)$  be the square of center  $x$  and sidelength  $r$  (with sides parallel to the axes of  $\mathbb{R}^2$ ). We recall the notation  $\text{Dis}$  for discrepancies: if  $\mathbf{X}$  is a point configuration and  $\Omega$  is a measurable subset, we let  $\text{Dis}(\mathbf{X}, \Omega) := \text{Pts}(\mathbf{X}, \Omega) - |\Omega|$ .

**Size of derivatives.** If  $M$  is a matrix we let  $\|M\|$  be its Euclidean norm. If  $\varphi$  is a bounded map we write  $|\varphi|_0$  for its sup-norm. If  $\varphi$  is differentiable, we let  $D\varphi$  be its differential and introduce the following notation:

- $|\varphi|_{1,\star}(x)$  is the size of  $D\varphi$  at a given point  $x$ ,  $|\varphi|_{1,\star}(x) := \|D\varphi(x)\|$ .
- $|\varphi|_{1,\text{loc}}(x)$  is the size of  $D\varphi$  around a given point  $x$ ,  $|\varphi|_{1,\text{loc}}(x) := \sup_{|x'-x| \leq 1} |\varphi|_{1,\star}(x')$
- $|\varphi|_{1,\Omega}$  is the size of  $D\varphi$  in a given domain  $\Omega$ ,  $|\varphi|_{1,\Omega} := \sup_{x \in \Omega} |\varphi|_{1,\star}(x)$ .
- Finally,  $|\varphi|_1$  is the sup-norm of  $D\varphi$ .

We define similarly  $|\varphi|_{k,\star}$ ,  $|\varphi|_{k,\text{loc}}$ ,  $|\varphi|_{k,\Omega}$  and  $|\varphi|_k$  for  $k \geq 2$ .

**Point configurations.** For all Borel subsets  $\Lambda$  of  $\mathbb{R}^2$ , we let  $\text{Conf}(\Lambda)$  be the space of locally finite<sup>3</sup> point configurations on  $\Lambda$ , endowed with the vague topology of Radon measures and the associated Borel  $\sigma$ -algebra. When  $\Lambda$  is not specified, we use the notation  $\text{Conf}$  for point configurations on  $\mathbb{R}^2$ . We will denote by  $\mathbf{X} \mapsto \mathbf{X} \cap \Lambda$  the natural projection  $\text{Conf} \rightarrow \text{Conf}(\Lambda)$ . We may write “ $x \in \mathbf{X}$ ” to express the fact that  $\mathbf{X}$  has an atom at a given point  $x \in \mathbb{R}^2$ . We say that a measurable function  $G$  on  $\text{Conf}$  is  $\Lambda$ -local when for all  $\mathbf{X}$  in  $\text{Conf}$  we have  $G(\mathbf{X}) = G(\mathbf{X} \cap \Lambda)$ . We say that a measurable subset (an “event”)  $\mathcal{E}$  of  $\text{Conf}$  is  $\Lambda$ -local when its indicator function  $\mathbf{1}_{\mathcal{E}}$  is  $\Lambda$ -local.

**Constants.** Unless specified otherwise,  $C$  denotes a universal constant (which may change from line to line) and  $C_\beta$  a constant that depends only on  $\beta$ . We will use  $\bar{C}$  for constants that may depend on  $\beta$  and the parameter  $\delta$  (as in (1.4)). We write  $A \preceq B$  or  $A = \mathcal{O}(B)$  if  $|A| \leq C|B|$ .

**Fluctuations.** If  $\varphi$  is a piece-wise continuous function on  $\mathbb{R}^2$  we define the *fluctuation of (the linear statistics associated to)  $\varphi$*  as the following random variable:

$$\text{Fluct}[\varphi] := \int_{\Sigma_N} \varphi(x) d\mathbf{f}_N(x).$$

### 2.2. Electric fields

We recall that  $-\log$  satisfies  $-\Delta(-\log) = 2\pi\delta_0$  on  $\mathbb{R}^2$  in the sense of distributions. In particular for all smooth enough test functions  $f$ , the following identity holds:

$$f(x) = \frac{-1}{2\pi} \int_{\mathbb{R}^2} -\log|x-y| \Delta f(y). \quad (2.1)$$

Let  $\Lambda \subset \mathbb{R}^2$  and let  $\mathbf{X}$  be a point configuration in  $\Lambda$ .

**Definition 2.1** (True electric potential and electric field). We let  $\mathfrak{h}^{\mathbf{X}}$  (resp.  $\nabla \mathfrak{h}^{\mathbf{X}}$ ) be the *true electric potential* (resp. *true electric field*) generated by  $\mathbf{X}$  (in  $\Lambda$ ), namely the map (resp. vector field) defined on  $\mathbb{R}^2$  by:

$$\mathfrak{h}^{\mathbf{X}}(x) := \int_{\Lambda} -\log|x-y| d(\mathbf{X} - \mathbf{m}_0)(y), \quad \text{resp.} \quad \nabla \mathfrak{h}^{\mathbf{X}}(x) = \int_{\Lambda} -\nabla \log|x-y| d(\mathbf{X} - \mathbf{m}_0)(y).$$

It is easy to check that  $\nabla \mathfrak{h}^{\mathbf{X}}$  is in  $\cap_{p \in [1,2)} L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$  but fails to be in  $L^2$  around each point charge, and that the following identity is satisfied on  $\Lambda$  in the sense of distributions:

$$-\Delta \mathfrak{h}^{\mathbf{X}} = -\text{div} \nabla \mathfrak{h}^{\mathbf{X}} = 2\pi(\mathbf{X} - \mathbf{m}_0).$$

<sup>3</sup>In fact all of the point configurations considered in this paper are finite.



**Definition 2.2** (Compatible electric fields). Let  $\mathbf{E}$  be a vector field in  $\cap_{p \in [1,2)} L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ . We say that  $\mathbf{E}$  is an *electric field compatible with  $\mathbf{X}$*  on  $\Lambda$  whenever we have:  $-\text{div } \mathbf{E} = 2\pi(\mathbf{X} - \mathbf{m}_0)$  on  $\Lambda$  in the sense of distributions.

Obviously the true electric field is a compatible electric field, however it is not the only one as can be seen by adding any divergence-free smooth vector field on  $\Lambda$  to  $\nabla \mathfrak{h}^{\mathbf{X}}$ .

In order to “take care” of the singularities, one often proceeds to a truncation of the fields near each point charge.

**Definition 2.3** (Truncation and spreading out Dirac masses). For  $\eta > 0$  we let  $f_\eta$  be the function:

$$f_\eta(x) := \max\left(-\log \frac{|x|}{\eta}, 0\right) = \begin{cases} -\log |x| + \log |\eta| & \text{if } x \leq \eta \\ 0 & \text{if } x \geq \eta \end{cases}.$$

For each point  $x$  of  $\mathbf{X}$ , let  $\eta(x)$  be a non-negative real number. The data of  $\vec{\eta} = \{\eta(x), x \in \mathbf{X}\}$  is called a *truncation vector*. If  $\mathfrak{h}^{\mathbf{X}}$  is the true electric potential, we let  $\mathfrak{h}^{\vec{\eta}}$  (resp.  $\nabla \mathfrak{h}^{\vec{\eta}}$ ) be the (true) *truncated* electric potential (resp. field) given by:

$$\mathfrak{h}^{\vec{\eta}} := \mathfrak{h}^{\mathbf{X}} - \sum_{x \in \mathbf{X}} f_{\eta(x)}(\cdot - x), \quad \text{resp. } \nabla \mathfrak{h}^{\vec{\eta}} = \nabla \mathfrak{h}^{\mathbf{X}} - \sum_{x \in \mathbf{X}} \nabla f_{\eta(x)}(\cdot - x).$$

We are thus effectively replacing  $-\log |x - \cdot|$  by  $-\log \eta$  near each point charge. Another way to think about  $f_\eta$  is that we are truncating the singularities by *smearing out* the point charge  $\delta_x$  *à la* Onsager. Indeed when computing the divergence of  $\nabla \mathfrak{h}^{\vec{\eta}}$  one finds that the atom at each point  $x \in \mathbf{X}$  has been replaced by a uniform measure of mass 1 on the circle of center  $x$  and radius  $\eta(x)$ . We refer to [AS21, Section 2.2 & Appendix B.1] or to [Ser20, Sec 3.1] for more details. The truncation procedure can be extended to define  $\mathbf{E}_{\vec{\eta}}$ , where  $\mathbf{E}$  is any electric field compatible with  $\mathbf{X}$ , by setting:

$$\mathbf{E}_{\vec{\eta}} := \mathbf{E} - \sum_{x \in \mathbf{X}} \nabla f_{\eta(x)}(\cdot - x).$$

For every point  $x$  of  $\mathbf{X}$  we define the “nearest-neighbor” distance  $r(x)$  as:

$$r(x) := \frac{1}{4} \min\left(\min_{y \in \mathbf{X}, y \neq x} |x - y|, 1\right). \quad (2.2)$$

In particular  $r(x)$  is always smaller than  $1/4$ . We let  $\vec{r} = (r(x), x \in \mathbf{X})$  be the associated truncation vector. We will sometimes use instead the vector  $s\vec{r}$  with  $s < 1$ .

### 2.3. Logarithmic energy and electric fields

The logarithmic interaction energy  $F_N$  defined in (1.1) can be expressed in terms of the true electric field generated by  $\mathbf{X}_N$  (Definition 2.1) as follows.

- Taking a uniform truncation vector  $\eta(x) = \eta > 0$  for all  $x$  in  $\mathbf{X}_N$ , we have the following equality in the limit as  $\eta \rightarrow 0$ :

$$F_N(\mathbf{X}_N) = \frac{1}{2} \lim_{\eta \rightarrow 0} \left( \frac{1}{2\pi} \int_{\mathbb{R}^2} |\nabla \mathfrak{h}^{\vec{\eta}}|^2 + N \log \eta \right).$$

(This quantity is “almost” non-decreasing as  $\eta \rightarrow 0$ , see e.g. [AS21, Lemma B.1].)

- On the other hand, taking for each  $x$  a truncation  $\eta(x) \leq r(x)$ , we get a non-asymptotic identity:

$$F_N(\mathbf{X}_N) = \frac{1}{2} \left( \frac{1}{2\pi} \int_{\mathbb{R}^2} |\nabla \mathfrak{h}^{\vec{\eta}}|^2 + \sum_{x \in \mathbf{X}_N} \log \eta(x) \right) - \sum_{x \in \mathbf{X}_N} \int_{\mathfrak{D}(x, \eta(x))} f_{\eta(x)}(t - x) dt. \quad (2.3)$$

The first formulation can be found in [SS12; RS16] and the second one in [LS18] or [AS21, Lemma 2.2]. Of course, expressing electrostatic interactions in terms of the electric field and “smearing out” point charges are both old ideas. One sees that in the small truncation limit there is a compensation between the explosion of  $\int_{\mathbb{R}^2} |\nabla \mathfrak{h}_{\vec{\eta}}^{\mathbf{X}_N}|^2$  and the very negative terms  $\sum_{x \in \mathbf{X}_N} \log \eta(x)$ , hence the name “renormalized energy” given in [SS15] (this renormalization procedure appeared in [BBH94] and the idea of using nearest-neighbor distances was borrowed from [GP77]). Despite their apparent mutual cancellation, in general the “positive” and “negative” parts can both be compared to  $F_N(\mathbf{X}_N)$  as explained e.g. in [AS21, Lemma B.2].

In the sequel we use the expression “electric energy” when referring to quantities of the type  $\int_{\Omega} |\mathbf{E}_{\vec{\eta}}|^2$  where  $\mathbf{E}$  is an electric field,  $\vec{\eta}$  a truncation vector and  $\Omega$  is some subset of  $\mathbb{R}^2$ , and we write:

$$\text{Ener}(\mathbf{X}_N, \Omega) := \int_{\Omega} |\nabla \mathfrak{h}_{\vec{r}}^{\mathbf{X}_N}|^2, \quad \text{Ener}_s(\mathbf{X}_N, \Omega) := \int_{\Omega} |\nabla \mathfrak{h}_{s\vec{r}}^{\mathbf{X}_N}|^2 \quad (\text{for } s \in (0, 1)). \quad (2.4)$$

From the identity (2.3) we deduce that for  $s < 1$ :

$$\text{Ener}_s(\mathbf{X}_N, \Omega) = \text{Ener}(\mathbf{X}_N, \Omega) - \text{Pts}(\mathbf{X}_N, \Omega) \log s, \quad (2.5)$$

For convenience, we will sometimes write  $\text{EnerPts}$  for the sum of  $\text{Ener}$  and  $\text{Pts}$  in a given domain.

## 2.4. Global and local laws

The following properties of  $F_N$  are well-known (see [AS21, Lemma 3.7]):

- It is bounded below: there exists a universal constant  $C$  such that for all  $N \geq 2$  and for all  $\mathbf{X}_N$ , we have  $F_N(\mathbf{X}_N) \geq -CN$ .
- It is typically of order  $N$  in the sense that for some constant  $C_{\beta}$  depending only on  $\beta$ :

$$\mathbb{E}_N^{\beta} \left[ \exp \left( \frac{\beta}{2} F_N(\mathbf{X}_N) \right) \right] \leq \exp(C_{\beta} N). \quad (2.6)$$

We refer to (2.6) as a “global law”, controlling the system at macroscopic scale. The next proposition expresses the fact that the system is also well-behaved down to some large microscopic scale. Let us first introduce two important distances that will play a role in Proposition 2.4 and in the rest of the paper.

**The smallest microscopic scale  $\rho_{\beta}$ .** We refer to a certain *minimal lengthscale* introduced in [AS21] and denoted by  $\rho_{\beta}$ . It is a positive constant that depends only on  $\beta$ , and corresponds to the length-scale above which good rigidity properties are proven.

**Distance to the boundary.** For technical (and possibly physical) reasons, properties of the full-system are easier to understand when one looks “away from the edge”, namely at some non-trivial distance of the boundary  $\partial\Sigma_N$ . For  $x$  a point in  $\Sigma_N$  and  $\ell > 0$ , we will say that “ $(x, \ell)$  satisfies (2.7)” when:

$$\text{dist}(\square(x, \ell), \partial\Sigma_N) \geq C_{\beta} N^{1/4}, \quad (2.7)$$

where  $C_{\beta}$  is some large enough constant introduced in [AS21, (1.16)].

Let us note that in the statement of Theorem 1 we assume that  $\text{dist}(\mathfrak{D}(x, R), \partial\Sigma_N) \geq \delta\sqrt{N}$  which is clearly a stronger assumption, at least for  $N$  large enough depending only on  $\delta, \beta$ .

**Proposition 2.4 (Local laws).** *There exists some universal constant  $C$ , and a “local laws” constant  $C_{LL}$  depending only on  $\beta$  such that the following holds. Let  $x$  be a point in  $\Sigma_N$  and  $\ell$  a lengthscale such that:*

1.  $\ell \geq \rho_{\beta}$  (the length-scale  $\ell$  is larger than the “minimal” one)
2.  $(x, \ell)$  satisfy (2.7) (we are sufficiently “far from the edge”).

*Then we control the electric energy in  $\square(x, \ell)$  in exponential moments:*

$$\log \mathbb{E}_N^{\beta} \left( \exp \left( \frac{\beta}{2} \text{Ener}(\mathbf{X}_N, \square(x, \ell)) \right) \right) \leq C_{LL} \beta \ell^2.$$

Moreover, we have the following control on the number of points:

$$\log \mathbb{E}_N^\beta \left( \exp \left( \frac{\beta}{C} \text{Pts}(\mathbf{X}_N, \square(x, \ell)) \right) \right) \leq C_{\text{LL}} \beta \ell^2, \quad (2.8)$$

together with a discrepancy bound:

$$\log \mathbb{E}_N^\beta \left( \exp \left( \frac{\beta}{C} \frac{\text{Dis}^2(\mathbf{X}_N, \square(x, \ell))}{\ell^2} \right) \right) \leq C_{\text{LL}} \beta. \quad (2.9)$$

*Proof of Proposition 2.4.* This is a subset of the statements in [AS21, Theorem 1], see Section B.1 for a technical discussion.  $\square$

As can be seen from Proposition 2.4,  $\text{Ener}(\mathbf{X}_N, \square(x, \ell))$ ,  $\text{Pts}(\mathbf{X}_N, \square(x, \ell))$  and  $\text{EnerPts}(\mathbf{X}_N, \square(x, \ell))$  are all expected to be of the same order as the area  $\ell^2$ .

**Remark 2.5.** For our purposes, we will repeatedly use the local laws under the following form: for  $C_\beta$  large enough, the probability of having more than  $C_\beta \ell^2$  points (or an energy higher than  $C_\beta \ell^2$ ) in a given square  $\square(x, \ell)$  is smaller than  $\exp(-\ell^2/C_\beta)$ .

One can replace squares by disks (or by any “reasonable” shape) in the previous statement, however it is worth observing that local laws do not directly yield interesting controls the energy (or number of points) on very thin strips, rectangles with diverging aspect ratio, boundaries of squares, thin annuli etc. In those situations, one must resort to splitting the region into squares, applying the local laws to each square and then using a union bound.

## 2.5. The electric energy controls fluctuations

The next lemma expresses how the electric energy controls linear statistics of Lipschitz functions.

**Lemma 2.6** (Bounds on fluctuations - Lipschitz case). *Let  $\mathbf{X}$  be a point configuration in  $\mathbb{R}^2$ , and let  $\varphi$  be a function in  $C^1(\mathbb{R}^2)$  with compact support. Let  $\mathbf{E}$  be any electric field compatible with  $\mathbf{X}$  on  $\text{supp } \varphi$  in the sense of Definition 2.2. Let  $\Omega$  be a domain containing a 1-neighborhood of  $\text{supp } \nabla \varphi$ . We have:*

$$\left| \int \varphi(x) \mathbf{d}(\mathbf{X} - \mathbf{m}_0)(x) \right| \leq \left( \int_{\mathbb{R}^2} |\nabla \varphi|^2 \right)^{\frac{1}{2}} \left( \int_{\Omega} |\mathbf{E}_{\vec{r}}|^2 \right)^{\frac{1}{2}} + |\varphi|_{1, \Omega} \text{Pts}(\mathbf{X}, \Omega) \quad (2.10)$$

**Localized case** *Assume that  $\tilde{\Omega}_1, \dots, \tilde{\Omega}_m$  cover  $\text{supp } \nabla \varphi$ , and that for each  $i$  the domain  $\Omega_i$  contains a 1-neighborhood of  $\tilde{\Omega}_i$ , then we can replace the right-hand side of (2.10) by:*

$$\sum_{k=1}^m |\varphi|_{1, \Omega_i} \times \left( \left( \int_{\Omega_i} |\mathbf{E}_{\vec{r}}|^2 \right)^{\frac{1}{2}} \times |\Omega_i|^{\frac{1}{2}} + \text{Pts}(\mathbf{X}, \Omega_i) \right) \quad (2.11)$$

**Remark 2.7.** Controls of the type (2.10) have appeared under various forms in previous works see e.g. [SS15, Lemma 5.1], [LS18, Proposition 2.5] or [AS21, Lemma B.5], they are usually phrased as: “the electric energy controls the fluctuations”. The electric energy in a given domain  $\Omega$  is typically of order  $|\Omega|$  and so is the number of points in  $\Omega$ , cf. Proposition 2.4, thus (2.10) bounds the typical fluctuations of  $\varphi$  by  $|\varphi|_1 |\text{supp } \nabla \varphi|$  whereas a naive  $L^\infty$  bound would rather give  $|\varphi|_0 \times |\text{supp } \varphi|$ . Since our test functions often live on some large lengthscale  $\ell$  with  $|\varphi|_1$  comparable to  $\ell^{-1} |\varphi|_0$ , there is indeed an improvement.

We give the proof of Lemma 2.6 in Section B.2. Compared to existing results, here we simply emphasize the role played by the support of the gradient (instead of the whole support of the test function), which yields more accurate estimates when  $\varphi$  is a sharp cut-off function. We will repeatedly use its “localized version” (2.11), whose proof is a straightforward of (2.10).

## 2.6. Finer bound on fluctuations for smooth test functions

For test functions with higher regularity the bound of Lemma 2.6 on fluctuations of linear statistics can be improved (see [LS18; BBNY19; Ser20] as well as [RV07; AHM11] for the  $\beta = 2$  case). In particular if  $\varphi_\ell(x) := \bar{\varphi}(x/\ell)$  for a fixed reference smooth test function  $\bar{\varphi}$ , then  $\varphi_\ell$  lives at scale  $\ell \gg 1$  yet its fluctuations remain *bounded* as  $\ell \rightarrow \infty$  (with high probability). One can refer e.g. to [Ser20, Thm. 1] and [Ser20, Cor. 2.1] for such results.

In this paper we will occasionally need a more specific statement valid in the radially symmetric case.

**Proposition 2.8** (Finer bound, the  $C^2$  radially symmetric case). *There exists a constant  $C_\beta$  depending only on  $\beta$  such that the following holds. Let  $x$  be a point in  $\Sigma_N$  and let  $\varphi$  be a test function which is radially symmetric around  $x$ , with compact support. Assume that  $\varphi$  is in  $C^2(\mathbb{R}^2)$  and let  $\mathcal{A}$  be an annulus containing (a 1-neighborhood of) the support of  $\Delta\varphi$ . Let  $s$  be a real number satisfying:*

$$|s| \leq \frac{\pi\beta}{4|\varphi|_2}. \quad (2.12)$$

Then the exponential moments of the fluctuations of  $\varphi$  satisfy:

$$\log \mathbb{E}_N^\beta [\exp(s\text{Fluct}[\varphi])] = \frac{s^2}{4\pi\beta} \int_{\mathbb{R}^2} |\nabla\varphi|^2 + \log \mathbb{E}_N^\beta [\exp(s|\varphi|_2 C_\beta (\text{EnerPts}(\mathbf{X}_N, \mathcal{A})))] .$$

Although it does not appear as such in the literature, Proposition 2.8 can be easily deduced from the tools of [Ser20], we give a proof in Section B.3.

## 2.7. Wegner's estimates

The recent paper [Tho22] provides upper bounds on the  $k$ -point correlation functions of Coulomb gases (in dimension 2 and higher). Among many other things, it states so-called ‘‘Wegner’s estimates’’ i.e. uniform controls of the form  $\rho_1 \leq C_\beta$ , where  $\rho_1$  is the one-point correlation function. We will use [Tho22, Theorem 1.6], which is valid even for  $r \ll 1$  (sub-microscopic scale):

**Lemma 2.9.** *There exists  $C_\beta$  such that for all  $x$  in  $\Sigma_N$  and for all  $r > 0$ , if  $(x, r)$  satisfies (2.7) we have:*

$$\mathbb{P}_N^\beta (\{\text{Pts}(\mathfrak{D}(x, r)) \geq 1\}) \leq C_\beta r^2. \quad (2.13)$$

## 3. Locating discrepancies near the boundary

Let  $0 \leq \varepsilon_R \leq 1$  (to be chosen later). The goal of this section to show that any ‘‘large’’ discrepancy  $\varepsilon_R R$  within a disk of radius  $R$  can be found (with high probability) *near the boundary of the disk*.

### 3.1. Cornering the discrepancy in an annulus close to the boundary

Let  $L \geq 100$  be a lengthscale to be chosen later. Let  $z$  be a point and  $R \geq \rho_\beta$  such that  $\mathfrak{D}(z, R)$  satisfies (2.7). In this section we sometimes simply write  $\mathfrak{D}_r$  (for  $r > 0$ ) instead of  $\mathfrak{D}(z, r)$ . Assume that the disk  $\mathfrak{D}(z, R)$  contains  $\varepsilon_R \cdot R$  too many<sup>4</sup> points. Compare two idealized situations: in the first one, the excess of charges is spread uniformly over  $\mathfrak{D}_R$ , in the sense that  $\text{Dis}(\mathbf{X}_N, \mathfrak{D}_r)$  behaves like  $\frac{r^2}{R^2} \times \varepsilon_R \cdot R$  for  $r \leq R$ . In the second situation, the excess is concentrated in a thin strip near  $\partial\mathfrak{D}_R$  and immediately compensated by a default on the other side of the boundary. According to [JLM93], the physically realistic picture is the second one: ‘‘*Macroscopic electrostatics of conductors implies that, for a given value of [the discrepancy]  $Q$ , the dominant configurations are such that  $Q$  is concentrated in a layer on the inner side of the boundary of the disk, while a charge  $-Q$  accumulates in a layer on the outer side.*’’. In order to give a corresponding mathematical statement, observe that the event

$$\bigcap_{R-2L \leq r \leq R-L} \left\{ \text{Dis}(\mathbf{X}_N, \mathfrak{D}(z, r)) \geq \frac{1}{2} \varepsilon_R \cdot R \right\}$$

<sup>4</sup>We mean  $\text{Pts}(\mathbf{X}_N, \mathfrak{D}_R) \geq \pi R^2 + \varepsilon_R \cdot R$ . points

expresses that some excess of charges is found in the disks  $D_r$  for a fairly wide region  $r \in [R - 2L, R - L]$  and that the discrepancy is thus not “concentrated in a layer on the inner side of the boundary of the disk”.

**Proposition 3.1.** *There exists a constant  $\mathfrak{C}_{3.1}(\beta)$  depending only on  $\beta$  such that the following holds. Let  $s$  be a real parameter. Assume that  $R, L, \varepsilon_R, s$  satisfy:*

$$\begin{cases} \mathfrak{C}_{3.1} \frac{1}{\varepsilon_R} \leq L \leq \frac{R}{10} \\ 0 \leq s \leq \frac{1}{\mathfrak{C}_{3.1}} \min\left(\frac{L^3}{R}, L\varepsilon_R\right). \end{cases} \quad (3.1)$$

Then we have:

$$\mathbb{P}_N^\beta \left[ \bigcap_{R-2L \leq r \leq R-L} \left\{ \text{Dis}(\mathbf{X}_N, \mathfrak{D}(z, r)) \geq \frac{1}{2} \varepsilon_R \cdot R \right\} \right] \leq \exp\left(-\frac{s\varepsilon_R \cdot R}{4}\right). \quad (3.2)$$

We postpone the proof of Proposition 3.1 to Section B.4. It relies on Proposition 2.8.

Next, we argue that if the complementary event occurs, then one can find an annulus of width  $\approx L$  near the boundary of  $\mathfrak{D}_R$  that carries a large discrepancy.

**Lemma 3.2** (Finding discrepancy in an annulus). *If  $\text{Dis}(\mathbf{X}_N, \mathfrak{D}_R) \geq \varepsilon_R \cdot R$  and if, in contrast to the event considered in (3.2), there exists a radius  $r$  with  $R - 2L \leq r \leq R - L$  such that:*

$$\text{Dis}(\mathbf{X}_N, \mathfrak{D}_r) < \frac{1}{2} \varepsilon_R \cdot R \quad (3.3)$$

then there exists an integer  $k$  with  $0 \leq k \leq R^2$  such that the discrepancy in the annulus  $\mathfrak{D}_R \setminus \mathfrak{D}_{R-2L+\frac{kL}{R^2}}$  is larger than  $\frac{1}{4} \varepsilon_R \cdot R$ .

*Proof.* Let  $r$  be such that (3.3) holds and let  $0 \leq k \leq R^2$  be the integer such that:

$$r_k := R - 2L + \frac{kL}{R^2} \leq r < R - 2L + \frac{(k+1)L}{R^2}.$$

It is easy to see that  $\text{Dis}(\mathbf{X}_N, \mathfrak{D}_{r_k}) \leq \frac{3}{4} \varepsilon_R \cdot R$  (a rough bound), indeed we have:

$$\text{Dis}(\mathbf{X}_N, \mathfrak{D}_{r_k}) \leq \text{Dis}(\mathbf{X}_N, \mathfrak{D}_r) + |\mathfrak{D}_r \setminus \mathfrak{D}_{r_k}|,$$

with equality if and only if there is no point in the annulus  $\mathfrak{D}_r \setminus \mathfrak{D}_{r_k}$ , whose area is of order  $\frac{RL}{R^2} \leq 1$ .

Since the total discrepancy in  $\mathfrak{D}_R$  is at least  $\varepsilon_R \cdot R$  and the one inside  $\mathfrak{D}_{r_k}$  is at most  $\frac{3}{4} \varepsilon_R \cdot R$  then the annulus  $\mathfrak{D}_R \setminus \mathfrak{D}_{r_k}$  must carry a discrepancy of size at least  $\frac{1}{4} \varepsilon_R \cdot R$ .  $\square$

**Remark 3.3.** If there is a negative discrepancy i.e. a default of points instead of an excess, one proceeds the same way, except that in Proposition 3.1 we bound the probability of  $\text{Dis}(\mathbf{X}_N, \mathfrak{D}_r) \leq \frac{-\varepsilon_R \cdot R}{2}$  for all  $R - 2L \leq r \leq R - L$  instead. The complementary event is that for some  $r$  with  $R - 2L \leq r \leq R - L$ , we have  $\text{Dis}(\mathbf{X}_N, \mathfrak{D}_r) > -\frac{\varepsilon_R \cdot R}{2}$ . Then in Lemma 3.2 instead of taking  $r_k$  a bit smaller than  $r$  we take it a bit larger and observe that the discrepancy cannot go down too much between  $\mathfrak{D}_r$  and  $\mathfrak{D}_{r_k}$ . Hence  $\text{Dis}(\mathbf{X}_N, \mathfrak{D}_{r_k}) \geq \frac{-3\varepsilon_R \cdot R}{4}$ , so the annulus  $\mathfrak{D}_R \setminus \mathfrak{D}_{r_k}$  carries a default of points at least  $\frac{\varepsilon_R \cdot R}{4}$  (in absolute value).

### 3.2. A well-separated family of boxes carrying the discrepancy

Let  $r$  be in  $[R - 2L, R - L]$ .

**Decomposition in boxes** We split the annulus  $\mathfrak{D}_R \setminus \mathfrak{D}_r$  into smaller parts that we call *boxes*.

**Definition 3.4** (Decomposition in boxes). For  $i \in \{0, \dots, \frac{R}{L} - 1\}$ , we let the  $i$ -th “box”  $\mathfrak{B}_i$  be the intersection of the annulus  $\mathfrak{D}_R \setminus \mathfrak{D}_r$  with a certain angular sector of center  $z$ :

$$\mathfrak{B}_i := (\mathfrak{D}_R \setminus \mathfrak{D}_r) \cap \left\{ z' \in \mathbb{R}^2 \simeq \mathbb{C}, i \leq \frac{\arg(z' - z)}{2\pi} \times \frac{R}{L} \leq i + 1 \right\}.$$

The boundary of each such box is made of two line segments of equal length in  $(L, 2L)$  and two concentric circular arcs which subtend the same angle at the center and whose arclengths are different but both in  $(\frac{L}{2}, L)$ . The shape is symmetric with respect to the straight line joining the midpoints of both arcs. We sometimes call “a box of size  $L$ ” any domain  $\mathfrak{B}$  that fits the previous description.

We let  $\omega_i$  be the center of mass of  $\mathfrak{B}_i$ , which only serves as a convenient reference point.

**Introducing the parameters  $M$  and  $T$**  Let  $T$  be a lengthscale, and  $M$  be an integer, both to be chosen at the end, such that:

$$T \geq 10L, \quad 100 \leq M \leq \frac{R}{L}, \quad T \leq \frac{ML}{100}, \quad T^2 \log T \leq ML. \quad (3.4)$$

(The fourth condition (which could be weakened), implies the third one but for clarity we write them all down.)

**Lemma 3.5** (Some well-separated boxes carry the discrepancy). *Assume that the discrepancy in  $\mathfrak{D}_R \setminus \mathfrak{D}_r$  is larger than  $\frac{1}{4}\varepsilon_R \cdot R$ . Then there exists  $l \in \{0, \dots, M - 1\}$  such that:*

$$\sum_{i \equiv l \pmod{M}} \text{Dis}(\mathbf{X}_N, \mathfrak{B}_i) \geq \frac{\varepsilon_R \cdot R}{4M}. \quad (3.5)$$

*Proof of Lemma 3.5.* It follows from a straightforward pigeonhole argument.  $\square$

From now on, we assume to have chosen such a  $l \in \{0, \dots, M - 1\}$  and we write the corresponding boxes as  $\mathfrak{B}_1, \dots, \mathfrak{B}_N$ , where  $N = \mathcal{O}\left(\frac{R}{ML}\right)$  is the number of box in each “well-separated” family. Moreover, we see each box  $\mathfrak{B}_i$  as being contained in a large disk  $\Lambda_i := \mathfrak{D}(\omega_i, T)$ , where  $T$  is as above.

Let  $d_{ij}$  be the distance between  $\Lambda_i$  and  $\Lambda_j$ .

**Claim 3.6.** *We have for each fixed  $i$ :*

$$\sum_{j, j \neq i} \frac{1}{d_{ij}} = \mathcal{O}\left(\frac{\log R}{ML}\right), \quad \sum_{j, j \neq i} \frac{1}{d_{ij}^2} = \mathcal{O}\left(\frac{1}{M^2 L^2}\right) \quad (3.6)$$

*Proof.* Let us observe that between two “consecutive” boxes in the family  $\{i \equiv l \pmod{M}\}$  considered in (3.5), there is a distance of order  $ML$  (since  $ML \geq 100T$  by assumption, this is also comparable to the distance between two consecutive *disks*). We can compare the sum to an harmonic sum (in the first case) or a converging Riemann series (in the second case).  $\square$

## Plan for the next two sections

We now want to treat each box  $\mathfrak{B}_i$  as living in its own smaller version of a 2DOCP contained in  $\Lambda_i$ , which leads us to the next two sections.

Think of a “sub-system” as the random collection of particles contained in a given sub-domain  $\Lambda \subset \Sigma_N$  with a reasonable shape (e.g. a square or a disk). These particles feel the influence of each other, but also of the full system in  $\Sigma_N$  because the logarithmic interaction is long-range. Hence sub-systems are typically not isolated and not independent from each other.

1. In Section 4 we observe that if we condition on the values of the discrepancies (or equivalently of the number of particles) in domains that are well-separated, then the corresponding sub-systems acquire a form of independence.
2. In Section 5 we show that “typical” sub-systems, seen as slight generalizations of the 2DOCP model introduced earlier (in (1.1), (1.2)), retains most of the properties of the *full* system mentioned in Section 2.

## 4. Approximate conditional independence for sub-systems

In this section, we consider a family  $\{\Lambda_i, 1 \leq i \leq \mathcal{N}\}$  of  $\mathcal{N} \geq 2$  disjoint disks in  $\Sigma_N$  (forming our “sub-systems”). We will eventually apply the results below to the  $\Lambda_i$ ’s chosen in Section 3.2, but the statements in the present section are general.

Let  $\text{Ext}$  be the complement  $\text{Ext} := \Sigma_N \setminus \cup_{i=1}^{\mathcal{N}} \Lambda_i$ . Let  $V^{\text{ext}}$  be the logarithmic potential generated by the system in  $\text{Ext}$ , namely:

$$V^{\text{ext}}(x) := \int_{\text{Ext}} -\log|x-y| \mathbf{d}\mathbf{f}_N(y). \quad (4.1)$$

The potential  $\text{Ext}$  is harmonic on all the  $\Lambda_i$ ’s and depends only on the configuration in  $\text{Ext}$ .

### 4.1. Decomposition of the interaction, conditional independence error

We let  $\text{Int}[\Lambda_1, \dots, \Lambda_{\mathcal{N}}]$  be the true logarithmic interaction between the sub-systems, namely:

$$\text{Int}[\Lambda_1, \dots, \Lambda_{\mathcal{N}}] := \frac{1}{2} \sum_{1 \leq i \neq j \leq \mathcal{N}} \iint_{\Lambda_i \times \Lambda_j} -\log|x-y| \mathbf{d}\mathbf{f}_N(x) \mathbf{d}\mathbf{f}_N(y).$$

By expanding the double integral defining  $F_N$ , we may write:

$$F_N(\mathbf{X}_N) = F_{\text{Ext}}(\mathbf{X}_N \cap \text{Ext}) + \text{Int}[\Lambda_1, \dots, \Lambda_{\mathcal{N}}] + \sum_{i=1}^{\mathcal{N}} \left( F_{\Lambda_i}(\mathbf{X}_N \cap \Lambda_i) + \int_{\Lambda_i} V^{\text{ext}}(x) \mathbf{d}\mathbf{f}_N(x) \right) \quad (4.2)$$

with  $F_{\text{Ext}}, F_{\Lambda_i}$  defined the obvious way (see (4.7)). On the other hand, for each  $1 \leq i \leq \mathcal{N}$ , let  $D_i$  be the discrepancy of  $\mathbf{X}_N$  in  $\Lambda_i$ , and let  $\widetilde{\text{Int}}[D_1, \dots, D_{\mathcal{N}}]$  be the approximation of  $\text{Int}[\Lambda_1, \dots, \Lambda_{\mathcal{N}}]$  given by:

$$\widetilde{\text{Int}}[D_1, \dots, D_{\mathcal{N}}] := \frac{1}{2} \sum_{1 \leq i \neq j \leq \mathcal{N}} -D_i D_j \log|\omega_i - \omega_j|,$$

where  $\omega_i$  denotes the center of  $\Lambda_i$ . We define the quantity  $\text{ErrorCI}$  as follows:

$$\text{ErrorCI}[\mathbf{X}_N | (\Lambda_1, \dots, \Lambda_{\mathcal{N}})] := \left| \text{Int}[\Lambda_1, \dots, \Lambda_{\mathcal{N}}] - \widetilde{\text{Int}}[D_1, \dots, D_{\mathcal{N}}] \right|, \quad (4.3)$$

and we use it below in order to measure a “conditional independence error”.

### 4.2. Bounds on the conditional independence error

For  $1 \leq i \neq j \leq \mathcal{N}$ , define the distance  $d_{ij}$  as  $d_{ij} := \text{dist}(\Lambda_i, \Lambda_j)$ . Assume that for all  $1 \leq i \leq \mathcal{N}$  the disk  $\Lambda_i$  has radius  $T$  and that:

$$\max_{1 \leq i \leq \mathcal{N}} n_i \leq 10T^2, \quad \min_{1 \leq i \neq j \leq \mathcal{N}} d_{ij} \geq 10T. \quad (4.4)$$

**Lemma 4.1** (The size of  $\text{ErrorCI}$ ). *We have, if (4.4) holds:*

$$|\text{ErrorCI}[\mathbf{X}_N | (\Lambda_1, \dots, \Lambda_{\mathcal{N}})]| = \mathcal{O}(T^5) \sum_{1 \leq i \neq j \leq \mathcal{N}} \frac{1}{d_{ij}}. \quad (4.5)$$

*Proof of Lemma 4.1.* For  $x \in \Lambda_i, y \in \Lambda_j$  (with  $i \neq j$ ) since the diameter of the squares is  $\mathcal{O}(T)$  and the mutual distances satisfy (4.4), a Taylor’s expansion yields:

$$\log|x-y| = \log|\omega_i - \omega_j| + \mathcal{O}\left(\frac{T}{|\omega_i - \omega_j|}\right), \quad (4.6)$$

with a universal implicit constant. Integrating (4.6) against the fluctuation measures in  $\Lambda_i$  and  $\Lambda_j$  yields:

$$\left| \iint_{\Lambda_i \times \Lambda_j} \log|x-y| \mathbf{d}\mathbf{f}_N(x) \mathbf{d}\mathbf{f}_N(y) - D_i D_j \log|\omega_i - \omega_j| \right| \leq (|\Lambda_i| + n_i) \cdot (|\Lambda_j| + n_j) \cdot \mathcal{O}\left(\frac{T}{d_{ij}}\right).$$

Using the bound on  $n_i, n_j$  given by (4.4) and summing over  $i \neq j$ , we get (4.5).  $\square$

**Remark 4.2.** It is possible to reduce the order of magnitude of  $\text{ErrorCI}$  by expanding the interaction in a more precise way.



### 4.3. 2DOCP's with harmonic external field

In the following we consider two-dimensional one-component plasmas whose energy takes into account the effect of an external field  $V^{\text{ext}}$  on each particle.

- Let  $V^{\text{ext}}$  (the *external field* on  $\Lambda$ ) be a lower semi-continuous function on  $\mathbb{R}^2$  that is harmonic on (the interior of)  $\Lambda$ . Let us emphasize that  $V^{\text{ext}}$  is harmonic hence very regular in the interior of  $\Lambda$  but we do not *a priori* control  $V^{\text{ext}}$  or its derivatives near  $\partial\Lambda$ . The situation is thus different from the choice of an external weight/potential as frequently found in the literature on log-gases.
- Let  $\mathbf{X}_n$  denote a  $n$ -tuple of points  $\mathbf{X}_n = (x_1, \dots, x_n)$  in  $\Lambda$ , let  $\mathbf{X}_n := \sum_{i=1}^n \delta_{x_i}$  be the associated atomic measure, and let  $\mathbf{f}_\Lambda := \mathbf{X}_n - \mathbf{m}_0 \mathbf{1}_\Lambda$  be the signed fluctuation measure on  $\Lambda$ . Let  $F_\Lambda(\mathbf{X}_n)$  be the logarithmic interaction energy:

$$F_\Lambda(\mathbf{X}_n) := \frac{1}{2} \iint_{(x,y) \in \Lambda \times \Lambda, x \neq y} -\log|x-y| \mathbf{d}\mathbf{f}_\Lambda(x) \mathbf{d}\mathbf{f}_\Lambda(y). \quad (4.7)$$

We define a probability density  $\mathbb{P}_{n,\Lambda,V^{\text{ext}}}^\beta$  on the space of  $n$ -tuples of points in  $\Lambda$  by setting:

$$\mathbf{d}\mathbb{P}_{n,\Lambda,V^{\text{ext}}}^\beta(\mathbf{X}_n) := \frac{\exp\left(-\beta\left(F_\Lambda(\mathbf{X}_n) + \int_\Lambda V^{\text{ext}}(x) \mathbf{d}\mathbf{f}_\Lambda(x)\right)\right)}{K_{n,\Lambda,V^{\text{ext}}}^\beta} \mathbf{d}\mathbf{X}_n, \quad (4.8)$$

where  $K_{n,\Lambda,V^{\text{ext}}}^\beta$  is the partition function, namely the normalizing constant:

$$K_{n,\Lambda,V^{\text{ext}}}^\beta := \int_{\Lambda^n} \exp\left(-\beta\left(F_\Lambda(\mathbf{X}_n) + \int_\Lambda V^{\text{ext}}(x) \mathbf{d}\mathbf{f}_\Lambda(x)\right)\right) \mathbf{d}\mathbf{X}_n, \quad (4.9)$$

and  $\mathbf{d}\mathbf{X}_n$  is the Lebesgue measure on  $\Lambda^n$ . We may now state the main result of this section.

### 4.4. Approximate conditional independence

**Proposition 4.3** (Approximate conditional independence).

- For  $1 \leq i \leq N$ , let  $G_i$  be a measurable function on  $\text{Conf}$  with non-negative real values and let  $\mathcal{E}_i$  be a measurable subset of  $\text{Conf}$ . Assume that  $G_i, \mathcal{E}_i$  are  $\Lambda_i$ -local in the sense of Section 2.1.
- Let  $\mathcal{E}_{\text{ext}}$  be a measurable subset of  $\text{Conf}$ , assume that  $\mathcal{E}_{\text{ext}}$  is  $\text{Ext}$ -local.
- Denote by  $\mathcal{E}_N$  the following event:  $\mathcal{E}_N := \mathcal{E}_{\text{ext}} \cap \bigcap_{i=1}^N \mathcal{E}_i$ .
- Finally, we say that a family  $\{\mathbf{n}_i\}_{1 \leq i \leq N}$  of integers is “admissible” (we write “ $\{\mathbf{n}_i\}$  adm.” below) when there exists  $\mathbf{X}_N \in \mathcal{E}_N$  such that  $\text{Pts}(\mathbf{X}_N, \Lambda_i) = \mathbf{n}_i$  for all  $1 \leq i \leq N$ .

We have:

$$\mathbb{E}_N^\beta \left[ \prod_{i=1}^N G_i(\mathbf{X}_N) \mathbf{1}_{\mathcal{E}_N} \right] \leq \exp\left(2\beta \sup_{\mathbf{X}_N \in \mathcal{E}_N} \text{ErrorCI}(\mathbf{X}_N)\right) \times \sup_{\mathbf{X}^{\text{ext}} \in \mathcal{E}_{\text{ext}}, \{\mathbf{n}_i\} \text{ adm.}} \prod_{i=1}^N \mathbb{E}_{\mathbf{n}_i, \Lambda_i, V^{\text{ext}}}^\beta \left[ G_i(\mathbf{X}) \middle| \mathcal{E}_i \right]. \quad (4.10)$$

Moreover, with the same assumptions, the following lower bound holds:

$$\mathbb{E}_N^\beta \left[ \prod_{i=1}^N G_i(\mathbf{X}_N) \mathbf{1}_{\mathcal{E}_N} \right] \geq \exp\left(-2\beta \sup_{\mathbf{X}_N \in \mathcal{E}_N} \text{ErrorCI}(\mathbf{X}_N)\right) \times \mathbb{P}_N^\beta(\mathcal{E}_N) \times \inf_{\mathbf{X}^{\text{ext}} \in \mathcal{E}_{\text{ext}}, \{\mathbf{n}_i\} \text{ adm.}} \prod_{i=1}^N \mathbb{E}_{\mathbf{n}_i, \Lambda_i, V^{\text{ext}}}^\beta \left[ G_i(\mathbf{X}) \middle| \mathcal{E}_i \right]. \quad (4.11)$$

Notice that one sup is now an inf, the error term  $\sup \text{ErrorCI}$  now appears in the exponent with a minus sign, and there is an extra factor  $\mathbb{P}_N^\beta(\mathcal{E}_N)$ .

*Proof of Proposition 4.3.* Let us start by using the definition (1.2) of  $\mathbb{P}_N^\beta$  and the decomposition (4.2) of the logarithmic interaction  $F_N(\mathbf{X}_N)$ . We obtain:

$$\begin{aligned} \mathbb{E}_N^\beta \left[ \prod_{i=1}^{\mathcal{N}} G_i(\mathbf{X}_N) \mathbf{1}_{\mathcal{E}_N} \right] &= \frac{1}{K_N^\beta} \int_{(\Sigma_N)^N} \exp(-\beta F_{\text{Ext}}(\mathbf{X}_N \cap \text{Ext})) \exp(-\beta \text{Int}[\Lambda_1, \dots, \Lambda_{\mathcal{N}}]) \\ &\quad \times \left[ \prod_{i=1}^{\mathcal{N}} \exp \left( -\beta \left( F_{\Lambda_i}(\mathbf{X}_N \cap \Lambda_i) + \int_{\Lambda_i} V^{\text{ext}}(x) d\mathbf{f}_N(x) \right) \right) \times G_i(\mathbf{X}_N) \right] \mathbf{1}_{\mathcal{E}_N}(\mathbf{X}_N) d\mathbf{X}_N. \end{aligned}$$

Next:

- We use a complete system of events by fixing the number  $n_i$  of points in each  $\Lambda_i$  (and thus also the number  $n_{\text{ext}}$  of points in  $\text{Ext}$ ). The knowledge of  $n_i$  is equivalent to fixing the discrepancy  $D_i$  in  $\Lambda_i$ . Since we are working under the event  $\mathcal{E}_N$ , the  $\{n_i\}$  must be admissible as defined in the statement of Proposition 4.3.
- Up to a combinatorial factor, we may then decompose the  $N$ -tuple  $\mathbf{X}_N$  into a  $n_{\text{ext}}$ -tuple  $\mathbf{X}^{\text{ext}}$  of points in  $\text{Ext}$ , and  $\mathbf{X}_i$  ( $1 \leq i \leq \mathcal{N}$ ), where each  $\mathbf{X}_i$  is a  $n_i$ -tuple of points in  $\Lambda_i$ .
- We decompose the Lebesgue measure  $d\mathbf{X}_N$  accordingly.
- We write  $\mathbf{X}^{\text{ext}}$  and  $\mathbf{X}_i$  ( $1 \leq i \leq \mathcal{N}$ ) for the associated atomic measures. We have the identities  $\mathbf{X}^{\text{ext}} = \mathbf{X}_N \cap \text{Ext}$  and  $\mathbf{X}_i = \mathbf{X}_N \cap \Lambda_i$  ( $1 \leq i \leq \mathcal{N}$ ).
- By our locality assumptions:  $\mathbf{1}_{\mathcal{E}_N}(\mathbf{X}_N) = \mathbf{1}_{\mathcal{E}_{\text{ext}}}(\mathbf{X}^{\text{ext}}) \times \prod_{i=1}^{\mathcal{N}} \mathbf{1}_{\mathcal{E}_i}(\mathbf{X}_i)$ .
- We use our locality assumption on  $G_i$  to write  $G_i(\mathbf{X}_N) = G_i(\mathbf{X}_i)$ .
- We introduce the measures  $\mathbf{f}_{\text{Ext}} := \mathbf{X}^{\text{ext}} - \mathbf{m}_0 \mathbf{1}_{\text{Ext}}$  and  $\mathbf{f}_{\Lambda_i} := \mathbf{X}_i - \mathbf{m}_0 \mathbf{1}_{\Lambda_i}$ .
- Finally, using the definition (4.3) we may replace  $\text{Int}[\Lambda_1, \dots, \Lambda_{\mathcal{N}}]$  by  $\widetilde{\text{Int}}[D_1, \dots, D_{\mathcal{N}}]$ , up to an error quantified by  $\text{ErrorCI}$ . The quantity  $\exp(-\beta \widetilde{\text{Int}}[D_1, \dots, D_{\mathcal{N}}])$  can be taken outside the integrals because it only depends on the data of  $\{n_i\}_{1 \leq i \leq \mathcal{N}}$ .

We obtain the following upper bound:

$$\begin{aligned} \mathbb{E}_N^\beta \left[ \prod_{i=1}^{\mathcal{N}} G_i(\mathbf{X}_N) \mathbf{1}_{\mathcal{E}_N} \right] &\leq \frac{1}{K_N^\beta} \sum_{\{n_i\} \text{ adm.}} \binom{N}{n_1 \dots n_{\mathcal{N}}} \exp(-\beta \widetilde{\text{Int}}[D_1, \dots, D_{\mathcal{N}}]) \\ &\quad \times \exp \left( \beta \sup_{\mathbf{X}_N \in \mathcal{E}_N} \text{ErrorCI}(\mathbf{X}_N) \right) \times \int_{(\text{Ext})^{n_{\text{ext}}}} \exp(-\beta F_{\text{Ext}}(\mathbf{X}^{\text{ext}})) \mathbf{1}_{\mathcal{E}_{\text{ext}}}(\mathbf{X}^{\text{ext}}) \\ &\quad \times \left[ \prod_{i=1}^{\mathcal{N}} \int_{(\Lambda_i)^{n_i}} \exp \left( -\beta \left( F_{\Lambda_i}(\mathbf{X}_i) + \int_{\Lambda_i} V^{\text{ext}}(x) d\mathbf{f}_{\Lambda_i}(x) \right) \right) \times G_i(\mathbf{X}_i) \mathbf{1}_{\mathcal{E}_i}(\mathbf{X}_i) d\mathbf{X}_i \right] d\mathbf{X}^{\text{ext}}. \quad (4.12) \end{aligned}$$

We may conveniently condense (4.12) using our notation. For  $1 \leq i \leq \mathcal{N}$ , in view of the definitions (4.8), (4.9), we write:

$$\begin{aligned} \int_{(\Lambda_i)^{n_i}} \exp \left( -\beta \left( F_{\Lambda_i}(\mathbf{X}_i) + \int_{\Lambda_i} V^{\text{ext}}(x) d\mathbf{f}_{\Lambda_i}(x) \right) \right) G_i(\mathbf{X}_i) \mathbf{1}_{\mathcal{E}_i}(\mathbf{X}_i) d\mathbf{X}_i \\ = \mathbb{E}_{n_i, \Lambda_i, V^{\text{ext}}}^\beta \left[ G_i(\mathbf{X}) \middle| \mathcal{E}_i \right] \times \mathbb{P}_{n_i, \Lambda_i, V^{\text{ext}}}^\beta \left[ \mathcal{E}_i \right] \times K_{n_i, \Lambda_i, V^{\text{ext}}}^\beta, \end{aligned}$$

and we may thus re-write (4.12) as:

$$\begin{aligned} \mathbb{E}_N^\beta \left[ \prod_{i=1}^{\mathcal{N}} G_i(\mathbf{X}_N) \mathbf{1}_{\mathcal{E}_N} \right] &\leq \frac{1}{K_N^\beta} \sum_{\{\mathbf{n}_i\} \text{ adm.}} \binom{N}{\mathbf{n}_1 \dots \mathbf{n}_N} \exp \left( -\beta \widetilde{\text{Int}}[\mathbf{D}_1, \dots, \mathbf{D}_N] \right) \\ &\times \exp \left( \beta \sup_{\mathbf{X}_N \in \mathcal{E}_N} \text{ErrorCI}(\mathbf{X}_N) \right) \times \int_{(\text{Ext})^{\mathbf{n}_{\text{ext}}}} \exp \left( -\beta F_{\text{Ext}}(\mathbf{X}^{\text{ext}}) \right) \mathbf{1}_{\mathcal{E}_{\text{ext}}}(\mathbf{X}^{\text{ext}}) \\ &\times \left[ \prod_{i=1}^{\mathcal{N}} \mathbb{E}_{\mathbf{n}_i, \Lambda_i, V^{\text{ext}}}^\beta \left[ G_i(\mathbf{X}) \middle| \mathcal{E}_i \right] \times \mathbb{P}_{\mathbf{n}_i, \Lambda_i, V^{\text{ext}}}^\beta \left[ \mathcal{E}_i \right] \times K_{\mathbf{n}_i, \Lambda_i, V^{\text{ext}}}^\beta \right] d\mathbf{X}^{\text{ext}}. \end{aligned} \quad (4.13)$$

In (4.12), (4.13), the converse inequalities hold up to adding a minus sign in front of ErrorCI in the exponent (recall that the  $G_i$ 's have non-negative values by assumption). Now, for all  $\mathbf{X}^{\text{ext}} \in \mathcal{E}_{\text{ext}}$  and all admissible  $\{\mathbf{n}_i\}$ , we have:

$$\prod_{i=1}^{\mathcal{N}} \mathbb{E}_{\mathbf{n}_i, \Lambda_i, V^{\text{ext}}}^\beta \left[ G_i(\mathbf{X}) \middle| \mathcal{E}_i \right] \leq \sup_{\mathbf{X}^{\text{ext}} \in \mathcal{E}_{\text{ext}} \{\mathbf{n}_i\} \text{ adm.}} \prod_{i=1}^{\mathcal{N}} \mathbb{E}_{\mathbf{n}_i, \Lambda_i, V^{\text{ext}}}^\beta \left[ G_i(\mathbf{X}) \middle| \mathcal{E}_i \right],$$

and the converse inequality holds up to trading the sup for an inf. We get:

$$\begin{aligned} \mathbb{E}_N^\beta \left[ \prod_{i=1}^{\mathcal{N}} G_i(\mathbf{X}_N) \mathbf{1}_{\mathcal{E}_N} \right] &\leq \sup_{\mathbf{X}^{\text{ext}} \in \mathcal{E}_{\text{ext}} \{\mathbf{n}_i\} \text{ adm.}} \prod_{i=1}^{\mathcal{N}} \mathbb{E}_{\mathbf{n}_i, \Lambda_i, V^{\text{ext}}}^\beta \left[ G_i(\mathbf{X}) \middle| \mathcal{E}_i \right] \times \exp \left( \beta \sup_{\mathbf{X}_N \in \mathcal{E}_N} \text{ErrorCI}(\mathbf{X}_N) \right) \\ &\times \frac{1}{K_N^\beta} \sum_{\{\mathbf{n}_i\} \text{ adm.}} \binom{N}{\mathbf{n}_1 \dots \mathbf{n}_N} \exp \left( -\beta \widetilde{\text{Int}}[\mathbf{D}_1, \dots, \mathbf{D}_N] \right) \int_{(\text{Ext})^{\mathbf{n}_{\text{ext}}}} \exp \left( -\beta F_{\text{Ext}}(\mathbf{X}^{\text{ext}}) \right) \mathbf{1}_{\mathcal{E}_{\text{ext}}}(\mathbf{X}^{\text{ext}}) \\ &\times \left[ \prod_{i=1}^{\mathcal{N}} \int_{(\Lambda_i)^{\mathbf{n}_i}} \mathbb{P}_{\mathbf{n}_i, \Lambda_i, V^{\text{ext}}}^\beta \left[ \mathcal{E}_i \right] \times K_{\mathbf{n}_i, \Lambda_i, V^{\text{ext}}}^\beta \right] d\mathbf{X}^{\text{ext}}, \end{aligned} \quad (4.14)$$

and the converse inequality holds up to trading the first sup for an inf and writing  $-\sup \text{ErrorCI}$  instead of  $\sup \text{ErrorCI}$  in the exponent. We now establish a bound on the partition function  $K_N^\beta$  in a similar fashion. By definition, we have:

$$\begin{cases} \frac{1}{K_N^\beta} \leq \left( \int_{(\Sigma_N)^N} \mathbf{1}_{\mathcal{E}_N} \exp \left( -\beta F_N(\mathbf{X}_N) \right) d\mathbf{X}_N \right)^{-1} \\ \frac{1}{K_N^\beta} = \left( \int_{(\Sigma_N)^N} \mathbf{1}_{\mathcal{E}_N} \exp \left( -\beta F_N(\mathbf{X}_N) \right) d\mathbf{X}_N \right)^{-1} \times \mathbb{P}_N^\beta(\mathcal{E}_N). \end{cases} \quad (4.15)$$

Arguing as above (taking all the  $G_i$ 's equal to the constant 1), we get:

$$\begin{aligned} &\int_{(\Sigma_N)^N} \mathbf{1}_{\mathcal{E}_N} \exp \left( -\beta F_N(\mathbf{X}_N) \right) d\mathbf{X}_N \\ &\leq \exp \left( \beta \sup_{\mathbf{X}_N \in \mathcal{E}_N} \text{ErrorCI}(\mathbf{X}_N) \right) \times \sum_{\{\mathbf{n}_i\} \text{ adm.}} \binom{N}{\mathbf{n}_1 \dots \mathbf{n}_N} \exp \left( -\beta \widetilde{\text{Int}}[\mathbf{D}_1, \dots, \mathbf{D}_N] \right) \\ &\times \int_{\text{Ext}^{\mathbf{n}_{\text{ext}}}} \exp \left( -\beta F_{\text{Ext}}(\mathbf{X}^{\text{ext}}) \right) \mathbf{1}_{\mathcal{E}_{\text{ext}}}(\mathbf{X}^{\text{ext}}) \left[ \prod_{i=1}^{\mathcal{N}} \int_{(\Lambda_i)^{\mathbf{n}_i}} \mathbb{P}_{\mathbf{n}_i, \Lambda_i, V^{\text{ext}}}^\beta \left[ \mathcal{E}_i \right] \times K_{\mathbf{n}_i, \Lambda_i, V^{\text{ext}}}^\beta \right] d\mathbf{X}^{\text{ext}}, \end{aligned} \quad (4.16)$$

and the converse inequality holds with  $-\sup \text{ErrorCI}$  instead of  $\sup \text{ErrorCI}$ . Combining (4.14), (4.15) and (4.16) we obtain (4.10) (note that many terms cancel out in the ratio). We have also proven the converse inequality (4.11), with an extra factor  $\mathbb{P}_N^\beta(\mathcal{E}_N)$  coming from (4.15).  $\square$

## 5. Generalized 2DOCP's arising as sub-systems

Let  $\Lambda$  be a disk of center  $\omega \in \mathbb{R}^2$  and radius  $T$ , and let  $\mathbf{n} \geq 1$  be an integer, corresponding to the number of points in  $\Lambda$ . In general we may have  $\mathbf{n} \neq |\Lambda|$ .

## 5.1. Good external potentials, good sub-systems

Let  $V^{\text{ext}}$  be an external field as in Section 4.3 (we will eventually use this for the particular choice (4.1)). We introduce two definitions to pinpoint “good situations” for the generalized 2DOCP measure  $\mathbb{P}_{n,\Lambda,V^{\text{ext}}}^\beta$  introduced in (4.8).

**Definition 5.1** (Good potential). We say that  $V^{\text{ext}}$  is a “good external potential on  $\Lambda$  with constant  $\bar{C}$ ” when the following holds:

**1. Control up to the edge.** There exists a function  $\tilde{V}^{\text{ext}}$  on  $\Lambda$  satisfying:

$$\tilde{V}^{\text{ext}}(x) = V^{\text{ext}}(x) \text{ if } \text{dist}(x, \partial\Lambda) \geq 1, \quad \tilde{V}^{\text{ext}}(x) \leq V^{\text{ext}}(x) + 100 \text{ for all } x \in \Lambda,$$

such that:

$$|\tilde{V}^{\text{ext}}(x) - \tilde{V}^{\text{ext}}(\omega)| \leq \bar{C} \times T \times \log^3 T.$$

**2. A technical decomposition.**  $V^{\text{ext}}$  can be decomposed as the sum

$$V^{\text{ext}} = \mathfrak{h}^\nu + R, \tag{5.1}$$

where  $\mathfrak{h}^\nu$  is the logarithmic potential generated by a positive measure supported on an annulus of width  $\hat{T} := \log T$  right outside  $\Lambda$ , and  $R$  is harmonic in  $\Lambda$ , such that we control the derivative of  $R$  up to the edge by:

$$|R|_{1,\Lambda} \leq \bar{C} \times \log^2 T,$$

and moreover we control the mass of  $\nu$  locally at scale  $\hat{T} = \log T$ :

$$\sup_{x \in \partial\Lambda} \nu(\mathfrak{D}(x, \log T)) \leq \bar{C} \log^2 T.$$

**Definition 5.2** (Good sub-system). We consider the event “ $\Lambda$  is a good-system” defined as the sub-set of all point configurations  $\mathbf{X} \in \text{Conf}$  such that:

1. The discrepancy  $\text{Dis}(\mathbf{X}, \Lambda)$  satisfies:  $|\text{Dis}(\mathbf{X}, \Lambda)| \leq T \log^2 T$ .
2.  $\mathbf{X}$  belongs to the event  $\mathcal{E}_\Lambda$  defined by:
  - a) There is absolutely no point in  $\Lambda$  at distance  $\leq e^{-\log^2 T}$  from  $\partial\Lambda$ .
  - b) There is no more than  $T \log T$  points in  $\Lambda$  at distance  $\leq 1$  from  $\partial\Lambda$ .

This event is of course  $\Lambda$ -local and even  $\{x \in \Lambda, \text{dist}(x, \partial\Lambda) \leq 1\}$ -local.

### The effective external potential is often good.

Let  $\Lambda_i := \mathfrak{D}(\omega_i, T)$  ( $1 \leq i \leq \mathcal{N}$ ) be the disks introduced at the end of Section 3.2. We define a common external potential for all  $\Lambda_i$ 's by setting (cf. (4.1)):

$$V^{\text{ext}}(x) := \int_{\Sigma_{\mathcal{N}} \setminus \cup_{i=1}^{\mathcal{N}} \Lambda_i} -\log|x-y| \mathbf{d}\mathbf{f}_{\mathcal{N}}(y). \tag{5.2}$$

**Proposition 5.3** (The effective external potential is often good). *There exists a constant  $\bar{C}$  depending only on  $\beta$  (and on the choice of  $\delta$  as in Assumption 1.4) such that the following holds. With  $\mathbb{P}_{\mathcal{N}}^\beta$ -probability greater than  $1 - \mathcal{N} \exp\left(-\frac{\log^2 T}{\bar{C}}\right)$ , for all  $i = 1, \dots, \mathcal{N}$  the external potential  $V^{\text{ext}}$  is a good external potential on  $\Lambda_i$  with constant  $\bar{C}$  (in the sense of Definition 5.1).*

The proof of Proposition 5.3 is elementary but cumbersome. We postpone it to Section C. Since the constant  $\bar{C}$  given by Proposition 5.3 depends only on  $\beta, \delta$ , let us keep in mind that if we say that some constant depends on the “good external potential constant”  $\bar{C}$  then in fact it itself only depends on  $\beta, \delta$ .

## The sub-systems are often good

**Lemma 5.4.** *For  $C_\beta$  large enough, if  $T \geq C_\beta$ , then with  $\mathbb{P}_N^\beta$ -probability  $\geq 1 - \mathcal{N} \exp\left(-\frac{\log^2 T}{C_\beta}\right)$ , for all  $i = 1, \dots, \mathcal{N}$ , the conditions of Definition 5.2 (expressing the fact that “ $\Lambda_i$  is a good sub-system”) are satisfied.*

*Proof.* Using the “discrepancy” part (2.9) of the local laws (Proposition 2.4) we see that if  $T$  is large enough (depending on  $\beta$ ) then for any fixed  $i$  we have:

$$\mathbb{P}_N^\beta (|\text{Dis}(\mathbf{X}_N, \Lambda_i)| \geq T \log T) \leq \exp\left(-\frac{\log^2 T}{C_\beta}\right).$$

Checking the conditions of  $\mathcal{E}_\Lambda$  requires more care.

**Claim 5.5** (The conditions of  $\mathcal{E}_\Lambda$  are often met). *For any fixed  $i$  we have:*

$$\mathbb{P}_N^\beta (\mathcal{E}_{\Lambda_i}) \geq 1 - \exp\left(-\frac{\log^2 T}{C_\beta}\right).$$

*Proof of Claim 5.5.* From the local laws, it is easy to see that the second condition in Definition 5.2 is often satisfied. Indeed we can cover the 1-neighborhood of the boundary  $\partial\Lambda_i$  by  $CT$  squares of sidelength 1, each of which contains at most  $\frac{\log T}{100C}$  points with probability  $\geq 1 - \exp\left(-\frac{\log^2 T}{C_\beta}\right)$  (for  $T$  large enough). We conclude with a union bound, which does not hinder the estimate.

The first condition of Definition 5.2 is more subtle and we rely on the “one-particle cluster” bound from [Tho22] mentioned in Lemma 2.9. Cover the region  $\Gamma_i := \{z \in \Lambda, \text{dist}(z, \partial\Lambda_i) \leq e^{-\log^2 T}\}$  by  $\mathcal{O}(Te^{\log^2 T})$  disks of radius  $r = 10e^{-\log^2 T}$ . For each disk, we know from (2.13) that the probability of it being occupied by at least one particle is smaller than  $C_\beta r^2$ . Then an union bound shows that the probability of at least one point falling anywhere in  $\Gamma_i$  is bounded by  $C_\beta T e^{-\log^2 T}$ , which concludes the proof of the claim.  $\square$

We use a union bound over  $\mathcal{N}$  such events to handle all the  $\Lambda_i$ ’s at once.  $\square$

Until the end of Section 5, we consider a “good external potential”  $V^{\text{ext}}$  with constant  $\bar{C}$  as in Definition 5.1 and we assume that  $|\mathfrak{n} - |\Lambda|| \leq T \log T$ .

## 5.2. 2DOCP’s with non-uniform neutralizing background

Instead of adding  $V^{\text{ext}}$ , let us consider 2DOCP’s in which the “neutralizing background” is no longer the uniform one, but a perturbation thereof.

- Let  $\mathbf{m}$  be a (non-negative) measure on  $\Lambda$  and let  $\zeta$  be some non-negative function on  $\mathbb{R}^2$ . Assume that  $\mathbf{m}$  is supported in  $\Lambda$  and that  $\zeta$  vanishes on the support of  $\mathbf{m}$ .
- Assume that the measure  $\mathbf{m}$  can be written as the sum of a measure which has a bounded density with respect to the Lebesgue measure  $\mathbf{m}_0$  on  $\Lambda$  and of a singular measure which has a bounded density with respect to the arc-length measure  $d\mathbf{s}$  on  $\partial\Lambda$ . This assumption will be justified later in Section D.2. In particular, it implies that  $\iint -\log|x - y| d\mathbf{m}(x) d\mathbf{m}(y)$  is finite.
- Let  $X_n$  be a  $n$ -tuple of points in  $\Lambda$  and  $\mathbf{X}_n$  be the associated point configuration, we let  $F_\Lambda(\mathbf{X}_n, \mathbf{m})$  be the logarithmic interaction energy computed with respect to  $\mathbf{m}$ , namely:

$$F_\Lambda(\mathbf{X}_n, \mathbf{m}) := \frac{1}{2} \iint_{(x,y) \in \Sigma_N \times \Sigma_N, x \neq y} -\log|x - y| d(\mathbf{X}_n - \mathbf{m})(x) d(\mathbf{X}_n - \mathbf{m})(y).$$

We define a probability density  $\mathbb{P}_{n,\Lambda}^\beta(\cdot, \mathbf{m}, \zeta)$  on the space of  $n$ -tuples of points in  $\Lambda$  by setting:

$$d\mathbb{P}_{n,\Lambda}^\beta(X_n, \mathbf{m}, \zeta) := \frac{\exp\left(-\beta (F_\Lambda(\mathbf{X}_n, \mathbf{m}) + n \sum_{i=1}^n \zeta(x_i))\right)}{K_{n,\Lambda}^\beta(\mathbf{m}, \zeta)} dX_n, \quad (5.3)$$

where  $K_{n,\Lambda}^\beta(\mathbf{m}, \zeta)$  is the partition function, namely the normalizing constant:

$$K_{n,\Lambda,\text{V}^{\text{ext}}}^\beta(\mathbf{m}, \zeta) := \int_{\Lambda^n} \exp\left(-\beta\left(F_\Lambda(\mathbf{X}_n, \mathbf{m}) + n \sum_{i=1}^n \zeta(x_i)\right)\right) d\mathbf{X}_n.$$

It is well-known that there is a way to pass from an external potential  $V^{\text{ext}}$  (which should be here treated as a *perturbation* of a reference potential) as in Section 4.3 to the appropriate “equilibrium” measure and vice-versa (we refer e.g. to the lecture notes [Ser15, Sec. 2&3]). However, there is here a specific difficulty due to the behavior of  $V^{\text{ext}}$  near  $\partial\Lambda$  and we postpone the necessary discussion (inspired by similar concerns in [BBNY17]) to Section D.1.

### Electric formalism for $F_\Lambda(\mathbf{X}_n, \mathbf{m})$

We extend here some of the formalism from Sections 2.2 and 2.3 in a fairly straightforward way. Define the “true electric potential/field” associated to a point configuration  $\mathbf{X}_n$  and the background measure  $\mathbf{m}$  as (cf. Definition 2.1):

$$\mathfrak{h}^{\mathbf{X}_n, \mathbf{m}} := -\log *(\mathbf{X}_n - \mathbf{m}), \quad \nabla \mathfrak{h}^{\mathbf{X}_n, \mathbf{m}} = -\nabla \log *(\mathbf{X}_n - \mathbf{m}),$$

their truncated versions being defined as in Definition 2.3. Then we have the following identity, which extends (2.3):

**Lemma 5.6.** *Assume that the total mass of  $\mathbf{m}$  is equal to  $n$ . Let  $\{\eta(x), x \in \mathbf{X}_n\}$  be a truncation vector with  $\eta(x) \leq r(x)$  (the nearest-neighbor distance introduced in (2.2)) for all  $x \in \mathbf{X}_n$ . We have:*

$$F_N(\mathbf{X}_n, \mathbf{m}) = \frac{1}{2} \left( \frac{1}{2\pi} \int_{\mathbb{R}^2} |\nabla \mathfrak{h}_{\eta}^{\mathbf{X}_n, \mathbf{m}}|^2 + \sum_{x \in \mathbf{X}_n} \log \eta(x) \right) - \sum_{x \in \mathbf{X}_n} \int_{\mathfrak{D}(x, \eta(x))} \mathfrak{f}_{\eta(x)}(t - x) d\mathbf{m}(t),$$

provided the disks  $\mathfrak{D}(x, \eta(x))$  do not intersect  $\partial\Lambda$ .

*Proof of Lemma 5.6.* Since we place ourselves away from the singular part of  $\mathbf{m}$ , the proof works exactly as when  $\mathbf{m}$  has a bounded density, see [AS21, Lemma 2.2].  $\square$

We also extend the notation  $\text{Ener}$  from (2.4) by setting:  $\text{Ener}(\mathbf{X}_n, \mathbf{m}, \Omega) := \int_{\Omega} |\nabla \mathfrak{h}_{\eta}^{\mathbf{X}_n, \mathbf{m}}|^2$ .

### 5.3. Good properties of sub-systems with good external potentials

If  $V^{\text{ext}}$  is a good external potential, there exists a probability measure  $\mu_W$  on  $\Lambda$  and a function  $\zeta_W$  such that:

$$\mathbb{P}_{n,\Lambda,\text{V}^{\text{ext}}}^\beta(\cdot) = \mathbb{P}_{n,\Lambda}^\beta(\cdot, n\mu_W, \zeta_W), \quad (5.4)$$

we refer to Section D.2 for a definition and precise study of  $\mu_W$  and to Lemma D.6 for a proof of (5.4). The key features of  $\mu_W$  are that:

1. It might be singular on  $\partial\Lambda$ .
2. It might have “holes” near  $\partial\Lambda$  (and  $\zeta_W > 0$  on these holes).
3. It has a constant, positive density  $\frac{1}{n}$  as soon as one looks at distance  $\geq \bar{C}' \log T$  from  $\partial\Lambda$  (with  $\bar{C}'$  depending on the “good external potential” constant  $\bar{C}$ ). On that region we have  $\zeta_W \equiv 0$ .

For good external potentials, it is thus equivalent to consider the “2DOCP with external background”  $\mathbb{P}_{n,\Lambda,\text{V}^{\text{ext}}}^\beta$  (as in (4.8)) or the “2DOCP with background measure”  $\mathbb{P}_{n,\Lambda}^\beta(\cdot, \mathbf{m}, \zeta)$  (with  $\mathbf{m} := n\mu_W$ ) and we will simply write  $\mathbb{P}_{n,\Lambda}^\beta$  for the corresponding Gibbs measure (and  $\mathbb{E}_{n,\Lambda}^\beta$  for expectations under  $\mathbb{P}_{n,\Lambda}^\beta$ ). We now compare the properties of this “generalized 2DOCP” to the ones of the original Gibbs measure  $\mathbb{P}_{n,\Lambda}^\beta$ . The “good properties” are easier to obtain away from the boundary of  $\Lambda$  and for simplicity we will often work in “the bulk”  $\Lambda^{\text{bulk}}$  defined as:

$$\Lambda^{\text{bulk}} := \mathfrak{D}(\omega, T/2) \subset \mathfrak{D}(\omega, T) = \Lambda.$$

## Global law

Recall that the event  $\mathcal{E}_\Lambda$  was introduced in Definition 5.2.

**Proposition 5.7** (Global law for sub-systems with good external potential). *There exists a constant  $\mathbf{C}_{\text{Global}}$  depending only on  $\beta$  and the “good potential” constant  $\bar{\mathbf{C}}$  such that:*

$$\log \mathbb{E}_{\mathbf{n}, \Lambda}^\beta \left[ \exp \left( \frac{\beta}{2} F_\Lambda(\mathbf{X}, \mathbf{n}\mu_W) \right) \middle| \mathcal{E}_\Lambda \right] \leq \mathbf{C}_{\text{Global}} T^2 \log^5 T. \quad (5.5)$$

We prove Proposition 5.7 in Section D. The bound (5.5) should of course be compared with (2.6) which is valid for the full system. When considering sub-systems with good external potentials we are (only) losing some power of  $\log T$ , which we have not tried to optimize.

## Local laws in the bulk

**Proposition 5.8** (Local laws for sub-systems with good external potential). *There exists a universal constant  $\mathbf{C}$  and a constant  $\mathbf{C}_{\text{Local}}$  (depending only on  $\beta$  and the “good potential” constant  $\bar{\mathbf{C}}$ ) such that if  $T$  is large enough (depending on  $\beta, \bar{\mathbf{C}}$ ) then for all  $\ell \geq \rho_\beta$ , for all  $x$  in  $\Lambda$ , provided that the square  $\square(x, \ell)$  is included in  $\Lambda^{\text{bulk}}$  we have:*

$$\log \mathbb{E}_{\mathbf{n}, \Lambda}^\beta \left[ \exp \left( \frac{\beta}{2} \text{Ener}(\mathbf{X}_n, \square(x, \ell)) \right) \middle| \mathcal{E}_\Lambda \right] \leq \mathbf{C}_{\text{Local}} \beta \ell^2,$$

where the “electric energy” is computed with respect to the background measure  $\mathbf{n}\mu_W$ , and also:

$$\log \mathbb{E}_{\mathbf{n}, \Lambda}^\beta \left[ \exp \left( \frac{\beta}{\mathbf{C}} \text{Pts}(\mathbf{X}_n, \square(x, \ell)) \right) \middle| \mathcal{E}_\Lambda \right] \leq \mathbf{C}_{\text{Local}} \beta \ell^2.$$

This should be compared to Proposition 2.4 for the full system.

*Proof of Proposition 5.8.* This follows from [Leb17; AS21], but it requires some explanation.

We wish to obtain local laws, namely good controls on the electric energy (controls on the number of points are obtained as a byproduct) that are proportional to the volume down to large enough microscopic scales as in [AS21, Thm. 1]. Only this time we are in presence of an external potential / a non-uniform background measure  $\mathbf{n}\mu_W$ , and we were only able to derive a *global bound* as in Proposition 5.7, with a global energy estimate that is slightly larger than the total volume. Our goal is thus twofold:

1. Extend the bootstrap in scales of [Leb17; AS21] to a situation where the background measure is not constant.
2. Show that the bootstrap in scale not only *propagates* estimates to smaller scales, but in fact *improves* (if needed) the estimates at each step, which allows to get rid of the logarithmic correction in (5.5)

The first point is, in fact, very simple. Indeed, although the methods of [Leb17; AS21] are *not* suited to situations where the background measure has singularities or holes (in fact this is the reason why local laws are not proven near the edge for the full system), they are local by design and work as soon as we look at distances  $\geq \mathbf{C} \log T$  from the boundary, because then the background measure  $\mathbf{n}\mu_W$  is equal to 1 (see Section 5.3). Since we only care about the bulk of  $\Lambda$ , this is fine.

The second point is more interesting. The basic tool for the proof of local laws is the fact that thanks to the screening construction of Serfaty *et al.* one can, up to some errors, decouple the system in a given region  $\Omega \subset \Lambda$  of characteristic length  $\mathcal{R}$  from the system in  $\Lambda \setminus \Omega$ . The main price to pay is the first energy error term in [AS21, (4.7)] which reads “ $\frac{\ell}{\mathcal{R}} S$ ”. In their framework the quantity  $\ell$  must be such that  $\ell^3 \geq \frac{S}{\mathcal{R}}$  ([AS21, (4.4)]) and  $\tilde{\ell}$  can be chosen<sup>5</sup> of order  $\mathcal{R}$ , so in fact we are paying a price of order  $\left(\frac{S}{\mathcal{R}}\right)^{4/3}$ . Moreover the quantity  $S$  can always be bounded by the energy in a domain  $2\Omega$  twice as large. So *if we know* that the energy  $E$  at scale  $2\mathcal{R}$  is typically  $E(2\mathcal{R})$ , then we pay a price  $\mathcal{O}\left(\frac{E(2\mathcal{R})}{\mathcal{R}}\right)^{4/3}$ . In [Leb17; AS21] (see

<sup>5</sup>The interested reader can find a list of conditions on  $\ell, \tilde{\ell}$  at the end of the proof of [AS21, Prop. 4.5]. The choice of  $\tilde{\ell}$  mentioned in [AS21, (4.27)] correspond to the *smallest* possible choice, but increasing  $\tilde{\ell}$  up to  $\mathcal{R}$  still gives a valid choice.



also [BBNY17]) one starts from an estimate  $E(2\mathcal{R}) \sim (2\mathcal{R})^2$  (the energy scales like the volume) and gets an error of order  $\mathcal{O}(\mathcal{R}^{4/3}) \ll \mathcal{R}^2$  which is indeed negligible (this can in fact be improved further down by choosing  $\ell, \tilde{\ell}$  more cleverly, see the proof of [AS21, Prop. 2.5]).

The key observation<sup>6</sup> is that even if we start from a poorer estimate, say with logarithmic corrections like  $E(2\mathcal{R}) \sim (2\mathcal{R})^2 \log^{100}(\mathcal{R})$  (cf. the one implied by our global law (5.5)) then the error  $\mathcal{O}\left(\frac{E(2\mathcal{R})}{\mathcal{R}}\right)^{4/3}$  remains much smaller than  $\mathcal{R}^2$ . In fact, a careful inspection of the proof (which we will not need here) shows that one could start with a “global law” as bad as  $\mathcal{O}(T^{5/2})$ , or even  $o(T^3)$  instead of (5.5) and still recover good local laws in the bulk.  $\square$

## 5.4. Sub-systems with good external potentials: consequences

### Discrepancy bounds

Once local laws hold (in the bulk), we retrieve all statements that rely purely on energy considerations. In particular the analogous of [AS21, (1.18)] is valid, namely:

**Lemma 5.9** (Discrepancy bounds in sub-systems). *If  $\square(x, \ell) \subset \Lambda^{\text{bulk}}$  then:*

$$\log \mathbb{E}_{\mathbf{n}, \Lambda}^{\beta} \left[ \exp \left( \frac{\beta \text{Dis}^2(\mathbf{X}_{\mathbf{n}}, \square(x, \ell))}{\mathcal{C} \ell^2} \right) \middle| \mathcal{E}_{\Lambda} \right] \leq \mathbf{C}_{\text{Local}}. \quad (5.6)$$

*In particular, we have an a priori Poisson-like bound on the number variance:*

$$\mathbb{E}_{\mathbf{n}, \Lambda}^{\beta} \left[ \text{Dis}^2(\mathbf{X}_{\mathbf{n}}, \square(x, \ell)) \middle| \mathcal{E}_{\Lambda} \right] \leq \mathbf{C} \mathbf{C}_{\text{Local}} \ell^2,$$

*together with a tail estimate: if  $\ell$  is larger than some constant depending on  $\beta, \bar{\mathbf{C}}$*

$$\mathbb{P}_{\mathbf{n}, \Lambda}^{\beta} \left[ |\text{Dis}(\mathbf{X}_{\mathbf{n}}, \square(x, \ell))| \geq \ell \log \ell \middle| \mathcal{E}_{\Lambda} \right] \leq \exp \left( -\frac{\log^2 \ell}{\mathbf{C}_{\beta}} \right).$$

*Proof of Claim 5.9.* The proof is as in [AS21], using an inequality that relates the presence of discrepancy to a certain energy cost, e.g. [AS21, Lemma B.4].  $\square$

**Remark 5.10.** As an inspection of the (short) proof of [AS21, Lemma B.4] quickly reveals, there is nothing specific to a square in the previous claim, and it also applies to a disk of radius  $\ell$ , or to any “reasonable” shape like the boxes introduced in Definition 3.4.

### Treating smooth linear statistics

Following exactly the same proof as in [LS18; Ser20] (or alternatively as in [BBNY19]), one would obtain a control on linear statistics of smooth enough test functions supported in the bulk  $\Lambda^{\text{bulk}}$ . We do not need it here, however it will be crucial for us to retrieve a specific property (the “smallness of the anisotropy”), but since this only serves as a tool for another result (the quantitative translation-invariance estimate presented in Section 6) we postpone the corresponding discussion to an appendix (see Section E.4). The only result that we will quote directly is one about *expectations* for fluctuations of linear statistics.

**Lemma 5.11** (Expectation of linear statistics in the bulk of subsystems). *There exists a constant  $\bar{\mathbf{C}}$  depending only on  $\beta$ , and on the “local laws” constant  $\mathbf{C}_{\text{Local}}$  (thus on the “good potential” constant) such that the following holds.*

*Let  $\varphi$  be a function of class  $C^2$ , compactly supported on a disk of radius  $\ell \geq \rho_{\beta}$  included in  $\Lambda^{\text{bulk}}$ . Then:*

$$\left| \mathbb{E}_{\mathbf{n}, \Lambda}^{\beta} \left[ \text{Fluct}[\varphi] \middle| \mathcal{E}_{\Lambda} \right] \right| \leq \bar{\mathbf{C}} |\varphi|_2 \ell^2. \quad (5.7)$$

<sup>6</sup>Having a poorer estimate is equivalent to having the parameter  $\mathcal{C}$  in [AS21, (4.25)] depend on  $\mathcal{R}$ . The conditions written at the end of the proof of [AS21, Proposition 4.5] can still be satisfied as long as  $\mathcal{C}$  is much smaller than  $\mathcal{R}$ , which corresponds to an initial energy estimate in  $o(\mathcal{R}^3)$ . If  $\mathcal{C}$  is smaller than  $\mathcal{R}^{3/2}$  then the error term will become smaller than  $\mathcal{R}^2$  in one step, otherwise one would need to apply the bootstrap in scales for some time (i.e. go down in scales) before reaching the desired local laws. Our situation corresponds to  $\mathcal{C} = \log^5 \mathcal{R}$ , cf. (5.5).

*Proof of Lemma 5.11.* This follows from the proofs of [LS18; Ser20] but is not explicitly written as such. Let  $t$  be a small parameter, we know from [LS18, Prop. 2.10] that:

$$\mathbb{E}_{n,\Lambda}^\beta \left[ e^{t\text{Fluct}[\varphi]} \Big| \mathcal{E}_\Lambda \right] = \frac{K_{n,\Lambda}(\mathbf{m}_s)}{K_{n,\Lambda}(\mathbf{m})} e^{\mathcal{O}(t^2)},$$

where  $\mathbf{m}_s$  is the measure  $\mathbf{m}_s := \mathbf{m} - s\Delta\varphi$ , where  $s = \frac{t}{2\pi\beta}$ . This is valid because  $\varphi$  is assumed to be supported in  $\Lambda^{\text{bulk}}$ , where  $\mathbf{m}$  has density 1. By e.g. [LS18, Lemma 3.6] we know that we can replace  $\mathbf{m}_s$  by the approximate measure  $\tilde{\mathbf{m}}_s := (\text{Id} + s\nabla\varphi)\#\mathbf{m}$  up to quadratic terms, i.e.:

$$\mathbb{E}_{n,\Lambda}^\beta \left[ e^{t\text{Fluct}[\varphi]} \Big| \mathcal{E}_\Lambda \right] = \frac{K_{n,\Lambda}(\tilde{\mathbf{m}}_s)}{K_{n,\Lambda}(\mathbf{m})} e^{\mathcal{O}(t^2)}.$$

Next, a consequence of [Ser20, Prop. 4.2] is that the ratio of partition functions can be written as:

$$\frac{K_{n,\Lambda}(\tilde{\mathbf{m}}_s)}{K_{n,\Lambda}(\mathbf{m})} = \exp \left( s\mathcal{O}(|\varphi|_2 \text{EnerPts}(\mathbf{X}_n, \text{supp } \nabla\varphi)) + \mathcal{O}(s^2) \right),$$

and thus taking the limit  $t \rightarrow 0$  (or equivalently  $s \rightarrow 0$ ) and using the local laws we get (5.7) after identifying the first order terms.  $\square$

## 6. Quantitative estimate on translation-invariance

In this section, we denote by  $\vec{u}$  be the vector  $\vec{u} := (0, 1)$  in  $\mathbb{R}^2$ .

### 6.1. The “spin wave” and its properties

**An auxiliary function** For  $\varepsilon \in (0, 1)$ , let  $f^{(\varepsilon)} : x = (x_1, x_2) \in \mathbb{R}^2 \rightarrow \mathbb{R}$  be defined as:

$$f^{(\varepsilon)}(x) := \begin{cases} x_1 & \text{if } |x| \leq 1 \\ x_1(1 - \varepsilon \log|x|) & \text{if } 1 \leq |x| \leq e^{1/\varepsilon} \\ 0 & \text{if } |x| \geq e^{1/\varepsilon}. \end{cases}$$

The function  $f^{(\varepsilon)}$  is continuous, piecewise  $C^2$  and compactly supported on the disk of radius  $e^{1/\varepsilon}$ . We have, by direct computations:

- On (the interior of) the unit disk,  $\partial_1 f^{(\varepsilon)} \equiv 1$ ,  $\partial_2 f^{(\varepsilon)} \equiv 0$  and the second (and third) derivatives of  $f^{(\varepsilon)}$  vanish.
- For  $1 \leq |x| \leq e^{1/\varepsilon}$ , we have :
  - $\partial_1 f^{(\varepsilon)}(x) = 1 - \varepsilon \log|x| - \varepsilon \frac{x_1^2}{|x|^2}$ ,  $\partial_2 f^{(\varepsilon)}(x) = -\varepsilon \frac{x_1 x_2}{|x|^2}$ .
  - The second derivatives of  $f^{(\varepsilon)}$  satisfy the pointwise bound  $|f^{(\varepsilon)}|_{2,\star}(x) \preceq \frac{\varepsilon}{|x|}$ .
  - The third derivatives of  $f^{(\varepsilon)}$  satisfy the pointwise bound  $|f^{(\varepsilon)}|_{3,\star}(x) \preceq \frac{\varepsilon}{|x|^2}$ .

In particular, observe that the first partial derivatives of  $f^{(\varepsilon)}$  are bounded by 1 with a jump of size  $\mathcal{O}(\varepsilon)$  along both  $\partial\mathfrak{D}(0, 1)$  and  $\partial\mathfrak{D}(0, e^{1/\varepsilon})$ , while the second partial derivatives have a jump of size  $\mathcal{O}(\varepsilon)$  along  $\partial\mathfrak{D}(0, 1)$  and of size  $\mathcal{O}(\varepsilon/e^{1/\varepsilon})$  along  $\partial\mathfrak{D}(0, e^{1/\varepsilon})$ .

Thus after applying a mollification to  $f^{(\varepsilon)}$  at scale  $\frac{1}{2}$  near  $\partial\mathfrak{D}(0, 1)$  and at scale  $\frac{1}{2}e^{1/\varepsilon}$  near  $\partial\mathfrak{D}(0, e^{1/\varepsilon})$  we may consider a function  $\bar{f}^{(\varepsilon)}$  which is smooth, compactly supported in  $\mathfrak{D}(0, 2e^{1/\varepsilon})$  and such that (for some universal constant  $\mathbf{C}$ ):

- $\bar{f}^{(\varepsilon)}(x) = x_1$  for  $|x| \leq \frac{1}{2}$ .
- The first derivatives of  $\bar{f}^{(\varepsilon)}$  are bounded by 2 on  $\mathbb{R}^2$ .
- $|\bar{f}^{(\varepsilon)}|_{2,\star}(x) \leq \mathbf{C}\varepsilon$  for  $|x| \leq 2$  and  $|\bar{f}^{(\varepsilon)}|_{2,\star}(x) \leq \mathbf{C}\frac{\varepsilon}{|x|}$  for  $2 \leq |x| \leq 2e^{1/\varepsilon}$ .
- $|\bar{f}^{(\varepsilon)}|_{3,\star}(x) \leq \mathbf{C}\varepsilon$  for  $|x| \leq 2$  and  $|\bar{f}^{(\varepsilon)}|_{3,\star}(x) \leq \mathbf{C}\frac{\varepsilon}{|x|^2}$  for  $2 \leq |x| \leq 2e^{1/\varepsilon}$ .

**Definition of the “spin wave”** Next, we define our “spin wave”<sup>7</sup>  $\mathbb{W}^{(\varepsilon)}$  as the vector field

$$\mathbb{W}^{(\varepsilon)} := \nabla^\perp \bar{f}^{(\varepsilon)} = \left( -\partial_2 \bar{f}^{(\varepsilon)}, \partial_1 \bar{f}^{(\varepsilon)} \right).$$

**Lemma 6.1.** *The following properties of  $\mathbb{W}^{(\varepsilon)}$  are straightforward:*

1.  $\mathbb{W}^{(\varepsilon)}$  is smooth, compactly supported on  $\mathfrak{D}(0, 2e^{1/\varepsilon})$  (because so is  $\bar{f}^{(\varepsilon)}$ ).
2.  $\mathbb{W}^{(\varepsilon)}(x) = \vec{u} = (0, 1)$  for  $|x| \leq \frac{1}{2}$  (because then  $\bar{f}^{(\varepsilon)}(x_1, x_2) = x_1$ ).
3.  $\operatorname{div} \mathbb{W}^{(\varepsilon)} = 0$  on  $\mathbb{R}^2$  (by definition of  $\mathbb{W}^{(\varepsilon)}$  as the perpendicular gradient of a smooth function).
4.  $|\mathbb{W}^{(\varepsilon)}|_0 \leq 2$  (it is bounded by the first derivative of  $\bar{f}^{(\varepsilon)}$ ),  $|\mathbb{W}^{(\varepsilon)}|_1 \leq C\varepsilon$  and more precisely:

$$\left| \mathbb{W}^{(\varepsilon)} \right|_{1, \star}(x) \leq C \times \begin{cases} \varepsilon & \text{for } |x| \leq 2 \\ \frac{\varepsilon}{|x|} & \text{for } 2 \leq |x| \leq 2e^{1/\varepsilon}. \end{cases}$$

5.  $|\mathbb{W}^{(\varepsilon)}|_{2, \star}(x) \leq C\varepsilon$  for  $|x| \leq 2$  and  $|\mathbb{W}^{(\varepsilon)}|_{2, \star}(x) \leq C \frac{\varepsilon}{|x|^2}$  for  $2 \leq |x| \leq 2e^{1/\varepsilon}$ .

We have thus constructed a smooth, divergence-free vector field which is constant near the origin and has an arbitrary small  $H^1$  norm (of order  $\varepsilon$ ). The downside is that the size of the support of  $\mathbb{W}^{(\varepsilon)}$  is exponential with respect to the parameter  $1/\varepsilon$ .

## 6.2. Slowly varying localized translations

For  $\ell > 0$ , let the vector field  $\mathbb{W}^{(\varepsilon, \ell)}$  be defined for  $x \in \mathbb{R}^2$  as  $\mathbb{W}^{(\varepsilon, \ell)}(x) := \mathbb{W}^{(\varepsilon)}(x/\ell)$ . Since  $\mathbb{W}^{(\varepsilon, \ell)}$  is continuous and compactly supported, it generates a global flow  $\{\Phi_t^{\varepsilon, \ell}\}_t$  with the following properties:

**Lemma 6.2.** *For all  $|t| \leq \frac{\ell}{10}$  we have:*

1.  $\Phi_t^{\varepsilon, \ell}$  is an area-preserving diffeomorphism of  $\mathbb{R}^2$ .
2. We have  $\Phi_t^{\varepsilon, \ell}(x) = x + t\vec{u}$  for  $|x| \leq \ell/4$ .
3. We have  $\Phi_t^{\varepsilon, \ell}(x) = x$  for  $|x| \geq 2\ell e^{1/\varepsilon}$ .

Thus for fixed  $|t| \leq \frac{\ell}{10}$  the diffeomorphism  $\Phi_t^{\varepsilon, \ell}$  coincides with the translation by  $t\vec{u}$  on the disk  $\mathfrak{D}(0, \ell/4)$  and with the identity outside  $\mathfrak{D}(0, 2e^{1/\varepsilon})$ , we call it a *localized translation* as in [Geo99]. The main difference with the construction of [Geo99] is that we have  $\int |\Phi_t^{\varepsilon, \ell} - \operatorname{Id}|_{1, \star}^2 = \mathcal{O}(\varepsilon)$  instead of  $\mathcal{O}(1)$ . Interestingly enough, a bounded (but not small)  $H^1$  norm for  $\Phi_t^{\varepsilon, \ell} - \operatorname{Id}$  (which induces a bounded, but not small energy cost, as we will show in Section E) is enough to prove translation-invariance in the infinite-volume setting, but fails to give anything valuable in finite-volume. However, according to a remark in [Sim14, Sec. III.7] “it appears that any model in which this weaker property is valid, the [possibility of finding a construction with arbitrarily small energy cost exists]”. This remains very intriguing to us.

*Proof of Lemma 6.2.* The fact that  $\Phi_t^{\varepsilon, \ell}$  is area-preserving follows from Liouville’s theorem, since the vector field  $\mathbb{W}^{(\varepsilon)}$  (and thus  $\mathbb{W}^{(\varepsilon, \ell)}$ ) is divergence-free by construction. Moreover, since we ensured that  $\mathbb{W}^{(\varepsilon)} \equiv \vec{u}$  on  $\mathfrak{D}(0, \frac{1}{2})$ , the rescaled vector field  $\mathbb{W}^{(\varepsilon, \ell)}$  coincides with  $\vec{u}$  on the disk  $\mathfrak{D}(0, \ell/2)$  hence we have  $\Phi_t^{\varepsilon, \ell}(x) = x + t\vec{u}$  as long as  $x + t\vec{u}$  remains in  $\mathfrak{D}(0, \ell/2)$ . In particular, if  $|x| \leq \ell/4$  and since  $|t| \leq \frac{\ell}{10}$  by assumption, we have  $\Phi_t^{\varepsilon, \ell}(x) = x + t\vec{u}$ . On the other hand,  $\mathbb{W}^{(\varepsilon, \ell)}$  vanishes identically outside  $\mathfrak{D}(0, 2\ell e^{1/\varepsilon})$  and thus the flow there coincides with the identity map.  $\square$

We will study  $\Phi_t^{\varepsilon, \ell}$  a bit more closely in Section E.1 for technical purposes.

<sup>7</sup>This terminology alludes to similar constructions used in the theory of continuous spin systems to prove so-called “Mermin-Wagner” theorems, see e.g. [FV17, Sec. 9.2] or [Sim14, Chapter 3].

### 6.3. Effect of localized translations on the energy

We now apply the “spin wave” / “localized translation” construction to a sub-system with good external potential. Let  $\Lambda$  (a disk of radius  $T$ ),  $\mathbf{n}$ ,  $V^{\text{ext}}$  be as in Section 5, with  $V^{\text{ext}}$  a good external potential, let  $\mathbf{m} = \mathbf{n}\mu_{\mathbb{W}}$  be the corresponding non-uniform background and assume that the properties listed in Section 5.3 hold. Let  $L$  be as in Section 3 and let us choose  $\varepsilon, \ell$  in such a way that:

$$e\sqrt{\log \ell} \leq L \leq \frac{\ell}{10}, \quad 5\ell e^{1/\varepsilon} \leq T \leq 10\ell e^{1/\varepsilon}, \quad \log \ell \leq \varepsilon^{-1}, \quad \varepsilon^{-1} \leq \ell^2 \leq \varepsilon^{-3}. \quad (6.1)$$

Let  $\text{ErrAve}(t, \varepsilon, \ell, \mathbf{X})$  denote the following “averaging error”:

$$F_{\Lambda}(\mathbf{X}, \mathbf{m}) = \frac{1}{2} (F_{\Lambda}(\Phi_t \mathbf{X}, \mathbf{m}) + F_{\Lambda}(\Phi_{-t} \mathbf{X}, \mathbf{m})) + \text{ErrAve}(t\vec{u}, \varepsilon, \ell, \mathbf{X}), \quad (6.2)$$

where  $\Phi_t \mathbf{X}$  denotes the configuration obtained after applying  $\Phi_t$  to all the points of  $\mathbf{X}$ .

**Proposition 6.3.** *There exists a constant  $\mathbf{C}$  depending on  $\beta$  on the “local laws” constant  $\mathbf{C}_{\text{Local}}$  (and thus on the good external potential constant) such that:*

$$\mathbb{P}_{\mathbf{n}, \Lambda}^{\beta} \left[ \sup_{|t| \leq \frac{\ell}{10}} |\text{ErrAve}(t\vec{u}, \varepsilon, \ell, \mathbf{X}_{\mathbf{n}})| \leq \mathbf{C}t^2 \varepsilon \log \varepsilon \left| \mathcal{E}_{\Lambda} \right. \right] \geq 1 - \exp(-\ell^2).$$

We prove Proposition 6.3 in Section E and use it next to “mollify” observables before taking expectations.

### 6.4. Effect of localized translations on expectations

**Proposition 6.4.** *Let  $\mathbf{G}$  be a measurable function on  $\text{Conf}(\Lambda)$ , let  $\mathcal{E} \subset \text{Conf}(\Lambda)$  be an event. Assume that the function  $\mathbf{G}$  is  $\mathfrak{D}(0, \ell/10)$ -local and that  $\mathcal{E}$  is  $\Lambda \setminus \Lambda^{\text{bulk}}$ -local. Recall that  $|t| \leq \ell/10$ .*

*For all  $\tau \in (0, 1)$  and for all  $\sigma > 0$ , we have:*

$$\mathbb{E}_{\mathbf{n}, \Lambda}^{\beta} \left[ \mathbf{G}(\mathbf{X}) \left| \mathcal{E} \right. \right] = \mathbb{E}_{\mathbf{n}, \Lambda}^{\beta} \left[ \frac{1}{2} (\mathbf{G}(\mathbf{X} + t\vec{u}) + \mathbf{G}(\mathbf{X} - t\vec{u})) \left| \mathcal{E} \right. \right] + \text{ErrorQI}(t\vec{u}, \varepsilon, \ell, \tau, \sigma, \mathbf{G}, \mathbf{X}), \quad (6.3)$$

with an error term  $\text{ErrorQI}$  bounded as follows:

$$\begin{aligned} |\text{ErrorQI}(t\vec{u}, \varepsilon, \ell, \tau, \sigma, \mathbf{G}, \mathbf{X})| &\leq 2 \left( \mathbb{E}_{\mathbf{n}, \Lambda}^{\beta} \left[ \mathbf{G}^2(\mathbf{X} \pm t\vec{u}) \left| \mathcal{E} \right. \right] + \sigma^2 \right)^{\frac{1}{2}} \times (e^{\beta\tau} - 1) \\ &\quad + 2e^{\beta\tau} \left( \mathbb{E}_{\mathbf{n}, \Lambda}^{\beta} \left[ \mathbf{G}^2(\mathbf{X} \pm \vec{u}) \left| \mathcal{E} \right. \right] + \sigma^2 \right)^{\frac{1}{2}} \times \left( \mathbb{P}_{\mathbf{n}, \Lambda}^{\beta} [|\mathbf{G}(\mathbf{X} \pm t\vec{u})| \geq \sigma] \right)^{\frac{1}{2}} \\ &\quad + \mathbb{E}_{\mathbf{n}, \Lambda}^{\beta} \left[ \mathbf{G}^2(\mathbf{X}) \left| \mathcal{E} \right. \right] \times \left( \mathbb{P}_{\mathbf{n}, \Lambda}^{\beta} [\text{ErrAve}(t\vec{u}, \varepsilon, \ell, \mathbf{X}) \geq \tau \left| \mathcal{E} \right. ] \right)^{\frac{1}{2}} \end{aligned}$$

We postpone the proof of Proposition 6.4 to Section E.5. There is of course nothing special about the vector  $\vec{u}$ , and we may replace it by any unit vector.

### 6.5. Application: expectation of discrepancies in sub-systems

**Proposition 6.5.** *Let  $\mathfrak{B}$  be a box of size  $L$  as introduced in Definition 3.4. Assume that it is contained in the disk  $\mathfrak{D}(0, \ell/10)$ . Then we have:*

$$\left| \mathbb{E}_{\mathbf{n}, \Lambda}^{\beta} \left[ \text{Dis}(\mathbf{X}_{\mathbf{n}}, \mathfrak{B}) \left| \mathcal{E}_{\Lambda} \right. \right] \right| \leq \bar{\mathbf{C}}L (\varepsilon \log \varepsilon \log L)^{1/3}. \quad (6.4)$$

The point of (6.4) is that if  $\varepsilon \log \varepsilon \log L$  is  $\ll 1$  then the bound on the average discrepancy is  $\ll L$ , and thus much better than the crude estimate via the standard deviation using Claim 5.9. This is crucial for us.

*Proof of Proposition 6.5.* Let us take  $\mathbf{G}(\mathbf{X}) := \text{Dis}(\mathbf{X}, \mathfrak{B})$ , which is clearly  $\mathfrak{D}(0, \ell/10)$ -local in view of our assumption on  $\mathfrak{B}$ . Moreover  $\mathcal{E}_\Lambda$  (by its Definition 5.2) is clearly  $\Lambda \setminus \Lambda^{\text{bulk}}$ -local. Fix the parameter  $\bar{t}$  as:

$$\bar{t} := (\varepsilon \log \varepsilon \log L)^{-1/3}. \quad (6.5)$$

We first check that  $\bar{t}$  has the correct range (using (6.1)):

$$\bar{t} \leq \varepsilon^{-1/3} \leq \ell^{2/3} \leq \frac{\ell}{10}.$$

In particular, for all vector  $v \in \mathbb{R}^2$  with  $\|v\| \leq \bar{t}$  we may apply the result of Proposition 6.4 to a localized translation in the direction  $\frac{v}{\|v\|}$  instead of  $\vec{u}$ .

Introduce a smooth cut-off function  $\bar{\chi}$  equal to 1 for  $|x| \leq \frac{\bar{t}}{2}$  and to 0 for  $|x| \geq \bar{t}$ , such that  $|\bar{\chi}|_k \leq \mathbf{C}\bar{t}^{-k}$  (for  $k = 1, 2$ ). We impose that  $\bar{\chi}$  be an even function. For all  $|v| \leq \bar{t}$  we have:

$$\mathbb{E}_{n,\Lambda}^\beta \left[ \text{Dis}(\mathbf{X}, \mathfrak{B}) \middle| \mathcal{E} \right] = \mathbb{E}_{n,\Lambda}^\beta \left[ \frac{1}{2} (\mathfrak{B}(\mathbf{X} + v, \mathfrak{B}) + \text{Dis}(\mathbf{X} - v, \mathfrak{B})) \middle| \mathcal{E} \right] + \text{ErrorQI}(v, \varepsilon, \ell, \tau, \sigma, \text{Dis}(\cdot, \mathfrak{B}), \mathbf{X}).$$

Integrating this against  $\frac{\bar{\chi}}{|\bar{\chi}|_{L^1}}$  (which has mass 1) and using the fact that  $\bar{\chi}$  is even we obtain:

$$\mathbb{E}_{n,\Lambda}^\beta \left[ \text{Dis}(\mathbf{X}_n, \mathfrak{B}) \middle| \mathcal{E}_\Lambda \right] = \frac{1}{|\bar{\chi}|_{L^1}} \int \bar{\chi}(v) \mathbb{E}_{n,\Lambda}^\beta \left( \text{Dis}(\mathbf{X} + v, \mathfrak{B}) \middle| \mathcal{E}_\Lambda \right) + \text{ErrorQI}(v) dv.$$

Observe that  $\text{Dis}(\mathbf{X} + v, \mathfrak{B}) = \text{Fluct}[1_{\mathfrak{B}}(\cdot + v)](\mathbf{X})$ , so that we may re-write the integral in the right-hand side as a convolution. Introducing the function  $\varphi := 1_{\mathfrak{B}} * \frac{\bar{\chi}}{|\bar{\chi}|_{L^1}}$  we get:

$$\left| \mathbb{E}_{n,\Lambda}^\beta \left[ \text{Dis}(\mathbf{X}_n, \mathfrak{B}) \middle| \mathcal{E}_\Lambda \right] - \mathbb{E}_{n,\Lambda}^\beta \left[ \text{Fluct}[\varphi] \middle| \mathcal{E}_\Lambda \right] \right| \leq \sup_{|v| \leq \bar{t}} \text{ErrorQI}(v, \varepsilon, \ell, \tau, \sigma, \text{Dis}(\cdot, \mathfrak{B}), \mathbf{X}_n). \quad (6.6)$$

By construction, the function  $\varphi$  is now a smooth cut-off function, equal to 1 on  $\{x \in \mathfrak{B}, \text{dist}(x, \partial\mathfrak{B}) \geq \bar{t}\}$  and to 0 for  $\{x \notin \mathfrak{B}, \text{dist}(x, \partial\mathfrak{B}) \geq \bar{t}\}$ . In particular, the support of  $\nabla\varphi$  has an area  $\mathcal{O}(L\bar{t})$ , and moreover we have  $|\varphi|_k \leq \mathbf{C}\bar{t}^{-k}$  for  $k = 1, 2$  (by Young's convolution inequality). From Lemma 5.11, we thus know:

$$\left| \mathbb{E}_{n,\Lambda}^\beta \left[ \text{Fluct}[\varphi] \middle| \mathcal{E}_\Lambda \right] \right| \leq \frac{1}{\bar{t}^2} \times L\bar{t} = \mathcal{O}\left(\frac{L}{\bar{t}}\right), \quad (6.7)$$

with multiplicative constants depending on  $\beta, \delta$ .

On the other hand, we know by Proposition 6.3 that choosing  $\bar{\mathbf{C}}$  large enough we can ensure:

$$\sup_{|v| \leq \bar{t}} \left( \mathbb{P}_{n,\Lambda}^\beta \left( \{ |\text{ErrAve}(v, \varepsilon, \ell, \mathbf{X})| \geq \bar{\mathbf{C}}\varepsilon \log \varepsilon \bar{t}^2 \} \middle| \mathcal{E}_\Lambda \right) \right) \leq \exp(-\ell^2/\bar{\mathbf{C}}).$$

Moreover  $\text{Dis}(\mathbf{X} + v, \mathfrak{B}) = \text{Dis}(\mathbf{X}, \mathfrak{B} - v)$  and we have by Lemma 5.9:  $\sup_{|v| \leq \bar{t}} \mathbb{E}_{n,\Lambda}^\beta \left[ \text{Dis}^2(\mathbf{X}, \mathfrak{B} - v) \middle| \mathcal{E}_\Lambda \right] \leq \bar{\mathbf{C}}L^2$  (because the translated object  $\mathfrak{B} - v$  remains a box within  $\Lambda^{\text{bulk}}$ ) and, still by Lemma 5.9, for  $\bar{\mathbf{C}}$  large enough we have:

$$\sup_{|v| \leq \bar{t}} \mathbb{P}_{n,\Lambda}^\beta \left( \{ |\text{Dis}(\mathbf{X} + v, \mathfrak{B})| \geq L \log L \} \right) \leq e^{-\frac{1}{\bar{\mathbf{C}}} \log^2 L}.$$

(Remark that the sup is outside  $\mathbb{P}_{n,\Lambda}^\beta, \mathbb{E}_{n,\Lambda}^\beta$  in those bounds - it would be significantly more challenging otherwise.)

Thus, choosing  $\tau = \bar{\mathbf{C}}\varepsilon \log \varepsilon \bar{t}^2$ ,  $\sigma = L \log L$  in the statement of Proposition 6.4, the error term **ErrorQI** reduces to:

$$\text{ErrorQI}(v, \varepsilon, \ell, \tau, \sigma, \text{Dis}(\mathbf{X}, \mathfrak{B}), \mathbf{X}) \leq \bar{\mathbf{C}}L \log L \left( \varepsilon \log \varepsilon \bar{t}^2 + \exp(-\ell^2/\bar{\mathbf{C}}) + \exp\left(-\frac{1}{\bar{\mathbf{C}}} \log^2 L\right) \right),$$

uniformly for  $|v| \leq \bar{t}$  and thus (using (6.1) and keeping only the dominant term):

$$\sup_{|v| \leq \bar{t}} \text{ErrorQI} \leq \bar{\mathbf{C}}L \log L \varepsilon \log \varepsilon \bar{t}^2. \quad (6.8)$$

Combining (6.6) with (6.7) and (6.8), and using our choice (6.5) for  $\bar{t}$ , we obtain (6.4).  $\square$

**Remark 6.6.** One can use the same argument to study expectations of discrepancies *in the full system*. Then one does not need to use  $\mathcal{E}_\Lambda$  since local laws are known to hold unconditionally, and one can take  $\varepsilon$  as small as  $\log^{-1} N$ .

**Corollary 6.7** (An application). *Assume that  $L, T$  satisfy the following relation:*

$$T = 100L \exp(10L).$$

Then we have:

$$\left| \mathbb{E}_{n,\Lambda}^\beta [\text{Dis}(\mathbf{X}, \mathfrak{B})] \right| \leq \bar{\mathbf{C}} L^{0.67}.$$

*Proof.* We can take  $\ell = 10L$  and  $\varepsilon = \ell^{-1} = \frac{1}{10L}$ , and  $T = 100L \exp(10L) = 10\ell e^{1/\varepsilon}$ . The conditions of (6.1) are clearly satisfied. The right-hand side of (6.4) is then bounded by  $\bar{\mathbf{C}} L^{2/3} \log L = \mathcal{O}(L^{0.67})$  (for  $L$  large enough).  $\square$

## 7. Conclusion: proof of Theorem 1

Let  $\delta > 0$  be fixed. Let  $x, R$  be such that  $\text{dist}(\mathfrak{D}(x, R), \partial\Sigma_N) \geq \delta\sqrt{N}$  as assumed in (1.4). Let  $\varepsilon_R$  be chosen as:

$$\varepsilon_R := \log^{-0.3}(R), \tag{7.1}$$

For simplicity, we will focus on the case of an *excess* of points, i.e. a positive discrepancy, the other case being treated similarly. Define the event  $\mathcal{A}_R$ :

$$\mathcal{A}_R := \{\text{Dis}(\mathbf{X}_N, \mathfrak{D}(x, R)) \geq \varepsilon_R R\}$$

The conclusion that we want to reach (as stated in (1.5)) is that for  $R, \bar{\mathbf{C}}$  large enough (depending on  $\beta$  and on the parameter  $\delta$ , but not on  $x$ ):

$$\mathbb{P}_N^\beta(\mathcal{A}_R) \leq \exp(-\log^{1.5} R). \tag{7.2}$$

We have not tried to optimize  $\varepsilon_R$  or the exponent in (7.2), what matters for us is that  $\varepsilon_R \rightarrow 0$  and that  $1.5 > 1$  so our probabilistic tail is better than algebraic. With the methods of the present paper, there is a hard limit on the smallness of  $\varepsilon_R$  - it has to be at least  $\log^{-1} R$ .

### Step 1. Choosing $L$ and cornering the discrepancy

Let  $L$  be chosen as:

$$L := \log^{0.99} R. \tag{7.3}$$

In particular, for  $R$  greater than some constant (depending only on  $\beta$ ) we have:

$$\mathbf{C}_{3.1} \frac{1}{\varepsilon_R} = \mathbf{C}_{3.1} \log^{0.3} R \leq \log^{0.99} R = L \leq R/10,$$

(where  $\mathbf{C}_{3.1}$  is the constant depending only on  $\beta$  introduced in Proposition 3.1) so the first condition of (3.1) is satisfied. Moreover let  $s$  be chosen as  $\frac{1}{\mathbf{c}_{3.1}} \min\left(\frac{L^3}{R}, L\varepsilon_R\right)$ , namely (for  $R$  large enough):  $s := \frac{1}{\mathbf{c}_{3.1}} \frac{L^3}{R}$ . By definition, the second condition of (3.1) is then satisfied. Let us compute:

$$\exp\left(-\frac{s\varepsilon_R R}{4}\right) = \exp\left(-\frac{\varepsilon_R L^3}{4\mathbf{c}_{3.1}}\right) \leq \exp\left(-\frac{\log^{-2.67} R}{\mathbf{c}_\beta}\right).$$

For each  $k$  with  $0 \leq k \leq R^2$ , let  $\mathcal{B}_R(k)$  be the event:

$$\mathcal{B}_R(k) := \left\{ \text{The discrepancy in the annulus } \mathfrak{D}_R \setminus \mathfrak{D}_{R-2L+\frac{kL}{R^2}} \text{ is larger than } \frac{1}{4}\varepsilon_R \cdot R \right\}$$

Combining Proposition 3.1 and Lemma 3.2 we obtain:

$$\mathbb{P}_N^\beta(\mathcal{A}_R) \leq \sum_{k=1}^{R^2} \mathbb{P}_N^\beta(\mathcal{B}_R(k)) + \exp\left(-\frac{\log^{-2.67} R}{C_\beta}\right), \quad (7.4)$$

and we now focus on bounding  $\mathcal{B}_R(k)$  (the index  $k$  plays no particular role, and for simplicity we forget about it).

For each  $k$ , as explained in Section 3.2, we can decompose the annulus  $\mathfrak{D}_R \setminus \mathfrak{D}_{R-2L+\frac{kL}{R^2}}$  into boxes  $\{\mathfrak{B}_i, i \in \{0, \dots, \frac{R}{L} - 1\}\}$  of size  $L$  as in Definition 3.4.

## Step 2. Choosing $T, M$ and a well-separated family

Let  $T$  be chosen (as in (6.7)) as:

$$T := 100 \times \log^{0.99} R \times \exp(10 \log^{0.99} R) = 100L \exp(10L), \quad (7.5)$$

and let  $M$  be chosen as:

$$M := T^6. \quad (7.6)$$

Since  $L = \log^{0.99} R$  (according to (7.3)), the conditions of (3.4) are clearly satisfied for  $R$  large enough.

Now, for  $l \in \{0, \dots, M-1\}$ , let  $\mathcal{C}(l)$  be the event:

$$\mathcal{C}(l) := \left\{ \sum_{i=l \bmod M} \text{Dis}(\mathbf{X}_N, \mathfrak{B}_i) \geq \frac{\varepsilon_R R}{4M} \right\} \quad (7.7)$$

By Lemma 3.5 we know that:

$$\mathbb{P}_N^\beta(\mathcal{B}_R) \leq \sum_{l=0}^{M-1} \mathbb{P}_N^\beta(\mathcal{C}(l)), \quad (7.8)$$

and we now focus on bounding  $\mathcal{C}(l)$  (again, the index  $l$  plays no role in the sequel). The index  $l$  being fixed, we only consider the boxes  $\mathfrak{B}_i$  for  $i \equiv l \pmod{M}$  and forget about the other boxes. We relabel those boxes as  $\mathfrak{B}_i$  for  $i \in \{1, \dots, \mathcal{N}\}$  where  $\mathcal{N}$  is the cardinality of that family of boxes, with:

$$\mathcal{N} = \mathcal{O}\left(\frac{R}{ML}\right). \quad (7.9)$$

As in Section 3.2 we let  $\Lambda_i$  be the disk  $\mathfrak{D}(\omega_i, T)$ , where  $\omega_i$  is the ‘‘center’’ of the box  $\mathfrak{B}_i$ , and we recall that  $d_{ij}$  denotes the distance between  $\Lambda_i$  and  $\Lambda_j$ . Using (3.6) and summing over  $i = 1, \dots, \mathcal{N} = \mathcal{O}(\frac{R}{ML})$ , we get:

$$\max_{1 \leq i \leq \mathcal{N}} \sum_{j \neq i} \frac{1}{d_{ij}} = \mathcal{O}\left(\frac{\log R}{ML}\right), \quad \sum_{1 \leq i \neq j \leq \mathcal{N}} \frac{1}{d_{ij}} = \mathcal{O}\left(\frac{R \log R}{(ML)^2}\right).$$

## A ‘‘good event’’

Let  $V^{\text{ext}}$  be the logarithmic potential generated by the system outside the  $\Lambda_i$ 's as in (5.2). For  $\bar{C} > 0$ , let  $\mathcal{E}_{\text{ext}}(\bar{C})$  be the event:

$$\mathcal{E}_{\text{ext}}(\bar{C}) := \{V^{\text{ext}} \text{ is a ‘‘good external potential’’ on each } \Lambda_i \text{ with constant } \bar{C}\}. \quad (7.10)$$

(See Definition 5.1). By Proposition 5.3 we know that if  $\bar{C}$  is chosen large enough and if  $T$  is large enough (i.e.  $R$  is large enough) depending only on  $\beta$  and on the parameter  $\delta$  from (1.4), then we have:

$$\mathbb{P}_N^\beta(\mathcal{E}_{\text{ext}}(\bar{C})) \geq 1 - \mathcal{N} \exp(-\log^2 T/\bar{C}).$$

Since  $\mathcal{N}$  is always smaller than  $R = e^{\log R}$  (see (7.9)) and since, by our choice (7.5) of  $T$  we have:

$$\exp(-\log^2 T/\bar{C}) \leq \exp(-\log^{1.98} R/\bar{C}) \ll \exp(-\log R),$$



we deduce that (for  $R, \bar{\mathcal{C}}$  large enough depending on  $\beta, \delta$ ):

$$\mathbb{P}_N^\beta (\mathcal{E}_{\text{ext}}(\bar{\mathcal{C}})) \geq 1 - \exp(-\log^2 T/\bar{\mathcal{C}}).$$

On the other hand, for each  $i = 1, \dots, \mathcal{N}$ , let  $\mathcal{E}_i$  be the event: “ $\Lambda_i$  is a good sub-system”, as in Definition 5.2. By Lemma 5.4, we know that:

$$\mathbb{P}_N^\beta \left( \bigcap_{i=1}^{\mathcal{N}} \mathcal{E}_i \right) \geq 1 - \mathcal{N} \exp(-\log^2 T/\mathcal{C}_\beta),$$

and by the same parameter comparison as above we get:  $\mathbb{P}_N^\beta \left( \bigcap_{i=1}^{\mathcal{N}} \mathcal{E}_i \right) \geq 1 - \exp(-\log^2 T/\mathcal{C}_\beta)$ . We thus have (for  $R, \bar{\mathcal{C}}$  large enough):

$$\mathbb{P}_N^\beta (\mathcal{C}) \leq \mathbb{P}_N^\beta \left( \mathcal{C} \cap \mathcal{E}_{\text{ext}}(\bar{\mathcal{C}}) \bigcap_{i=1}^{\mathcal{N}} \mathcal{E}_i \right) + \exp(-\log^2 T/\bar{\mathcal{C}}). \quad (7.11)$$

### Using approximate conditional independence

Let  $\omega$  be chosen as:

$$\omega := L^{-1.33}. \quad (7.12)$$

For  $1 \leq i \leq \mathcal{N}$ , let  $\mathbf{G}_i$  be the following function on Conf, which is clearly non-negative and  $\Lambda_i$ -local:

$$\mathbf{G}_i(\mathbf{X}) := e^{\omega \text{Dis}(\mathbf{X}, \mathfrak{B}_i)}.$$

If  $\mathbf{X}_N$  is in  $\mathcal{C}$  then by definition (see (7.7)) we have:

$$\prod_{i=1}^{\mathcal{N}} \mathbf{G}_i(\mathbf{X}_N) = e^{\omega \sum_{i=1}^{\mathcal{N}} \text{Dis}(\mathbf{X}_N, \mathfrak{B}_i)} \geq \exp\left(\frac{\omega \varepsilon_R R}{4M}\right),$$

and thus by Markov’s inequality (dropping the  $\bar{\mathcal{C}}$ -dependency):

$$\mathbb{P}_N^\beta \left( \mathcal{C} \cap \mathcal{E}_{\text{ext}} \bigcap_{i=1}^{\mathcal{N}} \mathcal{E}_i \right) \leq \exp\left(-\frac{\omega \varepsilon_R R}{4M}\right) \mathbb{E}_N^\beta \left[ \prod_{i=1}^{\mathcal{N}} \mathbf{G}_i(\mathbf{X}_N) \mathbf{1}_{\mathcal{E}_{\text{ext}} \cap \bigcap_{i=1}^{\mathcal{N}} \mathcal{E}_i} \right]. \quad (7.13)$$

We are in a position to use Proposition 4.3 with  $\mathcal{E}_N := \mathcal{E}_{\text{ext}} \cap \bigcap_{i=1}^{\mathcal{N}} \mathcal{E}_i$ . We obtain:

$$\begin{aligned} \mathbb{E}_N^\beta \left[ \prod_{i=1}^{\mathcal{N}} \mathbf{G}_i(\mathbf{X}_N) \mathbf{1}_{\mathcal{E}_{\text{ext}} \cap \bigcap_{i=1}^{\mathcal{N}} \mathcal{E}_i} \right] &\leq \exp\left(2\beta \sup_{\mathbf{X}_N \in \mathcal{E}_N} \text{ErrorCI}(\mathbf{X}_N)\right) \\ &\quad \times \sup_{\mathbf{X}^{\text{ext}} \in \mathcal{E}_{\text{ext}}, \{\mathbf{n}_i\} \text{ adm.}} \prod_{i=1}^{\mathcal{N}} \mathbb{E}_{\mathbf{n}_i, \Lambda_i, \mathbf{V}^{\text{ext}}}^\beta \left[ \mathbf{G}_i(\mathbf{X}) \middle| \mathcal{E}_i \right] \end{aligned} \quad (7.14)$$

Let us note that the event  $\bigcap_{i=1}^{\mathcal{N}} \mathcal{E}_i$  implies that the “admissible” number of points in each  $\Lambda_i$  is bounded by  $10T^2$  (see Definition 5.2) which checks the first condition of (4.4), and that the second condition of (4.4) is implied by the third condition<sup>8</sup> of (3.4). We may thus use Lemma 4.1 which, together with (3.6), implies that the conditional independence error ErrorCI between the  $\Lambda_i$ ’s is bounded by:

$$\sup_{\mathbf{X}_N \in \bigcap_{i=1}^{\mathcal{N}} \mathcal{E}_i} |\text{ErrorCI}[\mathbf{X}_N | \Lambda_1, \dots, \Lambda_{\mathcal{N}}]| = \mathcal{O}\left(\frac{T^5 R \log R}{(ML)^2}\right), \quad (7.15)$$

which controls the first term in the right-hand side of (7.14), and we now focus on the second one.

<sup>8</sup>The distance between two centers  $\omega_i, \omega_j$  is bounded below by  $ML$  and thus  $\text{dist}(\Lambda_i, \Lambda_j) \geq ML - 2T \geq 98T$ .

### Controlling expectations in each “sub-system”

Let us fix  $1 \leq i \leq \mathcal{N}$  and work in  $\Lambda_i$ . We can assume that  $V^{\text{ext}}$  is  $\bar{\mathcal{C}}$ -good on  $\Lambda_i$  because of (7.10).

**Claim 7.1.** *Let  $D_i := \text{Dis}(\mathbf{X}_n, \mathfrak{B}_i)$ . We have:*

$$\mathbb{E}_{\mathbf{n}_i, \Lambda_i}^\beta [e^{\omega D_i} | \mathcal{E}_{\Lambda_i}] = 1 + \mathcal{O}(L^{-0.66}) \leq e^{\mathcal{O}(L^{-0.66})} \quad (7.16)$$

*Proof.* We start by decomposing the expectation as:

$$\mathbb{E}_{\mathbf{n}_i, \Lambda_i}^\beta [e^{\omega D_i} | \mathcal{E}_{\Lambda_i}] = \mathbb{E}_{\mathbf{n}_i, \Lambda_i}^\beta [e^{\omega D_i} \mathbf{1}_{|\omega D_i| \leq \frac{1}{2}} | \mathcal{E}_{\Lambda_i}] + \mathbb{E}_{\mathbf{n}_i, \Lambda_i}^\beta [e^{\omega D_i} \mathbf{1}_{|\omega D_i| > \frac{1}{2}} | \mathcal{E}_{\Lambda_i}]. \quad (7.17)$$

Using a Taylor’s expansion, we may write the first term in the right-hand side of (7.17) as:

$$\begin{aligned} \mathbb{E}_{\mathbf{n}_i, \Lambda_i}^\beta [e^{\omega D_i} \mathbf{1}_{|\omega D_i| \leq \frac{1}{2}} | \mathcal{E}_{\Lambda_i}] &= \mathbb{E}_{\mathbf{n}_i, \Lambda_i}^\beta [(1 + \omega D_i + \mathcal{O}(\omega^2 D_i^2)) \mathbf{1}_{|\omega D_i| \leq \frac{1}{2}} | \mathcal{E}_{\Lambda_i}] \\ &\leq \mathbb{P}_{\mathbf{n}_i, \Lambda_i}^\beta [\mathbf{1}_{|\omega D_i| \leq \frac{1}{2}} | \mathcal{E}_{\Lambda_i}] + \omega \mathbb{E}_{\mathbf{n}_i, \Lambda_i}^\beta [D_i | \mathcal{E}_{\Lambda_i}] + \mathcal{O}(\omega^2 \mathbb{E}_{\mathbf{n}_i, \Lambda_i}^\beta [D_i^2 | \mathcal{E}_{\Lambda_i}]) + \mathbb{E}_{\mathbf{n}_i, \Lambda_i}^\beta [\omega | D_i | \mathbf{1}_{|\omega D_i| > \frac{1}{2}} | \mathcal{E}_{\Lambda_i}]. \end{aligned}$$

Using Corollary 6.7 to control  $\mathbb{E}_{\mathbf{n}_i, \Lambda_i}^\beta [D_i | \mathcal{E}_{\Lambda_i}]$  and Lemma 5.9 to control  $\mathbb{E}_{\mathbf{n}_i, \Lambda_i}^\beta [D_i^2 | \mathcal{E}_{\Lambda_i}]$ , and inserting into (7.17) we obtain:

$$\begin{aligned} \mathbb{E}_{\mathbf{n}_i, \Lambda_i}^\beta [e^{\omega D_i} | \mathcal{E}_{\Lambda_i}] &\leq 1 + \omega \mathcal{O}(L^{0.67}) + \omega^2 \mathcal{O}(L^2) \\ &\quad + \mathbb{E}_{\mathbf{n}_i, \Lambda_i}^\beta [e^{\omega D_i} \mathbf{1}_{|\omega D_i| > \frac{1}{2}} | \mathcal{E}_{\Lambda_i}] + \mathbb{E}_{\mathbf{n}_i, \Lambda_i}^\beta [\omega | D_i | \mathbf{1}_{|\omega D_i| > \frac{1}{2}}] - \mathbb{P}_{\mathbf{n}_i, \Lambda_i}^\beta [\mathbf{1}_{|\omega D_i| > \frac{1}{2}} | \mathcal{E}_{\Lambda_i}], \end{aligned} \quad (7.18)$$

and it remains to control the second line of (7.18), using Lemma 5.9. We write:

$$\begin{aligned} \mathbb{E}_{\mathbf{n}_i, \Lambda_i}^\beta [e^{\omega D_i} \mathbf{1}_{|\omega D_i| > \frac{1}{2}} | \mathcal{E}_{\Lambda_i}] &\leq \mathbb{E}_{\mathbf{n}_i, \Lambda_i}^\beta [e^{2\omega^2 L^2 \frac{D_i^2}{L^2}} \mathbf{1}_{|\omega D_i| > \frac{1}{2}} | \mathcal{E}_{\Lambda_i}] \\ &\leq \mathbb{E}_{\mathbf{n}_i, \Lambda_i}^\beta [e^{4\omega^2 L^2 \frac{D_i^2}{L^2}} | \mathcal{E}_{\Lambda_i}]^{\frac{1}{2}} \mathbb{E}_{\mathbf{n}_i, \Lambda_i}^\beta [\mathbf{1}_{|\omega D_i| > \frac{1}{2}} | \mathcal{E}_{\Lambda_i}]^{\frac{1}{2}} \leq e^{\bar{\mathcal{C}}\omega^2 L^2} \mathbb{P}_{\mathbf{n}_i, \Lambda_i}^\beta \left[ \left\{ |\omega D_i| > \frac{1}{2} \right\} | \mathcal{E}_{\Lambda_i} \right], \end{aligned}$$

where we have used Cauchy-Schwarz’s inequality, then the fact that  $\omega^2 L^2 \ll 1$  (see (7.12)) in order to apply Hölder’s inequality, and the exponential moment (5.6). Using (5.6) again we get that:

$$\mathbb{P}_{\mathbf{n}_i, \Lambda_i}^\beta \left[ \left\{ |\omega D_i| > \frac{1}{2} \right\} | \mathcal{E}_{\Lambda_i} \right] \leq \exp(-L^{0.66}/\bar{\mathcal{C}}),$$

using Markov’s inequality the fact that  $\frac{L^2}{\omega^2} = L^{0.66}$  (see (7.12)). We can thus write:

$$\mathbb{E}_{\mathbf{n}_i, \Lambda_i}^\beta [e^{\omega D_i} \mathbf{1}_{|\omega D_i| > \frac{1}{2}} | \mathcal{E}_{\Lambda_i}] = \mathcal{O}(e^{-L^{0.66}/\bar{\mathcal{C}}}).$$

The two other terms in the second line of (7.18) are handled the same way. We obtain:

$$\mathbb{E}_{\mathbf{n}_i, \Lambda_i}^\beta [e^{\omega D_i} | \mathcal{E}_{\Lambda_i}] \leq 1 + \omega \mathcal{O}(L^{0.67}) + \omega^2 \mathcal{O}(L^2) + \mathcal{O}(e^{-L^{0.66}/\bar{\mathcal{C}}}),$$

Inserting<sup>9</sup> the value  $\omega = L^{-1.33}$  we obtain (7.16). □

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<sup>9</sup>There might be a confusion between the *error* term  $\mathcal{O}(e^{-L^{0.66}/\bar{\mathcal{C}}})$  and the fact that we write  $1 + \mathcal{O}(L^{-0.66}) \leq e^{\mathcal{O}(L^{-0.66})}$ , but  $e^{-L^{0.66}}$  and  $e^{L^{-0.66}}$  are two different terms.

## Conclusion.

Inserting (7.15) and (7.16) into (7.14) we obtain:

$$\mathbb{E}_N^\beta \left[ \prod_{i=1}^{\mathcal{N}} \mathbf{G}_i(\mathbf{X}_N) \mathbf{1}_{\mathcal{E}_{\text{ext}} \cap \bigcap_{i=1}^{\mathcal{N}} \mathcal{E}_i} \right] \leq \exp \left( \mathcal{O} \left( \frac{T^5 R \log R}{(ML)^2} \right) + \mathcal{O} \left( \frac{R}{ML} L^{-0.66} \right) \right).$$

Since  $M = T^6$  (by (7.6)) it is easy to check that (for  $R$  large enough) the dominant term in the exponent is by far the second one because  $T \gg L$ . Returning to (7.13) we thus obtain:

$$\mathbb{P}_N^\beta \left( \mathcal{C} \cap \mathcal{E}_{\text{ext}}(\bar{\mathcal{C}}) \bigcap_{i=1}^{\mathcal{N}} \mathcal{E}_i \right) \leq \exp \left( -\frac{\omega \varepsilon_R R}{4M} \right) \exp \left( \mathcal{O} \left( \frac{R}{ML} L^{-0.66} \right) \right),$$

and since  $\omega = L^{-1.33}$  ((7.12)),  $L = \log^{0.99} R$  ((7.3)) and  $\varepsilon_R = \log^{-0.3} R$  ((7.1)) the first factor dominates the second one:

$$\mathbb{P}_N^\beta \left( \mathcal{C} \cap \mathcal{E}_{\text{ext}}(\bar{\mathcal{C}}) \bigcap_{i=1}^{\mathcal{N}} \mathcal{E}_i \right) \leq \exp \left( -\frac{\omega \varepsilon_R R}{8M} \right),$$

and thus after inserting the value of  $M$  (7.6) defined in terms of  $T$  as in (7.5) we obtain the following Markov-type inequality:

$$\mathbb{P}_N^\beta \left( \mathcal{C} \cap \mathcal{E}_{\text{ext}}(\bar{\mathcal{C}}) \bigcap_{i=1}^{\mathcal{N}} \mathcal{E}_i \right) \leq \exp \left( -\exp(\log R + o(\log R)) \right).$$

In particular, for  $R$  large enough this is smaller than  $\exp(-R^{1/2})$ . Returning successively to (7.11), (7.8), (7.4) and using the fact that  $\log T \geq \log^{0.99} R$  we obtain:

$$\mathbb{P}_N^\beta(\mathcal{C}) \leq \exp(-\log^{1.98} R/\bar{\mathcal{C}}), \quad \mathbb{P}_N^\beta(\mathcal{B}_R) \leq \exp(\mathcal{O}(\log R) - \log^{1.98} R/\bar{\mathcal{C}}) \leq \exp(-\log^{1.98} R/\bar{\mathcal{C}}),$$

and finally:

$$\mathbb{P}_N^\beta(\mathcal{A}_R) \leq \exp(\mathcal{O}(\log R) - \log^{1.98} R/\bar{\mathcal{C}}) \leq \exp(-\log^{1.98} R/\bar{\mathcal{C}})$$

which proves (7.2) and thus concludes the proof of Theorem 1.

## A. Discussion of the model

There are several slightly different ways to define the two-dimensional one-component plasma.

- Some papers work with an “infinitely extended equilibrium” Coulomb system, e.g. [MY80; Leb83; JLM93]. The mathematical existence of such infinite-volume limits is not yet clear for  $\beta \neq 2$ , see however [AS21, Corollary 1.1] for existence of infinite-volume *limit points* in the weak topology.
- In the statistical physics literature, it is common to place  $N$  particles in a “uniform neutralizing background of opposite charge” which occupies a certain domain  $\Sigma_N$  with constant density  $\rho_N := \frac{-N}{|\Sigma_N|}$ . There is *perfect confinement* in the sense that the particles are not allowed to live outside  $\Sigma_N$ . The domain is not always explicitly chosen, though it often ends up being a disk, mostly by default or for the convenience of symmetry<sup>10</sup>. Some authors state their results for different “reasonable shapes” as e.g. [SM76]. The influence of the background can be seen as applying some potential to each point charge, while they all interact with each other. Indeed, one may write:

$$F_N(\mathbf{X}_N) = \frac{1}{2} \iint -\log|x-y| d\mathbf{X}_N(x) d\mathbf{X}_N(y) + \int V_{\text{back}}(x) d\mathbf{X}_N(x) + C_N, \quad (\text{A.1})$$

where  $V_{\text{back}}$  is the logarithmic potential generated by the background, namely:

$$V_{\text{back}}(x) := \int_{\Sigma_N} -\log|x-y| \rho_N dy,$$

<sup>10</sup>There is also some interest for studying the 2DOCP on a sphere, which avoids having to deal with a boundary.

and  $C_N$  is a constant (the self-interaction of the background with itself) that does not depend on  $\mathbf{X}_N$  (only on  $N$ ) and can thus be absorbed in  $K_N^\beta$ . This model is sometimes called a *jellium*.

- In the mathematical physics literature around the planar Coulomb gas (e.g. [ZW06; AHM11; SS15]) the particles/eigenvalues  $\mathbf{X}_N$  are usually not confined *a priori* in a certain domain of the Euclidean space, but are rather subject to a certain external “confining” potential/field/weight  $V$  acting as  $V_{\text{back}}$  in (A.1) (this model is sometimes called a *(two-dimensional)  $\beta$ -ensemble* by analogy with well-known one-dimensional models coming from random matrix theory). Via a certain mean-field energy functional, the choice of  $V$  determines<sup>11</sup>:

1. A compact subset  $\Sigma_N$ , sometimes called the droplet.
2. An *equilibrium measure*  $\mu_N$  on  $\Sigma_N$ .
3. An effective confining potential  $\zeta_N$ .

After “splitting” (see [SS15, Lemma 3.1]), the energy takes the following form, cf. (1.1) and (A.1):

$$\tilde{F}_N(\mathbf{X}_N) = \frac{1}{2} \iint -\log|x-y| d(\mathbf{X}_N - \mu_N)(x) d(\mathbf{X}_N - \mu_N)(y) + \sum_{i=1}^N \zeta_N(x_i) + C_N. \quad (\text{A.2})$$

The confining potential  $\zeta_N$  vanishes on  $\Sigma_N$  and is positive outside of it - thus penalizing particles that leave  $\Sigma_N$ , and indeed the confinement is strong: few particles fall outside  $\Sigma_N$  and if they do they stay close by (see [Ame21] for a quantitative statement). Hence in typical situations most of the particles are located on  $\Sigma_N$  and arrange themselves according to the density  $\mu_N$ . The canonical choice is the quadratic potential  $V(x) = |x|^2$  for which  $\Sigma_N$  is a large disk of radius comparable to  $\sqrt{N}$  and  $\mu_N$  is the uniform measure on  $\Sigma_N$ .

We choose to work (a) in a disk (b) with a perfect confinement for the sake of convenience and simplicity of exposition, but one can certainly replace  $\Sigma_N$  by a square or by any “reasonable shape” without affecting the conclusions of Theorem 1.1 as our argument makes no use of the global geometry of the system. Our results are also valid without a perfect confinement: adding an effective confining potential  $\zeta_N$  to  $F_N$  as in (A.2) is transparent as our constructions always take place inside  $\Sigma_N$ , so Theorem 1.1 holds for the two-dimensional  $\beta$ -ensemble/Coulomb gas with quadratic potential. However, considering a non-uniform density (which might happen if  $V$  is not quadratic) would require some care.

## B. Auxiliary proofs of preliminary results

### B.1. Proof of Proposition 2.4

*Proof of Proposition 2.4.* The first item corresponds to the first statement in [AS21, Theorem 1], combined with their Lemma B.2. To be precise, the statement of [AS21, Theorem 1] involves the quantity  $F^{\square_R(x)}$  which is short for  $F^{\square_R(x)}(\mathbf{X}_N, U)$  with  $U = \mathbb{R}^2$  as defined in [AS21, Eq. (2.24)]. The potential  $u$  appearing there is defined in [AS21, Section 2.3] but as used in the main statement of [AS21] it coincides with the true potential (because, with their notation, “ $U = \mathbb{R}^2$ ” in this case). It remains to observe that Theorem 1 in [AS21] states a control on  $F^{\square_R(x)}$  which can be turned into a control on the electric energy only (as we write in Proposition 2.4), this is precisely the purpose of [AS21, Lemma B.2] and in particular their equation (B.8), which shows that one can indeed control the electric energy in terms of  $F^{\square_R(x)}$ . As a last technical comment for the careful reader: note that since here “ $U = \mathbb{R}^2$ ” (their notation) the various truncations  $r, \tilde{r}, \tilde{\tilde{r}}$  all coincide.

The second item of Proposition 2.4 is [AS21, (1.18)]. This, or [AS21, (1.19)], implies (2.8).  $\square$

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<sup>11</sup>Alternatively, one can associate to  $V$  a “thermal equilibrium measure” as in [AS21], which has unbounded support and plays the role of those three objects.

## B.2. Proof of Lemma 2.6

*Proof of Lemma 2.6.* For  $x$  in  $\mathbf{X}$  let us define the truncation  $\eta(x)$  as:

$$\eta(x) = \begin{cases} 0 & \text{if } x \notin \Omega \\ r(x) & \text{if } x \in \Omega, \end{cases}$$

where  $r$  is the “nearest-neighbor” distance as in (2.2). Let us recall that, by definition, we always have  $|r| \leq 1/4$ . Let  $\mathbf{E}$  be an electric field compatible with  $\mathbf{X}$  on  $\text{supp } \varphi$  and let  $\mathbf{E}_{\tilde{\eta}}$  be the electric field truncated accordingly. We have, in the sense of distributions on  $\text{supp } \nabla \varphi$ :

$$-\text{div } \mathbf{E}_{\tilde{\eta}} = 2\pi \left( \sum_{x \in \mathbf{X}} \delta_x^{(\eta(x))} - \mathbf{m}_0 \right),$$

where we replace the Dirac mass  $\delta_x$  by its “smeared out” version  $\delta_x^{(\eta(x))}$ , which is the uniform measure of mass 1 on the circle of center  $x$  and radius  $\eta(x)$ . Let us write  $\mathbf{X}_{\tilde{\eta}} := \sum_{x \in \mathbf{X}} \delta_x^{(\eta(x))}$ . The measures  $\mathbf{X}$  and  $\mathbf{X}_{\tilde{\eta}}$  coincide outside a 1-neighborhood of  $\text{supp } \nabla \varphi$  and the atoms there have been smeared out at a distance at most 1, thus:

$$\left| \int_{\mathbb{R}^2} \varphi(x) \mathbf{d}(\mathbf{X} - \mathbf{X}_{\tilde{\eta}})(x) \right| \leq |\varphi|_{1,\Omega} \times \text{Pts}(\mathbf{X}, \Omega),$$

which can be localized, if  $\tilde{\Omega}_1, \dots, \tilde{\Omega}_m$  cover  $\text{supp } \nabla \varphi$  we have:

$$\left| \int_{\mathbb{R}^2} \varphi(x) \mathbf{d}(\mathbf{X} - \mathbf{X}_{\tilde{\eta}})(x) \right| \leq \sum_{k=1}^m \sup_{x \in \Omega_i} |\nabla \varphi(x)| \times \text{Pts}(\mathbf{X}, \Omega_i),$$

where  $\Omega_i$  contains a 1-neighborhood of  $\tilde{\Omega}_i$ . Now, let us re-write the fluctuations of  $\varphi$  as:

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \varphi(x) \mathbf{d}(\mathbf{X} - \mathbf{m}_0)(x) \right| &\leq \left| \int_{\mathbb{R}^2} \varphi(x) \mathbf{d}(\mathbf{X}_{\tilde{\eta}}(x) - \mathbf{m}_0)(x) \right| + |\varphi|_{1,\Omega} \times \text{Pts}(\mathbf{X}, \Omega) \\ &= \frac{1}{2\pi} \left| \int_{\mathbb{R}^2} \varphi(x) \text{div } \mathbf{E}_{\tilde{\eta}} \right| + |\varphi|_{1,\Omega} \times \text{Pts}(\mathbf{X}, \Omega). \end{aligned}$$

Integrating by parts and using Cauchy-Schwarz’s inequality yields:

$$\left| \int_{\mathbb{R}^2} \varphi(x) \text{div } \mathbf{E}_{\tilde{\eta}} \right| = \left| \int_{\text{supp } \nabla \varphi} \nabla \varphi \cdot \mathbf{E}_{\tilde{\eta}} \right| \leq \left( \int_{\mathbb{R}^2} |\nabla \varphi|^2 \right)^{\frac{1}{2}} \left( \int_{\text{supp } \nabla \varphi} |\mathbf{E}_{\tilde{\eta}}|^2 \right)^{\frac{1}{2}}.$$

Finally, it remains to observe that with our choice of truncation  $\eta$  as above and the definition of  $\Omega$ , we have:

$$\int_{\text{supp } \nabla \varphi} |\mathbf{E}_{\tilde{\eta}}|^2 \leq \int_{\Omega} |\mathbf{E}_{\tilde{r}}|^2,$$

where  $\tilde{r}$  is the nearest-neighbor truncation, which concludes the proof of (2.10). This last step can also be localized by decomposing  $\text{supp } \nabla \varphi$  into several domains.  $\square$

## B.3. Proof of Proposition 2.8

*Proof of Proposition 2.8.* We start by rewriting the Laplace transform of the fluctuations as a ratio of partition functions, this is a standard trick that goes back (at least) to [Joh98]. In the two-dimensional context, it can be found e.g. in [LS18, Proposition 2.10]. Let  $\mathbf{m}_s$  be the signed measure defined by:

$$\mathbf{m}_s := \mathbf{m}_0 - \frac{s}{2\pi\beta} \Delta \varphi,$$

which coincides with  $\mathbf{m}_0$  outside of  $\mathcal{A}$  (let us recall that  $\mathcal{A}$  is some annulus containing  $\text{supp } \Delta \varphi$ ). Since  $\nabla \varphi$  is compactly supported, the total mass of  $\Delta \varphi$  (seen as a measure) is 0 so  $\mathbf{m}_s$  and  $\mathbf{m}_0$  have the same

total mass on  $\mathcal{A}$ . Moreover as soon as the parameter  $s$  satisfies the condition (2.12) ( $|s| \leq \frac{\pi\beta}{4|\varphi|_2}$ ) then  $\mathbf{m}_s$  has a non-negative density which is bounded between  $\frac{1}{2}$  and  $\frac{3}{2}$  on  $\mathcal{A}$ . We introduce the following notation:

$$F_N(\mathbf{X}_N, \mathbf{m}_s) := \frac{1}{2} \iint_{x \neq y} -\log|x-y| d(\mathbf{X}_N - \mathbf{m}_s)(x) d(\mathbf{X}_N - \mathbf{m}_s)(y).$$

**Claim B.1** (Laplace transform as ratio of partition functions). *The following identity holds:*

$$\mathbb{E}_N^\beta [\exp(s \text{Fluct}[\varphi])] = \exp\left(\frac{s^2}{4\pi\beta} \int_{\mathbb{R}^2} |\nabla\varphi|^2\right) \frac{\int_{\Sigma_N^N} \exp(-\beta F_N(\mathbf{X}_N, \mathbf{m}_s)) d\mathbf{X}_N}{\int_{\Sigma_N^N} \exp(-\beta F_N(\mathbf{X}_N, \mathbf{m}_0)) d\mathbf{X}_N} \quad (\text{B.1})$$

*Proof of Claim B.1.* We have by definition:

$$\mathbb{E}_N^\beta [\exp(s \text{Fluct}[\varphi])] = \frac{\int_{\Sigma_N^N} \exp\left(-\beta \left(F_N(\mathbf{X}_N, \mathbf{m}_0) - \frac{s}{\beta} \text{Fluct}[\varphi]\right)\right) d\mathbf{X}_N}{\int_{\Sigma_N^N} \exp(-\beta F_N(\mathbf{X}_N, \mathbf{m}_0)) d\mathbf{X}_N}.$$

Since  $\frac{1}{2\pi} \Delta \log = \delta_0$  (see (2.1)) one may write  $-\frac{s}{\beta} \text{Fluct}[\varphi]$  as:

$$-\frac{s}{\beta} \text{Fluct}[\varphi] := -\frac{s}{\beta} \int_{\Sigma_N} \varphi(x) d(\mathbf{X}_N - \mathbf{m}_0)(x) = \iint -\log|x-y| \left(\frac{s}{2\pi\beta} \Delta\varphi(y)\right) dy d(\mathbf{X}_N - \mathbf{m}_0)(x),$$

and then “complete the square”.  $\square$

In order to compare the partition functions in (B.1), it is common to introduce some sort of transportation map from  $\mathbf{m}_0$  to  $\mathbf{m}_s$ . The fact that the density of  $\mathbf{m}_0$  is constant and the one of  $\mathbf{m}_s$  has radial symmetry reduces the computation to a one-dimensional problem which can be solved exactly and explicitly (the general, non-radial case requires an abstract argument or resorting to an approximate transportation, which makes the analysis more involved). Proposition 2.8 is then a consequence of [Ser20, Prop. 4.2].

**Claim B.2** (Radial monotone rearrangement). *For convenience let us assume that  $\varphi$  is radially symmetric around 0, and let  $r_{\max}$  be the outer radius of the annulus  $\mathcal{A}$ . Let  $F_s : [0, r_{\max}] \rightarrow \mathbb{R}_+$  be the cumulative radial distribution function of the density  $\mathbf{m}_s$ , namely:*

$$F_s := r \mapsto \int_0^r \left(1 - \frac{s}{2\pi\beta} \Delta\varphi(\rho)\right) 2\pi\rho d\rho = \pi r^2 - \frac{s}{\beta} \int_0^r \Delta\varphi(\rho) \rho d\rho,$$

where we denote by  $\Delta\varphi(\rho)$  the value of  $\Delta\varphi(x)$  at any point  $x$  with  $|x| = \rho$ . Let  $\Phi_s$  be the transport map defined by:

$$\Phi_s(r) := F_s^{-1}(\pi r^2) \quad (r \geq 0). \quad (\text{B.2})$$

Finally, let  $\vec{\Phi}_s$  be the map  $\vec{\Phi}_s(x) := \Phi_s(|x|) \frac{x}{|x|}$ . Then:

1.  $\vec{\Phi}_s$  is a  $C^1$ -automorphism of the disk  $\mathcal{D}(0, r_{\max})$ , which transports  $\mathbf{m}_0$  onto  $\mathbf{m}_s$ .
2. The map  $\psi_s := x \mapsto \vec{\Phi}_s(x) - x$  is supported on the annulus  $\mathcal{A}$  and satisfies:

$$|\psi_s|_1 \leq s|\varphi|_2.$$

*Proof of Claim B.2.* Since  $s$  satisfies (2.12),  $F_s$  is strictly increasing and continuous, thus  $\Phi_s = F_s^{-1}$  is well-defined (it is the so-called “monotone rearrangement”). Since  $\mathbf{m}_0, \mathbf{m}_s$  are radially symmetric the map  $\vec{\Phi}_s$  transports  $\mathbf{m}_0$  onto  $\mathbf{m}_s$  (as two-dimensional measures), and we have  $\vec{\Phi}_s(x) = x$  for  $x \notin \mathcal{A}$ .

A computation in polar coordinates shows that  $|\psi_s|_1 \leq |\Phi_s' - 1|_0$ , on the other hand the derivatives of  $\Phi_s$  can be estimated in terms of the perturbation measure  $\Delta\varphi$  by using the transportation identity (B.2), which reads:

$$\Phi_s(r)^2 - r^2 = \frac{s}{\pi\beta} \int_0^{\Phi_s(r)} \Delta\varphi(\rho) \rho d\rho,$$

and taking derivatives. In particular one finds (after an integration by parts) that  $|\Phi_s(r) - r| \leq s|\varphi|_2 r$ , and then that:  $|\Phi_s'(r) - 1| \leq s|\varphi|_2$ .  $\square$

Next let us extend the notation  $\vec{\Phi}_s$  to  $\vec{\Phi}_s(\mathbf{X}_N) := \sum_{i=1}^N \delta_{\vec{\Phi}_s(x_i)}$ .

**Claim B.3** (Effect of the transportation on the energy). *We have:*

$$\left| F_N(\vec{\Phi}_s(\mathbf{X}_N), \mathbf{m}_s) - F_N(\mathbf{X}_N, \mathbf{m}_0) \right| \leq s|\varphi|_2 \text{EnerPts}(\mathbf{X}_N, \mathcal{A}).$$

*Proof of Claim B.3.* We apply [Ser20, Prop 4.2] with (in the notation of that paper)  $U_\ell = \mathcal{A}$ , which contains the support of  $\psi_s$  and thus of its derivative. Then [Ser20, (4.5)] states that the electric energy in  $\mathcal{A}$  stays bounded along the transport by  $\vec{\Phi}_s$ , and [Ser20, (4.6)] that in fact its derivative is bounded by  $|\psi_s|_1$  times the initial electric energy plus the number of points in  $\mathcal{A}$ .  $\square$

To conclude, we change variables in (B.1) using  $\vec{\Phi}_s$  and use Claim B.3 to estimate the effect on the energy. The Jacobian term that appears is again of order  $s|\varphi|_2$  times the number of points in the support of  $\psi_s$  and can be incorporated in the previous error term.  $\square$

#### B.4. Cornering the discrepancy: proof of Proposition 3.1

*Proof of Proposition 3.1.* Let  $\varphi$  be a nonnegative, non increasing, compactly supported  $C^2$  test function with radial symmetry around  $z$  such that:

1.  $\varphi \equiv 1$  on  $\mathfrak{D}_{R-2L}$ ,  $\varphi \equiv 0$  outside  $\mathfrak{D}_{R-L}$ ,  $\varphi$  takes values in  $[0, 1]$ .
2.  $|\varphi|_1 \leq \frac{100}{L}$ ,  $|\varphi|_2 \leq \frac{100}{L^2}$ .

Let  $\tilde{\varphi} : \mathbb{R}^2 \rightarrow \mathbb{R}$  be the function defined by  $\tilde{\varphi}(x) := \varphi(|x - z|)$ . Assume that:

$$\mathbf{X}_N \in \bigcap_{R-2L \leq r \leq R-L} \left\{ \text{Dis}(\mathbf{X}_N, \mathfrak{D}(z, r)) \geq \frac{1}{2} \varepsilon_R \cdot R \right\} \quad (\text{B.3})$$

**Claim B.4** ( $\tilde{\varphi}$  detects the discrepancy). *Under (B.3) we have:*

$$\text{Fluct}[\tilde{\varphi}] \geq \frac{\varepsilon_R \cdot R}{2}. \quad (\text{B.4})$$

*Proof of Claim B.4.* Without loss of generality we may assume that  $z = 0$ . We take advantage of the radial symmetry of  $\tilde{\varphi}$  and re-write  $\text{Fluct}[\tilde{\varphi}]$  as:

$$\text{Fluct}[\tilde{\varphi}] = \int_{\mathfrak{D}_{R-L}} \tilde{\varphi}(x) d\mathbf{f}_N(x) = \int_0^{R-L} \varphi(r) \left( \frac{d}{dr} \text{Dis}(\mathbf{X}_N, \mathfrak{D}_r) \right) dr.$$

Integrating by parts, we get:  $\text{Fluct}[\tilde{\varphi}] = - \int_{R-2L}^{R-L} \varphi'(r) \text{Dis}(\mathbf{X}_N, \mathfrak{D}_r) dr$ . By assumption (B.3) we have  $\text{Dis}(\mathbf{X}_N, \mathfrak{D}_r) \geq \frac{\varepsilon_R \cdot R}{2}$  for all  $r$  in the domain of integration, moreover by construction  $-\varphi' \geq 0$  and its integral is 1. We thus obtain (B.4).  $\square$

We have constructed a radially symmetric test function  $\tilde{\varphi}$  that is of class  $C^2$  and detects a fraction of the discrepancy. Let us compare (B.4) with the control on the size of  $\text{Fluct}[\tilde{\varphi}]$  given by Proposition 2.8.

**Claim B.5** (Fluctuations of  $\tilde{\varphi}$ ). *With  $s$  as in (3.1)*

$$\log \mathbb{E}_N^\beta \left[ e^{s \text{Fluct}[\tilde{\varphi}]} \right] \leq C_\beta (s^2 + s) \frac{R}{L}. \quad (\text{B.5})$$

*Proof of Claim B.5.* By construction,  $\tilde{\varphi}$  is a  $C^2$  test function with compact support and radial symmetry, and  $|\tilde{\varphi}|_2 \leq \frac{100}{L^2}$ . We assumed  $L \leq \frac{R}{10}$  and  $|s| \leq \frac{1}{c_{3.1}} \frac{L^3}{R}$  (see condition (3.1)), hence up to choosing the constant  $C_{3.1}$  large enough (depending on  $\beta$ ) we can ensure that  $s$  satisfies  $|s| \leq \frac{\pi\beta}{4|\varphi|_2}$ . Moreover the



support of  $\Delta\tilde{\varphi}$  is an annulus. In particular we may apply Proposition 2.8 and control the exponential moment of  $\text{Fluct}[\tilde{\varphi}]$  by:

$$\log \mathbb{E}_N^\beta [\exp (s \text{Fluct}[\tilde{\varphi}])] = \frac{s^2}{4\pi\beta} \int_{\mathbb{R}^2} |\nabla\tilde{\varphi}|^2 + \log \mathbb{E}_N^\beta [\exp (s|\tilde{\varphi}|_2 \mathcal{O}(\mathbf{C}_\beta \text{EnerPts}(\mathfrak{D}_{R-L} \setminus \mathfrak{D}_{R-2L})))] . \quad (\text{B.6})$$

The quantity  $\int_{\mathbb{R}^2} |\nabla\tilde{\varphi}|^2$  is readily bounded (up to some multiplicative constant) by  $RL$  (the area of the annulus where  $\Delta\tilde{\varphi}$  is supported) times  $\frac{1}{L^2}$  (the order of magnitude of  $|\varphi|_1^2$ ), and thus:

$$\frac{s^2}{4\pi\beta} \int_{\mathbb{R}^2} |\nabla\tilde{\varphi}|^2 = \mathcal{O}\left(\frac{s^2 R}{L}\right) . \quad (\text{B.7})$$

It remains to estimate the contribution of the following term:

$$\text{Rem} := \mathbb{E}_N^\beta [\exp (s|\tilde{\varphi}|_2 \mathcal{O}(\mathbf{C}_\beta \text{EnerPts}(\mathfrak{D}_{R-L} \setminus \mathfrak{D}_{R-2L})))] .$$

By construction we know that  $|\tilde{\varphi}|_2$  is of order  $\frac{1}{L^2}$ . Cover the annulus  $\mathfrak{D}_{R-L} \setminus \mathfrak{D}_{R-2L}$  by a family  $\{\square_i\}_{i \in I}$  of  $\#I = \mathcal{O}\left(\frac{R}{L}\right)$  squares of sidelength  $\mathcal{O}(L)$  and write:

$$\exp (s|\tilde{\varphi}|_2 \mathcal{O}(\mathbf{C}_\beta \text{EnerPts}(\mathfrak{D}_{R-L} \setminus \mathfrak{D}_{R-2L}))) \leq \exp \left( \frac{\mathbf{C}_\beta s R}{L^3} \frac{1}{\#I} \sum_{i \in I} \text{EnerPts}(\square_i) \right) .$$

By convexity we get:

$$\text{Rem} \leq \frac{1}{\#I} \sum_{i \in I} \mathbb{E}_N^\beta \left[ \exp \left( \frac{\mathbf{C}_\beta s R}{L^3} \text{EnerPts}(\square_i) \right) \right] . \quad (\text{B.8})$$

Up to choosing  $\mathbf{C}_{3.1}$  large enough we can ensure that the parameter  $\frac{\mathbf{C}_\beta s R}{L^3}$  in the right-hand side of (B.8) is smaller than any fixed constant. Then for each  $i \in I$  the local laws (Proposition 2.4) yield:

$$\log \mathbb{E}_N^\beta \left[ \exp \left( \frac{\mathbf{C}_\beta s R}{L^3} (\text{EnerPts}(\square_i)) \right) \right] \leq \frac{\mathbf{C}_\beta s R}{L^3} \times \mathbf{C}_{\text{LL}} L^2 = \mathcal{O}_\beta \left( \frac{s R}{L} \right) .$$

Taking an average over  $i \in I$  yields:

$$\log \text{Rem} = \mathcal{O}_\beta \left( \frac{s R}{L} \right) , \quad (\text{B.9})$$

and combining it with (B.6), (B.7) and (B.9), we obtain (B.5) as claimed.  $\square$

Next, applying Markov's inequality in exponential form to (B.5), we get:

$$\mathbb{P}_N^\beta \left( \text{Fluct}[\tilde{\varphi}] \geq \frac{\varepsilon_R \cdot R}{2} \right) \leq \exp \left( -\frac{s\varepsilon_R \cdot R}{2} + (s^2 + s)\mathbf{C}_\beta \left( \frac{R}{L} \right) \right) .$$

Since we assume  $L \geq \mathbf{C}_{3.1} \frac{1}{\varepsilon_R}$  and  $s \leq \frac{1}{\mathbf{C}_{3.1}} L\varepsilon_R$ , up to choosing  $\mathbf{C}_{3.1}$  large enough we can ensure that:

$$-\frac{s\varepsilon_R \cdot R}{2} + (s^2 + s)\mathbf{C}_\beta \left( \frac{R}{L} \right) \leq -\frac{s\varepsilon_R \cdot R}{4} ,$$

which concludes the proof.  $\square$

## C. Study of the external potential: proof of Proposition 5.3

*Proof of Proposition 5.3.* Let us fix an index  $i \in \{1, \dots, \mathcal{N}\}$  and study the external potential  $V^{\text{ext}}$  (as defined in (4.1)) on  $\Lambda_i$ . Let  $V_i$  be the potential generated on  $\Lambda_i$  by ‘‘everything outside  $\Lambda_i$ ’’, namely:

$$V_i(x) := \int_{\Sigma_N \setminus \Lambda_i} -\log|x-y| d\mathbf{f}_N(y), \quad x \in \Lambda_i . \quad (\text{C.1})$$

For all  $x$  in  $\Lambda_i$  we have, by definition:

$$V^{\text{ext}}(x) = V_i(x) - \sum_{j, j \neq i} \int_{\Lambda_j} -\log|x-y| \mathbf{d}\mathbf{f}_N(y). \quad (\text{C.2})$$

For convenience, we will first study  $V_i$  itself and then use a rough bound on the remaining terms in (C.2). In the rest of this section we choose an auxiliary length scale  $\hat{T}$  as:

$$\hat{T} := \log T.$$

## C.1. Decomposition and regularization

### Decomposition of $V_i$

Let us introduce a smooth cut-off function  $\chi_i$  equal to 1 on  $\mathfrak{D}(\omega_i, T + \hat{T})$  and to 0 outside  $\mathfrak{D}(\omega_i, T + 2\hat{T})$ , with  $|\chi_i|_k \leq C\hat{T}^{-k}$  and let:

- $R'_i$  be the field generated by the background measure in the annulus  $\mathcal{A} := \mathfrak{D}(\omega_i, T + 2\hat{T}) \setminus \mathfrak{D}(\omega_i, T)$ , weighted by  $\chi_i$ , namely:

$$R'_i := -\log * (-\chi_i \mathbf{1}_{\Sigma_N \setminus \Lambda_i} \mathbf{m}_0).$$

- $\mathfrak{h}^{\nu_i}$  be the field generated by the positive measure  $\nu_i$  corresponding to the point charges in  $\mathcal{A}$ , weighted by  $\chi_i$ , namely:

$$\mathfrak{h}^{\nu_i} := -\log * (\chi_i \mathbf{1}_{\Sigma_N \setminus \Lambda_i} \mathbf{X}_N).$$

- $R_i$  be the field generated by the signed measure  $(1 - \chi_i) \mathbf{f}_N$ , namely:

$$R_i := -\log * ((1 - \chi_i) \mathbf{f}_N). \quad (\text{C.3})$$

Of course, we have  $\mathfrak{h}^{\nu_i} + R'_i = -\log * (\chi_i \mathbf{f}_N)$  and thus the following decomposition holds (cf. (5.1)):

$$V_i = \mathfrak{h}^{\nu_i} + R'_i + R_i. \quad (\text{C.4})$$

**Remark C.1.** By Newton's theorem, the logarithmic potential  $R'_i$  is *constant* within the disk  $\Lambda_i$  thanks to its radial symmetry. So its derivative and its "interior" normal derivative  $\partial_n^- R'_i$  (see (D.7)) are identically 0 there, and it will play no further role in the proof.

### Regularization of $V_i$ near the edge.

Since there might be charges in  $\nu_i$  located very close to  $\partial\Lambda_i$ , we cannot expect in general to have an upper bound on  $V_i$  in  $\Lambda_i$  valid *up to the edge*. In order to study  $V_i$  near  $\partial\Lambda_i$ , we introduce a smooth cut-off function  $\chi_{\text{uv}} : \mathbb{R}^2 \rightarrow [0, 1]$  such that:

$$\chi_{\text{uv}}(z) = 0 \text{ if } |z| \leq \frac{1}{4}, \quad \chi_{\text{uv}}(z) = 1 \text{ if } |z| \geq 1, \quad |\chi_{\text{uv}}|_1 \leq 10,$$

and we use  $\chi_{\text{uv}}$  to define a *regularized* version of  $V_i$  as follows:

$$\tilde{V}_i(x) := \int_{\Sigma_N \setminus \Lambda_i} -\log|x-y| \chi_{\text{uv}}(x-y) \mathbf{d}\mathbf{f}_N(y), \quad x \in \Lambda_i.$$

**Claim C.2.** *We have:*

- $\tilde{V}_i(x) \leq V_i(x) + 100$  for all  $x \in \Lambda_i$ .
- $V_i(x) = \tilde{V}_i(x)$  for all  $x \in \Lambda_i$  with  $\text{dist}(x, \partial\Lambda_i) \geq 1$ .

*Proof of Claim C.2.* For the first point, observe that adding the short-distance cut-off  $\chi_{\text{uv}}$  diminishes the (non-negative) influence of point particles at distance less than 1, as well as the influence of the background in a small disk (which is negative but uniformly bounded). The second point is obvious.  $\square$

We introduce the (regularized) ‘‘boundary’’ term  $B_i$ :

$$B_i : x \mapsto B_i(x) := \int_{\Sigma_N \setminus \Lambda_i} \chi_i(y) \times \log|x - y| \chi_{\text{uv}}(x - y) \, \mathbf{d}\mathbf{f}_N(y). \quad (\text{C.5})$$

We can decompose  $\tilde{V}_i$  as (compare with (C.4)):

$$\tilde{V}_i = B_i + R_i, \quad (\text{C.6})$$

with  $R_i$  as above in (C.3) (observe that for  $x \in \Lambda_i$  and  $y$  in the support of  $1 - \chi_i$  we have  $\chi_{\text{uv}}(x - y) = 1$  i.e. the regularization by  $\chi_{\text{uv}}$  is transparent and does not appear in  $R_i$ ).

## C.2. Study of the boundary term

**Claim C.3.** *We have, with  $B_i$  as in (C.5):*

$$\mathbb{P}_N^\beta \left[ \sup_{x \in \Lambda_i} |\nabla B_i(x)| \leq \mathbf{C}_\beta \log^2 T \right] \geq 1 - \exp\left(-\frac{1}{\mathbf{C}_\beta} \log^2 T\right).$$

*Proof of Claim C.3.* For  $x \in \Lambda_i$  arbitrary and  $y$  in the support of  $\chi_i$  we have:

$$\|\nabla(\log|x - y| \chi_{\text{uv}}(x - y))\| \leq \frac{1}{1 + |x - y|}$$

(this is possible thanks to the regularization due to  $\chi_{\text{uv}}$  near  $\partial\Lambda_i$ ). To control  $|\nabla B_i(x)|$  it is thus enough to bound, for  $x \in \Lambda_i$ :

$$\int \mathbf{1}_{\Sigma_N \setminus \Lambda_i}(y) \chi_i(y) \frac{1}{1 + |x - y|} (\mathbf{d}\mathbf{X}_N(y) + \mathbf{d}y). \quad (\text{C.7})$$

By construction  $\chi_i \mathbf{1}_{\Sigma_N \setminus \Lambda_i}$  is supported in the annulus  $\mathfrak{D}(\omega_i, T + 2\hat{T}) \setminus \mathfrak{D}(\omega_i, T)$ . Let us cover this annulus by  $\mathcal{O}(\frac{T}{\hat{T}})$  squares of sidelength  $\hat{T}$ . In each square, by the local laws, we can ensure that there are at most  $\mathbf{C}_\beta \hat{T}^2$  points with probability  $1 - \exp\left(-\frac{1}{\mathbf{C}_\beta} \hat{T}^2\right)$  and so after an union bound we can ensure that *all of them* contain at most  $\mathbf{C}_\beta \hat{T}^2$  points with probability  $1 - \exp\left(-\frac{1}{\mathbf{C}_\beta} \hat{T}^2\right)$  ( $\hat{T}^2$  has been chosen in order to beat the combinatorial loss due to such union bounds). If so, then we have:

$$\int \mathbf{1}_{\Sigma_N \setminus \Lambda_i}(y) \chi_i(y) \frac{1}{1 + |x - y|} \mathbf{d}\mathbf{X}_N(y) \leq \hat{T} \int_{y \in \partial\Lambda_i} \frac{1}{1 + |x - y|} \mathbf{d}y,$$

for which a rough bound is  $\mathcal{O}(\log^2 T)$ . The continuous part of (C.7) is of the same order.  $\square$

We now focus on studying  $R_i$ . In the next sections the constants  $\mathbf{C}_\beta$  will also depend on  $\delta$  as in Assumption (1.4).

## C.3. Estimate on the first derivative at the center

**Lemma C.4.** *We have, with  $R_i$  as in (C.3):*

$$\mathbb{P}_N^\beta [|\nabla R_i(\omega_i)| \leq \mathbf{C}_\beta] \geq 1 - \exp\left(-\frac{\log^2 T}{\mathbf{C}_\beta}\right). \quad (\text{C.8})$$

*Proof of Lemma C.4.* Let us bound the first partial derivative of  $R_i$ . By definition of  $R_i$  we have:

$$\partial_1 R_i(x) = - \int_{\Sigma_N} (1 - \chi_i(y)) \partial_1 \log|x - y| \mathbf{d}\mathbf{f}_N(y).$$

**1. The first outer layer.** Let  $\sigma$  be a cut-off function with radial symmetry around  $\omega_i$  such that:

$$\sigma(z) = 1 \text{ if } |z - \omega_i| \leq 2T, \quad \sigma(z) = 0 \text{ if } |z - \omega_i| \geq 3T, \quad |\sigma|_k \leq CT^{-k} \text{ for } k \in \{1, 2\}.$$

We start by evaluating the contribution to  $R_i$  coming from the support of  $\sigma$ .

**Claim C.5.** Let  $A := \int (1 - \chi_i(y)) \sigma(y) \partial_1 \log |\omega_i - y| \mathbf{d}\mathbf{f}_N(y)$ . We have:

$$\mathbb{P}_N^\beta (|A| \leq \mathbf{C}_\beta) \geq 1 - \exp\left(-\frac{1}{\mathbf{C}_\beta} \hat{T}^2\right). \quad (\text{C.9})$$

*Proof of Claim C.5.* We are looking at the fluctuations of  $\varphi := y \mapsto (1 - \chi_i(y)) \sigma(y) \partial_1 \log |\omega_i - y|$  which is of class  $C^1$  and compactly supported within  $\mathfrak{D}(\omega_i, 3T) \setminus \mathfrak{D}(\omega_i, T + \hat{T})$ . Let us recall that:

- $(1 - \chi_i), \sigma$  are bounded and  $\partial_1 \log |\omega_i - y|$  is of order  $T^{-1}$  for  $y$  in the support of  $\varphi$ .
- $|\chi_i|_1$  is of order  $\hat{T}^{-1}$ ,  $|\sigma|_1$  is of order  $T^{-1}$  and  $|\partial_1 \log |\omega_i - \cdot||_{1, \text{supp } \varphi}$  is of order  $T^{-2}$ .

Moreover let us make the following observations:

- On the annulus  $\mathcal{A}_1 := \mathfrak{D}(\omega_i, T + 2\hat{T}) \setminus \mathfrak{D}(\omega_i, T + \hat{T})$ , of area  $\mathcal{O}(T\hat{T})$ ,  $\|\nabla\varphi\|$  is of order  $(T\hat{T})^{-1}$  (the dominant contribution comes from differentiating  $\chi_i$ ).
- On the annulus  $\mathcal{A}_2 := \mathfrak{D}(\omega_i, 3T) \setminus \mathfrak{D}(\omega_i, T + 2\hat{T})$ , of area  $\mathcal{O}(T^2)$ ,  $\|\nabla\varphi\|$  is of order  $T^{-2}$  ( $\chi_i$  does not play a role anymore).

Using Lemma 2.6 in its localized version (2.11) we are left to bound:

$$A_1 := (T\hat{T})^{-1} \times \left( \text{Ener}(\mathbf{X}_N, \mathcal{A}_1)^{\frac{1}{2}} \times (T\hat{T})^{\frac{1}{2}} + \text{Pts}(\mathbf{X}_N, \mathcal{A}_1) \right)$$

and

$$A_2 := T^{-2} \times \left( \text{Ener}(\mathbf{X}_N, \mathcal{A}_2)^{\frac{1}{2}} \times (T^2)^{\frac{1}{2}} + \text{Pts}(\mathbf{X}_N, \mathcal{A}_2) \right).$$

To control the first term  $A_1$ , we cover the annulus  $\mathcal{A}_1$  by  $\mathcal{O}(\frac{T}{\hat{T}})$  squares of sidelength  $\hat{T}$  and use the local laws. We obtain:

$$\mathbb{P}_N^\beta \left( \text{EnerPts}(\mathbf{X}_N, \mathcal{A}_1) \geq \mathbf{C}_\beta T\hat{T} \right) \leq \exp\left(-\frac{1}{\mathbf{C}_\beta} \hat{T}^2\right),$$

where we used a union bound on  $\exp(\mathcal{O}(\log T))$  events and the fact that  $\log T \ll \hat{T}^2$ . In particular this ensures that:

$$\mathbb{P}_N^\beta (|A_1| \leq \mathbf{C}_\beta) \geq 1 - \exp\left(-\frac{1}{\mathbf{C}_\beta} \hat{T}^2\right).$$

On the other hand, to control the second term  $A_2$  we may apply the local laws to the full disk  $\mathfrak{D}(\omega_i, 3T)$  and see that:

$$\mathbb{P}_N^\beta \left( \text{EnerPts}(\mathbf{X}_N, \mathfrak{D}(\omega_i, 3T)) \geq \mathbf{C}_\beta T^2 \right) \leq \exp\left(-\frac{T^2}{\mathbf{C}_\beta}\right),$$

which ensures that  $\mathbb{P}_N^\beta (|A_2| \leq \mathbf{C}_\beta) \geq 1 - \exp\left(-\frac{T^2}{\mathbf{C}_\beta}\right)$ . We deduce (C.9).  $\square$

**2. Dyadic scales up to the boundary.** Let  $K := \log_2\left(\frac{1}{10} \text{dist}(\Lambda_i, \partial\Sigma_N)\right)$ . By our assumption (1.4) we have  $K \geq \log_2\left(\frac{\delta\sqrt{N}}{10}\right)$ .

For  $\log_2(2T) \leq k \leq K$  we let  $\tau_k$  be a smooth cut-off function such that:

- $\tau_k$  has radial symmetry around  $\omega_i$  and is supported on the annulus  $\mathcal{A}'_k := \mathfrak{D}(\omega_i, 2^{k+1}) \setminus \mathfrak{D}(\omega_i, 2^k)$  of area  $\mathcal{O}(2^{2k})$ .
- We have  $|\tau_k|_1 \leq \mathbf{C}2^{-k}$  and  $|\tau_k|_2 \leq \mathbf{C}2^{-2k}$  (with  $\mathbf{C}$  independent of  $k$ ).
- We have the following ‘‘partition of unity’’-type of identity:

$$\sigma(z) + \sum_{k=\log_2(2T)}^K \tau_k(z) \equiv 1 \text{ for } |z - \omega_i| \leq 2^{K-1}.$$

For each  $k$  we let  $\phi_k$  be the following map:

$$\phi_k : y \mapsto -\tau_k(y)\partial_1 \log |\omega_i - y|.$$

One can check that:

$$|\phi_k|_1 \leq \mathbf{C}2^{-2k}, \quad |\phi_k|_2 \leq \mathbf{C}2^{-3k}.$$

**Claim C.6.** *We have, for  $k = \log_2(2T), \dots, K$ :*

$$\mathbb{P}_N^\beta \left[ |\text{Fluct}[\phi_k]| \geq \mathbf{C}_\beta 2^{-k/2} \right] \leq \exp \left( -\frac{1}{\mathbf{C}_\beta} 2^{k/2} \right).$$

*Proof of Claim C.6.* We use the ‘‘fine bounds’’ on fluctuations given by Proposition 2.8. We choose the parameter  $s = c2^k$  for some constant  $c$ , as allowed by (2.12), and observe that, in view of the bounds mentioned above:

$$s^2 \int_{\mathcal{A}'_k} |\nabla \phi_k|^2 \leq s^2 2^{2k} \times (2^{-2k})^2 = \mathcal{O}(c^2).$$

Moreover we may use the local laws to write (note that we deliberately stopped before reaching the boundary thanks to our choice of  $K$ ):

$$\log \mathbb{E}_N^\beta \left( e^{\mathbf{C}_\beta s |\phi_k|_2 (\text{EnerPts}(\mathbf{X}_N, \mathcal{A}'_k))} \right) \leq \log \mathbb{E}_N^\beta \left( e^{\mathbf{C}_\beta c 2^{-2k} (\text{EnerPts}(\mathbf{X}_N, \mathcal{A}'_k))} \right) \leq c\mathbf{C}_\beta.$$

In summary, we have:

$$\log \mathbb{E}_N^\beta \left[ e^{c2^k \text{Fluct}[\phi_k]} \right] = \mathcal{O}(c^2) + \mathcal{O}_\beta(c),$$

and Markov’s inequality yields the claim.  $\square$

Summing up the controls on  $\phi_k$  and using a union bound, we obtain:

$$\mathbb{P}_N^\beta \left[ \left| \sum_{k=\log_2(2T)}^K \text{Fluct}[\phi_k] \right| \geq \mathbf{C}_\beta T^{-1/2} \right] \leq \exp \left( -\frac{1}{\mathbf{C}_\beta} T^{1/2} \right) \ll \exp(-\log^2 T).$$

$\square$

**3. The rest of the system.** It remains to study the contribution to  $\nabla R_i$  coming from the part of the system far from  $\omega_i$ . Let us introduce an artificial cut-off  $\gamma$  supported on  $2\Sigma_N$ . We have

$$\begin{aligned} \int_{\Sigma_N} \left( 1 - \left( \sigma(y) + \sum_{k=\log_2(2T)}^K \tau_k(y) \right) \right) \partial_1 \log |y - \omega_i| \mathbf{d}\mathbf{f}_N(y), \\ = \int \gamma(y) \left( 1 - \left( \sigma(y) + \sum_{k=\log_2(2T)}^K \tau_k(y) \right) \right) \partial_1 \log |y - \omega_i| \mathbf{d}\mathbf{f}_N(y) \end{aligned}$$

and we may thus apply Lemma 2.6 to the function:

$$\varphi : y \mapsto \gamma(y) \left( 1 - \left( \sigma(y) + \sum_{k=\log_2(2T)}^K \tau_k(y) \right) \right) \partial_1 \log |y - \omega_i|,$$

whose derivative is of order at most  $\frac{1}{\delta^2 N}$  in view of Assumption 1.4 (this is where the dependency in  $\delta$  comes in). The support of  $\varphi$  has area  $N$ , the number of points is obviously bounded by  $N$  and the energy there is  $\mathcal{O}(N)$  with high probability by the *global* law. We obtain:

$$|\text{Fluct}[\varphi]| \leq \mathbf{C}_\beta \frac{1}{\delta^2}, \quad \text{with probability } 1 - \exp \left( -\frac{1}{\mathbf{C}_\beta} N \right).$$

Keeping the dominant contributions of these three steps, we obtain (C.8).  $\square$

## C.4. Estimate on the second derivative up to the boundary.

**Lemma C.7.** *We have, with  $R_i$  as in (C.3):*

$$\mathbb{P}_N^\beta \left[ \sup_{x \in \Lambda_i} |D^2 R_i(x)| \times (1 + \text{dist}(x, \partial \Lambda_i)) \leq \mathfrak{C}_\beta \right] \geq 1 - \exp \left( -\frac{1}{\mathfrak{C}_\beta} \hat{T}^2 \right).$$

*Proof of Lemma C.7.* We bound one of the four second partial derivatives (they can all be treated the same way) and use the same decomposition as in the proof of Lemma C.4.

**1. The first outer layer.** To control the first outer layer, we proceed like for Claim C.3. For  $x \in \Lambda_i$  arbitrary and  $y$  in the support of  $(1 - \chi_i)\sigma$  we have:

$$\|D^2 \log |x - y|\| \preceq \frac{1}{(\hat{T} + |x - y|)^2}$$

(the points  $x, y$  being at distance at least  $\hat{T}$  from each other). It is thus enough to bound:

$$\int \sigma(y) (1 - \chi_i(y)) \frac{1}{(\hat{T} + |x - y|)^2} (\mathbf{d}\mathbf{X}_N(y) + \mathbf{d}y). \quad (\text{C.10})$$

Assume as above that each square of sidelength  $\hat{T}$  covering the annulus  $\mathfrak{D}(\omega_i, T + 2\hat{T}) \setminus \mathfrak{D}(\omega_i, T)$  contains at most  $\mathfrak{C}\hat{T}^2$  points (an event that occurs with probability  $1 - \exp(-\frac{1}{\mathfrak{C}}\hat{T}^2)$  and does not depend on  $x$ ). We may then compare (C.10) to the following one-dimensional integral:

$$\hat{T} \int_{y \in \partial \Lambda_i} \frac{1}{(\hat{T} + |x - y|)^2} \mathbf{d}y,$$

for which a rough bound is  $\mathfrak{C} \frac{1}{1 + \text{dist}(x, \partial \Lambda_i)}$ .

**2. Dyadic scales up to the boundary.** We proceed as in the previous proof, but instead of using the fine bounds of Proposition 2.8 (which are only valid function-wise, so  $x$ -wise here) we use the rougher controls of Lemma 2.6 which allow for a uniform control. For each  $k$  we let  $\phi_k$  be the following map:

$$\phi_{k,x} : y \mapsto -\tau_k(y) \partial_{12} \log |x - y|.$$

If  $x$  is in  $\Lambda_i$  and  $y \in \mathcal{A}'_k$  for  $k \geq \log_2(2T)$  we have  $|\phi_{k,x}|_1 \leq \mathfrak{C}2^{-3k}$ . Hence using (2.10) we know that we can bound  $\text{Fluct}[\phi_{k,x}]$  by:

$$2^{-3k} \times (2^{2k} + \text{EnerPts}(\mathbf{X}_N, \mathcal{A}'_k)).$$

Using the local laws, we may control  $\text{EnerPts}(\mathbf{X}_N, \mathcal{A}'_k)$  by  $\mathfrak{C}_\beta 2^{2k}$  with probability  $1 - \exp(-\frac{1}{\mathfrak{C}_\beta} 2^{2k})$ . We deduce that:

$$\mathbb{P}_N^\beta \left[ \sup_{x \in \Lambda_i} |\text{Fluct}[\phi_{k,x}]| \geq \mathfrak{C}_\beta 2^{-k} \right] \leq \exp \left( -\frac{1}{\mathfrak{C}_\beta} 2^{2k} \right).$$

Summing again the contributions through an union bound, we get that the contribution to the second derivative of  $R_i$  due to the dyadic annuli is bounded by  $\mathfrak{C}T^{-1}$  with probability  $1 - \exp(-\frac{T^2}{\mathfrak{C}_\beta})$ .

**3. The rest of the system.** We argue as in the previous proof, but thanks to the additional derivative we gain a factor  $\frac{1}{\delta\sqrt{N}} \ll \text{dist}(x, \Lambda_i)^{-1}$ .  $\square$

## C.5. Summary and conclusion

### Bounding the contributions of other sub-systems

Returning to (C.1), (C.2), a rough bound on  $\|D \log |x - y|\|$  for  $x \in \Lambda_i$  and  $y \in \Lambda_j$ , together with the fact that each sub-system contains by assumption  $\mathcal{O}(T^2)$  points, ensures that (using (3.6)):

$$|V_i - V^{\text{ext}}|_{1, \Lambda_i} \leq \mathfrak{C}T^2 \sum_{j, j \neq i} \frac{1}{d_{ij}} = \mathcal{O} \left( T^2 \frac{\log R}{ML} \right). \quad (\text{C.11})$$

(Let us note that it is fairly easy to improve (C.11) by expanding the interaction in a more precise way and controlling some fluctuations, but it would add technicalities while not making a big difference in our final statement.)

## Summary

**1. Control up to the edge.** Combining Lemma C.4 and Lemma C.7 and integrating between  $\omega_i$  and any given  $x$  in  $\Lambda_i$  we deduce that  $|\nabla R_i(x)| \leq C \log T$  for all  $x$  in  $\Lambda_i$ . Combining this with the decomposition (C.6) and Claim C.3 we deduce that

$$\|\nabla \tilde{V}_i\| \leq C_\beta \log^2 T \text{ on } \Lambda_i \dots$$

...and thus after integrating between  $\omega_i$  and any given  $x$  in  $\Lambda_i$  we obtain:

$$\left| \tilde{V}_i(x) - \tilde{V}_i(\omega_i) \right| \leq C_\beta T \log^2 T$$

**2. Properties of the decomposition.** We have already bounded  $|R_i|_{1,\Lambda}$ , and we have used the control of  $\nu$  at scale  $\hat{T} : \log T$  several times in the argument above.

In conclusion, with probability  $1 - \exp\left(-\frac{\hat{T}^2}{C_\beta}\right)$  the potential  $V_i$  satisfies the requirements to be a “good external potential” on  $\Lambda_i$ . Thus in view of (C.11), since we enforce (3.4), with probability  $1 - \exp\left(-\frac{\hat{T}^2}{C_\beta}\right)$  the potential  $V^{\text{ext}}$  is a “good external potential with constant  $\bar{C}$ ”, where  $\bar{C}$  is some large enough constant depending only on  $\beta$  and  $\delta$ .

We conclude with a union bound on  $\mathcal{N}$  such events.

## D. Global laws for sub-systems: proof of Proposition 5.7

### D.1. Effect of an harmonic perturbation on the equilibrium measure

In this section, we revisit (parts of) the analysis of [BBNY17, Sec.3]. Broadly speaking, our goal is to understand the effect of the external potential  $V^{\text{ext}}$  on the typical repartition of charges within the subsystem. In particular, we want to use the fact that  $V^{\text{ext}}$  is *harmonic* on  $\Lambda$  and is generated by a “nice” external configuration. We start by recalling some elements of logarithmic potential theory.

#### D.1.1. Some potential theory

Let  $V : \mathbb{R}^2 \rightarrow \mathbb{R} \cup \{+\infty\}$  be a lower semi-continuous function satisfying, for some  $\varepsilon > 0$ , the following growth condition:

$$\lim_{|z| \rightarrow \infty} (V(z) - (2 + \varepsilon) \log |z|) = +\infty.$$

If  $V$  takes the value  $+\infty$ , assume that it is not too wild e.g. that  $V$  takes finite values on some open disk. Denote by  $\mathcal{I}_V$  the following functional defined on the space  $\mathcal{P}(\mathbb{R}^2)$  of probability measures on  $\mathbb{R}^2$ :

$$\mathcal{I}_V : \mu \mapsto \iint -\log |x - y| d\mu(x) d\mu(y) + \int V d\mu. \quad (\text{D.1})$$

The next proposition<sup>12</sup> covers well-known properties of the minimization problem associated to  $\mathcal{I}_V$ .

**Proposition D.1** (The equilibrium measure). *There exists a unique minimizer  $\mu_V$  of  $\mathcal{I}_V$ , which we call the “equilibrium measure” associated to  $V$ . Its support, denoted by  $S_V$ , is compact.*

<sup>12</sup>The statements of Proposition D.1 only hold “quasi everywhere” (q.e.), which means “up to a set of capacity zero”. This makes no difference for us and, for simplicity, we omit it.

The equilibrium measure  $\mu_V$  is characterized by the following fact: there exists a constant  $c_V$  (depending on  $V$ ) such that the logarithmic potential generated by  $\mu_V$ , namely  $\mathfrak{h}^{\mu_V}(x) := \int -\log|x-y|d\mu_V(y)$  satisfies Euler-Lagrange equations of the form:

$$\begin{cases} \mathfrak{h}^{\mu_V} + \frac{1}{2}V = c_V & \text{on } S_V \\ \mathfrak{h}^{\mu_V} + \frac{1}{2}V \geq c_V & \text{on } \mathbb{R}^2 \setminus S_V. \end{cases} \quad (\text{D.2})$$

Moreover, the equilibrium measure is connected to the solution of an obstacle problem in the following sense: if we define  $u_V : \mathbb{R}^2 \rightarrow \mathbb{R} \cup \{+\infty\}$  by setting:

$$u_V(z) := \sup \left\{ v(z), v \text{ is subharmonic, } v \leq \frac{V}{2} \text{ on } \mathbb{R}^2, \limsup_{|z| \rightarrow \infty} (v(z) - \log|z|) < \infty \right\}, \quad (\text{D.3})$$

and let the ‘‘coincidence set’’  $S_V^*$  be  $S_V^* := \{z \in \mathbb{R}^2, u_V(z) = \frac{1}{2}V(z)\}$ . Then  $S_V \subset S_V^*$  (in words: the support of the equilibrium measure is contained in the coincidence set of the obstacle problem) and we have:

$$u_V(z) = c_V - \mathfrak{h}^{\mu_V}(z) \text{ on } \mathbb{R}^2.$$

In particular the density of  $\mu_V$  is given equivalently by  $\frac{1}{2\pi}\Delta u_V$  or  $\frac{1}{4\pi}\Delta V$  on  $S_V$ .

*Proof of Proposition D.1.* The first part is a classical result by Frostman [Fro35]. The connection with the obstacle problem has been investigated in [HM13] and an exposition can be found in [Ser15, Sec.2]. See also [BBNY17, Sec. 2.1, 2.2] and the references therein.  $\square$

### D.1.2. Harmonic perturbations

We now study how the equilibrium measure reacts to certain perturbations of the external potential. Here we closely follow the exposition of [BBNY17, Sec. 3.3] while making several changes.

**The reference measure.** Let  $\rho \geq 1$  be some fixed radius, we work on the disk  $\rho\mathfrak{D} := \mathfrak{D}(0, \rho)$  (for convenience we adopt here the notation of [BBNY17]). Let us assume that the reference potential  $V^{\text{ref}}$  is given by:

$$V^{\text{ref}}(z) = \begin{cases} \frac{|z|^2}{\rho^2} & \text{for } |z| \leq \rho, \\ +\infty & \text{for } |z| > \rho. \end{cases} \quad (\text{D.4})$$

It satisfies the assumptions of Section D.1.1, and it is easy to check that the associated equilibrium measure  $\mu_{V^{\text{ref}}}$  is the uniform measure on the disk  $\rho\mathfrak{D}$  - in other words, we have  $\mu_V = \frac{1}{|\rho\mathfrak{D}|}\mathbf{m}_0$  on  $\rho\mathfrak{D}$ . Its density is  $\frac{1}{\pi\rho^2}$ , in particular the assumption that ‘‘ $\frac{1}{4\pi}\Delta V^{\text{ref}} \geq \alpha$  in  $\rho\mathfrak{D}$ ’’ made in [BBNY17, Sec. 3.3] is satisfied here but with  $\alpha$  proportional to  $\rho^{-2}$ .

**Class of perturbations.** We work with the same class of potentials  $W$  as in [BBNY17], namely we take  $W$  as:

$$W := \tau V^{\text{ref}} + \rho^{-2}(\mathfrak{h}^\nu + R), \text{ where:} \quad (\text{D.5})$$

- $V^{\text{ref}}$  is the reference potential of (D.4) and  $\tau$  is a real parameter. For us  $\tau$  will always be close to 1, and for convenience we can assume that  $|\tau - 1| \leq \frac{1}{10}$ .
- $\mathfrak{h}^\nu$  is the logarithmic potential generated by a *positive*, finite distribution  $\nu$  of charges located *outside* of the disk  $\rho\mathfrak{D}$ . For convenience (and since that is enough for us), we will assume that  $\nu$  is supported on an annulus  $\mathfrak{D}(0, \rho + \rho') \setminus \mathfrak{D}(0, \rho)$  for some  $\rho' \leq \frac{\rho}{100}$ . In [BBNY17, Prop. 3.3, 3.4] the controls are given in terms of the *total mass*  $\|\nu\|$  of  $\nu$ . We want more precise estimates and introduce to that aim the following quantity:

$$|\nu|_{\text{loc}}(\rho') := \sup_{x \in \partial\rho\mathfrak{D}} \nu(\mathfrak{D}(x, \rho')), \quad (\text{D.6})$$

which controls the mass of  $\nu$  in a *local* fashion, at scale  $\rho'$  along  $\partial\rho\mathfrak{D}$ .



- $R$  is some harmonic function on  $\rho\mathfrak{D}$  which is continuous up to the boundary. An important role is played by the *normal derivative of  $R$  on  $\partial\rho\mathfrak{D}$* . We introduce the notation:

$$\partial_n^- R(z) := \lim_{\varepsilon \rightarrow 0} \frac{R(z) - R(z - \varepsilon \vec{n})}{\varepsilon}, \quad \|\partial_n^- R\|_{\infty, \partial\rho\mathfrak{D}} := \sup_{z \in \partial\rho\mathfrak{D}} |\partial_n^- R(z)|, \quad (\text{D.7})$$

where for  $z \in \partial\rho\mathfrak{D}$  we denote by  $\vec{n}$  the outer unit normal vector (to the circle) at  $z$ .

Our scaling  $\rho^{-2}$  in front of the perturbation terms in (D.5) is not present in [BBNY17, (3.7)], but it is of course equivalent to taking the mass of  $\nu$  and the potential  $R$  to be of order  $\rho^{-2}$  in their statements. This scaling will compensate the fact that for us the quantity  $\alpha$  of [BBNY17] is of order  $\rho^{-2}$  (see above).

**Effect of the perturbation.** Passing from  $V^{\text{ref}}$  to  $W$  as in (D.5) has three effects on the equilibrium measure:

1. The support loses some parts located near the circle  $\partial\rho\mathfrak{D}$ .
2. A singular measure appears on the circle  $\partial\rho\mathfrak{D}$ .
3. The continuous density in the new support changes from  $\Delta V^{\text{ref}}$  to  $\tau \Delta V^{\text{ref}}$ .

In other words, the measure  $\mu_W$  can be written as:

$$\mu_W := \tau \mu_{V^{\text{ref}}} + (\eta - \tau \mu_{V^{\text{ref}}} \mathbf{1}_B),$$

with  $B$  a subset of  $\rho\mathfrak{D}$  located near  $\partial\rho\mathfrak{D}$  and  $\eta$  a measure that is absolutely continuous with respect to the arclength measure  $ds$  on  $\partial\rho\mathfrak{D}$ .

Of course the combination of all effects must have total mass 0 so that the resulting equilibrium measure  $\mu_W$  still be a probability measure. The point of the following propositions is to give quantitative controls on the location of  $B$  (first item) and the density of  $\eta$  (second item). The techniques are heavily borrowed from [BBNY17, Prop. 3.3 & 3.4], with some simplifications due to our specific context, and precisions thanks to our use of  $|\nu|_{\text{loc}}$  as in (D.6).

**Proposition D.2** (The support of  $\mu_W$  contains a large sub-disk). *The support of  $\mu_W$  contains the set of points  $z \in \rho\mathfrak{D}$  at distance at least  $\kappa$  from  $\partial\rho\mathfrak{D}$ , with  $\kappa$  such that:*

$$\kappa \geq \mathbf{c} \max \left( 20\rho', \frac{|\nu|_{\text{loc}}(\rho')}{\rho'}, |1 - \tau|\rho + \|\partial_n^- R\|_{\infty, \partial\rho\mathfrak{D}} \right). \quad (\text{D.8})$$

**Proposition D.3** (The singular component has a controlled density). *The Radon-Nikodym derivative of  $\eta$  with respect to the arclength measure on  $\partial\rho\mathfrak{D}$  is bounded by:*

$$\left\| \frac{d\eta}{ds} \right\|_{\infty} \leq \mathbf{c} \left( \frac{1}{\rho} \|\eta\| + \rho^{-2} \frac{|\nu|_{\text{loc}}(\rho')}{\rho'} + \rho^{-2} \|\partial_n^- R\|_{\infty, \partial\rho\mathfrak{D}} + |1 - \tau|\rho^{-1} \right). \quad (\text{D.9})$$

The next two subsections are devoted to the proofs of Propositions D.2 and D.3. We follow the proofs of the corresponding results in [BBNY17] while emphasizing the required modifications.

### D.1.3. Proof of Proposition D.2

**Importing an explicit construction.** The key ingredient for the proof of [BBNY17, Prop 3.3] is an explicit construction. To quote from [BBNY17, Proof of Prop 3.3]: “let  $D := \{z \in \rho\mathfrak{D}, \text{dist}(z, \partial\rho\mathfrak{D}) \geq \kappa\}$ , we show that  $D$  [is included in the support of  $\mu_W$ ] by exhibiting for every  $z_0 \in D$  a function  $v = v_{z_0}$  that satisfies  $v(z_0) = \frac{1}{2}W(z_0)$  and:

$$\begin{cases} v \text{ is sub-harmonic on } \mathbb{R}^2, \\ v \leq \frac{1}{2}W \text{ on } \mathbb{R}^2, \\ \lim_{|z| \rightarrow \infty} (v(z) - \log |z|) < \infty. \end{cases} \quad (\text{D.10})$$

Thus we have  $u_W = \frac{1}{2}W$  in  $D$  (cf. (D.3)) and since the perturbation  $W - V^{\text{ref}}$  is harmonic on  $\rho\mathfrak{D} \supset D$  we have<sup>13</sup> (as densities)  $\mu_W = \mu_{V^{\text{ref}}} > 0$  on  $D$  and thus  $D$  is contained in the support of  $\mu_W$ . It remains to construct such a function  $v$ , for which a crucial tool is the following lemma from [BBNY17]. For any  $r > 0$ , let  $l_r$  be the logarithmic potential generated by the uniform probability measure spread on  $\mathfrak{D}(0, r)$ .

**Lemma D.4** (An explicit construction). *Let  $z_0, w$  be two points in  $\mathbb{R}^2$  and let  $r$  such that  $r \leq \frac{1}{2}|z_0 - w|$ . Then there exists  $\tilde{z} \in \mathbb{R}^2$  and  $k \in \mathbb{R}$  (both depending on  $z_0$  and  $w$ ) such that:*

$$\frac{1}{2}(l_r(z_0 - \tilde{z}) + k) = -\frac{1}{2}\log|z_0 - w|, \quad \frac{1}{2}(l_r(z - \tilde{z}) + k) \leq -\frac{1}{2}\log|z - w| \text{ for } z \in \mathbb{R}^2. \quad (\text{D.11})$$

Moreover the point  $\tilde{z}$  can be found on the line segment between  $z_0$  and  $w$  with:

$$|\tilde{z} - z_0| = \frac{r^2}{|z_0 - w|}. \quad (\text{D.12})$$

*Proof of Lemma D.4.* This is [BBNY17, Lemma 3.6] with three minor differences. First of all, in their statement they assume that  $r \leq \frac{1}{2}|z_0 - w|$  and that  $r \in (0, 1)$  (which is always the case in their setting), whereas we take  $r$  arbitrary (still with the condition  $r \leq \frac{1}{2}|z_0 - w|$ ). There is in fact no additional generality in our statement: the case  $r \in (0, 1)$  extends to the general case by scaling. Secondly, they allow for any  $\sigma \geq \frac{1}{2}$  but we will only need  $\sigma = \frac{1}{2}$  (so “ $\sigma$ ” does not appear here). Finally, the estimate (D.12) is not written down in [BBNY17], however it is a straightforward consequence of the fact that  $|\tilde{z} - z_0| \leq r \leq \frac{1}{2}|z_0 - w|$  (which is given by their statement and assumption) and the first equality in [BBNY17, (3.14)] (with  $\sigma = \frac{1}{2}$ ).  $\square$

Next, we follow [BBNY17, Proof of Prop 3.3] with a slight adaptation to prove our Proposition D.2.

*Proof of Proposition D.2.* Contrarily to [BBNY17] we will not scale everything back to the unit circle. We fix  $z_0$  in  $D := \{z \in \rho\mathfrak{D}, \text{dist}(z, \partial\rho\mathfrak{D}) \geq \kappa\}$ , with  $\kappa$  satisfying the three conditions of (D.8) and seek to construct  $v$  satisfying (D.10). Let  $u_{V^{\text{ref}}}$  be the solution to the obstacle problem for the reference potential  $V^{\text{ref}}$  as in (D.3), let  $\tilde{R}$  be the harmonic extension of  $R$  outside  $\rho\mathfrak{D}$  as in [BBNY17, (3.17)], let  $G(z) = \max(0, \log|\rho^{-1}z|)$ . Let  $\gamma := 2\|\partial_{\bar{r}} R\|_{\infty, \partial\rho\mathfrak{D}}$  as defined in (D.7). Compared to [BBNY17] we always take  $\sigma = \frac{1}{2}$  (and  $\sigma$  does not appear here).

There remains to define one last term, for which we differ slightly from [BBNY17, (3.18)] as explained below. For each  $w$  in the support of  $\nu$ , we apply Lemma D.4 to  $z_0, w$  as above and (which is new compared to [BBNY17]) we choose the parameter  $r$  depending on  $w$  as follows:

$$r = r(w) = \frac{1}{10}|z_0 - w|,$$

which is valid choice as it is obviously smaller than  $\frac{1}{2}|z_0 - w|$ . We obtain a point<sup>14</sup>  $\tilde{z} = \tilde{z}(w)$  and a real number  $k = k(w)$  such that (D.11) are satisfied. Let us make the following important observation: we know by the construction of Lemma D.4 that  $\tilde{z}$  lies on the line segment between  $z_0$  and  $w$ , at distance  $\frac{1}{100}|z_0 - w|$  from  $z_0$ . Since by assumption we have on the one hand:

$$\text{dist}(z_0, \partial\rho\mathfrak{D}) \geq \kappa \geq 20\rho',$$

and on the other hand  $\text{supp } \nu \subset \mathfrak{D}(0, \rho + \rho') \setminus \mathfrak{D}(0, \rho)$  it is easy to check that:

$$|\tilde{z} - z_0| \leq \frac{1}{100}(\text{dist}(z_0, \partial\rho\mathfrak{D}) + \rho') \leq \frac{1}{4}\text{dist}(z_0, \partial\rho\mathfrak{D}),$$

and thus the point  $\tilde{z}$  remains in  $\rho\mathfrak{D}$ , in fact the entire disk  $\mathfrak{D}(\tilde{z}, r(w))$  is contained in  $\rho\mathfrak{D}$ .

This being done for all  $w \in \text{supp } \nu$  we may define a map  $L : \mathbb{R}^2 \rightarrow \mathbb{R}$  by:

$$L(z) := \frac{1}{2} \int_{\text{supp } \nu} (l_{r(w)}(z - \tilde{z}(w)) + k(w)) \nu(dw). \quad (\text{D.13})$$

Finally, as in [BBNY17, (3.18)] we form the map  $v = v_{z_0}$  by setting:

$$v : z \mapsto \tau u_{V^{\text{ref}}}(z) + \rho^{-2}(L(z) + \tilde{R}(z)) + \rho^{-1}\gamma G(z),$$

and claim that it satisfies (D.10).

<sup>13</sup>This follows from the last statement in Proposition D.1.

<sup>14</sup> $\tilde{z}$  depends also on  $z_0$  but we will not write down this dependency as we work for any fixed  $z_0$ .

**Sub-harmonicity** The distributional Laplacian is given by:

$$\Delta v = \tau \Delta u_{\text{Vref}} + \rho^{-2} \Delta L + \rho^{-2} (2\partial_n^- R + \gamma) \mathbf{d}s, \quad (\text{D.14})$$

the measure  $\mathbf{d}s$  being the arclength measure on  $\partial\rho\mathfrak{D}$  (we used the general formula of [BBNY17, (3.10)] and the fact that the jump of the normal derivatives of  $G$  along  $\partial\rho\mathfrak{D}$  is here  $\rho^{-2}$ ). In the interior of  $\mathfrak{D}$  we obtain (taking only the first two terms in the right-hand side of (D.14), using the last item of Proposition D.1 to evaluate  $\Delta u_{\text{Vref}}$ , and an explicit computation of  $\Delta l_r$  - we recall that  $l_r$  is the logarithmic potential generated by the uniform probability measure on the disk of radius  $r$ ):

$$\Delta v = \frac{\tau}{2\rho^2} - \rho^{-2} \left( \int_{\text{supp } \nu} \frac{1}{r(w)^2} \mathbf{1}_{\mathfrak{D}(\bar{z}(w), r(w))} \nu(\mathbf{d}w) \right)$$

(the differences with the analysis of [BBNY17] being that here  $\alpha$  (their notation) is of order  $\rho^{-2}$ ,  $\sigma$  (their notation) is always  $\frac{1}{2}$ , that  $L$  here is scaled by  $\rho^{-2}$  and most importantly that the distance  $r$  is here chosen *depending on  $w$*  as above). In order to check that  $\Delta v \geq 0$  (which is a requirement in (D.10)) we thus need to guarantee that:

$$\int_{\text{supp } \nu} \frac{1}{r(w)^2} \mathbf{1}_{\mathfrak{D}(\bar{z}(w), r(w))} \nu(\mathbf{d}w) \leq \frac{\tau}{2}, \quad \text{with } \frac{\tau}{2} \geq \frac{9}{20} \text{ because } |\tau - 1| \leq \frac{1}{10} \text{ by assumption.} \quad (\text{D.15})$$

Here we stop following the route of [BBNY17] and instead recall that by construction  $r(w) = \frac{1}{10}|z_0 - w|$ , thus we may write (bounding the indicator function by 1):

$$\int_{\text{supp } \nu} \frac{1}{r(w)^2} \mathbf{1}_{\mathfrak{D}(\bar{z}(w), r(w))} \nu(\mathbf{d}w) \leq 100 \int_{\text{supp } \nu} \frac{1}{|w - z_0|^2} \nu(\mathbf{d}w).$$

Cover the annulus  $\mathfrak{D}(0, \rho + \rho') \setminus \mathfrak{D}(0, \rho)$ , which by assumption contains the support of  $\nu$ , by  $\mathcal{O}(\rho/\rho')$  squares of sidelength  $\rho'$ . The quantity  $|\nu|_{\text{loc}}$  allows us to bound the mass of  $\nu$  on each such square. Then for  $z \in \rho\mathfrak{D}$ , if  $z$  is at distance at least  $20\rho'$  from the boundary then we can compare the integral  $\int \frac{1}{|z-w|^2} \nu(\mathbf{d}w)$  to

$$\frac{|\nu|_{\text{loc}}(\rho')}{\rho'} \times \int_{w \in \partial\mathfrak{D}} \frac{1}{|z-w|^2} \mathbf{d}w,$$

which is  $\mathcal{O}\left(|\nu|_{\text{loc}}(\rho') \times \frac{1}{\rho' \text{dist}(z, \partial\mathfrak{D})}\right)$  and thus smaller than  $\mathfrak{C} \frac{|\nu|_{\text{loc}}(\rho')}{\kappa\rho'}$  if  $z$  is at distance at least  $\kappa$  from the boundary. In particular, if  $\kappa$  is larger than some constant times  $\frac{|\nu|_{\text{loc}}(\rho')}{\rho'}$  we do have (D.15).

For the singular part on the boundary we need to ensure that  $2\partial_n^- R + \gamma \geq 0$ , which is true by definition of  $\gamma$ .

**Obstacle property** We want to check that  $v(z_0) = \frac{1}{2}W(z_0)$  and that  $v(z) \leq \frac{1}{2}W(z)$  on  $\mathbb{R}^2$  (in fact it is enough to check it on the disk as  $W$  is infinite outside). This step goes on exactly like in the original proof. We already know that  $u_{\text{Vref}}(z) = \frac{1}{2}V^{\text{ref}}(z)$  on  $\rho\mathfrak{D}$  (because  $u_{\text{Vref}}$  solves the ‘‘reference’’ obstacle problem), that  $\tilde{R}(z) = R(z)$  on  $\rho\mathfrak{D}$  (by definition), and that  $G$  vanishes on  $\rho\mathfrak{D}$ . Thus in view of (D.5) it suffices to have:

$$L(z_0) = \mathfrak{h}^\nu(z_0), \quad L(z) \leq \mathfrak{h}^\nu(z) \text{ for } z \neq z_0,$$

which was precisely guaranteed by construction of  $L$  using Lemma D.4, see (D.11) and (D.13).

**Growth at infinity** Here we proceed again a bit differently than in [BBNY17]. Recall that we have chosen our function as:

$$\tau u_{\text{Vref}}(z) + \rho^{-2} (L(z) + \tilde{R}(z)) + \rho^{-1} \gamma G(z).$$

As  $|z| \rightarrow \infty$  we have  $u_{\text{Vref}}(z) \sim \log|z|$ ,  $L(z) \sim -\|\nu\| \log|z|$ ,  $\tilde{R}(z) = \mathcal{O}(1)$  and  $G(z) \sim \log z$ . Thus the third condition of (D.10) is satisfied if we have:

$$\|\nu\| \geq \rho^2(1 - \tau) + \rho\gamma.$$

We argue that we may always assume that this is the case. Indeed, if not, then we may distribute<sup>15</sup> positive charges uniformly with density  $\gamma + \rho|1 - \tau|$  all around the circle  $\partial\rho\mathfrak{D}$  on an annulus of width 1, and redefine the perturbation  $R$  accordingly in order for the total potential to remain the same. This operation changes  $\nu$ : it increases its total mass to a convenient level and increases  $|\nu|_{\text{loc}}(1)$  by  $\gamma + \rho|1 - \tau|$ . In view of the ‘‘Sub-harmonicity’’ paragraph above, this is harmless as long as we assume that  $\kappa$  larger than some constant times  $\gamma + \rho|1 - \tau|$  in (D.8).

On the other hand, this operation does not affect the quantity  $\|\partial_n^- R\|_{\infty, \partial\rho\mathfrak{D}}$  as defined in (D.7), because placing a radially symmetric density of charges outside  $\rho\mathfrak{D}$  creates a *constant* potential within  $\rho\mathfrak{D}$ .

Hence without loss of generality we may assume that  $\|\nu\| \geq \rho^2(1 - \tau) + \rho\gamma$ . This concludes the proof of Proposition D.2.  $\square$

#### D.1.4. Proof of Proposition D.3

*Proof of Proposition D.3.* This time the changes are minor compared to [BBNY17, Proof of Prop. 3.4]. We follow their computations until the moment where they bound:

$$\int \frac{z - w}{|z - w|^2} \cdot \vec{n} \nu(dw) \leq \frac{1}{2} \|\nu\|,$$

where  $z$  is some fixed point on the circle and  $\vec{n}$  is the normal vector to the circle at  $z$ . We distinguish between  $|w - z| \leq 10\rho'$  and  $\geq 10\rho'$ . For the first contribution, we note<sup>16</sup> that  $\frac{z - w}{|z - w|^2} \cdot \vec{n}$  is always smaller than  $\frac{1}{2\rho}$  and thus:

$$\int_{|w - z| \leq 10\rho'} \frac{z - w}{|z - w|^2} \cdot \vec{n} \nu(dw) \leq \frac{C|\nu|_{\text{loc}}(\rho')}{2\rho}$$

We bound the rest of the integral by  $\int_{|w - z| \geq 10\rho'} \max\left(0, \frac{z - w}{|z - w|^2} \cdot \vec{n}\right) \nu(dw)$ , which we may compare to:

$$\frac{|\nu|_{\text{loc}}(\rho')}{\rho'} \int_{w \in \partial\mathfrak{D}_\rho, |w - z| \geq 10\rho'} \max\left(0, \frac{z - w}{|z - w|^2} \cdot \vec{n}\right) dw,$$

which is itself smaller than:

$$\frac{|\nu|_{\text{loc}}(\rho')}{\rho'} \int_{w \in \partial\mathfrak{D}_\rho} \max\left(0, \frac{z - w}{|z - w|^2} \cdot \vec{n}\right) dw = \frac{|\nu|_{\text{loc}}(\rho')}{\rho'} \times \mathcal{O}(1)$$

by one-dimensional scale-invariance. This is the dominant contribution.

The rest of the proof does not change, and we obtain their final estimate controlling the Radon-Nikodym derivative pointwise by:

$$\frac{1 - \tau}{2} \partial_n^- V + \int \frac{z - w}{|z - w|^2} \cdot \vec{n} \nu(dw) - \partial_n^- R + \frac{1}{2} \|\eta\|.$$

There we bound  $\partial_n^- V$  by  $\rho^{-1}$  (see (D.4)).  $\square$

## D.2. Application to a generalized 2DOCP with good external potential

We now apply the knowledge gained in Section D.1 to pass from a 2DOCP with good external potential to a 2DOCP with well-controlled non-uniform neutralizing background (see Sections 5.1 and 5.2).

Let  $\Lambda$  be a disk of radius  $\rho = T$  and  $n$  be the number of points in  $\Lambda$ . Let  $V^{\text{ext}}$  be a ‘‘good external potential’’ on  $\Lambda$ . Consider the potential  $W$  defined as:

$$W := \tau V_\Lambda^{\text{ref}} + \frac{2}{n} V^{\text{ext}}, \quad \tau := \frac{|\Lambda|}{n}, \quad (\text{D.16})$$

<sup>15</sup>This is inspired by a different but similar argument in the original proof.

<sup>16</sup>As in [BBNY17, Proof of Prop. 3.4], except that they scale everything back to  $\rho = 1$ .

where  $V_\Lambda^{\text{ref}}$  is as in (D.4) (with  $\rho = T$ ) and let  $\mu_W$  be the associated equilibrium measure, which we write (with the notation of Section D.1.2) as:

$$\mu_W = \tau \mu_{V_\Lambda^{\text{ref}}} + \left( \eta - \tau \mathbf{1}_B \mu_{V_\Lambda^{\text{ref}}} \right). \quad (\text{D.17})$$

We will use repeatedly below the fact that (see Definition 5.2):

$$\tau = 1 + \mathcal{O}\left(\frac{\log^2 T}{T}\right). \quad (\text{D.18})$$

### Some properties of the new equilibrium measure

**Claim D.5.** *For some constant  $C_\mu$  depending on the “good external potential”  $\bar{C}$  constant of  $V^{\text{ext}}$ , the following holds:*

*The measure  $\mu_W - \tau \mu_{V_\Lambda^{\text{ref}}}$  has mass  $1 - \tau = 1 - \frac{|\Lambda|}{n}$ , and is made of:*

1. *A continuous (negative) part of density  $-\frac{\tau}{|\Lambda|}$  supported on a subset  $B$  located at distance  $\leq \kappa$  from  $\partial\rho\mathfrak{D}$ , with  $\kappa$  as in (D.8). We can take:*

$$\kappa = C_\mu \times \log^2 T. \quad (\text{D.19})$$

2. *A singular component  $\eta$  living on  $\partial\Lambda$ , with a density  $\frac{d\eta}{ds}$  bounded as in (D.9). We have:*

$$\left\| \frac{d\eta}{ds} \right\| \leq C_\mu \frac{\log^2 T}{T^2}. \quad (\text{D.20})$$

*The total mass of the continuous part, of the singular part, and the total variation of  $\mu_W - \tau \mu_{V_\Lambda^{\text{ref}}}$  is bounded by  $C_\mu \frac{\log^2 T}{T}$ . Moreover, we have:*

$$n^2 \iint -\log|x-y| d\left(\eta - \tau \mu_{V_\Lambda^{\text{ref}}} \mathbf{1}_B\right)(x) d\left(\eta - \tau \mu_{V_\Lambda^{\text{ref}}} \mathbf{1}_B\right)(y) = \mathcal{O}\left(C_\mu^2 T^2 \log^5(T)\right). \quad (\text{D.21})$$

*Proof of Claim D.5.* Inserting Definition 5.1 of a “good external potential” into the bounds (D.8) and (D.9) with  $\rho' = \hat{T} = \log T$ , and using (D.18) we obtain (D.19) and (D.20).

The total mass of the continuous part is of order  $T \times C_\mu \log^2 T \times T^{-2} = \frac{\log^2 T}{T}$  (it is contained in a region at distance  $\leq C_\mu \log^2 T$  from the boundary of the disk of radius  $T$ , and its density is of order  $T^{-2}$ ).

The mass of the singular part matches the mass of the continuous part up to an error  $1 - \tau$ , but (D.18) holds. Hence the mass of the singular part is also bounded by  $C_\mu \frac{\log^2 T}{T}$ , and so is the total variation of  $\mu_W - \tau \mu_{V_\Lambda^{\text{ref}}}$ .

Scaling everything back to a disk of radius 1, and bounding the self-interaction using (D.19), (D.20), we get (D.21).  $\square$

### Inserting the equilibrium measure into the energy

Let  $\zeta_W$  be given by:

$$\zeta_W = \mathfrak{h}^{\mu_W} + \frac{1}{2}W - \mathcal{I}_W(\mu_W) + \frac{1}{2} \int W d\mu_W.$$

It is a standard fact of logarithmic potential theory that the function  $\zeta_W$  vanishes on the support of  $\mu_W$  and is non-negative outside of it, and we refer to  $\zeta_W$  as the “effective confining potential” (see [Ser15, Definition 2.18]).

**Lemma D.6.** *We have, using the notation of (4.8) and (5.3):  $\mathbb{P}_{n,\Lambda,V^{\text{ext}}}^\beta(\cdot) = \mathbb{P}_{n,\Lambda}^\beta(\cdot, n\mu_W, \zeta_W)$  thus (5.4) is justified.*

*Proof of Lemma D.6.* We start with the following claim:

**Claim D.7.** Let  $\mu_n := \frac{1}{n} \sum_{i=1}^n \delta_{x_i} = \frac{1}{n} \mathbf{X}$  be the empirical measure associated to a  $n$ -tuple in  $\Lambda$ . We have:

$$\begin{aligned} F_\Lambda(\mathbf{X}) + \int_\Lambda V^{\text{ext}}(x) d\mathbf{X}(x) &= \frac{1}{2} n^2 \left( \iint_{x \neq y} -\log|x-y| d\mu_n(x) d\mu_n(y) + \int \left( \frac{|\Lambda|}{n} V_\Lambda^{\text{ref}}(x) + \frac{2}{n} V^{\text{ext}}(x) \right) d\mu_n(x) \right) \\ &\quad - \frac{1}{2} \left( (2n|\Lambda| - |\Lambda|^2) \mathcal{I}_{\mu_{V_\Lambda^{\text{ref}}}}(\mu_{V_\Lambda^{\text{ref}}}) - |\Lambda|(n - |\Lambda|) \int V_\Lambda^{\text{ref}} d\mu_{V_\Lambda^{\text{ref}}} \right). \end{aligned}$$

*Proof.* The proof is elementary but it does require some care because of the possible non-neutrality and the various scalings involved. We start with expanding the definition (4.7) of  $F_\Lambda(\mathbf{X})$  as:

$$2F_\Lambda(\mathbf{X}) = \iint_{x \neq y} -\log|x-y| d\mathbf{X}(x) d\mathbf{X}(y) + \iint_{\Lambda \times \Lambda} -\log|x-y| dx dy - 2 \iint_{\Lambda \times \Lambda} -\log|x-y| d\mathbf{X}(x) dy.$$

Introducing the empirical measure  $\mu_n$  and the reference probability measure  $\mu_{V_\Lambda^{\text{ref}}}$  (uniform on  $\Lambda$ ) we get:

$$\begin{aligned} 2F_\Lambda(\mathbf{X}) &= n^2 \iint_{x \neq y} -\log|x-y| d\mu_n(x) d\mu_n(y) + |\Lambda|^2 \iint_{\Lambda \times \Lambda} -\log|x-y| d\mu_{V_\Lambda^{\text{ref}}}(x) d\mu_{V_\Lambda^{\text{ref}}}(y) \\ &\quad - 2n|\Lambda| \iint_{\Lambda \times \Lambda} -\log|x-y| d\mu_n(x) d\mu_{V_\Lambda^{\text{ref}}}(y). \end{aligned}$$

Let us recall the following standard identity valid on  $\Lambda$  (see (D.2) or e.g. [Ser15, Thm. 2.1]):

$$\int_\Lambda -\log|x-y| d\mu_{V_\Lambda^{\text{ref}}}(y) = -\frac{1}{2} V_\Lambda^{\text{ref}}(x) + \left( \mathcal{I}_{\mu_{V_\Lambda^{\text{ref}}}}(\mu_{V_\Lambda^{\text{ref}}}) - \frac{1}{2} \int V_\Lambda^{\text{ref}} d\mu_{V_\Lambda^{\text{ref}}} \right), \quad (\text{D.22})$$

we obtain after some computations:

$$\begin{aligned} 2F_\Lambda(\mathbf{X}) &= n^2 \left( \iint_{x \neq y} -\log|x-y| d\mu_n(x) d\mu_n(y) + \frac{|\Lambda|}{n} \int V_\Lambda^{\text{ref}} d\mu_n \right) \\ &\quad - (2n|\Lambda| - |\Lambda|^2) \mathcal{I}_{\mu_{V_\Lambda^{\text{ref}}}}(\mu_{V_\Lambda^{\text{ref}}}) + |\Lambda|(n - |\Lambda|) \int V_\Lambda^{\text{ref}} d\mu_{V_\Lambda^{\text{ref}}}. \end{aligned}$$

Dividing by 2 and inserting the contribution of  $V^{\text{ext}}$ , we obtain the claim.  $\square$

**Claim D.8.** We have:

$$F_\Lambda(\mathbf{X}) + \int_\Lambda V^{\text{ext}} d\mathbf{X} = F_\Lambda(\mathbf{X}, n\mu_W) + \int \zeta_W d\mathbf{X} + \text{Const.}(n, \Lambda, V^{\text{ext}})$$

*Proof of Claim D.8.* On the one hand, we know by Claim D.7 that:

$$F_\Lambda(\mathbf{X}) + \int_\Lambda V^{\text{ext}}(x) d\mathbf{X}(x) = \frac{1}{2} n^2 \left( \iint_{x \neq y} -\log|x-y| d\mu_n(x) d\mu_n(y) + \int W d\mu_n \right) + \text{Const.}(n, \Lambda).$$

On the other hand, using the “splitting formula” of Sandier-Serfaty (see [SS15, Lemma 2.1]) we have:

$$\frac{1}{2} n^2 \left( \iint_{x \neq y} -\log|x-y| d\mu_n(x) d\mu_n(y) + \int W d\mu_n \right) = \frac{1}{2} n^2 \mathcal{I}_W(\mu_W) + F_\Lambda(\mathbf{X}, n\mu_W) + n \int \zeta_W(x) d\mathbf{X}(x). \quad (\text{D.23})$$

Discarding quantities that do not depend on the configuration  $\mathbf{X}$ , we obtain the claim.  $\square$

We may thus equivalently use  $F_\Lambda(\mathbf{X}, n\mu_W) + \int \zeta_W d\mathbf{X}$  instead of  $F_\Lambda(\mathbf{X}) + \int_\Lambda V^{\text{ext}} d\mathbf{X}$  in the Boltzmann’s factor, up to some constant that gets absorbed in the partition function.  $\square$

### D.3. Proof of Proposition 5.7

*Proof of Proposition 5.7.* By definition of  $\mathbb{P}_{n,\Lambda}^\beta$  we have:

$$\begin{aligned} \log \mathbb{E}_{n,\Lambda}^\beta \left[ \exp \left( \frac{\beta}{2} F_\Lambda(\mathbf{X}, n\mu_W) \right) \middle| \mathcal{E}_\Lambda \right] &= \log \int_{\Lambda^n} \mathbf{1}_{\mathcal{E}_\Lambda}(\mathbf{X}) e^{-\frac{\beta}{2} (F_\Lambda(\mathbf{X}, n\mu_W) + 2 \int \zeta_W(x) d\mathbf{X}(x))} d\mathbf{X}_n \\ &\quad - \log \int_{\Lambda^n} \mathbf{1}_{\mathcal{E}_\Lambda}(\mathbf{X}) e^{-\beta (F_\Lambda(\mathbf{X}, n\mu_W) + \int \zeta_W(x) d\mathbf{X}(x))} d\mathbf{X}_n. \end{aligned} \quad (\text{D.24})$$

**1. Configuration-wise lower bound on the energy.** Since  $\zeta_W$  is non-negative, we have:

$$F_\Lambda(\bar{\mathbf{X}}, n\mu_W) + 2 \int \zeta_W d\mathbf{X} \geq F_\Lambda(\mathbf{X}, n\mu_W).$$

Because of the possible singularity of  $\mu_W$  along the boundary, we cannot directly use results like [AS21, Lemma B.2.] to bound  $F_\Lambda$  from below. Instead, for each  $i = 1, \dots, n$  we let  $\eta(x_i) := \frac{1}{4} \min(1, \text{dist}(x_i, \partial\Lambda))$  and use this as a “truncation vector”. By construction, the disks  $\mathfrak{D}(x_i, \eta(x_i))$  do not intersect the support of the singular part of  $\mu_W$ , we may thus use the monotonicity property of [AS21, Lemma B.1] (or Onsager’s lemma) and write that:

$$F_\Lambda(\mathbf{X}, n\mu_W) \geq \frac{1}{2} \left( \frac{1}{2\pi} \int_{\mathbb{R}^2} |\nabla \mathfrak{h}_{\bar{\eta}}^{\mathbf{X}, n\mu_W}|^2 + \sum_{i=1}^n \log \eta(x_i) \right) - \mathcal{O}(n).$$

The integral is obviously non-negative, and it remains to find a lower bound for the negative contribution coming from  $\sum_{i=1}^n \log \eta(x_i)$ . This is where we use our restriction to the event  $\mathcal{E}_\Lambda$ . Since  $\mathbf{X}$  is assumed to satisfy the conditions of Definition 5.2 we know that:

1. There is no index  $i$  such that  $\eta(x_i) \leq e^{-\log^2 T}$ .
2. There are at most  $T \log T$  indices  $i$  such that  $\eta(x_i) \leq \frac{1}{4}$ .

We thus have the rough lower bound:  $\sum_{i=1}^n \log \eta(x_i) \geq -n \log 4 - T \log T \times \log^2 T$ . Hence for  $\mathbf{X} \in \mathcal{E}_\Lambda$ , we have the configuration-wise lower bound:  $F_\Lambda(\mathbf{X}, n\mu_W) \geq -\mathcal{O}(T^2)$ , which we can integrate in order to obtain (with an implicit constant depending only on  $\beta$ ):

$$\log \int_{\Lambda^n} \mathbf{1}_{\mathcal{E}_\Lambda}(\mathbf{X}) e^{-\frac{\beta}{2} (F_\Lambda(\mathbf{X}, n\mu_W) + 2n \int \zeta_W d\mathbf{X})} d\mathbf{X}_n \leq \log |\mathcal{E}_\Lambda| + \mathcal{O}(T^2),$$

where  $|\mathcal{E}_\Lambda|$  denotes the volume of the event  $\mathcal{E}_\Lambda$  under the Lebesgue measure  $d\mathbf{X}_n$  on  $\Lambda^n$ .

**2. Lower bound on the partition function.** To find an upper bound on the second term in the right-hand side of (D.24), we rely on a “Jensen’s trick” inspired by [GZ19a] and write:

$$\begin{aligned} - \log \int_{\Lambda^n} \mathbf{1}_{\mathcal{E}_\Lambda}(\mathbf{X}) e^{-\beta (F_\Lambda(\mathbf{X}, n\mu_W) + n \int \zeta_W(x) d\mathbf{X}(x))} d\mathbf{X}_n \\ \leq - \log |\mathcal{E}_\Lambda| + \beta \int_{\Lambda^n} \mathbf{1}_{\mathcal{E}_\Lambda}(\mathbf{X}) \left( F_\Lambda(\mathbf{X}, n\mu_W) + n \int \zeta_W(x) d\mathbf{X}(x) \right) \frac{d\mathbf{X}_n}{|\mathcal{E}_\Lambda|}. \end{aligned} \quad (\text{D.25})$$

Let us start by doing computations without the indicator  $\mathbf{1}_{\mathcal{E}_\Lambda}(\mathbf{X})$ .

**Claim D.9.** *We have:*

$$\int_{\Lambda^n} \left( F_\Lambda(\mathbf{X}, n\mu_W) + n \int \zeta_W(x) d\mathbf{X}(x) \right) \frac{d\mathbf{X}_n}{|\Lambda^n|} \leq \mathcal{O}(T^2 \log^5 T). \quad (\text{D.26})$$

*Proof of Claim D.9.* Let us recall that the integrand can be written (see (D.23)) as:

$$F_\Lambda(\mathbf{X}, n\mu_W) + n \int \zeta_W(x) d\mathbf{X}(x) = \frac{1}{2}n^2 \left( \iint_{x \neq y} -\log|x-y| d\mu_n(x) d\mu_n(y) + \int W d\mu_n \right) - \frac{1}{2}n^2 \mathcal{I}_W(\mu_W).$$

By elementary computations, we obtain:

$$\frac{1}{2}n^2 \int_{\Lambda^n} \left( \iint_{x \neq y} -\log|x-y| d\mu_n(x) d\mu_n(y) \right) \frac{d\mathbf{X}_n}{|\Lambda|^n} = \frac{1}{2}n(n-1) \iint -\log|x-y| d\mu_{V_\Lambda^{\text{ref}}}(x) d\mu_{V_\Lambda^{\text{ref}}}(y)$$

and similarly:

$$\frac{1}{2}n^2 \int_{\Lambda^n} \left( \int W d\mu_n \right) \frac{d\mathbf{X}_n}{|\Lambda|^n} = \frac{1}{2}n^2 \int W(x) d\mu_{V_\Lambda^{\text{ref}}}(x),$$

and thus after some simplifications we get:

$$\begin{aligned} \int_{\Lambda^n} \left( F_\Lambda(\mathbf{X}, n\mu_W) + n \int \zeta_W(x) d\mathbf{X}(x) \right) \frac{d\mathbf{X}_n}{|\Lambda|^n} \\ = \frac{1}{2}n(n-1) \iint -\log|x-y| d\mu_{V_\Lambda^{\text{ref}}}(x) d\mu_{V_\Lambda^{\text{ref}}}(y) + \frac{1}{2}n^2 \int W d\mu_{V_\Lambda^{\text{ref}}} - \frac{1}{2}n^2 \mathcal{I}_W(\mu_W). \end{aligned}$$

We now insert the expressions (D.16), (D.17) for  $W$  and  $\mu_W$ , the definition (D.1) of  $\mathcal{I}_W(\mu_W)$ , expand and use the identity (D.22). We obtain:

$$\begin{aligned} \frac{1}{2}n(n-1) \iint -\log|x-y| d\mu_{V_\Lambda^{\text{ref}}}(x) d\mu_{V_\Lambda^{\text{ref}}}(y) + \frac{1}{2}n^2 \int W d\mu_{V_\Lambda^{\text{ref}}} - \frac{1}{2}n^2 \mathcal{I}_W(\mu_W) = \\ n^2 \iint -\log|x-y| d \left( \eta - \tau \mu_{V_\Lambda^{\text{ref}}} \mathbf{1}_B \right) (x) d \left( \eta - \tau \mu_{V_\Lambda^{\text{ref}}} \mathbf{1}_B \right) (y) - \frac{1}{2}n \iint -\log|x-y| d\mu_{V_\Lambda^{\text{ref}}}(x) d\mu_{V_\Lambda^{\text{ref}}}(y) \\ + n \int V^{\text{ext}}(x) \left( d\mu_{V_\Lambda^{\text{ref}}} - d\mu_W \right) (x). \quad (\text{D.27}) \end{aligned}$$

Using (D.21) we control the first term in the right-hand side of (D.27) by  $\mathcal{O}(T^2 \log^5(T))$ . On the other hand, by a direct estimate, we have  $n \iint -\log|x-y| d\mu_{V_\Lambda^{\text{ref}}}(x) d\mu_{V_\Lambda^{\text{ref}}}(y) = \mathcal{O}(T^2 \log T)$ . It remains to bound  $n \int V^{\text{ext}} \left( d\mu_{V_\Lambda^{\text{ref}}} - d\mu_W \right)$  from above. Since  $\mu_W, \mu_{V_\Lambda^{\text{ref}}}$  have the same mass, it is equivalent to bound:

$$n \int \left( V^{\text{ext}}(x) - V^{\text{ext}}(\omega) \right) \left( d\mu_{V_\Lambda^{\text{ref}}} - d\mu_W \right) (x),$$

where  $\omega$  is the center of  $\Lambda$ . Let us decompose the integral into two parts:

**Away from the boundary.** On  $\{\text{dist}(z, \partial\Lambda) \geq \kappa\}$  we know that  $\mu_W$  coincides with  $\tau \mu_{V_\Lambda^{\text{ref}}}$  and thus (since we take  $\kappa \geq 1$ ):

$$\left| n \int_{\{\text{dist}(z, \partial\Lambda) \geq \kappa\}} \left( V^{\text{ext}}(z) - V^{\text{ext}}(\omega) \right) \left( d\mu_W - d\mu_{V_\Lambda^{\text{ref}}} \right) \right| \leq n \times |1 - \tau| \times \sup_{\{\text{dist}(z, \partial\rho\mathfrak{D}) \geq 1\}} |V^{\text{ext}}(z) - V^{\text{ext}}(\omega)|.$$

We know that  $|1 - \tau| = \mathcal{O}\left(\frac{\log^2 T}{T}\right)$  and by assumption we have  $|V^{\text{ext}}(z) - V^{\text{ext}}(\omega)| = \mathcal{O}(T \log^3 T)$  on  $\{\text{dist}(z, \partial\rho\mathfrak{D}) \geq 1\}$ . Thus the contribution ‘‘away from the boundary’’ is bounded by  $\mathcal{O}(T^2 \log^5 T)$ .

**Near the boundary.** On  $\{\text{dist}(z, \partial\Lambda) \leq \kappa\}$  we control each contribution separately. On the one hand, we have, using the mean value formula for  $V^{\text{ext}}$  (which is harmonic on  $\Lambda$ ):

$$n \int_{\{\text{dist}(z, \partial\Lambda) \geq \kappa\}} \left( V^{\text{ext}}(z) - V^{\text{ext}}(\omega) \right) d\mu_{V_\Lambda^{\text{ref}}}(z) = 0.$$



On the other hand,  $\mu_W$  being a non-negative measure, we may write:

$$-n \int_{\{\text{dist}(z, \partial\Lambda) \geq \kappa\}} (\mathbf{V}^{\text{ext}}(z) - \mathbf{V}^{\text{ext}}(\omega)) \mathbf{d}\mu_W(z) \leq -n \int_{\{\text{dist}(z, \partial\Lambda) \geq \kappa\}} (\tilde{\mathbf{V}}^{\text{ext}}(z) - \tilde{\mathbf{V}}^{\text{ext}}(\omega)) \mathbf{d}\mu_W(z),$$

where we used the auxiliary function  $\tilde{\mathbf{V}}^{\text{ext}}$  from Definition 5.1. The mass of  $\mu_W$  near the boundary is bounded by  $\mathcal{O}\left(\frac{\kappa}{T} + T \|\frac{d\eta}{ds}\|_\infty\right)$ . Using the results of Claim D.5 and the definition of a “good external potential” we bound the contribution “near the boundary” by  $\mathcal{O}(T^2 \log^5 T)$  also.

This concludes the proof of the claim.  $\square$

It remains to argue that restricting our integrals to  $\mathcal{E}_\Lambda$  as in (D.25) instead of (D.26) has no consequence on the estimate, which is not totally obvious. We make a crucial use of the following observation: the “cluster bound” for independent points is not worse than the one mentioned in (2.13), and we thus have:

$$\frac{|\mathcal{E}_\Lambda|}{|\Lambda^n|} \geq 1 - \exp(-\log^2 T). \quad (\text{D.28})$$

**Claim D.10.** *We have:*

$$\frac{1}{2}n^2 \int_{\Lambda^n} 1_{\mathcal{E}_\Lambda}(\mathbf{X}) \left( \int \mathbf{W} \mathbf{d}\mu_n \right) \frac{\mathbf{dX}_n}{|\mathcal{E}_\Lambda|^n} \leq \frac{1}{2}n^2 \int_{\Lambda^n} \left( \int \mathbf{W} \mathbf{d}\mu_n \right) \frac{\mathbf{dX}_n}{|\Lambda|^n} + \exp(-\log^2 T). \quad (\text{D.29})$$

*Proof of Lemma D.10.* Let  $\mathbf{C}$  be some large constant, and let

$$\hat{\mathbf{W}} := \mathbf{W} - \mathbf{V}^{\text{ext}}(\omega) + \mathbf{C}T^2. \quad (\text{D.30})$$

Subtracting the same constant to both sides, we may write that:

$$\begin{aligned} \frac{1}{2}n^2 \int_{\Lambda^n} \left( \int \mathbf{W} \mathbf{d}\mu_n \right) \frac{\mathbf{dX}_n}{|\Lambda|^n} - \frac{1}{2}n^2 \int_{\Lambda^n} 1_{\mathcal{E}_\Lambda}(\mathbf{X}) \left( \int \mathbf{W} \mathbf{d}\mu_n \right) \frac{\mathbf{dX}_n}{|\mathcal{E}_\Lambda|^n} = \\ \frac{1}{2}n^2 \int_{\Lambda^n} \left( \int \hat{\mathbf{W}} \mathbf{d}\mu_n \right) \frac{\mathbf{dX}_n}{|\Lambda|^n} - \frac{1}{2}n^2 \int_{\Lambda^n} 1_{\mathcal{E}_\Lambda}(\mathbf{X}) \left( \int \hat{\mathbf{W}} \mathbf{d}\mu_n \right) \frac{\mathbf{dX}_n}{|\mathcal{E}_\Lambda|^n}. \end{aligned}$$

On the other hand, in view of the definition (D.16) of  $\mathbf{W}$  and the properties of the “good external potential”  $\mathbf{V}^{\text{ext}}$  as listed in Definition 5.1, we know that by choosing the constant  $\mathbf{C}$  large enough we can guarantee that  $\hat{\mathbf{W}} \geq 0$ , in which case it is clear that:

$$\frac{1}{2}n^2 \int_{\Lambda^n} 1_{\mathcal{E}_\Lambda}(\mathbf{X}) \left( \int \hat{\mathbf{W}} \mathbf{d}\mu_n \right) \frac{\mathbf{dX}_n}{|\mathcal{E}_\Lambda|^n} \leq \frac{1}{2}n^2 \int_{\Lambda^n} \left( \int \hat{\mathbf{W}} \mathbf{d}\mu_n \right) \frac{\mathbf{dX}_n}{|\Lambda|^n} \times \left( \frac{|\Lambda^n|}{|\mathcal{E}_\Lambda|^n} \right).$$

We thus see that:

$$\frac{1}{2}n^2 \int_{\Lambda^n} \left( \int \mathbf{W} \mathbf{d}\mu_n \right) \frac{\mathbf{dX}_n}{|\Lambda|^n} - \frac{1}{2}n^2 \int_{\Lambda^n} 1_{\mathcal{E}_\Lambda}(\mathbf{X}) \left( \int \mathbf{W} \mathbf{d}\mu_n \right) \frac{\mathbf{dX}_n}{|\mathcal{E}_\Lambda|^n} \leq \frac{1}{2}n^2 \int_{\Lambda^n} \left( \int \hat{\mathbf{W}} \mathbf{d}\mu_n \right) \frac{\mathbf{dX}_n}{|\Lambda|^n} \left( \frac{|\Lambda^n|}{|\mathcal{E}_\Lambda|^n} - 1 \right).$$

On the other hand, the same computation as in the proof of Claim D.9 yields:

$$\frac{1}{2}n^2 \int_{\Lambda^n} \left( \int \hat{\mathbf{W}} \mathbf{d}\mu_n \right) \frac{\mathbf{dX}_n}{|\Lambda|^n} = \frac{1}{2}n^2 \int \hat{\mathbf{W}}(x) \mathbf{d}\mu_{V_\Lambda^{\text{ref}}}(x),$$

which we can evaluate explicitly using our choice (D.30) for  $\hat{\mathbf{W}}$  and the expression (D.16) for  $\mathbf{W}$ . We obtain some polynomial in  $T$ , which gets absorbed by the sub-algebraic tail of (D.28). This concludes the proof of (D.29).  $\square$

We could proceed similarly for the other term in the integrand (which is easier because  $\mathbf{W}$  does not play any role).

This concludes the proof of (5.5).  $\square$

## E. Effect of localized translations on the energy: Proof of Proposition 6.3

In all this section,  $\varepsilon$  and  $\ell$  are fixed and we assume that  $|t| \leq \frac{\ell}{10}$ . To lighten notation, we will drop some dependencies with respect to  $\varepsilon$  and  $\ell$ .

### E.1. Some additional properties of localized translations

We decompose  $\Phi_t$  in two ways, either as:

$$\Phi_t(x) = x + \psi_t(x), \quad (\text{E.1})$$

which defines a vector field  $\psi_t$ , or alternatively as:

$$\Phi_t(x) = x + tW^{(\varepsilon, \ell)}(x) + \gamma_t(x), \quad (\text{E.2})$$

which defines a vector field  $\gamma_t$ .

**Proposition E.1.** 1. We have  $|\Phi_t - \text{Id}|_0 = |\psi_t|_0 \leq 2|t| \leq \frac{\ell}{5}$ .

2. For  $|x| \leq \ell/4$ , we have  $|\psi_t|_{1, \star}(x) = 0$ ,  $|\psi_t|_{2, \star}(x) = 0$ .

3. For  $\ell/4 \leq |x| \leq 2\ell e^{1/\varepsilon}$ , we have:

$$|\psi_t|_{k, \star}(x) \leq \frac{Ct\varepsilon}{|x|^k} \text{ for } k = 1, 2. \quad (\text{E.3})$$

4. For  $|x| \leq \ell/4$ , we have  $\gamma_t \equiv 0$ .

5. For  $\ell/4 \leq |x| \leq 2\ell e^{1/\varepsilon}$ , we have:

$$|\gamma_t|_{k, \star}(x) \leq \frac{Ct^2\varepsilon}{|x|^{k+1}}, \text{ for } k = 0, 1, 2, 3. \quad (\text{E.4})$$

*Proof of Proposition E.1.* Let us start by some general observations.

**Preliminary claims.** Since the vector field  $W^{(\varepsilon, \ell)}$  is bounded by 2 (see Lemma 6.1) the distance  $|\Phi_t(x) - x|$  is always smaller than  $2|t|$  and thus  $|\psi_t|_0 \leq 2|t|$ . This proves the first item. On the other hand, as stated in Lemma 6.2, the flow  $\Phi_t$  acts as a translation on  $\mathfrak{D}(0, \ell/4)$ , which means that  $\psi_t$  is a constant and thus the derivatives of  $\psi_t$  vanish identically there. This proves the second item, and also implies that the second-order correction  $\gamma_t$  vanishes identically on  $\mathfrak{D}(0, \ell/4)$ . In view of the bounds on  $W^{(\varepsilon)}$  given in Lemma 6.1 and the definition of  $W^{(\varepsilon, \ell)} := W^{(\varepsilon)}(\cdot/\ell)$ , after scaling we obtain for  $k = 1, 2$ :

$$|W^{(\varepsilon, \ell)}|_{k, \star}(x) \leq C\varepsilon \begin{cases} \frac{1}{|x|^k} & \text{for } |x| \geq 2\ell \\ \frac{1}{|\ell|^k} & \text{for } |x| \leq 2\ell. \end{cases}$$

Thus we may always write that  $|W^{(\varepsilon, \ell)}|_{k, \star}(x) \leq C\varepsilon \frac{1}{|x|^k}$ .

**Claim E.2.** If  $|x| \geq \frac{\ell}{4}$  then:

$$|W^{(\varepsilon, \ell)}|_{k, \star}(\Phi_t(x)) \leq C\varepsilon \frac{1}{|x|^k} \quad (\text{E.5})$$

*Proof.* Since  $|t| \leq \frac{\ell}{10}$  and  $|\Phi_t(x) - x| \leq 2|t|$ , we can ensure that if  $|x| \geq \frac{\ell}{4}$ , then  $|\Phi_t(x)| \geq \frac{1}{5}|x|$ , so in view of the previous estimates on  $W^{(\varepsilon, \ell)}$ , we obtain the claim.  $\square$

Let us end this paragraph with a simple general fact which will be useful to prove the remaining bounds:

$$\frac{d}{dt} \|f(t)\| \leq \left\| \frac{d}{dt} f(t) \right\|.$$

**Initial controls on  $D^k\Phi_t$ .**

**Claim E.3.** *We have, for  $|x| \geq \frac{\ell}{4}$ :*

$$\|D\Phi_t(x)\| \leq 1 + \mathbf{C} \frac{\varepsilon t}{|x|}, \quad \|D^k\Phi_t(x)\| \leq \mathbf{C} \frac{\varepsilon t}{|x|^k} \text{ for } k = 2, 3 \quad (\text{E.6})$$

*Proof of Claim E.3.* At  $t = 0$  we have  $\Phi_t = \text{Id}$  and thus  $D\Phi_t \equiv \text{Id}$ . Then we can compute:

$$\frac{d}{dt} \|D\Phi_t(x)\| \leq \left\| \frac{d}{dt} D\Phi_t(x) \right\| = \|D(\mathbf{w}^{(\varepsilon, \ell)} \circ \Phi_t(x))\| \leq \|D\mathbf{w}^{(\varepsilon, \ell)} \circ \Phi_t(x)\| \times \|D\Phi_t(x)\|.$$

Using (E.5) and an elementary differential inequality, we see that:

$$\|D\Phi_t(x)\| \leq e^{\frac{\varepsilon t}{|x|}} = 1 + \mathbf{C}' \frac{\varepsilon t}{|x|}.$$

Arguing similarly, we have:  $D^2\Phi_t = 0$  at  $t = 0$ , and:

$$\frac{d}{dt} \|D^2\Phi_t(x)\| \leq \left\| \frac{d}{dt} D^2\Phi_t(x) \right\| = \|D^2(\mathbf{w}^{(\varepsilon, \ell)} \circ \Phi_t(x))\|.$$

Applying Leibniz's rule, using the previous bound under the rough form  $\|D\Phi_t\| \leq \mathbf{C}$  and inserting (E.5) again, we obtain:

$$\frac{d}{dt} \|D^2\Phi_t(x)\| \leq \frac{\mathbf{C}\varepsilon}{|x|^2} + \frac{\mathbf{C}\varepsilon}{|x|} \times \|D^2\Phi_t(x)\|.$$

Solving again the differential inequality with initial condition 0, we obtain:

$$\|D^2\Phi_t(x)\| \leq \mathbf{C} \frac{\varepsilon t}{|x|^2}$$

We proceed the same way for the third derivative. □

**Application 1: properties of  $\psi_t$**  Returning to the definition (E.1) of  $\psi_t$ , we see that  $D\psi_t = 0$  at  $t = 0$  and we may then compute:

$$\frac{d}{dt} \|D\psi_t(x)\| \leq \left\| \frac{d}{dt} D\psi_t(x) \right\| = \|D(\mathbf{w}^{(\varepsilon, \ell)} \circ \Phi_t(x))\| \leq \|D\mathbf{w}^{(\varepsilon, \ell)} \circ \Phi_t(x)\| \times \|D\Phi_t(x)\|.$$

By the same computations as above, we find  $\frac{d}{dt} \|D\psi_t(x)\| \leq \mathbf{C} \frac{\varepsilon}{|x|}$  which implies the bound (E.3) for  $k = 1$ .

To study  $|\psi_t|_2$ , observe that  $D^2\Phi_t = D^2\psi_t$  and use (E.6).

**Application 2: properties of  $\gamma_t$ .** If we want to control, say  $|\gamma_t|_0$ , we compute:

$$\left| \frac{d}{dt} \gamma_t \right| = \left| \mathbf{w}^{(\varepsilon, \ell)} \circ \Phi_t(x) - \mathbf{w}^{(\varepsilon, \ell)}(x) \right| \leq \mathbf{C} \frac{\varepsilon |\psi_t|_0}{|x|} \leq \mathbf{C} \frac{\varepsilon t}{|x|},$$

where we used the bound on  $|\psi_t|_0$  stated as the first item of Proposition E.1 together with Claim E.2 in order to bound the Lipschitz constant of  $\mathbf{w}^{(\varepsilon, \ell)}$  between  $x$  and  $\Phi_t(x)$ . Integrating on  $t$  yields  $|\gamma_t|_0 \leq \varepsilon t^2 |x|^{-1}$  as claimed.

Higher derivatives are controlled the same way, using Leibniz's rule together with Claims E.2 and E.3. □

**Upper bounds on the derivatives of  $\psi_t$ .** Let us introduce two functions  $\Psi_1, \Psi_2$ :

$$\Psi_1 : x \mapsto \frac{\mathbf{C}t\varepsilon}{(\ell + |x|)}, \quad \Psi_2 : x \mapsto \frac{\mathbf{C}t\varepsilon}{(\ell + |x|)^2} \quad (\text{E.7})$$

Choosing the constant  $\mathbf{C}$  suitably we have, in view of Proposition E.1, the pointwise bounds:

$$|\psi_t|_{1, \star}(x) \leq \Psi_1(x), \quad |\psi_t|_{2, \star}(x) \leq \Psi_2(x), \text{ for all } x \in \mathbb{R}^2,$$

thus when looking for *upper bounds* we may replace occurrences of  $|\psi_t|_{1, \star}, |\psi_t|_{2, \star}$  by  $\Psi_1, \Psi_2$ . The upside of working with  $\Psi_1, \Psi_2$  is that they enjoy the following properties deduced from elementary calculus:

- (Slow variation at scale  $\ell$ .) For all  $x$  we have:

$$\sup_{y, |x-y| \leq \ell} \frac{|\Psi_1(y)|}{|\Psi_1(x)|} \leq 2, \quad \sup_{y, |x-y| \leq \ell} \frac{|\Psi_2(y)|}{|\Psi_2(x)|} \leq 2. \quad (\text{E.8})$$

- (Slow variation.) For all  $x$  we have:

$$\sup_{y, |x-y| \leq \frac{1}{2}|x|} \frac{|\Psi_1(y)|}{|\Psi_1(x)|} \leq 2, \quad \sup_{y, |x-y| \leq \frac{1}{2}|x|} \frac{|\Psi_2(y)|}{|\Psi_2(x)|} \leq 2. \quad (\text{E.9})$$

Both (E.8) and (E.9) will be convenient to simplify computations later on. We also record the following simple facts:

**Claim E.4.** *If  $\varepsilon$  is chosen smaller than some universal constant, then:*

- $|\psi_t|_1 \leq \frac{1}{5}$  (globally).
- For all  $x, x'$  we have:

$$\frac{1}{2}|x - x'| \leq |\Phi_t(x) - \Phi_t(x')| \leq 2|x - x'|.$$

- For all  $x$  we have:

$$|\psi_t|_{1,\star}(\Phi_t(x)) \leq 2\Psi_1(x), \quad |\psi_t|_{2,\star}(\Phi_t(x)) \leq 2\Psi_2(x). \quad (\text{E.10})$$

*Proof of Claim E.4.* The first item follows directly from Proposition E.1 (see in particular (E.3)). It implies the second item straightforwardly. To prove (E.10) let us recall that  $|\Phi_t(x) - x| \leq 2|t| \leq \frac{\ell}{5}$  (by the first item of Proposition E.1 and the assumption  $|t| \leq \frac{\ell}{10}$ ) and that  $|\psi_t|_{1,\star}(\Phi_t(x)) \leq \Psi_1(\Phi_t(x))$  by construction (similarly for  $|\psi_t|_{2,\star}, \Psi_2$ ). We thus have:

$$|\psi_t|_{1,\star}(\Phi_t(x)) \leq \Psi_1(\Phi_t(x)) \leq \sup_{|y-x| \leq \frac{\ell}{10}} \Psi_1(y),$$

which is smaller than  $2\Psi_1(x)$  according to (E.8), as desired.  $\square$

## E.2. The “well-spread” event

**Definition E.5** (The WellSpread event). Let  $\Omega$  be some subset of  $\mathbb{R}^2$ , let  $\ell \geq 1$  be a length-scale, let  $K \geq 10$ . We define the event  $\text{WellSpread}(\Omega, \ell, K)$  as follows:

$$\text{WellSpread}(\Omega, \ell, K) := \bigcap_{x \in (\ell\mathbb{Z})^2 \cap \Omega} \{ \text{Pts}(\mathbf{X}, \square(x, \ell)) \leq K\ell^2 \} \cap \{ \text{Ener}(\mathbf{X}, \square(x, \ell)) \leq K\ell^2 \}.$$

Saying that  $\mathbf{X} \in \text{WellSpread}(\Omega, \ell, K)$  essentially means that if we cover  $\Omega$  by squares of side-length  $\ell$ , then the number of points and the electric energy in each square are of order  $\ell^2$ , which is what we expect in view of the local laws. It has the following consequences:

1. If  $\mathbf{X} \in \text{WellSpread}(\Omega, \ell, K)$  then for each  $x \in (\ell\mathbb{Z})^2 \cap \Omega$  the quantity  $\text{Ener}_s(\mathbf{X}, \square(x, \ell))$  (as defined in (2.4)) is bounded by  $K\ell^2(1 + \log s)$  for  $s \in (0, 1)$ .
2. Let us say that a function  $f$  varies slowly at scale  $\ell$  when:

$$\sup_{|x'-x| \leq 2\ell} |f(x')| \leq 10|f(x)|. \quad (\text{E.11})$$

Then if  $f$  satisfies (E.11) and is supported in  $\Omega$ , assuming that  $\mathbf{X} \in \text{WellSpread}(\Omega, \ell, K)$  allows us to make two computational simplifications:

- a) An energy density upper bound, namely (for  $s \in (0, 1)$ ):

$$\left| \int f(x) |\nabla h_{s\bar{r}}|^2 dx \right| \leq K \|f\|_{L^1} (1 + |\log s|). \quad (\text{E.12})$$

b) Sum-integral comparisons:

$$\left| \sum_{x \in \mathbf{X}} f(x) dx \right| \preceq K \|f\|_{L^1}. \quad (\text{E.13})$$

(To prove (E.12) and (E.13), cover the support of  $f$  by squares of sidelength  $\ell$  and use (E.11)).

**Lemma E.6** (The Well-Spread event is frequent). *For  $K$  larger than some constant (depending on  $\beta$  and the “good external potential” constant  $\bar{\mathfrak{C}}$ ) and if  $\varepsilon \ell^2 \geq 1$  (which is guaranteed by (6.1)), we have:*

$$\mathbb{P}_{\mathbf{n}, \Lambda}^\beta (\text{WellSpread}(\Lambda^{\text{bulk}}, \ell, K) | \mathcal{E}_\Lambda) \geq 1 - \exp(-\ell^2) \quad (\text{E.14})$$

*Proof of Lemma E.6.* Covering  $\Lambda^{\text{bulk}}$  by  $\mathcal{O}(T^2 \ell^{-2})$  squares of sidelength  $\ell$  and using the local laws in  $\Lambda^{\text{bulk}}$  on each one, together with a union bound, we get:

$$\mathbb{P}_{\mathbf{n}, \Lambda}^\beta (\text{WellSpread}(\Lambda^{\text{bulk}}, \ell, K) | \mathcal{E}_\Lambda) \geq 1 - \mathfrak{C} \frac{T^2}{\ell^2} \exp\left(-\frac{K \ell^2}{\mathfrak{C}_\beta}\right).$$

Using the relation between  $T, \ell$  and  $\varepsilon$  as in (6.1), we obtain:

$$\mathbb{P}_{\mathbf{n}, \Lambda}^\beta (\text{WellSpread}(\Lambda^{\text{bulk}}, \ell, K) | \mathcal{E}_\Lambda) \geq 1 - \mathfrak{C} \exp\left(\frac{2}{\varepsilon} - \frac{K \ell^2}{\mathfrak{C}_\beta}\right),$$

and thus if we choose  $K$  larger than some constant (depending on  $\beta$  and  $\bar{\mathfrak{C}}$ ) and impose that  $\varepsilon \ell^2 \geq 1$  then (E.14) holds.  $\square$

### E.3. The measure-preserving case

Let  $|t| \leq \frac{\ell}{10}$  and let  $\Phi = \Phi_t^{\varepsilon, \ell}$  be the localized translation constructed in Section 6.2. Let  $\psi = \psi_t^{\varepsilon, \ell} = \Phi - \text{Id}$  as studied in Proposition E.1. In this section, we carefully inspect the proof of [Ser20, Prop 4.2] in order to obtain the following:

**Proposition E.7** (Energy comparison along a measure-preserving map). *Let  $\mathbf{X}$  be a point configuration with  $\mathbf{n}$  points in  $\Lambda$ . Assume that  $\mathbf{X}$  belongs to  $\text{WellSpread}(\Lambda^{\text{bulk}}, \ell, K)$  for a certain  $K > 1$ . Then:*

$$F_\Lambda(\Phi \cdot \mathbf{X}) = F_\Lambda(\mathbf{X}) + A_1[\mathbf{X}, \psi] + \mathcal{O}(K^2 t^2 \varepsilon \log \varepsilon).$$

The quantity  $A_1$  appears in the proof as a black box (see (E.31), we refer to Section E.4 for more details.

**Remark E.8** (Comparison with existing statements). Compared to the result of [Ser20] there are two modifications:

1. We estimate the energy cost of transporting by  $\Phi$  through the *local* density of electric energy density and points instead of using the global one (denoted by  $\Xi(t)$  in [Ser20]).
2. The quantity  $|\psi|_{L^\infty} |\psi|_{C^2}$  present in the control on the second derivative of the energy in [Ser20, Prop. 4.2] does not appear in our computations.

Both items are crucial for us. The localization allows us to bound the error in terms of  $\int_\Lambda |\psi|_{1, \star}^2$  instead of  $|\psi|_1^2 \times |\Lambda|$  - in the case of our localized translation the former is  $\mathcal{O}(\varepsilon)$  while the latter is gigantic. On the other hand, even after localizing, the contribution of  $|\psi|_{L^\infty} |\psi|_{C^2}$  would be of order 1 but not small<sup>17</sup> so it was necessary to get rid of it.

Obtaining these two refinements requires significant adaptations. On the other hand, the “measure-preserving” character of  $\Phi$  will bring several small simplifications: the background measure is not affected by the transport so all distinctions between  $\mu$  and  $\nu := \Phi \# \mu$  (using the notation of [Ser20]) are void. We will in particular repeatedly use the fact that (for various functions  $f$ ):

$$\int f(x) (\mathfrak{d}(\Phi \cdot \mathbf{X})(x) - dx) = \int f(\Phi(x)) (\mathfrak{d}\mathbf{X}(x) - dx).$$

<sup>17</sup>It is in fact impossible to make it small by choosing a different “localized translation”, because in dimension 2 the (homogeneous) Sobolev space  $\dot{W}^{2,1}$  is embedded in  $L^\infty$ . This is, fortunately, not the case for  $W^{1,2}$ .

Since we are working on a disk with a constant background, the logarithmic potential generated by said background is explicitly computable and given by:

$$V_\Lambda(x) := \int_\Lambda -\log|x-y|dy = \frac{|x|^2}{4}. \quad (\text{E.15})$$

We use the explicit expression (E.15) for simplicity a couple times below, although one could work with a more general shape and proceed to a more careful analysis instead.

*Proof of Proposition E.7.* We follow the steps of [Ser20, Appendix A] while making several important changes. We will only use a couple technical results as “black boxes” and copy or adapt all the main arguments and computations.

### E.3.1. Transporting vector fields

**Definition E.9** (Transport of vector fields). If  $v$  is a vector field on  $\mathbb{R}^2$  we define  $\Phi\#v$ , the “transport of  $v$  by  $\Phi$ ”, as:

$$\Phi\#v := (\text{D}\Phi \circ \Phi^{-1})^T v \circ \Phi^{-1}. \quad (\text{E.16})$$

The point is that when  $\text{div } v$  is a measure we have:

$$\text{div } \Phi\#v = \Phi\#(\text{div } v), \quad (\text{E.17})$$

where on the left-hand side there is a *transport of vector fields* while the right-hand side is a *push-forward of measures*. The identity (E.17) is a special case of [Ser20, Lemma A.3]. On top of it, we make the following simple observation:

**Claim E.10.** *If  $v = \nabla h$  is a gradient, then:*

$$\Phi\#\nabla h = (\text{D}\Phi \circ \Phi^{-1})^T \nabla (h \circ \Phi^{-1}) (\text{D}\Phi \circ \Phi^{-1}).$$

Thus if  $v$  is a gradient and  $\Phi$  is close to the identity map, then  $\Phi\#v$  is “almost” a gradient. The proof of Claim E.10 is straightforward using the definition (E.16) and some calculus.

### E.3.2. Setting up the energy comparison

We want to compare  $F_\Lambda(\mathbf{Y})$  to  $F_\Lambda(\mathbf{X})$ . We start by recalling known expressions for both quantities. Let  $\mathfrak{h}_{\vec{\eta}}^{\mathbf{X}}, \mathfrak{h}_{\vec{\eta}}^{\mathbf{Y}}$  be the true electric potentials generated by  $\mathbf{X}, \mathbf{Y}$  in  $\Lambda$  in the sense of Definition 2.1. As a truncation vector, let us choose<sup>18</sup>

$$\vec{\eta} := s\vec{r} \text{ with } s = \varepsilon^2 \quad (\text{E.18})$$

(in particular  $s \leq \frac{1}{10}$ ), the distances  $\vec{r}$  being computed with respect to the configuration  $\mathbf{X}$ . We recall that, by Claim E.4:

$$\frac{1}{2}|x_i - x_j| \leq |y_j - y_i| \leq 2|x_i - x_j|, \quad (\text{E.19})$$

and thus if we compute the nearest-neighbor distances  $\vec{r}$  with respect to  $\mathbf{Y}$  instead of  $\mathbf{X}$  we still have  $\vec{\eta} \leq 2s\vec{r} \leq \frac{1}{5}\vec{r}$ .

From Lemma 5.6 we know that:

$$F_\Lambda(\mathbf{X}) - F_\Lambda(\mathbf{Y}) = \frac{1}{4\pi} \left( \int |\nabla \mathfrak{h}_{\vec{\eta}}^{\mathbf{X}}|^2 - \int |\nabla \mathfrak{h}_{\vec{\eta}}^{\mathbf{Y}}|^2 \right),$$

where  $\mathfrak{h}^{\mathbf{X}} = \mathfrak{h}^{\mathbf{X}, \mathbf{m}}$ ,  $\mathfrak{h}^{\mathbf{Y}} := \mathfrak{h}^{\mathbf{Y}, \mathbf{m}}$  (the background will be  $\mathbf{m} = n\mu_W$  everywhere and we omit it) and  $\vec{\eta}$  is as in (E.18). We introduce two additional vector fields, using the notation of (E.16) for the first one:

$$E_{\vec{\eta}} := \Phi\#\nabla \mathfrak{h}_{\vec{\eta}}^{\mathbf{X}}, \quad \nabla \hat{h} := \nabla(-\log) * \left( \sum_{i=1}^n \Phi\#\delta_{x_i}^{(\eta_i)} - \mathbf{m} \right),$$

<sup>18</sup>Such a choice of a very small truncation parameter appears in [LS18; Ser20]. It might seem “unphysical” but it is very convenient to get rid of annoying error terms, while only costing  $\log s$  in view of (2.5).

(we recall that  $\delta_{x_i}^{(\eta_i)}$  denotes the uniform measure of mass 1 spread on the circle of center  $x_i$  and radius  $\eta_i$ ). To summarize we have, besides  $\nabla \mathfrak{h}_{\bar{\eta}}^{\mathbf{X}}$  which is the true electric field generated by  $\mathbf{X}$ ,

$$\begin{aligned}\nabla \mathfrak{h}_{\bar{\eta}}^{\mathbf{Y}} &= \nabla(-\log) \star \left( \sum_{i=1}^n \delta_{\Phi(x_i)}^{(\eta_i)} - \mathbf{m} \right) \\ \nabla \hat{h} &= \nabla(-\log) \star \left( \sum_{i=1}^n \Phi \# \delta_{x_i}^{(\eta_i)} - \mathbf{m} \right) \\ \mathbb{E}_{\bar{\eta}} &= \Phi \# \nabla(-\log) \star \left( \sum_{i=1}^n \delta_{x_i}^{(\eta_i)} - \mathbf{m} \right).\end{aligned}$$

There are subtle differences between these three vector fields:

- $\nabla \mathfrak{h}_{\bar{\eta}}^{\mathbf{Y}}$  is a gradient, it is the true electric field generated by  $\mathbf{Y}$ , and in its divergence the charges are spread along a circle with centers  $y_i = \Phi(x_i)$  ( $1 \leq i \leq n$ ).
- $\nabla \hat{h}$  is also a gradient, but the charges are spread along deformed circles  $\Phi \# \delta_{x_i}^{(\eta_i)}$  which are (approximately) ellipses of center  $y_i$ .
- $\mathbb{E}_{\bar{\eta}}$  is not a gradient in general, but it is obtained by transporting  $\nabla \mathfrak{h}_{\bar{\eta}}^{\mathbf{X}}$  (which is a gradient and for which the charges are spread along circles around the original points  $x_i$ 's) according to (B.2).

By the identity (E.17),  $\nabla \hat{h}$  and  $\mathbb{E}_{\bar{\eta}}$  have the same divergence, and as in [Ser20, (A.23)] we get the decomposition:

$$\int |\mathbb{E}_{\bar{\eta}}|^2 = \int |\nabla \hat{h}|^2 + \int |\mathbb{E}_{\bar{\eta}} - \nabla \hat{h}|^2.$$

**Bounding the difference  $\mathbb{E}_{\bar{\eta}} - \nabla \hat{h}$ .** We control the second term in the right-hand side immediately. Using Claim E.10 we see that:

$$\mathbb{E}_{\bar{\eta}} = (\mathbb{D}\Phi \circ \Phi^{-1})^T \nabla (\mathfrak{h}_{\bar{\eta}}^{\mathbf{X}} \circ \Phi^{-1}) (\mathbb{D}\Phi \circ \Phi^{-1}),$$

and thus since  $\Phi = \text{Id} + \psi$  with  $|\psi|_1$  smaller than  $\frac{1}{5}$  we have the pointwise bound:

$$|\mathbb{E}_{\bar{\eta}} - \nabla (\mathfrak{h}_{\bar{\eta}}^{\mathbf{X}} \circ \Phi^{-1})| \leq \mathfrak{C} \|\mathbb{D}\psi \circ \Phi^{-1}\| \times |\nabla \mathfrak{h}_{\bar{\eta}}^{\mathbf{X}} \circ \Phi^{-1}|,$$

which implies (after integrating the previous inequality and changing variables by  $\Phi$ ):

$$\int |\mathbb{E}_{\bar{\eta}} - \nabla (\mathfrak{h}_{\bar{\eta}}^{\mathbf{X}} \circ \Phi^{-1})|^2 \leq \mathfrak{C} \int |\psi|_{1,\star}^2 |\nabla \mathfrak{h}_{\bar{\eta}}^{\mathbf{X}}|^2.$$

This provides an upper bound on the  $L^2$ -distance between  $\mathbb{E}_{\bar{\eta}}$  and the space of gradients, and thus since  $\nabla \hat{h}$  is its projection onto that space we get:

$$\int |\mathbb{E}_{\bar{\eta}} - \nabla \hat{h}|^2 \leq \mathfrak{C} \int_{\Lambda} |\psi|_{1,\star}^2(x) |\nabla \mathfrak{h}_{\bar{\eta}}^{\mathbf{X}}|^2(x) dx.$$

This is the first moment where we will use the notation and simple facts of Section 6.2 combined with our WellSpread assumption. First, replacing  $|\psi|_{1,\star}$  by  $\Psi_1$  (as in (E.7)) provides an upper bound. Next, since  $\Psi_1$  has slow variations at scale  $\ell$  (see (E.8)) and we are working on  $\text{WellSpread}(\Lambda, \ell, K)$ , we may apply the energy density upper bound (E.12) (we will apply a similar chain of argument repeatedly in the rest of the proof). Here in conclusion, we have:

$$\int_{\Lambda} |\psi|_{1,\star}^2(x) |\nabla \mathfrak{h}_{\bar{\eta}}^{\mathbf{X}}|^2(x) dx \leq \int_{\Lambda} \Psi_1^2(x) |\nabla \mathfrak{h}_{\bar{\eta}}^{\mathbf{X}}|^2(x) dx \preceq K \left( \int_{\Lambda} \Psi_1^2(x) dx \right) (1 + |\log s|), \quad (\text{E.20})$$

which finally implies, after a direct estimate of the  $L^2$  norm of  $\Psi_1$  (see (E.7)), that:

$$\int |\mathbb{E}_{\bar{\eta}} - \nabla \hat{h}|^2 \leq \mathfrak{C} K t^2 \varepsilon (1 + |\log s|) = \mathcal{O}(K t^2 \varepsilon \log \varepsilon), \quad (\text{E.21})$$

because we have chosen  $s = \varepsilon^2$  in (E.18). Going back to [Ser20, (A.25)], and inserting (E.21) we write<sup>19</sup>:

$$F_\Lambda(\mathbf{Y}) - F_\Lambda(\mathbf{X}) = \text{Main} + \text{Rem} + \mathcal{O}(Kt^2\varepsilon \log \varepsilon)$$

where Main, Rem are given by:

$$\text{Main} := \frac{1}{4\pi} \left( \int |\mathbf{E}_{\vec{\eta}}|^2 - \int |\nabla \mathfrak{h}_{\vec{\eta}}^{\mathbf{X}}|^2 \right), \quad \text{Rem} := \frac{1}{4\pi} \left( \int |\nabla \mathfrak{h}_{\vec{\eta}}^{\mathbf{Y}}|^2 - \int |\nabla \hat{h}|^2 \right).$$

### E.3.3. The Main term

For the term Main, a direct expansion of  $\mathbf{E}_{\vec{\eta}} = \Phi \# \nabla \mathfrak{h}_{\vec{\eta}}^{\mathbf{X}}$  using the definition (E.16) gives:

$$\text{Main} = \frac{1}{4\pi} \int \langle \nabla \mathfrak{h}_{\vec{\eta}}^{\mathbf{X}}, 2\mathbf{D}\psi \nabla \mathfrak{h}_{\vec{\eta}}^{\mathbf{X}} \rangle + \int \mathcal{O}(|\psi|_{1,\star}^2) |\nabla \mathfrak{h}_{\vec{\eta}}^{\mathbf{X}}|^2. \quad (\text{E.22})$$

This is consistent with [Ser20, (A.31)], the improvement being that we have no contribution of the form  $|\psi|_{L^\infty} |\psi|_{C^2}$  in the second order term thanks to the fact that  $\Phi$  is measure-preserving. Let us also note that although  $\Phi$  is measure-preserving, we may not have  $\text{div} \psi = 0$ , however it is true that  $\text{div} \psi = \mathcal{O}(|\psi|_{1,\star}^2)$  pointwise and thus the  $-\text{div} \psi$  term appearing in [Ser20, (A.32)] can be absorbed in our second order correction as in (E.22). Arguing as in (E.20) we may re-write the error term in (E.22) and get:

$$\text{Main} = \frac{1}{4\pi} \int \langle \nabla \mathfrak{h}_{\vec{\eta}}^{\mathbf{X}}, 2\mathbf{D}\psi \nabla \mathfrak{h}_{\vec{\eta}}^{\mathbf{X}} \rangle + \mathcal{O}(Kt^2\varepsilon \log \varepsilon).$$

In this step we see how the  $H^1$  norm of  $\psi$  appears as a second-order contribution to the energy change, and since by construction we have  $\int |\psi|_{1,\star}^2 = \mathcal{O}(\varepsilon)$  we may indeed hope (if Main is indeed the “main” term) to have a *small* energy cost.

### E.3.4. The Rem term

The Rem term is due to the difference between the electric fields  $\nabla \mathfrak{h}_{\vec{\eta}}^{\mathbf{Y}}$  (for which charges are spread along circles) and  $\nabla \hat{h}$  (for which charges are spread along approximate ellipses). The fact that we can choose *small* truncations via the parameter  $s$  (we which recall to have chosen as  $s = \varepsilon^2$  in (E.18), we could even have used an higher power of  $\varepsilon$ ) will turn out to be crucial in order to control those errors. Let us keep the notation of [Ser20] and use:

- $\delta_{y_i}^{(\eta_i)}$  to denote the charge spread uniformly on the circle of center  $y_i = \Phi(x_i)$  and radius  $\eta_i$ ,
- $\hat{\delta}_{y_i}$  to denote the push-forward by  $\Phi$  of the measure  $\delta_{x_i}^{(\eta_i)}$ .

We also write  $v_i$  for the function

$$v_i := -\log \star \left( \hat{\delta}_{y_i} - \delta_{y_i}^{(\eta_i)} \right). \quad (\text{E.23})$$

As in [Ser20, (A.41)] we decompose Rem as  $\text{Rem}_1 + \text{Rem}_2 + \text{Rem}_3$  and analyse each term separately.

**The  $\text{Rem}_1$  term.**  $\text{Rem}_1$  is defined as:

$$\text{Rem}_1 = -\frac{1}{2} \sum_{i=1}^n \int v_i \left( \hat{\delta}_{y_i} + \delta_{y_i}^{(\eta_i)} \right).$$

We write as in [Ser20, (A.42)] (with  $\vec{n}$  the unit normal vector to the circle)

$$\text{Rem}_1 = \frac{1}{2} \sum_{i=1}^n \frac{1}{\eta_i} \int_{\partial \mathfrak{D}(y_i, \eta_i)} (\psi(y) - \psi(y_i)) \cdot \vec{n} + \mathcal{O} \left( \sum_{i=1}^n |\psi|_{1,\text{loc}}^2(y_i) \right).$$

<sup>19</sup>Let us observe that the additional term Err appearing in [Ser20, (A.23)] is 0 in our case because  $\Phi$  is measure-preserving and thus, with the notation of [Ser20],  $\nu = \mu$ .



Let us use  $\Psi_1$  as an upper bound to the derivative of  $\psi$ , we have:

$$\sum_{i=1}^n |\psi|_{1,\text{loc}}^2(y_i) \preceq \sum_{i=1}^n (\Psi_1)^2(x_i),$$

Moreover, a Taylor's expansion yields:

$$\psi(y) - \psi(y_i) = \mathbf{D}\psi(y_i)(y - y_i) + \mathcal{O}(|\psi|_{2,\text{loc}}(y_i)\eta_i^2),$$

but since  $\Phi$  is measure-preserving we know that  $\text{div } \psi(y_i) = \mathcal{O}(|\psi|_{1,\star}^2(y_i))$ , thus:

$$\frac{1}{2} \sum_{i=1}^n \frac{1}{\eta_i} \int_{\partial \mathfrak{D}(y_i, \eta_i)} (\psi(y) - \psi(y_i)) \cdot \vec{n} = \mathcal{O}(s\Psi_2(x_i) + \Psi_1^2(x_i)).$$

Since  $\Psi_1, \Psi_2$  have slow variations and since we are working under the `WellSpread` assumption we may compare sums to integrals as in (E.13), hence:

$$\sum_{i=1}^n (\Psi_1)^2(x_i) + s\Psi_2(x_i) \leq K \int \Psi_1^2(x) dx + Kt^2s = \mathcal{O}(Kt^2\varepsilon).$$

We thus obtain:

$$\text{Rem}_1 = \mathcal{O}(Kt^2\varepsilon).$$

**The  $\text{Rem}_2$  term.**  $\text{Rem}_2$  is defined as (see (E.23))

$$\text{Rem}_2 := \frac{1}{2} \sum_{1 \leq i \neq j \leq n} \int v_i \left( \hat{\delta}_{y_j} - \delta_{y_j}^{(\eta_j)} \right).$$

We write as in [Ser20, (A.45)]:

$$\left| \sum_{1 \leq i \neq j \leq n} \int v_i \left( \hat{\delta}_{y_j} - \delta_{y_j}^{(\eta_j)} \right) \right| \leq \sum_{1 \leq i \neq j \leq n} \eta_i^2 \eta_j \left( \frac{|\psi|_{1,\text{loc}}^2(x_i)}{|y_i - y_j|^3} + \frac{|\psi|_{1,\text{loc}}(x_i) |\psi|_{2,\text{loc}}(x_i)}{|y_i - y_j|^2} \right),$$

and then proceed a bit differently.

First, we use  $\Psi_1, \Psi_2$  as upper bounds to the derivatives of  $\psi$ , use (E.19) and (E.9) in order to replace the  $y_i, y_j$ 's in the right-hand side by the corresponding  $x_i, x_j$ 's up to some multiplicative constant. Next, we decompose (as in [Ser20]) the sum between contributions coming from ‘‘close’’ and ‘‘far away’’ pairs of points.

**Distances smaller than  $10\ell$ .** Since we always have  $|x_i - x_j|^3 \geq r_i^2 r_j$  and since we take the truncation  $\eta_i = sr_i$  we may write for each fixed  $i$

$$\begin{aligned} \sum_{1 \leq j \leq n, j \neq i, |x_j - x_i| \leq 10\ell} \eta_i^2 \eta_j \left( \frac{\Psi_1^2(x_i)}{|x_i - x_j|^3} + \frac{\Psi_1(x_i) \Psi_2(x_i)}{|x_i - x_j|^2} \right) \\ \leq s^3 (\Psi_1^2(x_i) + \Psi_1(x_i) \Psi_2(x_i)) \times \#\{j, |x_j - x_i| \leq 10\ell\}. \end{aligned}$$

Since we condition on `WellSpread`( $\Lambda^{\text{bulk}}, \ell, K$ ) we may bound  $\#\{j, |x_j - x_i| \leq 10\ell\}$  by  $CK\ell^2$ , and thus:

$$s^3 (\Psi_1^2(x_i) + \Psi_1(x_i) \Psi_2(x_i)) \times \#\{j, |x_j - x_i| \leq 10\ell\} \preceq s^3 K\ell^2 (\Psi_1^2(x_i) + \Psi_1(x_i) \Psi_2(x_i)).$$

Next we compare the sum (over  $i$ ) of the previous quantity to an integral using (E.13) and get:

$$\sum_{1 \leq j \leq n, j \neq i, |x_j - x_i| \leq 10\ell} \eta_i^2 \eta_j \left( \frac{\Psi_1^2(x_i)}{|x_i - x_j|^3} + \frac{\Psi_1(x_i) \Psi_2(x_i)}{|x_i - x_j|^2} \right) \leq Cs^3 K^2 \ell^2 \int (\Psi_1^2(x) + \Psi_1(x) \Psi_2(x)) dx,$$

and the right-hand side can be evaluated using (E.7)

$$s^3 K^2 \ell^2 \int (\Psi_1^2(x) + \Psi_1(x) \Psi_2(x)) dx = \mathcal{O}(s^3 K^2 \ell^2 \varepsilon t^2). \quad (\text{E.24})$$

**Remark E.11.** We do lose some information when replacing all distances by the smallest one over a large zone of size  $\ell$ , but for technical reasons it seemed hard to do much better, and it works well enough for us.

**Distances larger than  $10\ell$ .** The function:  $z \mapsto \frac{1}{|z|^3}$  “varies slowly at scale  $\ell$ ” (in the sense of (E.11)) on  $\{|z| \geq 10\ell\}$  thus using (E.13) for each fixed  $i$  we can compare

$$\sum_{1 \leq j \leq n, j \neq i, |x_j - x_i| \geq 10\ell} \frac{1}{|x_i - x_j|^3}$$

to  $K$  times the corresponding integral, namely:

$$\int_{\{|x - x_i| \geq 10\ell\}} \frac{1}{|x_i - x|^3} dx = \mathcal{O}(1).$$

Similarly the sum  $\sum_{1 \leq j \leq n, |x_j - x_i| \geq 10\ell} \frac{1}{|x_i - x_j|^2}$  can be compared to:

$$K \times \int_{\Lambda \cap \{|x - x_i| \geq 10\ell\}} \frac{1}{|x_i - x|^2} dx = K \times \mathcal{O}(\log T).$$

Using the obvious bound  $\eta_i^2 \eta_j \leq s^3$  we thus obtain:

$$\begin{aligned} A := \sum_{1 \leq j \leq n, |x_j - x_i| \geq 10\ell} \eta_i^2 \eta_j & \left( \frac{|\psi|_{1,\text{loc}}^2(x_i)}{|x_i - x_j|^3} + \frac{|\psi|_{1,\text{loc}}(x_i) |\psi|_{2,X}(x_i)}{|x_i - x_j|^2} \right) \\ & \leq s^3 K \sum_{1 \leq i \leq n} |\psi|_{1,\text{loc}}^2(x_i) + |\psi|_{1,\text{loc}}(x_i) |\psi|_{2,\text{loc}}(x_i) \log T. \end{aligned}$$

Using again  $\Psi_1, \Psi_2$  instead and comparing again the sum to an integral, we obtain:

$$A \leq s^3 K^2 \left( \int \Psi_1^2(x) + \log T \Psi_1(x) \Psi_2(x) dx \right),$$

which can be evaluated using (E.7) (and (6.1)):

$$A \leq s^3 K^2 \left( \int \Psi_1^2(x) + \log T \Psi_1(x) \Psi_2(x) dx \right) \leq s^3 K^2 (\varepsilon t^2 + \varepsilon^2 t^2 \log T) = \mathcal{O}(s^3 K^2 \varepsilon \log \varepsilon t^2) \quad (\text{E.25})$$

**Conclusion for  $\text{Rem}_2$ .** Combining (E.24) and (E.25) and discarding negligible terms we get:

$$\text{Rem}_2 = \mathcal{O}(s^3 K^2 \ell^2 \varepsilon \log \varepsilon t^2) = \mathcal{O}(K^2 \varepsilon \log \varepsilon t^2),$$

where we have used (6.1) to simplify the expression.

**The  $\text{Rem}_3$  term.**  $\text{Rem}_3$  is defined as ( $\mathbf{m}$  is outside of the sum over  $j$ ):

$$\text{Rem}_3 := - \sum_{1 \leq i \leq n} \int_{\Lambda} v_i \left( \sum_{1 \leq j \leq n, j \neq i} \hat{\delta}_{y_j} - \mathbf{m} \right).$$

Writing  $\hat{\delta}_{y_j} - \mathbf{m} = \hat{\delta}_{y_j} - \delta_{y_j}^{(\eta_j)} + \delta_{y_j}^{(\eta_j)} - \mathbf{m}$  and recalling that  $v_i := -\log \star (\hat{\delta}_{y_i} - \delta_{y_i}^{(\eta_i)})$ , one can express  $\text{Rem}_3$  (as in [Ser20, Substep (5.3)]) as:

$$\text{Rem}_3 = \sum_{1 \leq i \leq n} \int \tilde{\mathfrak{h}}_i^Y \left( \delta_{y_i}^{(\eta_i)} - \hat{\delta}_{y_i} \right).$$

We recall that the truncated field  $\tilde{\mathfrak{h}}_{\bar{\eta}}$  coincides with  $\tilde{\mathfrak{h}}_i$  inside the  $i^{\text{th}}$  spread out charge.

The analysis of [Ser20, Step 4.] shows that, for each  $i$  we have, as in [Ser20, Substep 5.3]:

$$\int \tilde{\mathfrak{h}}_i^Y \left( \delta_{y_i}^{(\eta_i)} - \hat{\delta}_{y_i} \right) = \int_{|u|=\eta_i} \nabla \tilde{\mathfrak{h}}_i^Y(y_i + u) \cdot (\psi(x_i + u) - \psi(x_i)) \, \mathbf{d}u + \mathcal{O} \left( |\tilde{\mathfrak{h}}_i^Y|_{C^2(B(y_i, 2\eta_i))} \eta_i^2 |\psi|_{1, \text{loc}}^2(x_i) \right). \quad (\text{E.26})$$

Here and below we temporarily borrow the notation of [Ser20] for local controls on derivatives of  $\tilde{\mathfrak{h}}_i^Y$ , namely:

$$|\tilde{\mathfrak{h}}_i^Y|_{C^k(B(x, r))} = L^\infty \text{ norm of the } k\text{-th derivative of } \tilde{\mathfrak{h}}_i^Y \text{ over the ball/disk of center } x \text{ and radius } r.$$

We cannot use (E.26) as such because of the various dependencies in  $Y$ , which we need to analyse.

### Step 1: Preliminary claims

**Claim E.12.** [Variation of the first derivatives] For all  $a \in \{1, 2\}$ ,  $1 \leq i \leq n$  and  $|u| \leq \eta_i$  we have:

$$|\partial_a \tilde{\mathfrak{h}}_i^Y(y_i + u) - \partial_a \tilde{\mathfrak{h}}_i^X(x_i + u)| \leq K \left( \frac{1}{r_i} \Psi_1(x_i) \ell^2 + |\psi|_0 \log T \right)$$

*Proof of Claim E.12.* We have by definition:

$$\begin{aligned} \partial_a \tilde{\mathfrak{h}}_i^Y(y_i + u) - \partial_a \tilde{\mathfrak{h}}_i^X(x_i + u) &= \sum_{1 \leq j \leq n, j \neq i} \partial_a(-\log)(y_j - (y_i + u)) - \partial_a(-\log)(x_j - (x_i + u)) \\ &\quad - \partial_a V_\Lambda(y_i + u) + \partial_a V_\Lambda(x_i + u), \end{aligned}$$

where  $V_\Lambda$  is the logarithmic potential generated by the background measure on  $\Lambda$ , whose expression is given in (E.15). Since  $y_i = x_i + \psi(x_i)$  the difference  $\partial_a V_\Lambda(y_i + u) - \partial_a V_\Lambda(x_i + u)$  is easily bounded by  $|\psi|_0$ . We now focus on the contribution coming from the point particles. We have:

$$|\partial_a \tilde{\mathfrak{h}}_i^Y(y_i + u) - \partial_a \tilde{\mathfrak{h}}_i^X(x_i + u)| \leq \sum_{1 \leq j \leq n, j \neq i} \frac{|\psi(x_i) - \psi(x_j)|}{|x_i - x_j|^2}.$$

Let us split the sum into two parts corresponding to ‘‘close’’ and ‘‘far away’’ pairs of points.

**Distances smaller than  $10\ell$ .** On the one hand we have:

$$\sum_{j, |x_j - x_i| \leq 10\ell, j \neq i} \frac{|\psi(x_i) - \psi(x_j)|}{|x_i - x_j|^2} \leq \frac{1}{r_i} \Psi_1(x_i) \times \# \{j, |x_j - x_i| \leq 10\ell\} \leq K \frac{1}{r_i} \Psi_1(x_i) \ell^2.$$

To be precise, here we have used a mean value argument to argue that:

$$\frac{|\psi(x_i) - \psi(x_j)|}{|x_i - x_j|} \leq \sup_{x \in [x_i, x_j]} |\psi|_{1, \star}(x),$$

then we wrote  $|\psi|_{1, \star}(x) \leq \Psi_1(x)$  and finally we used (E.8). Then we applied the local control on the number of points implied by  $\text{WellSpread}(\Lambda^{\text{bulk}}, \ell, K)$ .

**Distances larger than  $10\ell$ .** On the other hand:

$$\sum_{|x_j - x_i| \geq 10\ell} \frac{|\psi(x_i) - \psi(x_j)|}{|x_i - x_j|^2} \leq K |\psi|_0 \int_{|x - x_i| \geq 10\ell, x \in \Lambda} \frac{1}{|x - x_i|^2} \, \mathbf{d}x \leq K |\psi|_0 \log T.$$

This time we simply bounded  $|\psi(x_i) - \psi(x_j)|$  by  $2|\psi|_0$  and used the fact that  $z \mapsto \frac{1}{|z|^2}$  ‘‘varies slowly at scale  $\ell$ ’’ (in the sense of (E.11)) on  $\{|z| \geq 10\ell\}$  in order to compare the sum to an integral as in (E.13).

Combining those estimates proves the claim.  $\square$

**Claim E.13** (Variation of the second derivatives). *For all  $a, b \in \{1, 2\}$ ,  $1 \leq i \leq n$  and  $|u| \leq \eta_i$  we have:*

$$|\partial_{ab}\tilde{h}_i^Y(y_i + u) - \partial_{ab}\tilde{h}_i^X(x_i + u)| \leq K \left( \frac{1}{r_i^2} \ell^2 \Psi_1(x_i) + \Psi_1(x_i) \log T + |\psi|_0 \frac{1}{\ell + |x_i|} \right).$$

*Proof of Claim E.13.* The proof is similar to Claim E.12. We have by definition:

$$\begin{aligned} \partial_{ab}\tilde{h}_i^Y(y_i + u) - \partial_{ab}\tilde{h}_i^X(x_i + u) &= \sum_{1 \leq j \leq n, j \neq i} \partial_{ab}(-\log)(y_j - (y_i + u)) - \partial_{ab}(-\log)(x_j - (x_i + u)) \\ &\quad - \partial_{ab}V_\Lambda(y_i + u) + \partial_{ab}V_\Lambda(x_i + u), \end{aligned}$$

however the second derivatives of  $V_\Lambda$  are *constant* (see (E.15)) so we can discard those terms. Writing  $y = \Phi(x) = x + \psi(x)$ , we get:

$$|\partial_{ab}\tilde{h}_i^Y(y_i + u) - \partial_{ab}\tilde{h}_i^X(x_i + u)| \leq \sum_{1 \leq j \leq n, j \neq i} \frac{|\psi(x_i) - \psi(x_j)|}{|x_i - x_j|^3}.$$

We now split the sum into three parts:  $|x_j - x_i| \leq \ell$ ,  $|x_j - x_i| \leq \frac{1}{2}|x_i|$  and  $|x_j - x_i| \geq (\ell \cup \frac{1}{2}|x_i|)$ .

**Distances smaller than  $10\ell$ .** Arguing as in the proof of Claim E.12 we get:

$$\sum_{1 \leq j \leq n, j \neq i} \frac{|\psi(x_i) - \psi(x_j)|}{|x_i - x_j|^3} \leq K \frac{1}{r_i^2} \ell^2 \Psi_1(x_i).$$

**Distances between  $10\ell$  and  $\frac{1}{2}|x_i|$ .** For  $10\ell \leq |x_j - x_i| \leq \frac{1}{2}|x_i|$  we write by a mean value argument:

$$\frac{|\psi(x_i) - \psi(x_j)|}{|x_i - x_j|^3} \leq \sup_{x \in [x_i, x_j]} |\psi|_{1, \star}(x) \times \frac{1}{|x_j - x_i|^2},$$

then we may again replace  $|\psi|_{1, \star}(x)$  by  $\Psi_1(x)$  and use property (E.9) to bound it by  $\Psi_1(x_i)$  up to some multiplicative constant. Next, comparing a sum to an integral, we have:

$$\sum_{j, |x_j - x_i| \geq 10\ell} \frac{1}{|x_i - x_j|^2} \leq K \log T.$$

In conclusion, we get for fixed  $i$ :

$$\sum_{1 \leq j \leq n, j \neq i, 10\ell \leq |x_j - x_i| \leq \frac{1}{2}|x_i|} \frac{|\psi(x_i) - \psi(x_j)|}{|x_i - x_j|^3} \leq K \Psi_1(x_i) \log T.$$

**Large distances.** To estimate the remaining contribution, we write:

$$\sum_{1 \leq j \leq n, |x_j - x_i| \geq \max(10\ell, \frac{1}{2}|x_i|)} \frac{|\psi(x_i) - \psi(x_j)|}{|x_i - x_j|^3} \leq K |\psi|_0 \int_{|x - x_i| \geq \max(10\ell, \frac{1}{2}|x_i|)} \frac{1}{|x - x_i|^3} dx \leq K |\psi|_0 \frac{1}{|x_i| + \ell},$$

where we compared the sum to an integral using (E.13).

Combining all three estimates proves the claim.  $\square$

**Remark E.14.** In Claims E.12 and E.13 we have studied the contributions coming from the point particles and the background separately. Taking cancellations between those terms into account would yield more accurate estimates, but we do not need them here.

**Step 2: Studying the first-order term.** Let us recall that we are still trying to express  $\text{Rem}_3$  purely in terms of the original points  $(x_1, \dots, x_n)$ .

Going back to (E.26), let us write the first-order term as:

$$\begin{aligned} \int_{|u|=\eta_i} \nabla \tilde{h}_i^Y(y_i + u) \cdot (\psi(x_i + u) - \psi(x_i)) \, \mathbf{d}u &= \int_{|u|=\eta_i} \nabla \tilde{h}_i^X(x_i + u) \cdot (\psi(x_i + u) - \psi(x_i)) \, \mathbf{d}u \\ &+ \int_{|u|=\eta_i} (\nabla \tilde{h}_i^Y(y_i + u) - \nabla \tilde{h}_i^X(x_i + u)) \cdot (\psi(x_i + u) - \psi(x_i)) \, \mathbf{d}u. \end{aligned}$$

We keep the first term in the right-hand side as such and we focus on the second one, which we decompose as:

$$\begin{aligned} \int_{|u|=\eta_i} (\nabla \tilde{h}_i^Y(y_i + u) - \nabla \tilde{h}_i^X(x_i + u)) \cdot (\psi(x_i + u) - \psi(x_i)) \, \mathbf{d}u \\ = \int_{|u|=\eta_i} (\nabla \tilde{h}_i^Y(y_i) - \nabla \tilde{h}_i^X(x_i)) \cdot (\psi(x_i + u) - \psi(x_i)) \, \mathbf{d}u \\ + \int_{|u|=\eta_i} ((\nabla \tilde{h}_i^Y(y_i + u) - \nabla \tilde{h}_i^X(x_i + u)) - (\nabla \tilde{h}_i^Y(y_i) - \nabla \tilde{h}_i^X(x_i))) \cdot (\psi(x_i + u) - \psi(x_i)) \, \mathbf{d}u. \quad (\text{E.27}) \end{aligned}$$

Let us also write that  $\psi(x_i + u) - \psi(x_i) = \mathbf{D}\psi(x_i) \times u + \mathcal{O}(|\psi|_{2,\text{loc}}(x_i)\eta_i^2)$ , and observe that according to Claim E.13 we have (for  $|u| = \eta_i$ )

$$|(\nabla \tilde{h}_i^Y(y_i + u) - \nabla \tilde{h}_i^X(x_i + u)) - (\nabla \tilde{h}_i^Y(y_i) - \nabla \tilde{h}_i^X(x_i))| \preceq \eta_i \times K \left( \frac{1}{r_i^2} \ell^2 \Psi_1(x_i) + \Psi_1(x_i) \log T + |\psi|_0 \frac{1}{\ell + |x_i|} \right).$$

We may thus re-write the right-hand side of (E.27) as:

$$\begin{aligned} \int_{|u|=\eta_i} (\nabla \tilde{h}_i^Y(y_i) - \nabla \tilde{h}_i^X(x_i)) \cdot \mathbf{D}\psi(x_i) u \, \mathbf{d}u + \int_{|u|=\eta_i} (\nabla \tilde{h}_i^Y(y_i) - \nabla \tilde{h}_i^X(x_i)) \times \mathcal{O}(|\psi|_{2,\text{loc}}(x_i)\eta_i^2) \, \mathbf{d}u \\ + \mathcal{O} \left( \eta_i \times K \left( \frac{1}{r_i^2} \ell^2 \Psi_1(x_i) + \Psi_1(x_i) \log T + |\psi|_0 \frac{1}{\ell + |x_i|} \right) \right) \times |\psi|_{1,\text{loc}}(x_i) \times \eta_i. \end{aligned}$$

The first term vanishes by symmetry, and we can bound the second term further using Claim E.12. In conclusion, we obtain (using that  $\eta_i \leq sr_i \leq s$ ):

$$\begin{aligned} \int_{|u|=\eta_i} \nabla \tilde{h}_i^Y(y_i + u) \cdot (\psi(x_i + u) - \psi(x_i)) \, \mathbf{d}u &= \int_{|u|=\eta_i} \nabla \tilde{h}_i^X(x_i + u) \cdot (\psi(x_i + u) - \psi(x_i)) \, \mathbf{d}u \\ &+ K \left( s^2 \Psi_1(x_i) \Psi_2(x_i) \ell^2 + s^2 |\psi|_0 \Psi_2(x_i) \log T + s^2 \Psi_1^2(x_i) (\ell^2 + \log T) + s^2 |\psi|_0 \Psi_1(x_i) \frac{1}{\ell + |x_i|} \right). \quad (\text{E.28}) \end{aligned}$$

Summing the error term in (E.28) over  $i$  yields:

$$\begin{aligned} \sum_{i=1}^n K \left( s^2 \Psi_1(x_i) \Psi_2(x_i) \ell^2 + s^2 |\psi|_0 \Psi_2(x_i) \log T + s^2 \Psi_1^2(x_i) (\ell^2 + \log T) + s^2 |\psi|_0 \Psi_1(x_i) \frac{1}{\ell + |x_i|} \right) \\ \leq K^2 \left( s^2 \ell^2 \int \Psi_1(x) \Psi_2(x) \, \mathbf{d}x + s^2 |\psi|_0 \log T \int \Psi_2(x) \, \mathbf{d}x + s^2 (\ell^2 + \log T) \int \Psi_1^2(x) \, \mathbf{d}x + s^2 |\psi|_0 \int \Psi_1(x) \frac{1}{\ell + |x|} \, \mathbf{d}x \right) \\ \leq K^2 (s^2 \ell^2 \varepsilon^2 t^2 + s^2 t^2 \log T + s^2 (\ell^2 + \log T) \varepsilon t^2 + s^2 t^2) = \mathcal{O}(K^2 \varepsilon t^2), \end{aligned}$$

where we have used (6.1) to simplify the expression involving  $T, \ell, \varepsilon$  and  $s$  (we recall that  $s = \varepsilon^2$ ).

**Step 3: Re-writing the error term.** The error term in (E.26) involves  $|\tilde{h}_i^Y|_{C^2(B(y_i, 2\eta_i))}$ , which is expressed in terms of the *transported* points and thus remains an issue for us. Using Claim E.13 however, we see that:

$$|\tilde{h}_i^Y|_{C^2(B(y_i, 2\eta_i))} \leq |\tilde{h}_i^X|_{C^2(B(x_i, 2\eta_i))} + K \mathcal{O} \left( \frac{1}{r_i^2} \ell^2 |\psi|_{1,\text{loc}}(x_i) + |\psi|_{1,\text{loc}}(x_i) \log T + |\psi|_0 \frac{1}{\ell + |x_i|} \right). \quad (\text{E.29})$$

Now, the analysis of [Ser20, Lemma A.2] gives:

$$|\tilde{h}_i^X|_{C^2(B(x_i, 2\eta_i))} \preceq \frac{1}{r_i^2} \left( 1 + \int_{\mathfrak{D}(x_i, r_i)} |\nabla \tilde{h}_i^X|^2 \right). \quad (\text{E.30})$$

**Remark E.15.** The proof of (E.30) uses the fact that  $\tilde{h}_i^X$  is almost harmonic on the disk  $\mathfrak{D}(x_i, r_i)$ , and is simplified by the fact that the background measure ( $\mu$  in the notation of [Ser20]) is here constant (beware: when reading [Ser20, (A.5), (A.6)], our background measure corresponds to  $N\mu$  and not  $\mu$  - compare [Ser20, (3.1)] with our Definition 2.1. Also, the nearest-neighbor distances in [Ser20] are of order  $N^{-1/2}$  where ours are of order one, this is due to a different choice of scaling, but [Ser20, Lemma A.2] is valid in any case).

Combining (E.29) and (E.30) the error term appearing in (E.26) can be re-written as:

$$\begin{aligned} |\tilde{h}_i^Y|_{C^2(B(y_i, 2\eta_i))} \eta_i^2 |\psi|_{1, \text{loc}}^2(x_i) &\preceq \Psi_1^2(x_i) \frac{\eta_i^2}{r_i^2} \left( 1 + \int_{\mathfrak{D}(x_i, r_i)} |\nabla \tilde{h}_i^X|^2 \right) \\ &\quad + K \Psi_1^2(x_i) \eta_i^2 \left( \frac{1}{r_i^2} \ell^2 \Psi_1(x_i) + \Psi_1(x_i) \log T + |\psi|_0 \frac{1}{\ell + |x_i|} \right), \end{aligned}$$

Summing over  $1 \leq i \leq n$ , using the WellSpread assumption, comparing to an integral and using again that  $\eta_i \leq sr_i$  we obtain:

$$\begin{aligned} \sum_{1 \leq i \leq n} |\tilde{h}_i^Y|_{C^2(B(y_i, 2\eta_i))} \eta_i^2 |\psi|_{1, \text{loc}}^2(x_i) \\ \preceq K(1 + |\log s|) s^2 \int \Psi_1^2(x) dx + K^2 s^2 \left( (\ell^2 + \log T) \int_{\Lambda} \Psi_1^3(x) dx + |\psi|_0 \int_{\Lambda} \Psi_1^2(x) \frac{1}{\ell + |x|} dx \right) \end{aligned}$$

Estimating everything explicitly and keeping only the dominant term we may thus control the error term in (E.26) by:

$$\sum_{1 \leq i \leq n} |\tilde{h}_i^Y|_{C^2(B(y_i, 2\eta_i))} \eta_i^2 |\psi|_{1, \text{loc}}^2(x_i) = \mathcal{O}(K^2(1 + \log |s|) s^2 \ell^2 \varepsilon t^2) = \mathcal{O}(K^2 \varepsilon \log \varepsilon t^2),$$

where we have used that  $s = \varepsilon^2$  and (6.1) again.

**Step 4: Conclusion for Rem<sub>3</sub>.** In conclusion we obtained that:

$$\text{Rem}_3 = \sum_{1 \leq i \leq n} \int_{|u|=\eta_i} \nabla \tilde{h}_i^X(x_i + u) \cdot (\psi(x_i + u) - \psi(x_i)) du + \mathcal{O}(K^2 \varepsilon \log \varepsilon t^2).$$

### E.3.5. Concluding the proof of Proposition E.7

Combining all the previous steps, we find that:

$$F_{\Lambda}(\Phi \cdot \mathbf{X}) = F_{\Lambda}(\mathbf{X}) + A_1(\mathbf{X}, \psi) + \mathcal{O}(K^2 \varepsilon \log \varepsilon t^2),$$

where the ‘‘anisotropy’’ term  $A_1$  is defined as the sum of the linear (in  $\psi$ ) terms obtained in Main and Rem<sub>3</sub>, namely:

$$A_1(\mathbf{X}, \psi) := \frac{1}{4\pi} \int \langle \nabla \tilde{h}_{\tilde{\eta}}^X, 2D\psi \nabla \tilde{h}_{\tilde{\eta}}^X \rangle + \sum_{1 \leq i \leq n} \int_{|u|=\eta_i} \nabla \tilde{h}_i^X(x_i + u) \cdot (\psi(x_i + u) - \psi(x_i)) du. \quad (\text{E.31})$$

There was a linear term appearing in Rem<sub>1</sub> but it was found to be negligible, and comparing (E.31) to [Ser20, (4.8)], the reader might observe that there is another term missing (the last term in [Ser20, (4.8)]), in fact for us this term is  $\mathcal{O}(K\varepsilon t^2)$  and can thus be incorporated in the error term. This is due to the fact that our  $\Phi$  is measure-preserving.

This concludes the proof of Proposition E.7.  $\square$

## E.4. Smallness of the anisotropy

Applying the result of Proposition E.7 to  $\Phi_t$  and  $\Phi_{-t}$  we obtain that if  $\mathbf{X}$  is in  $\text{WellSpread}(\Lambda^{\text{bulk}}, \ell, K)$  we have:

$$\frac{1}{2} (F_\Lambda(\Phi_t \mathbf{X}, \mathbf{m}) + F_\Lambda(\Phi_{-t} \mathbf{X}, \mathbf{m})) - F_\Lambda(\mathbf{X}, \mathbf{m}) = \frac{1}{2} A_1[\mathbf{X}, \psi_t + \psi_{-t}] + \mathcal{O}(K^2 \varepsilon \log \varepsilon t^2). \quad (\text{E.32})$$

Let us decompose  $\psi_t, \psi_{-t}$  as in (E.1), (E.2). The terms of first order in  $t$  cancel each other, and it remains to bound  $A_1[\mathbf{X}, \gamma_t + \gamma_{-t}]$ . At this point, using the “rough” bounds of [AS21, Prop. 4.2] on the anisotropy  $A_1$ , even in a localized way, would yield a bounded, *but not small*, error term - which would make the whole approach pointless. We thus need to rely on a finer understanding of anisotropy terms as put forward in [LS18; Ser20] (see also [BBNY19] for similar concerns about their “angle term”).

For simplicity, we will focus on the context of interest for us, namely when the background measure  $\mathbf{m}$  is given by  $n\mu_W$  as above - in particular it is constant on  $\Lambda^{\text{bulk}}$ .

### Anisotropy.

Let  $\psi$  be a continuous vector field supported on  $\Lambda^{\text{bulk}}$  and let  $U$  be a neighborhood of  $\text{supp } \psi$ . We define  $A_1(\mathbf{X}, \mathbf{m}, \psi)$  as:

$$A_1(\mathbf{X}, \mathbf{m}, \psi) := \iint_{(x,y) \in \Lambda \times \Lambda, x \neq y} \psi(x) \cdot \nabla \log |x - y| d(\mathbf{X}_n - \mathbf{m})(x) d(\mathbf{X} - \mathbf{m})(y). \quad (\text{E.33})$$

(It has an alternative expression using the electric field, which is the one that we used above in (E.31), see [Ser20, (4.14)]). The expression (E.33) makes sense because  $\mathbf{m}$  has a continuous density near  $\text{supp } \psi$  and is such that  $\iint -\log |x - y| d\mathbf{m}(x) d\mathbf{m}(y)$  is finite. Thanks to a clever integration by parts, one can control  $A_1$  as follows (see [Ser20, (4.10)]):

$$A_1(\mathbf{X}, \mathbf{m}, \psi) \leq C |\psi|_1 \times (\text{EnerPts}(\mathbf{X}, \text{supp } \psi)). \quad (\text{E.34})$$

If  $\psi$  lives on a disk of radius  $\ell$ , if  $|\psi|_1 = \mathcal{O}(\ell^{-2})$  and if local laws hold then we can expect the anisotropy to typically be  $\mathcal{O}(1)$ . Let us now explain how  $A_1$  shows up in the computations.

### Energy comparison.

Assume that  $\psi$  is a vector field of class  $C^3$ , supported on a disk of radius  $\ell$  within  $\Lambda^{\text{bulk}}$ , and such that:

$$|\psi|_k \leq C_\psi \ell^{-k-1}, \text{ for } k = 1, 2, 3. \quad (\text{E.35})$$

Let  $\tau$  be a real parameter such that (for some universal  $C$  large enough):

$$|\tau| |\psi|_1 \leq \frac{1}{C}, \quad |\tau| |\psi|_2 \leq \log \ell \frac{1}{C} \quad (\text{E.36})$$

For all  $\tau$  such that (E.36) is satisfied, let  $\Phi_\tau := \text{Id} + \tau\psi$  and  $\mathbf{m}_\tau := \Phi_\tau \# \mathbf{m}$ .

**Lemma E.16.** *We have the following expansion for all  $\mathbf{X}$*

$$F_\Lambda(\Phi_\tau \cdot \mathbf{X}, \mathbf{m}_\tau) = F_\Lambda(\mathbf{X}_n, \mathbf{m}) + \tau A_1(\mathbf{X}, \mathbf{m}, \psi) + \tau^2 \text{ErrEnerTrans}(\mathbf{X}, \psi), \quad (\text{E.37})$$

where  $\text{ErrEnerTrans}(\mathbf{X}, \psi)$  is controlled by:

$$\text{ErrEnerTrans}(\mathbf{X}, \psi) \leq C_\psi^2 \frac{\log \ell}{\ell^4} \text{EnerPts}(\mathbf{X}, \text{supp } \psi),$$

the energy being computed with  $\mathbf{m}$  as neutralizing background.

*Proof of Lemma E.16.* This follows from the second-order expansion of the energy as found in [Ser20, Lemma 4.1, Prop 4.2]. There is some care required in order to check that [Ser20, (4.11)] does indeed yield the claimed second-order correction, but this is made simpler by our assumption (E.35) and the fact that  $\mathbf{m}$  is constant on  $\Lambda^{\text{bulk}}$ .  $\square$

Since  $\text{supp } \psi$  has volume  $\mathcal{O}(\ell^2)$ , in view of the local laws we expect  $\text{ErrEnerTrans}$  to be  $\mathcal{O}(\ell^{-2} \log \ell)$  (with a constant depending on  $C_\psi$ ). The anisotropy is thus the first-order contribution to the energy change induced by a transport which is a small perturbation of the identity map.

$A_1$  **versus** Ani. What is called the “anisotropy” in [LS18] and [Ser20] is not exactly the same term, in fact [LS18] refers to Ani and [Ser20] to  $A_1$ , where the latter is as defined above and the former corresponds to:

$$\text{Ani}(\mathbf{X}, \mathbf{m}, \psi) := A_1(\mathbf{X}, \mathbf{m}, \psi) - \frac{1}{4} \int \text{div } \psi(x) d\mathbf{X}(x). \quad (\text{E.38})$$

So in fact “ $A_1$ ” contains a possibly non-vanishing contribution  $\frac{1}{4} \int \text{div } \psi(x) d\mathbf{X}(x) \approx \frac{1}{4} \int \text{div } \psi(x) d\mathbf{m}(x)$  which we need to subtract in order to obtain “Ani” which is the term that will eventually be found to be negligible. The term appearing in the energy expansion (E.37) is  $A_1$ .

Both  $A_1$  and Ani satisfy the same control (E.34). In most relevant cases,  $\psi$  happens to be such that  $|\psi|_1 \times |\text{supp } \psi| = \mathcal{O}(1)$ . Since (by local laws) the electric energy is typically proportional to the volume, we deduce that (for such “usual”  $\psi$ 's) *the anisotropy Ani is (at most) of order 1*. Analytically speaking, i.e. as far as deterministic, function-wise bounds are concerned, it is very challenging to do better. However, a key result underlying some of the recent progress in the study of 2DOCP's is that Ani is, so to speak, *often smaller than it seems*. This is a *probabilistic* statement found in [LS18, Corollary 4.4], [Ser20, Lemma 7.2.], see also (with a different formalism) [BBNY19, Section 8.5]. Let us sketch the proof of this fact.

### Smallness of the anisotropy via “Serfaty’s trick”.

Let  $K_{n,\Lambda}^\beta(\mathbf{m}_\tau)$  be the partition function associated to the background  $\mathbf{m}_\tau$  (we keep the same “effective confinement”  $\zeta_W$  for all  $\tau$ ) namely:

$$K_{n,\Lambda}^\beta(\mathbf{m}_\tau) := \int_{\Lambda^n} \exp \left( -\beta \left( F_\Lambda(\mathbf{X}_n, \mathbf{m}_\tau) + n \sum_{i=1}^n \zeta_W(x_i) \right) \right) d\mathbf{X}_n.$$

The key point is that there are two ways to evaluate the ratio  $\frac{K_{n,\Lambda}^\beta(\mathbf{m}_\tau)}{K_{n,\Lambda}^\beta(\mathbf{m})}$ :

1. By the transportation approach of [LS18; Ser20], involving a change of variables  $(x_1, \dots, x_N) = (\Phi_\tau(x'_1), \dots, \Phi_\tau(x'_N))$  in the very definition of  $K_{n,\Lambda}^\beta(\mathbf{m}_\tau)$  and an analysis of its effect on the energy. As seen in Lemma E.16, the anisotropy of  $\psi$  appears there as one of the contributions.
2. By using “free energy expansions”, i.e. explicit expressions of (the logarithm of) the two partition functions up to some error term that has to be negligible. This was done in [LS18] with a non-quantitative error term originating in the analysis of [LS17], in [BBNY19] using their own expansion, and much improved in [Ser20] using the error bounds of [AS21].

This gives two expressions for the same quantity, and since the anisotropy appears only (in exponential moments) in the first one, then it must be confounded with some error terms of the second one. This is fruitful for  $\tau$  large (but not too large), and thus also for smaller values of  $\tau$  by Hölder’s inequality.

This “trick”, which yields a form of “smallness of the anisotropy”, is used in [LS18; BBNY19; Ser20] as a tool to prove central limit theorems for fluctuations of smooth linear statistics. Unfortunately, it is hard to pinpoint a clear general statement in the literature, so we recall the proof in the next paragraphs. Recall that we take  $\mathbf{m} = n\mu_W$  as background measure, which has constant density 1 in the bulk.

**1. Comparison along a transport.** Here for technical reasons we need to work with partition functions restricted to  $\mathcal{E}_\Lambda$ , and we write:

$$K_{n,\Lambda}^\beta(\mathbf{m}_\tau | \mathcal{E}_\Lambda) := \int_{\Lambda^n} \mathbf{1}_{\mathcal{E}_\Lambda}(\mathbf{X}_n) \exp \left( -\beta \left( F_\Lambda(\mathbf{X}_n, \mathbf{m}_\tau) + n \sum_{i=1}^n \zeta_W(x_i) \right) \right) d\mathbf{X}_n.$$

We have the following “comparison of partition functions”:

**Claim E.17.**

$$\begin{aligned} \log \frac{K_{n,\Lambda}^\beta(\mathbf{m}_\tau | \mathcal{E}_\Lambda)}{K_{n,\Lambda}^\beta(\mathbf{m} | \mathcal{E}_\Lambda)} &= \left( \frac{\beta}{4} - 1 \right) \left( \int \log \mathbf{m}_\tau d\mathbf{m}_\tau - \int \log \mathbf{m} d\mathbf{m} \right) \\ &+ \log \mathbb{E}_{n,\Lambda}^\beta \left[ \exp \left( \tau \text{Ani}[\mathbf{X}, \mathbf{m}, \psi] + (\tau |\psi|_2 + \tau^2 |\psi|_1^2) \text{EnerPts}(\text{supp } \psi) + \tau^2 \text{ErrEnerTrans} \right) | \mathcal{E}_\Lambda \right]. \quad (\text{E.39}) \end{aligned}$$



*Proof of Claim E.17.* We follow the same steps as in [LS18, Prop 4.3]. First we write  $K_{n,\Lambda}^\beta(\mathbf{m}_\tau|\mathcal{E}_\Lambda)$  as:

$$K_{n,\Lambda}^\beta(\mathbf{m}_\tau|\mathcal{E}_\Lambda) = \int_{\Lambda^n} \mathbf{1}_{\mathcal{E}_\Lambda}(\mathbf{Y}_n) \exp\left(-\beta\left(F_\Lambda(\mathbf{Y}_n, \mathbf{m}_\tau) + n \int \zeta_W(x) d\mathbf{Y}_n(x)\right)\right) d\mathbf{Y}_n$$

and perform the change of variables  $\mathbf{Y}_n = \Phi_\tau \cdot \mathbf{X}_n$ . By construction we have  $\Phi_\tau = \text{Id}$  outside  $\Lambda^{\text{bulk}}$  and in particular the term  $n \int \zeta_W(x) d\mathbf{Y}_n(x)$  (which only detects points outside the support of  $\mu_W$ ) is not affected by this, nor is the indicator  $\mathcal{E}_\Lambda$  which only cares about points very close to  $\partial\Lambda$ . We obtain:

$$\begin{aligned} & K_{n,\Lambda}^\beta(\mathbf{m}_\tau|\mathcal{E}_\Lambda) \\ &= \int_{\mathcal{E}_\Lambda} \exp\left(-\beta\left(F_\Lambda(\Phi_\tau \cdot \mathbf{X}_n, \Phi_\tau \# \mathbf{m}) + n \int \zeta_W(x) d\mathbf{X}_n(x)\right) + \int \log \det D\Phi_\tau(x) d\mathbf{X}_n(x)\right) d\mathbf{X}_n, \end{aligned} \quad (\text{E.40})$$

where the last term in the integrand is the Jacobian of the transformation. Using (E.37) we may compare the energy before and after the transport.

$$F_\Lambda(\Phi_\tau \cdot \mathbf{X}_n, \Phi_\tau \# \mathbf{m}) = F_\Lambda(\mathbf{X}_n, \mathbf{m}) + \tau A_1[\mathbf{X}, \mathbf{m}, \psi] + \tau^2 \text{ErrEnerTrans},$$

which we can re-write (using (E.38)) as:

$$F_\Lambda(\Phi_\tau \cdot \mathbf{X}_n, \Phi_\tau \# \mathbf{m}) = F_\Lambda(\mathbf{X}_n, \mathbf{m}) + \frac{\tau}{4} \int \text{div } \psi(x) d\mathbf{X}_n(x) + \tau \text{Ani}[\mathbf{X}, \mathbf{m}, \psi] + \tau^2 \text{ErrEnerTrans}.$$

The terms  $\int \log \det D\Phi_\tau(x) d\mathbf{X}_n(x)$  and  $\tau \int \text{div } \psi(x) d\mathbf{X}_n(x)$  are both equal to a deterministic quantity up to small error terms. On the one hand, we have:

$$\int \log \det D\Phi_\tau(x) d\mathbf{X}_n(x) = \int \log \det D\Phi_\tau(x) dx + \text{Fluct}[\log \det D\Phi_\tau]$$

and on the other hand:  $\tau \int \text{div } \psi(x) d\mathbf{X}_n(x) = \int \log \det D\Phi_\tau(x) d\mathbf{X}_n(x) + \tau^2 \sum_{i=1}^n |\psi(x_i)|_1^2$ . The quantity  $\int \log \det D\Phi_s(x) dx$  coincides (see. [LS18, (4.11)–(4.13)]) with:

$$\int \log \det D\Phi_s(x) dx = \int \log \mathbf{m} d\mathbf{m} - \int \log \mathbf{m}_\tau d\mathbf{m}_\tau, \quad (\text{E.41})$$

hence we obtain:

$$\begin{aligned} F_\Lambda(\Phi_\tau \cdot \mathbf{X}_n, \Phi_\tau \# \mathbf{m}) &= F_\Lambda(\mathbf{X}_n, \mathbf{m}) + \frac{1}{4} \left( \int \log \mathbf{m} d\mathbf{m} - \int \log \mathbf{m}_\tau d\mathbf{m}_\tau \right) \\ &\quad + \text{Fluct}[\log \det D\Phi_\tau] + \tau^2 \sum_{i=1}^n |\psi(x_i)|_1^2 + \tau^2 \text{ErrEnerTrans}. \end{aligned}$$

Using Lemma 2.6 we can control:  $\text{Fluct}[\log \det D\Phi_\tau] \preceq \tau |\psi|_2 \text{EnerPts}(\text{supp } \psi)$  and on the other hand we have:  $\tau^2 \sum_{i=1}^n |\psi(x_i)|_1^2 \leq \tau^2 |\psi|_1^2 \text{EnerPts}(\text{supp } \psi)$ , thus we can write:

$$\begin{aligned} F_\Lambda(\Phi_\tau \cdot \mathbf{X}_n, \Phi_\tau \# \mathbf{m}) &= F_\Lambda(\mathbf{X}_n, \mathbf{m}) + \frac{1}{4} \left( \int \log \mathbf{m} d\mathbf{m} - \int \log \mathbf{m}_\tau d\mathbf{m}_\tau \right) \\ &\quad + (\tau |\psi|_2 + \tau^2 |\psi|_1^2) \text{EnerPts}(\text{supp } \psi) + \tau^2 \text{ErrEnerTrans}, \end{aligned}$$

and inserting this in (E.40) yields (E.39).  $\square$

**Remark E.18.** Each term in the right-hand side of (E.41) might be infinite when taken separately (because e.g.  $\mathbf{m}$  may have a singularity on  $\partial\Lambda$  and hence infinite entropy) but the difference makes sense as the two measures coincide outside  $\Lambda^{\text{bulk}}$ .

## Free energy comparisons

As we have seen above, one can compare two partition functions using a “transportation” approach. On the other hand, we have the following.

**Claim E.19** (Free energy comparison, the “direct” approach). *Assume that the support of  $\tilde{\mathbf{m}} - \mathbf{m}$  is contained in a square  $\Omega$  of sidelength  $\hat{\ell}$  included in  $\Lambda^{\text{bulk}}$ .*

$$\left| \log \frac{K_{n,\Lambda}^\beta(\tilde{\mathbf{m}}|\mathcal{E}_\Lambda)}{K_{n,\Lambda}^\beta(\mathbf{m}|\mathcal{E}_\Lambda)} \right| = \left( \frac{\beta}{4} - 1 \right) \left( \int \log \tilde{\mathbf{m}} d\tilde{\mathbf{m}} - \int \log \mathbf{m} d\mathbf{m} \right) + \mathcal{O}(\hat{\ell} \log \hat{\ell})$$

*Proof of Claim E.19.* This is essentially the result of [Ser20, Proposition 6.4], except that our reference measure does not necessarily have a  $C^1$  density near the edge of  $\Lambda$ . This is in fact not a problem as long as we are doing comparisons inside  $\Lambda^{\text{bulk}}$  (i.e. as long as the other measure coincides with  $\mathbf{m}$  outside  $\Lambda^{\text{bulk}}$ ), but it requires an explanation.

The proof of [Ser20, Proposition 6.4] relies on two ingredients:

1. [Ser20, Proposition 6.3] (free energy expansion for general density in a rectangle). This we can import directly as it has nothing to do with our specific setup.
2. [Ser20, Proposition 3.6] (almost additivity of the free energy). It says that one can decompose  $K_{n,\Lambda}^\beta(\mathbf{m})$  into two parts: inside/outside  $\Omega$ , with a small error. This is proven by two inequalities: one is easy and corresponds to the sub-additivity of Neumann energies whereas the other one uses the screening procedure and the local laws on (a neighborhood of)  $\Omega$  in order to control the screening error terms (see [Ser20, Prop. 3.6]). Since the screening procedure takes place in a neighborhood of  $\Omega$ , the possible singularities near  $\partial\Lambda$  are irrelevant. The only adaptation needed is to replace the local laws used in [Ser20] by ours (which is the reason why we “condition” on the event  $\mathcal{E}_\Lambda$ ).

□

## Conclusion 1: smallness of Ani

We may now apply “Serfaty’s trick”. Comparing the statements of Claim E.17 and Claim E.19 we see that necessarily:

$$\log \mathbb{E}_{n,\Lambda}^\beta \left[ \exp(\tau \text{Ani}[\psi, \mathbf{X}, \mathbf{m}] + (\tau|\psi|_2 + \tau^2|\psi|_1^2) \text{EnerPts}(\text{supp } \psi) + \tau^2 \text{ErrEnerTrans}) \mid \mathcal{E}_\Lambda \right] = \mathcal{O}(\hat{\ell} \log \hat{\ell}).$$

Using the local laws and our assumptions (E.35) on  $\psi$  we may control the exponential moments of the error terms:

$$\begin{aligned} \log \mathbb{E}_{n,\Lambda}^\beta \left[ \exp((\tau|\psi|_2 + \tau^2|\psi|_1^2) \text{EnerPts}(\text{supp } \psi) + \tau^2 \text{ErrEnerTrans}) \mid \mathcal{E}_\Lambda \right] \\ = \mathcal{O} \left( \tau \mathbf{C}_\psi \hat{\ell}^{-1} + \tau^2 \mathbf{C}_\psi^2 \hat{\ell}^{-2} + \mathbf{C}_\psi^2 \tau^2 \hat{\ell}^{-2} \log \hat{\ell} \right). \end{aligned}$$

This is valid for all  $\tau$  smaller than  $\frac{\hat{\ell}^2}{\mathbf{C}_\psi \log \hat{\ell}}$  so that (E.36) are satisfied. Taking  $\tau = \hat{\ell}^{3/2}$ , we obtain the following statement:

**Lemma E.20** (“The anisotropy is small”). *If  $\psi$  is a  $C^3$  vector field compactly supported on a disk of radius  $\hat{\ell}$  within  $\Lambda^{\text{bulk}}$ , and satisfying (E.35), then we have, for all  $\hat{\ell}$  large enough (depending on the constant  $\mathbf{C}_\psi$ )*

$$\mathbb{E}_{n,\Lambda}^\beta \left[ \exp \left( \hat{\ell}^{3/2} \text{Ani}(\psi, \mathbf{X}_n, \mathbf{m}) \right) \mid \mathcal{E}_\Lambda \right] \leq e^{\mathcal{O}(\hat{\ell} \log \hat{\ell}) + \mathcal{O}(\mathbf{C}_\psi^2 \hat{\ell} \log \hat{\ell})},$$

*with an implicit constant depending on  $\beta$  and the “local laws” constant  $\mathbf{C}_{\text{Local}}$ . In particular, we have the following tail estimate on the distribution of  $\text{Ani}(\psi)$ : for  $\hat{\ell}$  large enough,*

$$\mathbb{P}_{n,\Lambda}^\beta \left[ |\text{Ani}(\psi, \mathbf{X}_n, \mathbf{m})| \geq \frac{\log^2 \hat{\ell}}{\hat{\ell}^{1/2}} \mid \mathcal{E}_\Lambda \right] \leq \exp(-\hat{\ell} \log^2 \hat{\ell}).$$

**Remark E.21.** The same analysis holds for the full system, with no need for a conditioning on  $\mathcal{E}_\Lambda$  and as soon as (2.7) is satisfied, so for a broader notion of “bulk”.

## Conclusion 2: proof of Proposition 6.3

We may now conclude the proof of Proposition 6.3.

*Proof of Proposition 6.3.* Let us return to (E.32) and use the fact that by definition (see (E.2)):

$$\psi_t = tW^{(\varepsilon, \ell)}(x) + \gamma_t(x), \quad \psi_{-t} = -tW^{(\varepsilon, \ell)}(x) + \gamma_{-t}(x),$$

we obtain (for  $\mathbf{X}$  in  $\text{WellSpread}(\Lambda^{\text{bulk}}, \ell, K)$  and thus with probability  $1 - \exp(-\ell^2)$  up to choosing  $K$  large enough as in Lemma E.6):

$$\frac{1}{2} (F_\Lambda(\Phi_t \mathbf{X}, \mathbf{m}) + F_\Lambda(\Phi_{-t} \mathbf{X}, \mathbf{m})) - F_\Lambda(\mathbf{X}, \mathbf{m}) = \frac{1}{2} \text{Ani}[\mathbf{X}, \gamma_t + \gamma_{-t}] + \mathcal{O}(K^2 \varepsilon \log \varepsilon t^2). \quad (\text{E.42})$$

Let us introduce dyadic length scales  $\ell_i := 2^i$  for  $0 \leq i \leq \log(T/2)$  and associated cut-off functions  $\chi_i$ . We decompose  $\gamma_t$  as:  $\gamma_t = \sum_i \chi_i \gamma_t$ . Using Proposition E.1 and in particular (E.4) we see that the vector field  $\tilde{\psi}^{(i)} := \frac{1}{\varepsilon \ell^2} \chi_i \gamma_t$  satisfies  $|\tilde{\psi}^{(i)}|_k \preceq \ell_i^{-k-1}$ . Using Lemma E.20, we know that for each  $i$  large enough:

$$\mathbb{P}_{n, \Lambda}^\beta \left[ |\text{Ani}(\tilde{\psi}^{(i)})| \geq \frac{\log^2 \ell_i}{\ell_i^{\frac{1}{2}}} | \mathcal{E}_\Lambda \right] \leq \exp(-\ell_i \log^2 \ell_i).$$

Since this bound is only interesting (probabilistically speaking) for large enough  $i$ , we use it for  $i \geq \log^2 \ell$ , in which case we have:

$$\sum_{i=\log^2 \ell}^{\log(T/2)} \text{Ani}(\tilde{\psi}^{(i)}) \leq \sum_{i=\log^2 \ell}^{\log(T/2)} \frac{\log^2 \ell_i}{\ell_i^{\frac{1}{2}}} = \mathcal{O}(\varepsilon t^2)$$

with probability  $\geq 1 - \sum_{i=\log^2 \ell}^{\log(T/2)} \exp(-\ell_i \log^2 \ell_i) \geq 1 - \exp(-\log^2 \ell)$ . The first contributions (for  $0 \leq i \leq \log^2 \ell$ ) are controlled using the ‘‘rougher’’ control (E.34), we get:

$$\sum_{i=0}^{\log^2 \ell} \text{Ani}(\tilde{\psi}^{(i)}) = \mathcal{O}(\log \ell^2).$$

By (6.1) we know that  $\log \ell^2$  is comparable to  $\log \varepsilon$ . In conclusion, we obtain:

$$\mathbb{P}_{n, \Lambda}^\beta (|\text{Ani}[\mathbf{X}, \gamma_t + \gamma_{-t}]| \leq C \varepsilon \log \varepsilon t^2 | \mathcal{E}_\Lambda) \geq 1 - \exp(-\ell^2),$$

with a constant  $C$  depending on  $\beta$  and the constant  $C_{\text{Local}}$ .

Combined with (E.42) which is valid under an event of comparable probability (see Lemma E.6), we conclude the proof of Proposition 6.3.  $\square$

## E.5. Effect on expectations: proof of Proposition 6.4

*Proof of Proposition 6.4.* For  $\tau \in (0, 1)$ , let us introduce the event  $\mathcal{F}_\tau$  as:

$$\mathcal{F}_\tau := \{\text{ErrAve} \leq \tau\} \quad (\text{E.43})$$

(recall the definition (6.2) of  $\text{ErrAve}$ ). We split the proof into several steps.

### The case of non-negative functions

In this paragraph, we make the additional assumption that  $G$  is non-negative. We can obviously decompose  $\mathbb{E}_{n, \Lambda}^\beta[G(\mathbf{X}) | \mathcal{E}]$  as:

$$\mathbb{E}_{n, \Lambda}^\beta[G(\mathbf{X}) | \mathcal{E}] = \mathbb{E}_{n, \Lambda}^\beta[G(\mathbf{X}) \mathbf{1}_{\mathcal{F}_\tau} | \mathcal{E}] + \mathbb{E}_{n, \Lambda}^\beta[G(\mathbf{X}) \mathbf{1}_{\overline{\mathcal{F}_\tau}} | \mathcal{E}].$$

**Claim E.22.**

$$\mathbb{E}_{n, \Lambda}^\beta[G(\mathbf{X}) \mathbf{1}_{\mathcal{F}_\tau}(\mathbf{X}) | \mathcal{E}] \leq \mathbb{E}_{n, \Lambda}^\beta \left[ \frac{1}{2} (G(\Phi_t \cdot \mathbf{X}) + G(\Phi_{-t} \cdot \mathbf{X})) | \mathcal{E} \right] \times e^{\beta \tau}. \quad (\text{E.44})$$

*Proof of Claim E.22.* According to the definition of  $\mathbb{E}_{n,\Lambda}^\beta$  in Section 5.2, the expectation  $\mathbb{E}_{n,\Lambda}^\beta[\mathbf{G}(\mathbf{X})|\mathcal{E}]$  admits the following expression:

$$\frac{\int_{\Lambda^n} \mathbf{1}_{\mathcal{E}}(\mathbf{X}) \mathbf{1}_{\mathcal{F}_\tau}(\mathbf{X}) \mathbf{G}(\mathbf{X}) e^{-\beta(F_\Lambda(\mathbf{X}, \mathbf{m}) + n \sum_{i=1}^n \zeta(x_i))} d\mathbf{X}}{\int_{\Lambda^n} \mathbf{1}_{\mathcal{E}}(\mathbf{X}) e^{-\beta(F_\Lambda(\mathbf{X}, \mathbf{m}) + n \sum_{i=1}^n \zeta(x_i))} d\mathbf{X}}.$$

Since the function  $\zeta$  vanishes identically on  $\Lambda^{\text{bulk}}$  and since our localized translations act as the identity outside  $\Lambda^{\text{bulk}}$ , the term  $\sum_{i=1}^n \zeta(x_i)$  will not be affected by the operations below. For simplicity, we omit it altogether. Let us focus on the integral appearing in the numerator. By definition of  $\text{ErrAve}$  and of  $\mathcal{F}_\tau$  as in (6.2), (E.43), we may write:

$$\mathbf{1}_{\mathcal{F}_\tau}(\mathbf{X}) e^{-\beta F_\Lambda(\mathbf{X}, \mathbf{m})} = \mathbf{1}_{\mathcal{F}_\tau}(\mathbf{X}) e^{-\beta \frac{1}{2}(F_\Lambda(\Phi_t \cdot \mathbf{X}, \mathbf{m}) + F_\Lambda(\Phi_{-t} \cdot \mathbf{X}, \mathbf{m})) + \beta \text{ErrAve}} \leq e^{\beta \tau} e^{-\beta \frac{1}{2}(F_\Lambda(\Phi_t \cdot \mathbf{X}, \mathbf{m}) + F_\Lambda(\Phi_{-t} \cdot \mathbf{X}, \mathbf{m}))},$$

where we have bounded the indicator function by 1 in the right-hand side. Using the convexity of  $x \mapsto \exp(-\beta x)$  and the fact that  $\mathbf{G}$  is assumed to be non-negative, we deduce that:

$$\begin{aligned} \int_{\Lambda^n} \mathbf{1}_{\mathcal{E}}(\mathbf{X}) \mathbf{1}_{\mathcal{F}_\tau}(\mathbf{X}) \mathbf{G}(\mathbf{X}) e^{-\beta F_\Lambda(\mathbf{X}, \mathbf{m})} d\mathbf{X} \\ \leq \frac{1}{2} \left[ \int_{\Lambda^n} \mathbf{1}_{\mathcal{E}}(\mathbf{X}) \mathbf{G}(\mathbf{X}) e^{-\beta F_\Lambda(\Phi_t \cdot \mathbf{X}, \mathbf{m})} d\mathbf{X} + \int_{\Lambda^n} \mathbf{1}_{\mathcal{E}}(\mathbf{X}) \mathbf{G}(\mathbf{X}) e^{-\beta F_\Lambda(\Phi_{-t} \cdot \mathbf{X}, \mathbf{m})} d\mathbf{X} \right] \times e^{\beta \tau}. \end{aligned}$$

By construction, the change of variable  $(x_1, \dots, x_n) \mapsto (\Phi_t(x_1), \dots, \Phi_t(x_n))$ , which maps  $\mathbf{X}$  to  $\Phi_t \mathbf{X}$ , has a Jacobian equal to 1. We thus have, looking at the first integral on the right-hand side:

$$\int_{\Lambda^n} \mathbf{1}_{\mathcal{E}}(\mathbf{X}) \mathbf{G}(\mathbf{X}) e^{-\beta F_\Lambda(\Phi_t \cdot \mathbf{X}, \mathbf{m})} d\mathbf{X} = \int_{\Lambda^n} \mathbf{1}_{\mathcal{E}}(\Phi_{-t} \mathbf{X}) \mathbf{G}(\Phi_{-t} \mathbf{X}) e^{-\beta F_\Lambda(\mathbf{X}, \mathbf{m})} d\mathbf{X}.$$

Moreover, since  $\Phi_{-t} \equiv \text{Id} - t\vec{u}$  on the disk  $\mathfrak{D}(0, \ell/4)$  (see Lemma 6.2), since  $\mathbf{G}$  is assumed to be  $\mathfrak{D}(0, \ell/10)$ -local, and since  $|t|$  is taken smaller than  $\ell/10$ , we have  $\Phi_{-t}(\mathfrak{D}(0, \ell/10)) \subset \mathfrak{D}(0, \ell/4)$  and thus:

$$\mathbf{G}(\Phi_{-t}(\mathbf{X})) = \mathbf{G}(\mathbf{X} - t\vec{u}).$$

On the other hand since by construction  $\Phi_t$  coincides with the identity map outside  $\Lambda^{\text{bulk}}$  and since the event  $\mathcal{E}$  is assumed to be  $\Lambda \setminus \Lambda^{\text{bulk}}$ -local, we have:

$$\mathbf{1}_{\mathcal{E}}(\Phi_{-t}(\mathbf{X})) = \mathbf{1}_{\mathcal{E}}(\mathbf{X}). \quad (\text{E.45})$$

Hence we can ensure that:

$$\int_{\Lambda^n} \mathbf{1}_{\mathcal{E}}(\Phi_{-t} \mathbf{X}) \mathbf{G}(\Phi_{-t} \mathbf{X}) e^{-\beta F_\Lambda(\mathbf{X}, \mathbf{m})} d\mathbf{X} = \int_{\Lambda^n} \mathbf{1}_{\mathcal{E}}(\mathbf{X}) \mathbf{G}(\mathbf{X} - t\vec{u}) e^{-\beta F_\Lambda(\mathbf{X}, \mathbf{m})} d\mathbf{X},$$

and similarly for the other term (reversing the roles of  $-t$  and  $t$ ). In conclusion, we obtain:

$$\int_{\Lambda^n} \mathbf{1}_{\mathcal{E}}(\mathbf{X}) \mathbf{1}_{\mathcal{F}_\tau}(\mathbf{X}) \mathbf{G}(\mathbf{X}) e^{-\beta F_\Lambda(\mathbf{X}, \mathbf{m})} d\mathbf{X} \leq \left[ \int_{\Lambda^n} \mathbf{1}_{\mathcal{E}}(\mathbf{X}) \frac{1}{2} (\mathbf{G}(\mathbf{X} + t\vec{u}) + \mathbf{G}(\mathbf{X} - t\vec{u})) e^{-\beta F_\Lambda(\mathbf{X}, \mathbf{m})} d\mathbf{X} \right] \times e^{\beta \tau}.$$

Dividing back by the partition function, we obtain (E.44). On the other hand, we have by Cauchy-Schwarz's inequality:

$$\mathbb{E}_{n,\Lambda}^\beta[\mathbf{G}(\mathbf{X}) \mathbf{1}_{\mathcal{F}_\tau}(\mathbf{X}) | \mathcal{E}] \leq \left( \mathbb{E}_{n,\Lambda}^\beta[\mathbf{G}^2(\mathbf{X}) | \mathcal{E}] \right)^{\frac{1}{2}} \times \left( \mathbb{P}_{n,\Lambda}^\beta(\{\text{ErrAve} \geq \tau\} | \mathcal{E}) \right)^{\frac{1}{2}}.$$

In summary, we have obtained *under the extra assumption that  $\mathbf{G}$  is non-negative*, and for all  $\tau \in (0, 1)$ :

$$\begin{aligned} \mathbb{E}_{n,\Lambda}^\beta[\mathbf{G}(\mathbf{X}) | \mathcal{E}] &\leq \mathbb{E}_{n,\Lambda}^\beta \left[ \frac{1}{2} (\mathbf{G}(\Phi_t \mathbf{X}) + \mathbf{G}(\Phi_{-t} \mathbf{X})) | \mathcal{E} \right] \times e^{\beta \tau} \\ &\quad + \left( \mathbb{E}_{n,\Lambda}^\beta[\mathbf{G}^2(\mathbf{X}) | \mathcal{E}] \right)^{\frac{1}{2}} \times \left( \mathbb{P}_{n,\Lambda}^\beta(\{\text{ErrAve} \geq \tau\} | \mathcal{E}) \right)^{\frac{1}{2}}. \end{aligned}$$

□

### E.5.1. The general case.

We no longer assume that  $\mathbf{G} \geq 0$ . For  $\sigma > 0$ , let us introduce the event  $\mathcal{G}_\sigma := \{\mathbf{G} + \sigma \geq 0\}$ . We can write:

$$\mathbb{E}_{n,\Lambda}^\beta[\mathbf{G}(\mathbf{X}) + \sigma | \mathcal{E}] = \mathbb{E}_{n,\Lambda}^\beta[\mathbf{G}(\mathbf{X}) | \mathcal{E}] + \sigma = \mathbb{E}_{n,\Lambda}^\beta[(\mathbf{G}(\mathbf{X}) + \sigma) \mathbf{1}_{\mathcal{G}_\sigma}(\mathbf{X}) | \mathcal{E}] + \mathbb{E}_{n,\Lambda}^\beta[(\mathbf{G}(\mathbf{X}) + \sigma) \mathbf{1}_{\overline{\mathcal{G}_\sigma}}(\mathbf{X}) | \mathcal{E}]. \quad (\text{E.46})$$

Since  $\mathbf{X} \mapsto (\mathbf{G}(\mathbf{X}) + \sigma) \mathbf{1}_{\mathcal{G}_\sigma}(\mathbf{X})$  is non-negative by construction and has the same local character as  $\mathbf{G}$ , we may apply the conclusions of the previous paragraph and write:

$$\begin{aligned} & \mathbb{E}_{n,\Lambda}^\beta[(\mathbf{G}(\mathbf{X}) + \sigma) \mathbf{1}_{\mathcal{G}_\sigma}(\mathbf{X}) | \mathcal{E}] \\ & \leq \mathbb{E}_{n,\Lambda}^\beta \left[ \frac{1}{2} (\mathbf{G}(\mathbf{X} + t\vec{u}) + \sigma) \mathbf{1}_{\mathcal{G}_\sigma}(\mathbf{X} + t\vec{u}) + \frac{1}{2} (\mathbf{G}(\mathbf{X} - t\vec{u}) + \sigma) \mathbf{1}_{\mathcal{G}_\sigma}(\mathbf{X} - t\vec{u}) | \mathcal{E} \right] \times e^{\beta\tau} \\ & \quad + \mathbb{E}_{n,\Lambda}^\beta \left[ (\mathbf{G}(\mathbf{X}) + \sigma)^2 | \mathcal{E} \right] \times \left( \mathbb{P}_{n,\Lambda}^\beta \left( \{\text{ErrAve} \geq \tau\} | \mathcal{E} \right) \right)^{\frac{1}{2}}, \end{aligned}$$

where in the last expectation term we have bounded an indicator function by 1. We would like to get rid of the two remaining indicator functions in the right-hand side.

Using again Cauchy-Schwarz's inequality (and the definition of  $\mathcal{G}_\sigma$ ) we see that:

$$\left| \mathbb{E}_{n,\Lambda}^\beta[(\mathbf{G}(\mathbf{X} + t\vec{u}) + \sigma) \mathbf{1}_{\overline{\mathcal{G}_\sigma}}(\mathbf{X} + t\vec{u}) | \mathcal{E}] \right| \leq 2 \left( \mathbb{E}_{n,\Lambda}^\beta \left[ \mathbf{G}^2(\mathbf{X} + t\vec{u}) | \mathcal{E} \right] + \sigma^2 \right)^{\frac{1}{2}} \times \left( \mathbb{P}_{n,\Lambda}^\beta [|\mathbf{G}(\mathbf{X} + t\vec{u})| \geq \sigma] \right)^{\frac{1}{2}},$$

and similarly after replacing  $t$  by  $-t$ . Returning to (E.46), we thus obtain:

$$\begin{aligned} \mathbb{E}_{n,\Lambda}^\beta[\mathbf{G}(\mathbf{X}) | \mathcal{E}] + \sigma & \leq \left( \mathbb{E}_{n,\Lambda}^\beta \left[ \frac{1}{2} (\mathbf{G}(\mathbf{X} + t\vec{u}) + \mathbf{G}(\mathbf{X} - t\vec{u})) | \mathcal{E} \right] + \sigma \right) \times e^{\beta\tau} \\ & \quad + 2e^{\beta\tau} \left( \mathbb{E}_{n,\Lambda}^\beta \left[ \mathbf{G}^2(\mathbf{X} \pm \vec{u}) | \mathcal{E} \right] + \sigma^2 \right)^{\frac{1}{2}} \times \left( \mathbb{P}_{n,\Lambda}^\beta [|\mathbf{G}(\mathbf{X} \pm t\vec{u})| \geq \sigma] \right)^{\frac{1}{2}} \\ & \quad + \mathbb{E}_{n,\Lambda}^\beta \left[ \mathbf{G}^2(\mathbf{X}) | \mathcal{E} \right] \times \left( \mathbb{P}_{n,\Lambda}^\beta [\text{ErrAve} \geq \tau | \mathcal{E}] \right)^{\frac{1}{2}}. \end{aligned}$$

We may subtract  $\sigma$  on both sides, and observe that we have:

$$\begin{aligned} & \left( \mathbb{E}_{n,\Lambda}^\beta \left[ \frac{1}{2} (\mathbf{G}(\mathbf{X} + t\vec{u}) + \mathbf{G}(\mathbf{X} - t\vec{u})) | \mathcal{E} \right] + \sigma \right) \times e^{\beta\tau} - \sigma = \mathbb{E}_{n,\Lambda}^\beta \left[ \frac{1}{2} (\mathbf{G}(\mathbf{X} + t\vec{u}) + \mathbf{G}(\mathbf{X} - t\vec{u})) | \mathcal{E} \right] \\ & \quad + \mathbb{E}_{n,\Lambda}^\beta \left[ \frac{1}{2} (\mathbf{G}(\mathbf{X} + t\vec{u}) + \mathbf{G}(\mathbf{X} - t\vec{u})) + \sigma | \mathcal{E} \right] (e^{\beta\tau} - 1). \end{aligned}$$

Using Cauchy-Schwarz's inequality one more time to bound the second line, we obtain:

$$\begin{aligned} \mathbb{E}_{n,\Lambda}^\beta[\mathbf{G}(\mathbf{X}) | \mathcal{E}] & \leq \mathbb{E}_{n,\Lambda}^\beta \left[ \frac{1}{2} (\mathbf{G}(\mathbf{X} + t\vec{u}) + \mathbf{G}(\mathbf{X} - t\vec{u})) | \mathcal{E} \right] + 2 \left( \mathbb{E}_{n,\Lambda}^\beta \left[ \mathbf{G}^2(\mathbf{X} \pm t\vec{u}) | \mathcal{E} \right] + \sigma^2 \right)^{\frac{1}{2}} \times (e^{\beta\tau} - 1) \\ & \quad + 2e^{\beta\tau} \left( \mathbb{E}_{n,\Lambda}^\beta \left[ \mathbf{G}^2(\mathbf{X} \pm \vec{u}) | \mathcal{E} \right] + \sigma^2 \right)^{\frac{1}{2}} \times \left( \mathbb{P}_{n,\Lambda}^\beta [|\mathbf{G}(\mathbf{X} \pm t\vec{u})| \geq \sigma] \right)^{\frac{1}{2}} \\ & \quad + \mathbb{E}_{n,\Lambda}^\beta \left[ \mathbf{G}^2(\mathbf{X}) | \mathcal{E} \right] \times \left( \mathbb{P}_{n,\Lambda}^\beta [\text{ErrAve} \geq \tau | \mathcal{E}] \right)^{\frac{1}{2}}. \end{aligned}$$

Replacing  $\mathbf{G}$  by  $-\mathbf{G}$  (which is possible as there is no more a sign constraint on  $\mathbf{G}$ ), we obtain the converse inequality, which yields (6.3).  $\square$

**Remark E.23.** The identity (E.45) requires that  $\mathcal{E}$  be  $\Lambda \setminus \Lambda^{\text{bulk}}$ -local. One could try to “pass down” as much information as possible but if  $\mathcal{E}$  is “too rich” then it risks to be perturbed by our localized translation.

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