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***Abstract*— Cooperative localization is a promising solution to improve the accuracy and overcome the shortcomings of GNSS. Cooperation is often achieved by measuring the distance between users. To optimally integrate a distance measurement between two users into a navigation filter, the correlation between their position estimates must be known. Unfortunately, in large scale networks the agents cannot compute their correlations with other agents and must use consistent filters. A consistent filter provides an upper bound on the covariance of the estimator taking into account all the possible correlations. In this paper, a consistent linear filter for integrating a distance measurement is derived using Split Covariance Intersection. Its analysis shows that a distance measurement between two agents can only benefit one of them, i.e., only one of the two can use it to improve its estimator. Furthermore, in some cases, none can. A necessary condition for an agent to benefit from the measurement is given for a general class of objective functions. When the objective function is the trace or the determinant, necessary and sufficient conditions are given.**

***Index Terms*— Filtering, sensor network, cooperative control.**

I. Introduction

Accurate positioning is a key challenge for numerous strategic applications. Global Navigation Satellite Systems (GNSSs) provide a low-cost and effective solution for achieving satisfactory accuracy in open-sky environments. However, in GNSS-denied environments such as urban canyons, indoors or underwater, other methods must be used. Cooperative localization is a promising alternative because it can slow down the loss of accuracy in the absence of GNSS signal [20]. A simple form of cooperation is the measurement of distances between the users, hereafter called *agents*. Inter-agent distances can be inexpensively determined by measuring the Received Signal Strength or the Time-of-Flight of a signal. For example, cooperative localization has been used for terrestrial vehicles [10] and UAVs [5], [7]. Recent research has demonstrated the effectiveness of cooperation in significantly improving positioning accuracy while using simple algorithms [15].

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One of the challenges in cooperative systems when integrating inter-agent distance measurements is dealing with the correlations between agents' estimators. The use of inter-agent measurements tends to correlate their estimators. If these cross-covariances are not taken into account, they lead to an underestimation of the errors and a potential divergence of the filter [3]. These problems can be avoided by calculating and storing the cross-covariances between each pair of estimators. In centralized systems, the estimators of all agents are stacked into a global estimator whose covariance includes all the cross-covariances. This global estimator is then updated in a centralized manner (at a computing station), or in a distributed manner [19]. However, such solutions are difficult to implement for networks with a large number of agents due to the computational and transmission costs. Another solution to avoid correlation problems involves ensuring that no redundant information is used. In [4], the authors propose to keep a table of all past interactions to ensure that only independent information is used in the filter.

In order to use two-way cooperation when no cross-covariance information is available, *consistent* filters should be used. The literature also refers to these as "*conservative* filters", see, e.g., [8]. *Consistency* ensures that the estimator is not overconfident. However, the covariance of the estimator is generally not computable due to the lack of information, and only an upper bound is provided. Several methods have been developed to *fuse* different estimators with unknown covariances. The three main methods are referred to in the literature as Covariance Intersection (CI) [12], Inverse Covariance Intersection [16] and Ellipsoid Intersection [21]. A recent paper [8] unifies these concepts in a general framework called the Conservative Linear Unbiased Estimator (CLUE). Among these methods, CI and its derivation the Split Covariance Intersection (SCI) [11] are particularly efficient: several papers, see, e.g., [6], have described the efficiency of CI, and more recently CI has been shown to be the best method to fuse two estimators under unknown correlation [18]. Even if SCI has not been showed to be optimal, its simplicity makes it a widely used method for decentralized cooperative positioning [13], [14], [17]. However, to the best of our knowledge, no specific study on the integration of a distance measurement between two agents with an unknown correlation using SCI has been performed in the literature.

In this paper, we consider cooperative networks where agents estimate their positions and store their own estimators. Here, cooperation occurs through distance measurements be-

tween agents. We address the problem of integrating, through linear filtering, a distance measurement between two agents who have *consistent* estimators of their states but the cross-covariance between these estimators is unknown. Throughout the paper, we only consider the integration of a distance measurement. In particular, the dynamics of the system are beyond the scope of this paper and is the subject of further work. Our first contribution is the derivation of the optimal SCI filter for this problem, optimal w.r.t. a given increasing cost function. In practice, this function is often the trace or the determinant of the covariance of the estimate. Our second contribution is the analysis of the SCI filter which shows whether a distance measurement can be used to improve the estimation. We provide a necessary condition in the general case, and necessary and sufficient conditions for the cases where the cost function is the trace or the determinant.

The paper is organized as follows. Section II introduces the problem of optimal filtering, then Section III derives a candidate: the SCI filter. Section IV investigates the usefulness of such a filter, and Section V discusses the results, in particular in simulations. Finally, Section VI provides some perspectives and future directions.

Notation. In the sequel, vectors are underlined e.g., $\underline{x} \in \mathbb{R}^n$, random variables are denoted in lowercase boldface letters e.g., $\boldsymbol{z}$ for a scalar or $\underline{\boldsymbol{x}}$ for a vector, and matrices are denoted in uppercase boldface variables e.g., $\boldsymbol{M} \in \mathbb{R}^{n\times n}$. The notation $\mathrm{E}[\cdot]$ denotes expected value of a random variable and $\|\cdot\|$ the Euclidean norm of a vector. The trace, the determinant, the inverse and the transpose of a matrix $\boldsymbol{M}$ and the identity matrix are denoted as $\operatorname{tr}\boldsymbol{M}$, $|\boldsymbol{M}|$, $\boldsymbol{M}^{-1}$, $\boldsymbol{M}^\intercal$ and $\boldsymbol{I}$ respectively. For two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, $\boldsymbol{A} \preceq \boldsymbol{B}$ means that the difference $\boldsymbol{B}-\boldsymbol{A}$ is positive semi-definite. A positive definite matrix $\boldsymbol{P}$ is represented on the figures by the ellipsoid $\mathcal{E}_{\boldsymbol{P}} = \left\{\underline{x} \mid \underline{x}^\intercal \boldsymbol{P}^{-1}\underline{x} \leq 1\right\}$.

II. Problem statement

Let us first recall the definition of a consistent estimator.

Definition 1. *An estimator $(\underline{\hat{\boldsymbol{x}}}, \boldsymbol{P})$ of a random variable $\underline{\boldsymbol{x}}$ is said to be* consistent *if $\mathrm{E}[\underline{\tilde{\boldsymbol{x}}}] = 0$ and $\tilde{\boldsymbol{P}} \preceq \boldsymbol{P}$ where $\underline{\tilde{\boldsymbol{x}}} = \underline{\boldsymbol{x}} - \underline{\hat{\boldsymbol{x}}}$ denotes the error and $\tilde{\boldsymbol{P}} = \mathrm{E}[\underline{\tilde{\boldsymbol{x}}}\underline{\tilde{\boldsymbol{x}}}^\intercal]$ the mean-squared error (MSE).*

In other words, a consistent estimator is unbiased and does not underestimate the covariance.

Consider two agents, denoted A and B, characterized by their states $\underline{\boldsymbol{x}}_A$ and $\underline{\boldsymbol{x}}_B$ in $\mathbb{R}^n$. For the sake of simplicity and without loss of generality, the states are assumed to contain only the positions of the agents; in practice, they may also contain their orientations or their velocities, for example. The two agents have consistent estimators of their states denoted $(\underline{\hat{\boldsymbol{x}}}_A, \boldsymbol{P}_A)$ and $(\underline{\hat{\boldsymbol{x}}}_B, \boldsymbol{P}_B)$. Notice that the true (centralized) covariances of the errors, defined as,

$$\tilde{\boldsymbol{P}} = \mathrm{E}\left[\begin{pmatrix}\underline{\tilde{\boldsymbol{x}}}_A \\ \underline{\tilde{\boldsymbol{x}}}_B\end{pmatrix}\begin{pmatrix}\underline{\tilde{\boldsymbol{x}}}_A \\ \underline{\tilde{\boldsymbol{x}}}_B\end{pmatrix}^\intercal\right] = \begin{bmatrix}\tilde{\boldsymbol{P}}_A & \tilde{\boldsymbol{P}}_{AB} \\ \tilde{\boldsymbol{P}}_{AB}^\intercal & \tilde{\boldsymbol{P}}_B\end{bmatrix}, \tag{1}$$

is unknown, i.e., $\tilde{\boldsymbol{P}}_A$, $\tilde{\boldsymbol{P}}_B$, and $\tilde{\boldsymbol{P}}_{AB}$ are unknown. However, consistency restrains the set of possible $\tilde{\boldsymbol{P}}$ to:

$$\boldsymbol{\mathcal{P}} = \left\{\begin{bmatrix}\tilde{\boldsymbol{P}}_A & \tilde{\boldsymbol{P}}_{AB} \\ \tilde{\boldsymbol{P}}_{AB}^\intercal & \tilde{\boldsymbol{P}}_B\end{bmatrix} \succeq \boldsymbol{0} \mid \tilde{\boldsymbol{P}}_A \preceq \boldsymbol{P}_A, \tilde{\boldsymbol{P}}_B \preceq \boldsymbol{P}_B\right\}. \tag{2}$$

Furthermore, consider $\boldsymbol{z} = \|\underline{\boldsymbol{x}}_A - \underline{\boldsymbol{x}}_B\| + \tilde{\boldsymbol{z}}$ a measurement of the distance between A and B where $\tilde{\boldsymbol{z}}$ denotes the error. The error $\tilde{\boldsymbol{z}}$ is assumed centered, $\mathrm{E}[\tilde{\boldsymbol{z}}] = 0$, with variance $\mathrm{E}\left[\tilde{\boldsymbol{z}}^2\right] = \sigma_m^2$, and independent of the errors of the estimators $\underline{\tilde{\boldsymbol{x}}}_A$ and $\underline{\tilde{\boldsymbol{x}}}_B$. Finally, consider the agents sufficiently far from each other to assume that the unit-length director vectors $\underline{u}_{BA} = \frac{\underline{x}_A - \underline{x}_B}{\|\underline{x}_A - \underline{x}_B\|}$ and $\underline{\hat{u}}_{BA} = \frac{\underline{\hat{x}}_A - \underline{\hat{x}}_B}{\|\underline{\hat{x}}_A - \underline{\hat{x}}_B\|}$ are equal and allow the linearization of the measurement as:

$$\boldsymbol{z} = \underline{u}_{BA}^\intercal\left(\underline{\boldsymbol{x}}_A - \underline{\boldsymbol{x}}_B\right) + \tilde{\boldsymbol{z}}. \tag{3}$$

The objective is to *improve* the estimator of A using the estimator of B and the distance measurement to create a *better* estimator $(\underline{\hat{\boldsymbol{x}}}_F, \boldsymbol{P}_F)$ of the state of A. Throughout this paper, the estimators are compared w.r.t. an increasing cost function J (increasing in the sense of the Loewner ordering i.e., $\boldsymbol{P} \prec \boldsymbol{Q} \implies J(\boldsymbol{P}) < J(\boldsymbol{Q})$). For two consistent estimators $(\underline{\hat{\boldsymbol{x}}}_1, \boldsymbol{P}_1)$ and $(\underline{\hat{\boldsymbol{x}}}_2, \boldsymbol{P}_2)$ of the same random variable $\underline{\boldsymbol{x}}$, $(\underline{\hat{\boldsymbol{x}}}_1, \boldsymbol{P}_1)$ is said to be *better* than $(\underline{\hat{\boldsymbol{x}}}_2, \boldsymbol{P}_2)$ if $J(\boldsymbol{P}_1) < J(\boldsymbol{P}_2)$. The estimator $(\underline{\hat{\boldsymbol{x}}}_F, \boldsymbol{P}_F)$ is designed as an unbiased[1] *linear filter* defined as follows.

Definition 2. *A* linear filter *for the state of A is an estimator $(\underline{\hat{\boldsymbol{x}}}_F, \boldsymbol{P}_F)$ of $\underline{\boldsymbol{x}}_A$ where $\underline{\hat{\boldsymbol{x}}}_F$ is a linear combination of $\underline{\hat{\boldsymbol{x}}}_A$, $\underline{\hat{\boldsymbol{x}}}_B$ and $\boldsymbol{z}$. It is defined by two matrices $\boldsymbol{K}_F, \boldsymbol{L}_F$ and a vector $\underline{w}_F$ such that:*

$$\underline{\hat{\boldsymbol{x}}}_F = \boldsymbol{K}_F\underline{\hat{\boldsymbol{x}}}_A + \boldsymbol{L}_F\underline{\hat{\boldsymbol{x}}}_B + \underline{w}_F\boldsymbol{z}. \tag{4}$$

To have $\underline{\hat{\boldsymbol{x}}}_F$ unbiased, since $\mathrm{E}[\boldsymbol{z}] = \underline{u}_{BA}^\intercal\left(\mathrm{E}\left[\underline{\boldsymbol{x}}_A\right] - \mathrm{E}\left[\underline{\boldsymbol{x}}_B\right]\right)$ the gains must be dependent and satisfy:

$$\boldsymbol{K}_F = \boldsymbol{I} - \underline{w}_F\underline{u}_{BA}^\intercal, \qquad \boldsymbol{L}_F = \underline{w}_F\underline{u}_{BA}^\intercal. \tag{5}$$

An unbiased linear filter is then defined by only $\underline{w}_F$ as:

$$\underline{\hat{\boldsymbol{x}}}_F = \left(\boldsymbol{I} - \underline{w}_F\underline{u}_{BA}^\intercal\right)\underline{\hat{\boldsymbol{x}}}_A + \underline{w}_F\underline{u}_{BA}^\intercal\underline{\hat{\boldsymbol{x}}}_B + \underline{w}_F\boldsymbol{z}. \tag{6}$$

For a possible $\tilde{\boldsymbol{P}} \in \boldsymbol{\mathcal{P}}$, the covariance of the error $\underline{\tilde{\boldsymbol{x}}}_F = \underline{\boldsymbol{x}}_A - \underline{\hat{\boldsymbol{x}}}_F$ is:

$$\begin{aligned}\tilde{\boldsymbol{P}}_F = \tilde{\boldsymbol{P}}_A &+ \left(\tilde{\sigma}_A^2 + \tilde{\sigma}_B^2 - 2\tilde{\gamma} + \sigma_m^2\right)\underline{w}_F\underline{w}_F^\intercal \\ &- \left(\tilde{\boldsymbol{P}}_A - \tilde{\boldsymbol{P}}_{AB}\right)\underline{u}_{BA}\underline{w}_F^\intercal - \underline{w}_F\underline{u}_{BA}^\intercal\left(\tilde{\boldsymbol{P}}_A - \tilde{\boldsymbol{P}}_{AB}^\intercal\right)\end{aligned} \tag{7}$$

where

$$\tilde{\sigma}_A^2 = \underline{u}_{BA}^\intercal\tilde{\boldsymbol{P}}_A\underline{u}_{BA}, \qquad \tilde{\sigma}_B^2 = \underline{u}_{BA}^\intercal\tilde{\boldsymbol{P}}_B\underline{u}_{BA}, \tag{8}$$

$$\tilde{\gamma} = \underline{u}_{BA}^\intercal\tilde{\boldsymbol{P}}_{AB}\underline{u}_{BA}. \tag{9}$$

If the true covariance $\tilde{\boldsymbol{P}}$ were known, the *optimal* linear filter would be calculable. It would be derived from the classical Kalman Filter equations [2]:

$$\underline{\hat{\boldsymbol{x}}}_F^* = \underline{\hat{\boldsymbol{x}}}_A + \frac{\boldsymbol{z} - \underline{u}_{BA}^\intercal\left(\underline{\hat{\boldsymbol{x}}}_A - \underline{\hat{\boldsymbol{x}}}_B\right)}{\tilde{\sigma}_A^2 + \tilde{\sigma}_B^2 - 2\tilde{\gamma} + \sigma_m^2}(\tilde{\boldsymbol{P}}_A - \tilde{\boldsymbol{P}}_{AB})\underline{u}_{BA}, \tag{10a}$$

$$\tilde{\boldsymbol{P}}_F^* = \tilde{\boldsymbol{P}}_A - \frac{(\tilde{\boldsymbol{P}}_A - \tilde{\boldsymbol{P}}_{AB})\underline{u}_{BA}\underline{u}_{BA}^\intercal(\tilde{\boldsymbol{P}}_A - \tilde{\boldsymbol{P}}_{AB}^\intercal)}{\tilde{\sigma}_A^2 + \tilde{\sigma}_B^2 - 2\tilde{\gamma} + \sigma_m^2}. \tag{10b}$$

[1] A bias would increase the MSE $\tilde{\boldsymbol{P}}_F$ in the Loewner ordering sense.

In this case, $\tilde{\boldsymbol{P}}_F^*$ is the minimum (in the Loewner ordering) of the $\tilde{\boldsymbol{P}}_F$, and thus is optimal w.r.t. any increasing function J.

However, as $\tilde{\boldsymbol{P}}$ is not known, such a consideration is not possible, and the linear filter must be consistent, i.e., it must satisfy $\boldsymbol{P}_F \succeq \tilde{\boldsymbol{P}}_F$ for every possible covariance. We are now in a position to state the main problem.

Problem 1. *Find a gain $\underline{w}_F$ and a covariance $\boldsymbol{P}_F$ such that the estimator $(\hat{\underline{\boldsymbol{x}}}_F, \boldsymbol{P}_F)$ defined by* (6) *is consistent and optimal w.r.t. J. In other words:*

$$\begin{cases} \underset{\underline{w}_F, \boldsymbol{P}_F}{\text{minimize}} & J(\boldsymbol{P}_F) \\ \textit{subject to} & \forall \tilde{\boldsymbol{P}} \in \boldsymbol{\mathcal{P}}, \quad \boldsymbol{P}_F \succeq \tilde{\boldsymbol{P}}_F \end{cases} \tag{P$_1$}$$

where $\tilde{\boldsymbol{P}}_F$ is given by (7)*.*

The main result of this paper is the design of a candidate filter solving Problem 1 using SCI and its analysis.

III. SCI FOR A DISTANCE MEASUREMENT

As a preliminary remark, (6) can be rewritten as:

$$\hat{\underline{\boldsymbol{x}}}_F = \hat{\underline{\boldsymbol{x}}}_A + \underline{w}_F \left[\boldsymbol{z} - \underline{u}_{BA}^\intercal \left(\hat{\underline{\boldsymbol{x}}}_A - \hat{\underline{\boldsymbol{x}}}_B\right)\right] \tag{11}$$

$$= \left(\boldsymbol{I} - \underline{w}_F \underline{u}_{BA}^\intercal\right) \hat{\underline{\boldsymbol{x}}}_A + \underline{w}_F \underline{u}_{BA}^\intercal \left(\hat{\underline{\boldsymbol{x}}}_B + \boldsymbol{z}\underline{u}_{BA}\right). \tag{12}$$

These two expressions highlight the equivalence between linear filtering and fusion. Equation (11) represents the usual form of the Kalman correction step: $\underline{w}_F$ is a gain and $\boldsymbol{z} - \underline{u}_{BA}^\intercal \left(\hat{\underline{\boldsymbol{x}}}_A - \hat{\underline{\boldsymbol{x}}}_B\right)$ is the innovation on the measurement. Equation (12) on the other hand represents a fusion: the term $\hat{\underline{\boldsymbol{x}}}_B + \boldsymbol{z}\underline{u}_{BA}$ is another estimator of the state of A. Let $\hat{\underline{\boldsymbol{x}}}'_A$ denote this estimator. Equation (12) corresponds to the fusion of $\hat{\underline{\boldsymbol{x}}}_A$ and the observation of $\hat{\underline{\boldsymbol{x}}}'_A$ through $\underline{u}_{BA}$.

Since the estimators $\hat{\underline{\boldsymbol{x}}}_A$ and $\hat{\underline{\boldsymbol{x}}}_B$ are correlated to an unknown degree and $\boldsymbol{z}$ is independent of $\hat{\underline{\boldsymbol{x}}}_A$, the fusion of $\hat{\underline{\boldsymbol{x}}}_A$ and $\hat{\underline{\boldsymbol{x}}}'_A$ respects the assumptions of SCI [11].

SCI fuses two estimators using a linear combination whose weights depend on their covariances. For any $\omega \in [0, 1)$, SCI provides a consistent estimator $(\hat{\underline{\boldsymbol{x}}}_{\text{SCI}}(\omega), \boldsymbol{P}_{\text{SCI}}(\omega))$ for the state of A, hereafter called an SCI filter. It is defined as:

$$\boldsymbol{P}_{\text{SCI}}(\omega) = \left[(1-\omega)\boldsymbol{P}_A^{-1} + \omega \frac{\underline{u}_{BA}\underline{u}_{BA}^\intercal}{\sigma_B^2 + \omega\sigma_m^2}\right]^{-1}, \tag{13a}$$

$$\hat{\underline{\boldsymbol{x}}}_{\text{SCI}}(\omega) = (1-\omega)\boldsymbol{P}_{\text{SCI}}(\omega)\boldsymbol{P}_A^{-1}\hat{\underline{\boldsymbol{x}}}_A + \omega\boldsymbol{P}_{\text{SCI}}(\omega)\frac{\underline{u}_{BA}\underline{u}_{BA}^\intercal}{\sigma_B^2 + \omega\sigma_m^2}\left(\hat{\underline{\boldsymbol{x}}}_B + \boldsymbol{z}\underline{u}_{BA}\right), \tag{13b}$$

where $\sigma_B^2 = \underline{u}_{BA}^\intercal \boldsymbol{P}_B \underline{u}_{BA}$. The expression of the SCI filter can be simplified. The covariance (13a) is calculable using the Sherman-Morrison formula of the inverse [9, Chap. 0.7], and the estimator (13b) is re-expressed in the form of a correction as (11). Thus, (13) becomes:

$$\hat{\underline{\boldsymbol{x}}}_{\text{SCI}}(\omega) = \hat{\underline{\boldsymbol{x}}}_A + \frac{\omega\left[\boldsymbol{z} - \underline{u}_{BA}^\intercal(\hat{\underline{\boldsymbol{x}}}_A - \hat{\underline{\boldsymbol{x}}}_B)\right]}{\omega\sigma_A^2 + (1-\omega)\left(\sigma_B^2 + \omega\sigma_m^2\right)}\boldsymbol{P}_A\underline{u}_{BA}, \tag{14a}$$

$$\boldsymbol{P}_{\text{SCI}}(\omega) = \frac{1}{1-\omega}\left[\boldsymbol{P}_A - \frac{\omega\boldsymbol{P}_A\underline{u}_{BA}\underline{u}_{BA}^\intercal\boldsymbol{P}_A}{\omega\sigma_A^2 + (1-\omega)\left(\sigma_B^2 + \omega\sigma_m^2\right)}\right], \tag{14b}$$

where $\sigma_A^2 = \underline{u}_{BA}^\intercal \boldsymbol{P}_A \underline{u}_{BA}$. The gain of the SCI filter is $\underline{w}_{\text{SCI}}(\omega) = \frac{\omega}{\omega\sigma_A^2 + (1-\omega)\left(\sigma_B^2 + \omega\sigma_m^2\right)}\boldsymbol{P}_A\underline{u}_{BA}$. The optimal filter w.r.t. J within the family of SCI filter is called the *optimal SCI filter*. To find the optimal SCI filter, the parameter ω is chosen to minimize the cost function J:

$$\omega^* = \underset{0\leq\omega\leq 1}{\arg\min}\, J(\boldsymbol{P}_{\text{SCI}}(\omega)). \tag{15}$$

We have proved the following result.

Theorem 1. *The optimal SCI filter for the state of A is $(\hat{\underline{\boldsymbol{x}}}_{SCI}(\omega^*), \boldsymbol{P}_{SCI}(\omega^*))$ given by* (14) *where the parameter ω^* is defined by* (15)*. This estimator is denoted $(\hat{\underline{\boldsymbol{x}}}^*_{SCI}, \boldsymbol{P}^*_{SCI})$.*

SCI has not been shown to be optimal for the fusion, therefore the optimal SCI filter $(\hat{\underline{\boldsymbol{x}}}^*_{\text{SCI}}, \boldsymbol{P}^*_{\text{SCI}})$ may be suboptimal for Problem 1. However, since it is very simple to implement, it is widely used in practice. Furthermore, when the measurement is very accurate, i.e., when σ_m^2 tends to 0, SCI becomes CI which is optimal (only when $\sigma_m^2 = 0$) [18]. These two reasons lead us to consider this filter and analyze its usefulness.

IV. USEFULNESS OF THE FILTERING

A. General increasing cost function

Theorem 1 gives the expression of the optimal SCI filter. Setting $\omega = 0$ in the fusion ensures that the estimator $(\hat{\underline{\boldsymbol{x}}}^*_{\text{SCI}}, \boldsymbol{P}^*_{\text{SCI}})$ is at least as good as the original estimator $(\hat{\underline{\boldsymbol{x}}}_A, \boldsymbol{P}_A)$. In fact, setting $\omega = 0$ corresponds to keeping the original estimator and ignoring the measurement. However, there is no reason why there should be a better estimator, the optimal parameter could be $\omega^* = 0$. In such a case, SCI filters cannot improve the estimator of A. We call *pertinent* a linear filter for the state of A that provides a better estimator than $(\hat{\underline{\boldsymbol{x}}}_A, \boldsymbol{P}_A)$. This section characterizes the pertinence of SCI filters.

Definition 3. *A* pertinent *linear filter for the state of A is a linear filter $(\hat{\underline{\boldsymbol{x}}}_F, \boldsymbol{P}_F)$ such that $J(\boldsymbol{P}_F) < J(\boldsymbol{P}_A)$.*

By definition of the optimal SCI filter, there is a pertinent SCI filter if and only if the optimal SCI filter is pertinent. The underlying question is therefore: Is the optimal SCI filter $(\hat{\underline{\boldsymbol{x}}}^*_{\text{SCI}}, \boldsymbol{P}^*_{\text{SCI}})$ pertinent?

Let us start with two corollaries of Theorem 1 that provide a necessary condition for the existence of a pertinent SCI filter.

Corollary 1. *If $\sigma_A^2 \leq \sigma_B^2$, there is not any pertinent SCI filter for the state of A.*

Proof. Assume that $\sigma_A^2 \leq \sigma_B^2$ and let $\omega \in [0, 1)$. We will show that $\boldsymbol{P}_{\text{SCI}}(\omega) \succeq \boldsymbol{P}_A$ which is sufficient since J is increasing. In (13a), $\boldsymbol{P}_{\text{SCI}}^{-1}(\omega)$ is expressed as a convex combination of $\boldsymbol{P}_A^{-1}$ and $\boldsymbol{Q} = \frac{\underline{u}_{BA}\underline{u}_{BA}^\intercal}{\sigma_B^2 + \omega\sigma_m^2}$. Therefore, if $\boldsymbol{Q} \preceq \boldsymbol{P}_A^{-1}$, then $\boldsymbol{P}_{\text{SCI}}^{-1}(\omega) \preceq \boldsymbol{P}_A^{-1}$. Let us prove that $\boldsymbol{Q} \preceq \boldsymbol{P}_A^{-1}$, by proving that $\forall \underline{v}$, $\underline{v}^\intercal(\boldsymbol{P}_A^{-1} - \boldsymbol{Q})\underline{v} \geq 0$. Write $\underline{v} = \alpha\underline{u}_{BA} + \underline{w}$ with $\alpha \in \mathbb{R}$ and $\underline{w}^\intercal\underline{u}_{BA} = 0$. Then, $\underline{v}^\intercal\boldsymbol{Q}\underline{v} = \alpha^2(\sigma_B^2 + \omega\sigma_m^2)^{-1} \leq \alpha^2/\sigma_A^2$. Finally, let us prove that $\underline{v}^\intercal\boldsymbol{P}_A^{-1}\underline{v} \geq \alpha^2/\sigma_A^2$.

Consider an orthonormal basis $\mathcal{B} = (\underline{u}_{BA}, \underline{u}_2, \ldots, \underline{u}_n)$ containing $\underline{u}_{BA}$ and the orthogonal matrix $\boldsymbol{R} =$

$\begin{bmatrix}\underline{u}_{BA} & \underline{u}_2 & \cdots & \underline{u}_n\end{bmatrix}$. In this basis, the covariance of the estimator of A and the vector $\underline{v}$ become:

$$\boldsymbol{P}_A = \boldsymbol{R}\begin{bmatrix}\sigma_A^2 & \underline{b}^\intercal \\ \underline{b} & \boldsymbol{C}\end{bmatrix}\boldsymbol{R}^\intercal, \qquad \underline{v} = \boldsymbol{R}\begin{pmatrix}\alpha \\ \underline{w}'\end{pmatrix}$$

where $\underline{b}, \underline{w}' \in \mathbb{R}^{n-1}$ and $\boldsymbol{C} \in \mathbb{R}^{(n-1)\times(n-1)}$. Let $\boldsymbol{S} = \boldsymbol{C} - \underline{b}\underline{b}^\intercal/\sigma_A^2$ be the Schur complement of the first entry. $\boldsymbol{S}$ is invertible and $\boldsymbol{S}^{-1} \succeq \boldsymbol{0}$, see, e.g., [9, Chap. 7]. The inverse of $\boldsymbol{P}_A$ and $\underline{v}^\intercal \boldsymbol{P}_A^{-1}\underline{v}$ become:

$$\boldsymbol{P}_A^{-1} = \boldsymbol{R}\begin{bmatrix}\sigma_A^{-2} + \sigma_A^{-4}\underline{b}^\intercal\boldsymbol{S}^{-1}\underline{b} & -\sigma_A^{-2}\underline{b}^\intercal\boldsymbol{S}^{-1} \\ -\sigma_A^{-2}\boldsymbol{S}^{-1}\underline{b} & \boldsymbol{S}^{-1}\end{bmatrix}\boldsymbol{R}^\intercal,$$

$$\underline{v}^\intercal\boldsymbol{P}_A^{-1}\underline{v} = \frac{\alpha^2}{\sigma_A^2} + \left(\frac{\alpha}{\sigma_A^2}\underline{b} - \underline{w}'\right)^\intercal \boldsymbol{S}^{-1}\left(\frac{\alpha}{\sigma_A^2}\underline{b} - \underline{w}'\right).$$

Hence, $\underline{v}^\intercal\boldsymbol{P}_A^{-1}\underline{v} \geq \frac{\alpha^2}{\sigma_A^2}$. □

Corollary 2. *If there is a pertinent SCI filter for the state of A, then there is not any pertinent SCI filter for the state of B.*

Proof. If there is a pertinent SCI filter for the state of A, Corollary 1 implies $\sigma_B^2 < \sigma_A^2$. Similarly, if there is a pertinent SCI filter for the state of B, then $\sigma_A^2 < \sigma_B^2$. Both inequalities cannot hold simultaneously. □

Corollary 1 provides only a necessary condition for the existence of pertinent SCI filters. In the following, we extend the property to provide a necessary and sufficient condition for the two most commonly used cost functions: the trace and the determinant. It is based on the following result.

Theorem 2. *Assume that the function $f : \omega \mapsto J(\boldsymbol{P}_{SCI}(\omega))$, with $\boldsymbol{P}_{SCI}(\omega)$ given by* (14b)*, is convex on $[0, 1)$. Then, the optimal SCI filter $(\hat{\underline{\boldsymbol{x}}}^*_{SCI}, \boldsymbol{P}^*_{SCI})$ is pertinent if and only if $f'(0) < 0$.*

Proof. By definition, $(\hat{\underline{\boldsymbol{x}}}^*_{\text{SCI}}, \boldsymbol{P}^*_{\text{SCI}})$ is pertinent if and only if $f(\omega^*) < f(0) = J(\boldsymbol{P}_A)$. If $f'(0) < 0$, then there is a $\omega_0 > 0$ such that $f(\omega_0) < f(0)$ and therefore $f(\omega^*) < f(0)$. If $f'(0) \geq 0$, since f is convex on $[0, 1)$, $\forall\omega \in [0, 1)$, $f(\omega) \geq f(0)$ and therefore, $f(\omega^*) = f(0)$. □

B. Particular case of the trace

In this paragraph, consider $J(\cdot) = \operatorname{tr}\cdot$ and let $g : \omega \mapsto \operatorname{tr}\boldsymbol{P}_{\text{SCI}}(\omega)$ be the cost function to be optimized. According to (14b), the cost function is:

$$g(\omega) = \frac{\operatorname{tr}\boldsymbol{P}_A}{1-\omega}\left[1 - \frac{\omega r_A\sigma_A^2}{\omega\sigma_A^2 + (1-\omega)(\sigma_B^2 + \omega\sigma_m^2)}\right] \tag{16}$$

where:

$$r_A = \frac{\frac{1}{\sigma_A^2}\left\|\boldsymbol{P}_A\underline{u}_{BA}\right\|^2}{\operatorname{tr}\boldsymbol{P}_A}. \tag{17}$$

The meaning of r_A is discussed in Section V. In order to apply Theorem 2, let us prove the following.

Lemma 1. *The cost function g is convex on $[0, 1)$.*

Proof. Let us first prove that $0 < r_A \leq 1$. Consider again the orthonormal basis $\mathcal{B} = (\underline{u}_{BA}, \underline{u}_2, \ldots, \underline{u}_n)$ containing $\underline{u}_{BA}$ and the orthogonal matrix $\boldsymbol{R} = \begin{bmatrix}\underline{u}_{BA} & \underline{u}_2 & \cdots & \underline{u}_n\end{bmatrix}$. In this basis, the covariance of the estimator of A becomes:

$$\boldsymbol{R}^\intercal\boldsymbol{P}_A\boldsymbol{R} = \begin{bmatrix}\sigma_A^2 & \rho_2\sigma_A\sigma_2 & \cdots & \rho_n\sigma_A\sigma_n \\ \rho_2\sigma_A\sigma_2 & \sigma_2^2 & * & * \\ \vdots & * & \ddots & * \\ \rho_n\sigma_A\sigma_n & * & * & \sigma_n^2\end{bmatrix} \tag{18}$$

where only the diagonal coefficients and the correlations with the first component have been labeled. With these notations, r_A is developed as:

$$r_A = \frac{\frac{1}{\sigma_A^2}\left\|\boldsymbol{P}_A\underline{u}_{BA}\right\|^2}{\operatorname{tr}\boldsymbol{P}_A} = \frac{\sigma_A^2 + \sum_{i=2}^n \rho_i^2\sigma_i^2}{\sigma_A^2 + \sum_{i=2}^n \sigma_i^2}. \tag{19}$$

Since the correlations satisfy $|\rho_i| \leq 1$, the ratio satisfies $0 < r_A \leq 1$.

To prove the convexity of g, let us first assume that $r_A < 1$. By putting all the terms in the same fraction, the cost function g is a rational function:

$$\begin{aligned} g(\omega) &= \frac{\operatorname{tr}\boldsymbol{P}_A}{1-\omega}\frac{(1-r_A)\omega\sigma_A^2 + (1-\omega)(\sigma_B^2 + \omega\sigma_m^2)}{\omega\sigma_A^2 + (1-\omega)(\sigma_B^2 + \omega\sigma_m^2)} \\ &= \frac{\operatorname{tr}\boldsymbol{P}_A}{1-\omega}\frac{\sigma_B^2 + [(1-r_A)\sigma_A^2 - \sigma_B^2 + \sigma_m^2]\omega - \sigma_m^2\omega^2}{\sigma_B^2 + (\sigma_A^2 - \sigma_B^2 + \sigma_m^2)\omega - \sigma_m^2\omega^2}. \end{aligned}$$

Let $P : \omega \mapsto \sigma_B^2 + (\sigma_A^2 - \sigma_B^2 + \sigma_m^2)\omega - \sigma_m^2\omega^2$ and $Q : \omega \mapsto \sigma_B^2 + [(1-r_A)\sigma_A^2 - \sigma_B^2 + \sigma_m^2]\omega - \sigma_m^2\omega^2$ be the polynomials of degree 2 at the denominator and numerator. Since $P(0) = \sigma_B^2$ and $P(1) = \sigma_A^2$, P has two roots b and d which satisfy: $b < 0 < 1 < d$. Similarly, by noting that $Q(\omega) = P(\omega) - r_A\sigma_A^2\omega$, $Q(b) = -r_Ab\sigma_A^2 > 0$, $Q(1) = (1-r_A)\sigma_A^2 > 0$ and $Q(d) = -r_Ad\sigma_A^2 < 0$, Q has two roots a and c which satisfy:

$$a < b < 0 < 1 < c < d.$$

The cost function is therefore:

$$g(\omega) = -\frac{\operatorname{tr}\boldsymbol{P}_A(\omega-a)(\omega-c)}{(\omega-1)(\omega-b)(\omega-d)}.$$

Then, using partial fraction decomposition:

$$g(\omega) = \operatorname{tr}\boldsymbol{P}_A\left(\frac{A}{\omega-1} + \frac{B}{\omega-b} + \frac{C}{\omega-d}\right) \tag{20}$$

with:

$$A = -\frac{(1-a)(1-c)}{(1-b)(1-d)} < 0, \quad B = -\frac{(b-a)(b-c)}{(b-1)(b-d)} > 0,$$
$$C = -\frac{(d-a)(d-c)}{(d-1)(d-b)} < 0.$$

Since $\operatorname{tr}\boldsymbol{P}_A > 0$, the three terms in (20) are convex on $(b, 1)$. Therefore, since $b < 0$, the cost function g is convex on $[0, 1)$.

Finally, if $r_A = 1$, then $c = 1$ which simplifies the expression of g. By the same logic, g is convex on (b, d), and thus on $[0, 1]$. Note that $r_A = 1$ corresponds to the one-dimensional case. □

We are now in a position to apply Theorem 2 to the trace.

Corollary 3. *The optimal SCI filter for the state of A w.r.t. to the trace is pertinent if and only if:*

$$\sigma_B^2 < r_A\sigma_A^2. \tag{21}$$

Proof. Since g is convex on $[0,1)$ as stated in Lemma 1, by Theorem 2, the optimal SCI filter is pertinent if and only if $g'(0) < 0$. Let us calculate $g'(0)$.

$$g'(0) = \operatorname{tr} P_A \left(1 - r_A \frac{\sigma_A^2}{\sigma_B^2}\right)$$

Since $\operatorname{tr} P_A > 0$, $g'(0) < 0$ if and only of $\sigma_B^2 < r_A \sigma_A^2$. □

C. Particular case of the determinant

In this paragraph, consider $J(\cdot) = |\cdot|$ and let $h : \omega \mapsto |\boldsymbol{P}_{\text{SCI}}(\omega)|$ be the cost function to be optimized. According to (14b), the cost function is:

$$h(\omega) = \frac{|\boldsymbol{P}_A|}{(1-\omega)^n} \left| \boldsymbol{I} - \frac{\omega \boldsymbol{P}_A^{1/2} \underline{u}_{BA} \underline{u}_{BA}^\intercal \boldsymbol{P}_A^{1/2}}{\omega\sigma_A^2 + (1-\omega)(\sigma_B^2 + \omega\sigma_m^2)} \right| \quad (22)$$

where $\boldsymbol{P}_A^{1/2}$ denotes the square root of $\boldsymbol{P}_A$ and n is the dimension of the state. Using that $|I + \underline{u}\underline{v}^\intercal| = 1 + \underline{u}^\intercal \underline{v}$, (22) becomes:

$$h(\omega) = \frac{|\boldsymbol{P}_A|}{(1-\omega)^{n-1}} \frac{\sigma_B^2 + \omega\sigma_m^2}{\omega\sigma_A^2 + (1-\omega)[\sigma_B^2 + \omega\sigma_m^2]}. \quad (23)$$

As for the trace, the convexity of h is proven in the following result.

Lemma 2. *The cost function h is convex on $[0,1)$.*

Proof. This proof is very similar to the proof of Lemma 1. First, let $P : \omega \mapsto \sigma_B^2 + (\sigma_A^2 - \sigma_B^2 + \sigma_m^2)\omega - \sigma_m^2\omega^2$. As $P(0) > 0$, $P(1) > 0$ and $P(-\sigma_m^2/\sigma_B^2) < 0$, The cost function h is a rational function which can be expressed as follows.

$$h(\omega) = -\frac{|\boldsymbol{P}_A|\,(\omega + \sigma_m^2/\sigma_B^2)}{(1-\omega)^{n-1}(\omega - a)(\omega - b)}$$

where the zeros of P a and b satisfy:

$$-\sigma_m^2/\sigma_B^2 < a < 0 < 1 < b.$$

Then, using partial fraction decomposition:

$$h(\omega) = \frac{|\boldsymbol{P}_A|\,A}{(\omega - a)(1-\omega)^{n-1}} + \frac{|\boldsymbol{P}_A|\,B}{(\omega - b)(1-\omega)^{n-1}} \quad (24)$$

with:

$$A = \frac{-(a + \sigma_m^2/\sigma_B^2)}{(a-b)} > 0, \quad B = \frac{-(b + \sigma_m^2/\sigma_B^2)}{(b-a)} < 0.$$

Both terms in (24) are convex on $[0,1)$. □

We are now in a position to apply Theorem 2 to the determinant.

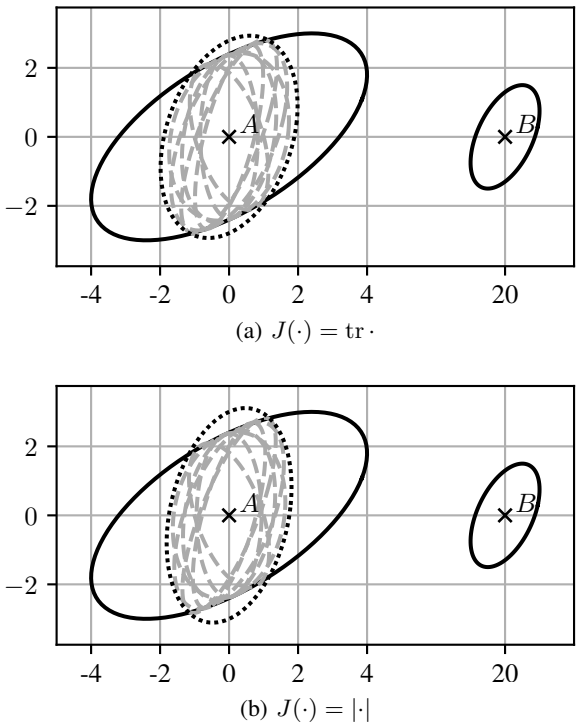
Corollary 4. *The optimal SCI filter for the state of A w.r.t. to the determinant is pertinent if and only if:*

$$\sigma_B^2 < \frac{1}{n}\sigma_A^2. \quad (25)$$

Proof. Since h is convex on $[0,1)$ as stated in Lemma 2, the optimal SCI filter is pertinent if and only if $h'(0) < 0$. Let us calculate $g'(0)$.

$$h'(0) = |P_A| \left(n - \frac{\sigma_A^2}{\sigma_B^2}\right)$$

Since $|P_A| > 0$, $h'(0) < 0$ if and only if $n\sigma_B^2 < \sigma_A^2$. □



(a) $J(\cdot) = \operatorname{tr} \cdot$

(b) $J(\cdot) = |\cdot|$

Fig. 1: Example of the optimal SCI filters for two different cost functions J. The solid ellipses represent the covariance of the estimators $\boldsymbol{P}_A$ and $\boldsymbol{P}_B$. The dotted ellipse represents the covariance of the optimal SCI filter $\boldsymbol{P}^*_{\text{SCI}}$. The dashed ellipses represent possible covariances $\tilde{\boldsymbol{P}}_F$ obtained by (7) with $\underline{w}_F = \underline{w}_{\text{SCI}}(\omega^*)$ and different $\tilde{\boldsymbol{P}} \in \boldsymbol{\mathcal{P}}$. The measurement variance is $\sigma_m^2 = 1$.

V. Discussion and numerical simulations

Illustrations of the optimal SCI filters for the trace and the determinant are shown in Fig. 1. In this figure, the consistency of the filter has been illustrated by generating several possible $\tilde{\boldsymbol{P}} \in \boldsymbol{\mathcal{P}}$ and plotting the resulting MSE given by (7). As observed, the two optimal SCI filters are different. This means that they depend on the cost function which is a first difference with the usual case where $\tilde{\boldsymbol{P}}$ is known.

The main implication of Theorem 1 and Corollary 1 is that it is not always possible to improve an estimator using SCI. Improvements require that the *helping* agent, here Agent B, has a sufficiently good precision (w.r.t. Agent A) in the direction of the measure, i.e., that $\sigma_B^2 < \sigma_A^2$ as stated in Corollary 1. Furthermore, this condition is only necessary, harder constraints should be satisfied in practice, they depend on the cost function J. It can happen that neither agent can improve its estimator, for example when considering the trace, if $r_A\sigma_A^2 < \sigma_B^2 < \sigma_A^2$ as shown in Corollary 3. This is an important difference from the usual case. If $\tilde{\boldsymbol{P}}$ were known, it would *almost* always be possible to improve the estimator: the optimal linear filter was recalled in (10) (Section II). In this case, even extremely poor precision on the helping agent would provide (tiny) improvements.

The expression of the SCI filter (14) involves the whole

statistic of the estimate of Agent A but only the variance of the estimate of Agent B in the direction of the measurement. This asymmetry reduces the communication cost of the filter: the agents only need to send their estimate and their variance in the direction of the measurement (but not their full covariance). The asymmetry is due to the linearization assumption.

Moreover, for the two commonly used cost functions, the trace and the determinant, the necessary and sufficient conditions given in Corollary 3 and Corollary 4 allow to check efficiently the pertinence of the SCI filter before doing the measurement. In practice, this criterion can save energy by avoiding useless measurements. Furthermore, the precision of the measurement σ_m^2 does not appear in these conditions (21) or (25). This means that if the property is not satisfied, even a perfect measurement cannot improve the estimate. However, the variance σ_m^2 does affect the improvement (if it occurs). As (14) suggests, the larger σ_m^2, the smaller the improvement. In addition, it seems that the condition for the existence of a pertinent filter becomes more difficult to satisfy as the dimension increases. This is clear for the determinant because of the factor $1/n$ in (25). For the trace, the factor r_A in (21) represents the ratio of the information contained in the direction $\underline{u}_{BA}$ to the total amount of information. As expressed in (19), if $r_A = 1$, then the components of $\hat{\underline{\boldsymbol{x}}}_A$ are all perfectly correlated. Therefore, if $r_A = 1$ the correction would correct the entire estimate. On the other hand, if $r_A \ll 1$, $\hat{\underline{\boldsymbol{x}}}_A$ is poorly correlated with its component in the direction $\underline{u}_{BA}$. The impact of the correction on the other components cannot be inferred and the correction should be more cautious. As the dimension increases, r_A decreases requiring smaller σ_B^2.

Finally, it is worth mentioning that the optimal parameter ω^* can be calculated analytically for the cases of the trace and the determinant. Since the cost functions g and h are convex, the minimum is reached either at 0, at 1 (only possible when $r_A = 1$), or when $g'(\omega) = 0$ or $h'(\omega) = 0$. From their decompositions (20) and (24), solving $g'(\omega) = 0$ requires finding the roots of a polynomial of degree 4, and solving $h'(\omega) = 0$ requires finding the roots of a polynomial of degree 3, both of which can be done analytically.

VI. Concluding remarks

The SCI filter is a candidate for the optimal filtering problem. However, the optimal SCI filter proposed here may not be the solution to Problem 1: a better linear filter may exist outside the family of SCI filters. We conjecture, after numerous simulations, that the optimal SCI filter is indeed the solution to Problem 1. The conjecture is based on the facts that SCI is the natural extension of CI when the estimators have independent components, and that CI is optimal for the problem it considers [18].

Furthermore, we only studied the integration of one distance measurement. Future works will focus on the simultaneous integration of distance measurements with multiple helping agents. They will consider filters such as:

$$\hat{\underline{\boldsymbol{x}}}_F = \boldsymbol{K}\hat{\underline{\boldsymbol{x}}}_A + \boldsymbol{L}_1\hat{\underline{\boldsymbol{x}}}_1 + \boldsymbol{z}_1\underline{w}_1 + \cdots + \boldsymbol{L}_m\hat{\underline{\boldsymbol{x}}}_m + \boldsymbol{z}_m\underline{w}_m.$$

Most of the concepts introduced in this paper can be adapted, but as [1] explains, CI is not optimal for more than two estimators, therefore SCI should also be suboptimal.

References

[1] J. Ajgl and O. Straka. Fusion of multiple estimates by covariance intersection: Why and howit is suboptimal. *International Journal of Applied Mathematics and Computer Science*, 28(3):521–530, 2018.

[2] B.D.O. Anderson and J.B. Moore. *Optimal filtering*. Courier Corporation, 1979.

[3] P.O. Arambel, C. Rago, and R.K. Mehra. Covariance intersection algorithm for distributed spacecraft state estimation. In *Proceedings of the 2001 American Control Conference. (Cat. No.01CH37148)*, volume 6, pages 4398–4403 vol.6, 2001.

[4] A. Bahr, M.R. Walter, and J.J. Leonard. Consistent cooperative localization. In *2009 IEEE International Conference on Robotics and Automation*, pages 3415–3422, 2009.

[5] F. Causa and G. Fasano. Improving navigation in gnss-challenging environments: Multi-uas cooperation and generalized dilution of precision. *IEEE Transactions on Aerospace and Electronic Systems*, 57(3):1462–1479, 2021.

[6] L. Chen, P.O. Arambel, and R.K. Mehra. Estimation under unknown correlation: covariance intersection revisited. *IEEE Transactions on Automatic Control*, 47(11):1879–1882, 2002.

[7] R. Chen, B. Yang, and W. Zhang. Distributed and collaborative localization for swarming uavs. *IEEE Internet of Things Journal*, 8(6):5062–5074, 2020.

[8] R. Forsling, A. Hansson, F. Gustafsson, Zoran Sjanic, Johan L., and G. Hendeby. Conservative linear unbiased estimation under partially known covariances. *IEEE Transactions on Signal Processing*, 70:3123–3135, 2022.

[9] Roger A Horn and Charles R Johnson. *Matrix analysis*. Cambridge university press, 2012.

[10] M.A. Hossain, I. Elshafiey, and A. Al-Sanie. Cooperative vehicular positioning with vanet in urban environments. In *2016 IEEE Asia-Pacific Conference on Applied Electromagnetics (APACE)*, pages 393–396, 2016.

[11] S. Julier and J.K. Uhlmann. General decentralized data fusion with covariance intersection (CI). *Handbook of Multisensor Data Fusion*, 2001.

[12] S.J. Julier and J.K. Uhlmann. A non-divergent estimation algorithm in the presence of unknown correlations. In *Proceedings of the 1997 American Control Conference (Cat. No.97CH36041)*, volume 4, pages 2369–2373 vol.4, 1997.

[13] H. Li and F. Nashashibi. Cooperative multi-vehicle localization using split covariance intersection filter. *IEEE Intelligent transportation systems magazine*, 5(2):33–44, 2013.

[14] A. Lima, P. Bonnifait, V. Cherfaoui, and J. Al Hage. Data fusion with split covariance intersection for cooperative perception. In *2021 IEEE International Intelligent Transportation Systems Conference (ITSC)*, pages 1112–1118. IEEE, 2021.

[15] A. Minetto, G. Falco, and F. Dovis. On the trade-off between computational complexity and collaborative gnss hybridization. In *2019 IEEE 90th Vehicular Technology Conference (VTC2019-Fall)*, pages 1–5, 2019.

[16] B. Noack, J. Sijs, M. Reinhardt, and U.D. Hanebeck. Decentralized data fusion with inverse covariance intersection. *Automatica*, 79:35–41, 2017.

[17] C. Pierre, R. Chapuis, R. Aufrère, J. Laneurit, and C. Debain. Range-only based cooperative localization for mobile robots. In *2018 21st International Conference on Information Fusion (FUSION)*, pages 1933–1939, 2018.

[18] M. Reinhardt, B. Noack, P.O. Arambel, and U.D. Hanebeck. Minimum covariance bounds for the fusion under unknown correlations. *IEEE Signal Processing Letters*, 22(9):1210–1214, 2015.

[19] S.I. Roumeliotis and G.A. Bekey. Distributed multirobot localization. *IEEE Transactions on Robotics and Automation*, 18(5):781–795, 2002.

[20] S.I. Roumeliotis and I.M. Rekleitis. Analysis of multirobot localization uncertainty propagation. In *Proceedings 2003 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS 2003) (Cat. No.03CH37453)*, volume 2, pages 1763–1770 vol.2, 2003.

[21] J. Sijs, M. Lazar, and P. Bosch. State fusion with unknown correlation: Ellipsoidal intersection. In *Proceedings of the 2010 American Control Conference*, pages 3992–3997. IEEE, 2010.