Bifurcation results for quasi-linear operators from the Fučik spectrum of the Laplacian

Emmanuel Wend-Benedo Zongo

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Abstract

In this paper, we analyze an eigenvalue problem for a quasi-linear elliptic operators involving Dirichlet boundary condition in an open smooth bounded set of $\mathbb{R}^N$. We investigate a bifurcation results (from trivial solution and from infinity) of an eigenvalue problem involving the $(p,2)$-Laplace operator, from the Fučik spectrum of the Laplacian.

Keywords: quasi-linear operators, Fučik spectrum, bifurcation from trivial solution, bifurcation from infinity, half-eigenvalues.
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Contents

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1 Introduction

Assume $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is an open bounded domain with smooth boundary $\partial\Omega$. A classical result in the theory of eigenvalue problems guarantees that the problem

$$\begin{cases} -\Delta u = \lambda u \ \text{ in } \Omega, \\ u = 0 \qquad \text{on } \partial\Omega \end{cases} \tag{1.1}$$

possesses a nondecreasing sequence of eigenvalues $0 < \lambda_1 < \lambda_2 \leq \dots$ and a sequence of corresponding eigenfunctions which define a Hilbert basis in $L^2(\Omega)$, (see [3]). Moreover, it is known that the first eigenvalue of problem (1.1) is characterized in the variational point of view by,

$$\lambda_1 := \inf_{u \in W_0^{1,2}(\Omega)\setminus\{0\}} \left\{ \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega u^2 \, dx} \right\}.$$

In [7], the authors investigated the asymptotic behavior of the spectrum and the existence of multiple solutions of the following nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u - \Delta u = \lambda u \ \text{ on } \ \Omega, \\ u = 0 \quad \text{ on } \ \partial\Omega, \end{cases} \tag{1.2}$$

where $-\Delta_p u = -\text{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplace operator. In [7] it was shown that for $p > 2$ there exist eigenvalue branches emanating from $(\lambda_k, 0)$, and for $1 < p < 2$ emanating from (λ_k, ∞). A nonlinear generalization of the spectrum of problem (1.1) is the Fučik spectrum, given by the values $(\lambda_+, \lambda_-) \in \mathbb{R}^+ \times \mathbb{R}^+$ for which the problem $-\Delta u = \lambda_+ u^+ - \lambda_- u^-$ with Dirichlet boundary condition has nontrivial solution. The values λ_+ and λ_- which are such that $-\Delta u = \lambda_+ u^+ - \lambda_- u^-$ has nontrivial solution $u \in W_0^{1,2}(\Omega)$ will be called half-eigenvalues, while the corresponding solutions u will be called half-eigenfunctions. The Fučik spectrum was introduced by S. Fučik [13] and N. Dancer [10] in the 70's, mainly motivated by the works of A. Ambrosetti and G. Prodi [2] in connection with what is known in the literature as "asymmetric problems".

In this paper we consider the following nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u - \Delta u = \lambda_+ u^+ - \lambda_- u^- \ \text{ on } \ \Omega, \\ u = 0 \quad \text{ on } \ \partial\Omega. \end{cases} \tag{1.3}$$

The operator $-\Delta_p - \Delta$ appears in quantum field theory (see, [4]), where it arises in the mathematical description of propagation phenomena of solitary waves. The main purpose of this paper is to study the asymptotic behavior of the spectrum of problem (1.3) for all $p > 2$ and $1 < p < 2$. We intend to study problem (1.3) as a bifurcation in the Fučik

eigenvalues from trivial solution if $p > 2$ and a bifurcation from infinity if $1 < p < 2$.

In [6], the author has proved that the equation

$$Au - \gamma u^- = \lambda u + N(\lambda, u) \tag{1.4}$$

has global bifurcation branches emanating from the points $(\lambda_k^1, 0)$ and $(\lambda_k^2, 0)$ in $\mathbb{R} \times L^2(\Omega)$ for any $k \geq 1$. In equation (1.4), the operator $A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$ is an unbounded self-adjoint operator with compact resolvent and with eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots$, repeated according to their multiplicity, and $\gamma \in \mathbb{R}^+$ satisfies $0 < \gamma < \min\{\lambda_k - \lambda_{k-1}, \lambda_{k+1} - \lambda_k\}$ $(k \geq 2)$ and $N(\lambda, u) = o(\|u\|)$ for u near zero uniformly on bounded λ intervals. The points λ_k^1 and λ_k^2 are called split eigenvalues of the operator $A - \gamma(\cdot)^-$ and their existence was proved by B. Ruf in [5]. Our work is inspired by the work of [6]. In the present article, the point $(\lambda_k^1, 0)$ will be a bifurcation point for positive solutions and the point $(\lambda_k^2, 0)$ will be a bifurcation point for negative solutions of problem (1.3).

A similar structure as (1.4) was proved in [8] concerning bifurcation problems for Pucci's operators. More precisely in [8], the authors studied the following problem as a bifurcation problem from the trivial solution

$$\begin{cases} -\mathcal{M}^+_{\lambda,\Lambda}(D^2)u = \mu u + f(u) & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

(respectively $\mathcal{M}^-_{\lambda,\Lambda}$) where Ω is a bounded regular domain, and $\mathcal{M}^{\pm}_{\lambda,\Lambda}$ are the extremal Pucci's operators with parameters $0 < \lambda \leq \Lambda$. It is shown that bifurcation occurs form the "half-eigenvalues" μ_k^+ and μ_k^-.

Our main results are: let $\gamma \in \mathbb{R}^+$ that satisfies $0 < \gamma < \min\{\lambda_k - \lambda_{k-1}, \lambda_{k+1} - \lambda_k\}$, $k \geq 2$

- For $p > 2$, we obtain bifurcation branches emanating from the points $(\lambda_k^1, 0)$ and $(\lambda_k^2, 0)$ in $\mathbb{R} \times L^2(\Omega)$ for the operator $-\Delta_p - \Delta - \gamma(\cdot)^-$, where λ_k^1 and λ_k^2 are the split eigenvalues of $-\Delta - \gamma(\cdot)^-$,
- For $1 < p < 2$, we obtain bifurcation branches emanating from the points (λ_k^1, ∞) and (λ_k^2, ∞) in $\mathbb{R} \times L^2(\Omega)$ for the operator $-\Delta_p - \Delta - \gamma(\cdot)^-$.

This article is organized as follows. In section 2, we present the Fučik spectrum of problem (1.3) and in section 3 we show that are solutions branches of problem (1.3) that bifurcate from the Fučik eigenvalues of problem (2.2). In section 3.3, we discuss bifurcation from infinity of problem (1.3). For a further discussion on this topic, we refer the interested reader to the work of [8] and the references therein.

We recall that if $1 < p < q$, then $L^q(\Omega) \subset L^p(\Omega)$ and as a consequence, one has $W_0^{1,q}(\Omega) \subset W_0^{1,p}(\Omega)$. In what follows, we denote by $\|.\|_{1,p}$ and $\|.\|_2$ the norms on $W_0^{1,p}(\Omega)$ and $L^2(\Omega)$ defined respectively by

$$\|u\|_{1,p} = \left(\int_\Omega |\nabla u|^p \, dx\right)^{\frac{1}{p}} \quad \text{and} \quad \|u\|_2 = \left(\int_\Omega |u|^2 \, dx\right)^{\frac{1}{2}}.$$

2 Fučik spectrum and nonlinear spectrum

An eigenvalue $\lambda \in \mathbb{R}$ of the quasi-linear elliptic operator $-\Delta_p - \Delta$ on Ω is such that the following problem

$$\begin{cases} -\Delta_p u - \Delta u = \lambda u & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

has a nontrivial solution for $p \in (1,\infty)\backslash\{2\}$. We denote by Σ the spectrum of problem (2.1). The Fučik spectrum of $-\Delta$ on Ω with Dirichlet boundary conditions is defined as the set $\mathcal{S}_{(\lambda_+,\lambda_-)}$ of pairs $(\lambda_+,\lambda_-) \in \mathbb{R}^2$ such that the problem

$$\begin{cases} -\Delta u = \lambda_+ u^+ - \lambda_- u^- & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.2}$$

has a nontrivial solution. Then, the Fučik spectrum of $-\Delta_p - \Delta$ on Ω with Dirichlet boundary conditions is defined as the set $\Sigma_{(\lambda_+,\lambda_-)}$ of pairs $(\lambda_+,\lambda_-) \in \mathbb{R}^2$ such that the problem

$$\begin{cases} -\Delta_p u - \Delta u = \lambda_+ u^+ - \lambda_- u^- & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.3}$$

has a nontrivial solution for $p \in (1,\infty)\backslash\{2\}$. Clearly $\Sigma_{(\lambda_+,\lambda_-)}$ generalizes the notion of the spectrum of problem (2.1), since Σ consists of all values $\lambda \in \mathbb{R}$ such that problem (2.3) for $\lambda_+ = \lambda_- = \lambda$ has a nontrivial solution. Here we write $u^\pm = \max\{\pm u, 0\}$ in Ω, where u^+ and u^- denote respectively the positive and the negative part of u. We also recall that $u = u^+ - u^-$.

2.1 Spectrum of problem (2.3)

Definition 2.1. *We say that $u \in W_0^{1,p}(\Omega)$ (if $p > 2$) or $u \in W_0^{1,2}(\Omega)$ (if $1 < p < 2$) with $u \not\equiv 0, \lambda_\pm \in \mathbb{R}$, is a weak solution of (2.3) if and only if*

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla \varphi \, dx + \int_\Omega \nabla u \cdot \nabla \varphi \, dx = \lambda_+ \int_\Omega u^+ \varphi \, dx - \lambda_- \int_\Omega u^- \varphi \, dx \tag{2.4}$$

for all $\varphi \in W_0^{1,p}(\Omega)$ *(if* $p > 2$ *),* $\varphi \in W_0^{1,2}(\Omega)$ *(if* $1 < p < 2$ *).*

The function u is called an half-eigenfunction associated with the pair $(\lambda_+, \lambda_-) \in \mathbb{R}^2$.

Remark 2.2. *For* $\lambda_+ \leq \lambda_1$ *and* $\lambda_- \leq \lambda_1$*, equation (2.3) has only zero solution.*

2.2 Invertibility property

Lemma 2.3. *Let* $p > 2$*. Then operator* $-\Delta_p - \Delta - \gamma(\cdot)^-$ *is a global homeomorphism between* $W_0^{1,p}(\Omega)$ *and its dual space for a fixed* $\gamma \geq 0$*.*

For the proof of this lemma, we need the following lemma (see [12] for a proof).

Lemma 2.4. *Let* $p > 2$*. Then there exist two positive constants* c_1, c_2 *such that, for all* $x_1, x_2 \in \mathbb{R}^n$*, we have :*

(a) $(x_2 - x_1) \cdot (|x_2|^{p-2}x_2 - |x_1|^{p-2}x_1) \geq c_1|x_2 - x_1|^p$

(b) $\big||x_2|^{p-2}x_2 - |x_1|^{p-2}x_1\big| \leq c_2(|x_2| + |x_1|)^{p-2}|x_2 - x_1|$

Proof of Lemma 2.3. Define the quasi-linear operator $\mathcal{M} : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ (with $1/p + 1/p' = 1$) by
$\langle \mathcal{M}u, v\rangle = \displaystyle\int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx - \gamma \int_\Omega u^- v dx$ for all $u, v \in W_0^{1,p}(\Omega)$.
To show that $-\Delta_p - \Delta - \gamma(\cdot)^-$ is a homeomorphism, it is enough to show that $\mathcal{M}$ is a continuous strongly monotone operator, (see [9, Corollary 2.5.10]).
For $p > 2$, for all $u, v \in W_0^{1,p}(\Omega)$, by (a), we get

$$\begin{aligned}
\langle \mathcal{M}u - \mathcal{M}v, u - v\rangle &= \int_\Omega |\nabla(u - v)|^2 dx + \int_\Omega \Big(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\Big) \cdot \nabla(u - v) \, dx \\
&\quad - \gamma \int_\Omega (u^- - v^-)(u - v) dx \\
&\geq \int_\Omega |\nabla(u - v)|^2 dx + c_1 \int_\Omega |\nabla(u - v)|^p dx - \gamma \int_\Omega (u^- - v^-)(u - v) dx \\
&\geq \int_\Omega |\nabla(u - v)|^2 dx + c_1 \int_\Omega |\nabla(u - v)|^p dx \\
&\geq c_1 \|u - v\|_{1,p}^p.
\end{aligned}$$

Thus $\mathcal{M}$ is a strongly monotone operator. We claim that $\mathcal{M}$ is a continuous operator from $W_0^{1,p}(\Omega)$ to $W^{-1,p'}(\Omega)$. Indeed, assume that $u_n \to u$ in $W_0^{1,p}(\Omega)$. We have to show that $\|\mathcal{M}u_n - \mathcal{M}u\|_{W^{-1,p'}(\Omega)} \to 0$ as $n \to \infty$. Using (b), Hölder's inequality and the Sobolev

embedding theorem, one has

$$\begin{aligned}
|\langle \mathcal{M}u_n - \mathcal{M}u, v\rangle| &\leq \int_\Omega \left| |\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u \right| |\nabla v| dx + \int_\Omega |\nabla(u_n - u)||\nabla v| dx \\
&+ \int_\Omega \gamma |u_n^- - u^-||v| dx \\
&\leq c_2 \int_\Omega (|\nabla u_n| + |\nabla u|)^{p-2} |\nabla(u_n - u)||\nabla v| dx + \int_\Omega |\nabla(u_n - u)||\nabla v| dx \\
&+ \int_\Omega \gamma |u_n - u||v| dx \\
&\leq c_2 \left(\int_\Omega (|\nabla u_n| + |\nabla u|)^p dx \right)^{(p-2)/p} \left(\int_\Omega |\nabla(u_n - u)|^p dx \right)^{1/p} \left(\int_\Omega |\nabla v|^p dx \right)^{1/p} \\
&+ c_3 \|u_n - u\|_{1,2} \|v\|_{1,2} \\
&\leq c_4 (\|u_n\|_{1,p} + \|u\|_{1,p})^{p-2} \|u_n - u\|_{1,p} \|v\|_{1,p} + c_5 \|u_n - u\|_{1,p} \|v\|_{1,p}.
\end{aligned}$$

Thus $\|\mathcal{M}u_n - \mathcal{M}u\|_{W^{-1,p'}(\Omega)} \to 0$, as $n \to +\infty$, and hence $\mathcal{M}$ is a homeomorphism. □

3 Bifurcation results

Here we show bifurcation results from trivial solutions and from infinity of problem (2.1) for $p > 2$ and $1 < p < 2$ respectively.

3.1 Preliminary results toward the bifurcation results

Let $p > 2$, then for $u \in W_0^{1,p}(\Omega)$, we have

$$-\Delta_p u - \Delta u = \lambda_+ u^+ - \lambda_- u^- \Leftrightarrow -\Delta_p u - \Delta u - (\lambda_+ - \lambda_-)u^- = \lambda_+ u.$$

We fix $\gamma = \lambda_+ - \lambda_- > 0$ and we consider $\lambda := \lambda_+$ as a parameter of the equation

$$\begin{cases} -\Delta_p u - \Delta u - \gamma u^- = \lambda u & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Using Lemma 2.3, we write equation (3.1) in the following form:

$$u = \lambda(-\Delta)^{-1}u + \left\{ (-\Delta_p - \Delta - \gamma(\cdot)^-)^{-1} - (-\Delta)^{-1} \right\}(\lambda u), \tag{3.2}$$

where we consider

$$(-\Delta_p - \Delta - \gamma(\cdot)^-)^{-1} : L^2(\Omega) \subset W^{-1,p'}(\Omega) \to W_0^{1,p}(\Omega) \subset\subset L^2(\Omega),$$

and

$$(-\Delta)^{-1} : L^2(\Omega) \subset W^{-1,2}(\Omega) \to W_0^{1,2}(\Omega) \subset\subset L^2(\Omega).$$

So, for $p > 2$ the mapping

$$(-\Delta_p - \Delta - \gamma(\cdot)^-)^{-1} - (-\Delta)^{-1} : L^2(\Omega) \subset W^{-1,p'}(\Omega) \to W_0^{1,p}(\Omega) \subset\subset L^2(\Omega)$$

is a compact mapping thanks to the Rellich-Kondrachov theorem. For simplicity, we write equation (3.2) as follows

$$u = \lambda Au + T_\lambda(u), \tag{3.3}$$

with $Au = (-\Delta)^{-1}$ and $T_\lambda(u) = \left\{(-\Delta_p - \Delta - \gamma(\cdot)^-)^{-1} - (-\Delta)^{-1}\right\}(\lambda u)$.

The $L^2(\Omega)$ space with norm $\|\cdot\|_2$ admits the orthogonal decomposition

$$L^2(\Omega) = [e_k] \oplus [e_k]^\perp, \tag{3.4}$$

where e_k is the normalized k-th eigenfunction associated to a simple eigenvalue λ_k of the Dirichlet Laplace operator (see [5]), and $[e_k]$ denotes the eigenspace of λ_k. In view of (3.4), we set $H_k = [e_k]^\perp$ and $E_k = [e_k]$ for simplicity. An element of $u \in L^2(\Omega)$ in (3.4) will be seen as $u = \alpha e_k + v$, with $\alpha \in \mathbb{R}$ and $v \in H_k$.

The following result was obtained by B. Ruf in [5]. If $\gamma \geq 0$ satisfies

$$0 < \gamma < \min\{\lambda_k - \lambda_{k-1}, \lambda_{k+1} - \lambda_k\}, \tag{3.5}$$

then the following nonlinear eigenvalue problem

$$-\Delta u - \gamma u^- = \lambda u,$$

has a unique split eigenvalue $(\lambda_k^1, \lambda_k^2) \in [\lambda_k, \lambda_{k+1}] \times [\lambda_k, \lambda_{k+1}]$ of $-\Delta - \gamma(\cdot)^-$ and uniquely determined eigenfunctions v_k^1, v_k^2 in $L^2(\Omega)$ such that

$$-\Delta v_k^i - \gamma(v_k^i)^- = \lambda_k^i v_k^i, \quad i = 1, 2.$$

For $\gamma = 0$ the k-th split eigenvalue reduces to the eigenvalue λ_k and v_k^1, v_k^2 to the eigenfunctions e_k, $-e_k$ respectively.

Consider

$$\begin{cases} -\Delta u - \gamma u^- = \lambda u \quad \text{on } \Omega, \\ u = 0 \qquad\quad \text{on } \partial\Omega, \quad \text{with } \gamma > 0. \end{cases} \tag{3.6}$$

We say that $u \in W_0^{1,2}(\Omega)$ with $u \not\equiv 0$ is a weak solution of (3.6) if and only if

$$\int_\Omega \nabla u \cdot \nabla v \, dx - \gamma \int_\Omega u^- v \, dx = \lambda \int_\Omega uv \, dx, \tag{3.7}$$

for all $v \in W_0^{1,2}(\Omega)$.

Remark 3.1. *Suppose that u is an eigenfunction associated to λ in (3.7). Then αu is also an eigenfunction associated to λ for $\alpha > 0$.*

From this remark, we aim to show that problem (3.1) has bifurcation branches emanating from the points $(\lambda_k^1, 0)$ and $(\lambda_k^2, 0)$ in $\mathbb{R} \times L^2(\Omega)$ in the direction $\alpha > 0$ when $p > 2$. In what follows, we assume that (3.5) holds. Let $\eta := \frac{1}{2}\min\{|(e_k, v_k^1(\gamma))_2|, |(e_k, v_k^2(\gamma))_2|\} > 0$, and define for $\rho \in \mathbb{R}$, $\rho > 0$,

$$\boldsymbol{K}^+_{\boldsymbol{\rho},\boldsymbol{\eta}}(\gamma) := \{(\lambda, u) \in \mathbb{R} \times L^2(\Omega) : \ |\lambda - \lambda_k^1(\gamma)| < \rho, \ \ (e_k, u)_2 > \eta \|u\|_2\},$$

$$\boldsymbol{K}^-_{\boldsymbol{\rho},\boldsymbol{\eta}}(\gamma) := \{(\lambda, u) \in \mathbb{R} \times L^2(\Omega) : \ |\lambda - \lambda_k^2(\gamma)| < \rho, \ \ (e_k, u)_2 < -\eta \|u\|_2\},$$

where $(\cdot, \cdot)_2$ denotes the $L^2(\Omega)$ inner product. Let

$$B_\rho := \{(\lambda, u) \in \mathbb{R} \times L^2(\Omega) : \ (|\lambda|^2 + \|u\|_2^2)^{1/2} < \rho\}.$$

We see that both $\boldsymbol{K}^+_{\boldsymbol{\rho},\boldsymbol{\eta}}(\gamma)$ and $\boldsymbol{K}^-_{\rho,\eta}(\gamma)$ are two disjoints convex cones.
For $\gamma > 0$ fixed, let Σ_γ denote the closure of the set of nontrivial solutions of problem (3.1). The following lemma localizes the possible solutions of equation (3.1). We refer to [11, Lemma 1.24] for similar results.

Lemma 3.2. *There exists $\rho_0 > 0$ such that for all $0 < \rho < \rho_0$:*

$$\left(\Sigma_\gamma \backslash \{(\lambda_k^1, 0)\} \cap B_\rho\right) \subset \boldsymbol{K}^+_{\boldsymbol{\rho},\boldsymbol{\eta}}(\gamma).$$

If $(\lambda, u) \in \left(\Sigma_\gamma \backslash \{(\lambda_k^1, 0)\} \cap B_\rho\right)$, then $u = \alpha e_k + v$, where $|\alpha| > \eta\|u\|_2$, and $|\lambda - \lambda_k^1| = o(1), v = o(\alpha)$ for α near zero. Moreover $\alpha > 0$. The same result holds for $\boldsymbol{K}^-_{\boldsymbol{\rho},\boldsymbol{\eta}}(\gamma)$.

Proof. Suppose by contradiction that there is no ρ_0 as in Lemma 3.2. Then there exist two sequences ρ_n and $(\lambda_n, u_n) \in \left(\Sigma_\gamma \backslash \{(\lambda_k^1, 0)\} \cap B_{\rho_n}\right)$ such that $\rho_n \to 0$, $|\lambda_n - \lambda_k^1| \leq \rho_n$, $u_n \to 0$ and $|(e_k, u_n)_2| \leq \eta\|u_n\|_2$ as $n \to \infty$. Since $u_n \not\equiv 0$, we introduce the following change of variable $\bar{u}_n = u_n / \|u_n\|_2$ in the following equation

$$-\Delta_p u_n - \Delta u_n - \gamma(u_n)^- = \lambda_n u_n.$$

Then it follows that $-\|u_n\|_2^{p-1}\Delta_p \bar{u}_n - \Delta \bar{u}_n - \gamma(\bar{u}_n)^- = \lambda_n \bar{u}_n$, for $p > 2$. Let $\mu \in \varrho(-\Delta)$ (the resolvent set of $-\Delta$), hence

$$\bar{u}_n = (-\Delta - \mu)^{-1}[(\lambda_n - \mu)\bar{u}_n + \gamma(\bar{u}_n)^- + \|u_n\|_2^{p-1}\Delta_p \bar{u}_n].$$

Using the fact that $(-\Delta - \mu)^-$ is compact, then for a subsequence, $\bar{u}_n \to \bar{u}$ in $L^2(\Omega)$ as $n \to \infty$, and with the continuity of Δ_p we find that

$$\bar{u} = (-\Delta - \mu)^{-1}[(\lambda_k^1 - \mu)\bar{u} + \gamma(\bar{u})^-],$$

since $\lambda_n \to \lambda_k^1$ and $u_n \to 0$ as $n \to \infty$. This implies that $\bar{u} = v_k^1$ (or possibly $\bar{u} = v_k^2$ if $\lambda_k^1 = \lambda_k^2$). Thus we get $2\eta \leq |(e_k, \bar{u})| \leq \eta$ since $|(e_k, u_n)_2| \leq \eta \|u_n\|_2$. This is a contradiction. Hence there exists ρ_0 as above and for $0 < \rho < \rho_0$, if $(\lambda, u) \in \left(\Sigma_\gamma \backslash \{(\lambda_k^1, 0)\} \cap B_\rho\right)$, then $u = \alpha e_k + v$ with $|\alpha| > \eta \|u\|_2$. We have $\|v\|_2 \leq |\alpha| + \|u\|_2 \leq |\alpha| + |\alpha|/\eta$ and $|\lambda - \lambda_k^1| < \rho \to 0$. So, for α near zero, we have $|\lambda - \lambda_k^1| = o(1)$ and $v = o(\alpha)$. □

Remark 3.3. *The ρ_0 in Lemma 3.2 can be chosen indepently of $\gamma \in [0, \bar{\gamma}]$, where $\bar{\gamma}$ satisfies (3.5) if we set*

$$\eta := \frac{1}{2} \inf_{\gamma \in [0, \bar{\gamma}]} \{|(e_k, v_k^1(\gamma))_2|, |(e_k, v_k^2(\gamma))_2|\}.$$

We employ a Lyapunov-Schmidt method, that is, we write equation (3.3) as equivalent system in $\mathbb{R} \times H_k$ and $\mathbb{R} \times E_k$. We denote by P_k and Q_k the orthogonal projections onto H_k and E_k respectively, that is $P_k(\alpha e_k + v) = v$ and $Q_k(\alpha e_k + v) = \alpha e_k$. We write equation (3.3) in $\mathbb{R} \times H_k$ and $\mathbb{R} \times E_k$ as

$$\begin{cases} v = \lambda A v + P_k(T_\lambda(\alpha e_k + v)), \\ \alpha e_k = \lambda \alpha A e_k + Q_k(T_\lambda(\alpha e_k + v)). \end{cases} \tag{3.8}$$

This is equivalent to

$$v = \lambda A v + P_k(T_\lambda(\alpha e_k + v)) \tag{3.9}$$

and

$$\alpha = \frac{\alpha \lambda}{\lambda_k} + (Q_k(T_\lambda(\alpha e_k + v)), e_k)_2. \tag{3.10}$$

We set now,

$$\begin{aligned} R_\gamma(\alpha, \lambda, v) &= \frac{\alpha \lambda}{\lambda_k} + (Q_k(T_\lambda(\alpha e_k + v)), e_k)_2 - \alpha + \lambda, \\ S_\gamma(\alpha, \lambda, v) &= (\lambda - \lambda_k^1) A v + P_k(\tilde{T}_\lambda(\alpha e_k + v)) \end{aligned} \tag{3.11}$$

We then define

$$\psi_\gamma(\alpha, \lambda, v) := (\lambda - R_\gamma(\alpha, \lambda, v), v - S_\gamma(\alpha, \lambda, v)).$$

Then the solutions of equation (3.3) are the zeros of ψ_γ and conversely. The trivial solutions of (3.3) correspond to the solutions $(0, \lambda, 0)$, $\lambda \in \mathbb{R}$ of equations (3.10) and (3.11). Since ψ_γ is continuous and compact with respect to (λ, v) for fixed $\alpha > 0$, for fixed α it has the appropriate form for the use of the theory of Leray-Schauder degree.

3.2 Bifurcation from trivial solutions

Here, we prove our bifurcation result.
Let $\sigma(-\Delta - \gamma(\cdot)^-) := \{\lambda \in \mathbb{R} : \ \exists u \neq 0 \text{ in } L^2(\Omega) : \ -\Delta u - \gamma(u)^- = \lambda u\}$.

Theorem 3.4. *There exists a solution branch $\mathcal{C}_{\lambda_k^1}$ in Σ_γ such that $(\lambda_k^1, 0) \in \mathcal{C}_{\lambda_k^1}$. Then $\mathcal{C}_{\lambda_k^1}$ either*

(a) meets infinity in $\mathbb{R} \times L^2(\Omega)$, or

(b) meets $(\mu, 0)$, where $\mu \in \sigma(-\Delta - \gamma(\cdot)^-) \backslash \{\lambda_k^1, \lambda_k^2\}$.

The analogous statement holds for $\mathcal{C}_{\lambda_k^2}$.

This bifurcation result is explained by Figure 1 below.

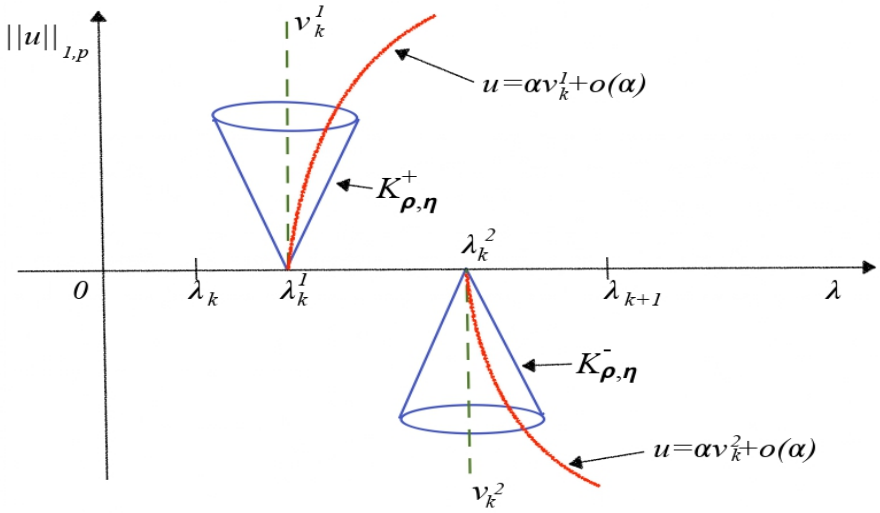

Figure 1: Bifurcation from trivial solutions.

Proof. Let $\mathscr{U}_\gamma = \{(\lambda, v) \in \mathbb{R} \times H_k \ / |\lambda - \lambda_k^1(\gamma)| < \varepsilon_1, \ \ \varrho_1 > \|v\|\}$, where $\gamma \in [0, \bar{\gamma}]$ with $\bar{\gamma}$ arbitrary but fixed and satisfying $0 < \bar{\gamma} < \lambda_{k+1} - \lambda_k$. By Lemma 3.2 all the nontrivial solutions (α, λ, v) of equations (3.9) and (3.10) near $(0, \lambda_k^1, 0)$ satisfy $(\lambda, v) \in \mathscr{U}_\gamma$ if $|\alpha| < \alpha_1$ provided that $\alpha_1, \varepsilon_1, \varrho_1$ are sufficiently small and $\gamma \in [0, \bar{\gamma}]$. Therefore $\psi_\gamma(\alpha, \lambda, v) \neq 0$ on $\partial\mathscr{U}_\gamma$ for $0 < \alpha < \alpha_1$ and $\gamma \in [0, \bar{\gamma}]$. Consequently the Leray-Schauder degree $\deg(\psi_\gamma(\alpha, \lambda, v), \mathscr{U}_\gamma, (0,0))$ is well defined for $0 < |\alpha| < \alpha_1$. Since λ_k^1 is continuously dependent on γ (see [5, Lemma 2.4], $\bigcup_{[0,\bar{\gamma}]} \{\gamma\} \times \mathscr{U}_\gamma$ is a bounded open sets in $[0, \bar{\gamma}] \times \mathbb{R} \times H_k$. By the homotopy invariance of the Leray-Schauder degree, one has

$$\deg(\psi_\gamma(\alpha, \lambda, v), \mathscr{U}_\gamma, (0,0)) = c_\pm, \ \text{ for } \ 0 < |\alpha| < \alpha_1.$$

To evaluate $c_\pm$, we consider the family of operators Θ_τ defined as

$$\Theta_\tau(\alpha, \lambda, v) := \left(\lambda - (\frac{\alpha}{\lambda_k} + 1)\lambda - \frac{\tau}{\lambda_k}(Q_k(T_\lambda(\alpha e_k + v)), e_k)_2 + \alpha, v - \tau S_\gamma(\alpha, \lambda, v)\right)$$

for $\tau \in [0,1]$. As above, we can assume that $\Theta_\tau(\alpha, \lambda, v) \neq 0$ on $\partial\mathscr{U}_\gamma$ for $\tau \in [0,1]$ and $\alpha \in (-\alpha_1, \alpha_1)$. Using again the homotopy invariance of the degree implies that

$$\deg(\Theta_\tau(\alpha, \lambda, v), \mathscr{U}_\gamma, (0,0)) = c_\pm, \ \text{ for } \ 0 < |\alpha| < \alpha_1 \ \text{ and } \ \tau \in [0,1]. \tag{3.12}$$

Therefore to evaluate $c_\pm$, it suffices to take $\tau = 0$, and

$$\Theta_0(\alpha, \lambda, v) = \left(\lambda - (\frac{\alpha}{\lambda_k} + 1)\lambda + \alpha, v\right).$$

The only zero that Θ_0 possesses is $v = 0$, $\lambda = \lambda_k$. We note that $\Theta_0(\alpha, \lambda, v)$ is an isomorphism on $\mathbb{R} \times H_k$, and hence

$$c_\pm = \mathrm{ind}(\Theta_0(\alpha, \lambda, v), (\lambda_k, 0)) = \pm 1.$$

We observe that

$$\mathrm{ind}(\Theta_0(\alpha, \lambda, v), (\lambda_k, 0)) = \deg(\hat{\Theta}_0(\alpha, \lambda, v), \mathscr{U}_\gamma, (0,0)) = \deg(\Theta_0(\alpha, \lambda, v), \mathscr{U}_\gamma, (-\alpha, 0)), \tag{3.13}$$

where $\hat{\Theta}_0(\alpha, \lambda, v) = \left(\lambda - (\frac{\alpha}{\lambda_k} + 1)\lambda + \alpha, v\right)$, which is the homogeneous (in α) linear operator corresponding to $\Theta_0(\alpha, \lambda, v)$. For $\alpha \neq 0$, $\hat{\Theta}_0(\alpha, \lambda, v)$ is an isomorphism and then

$$\deg(\hat{\Theta}_0(\alpha, \lambda, v), \mathscr{U}_\gamma, (-\alpha, 0)) = \mathrm{ind}(\hat{\Theta}_0(\alpha, \lambda, v), (-\alpha, 0)) = \mathrm{ind}(\hat{\Theta}_0(\alpha, \lambda, v), (0,0)). \tag{3.14}$$

Since $\left(\lambda - \gamma(\frac{\alpha}{\lambda_k} + 1)\lambda + \alpha, v\right)$ has no characteristic values γ in $(0,1)$ for $\alpha < 0$ and one characteristic value in the interval $(0,1)$ for $\alpha > 0$, the theorem on change on index and

relation (3.12) and (3.14) imply that $c_\pm = \pm 1$. On the other hand, by Lemma 3.2 we have that

$$(-\Delta_p - \Delta)(\alpha e_k + v) - \bar{\gamma}(\alpha e_k + v)^- = \lambda(\alpha e_k + v), \;\; v \in H_k$$

has no solution for $-\alpha_1 < \alpha < 0$ and $|\lambda - \lambda_k^1(\bar{\gamma})| < \rho_0$. Since this equation is equivalent to equation (3.9), we infer to

$$\deg(\psi_{\bar{\gamma}}(\alpha, \lambda, v), \mathscr{U}_{\bar{\gamma}}, (0,0)) = 0, \;\; \text{for} \;\; -\alpha < \alpha < 0.$$

With this jump in the degree, we obtain the existence of the solution branch $\mathcal{C}_{\lambda_k^1}$. The branch $\mathcal{C}_{\lambda_k^2}$ is obtained similarly. □

3.3 Bifurcation from infinity

Here, we prove a bifurcation from infinity result. As in [7], we introduce the following change of variable: for $u \in W_0^{1,2}(\Omega)$, $u \neq 0$, we set $v = u/\|u\|_{1,2}^{2-\frac{1}{2}p}$ for all $1 < p < 2$. We have $\|v\|_{1,2} = \frac{1}{\|u\|_{1,2}^{1-\frac{1}{2}p}}$ and

$$|\nabla v|^{p-2}\nabla v = \frac{1}{\|u\|_{1,2}^{(2-\frac{1}{2}p)(p-1)}} |\nabla u|^{p-2}\nabla u.$$

Introducing this change of variable in (2.4), we find that,

$$\|u\|_{1,2}^{(2-\frac{1}{2}p)(p-2)} \int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla \varphi \, dx + \int_\Omega \nabla v \cdot \nabla \varphi \, dx = \lambda_+ \int_\Omega v^+ \varphi \, dx - \lambda_- \int_\Omega v^- \varphi \, dx \tag{3.15}$$

for every $\varphi \in W_0^{1,2}(\Omega)$. But, on the other hand, we have

$$\|v\|_{1,2}^{p-4} = \frac{1}{\|u\|_{1,2}^{(1-\frac{1}{2}p)(p-4)}} = \frac{1}{\|u\|_{1,2}^{(2-\frac{1}{2}p)(p-2)}}.$$

Consequently it follows that equation (3.15) is equivalent to

$$\|v\|_{1,2}^{4-p} \int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla \varphi \, dx + \int_\Omega \nabla v \cdot \nabla \varphi \, dx = \lambda_+ \int_\Omega v^+ \varphi \, dx - \lambda_- \int_\Omega v^- \varphi \, dx, \tag{3.16}$$

for every $\varphi \in W_0^{1,2}(\Omega)$. This leads to the following nonlinear eigenvalue problem ($1 < p < 2$)

$$\begin{cases} -\|v\|_{1,2}^{4-p}\Delta_p v - \Delta v = \lambda_+ v^+ - \lambda_- v^- \;\; \text{in } \Omega, \\ v = 0 \qquad\qquad\qquad\quad \text{on } \partial\Omega. \end{cases} \tag{3.17}$$

Remark 3.5. *With this transformation, we have that the pair (λ_k^1, ∞) or (λ_k^2, ∞) is a bifurcation point for the problem (1.3) if and only if the pair $(\lambda_k^1, 0)$ or $(\lambda_k^2, 0)$ is a bifurcation point for the problem (3.17).*

For $v \in W_0^{1,2}(\Omega)$, we have

$$-\|v\|_{1,2}^{4-p}\Delta_p v - \Delta v = \lambda_+ v^+ - \lambda_- v^- \Leftrightarrow -\|v\|_{1,2}^{4-p}\Delta_p v - \Delta v - (\lambda_+ - \lambda_-)v^- = \lambda_+ v.$$

We fix $\gamma = \lambda_+ - \lambda_- > 0$ and we consider $\lambda := \lambda_+$ as a parameter of the equation

$$\begin{cases} -\|v\|_{1,2}^{4-p}\Delta_p v - \Delta v - \gamma v^- = \lambda v \quad \text{on } \Omega, \\ v = 0 \qquad\qquad \text{on } \partial\Omega. \end{cases} \tag{3.18}$$

Let us consider a small ball $B_r := \{\ w \ \in W_0^{1,2}(\Omega) : \quad \|w\|_{1,2} < r\ \}$, and consider the operator

$$A := -\|\cdot\|_{1,2}^{4-p}\Delta_p - \Delta - \gamma(\cdot)^- : W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega).$$

Proposition 3.6. *Let $1 < p < 2$. There exists $r > 0$ such that the mapping $A : B_r \subset W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ is invertible, with a continuous inverse.*

We first remark that for $\gamma > 0$, $-\gamma(u^- - v^-, u - v)_2 \geq 0$. Indeed, we have

$$\begin{aligned} (u^- - v^-, u - v)_2 &= \int_\Omega (u^- - v^-)(u - v)dx \\ &= \int_\Omega (u^- - v^-)(u^+ - v^+)dx - \int_\Omega |u^- - v^-|^2 dx \\ &= \frac{1}{4}\left[(|u| - |v|)^2 - \|u - v\|_2^2\right] - \|u^- - v^-\|_2^2 \leq 0, \end{aligned}$$

using the definition of $u^\pm$ and $v^\pm$.

Proof of Proposition 3.6. In order to prove that the operator A is invertible with a continuous inverse, it is enough to prove that A is continuous strongly monotone operator. We show that there exists $\delta > 0$ such that

$$\langle A(u) - A(v), u - v\rangle \geq \delta\|u - v\|_{1,2}^2, \ \text{ for } \ u, v \in B_r(0) \subset W_0^{1,2}(\Omega)$$

with $r > 0$ sufficiently small.
Indeed, using that $-\Delta_p$ is strongly monotone on $W_0^{1,p}(\Omega)$ on the one hand and the Hölder inequality on the other hand, we have

$$\begin{aligned} \langle A(u) - A(v), u - v\rangle &= \|\nabla u - \nabla v\|_2^2 + \left(\|u\|_{1,2}^{4-p}(-\Delta_p u) - \|v\|_{1,2}^{4-p}(-\Delta_p v), u - v\right)_2 \\ &- \gamma(u^- - v^-, u - v)_2 \\ &= \|u - v\|_{1,2}^2 + \|u\|_{1,2}^{4-p}\left((-\Delta_p u) - (-\Delta_p v), u - v\right)_2 \\ &+ \left(\|u\|_{1,2}^{4-p} - \|v\|_{1,2}^{4-p}\right)\left(-\Delta_p v, u - v\right)_2 - \gamma(u^- - v^-, u - v)_2 \\ &\geq \|u - v\|_{1,2}^2 - \left|\|u\|_{1,2}^{4-p} - \|v\|_{1,2}^{4-p}\right| \|\nabla v\|_p^{p-1}\|\nabla(u - v)\|_p \\ &\geq \|u - v\|_{1,2}^2 - \left|\|u\|_{1,2}^{4-p} - \|v\|_{1,2}^{4-p}\right| C\|v\|_{1,2}^{p-1}\|u - v\|_{1,2}. \end{aligned} \tag{3.19}$$

Now, we obtain by the Mean Value Theorem that there exists $\theta \in [0,1]$ such that

$$\begin{aligned}
\left| \|u\|_{1,2}^{4-p} - \|v\|_{1,2}^{4-p} \right| &= \left| \frac{d}{dt} \left(\|u + t(v-u)\|_{1,2}^2 \right)^{2-\frac{1}{2}p} |_{t=\theta} (v-u) \right| \\
&= \left| (2 - \frac{1}{2}p) \left(\|u + \theta(v-u)\|_{1,2}^2 \right)^{1-\frac{1}{2}p} 2 \left(u + \theta(v-u), v-u \right)_{1,2} \right| \\
&\leq (4-p) \|u + \theta(v-u)\|_{1,2}^{2-p} \|u + \theta(v-u)\|_{1,2} \|u - v\|_{1,2} \\
&= (4-p) \|u + \theta(v-u)\|_{1,2}^{3-p} \|u - v\|_{1,2} \\
&\leq (4-p) \left((1-\theta)\|u\|_{1,2} + \theta \|v\|_{1,2} \right)^{3-p} \|u - v\|_{1,2} \\
&\leq (4-p) r^{3-p} \|u - v\|_{1,2}.
\end{aligned}$$

Hence, continuing with the estimate of equation (3.19), we get

$$\langle A(u) - A(v), u - v \rangle \geq \|u - v\|_{1,2}^2 (1 - (4-p) r^{3-p} C r^{p-1}) = \|u - v\|_{1,2}^2 (1 - C' r^2),$$

and thus the claim, for $r > 0$ small enough. Hence, the operator A is strongly monotone on B_r and it is continuous, and hence the claim follows. □

Using Proposition 3.6, we write equation (3.18) as the following form:

$$v = \lambda (-\Delta)^{-1} v + \left\{ (-\| \cdot \|_{1,2}^{4-p} \Delta_p - \Delta - \gamma(\cdot)^-)^{-1} - (-\Delta)^{-1} \right\} (\lambda v), \tag{3.20}$$

where we consider

$$(-\| \cdot \|_{1,2}^{4-p} \Delta_p - \Delta - \gamma(\cdot)^-)^{-1} : L^2(\Omega) \to W_0^{1,2}(\Omega) \subset\subset L^2(\Omega),$$

and

$$(-\Delta)^{-1} : L^2(\Omega) \subset W^{-1,2}(\Omega) \to W_0^{1,2}(\Omega) \subset\subset L^2(\Omega).$$

So, for $1 < p < 2$ the mapping

$$(-\| \cdot \|_{1,2}^{4-p} \Delta_p - \Delta - \gamma(\cdot)^-)^{-1} - (-\Delta)^{-1} : L^2(\Omega) \subset W^{-1,2}(\Omega) \to W_0^{1,2}(\Omega) \subset\subset L^2(\Omega)$$

is a compact mapping thanks to the Rellich-Kondrachov theorem.

For $\gamma > 0$ fixed, let $\mathcal{P}_\gamma$ denote the closure of the set of nontrivial solutions of problem (3.18). The following lemma localizes the possible solutions of equation (3.18).

Lemma 3.7. *There exists $\rho_0 > 0$ such that for all $0 < \rho < \rho_0$:*

$$\left(\mathcal{P}_\gamma \backslash \{(\lambda_k^1, 0)\} \cap B_\rho \right) \subset \boldsymbol{K}_{\rho,\eta}^+(\gamma).$$

If $(\lambda, u) \in \left(\mathcal{P}_\gamma \backslash \{(\lambda_k^1, 0)\} \cap B_\rho \right)$, then $u = \alpha e_k + v$, where $|\alpha| > \eta \|u\|_2$, and $|\lambda - \lambda_k^1| = o(1), v = o(1)$ for α near zero. Moreover $\alpha > 0$. The same statement hold for $\boldsymbol{K}_{\rho,\eta}^-(\gamma)$.

The proof of this lemma is similar to the proof of Lemma 3.2 and it is left to the reader.

Theorem 3.8. *There exists a solution branch $\mathcal{S}_{\lambda_k^1}$ in $\mathcal{P}_\gamma$ such that $(\lambda_k^1, 0) \in \mathcal{S}_{\lambda_k^1}$. Then $\mathcal{S}_{\lambda_k^1}$ either*

(a) meets infinity in $\mathbb{R} \times L^2(\Omega)$, or

(b) meets $(\mu, 0)$, where $\mu \in \sigma(-\Delta - \gamma(\cdot)^-) \backslash \{\lambda_k^1, \lambda_k^2\}$.

The analogous statement holds for $\mathcal{S}_{\lambda_k^2}$.

One can adapt the proof of Theorem 3.4 to this theorem. Theorem 3.8 can be illustrate as in Figure 1, where the norm $\|u\|_{1,p}$ in the y-axis has to be replaced by $\|u\|_{1,2}$.

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