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# Modelling and Optimal Control of MIMO System - France Macroeconomic Model Case

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**Abstract**—In this paper, we focus on the French Macroeconomic model. We use real economic data, available as time series, starting from 1980s and openly provided by the INSEE, the national statistics bureau of France. Variables such as Gross Domestic Production, Exportation, Importation, Household Consumption, Gross Fixed Capital Formation and Public expenditure are included in the analysis. Our objective is to maintain a constant economic growth rate according to the available resources. We implement an optimal control policy via Linear-Quadratic Regulator (LQR) to achieve that. Since we aim to maintain a constant growth rate, the control system is modified for this purpose. We prove the efficiency with three experiments based on real data, and we test the method robustness with respect to: (1) variation of LQR parameters, (2) realistic constraints on inputs, and (3) perturbations on outputs. Results show that our designed control system can guide the output to the desired growth rate. Varying parameters of LQR can change the convergence speed of the system. Constraints on specific inputs can delay the system convergence and stimulate other inputs to compensate for the constraints. Perturbations on outputs experiment show that our control law can help the system to recover to the original state-space.

**Index Terms**—MIMO model, LQR, Optimal control, Macroeconomic data.

## I. INTRODUCTION

Control theory has a long history of implementation in economic domain. [8] summarizes the development of stochastic control theory in macroeconomic policy analysis in three periods. The first is pre-1970 when the major ideas of policy analysis and of optimization were formed [10]. The second is the early and middle 1970s when formal stochastic control theory was rapidly developed for and applied to the study of macroeconomic policy [7], [12]. The third period, beginning in the late 1970s, was stimulated by the introduction of the idea of rational expectations in economic analysis [4].

Recent works focus more on applications. [15] proposes a general class of PID-based monetary policy rules, the feedback rules let the model use a control signal (e.g. central bank's policy interest rate) responds to movements in a small number of macroeconomic factors, such as the current amount of labor market slack and the deviation of the rate of inflation from its target. Under an optimal control monetary policy [5], the current and expected future path of the policy is instead typically calculated with a procedure that minimizes a cost function subject to certain constraints. For fiscal policy [18] and resource allocation [9], [16] problems, they follow

the same ideas of optimal control, differences are the cost function and the constraints. To estimate the asset holdings of a portfolio, [13] uses algorithms applied to nonlinear dynamic systems to estimate the state with a discrete-time observer.

In this paper, we will apply control techniques to a French macroeconomic model using real economic data (available as time series) starting from 1980s. Our objective would be to design a meaningful control policy that would allow a constant growth rate of the Gross Domestic Product. After choosing the appropriate variables as inputs and outputs, we will first model this system as Multiple-Input and Multiple-Output (MIMO) system. After data pre-processing to make the time series stationary, the orders and parameters of MIMO system are estimated and validated before transforming it into a state-space model. The model we estimate in this paper is an autoregressive model with external inputs. To satisfy our objective, an optimal control solution: Linear-Quadratic regulator (LQR) is designed and implemented. By varying parameters of LQR, we can control the converging speed of output to state-space. A further simulation result shows that if we put constraints on the level of one of input signals, we can observe the compensation effect from other inputs. But even there is no limitation on the amplitude of inputs variables, the level of output and its converging speed to state-space is still slightly worse than the case without constraints. Perturbations on outputs are also studied, which simulates a scenario of an economic crisis. The experiments are also developed with and without constraints on input signals. Results show that our system can quickly recover from the disturbance, and constraints on input signals delay the recovery.

## II. SYSTEM IDENTIFICATION

In this section, the process of estimating the economic model is given. After the data preparation (to make it stationary), a MIMO model is computed. The parameters of the model are then estimated and validated.

### A. Data Preparation

All the data used throughout this paper are obtained from the INSEE (Institut National de la Statistique et des Études Économiques) which regroups all the official economic French Data <sup>1</sup>. A plethora of data is available, which for a non-expert makes it impossible to decide which makes sense in our

<sup>1</sup><https://www.insee.fr/fr/accueil>

particular case. After an in depth analysis on the economical meaning of each variable and long discussions with experts, we decide to study 6 time series namely: Gross Domestic Production (GDP), Exportation (EXP), Importation (IMP), Household Consumption (HC), Gross Fixed Capital Formation (GFCF) and Public Expenditure (PE). Moreover, these variables have a causal relationship between each other and a meaningful influence on the GDP. In the INSEE database, all the data are quarterly ranged from the first quarter of 1980 to the fourth quarter of 2018, 1980T1 to 2018T4, where T1 and T4 denote first and fourth quarter of the year<sup>2</sup> to have a meaningful analysis we decide to take all the data. Following a similar analysis as the one performed in [21] we decide the inputs and outputs of the system. The inputs of the model would be variables on which the appropriate governmental structures can act and incite their modification, namely Household Consumption, Gross Fixed Capital Formation and Public Expenditure. On the other side, the outputs of the model are variables on which we suppose that, on a regular basis, are only measured but the government can not directly act on: Gross Domestic Production, Exportation, Importation.

The reader should note that the original data from INSEE are presented on the values of current price and make no adjustment for inflation. This is problematic as the current price measure measures for example GDP, inflation or asset prices using the actual prices we notice in the economy not the real, deflated, one. Before anything else, we therefore need to deflate it by using France GDP deflator (base year: 2014) obtained from the World Bank. Deflated time series are showed in Fig. 1.

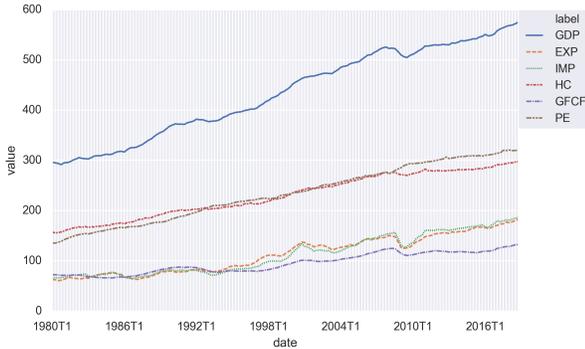


Fig. 1: Original data (Value Unit: Billion Euro)

As we deal with time series having an economic meaning, we prefer to use natural logarithm to better linearize them. Moreover, as we will use these time series to do linear regression, we must ensure that all these series are stationary.

Augmented Dickey–Fuller (ADF) test [6] is a commonly used statistical unit root test to examine whether a given time series is stationary or not. In our case, we would test the stationary for the natural logarithm of our original time series

<sup>2</sup>The reader must note that especially due to the arrival of Covid-19 pandemics, data from 2019 were not taken into account in this study.

TABLE I: Augmented Dickey–Fuller test for unit root. Null hypotheses: Variable contains an unit root

Variables	ADF Level	ADF First Difference
LGDP	-1.10548(1)	-7.66072(0)***
LEXP	-3.28665(2)	-6.29252(3)***
LIMP	-2.87331(4)	-6.20372(3)***
LHC	-1.45805(3)	-5.59492(2)***
LGFCF	-3.60257(2)	-4.01801(1)***
LPE	-1.92627(1)	-5.69958(1)***

Notes: (1) \*\*\*, \*\* and \* denote significance at 1%, 5%, and 10% levels. (2) Figures in parentheses are the number of lags (delays) used.

as well as the first difference of natural logarithm. Results are shown in Tab. I where LGDP, LEXP, LIMP, LHC, LGFCF and LPE denote the natural logarithm of our 6 time series introduced before: GDP, EXP, IMP, HC, GFCF and PE. From Tab. I we can therefore observe that the test, in the case of the natural logarithm of the original data, cannot reject the null hypothesis that the variable contains a unit root, which implies that the original time series are therefore not stationary. Nevertheless, we can notice that the results for the first difference of natural logarithm of all the series rejects the null hypothesis at 99% (i.e., note the \*\*\* following the values as indicator). We can therefore infer that these time series using the first difference of natural logarithm are stationary. Moreover, in later analysis we decide to use the first difference of natural logarithm of GDP, EXP, IMP, HC, GFCF and PE (hereinafter referred to as DLGDP - for difference logarithm of GDP, DLEXP, DLIMP, DLHC, DLGFCF and DLPE).

### B. Selection of MIMO model order

In order to introduce our economic model, we first define the input and output vectors as follows:

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \text{DLGDP} \\ \text{DLEXP} \\ \text{DLIMP} \end{bmatrix} \quad u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \text{DLHC} \\ \text{DLGFCF} \\ \text{DLPE} \end{bmatrix} \quad (1)$$

where  $y$  is the output (endogenous variables) of our model,  $u$  is the input (exogenous variables) of our model. Note that the selections of  $y$  and  $u$  are done according to their economic interpretation and attributes. Input (exogenous) variables are therefore the factors we could manipulate in an economic system (e.g., increasing public expenditure for example) while output (endogenous) variables are the consequences that we could only observe in our case but not directly interfere with (e.g., the variation of importations for example).

Consider an "m-input-p-output" system represented by a canonical input-output representation [14], for  $i = 1, 2, \dots, p$ :

$$y_i(k) = \sum_{j=1}^p \sum_{q=1}^{n_{ij}} a_{ijq} y_j(k+q-n_i-1) + \sum_{j=1}^m \sum_{q=1}^{n_j} b_{ijq} u_j(k+q-n_j-1) + e_i(k) \quad (2)$$

where  $p$  and  $m$  is the numbers of outputs and inputs,  $y_i(k)$  denotes the value of output  $y_i$  at time  $k$ ,  $a_{ijq}$  and  $b_{ijq}$  are the coefficients of  $y_i(k+q+n_i-1)$  and  $u_i(k+q-n_j-1)$ ,  $e_i(k)$

is the white noise at time  $k$ . Moreover,  $n_i$  are the observability indices and  $n_{ij}$  are given by:

$$n_{ij} = \min\{n_i, n_j\}, \text{ if } i \leq j \quad (3)$$

and

$$n_{ij} = \min\{n_i + 1, n_j\}, \text{ if } i > j \quad (4)$$

We apply this method for each output using the technique of "instrumental variables" (see for example [17]), and implementing a criterion which penalizes the model complexity as we want to estimate models of reduced order (see for example [11], [17] among others). Fig. 2 shows the order selection process for  $y_1$ ,  $y_2$  and  $y_3$  while *Criteria* variable calculates the average of estimation errors. The *Criteria*, (namely  $J(\hat{n})$ ) is defined as follows:

$$J(\hat{n}) = \min_{\hat{\theta}} \frac{1}{N} \|Y(t) - R(\hat{n})\hat{\theta}\|^2 \quad (5)$$

where  $\hat{n}$  is the estimated system order,  $N$  is the number of data,  $\hat{\theta}$  is the estimation of parameters.  $Y(t)$  is the real value of the data and  $R(\hat{n})\hat{\theta}$  is the estimated one.  $S(\hat{n}, N)$  is the part served as *penalty* on the order of the model and defined as follows:

$$S(\hat{n}, N) = \frac{2\hat{n}\log(N)}{N} \quad (6)$$

According to [17],  $J(\hat{n})$  goes towards 0 as the estimated order approaches the true one. Therefore, as order increases, we can see from Fig. 2 that *criteria* decreases to 0. But  $S(\hat{n}, N)$  increases with the increase of the chosen order.

As it can be seen from Fig. 2, the estimated order of the model between the inputs and each of the three outputs is: 5 for  $y_1$ , 4 for  $y_2$  and 5 for  $y_3$ . One should notice that these estimated orders are not definitive as we still need to pass the validation process to decide the final order.

### C. Estimation and Validation of parameters

After finding the model order, Least Squares method is used to estimate the parameters of the 3 equations for  $y_1$ ,  $y_2$  and  $y_3$ . After each estimation, a whiteness test (autocorrelation test) will be applied to make sure the residuals from the estimated equation are white noise, which means the estimated model extracted all the knowledge from training data (this problem is well studied in both economic [2] and control engineering [17] domain). The algorithms of calculation vary through these studies, but the goal remains the same: testing if values are mutually uncorrelated. In this paper, we choose to use the implementation of autocorrelation test function from [3].

Consider  $T$  as the total number of data points in the dataset, we can therefore conclude that all the autocorrelation values should be in the range  $(0 \pm \frac{1.96}{\sqrt{0.5}})$ . In our case, from 1980T1 to 2018T4 we have 156 data points, so the limit is  $\pm 0.157$  here. Fig. 3 shows the final autocorrelation test for estimation residuals of  $y_1$ ,  $y_2$  and  $y_3$  where the limit of  $\pm 0.157$  is manifested by the horizontal blue lines.

Please recall that our objective is to find the minimal model, which means a model with the lowest order (minimum number of variables), but still valid. Therefore, once we find a valid candidate model we would still try to reduce the order by eliminating the variables whose coefficient is much smaller than the others by comparing their absolute value. Nevertheless this can pose problems concerning the validity of the mode (for example the residual not being a white noise anymore). Therefore, every time we delete one variable we need to re-estimate all the parameters then re-do the whiteness test on residuals in order to check the validity of the new model. We continue to remove variables until none can be removed from the equation without compromising the validity of the model.

The minimal model can be written explicitly as follows:

$$\begin{aligned} y_1(k) = & a_{115}y_1(k-1) + a_{114}y_1(k-2) \\ & + a_{113}y_1(k-3) + a_{112}y_1(k-4) + a_{111}y_1(k-5) \\ & + a_{124}y_2(k-2) + a_{123}y_2(k-3) + a_{122}y_2(k-4) \\ & + a_{135}y_3(k-1) + a_{134}y_3(k-2) + a_{133}y_3(k-3) \\ & + b_{111}u_1(k-1) + b_{131}u_3(k-1) + e_1(k) \end{aligned} \quad (7)$$

$$\begin{aligned} y_2(k) = & a_{214}y_1(k-1) + a_{221}y_2(k-4) \\ & + a_{222}y_2(k-3) + a_{223}y_2(k-2) + a_{224}y_2(k-1) \\ & + a_{234}y_3(k-1) + b_{211}u_1(k-1) + b_{221}u_2(k-1) \\ & + e_2(k) \end{aligned} \quad (8)$$

$$\begin{aligned} y_3(k) = & a_{314}y_1(k-1) + a_{313}y_1(k-2) \\ & + a_{324}y_2(k-1) + a_{323}y_2(k-2) + a_{334}y_3(k-1) \\ & + a_{333}y_3(k-2) + a_{332}y_3(k-3) + a_{331}y_3(k-4) \\ & + b_{321}u_2(k-1) + e_3(k) \end{aligned} \quad (9)$$

As we can see, the final orders of the reduced model are 5 for  $y_1$ , 4 for  $y_2$  and 4 for  $y_3$ . The numerical values of the parameters are detailed in below

If we define our state vector  $X$  as:

$$\begin{aligned} X(k) = & [y_1(k-1) \ y_1(k-2) \ y_1(k-3) \ y_1(k-4) \ y_1(k-5) \\ & y_2(k-1) \ y_2(k-2) \ y_2(k-3) \ y_2(k-4) \\ & y_3(k-1) \ y_3(k-2) \ y_3(k-3) \ y_3(k-4)]^T \end{aligned} \quad (10)$$

the Eq. (7), (8), (9) can be written as a discrete-time state-space model in the following form:

$$\begin{aligned} X(k+1) &= A \cdot X(k) + B \cdot u(k) \\ Y(k) &= C \cdot X(k) + D \cdot u(k) \end{aligned} \quad (11)$$

where  $k \in \mathbb{Z}^+$ , output  $y$  and control  $u$  vectors are given by equation 1,  $A \in \mathbb{R}^{13 \times 13}$ ;  $B \in \mathbb{R}^{13 \times 3}$ ;  $C \in \mathbb{R}^{1 \times 13}$ ;  $D \in \mathbb{R}^{1 \times 3}$ . The values of  $A, B, C, D$  are showed in equation (12) below, where  $O_{i \times j}$  is the zero matrix of size  $i \times j$ .

Eigenvalues of the state matrix  $A$  are checked and all are within the unit circle which means the open-loop model is

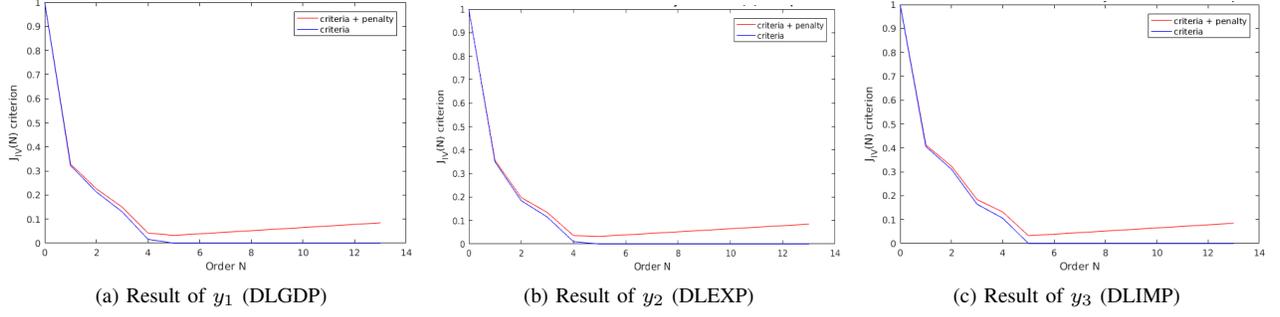


Fig. 2: Order Selection

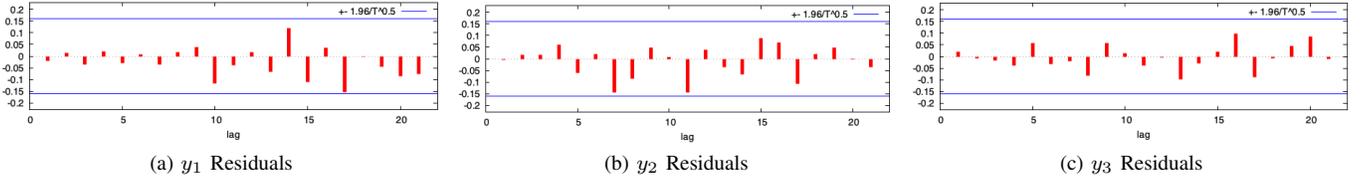


Fig. 3: Autocorrelation Test for the estimation of residuals

$$A = \begin{bmatrix}
 0.397 & 0.06 & -0.126 & 0.193 & 0.066 & 0.037 & 0.05 & 0 & 0 & 0 & 0 & -0.29 & -0.038 \\
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0.264 & 0.072 & -1.055 & 0.958 & 0 & 0.225 & 0.331 & 0.008 & -0.223 & 0.208 & -0.185 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 -1.0538 & 0.97 & -0.6 & 0 & 0 & 0.356 & 0.399 & 0 & 0 & 0.05 & -0.315 & 0.035 & -0.218 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
 \end{bmatrix}$$

$$B = \begin{bmatrix}
 0.177 & 0 & 0.118 \\
 \dots & \mathcal{O}_{4 \times 3} & \dots \\
 0.342 & 0 & 0.364 \\
 \dots & \mathcal{O}_{3 \times 3} & \dots \\
 0 & 0.869 & 0.765 \\
 \dots & \mathcal{O}_{3 \times 3} & \dots
 \end{bmatrix}; C = [1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0]; D = [0 \ 0 \ 0] \quad (12)$$

stable as we expected. Controllability and observability of the system are also tested, the rank of the *observability* and *controllability* matrices are equal to the number of states, which suggests that our system is controllable and observable.

### III. OPTIMAL CONTROL POLICY

The theory of optimal control is concerned with operating a dynamic system at minimum cost. The case where the system dynamics are described by a set of linear differential equations and the cost is described by a quadratic function is called the LQ problem [1].

In this section, we will first introduce Linear–Quadratic Regulator (LQR) and then, according to the physical nature of our input and output, we develop several modifications of the method.

First, consider a discrete-time linear system given by:

$$X(k+1) = A \cdot X(k) + B \cdot u(k)$$

the cost function of discrete time LQR in finite horizon can be presented as follows:

$$J = X(N)^T Q X(N) + \sum_{k=0}^{N-1} (X(k)^T Q X(k) + u(k)^T R u(k) + 2x(k)^T N u(k)) \quad (13)$$

where Q, R are the weighting matrices for state and input while the cross term matrix N is set to 0 in our case as the states and input vectors do not have the same economic meaning in order to be multiplied together.

The control law that minimizes the cost function  $J$  is:

$$u(k) = -K \cdot X(k) \quad (14)$$

where  $K$  is given by:

$$K = (R + BP^T B)^T (B^T P A + N^T) \quad (15)$$

and  $P$  is the unique positive definite solution to the discrete time algebraic Riccati equation (DARE):

$$P = A^T P A - (A^T P B + N)(R + B^T P B)^{-1} (B^T P A + N^T) + Q \quad (16)$$

More details about the practical implementation of LQR in our case are given in Sec. IV-A below.

#### A. Reference Input

As we will focus on controlling GDP, then we need to implement our designed reference to let output reach the desirable value. Recall that  $y_1$  from Eq. 1 is the first difference of natural logarithm of GDP, illustrated below:

$$y_1(k+1) = \ln(\text{GDP}(k)) - \ln(\text{GDP}(k-1)) = \ln\left(\frac{\text{GDP}(k)}{\text{GDP}(k-1)}\right) \quad (17)$$

If we want our GDP to have constant  $p$  percent (%) increasing, we have:

$$\frac{\text{GDP}(k)}{\text{GDP}(k-1)} = 1 + \frac{p}{100}$$

and then the quarterly GDP increasing ratio  $p$  can be interpreted as:

$$p = (e^{y_1(k)} - 1) \times 100 \quad (18)$$

where  $e$  is the base of the natural logarithm. One can remark that as we want a constant increasing ratio, i.e.  $p$  constant, the  $y_1$  needs also to be constant.

#### B. Overall control system

In the state-space, the state and input vectors are constant. If we can define the optimal state-space  $X_r$ , the corresponding input vector as  $u_r$  and the desired output as  $Y_r$ , at equilibrium all these variables are satisfying the following equations according to Eq. (11):

$$\begin{aligned} X_r(k) &= A \cdot X_r(k) + B \cdot u_r(k) \\ Y_r(k) &= C \cdot X_r(k) + D \cdot u_r(k) \end{aligned} \quad (19)$$

We see from Fig. 4 that, different from general control method (e.g. PID), the Reference in LQR does not directly apply feedback from the output but reacts on feedback control law  $u = -K \cdot X$ . Nevertheless, as our state vector contains only the previous values of the output we do not need an observer before implementing our control law. To drive the error between  $X(k)$  and  $X_r(k)$  to 0, we make a change on Eq. (11), namely:

$$\begin{aligned} X(k) - X_r(k) &= A \cdot X(k) + B \cdot u(k) - X_r(k) \\ &= A \cdot X(k) + B \cdot u(k) - A \cdot X_r(k) - B \cdot u_r(k) \\ &= A \cdot (X(k) - X_r(k)) + B \cdot (u(k) - u_r(k)) \end{aligned} \quad (20)$$

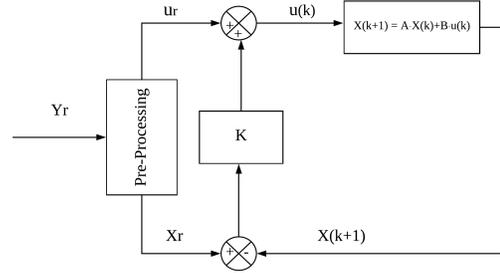


Fig. 4: Modified LQR Control problem

if we define  $\Delta X(k) = X(k) - X_r(k)$  and  $\Delta u(k) = u(k) - u_r(k)$ , then we have a new linear system:

$$\Delta X(k) = A \cdot (\Delta X(k)) + B \cdot (\Delta u(k)) \quad (21)$$

and re-write the cost function of LQR (13) as:

$$\begin{aligned} J_r &= \sum_{k=0}^{N-1} ((\Delta X(k))^T Q (\Delta X(k)) + (\Delta u(k))^T R (\Delta u(k))) \\ &\quad + 2(\Delta X(k))^T N (\Delta u(k)) + (\Delta X(N))^T Q (\Delta X(N)) \end{aligned} \quad (22)$$

the feedback control law that minimizes  $J_r$  can be written as:

$$\Delta u(k) = -K \cdot \Delta X(k) \quad (23)$$

where  $K = (R + BP^T B)^T (B^T P A + N^T)$  is independent of state and input vectors. We can notice therefore that the  $K$  in (14) and (23) does not change.

The control law (23) can also be written as:

$$u(k) = -K \cdot (X(k) - X_r(k)) + u_r(k) \quad (24)$$

According to Eq. (24) we can not directly apply designed output into the feedback, we will need a pre-processing function to transfer the desired output  $Y_r$  to the desired state vector  $X_r$  and input vector  $u_r$  when the system reaches the state-space. One thing to notice that there will not be only one pair of  $X_r$  and  $u_r$  to satisfy the pre-processing function condition, it will be a range for both value, we will let experts to choose the values which make more sense in real world.

Recall the cost function of LQR (13). where  $N$  is set to 0. In this case, we set the weight matrix  $Q = C^T Q' C$ , since  $Y(k) = C \cdot X(k)$ , and the auxiliary matrix  $Q'$  weights the plant output [20]. We find therefore the usual conclusions: when  $R \gg C^T Q' C$ , the cost function is dominated by the control effort  $u$ , and so the controller minimizes the control action itself, this control strategy is used when the control signal is constrained; when  $R \ll C^T Q' C$ , the cost function is dominated by the output  $Y$ , and there is less penalty for using large  $u$ .

## IV. SYSTEM EVALUATION

### A. Setting-up the evaluation

According to the reality, a yearly GDP growth ratio of 3.2% is interesting to study<sup>3</sup>. To reach this level, the quarterly GDP increasing ratio  $p$  is around 0.8% (i.e.  $(1.008)^4 \approx 1.032$ ).

Consider  $Y_r = [y_{1r} \ y_{2r} \ y_{3r}]$  as the reference vector signal. From (18) we know:

$$y_{1r} = \ln\left(1 + \frac{p}{100}\right) = \ln(1.008) = 0.007968 \approx 0.008$$

Following the same computation process as above, we can find  $y_{2r} = 0.0175$  and  $y_{3r} = 0.0086$ .

Recall the relations between  $Y_r$ ,  $X_r$  and  $u_r$  in (19), in our experiment, since the output is identical to the first entry of the state vector, the matrix  $D$  will be 0 in our case. These equations can be re-written as:

$$\begin{aligned} X_r(k) &= (I - A)^{-1} \cdot B \cdot u_r(k) \\ Y_r(k) &= C \cdot X_r(k) \end{aligned} \quad (25)$$

where  $I$  is the identity matrix of suitable dimensions. As we explained in the end of Sec. III-B, there is not only one pair of  $X_r$  and  $u_r$  that satisfy (25). for example, one reasonable pair of  $X_r$  and  $u_r$  is:

$$X_r(k) = [0.008, 0.008, 0.008, 0.008, 0.008, 0.0173, 0.0173, 0.0173, 0.0173, 0.0086, 0.0086, 0.0086, 0.0086]^T$$

$$u_r(k) = [0.024, 0.003, 0.003]^T$$

Knowing about the constraint of  $u$ , diagonal weights [19] of  $Q$  and  $R$  are used.

$$Q = \begin{bmatrix} q_1 & & \\ & \ddots & \\ & & q_{n_q} \end{bmatrix}; R = \rho \begin{bmatrix} r_1 & & \\ & \ddots & \\ & & r_{n_r} \end{bmatrix} \quad (26)$$

where  $n_q = \text{rank}(A) = 13$ ,  $n_r = \text{rank}(B) = 3$ . For the sake of simplicity, we will let  $q_i = 1$  for all  $i \in [1, 13]$ , and  $r_j = 1$  for all  $j \in [1, 3]$ , we will use  $\rho$  to adjust the input/state balance. We choose  $\rho \in \{1, 10, 100\}$  to implement the experiments to compare the converging speed and observe the input signal range.

### B. Experimental Evaluation

For the evaluation of the proposed control strategy, we implement the system detailed in Fig. 4.

The initial state vector  $X_0$  of the state-space model is set by a linear regression over the real data only from 1980T1 to 1981T1. The resulting values are given below:

$$X_0 = [0.001, 0.012, -0.008, -0.004, -0.004, 0.036, 0.032, 0.013, -0.03, 0.007, 0.017, 0.006, -0.002]^T$$

<sup>3</sup>Actually, the recent 10 years (2010-2019) average GDP growth ratio of France is 1.38%, but if we look back 25 years ago, the highest GDP growth ratio are showing during 1998-2001, which the average ratio is around 3.2%. Therefore, we want to study what measurements should be implemented to sustain this growth ratio.

TABLE II: Output behavior. Please note that the state-space error is always 0 due to the type of the controller we consider.

$\rho$	response time (unit in quarter)	output variation range
$\rho = 1$	18	(0.001, 0.008)
$\rho = 10$	32	(-0.001, 0.008)
$\rho = 100$	37	(-0.002, 0.008)
$\rho = 1$ and input constraint	20	(0.001, 0.008)

Moreover, due to the meaning of our variables we have the following constraints regarding the three inputs:  $u_1 \in [0, 0.3]$ ,  $u_2 \leq [0, 0.008]$ ,  $u_3 \leq [0, 0.008]$ . Details about these limits are given below.

For the sake of better presenting our results, throughout all the evaluation procedure we will consider the same simulation time, namely 50 time points (i.e. quarters).

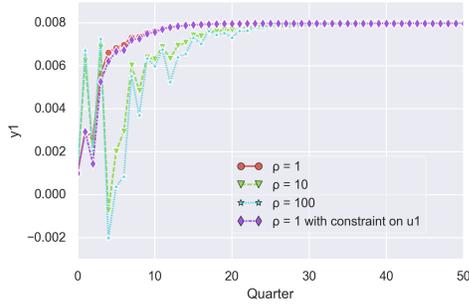
#### 1) Variation of $\rho$ without Constraints on Input Signals:

Let us first discard the limitations on the control signal in order to observe the behavior of the control system as well as its limitations for different values of  $\rho$ . The output (i.e.  $y_1$ ) is illustrated in Fig. 5a for  $\rho \in \{1, 10, 100\}$ . A more detailed result is showed in Tab. II where the settling time as well as the variation range of the output are given. For the time being, the reader can just ignore the result of "1 with constraint" which will be detailed later on. From these results, we can clearly conclude that increasing the  $\rho$  value will also increase the convergence time to the state-space. Nevertheless, this comes with a significant impact on the inputs.

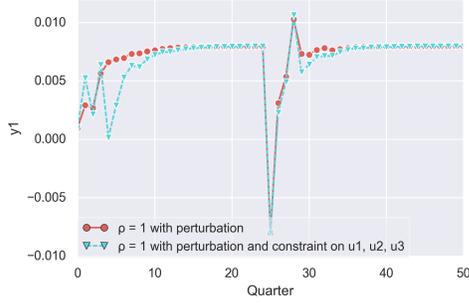
As we expected, the benefits of decreasing  $\rho$  come with cost, translated by the fact that the input exceeds the maximum allowed limits. Fig. 6a, 6b and 6c show the input signals evolution during the control. From the figures and table, we can see that when  $\rho = 1$ , the input signal ranges are much larger than the other two comparisons, and the maximum values are also always higher.

2) Constraints on Input Signals: In our first experiments, we do not impose constraints on input signals, but in reality, there are some levels that input signals cannot reach. Therefore, in this experiment, we set  $\rho = 1$  as well as a maximum limit 0.03 on signal  $u_1$ :  $u_1 \leq 0.03$  (which is strictly lower than the maximum value of  $u_1$  in the non-constrained case). Input signals results of " $\rho = 1$  with constraint on  $u_1$ " are showed in Fig. 6a, 6b and 6c and Tab. III. Comparing to only  $\rho = 1$  result, we can see for signal  $u_1$ , the minimum value of the range is still 0.009, but the maximum value of the range becomes 0.03 (which is the limit) instead of 0.038. For  $u_2$  and  $u_3$ , the maximum value increases as they are being used to compensate for the insufficiency of  $u_1$ . The reader should also note that for the sake of readability, all the input behavior is also detailed in Table III.

From Fig. 5a, 6a, 6b and 6c, we can observe that even  $u_2$  and  $u_3$  react to compensate limitation of  $u_1$ , at 1<sup>st</sup> and 4<sup>th</sup> quarter, new results are slightly lower than the result without constraint. As for the converging speed, from Tab. II, we can also notice that experiment with constraint converges slower than the experiment without constraint. But still better than



(a) Output  $y_1$  under different  $\rho$  values



(b) Output  $y_1$  under different  $\rho$  values

Fig. 5: Output  $y_1$  under different  $\rho$  values

the results of  $\rho = 10$  or  $100$ .

Moreover, one thing to note is that the curves of " $\rho = 1$  with perturbation" and " $\rho = 1$  with perturbation and constraint" are overlapped during the quarter from 0 to 25. And the curves of  $\rho = 1, 10, 100$  and " $\rho = 1$  with constraint" are overlapped during the quarter from 25 to 50. The signal ranges are showed in Tab. III.

3) *Perturbation on Output Signals*: In this experiment, we still keep  $\rho = 1$ , and we add perturbation on output signals to simulate economic crisis, to see how the system will react. From the reference signals setting in Sec. IV-A, we know that when the system is stable,  $y_1, y_2, y_3$  should equal to 0.008, 0.0173 and 0.0086. And  $u_1, u_2, u_3$  should equal 0.024, 0.003 and 0.003.

From Fig. 5b, 6a, 6b and 6c, we can see that at 24th quarter, the system has converged to a stable state. **Then we add a negative perturbation pulse signal -0.16 on  $y_1, y_2$  and  $y_3$  at 25th quarter.** The " $\rho = 1$  with perturbation and constraint on  $u_i$ " curves are the scenario where we not only implement the perturbation, but also implement constraints on all the input signals. Besides enforcing all to be positive,  $u_1$  is limited to 0.03 while  $u_2$  and  $u_3$  are limited to 0.008:  $u_1 \in [0, 0.3]$ ,  $u_2 \leq [0, 0.008]$ ,  $u_3 \leq [0, 0.008]$ .

Fig. 5b shows that after about 20 quarters after 25th quarter, the two systems totally recover from the perturbation, apparently the curve without constraint recovers faster than the other. From Fig. 6a 6b and 6c, we can observe that the amplitude of all inputs for the experiment of " $\rho = 1$  with perturbation" is higher than " $\rho = 1$  with perturbation and

TABLE III: Summary of Input variation

$\rho$	input signal	signal variation range
1	$u_1$	(0.0163, 0.0375)
10	$u_1$	(0.0218, 0.0274)
100	$u_1$	(0.0237, 0.0245)
1 with constraint	$u_1$	(0.0163, 0.03)
1 with perturbation	$u_1$	(0.0146, 0.0375)
1 with perturbation and constraint	$u_1$	(0.0153, 0.03)
1	$u_2$	(-0.0095, 0.003)
10	$u_2$	(-0.0047, 0.0034)
100	$u_2$	(0.0017, 0.0032)
1 with constraint	$u_2$	(-0.0095, 0.0031)
1 with perturbation	$u_2$	(-0.0095, 0.0087)
1 with perturbation and constraint	$u_2$	(0.0, 0.0085)
1	$u_3$	(-0.0196, 0.0068)
10	$u_3$	(-0.0065, 0.0051)
100	$u_3$	(0.0014, 0.0034)
1 with constraint	$u_3$	(-0.0196, 0.008)
1 with perturbation	$u_3$	(-0.0196, 0.0103)
1 with perturbation and constraint	$u_3$	(0.0, 0.0099)

constraint on  $u_i$ " (for each input). The input signal ranges shown in Tab. III also confirm that. One interesting point in Fig. 6b and 6c reveal that if we do not impose constraint on input signals, inputs  $u_2$  and  $u_3$  can be negative, recall that  $u_2$  represents the first difference of logarithm of Gross Fixed Capital Formation (also called investment). A negative signal means instead of investing during the crisis, we should sell our assets. As  $u_3$  represents the first difference of logarithm of Public Expenditure, negative means we need to reduce government spending. Nevertheless, all these conclusions, although correct from the engineering point of view, need to be coordinated with expert's advice.

## V. CONCLUSION

Applying control theory to economic problems has been successfully studied in many cases, resource allocation is one of the well-established problems in this area. It demands dynamically choosing available resources with constraints over time to maximize or minimize an objective function.

In this paper, to apply optimal control, French macroeconomic quarterly data from 1980T1 to 2018T4 are used, more precisely we use 6 variables which are representative from an economic point of view: GDP, EXP, IMP, HC, GFCF and PE. After estimating the model, an optimal LQR control solution is designed for our problem, which is to maintain a constant GDP increase ratio. Simulations using different realistic constraints and perturbations are performed.

The control structure designed in this paper has good applicability and extensibility for other economic systems as well. This work can be further extended by considering more variables or by imposing certain characteristic constraints on inputs/outputs.

## ACKNOWLEDGMENTS

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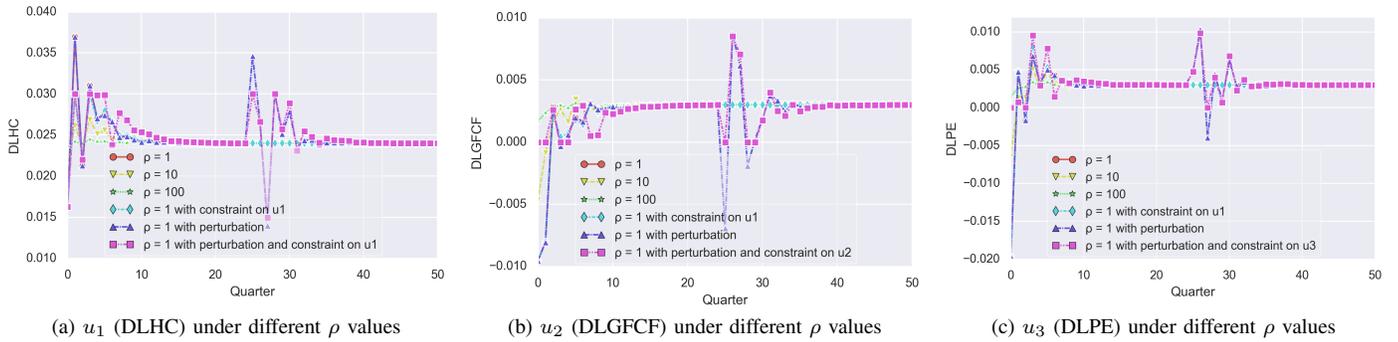


Fig. 6: Input traces under different  $\rho$  values

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