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1                   **ANALYSIS OF SCHWARZ METHODS FOR CONVECTED**  
2                   **HELMHOLTZ LIKE EQUATIONS**

3                   M.J. GANDER, AND A. TONNOIR

4       **Abstract.** We present and analyze Schwarz domain decomposition methods for a general diffu-  
5 sion problem with complex advection. The complex advection term changes completely the nature  
6 of the solution and makes it more Helmholtz like. We analyze in detail the influence of the outer  
7 boundary conditions on the performance of the Schwarz algorithm, including PML conditions to  
8 emulate free space problems, and optimized transmission conditions, also for multiple subdomains.  
9 Our results show that the performance of Schwarz methods for such Helmholtz like problems is  
10 much better on free space configurations than in waveguides or closed cavities. Equations with com-  
11 plex advection appear in diverse applications, for example the convected Helmholtz equation, the  
12 Gross-Pitaevskii equation, Schrödinger equations, and also as important component in the wave-ray  
13 multigrid algorithm for Helmholtz problems. We show as an example the performance of our Schwarz  
14 methods for a potential flow around a schematic submarine.

15       **Key words.** Complex advection, convected Helmholtz equation, Schwarz methods.

16       **MSC codes.** 65M55, 65N55, 65F10.

17       **1. Introduction.** We are interested in solving numerically a partial differential  
18 equation (PDE) with a complex (!) advection term of the form

19 (1.1)                $-\operatorname{div}(A\nabla u) + i\mathbf{a} \cdot \nabla u + \mu u = f \quad \text{in } \Omega, \quad i := \sqrt{-1},$

20 where  $\Omega$  is a subset of  $\mathbb{R}^2$ ,  $A$  is a  $2 \times 2$  positive definite matrix function,  $\mu$  is a real  
21 function and  $\mathbf{a}$  is a vector function in  $\mathbb{R}^2$ . We will assume that the source term  $f$  is  
22 compactly supported, and (1.1) must be equipped with appropriate boundary condi-  
23 tions that we will specify later. Equation (1.1) is very different from a classical  
24 advection diffusion equation with real advection term, and can have Helmholtz char-  
25 acter even when  $\mu$  has the good sign, i.e.  $\mu \geq 0$ . Equation (1.1) appears in various  
26 contexts:

- 27       • **The convected Helmholtz equation:** in this case,  $\mu = -\omega^2$  with  $\omega$  the  
28 pulsation of the wave,  $\mathbf{a} = -2\omega\mathbf{v}$  with  $\mathbf{v}$  the underlying flow (with convention  
29  $e^{-i\omega t}$  for the time variable), and the solution  $u$  represents a pressure field. If  
30 the underlying flow is assumed to be incompressible, then we have

31                               
$$A = c_0^2 \operatorname{Id} - \mathbf{v} \mathbf{v}^T,$$

32 with  $c_0 > 0$  the sound speed, see e.g. [36, 3, 5]. Note that to ensure that  
33 the matrix  $A$  is positive definite, the flow speed  $\mathbf{v}$  must be small enough with  
34 respect to the sound speed  $c_0$  (under mach 1).

- 35       • **The Gross-Pitaevskii equation:** equation (1.1) also appears as an inter-  
36 mediate problem for computing ground states of the Gross-Pitaevskii equa-  
37 tion (which consists in solving a minimization problem), see [11, p.1107] or  
38 [2]. Solving equation (1.1) is an essential ingredient to compute the Sobolev  
39 gradient of the cost functional.
- 40       • **The linearized Schrödinger equation:** when looking for traveling wave  
41 solutions of the form  $\psi(t, \mathbf{x}) = u(\mathbf{x} - \mathbf{a}t)$  to the linearized Schrödinger equation

42                               
$$i\partial_t \psi + \frac{1}{2} \Delta \psi - V\psi = 0,$$

43 see [7] or [10, p.198], equation (1.1) appears with  $A = \frac{1}{2} \operatorname{Id}$  and  $\mu = V$ .

- **The ray equation:** equation (1.1) also appears as a fundamental ingredient in the wave-ray multigrid method for solving the Helmholtz equation [9, 33, 34, 38],

$$-\Delta \tilde{u} - \omega^2 \tilde{u} = 0,$$

when seeking the ray component of the form  $\tilde{u}(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}}u$ , where  $\mathbf{k}$  is a given direction in  $\mathbb{R}^2$  satisfying the dispersion relation  $\|\mathbf{k}\|_2^2 = \omega^2$ . Then,  $u$  satisfies equation (1.1) with  $A = \text{Id}$ ,  $\mathbf{a} = -2\mathbf{k}$  and  $\mu = 0$ .

When equipped with classical Dirichlet, Neumann or Robin (impedance) boundary conditions (BCs), one can show that problem (1.1) is of Fredholm type, since the operator  $-\text{div}(A\nabla\cdot) + \cdot$  is coercive. We deduce then that the problem is of type *coercive + compact*, see [3, p.6], and therefore admits a unique solution, except for at most a countable set of parameters.

Depending on the situation of interest from the list above, we will consider Dirichlet, Neumann or Robin BCs. Furthermore, we will also consider the case where we have Perfectly Matched Layers (PMLs) surrounding the domain of interest, which is important for wave-like problems on unbounded domains. The derivation of the PML formulation is not straightforward for equations of the type (1.1), see for instance [5, 36] for the convected Helmholtz equation, and we will briefly recall the PML construction hereafter.

Our goal is to analyze convergence properties of a Schwarz Domain Decomposition Method (DDM) with overlap using classical Fourier analysis, see [23, 15]. In particular, we wish to emphasize the impact of considering PML to truncate the computational domain. There is an important body of literature dedicated to the study of Schwarz methods for the Helmholtz equation, see [25, 27, 29, 24, 28, 23, 21, 22, 13] and reference therein. However, only few results exist for the convected Helmholtz equation; an exception is the recent paper [32], in which the authors study a non-overlapping DDM for the convected Helmholtz equation. In fact, in the case of constant parameters  $A$ ,  $\mathbf{a}$  and  $\mu$ , one can reformulate, as we explain hereafter, equation (1.1) as a classical Helmholtz equation, using an appropriate change of variables. This shows in particular that we will clearly face the same difficulties as in the Helmholtz case [17] for solving (1.1), but we can also benefit from the results known for the Helmholtz case.

The rest of our paper is organized as follows: First in section 2 we recall the link between the classical Helmholtz equation and equation (1.1), and explain how we can derive a stable PML formulation and first order Absorbing BCs (ABCs). Then, in section 3 we present a Fourier analysis of a Schwarz DDM considering vertical slicing and Robin transmission conditions. We study the impact of various outer PML truncations on the performance of the method, and explain how to properly take them into account in the implementation. Finally, in section 4 we give some concluding remarks.

*Remark 1.1.* Schwarz methods have been intensively studied for a formally similar equation, namely the advection-diffusion equation, see [1, 26, 16], but the mathematical character of this equation with real advection term is very different from our equation (1.1). Also, the anisotropic aspect of diffusion was studied for Schwarz methods in [20, 19], but again without the fundamentally character changing term of the complex advection in (1.1).

**2. Reformulation as a Helmholtz equation and related results.** In this section, as well as for the analysis in the next section, we will suppose that  $A$ ,  $\mathbf{a}$  and

92  $\mu$  are constant parameters. Note that for the construction of ABCs and for the PML  
 93 formulation, a generalization to locally perturbed parameters is possible.

94 **2.1. Link with the Helmholtz equation.** For the convected Helmholtz equa-  
 95 tion, in the case of constant parameters, it is well-known that there exist coordi-  
 96 nate transformations that map the convected Helmholtz equation into the Helmholtz  
 97 equation, see [36, 30]. Let us explain a similar idea for (1.1), namely to consider  
 98  $u(\mathbf{x}) = v(\mathbf{x})e^{i\mathbf{k}\cdot\mathbf{x}}$  with  $\mathbf{k}$  to be suitably chosen later. Then, we have

$$99 \quad \nabla u = (\nabla v)e^{i\mathbf{k}\cdot\mathbf{x}} + ve^{i\mathbf{k}\cdot\mathbf{x}}i\mathbf{k},$$

100 and

$$101 \quad \operatorname{div}(A\nabla u) = \operatorname{div}(A\nabla v)e^{i\mathbf{k}\cdot\mathbf{x}} + 2iA\nabla v \cdot \mathbf{k}e^{i\mathbf{k}\cdot\mathbf{x}} - v\|\mathbf{k}\|_A^2 e^{i\mathbf{k}\cdot\mathbf{x}},$$

102 where we denote by  $\|\mathbf{k}\|_A^2 := A\mathbf{k} \cdot \mathbf{k}$ . Introducing these results into (1.1), we get that  
 103  $v$  satisfies

$$104 \quad (2.1) \quad -\operatorname{div}(A\nabla v) + \nabla v \cdot (i\mathbf{a} - 2iA\mathbf{k}) + (\mu + \|\mathbf{k}\|_A^2 - \mathbf{a} \cdot \mathbf{k})v = e^{-i\mathbf{k}\cdot\mathbf{x}}f.$$

105 If we choose now  $\mathbf{k} := \frac{1}{2}A^{-1}\mathbf{a}$ , equation (2.1) simplifies to

$$106 \quad (2.2) \quad -\operatorname{div}(A\nabla v) + \left(\mu - \frac{\|\mathbf{a}\|_{A^{-1}}^2}{4}\right)v = e^{-iA^{-1}\frac{\mathbf{a}}{2}\cdot\mathbf{x}}f.$$

107

108 *Remark 2.1.* Note that (2.2) is a classical Helmholtz problem if  $\mu - \frac{\|\mathbf{a}\|_{A^{-1}}^2}{4} < 0$ ,  
 109 so even for  $\mu \geq 0$ , (1.1) has a Helmholtz character if  $\frac{\|\mathbf{a}\|_{A^{-1}}^2}{4}$  is large enough: the  
 110 complex convection term is the reason for this Helmholtz character of equation (1.1).  
 111 Now, from (2.2), using an appropriate linear coordinate transformation of the form

$$112 \quad (2.3) \quad \mathbf{x}' = S\mathbf{x},$$

113 we can rewrite the operator  $\operatorname{div}(A\nabla\cdot)$  as a Laplace operator, because

$$114 \quad (2.4) \quad \nabla\cdot = S^T\nabla'\cdot \implies \operatorname{div}(A\nabla\cdot) = \operatorname{div}'(SAS^T\nabla'\cdot),$$

115 and since  $A$  is symmetric positive definite, we can use the Cholesky decomposition  
 116  $A = GG^T$  and thus take  $S = G^{-1}$  to simplify (2.4). Equation (2.2) then simplifies to

$$117 \quad (2.5) \quad -\Delta'v' + \left(\mu - \frac{1}{4}\|\mathbf{a}\|_{A^{-1}}^2\right)v' = \tilde{f}',$$

118 where  $v'(\mathbf{x}') = v(\mathbf{x})$  and  $\tilde{f}'(\mathbf{x}') = e^{-iA^{-1}\frac{\mathbf{a}}{2}\cdot S^{-1}\mathbf{x}'}f(S^{-1}\mathbf{x}')$ .

119 *Remark 2.2.* In our case, the change of variables (2.3) preserves the vertical  
 120 boundaries, because the matrix  $S = G^{-1}$  is lower triangular, so that

$$121 \quad \{x = \alpha\} \iff \{x' = s_{11}\alpha\},$$

122 where  $s_{ij} = [S]_{i,j}$ . In contrast, horizontal boundaries are deformed into oblique  
 123 boundaries (if  $a_{12} \neq 0$ ), in the same spirit as in [37].

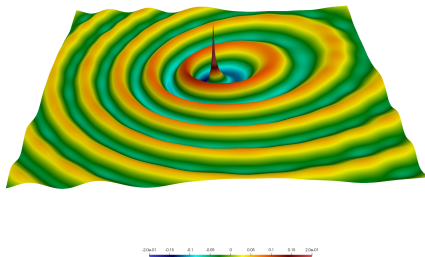


FIG. 1. Example of using the ABC (2.6) with a point source in the center of the domain.

124 **2.2. Derivation of a simple Absorbing Boundary Condition.** The reformulation (2.5) as a Helmholtz equation is used in the literature for constructing ABCs  
 125 for the convected Helmholtz equation, see [4], or PMLs [5, 36], albeit using a different  
 126 change of variables (the choice is not unique). Suppose that<sup>1</sup>  $\mu - \frac{1}{4}\|\mathbf{a}\|_{A^{-1}}^2 < 0$ , then  
 127 we can easily deduce the equivalent of the classical ABC for the Helmholtz equation  
 128 (taking the convention  $e^{-i\omega t}$  for the time variable):  
 129

$$\begin{aligned}
 \nabla' v' \cdot \mathbf{n}' - i\tilde{\omega}v' = 0 &\iff S^{-T}\nabla v \cdot \mathbf{n}' - i\tilde{\omega}v = 0, \\
 &\iff A\nabla v \cdot \underbrace{(G^{-T}\mathbf{n}')}_{=\mathbf{n}/\|G^T\mathbf{n}\|} - i\tilde{\omega}v = 0, \\
 130 \quad (2.6) & \\
 &\iff A\nabla u \cdot \mathbf{n} - i\frac{1}{2}\mathbf{a} \cdot \mathbf{n}u - i\tilde{\omega}\|\mathbf{n}\|_A u = 0,
 \end{aligned}$$

131 where  $\tilde{\omega} := \sqrt{-\mu + \frac{1}{4}\|\mathbf{a}\|_{A^{-1}}^2}$ , and  $\mathbf{n}$  is the normal on a given boundary surrounding  
 132 the domain of computations. Note that this ABC for equation (1.1) is valid for any  
 133 straight boundary since the classical ABC for the Helmholtz equation is valid on any  
 134 straight line, no matter the orientation. If on the contrary we consider this condition  
 135 on a circular boundary for the Helmholtz equation, then the boundary for equation  
 136 (1.1) is no more a circle, see for instance [4, 35] for more details and higher order  
 137 ABCs.

138 As a numerical illustration, we consider the convected Helmholtz problem

$$\begin{aligned}
 139 \quad (2.7) \quad -\operatorname{div}(A\nabla u) - 2i\omega\mathbf{v} \cdot \nabla u - \omega^2 u &= \delta \quad \text{in } \Omega = (0, 4)^2, \\
 A\nabla u \cdot \mathbf{n} + i\omega\mathbf{v} \cdot \mathbf{n}u - i\tilde{\omega}\|\mathbf{n}\|_A u &= 0 \quad \text{on } \partial\Omega,
 \end{aligned}$$

140 where  $\delta$  is the Dirac source term, and  $A = \operatorname{Id} - \mathbf{v}\mathbf{v}^T$ . We show in Figure 1 the  
 141 solution  $u$  we obtain using the ABC (2.6) for the model problem (2.7) with problem  
 142 parameters

$$143 \quad (2.8) \quad \mathbf{v} := \operatorname{Ma} \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}, \quad \theta = \frac{\pi}{4}, \quad \operatorname{Ma} = \frac{1}{2}, \quad \omega = 10.$$

144 **2.3. Derivation of a Cartesian PML formulation.** In this subsection, we  
 145 will also assume that  $A$  is a diagonal matrix so that the coordinate transformation

<sup>1</sup>In what follows, we will always assume that  $\mu - \frac{1}{4}\|\mathbf{a}\|_{A^{-1}}^2 < 0$ , since otherwise, the problem is coercive and has lost its difficult Helmholtz character.

146 (2.3) preserves both horizontal and vertical boundaries, since the matrix  $S$  is diag-  
 147 onal, which simplifies the construction of a Cartesian PML. We refer to [12] for the  
 148 construction of a PML on a polygonal domain. We emphasize that the assumptions  
 149 that the parameters are constant and  $A$  is diagonal are necessary only in the PML  
 150 region.

151 Under these hypotheses, it is well-known that the PML formulation for the  
 152 Helmholtz equation in  $(x', y')$  coordinates reads

$$153 \quad (2.9) \quad -\operatorname{div}'(D'_{PML}\nabla'v') + s'_{x'}s'_{y'}(\mu - \frac{1}{4}\|\mathbf{a}\|_{A^{-1}}^2)v' = \tilde{f}',$$

154 where

$$155 \quad D'_{PML} = \begin{bmatrix} s'_{y'}/s'_{x'} & 0 \\ 0 & s'_{x'}/s'_{y'} \end{bmatrix}.$$

156 The complex valued functions  $s'_{x'}$  and  $s'_{y'}$  are defined by

$$157 \quad (2.10) \quad s'_{x'}(x') := \begin{cases} 1 & \text{if } x' \in (a' + \ell', b' - \ell'), \\ 1 + \imath\sigma_x & \text{otherwise,} \end{cases}$$

158 and

$$159 \quad (2.11) \quad s'_{y'}(y') := \begin{cases} 1 & \text{if } y' \in (c' + \ell', d' - \ell'), \\ 1 + \imath\sigma_y & \text{otherwise,} \end{cases}$$

160 where  $\sigma_x > 0$  and  $\sigma_y > 0$  are the strength of the PML in each direction, and  $\ell' > 0$  is  
 161 the depth of the PML. In  $(x', y')$  coordinates, the computational domain would then  
 162 be the square  $(a', b') \times (c', d')$ . Returning to the  $(x, y)$  coordinates, we get

$$163 \quad (2.12) \quad \begin{aligned} & -\operatorname{div}'(D'_{PML}\nabla'v') + s'_{x'}s'_{y'}(\mu - \frac{1}{4}\|\mathbf{a}\|_{A^{-1}}^2)v' = \tilde{f}', \\ \iff & -\operatorname{div}(A_{PML}\nabla v) + s_x s_y (\mu - \frac{1}{4}\|\mathbf{a}\|_{A^{-1}}^2)v = \tilde{f}, \end{aligned}$$

164 where

$$165 \quad A_{PML} = \begin{bmatrix} a_{11}s_y/s_x & 0 \\ 0 & a_{22}s_x/s_y \end{bmatrix},$$

166 and  $s_x(x) = s'_{x'}(x')$  and  $s_y(y) = s'_{y'}(y')$ . Recalling that  $v(\mathbf{x}) = u e^{-i\frac{1}{2}A^{-1}\mathbf{a}\cdot\mathbf{x}}$ , we get

$$167 \quad A_{PML}\nabla v = \left( A_{PML}\nabla u - u\frac{1}{2}A_{PML}A^{-1}\mathbf{a} \right) e^{-i\frac{1}{2}A^{-1}\mathbf{a}\cdot\mathbf{x}},$$

168 and, observing that  $A_{PML}$  depends on  $(x, y)$ ,

$$169 \quad \operatorname{div}(A_{PML}\nabla v) = \left( \operatorname{div}(A_{PML}\nabla u) - \imath\frac{1}{2}\mathbf{a}_{PML} \cdot \nabla u \right. \\ \left. - \imath\frac{1}{2}\operatorname{div}(u\mathbf{a}_{PML}) - u\frac{1}{4}\|\mathbf{a}\|_{A_{PML}^{-1}}^2 \right) e^{-i\frac{1}{2}A^{-1}\mathbf{a}\cdot\mathbf{x}},$$

170 where

$$171 \quad \tilde{A}_{PML}^{-1} := A^{-1}A_{PML}A^{-1}, \quad \mathbf{a}_{PML} := A_{PML}A^{-1}\mathbf{a}.$$

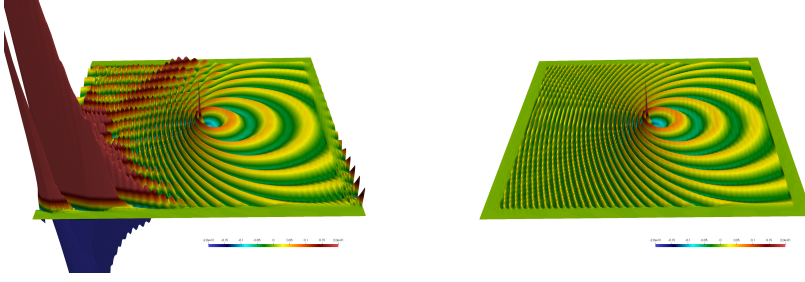


FIG. 2. Solution obtained using (on the left) the classical PML and (on the right) the modified PML.

172 Thus, inserting this expression into (2.12), we obtain the PML formulation for (1.1),  
 173 namely

$$\begin{aligned}
 174 \quad (2.13) \quad & -\operatorname{div}(A_{\text{PML}} \nabla u) + i \frac{1}{2} \mathbf{a}_{\text{PML}} \cdot \nabla u + i \frac{1}{2} \operatorname{div}(u \mathbf{a}_{\text{PML}}) \\
 & + s_x s_y \left( \mu - \frac{1}{4} \|\mathbf{a}\|_{A^{-1}}^2 + \frac{1}{4 s_x s_y} \|\mathbf{a}\|_{A_{\text{PML}}^{-1}}^2 \right) u = f.
 \end{aligned}$$

175 Note that the PDE remains unchanged from the initial one in (1.1) in the physical  
 176 region.

177 As a numerical illustration, let us consider once again problem (2.7), but this  
 178 time with the problem parameters

$$179 \quad (2.14) \quad \mathbf{v} := \mathcal{M} \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}, \quad \theta := 0, \quad \mathcal{M} := \frac{4}{5}, \quad \omega := 20.$$

180 In Figure 2, we show the real part of the solution computed using a naive classical  
 181 PML (left), which is known to have instabilities in some configurations, as we can  
 182 clearly see here, and the PML formulation (2.13) (right), which works perfectly.

183 **3. Fourier analysis of a classical Schwarz algorithm.** We now present and  
 184 analyze Schwarz domain decomposition methods for (1.1) in a specific geometry: find  
 185  $u \in H^1(\Omega)$  satisfying

$$186 \quad (3.1) \quad -\operatorname{div}(A \nabla u) + i \mathbf{a} \cdot \nabla u + \mu u = f \quad \text{in } \Omega = (a, b) \times (c, d),$$

187 where  $A$  is a diagonal matrix,  $a < b$  and  $c < d$  with  $\{a, b, c, d\} \in \mathbb{R}$ . For the boundary  
 188 conditions, we will consider four configurations:

- 189 • *Dirichlet-Dirichlet*: we impose homogeneous Dirichlet boundary conditions  
 190 on both vertical and horizontal boundaries,
- 191 • *Dirichlet-PML*: we impose also homogeneous Dirichlet boundary conditions  
 192 on the left and right but a PML on the bottom and top boundaries (which  
 193 terminates with a homogeneous Dirichlet boundary condition),
- 194 • *PML-Dirichlet*: the same idea but with PML on the vertical boundaries,
- 195 • *PML-PML*: imposing PML on all sides of the domain.

196 The first case models a bounded domain, the second and third cases a waveguide  
 197 with different orientation, and the last case a free space problem. We decompose  
 198 the domain  $\Omega$  first into two overlapping subdomains  $\Omega_1 := (a, \beta) \times (c, d)$  and  $\Omega_2 :=$   
 199  $(\alpha, b) \times (c, d)$  with  $\alpha \leq \beta$ , see Figure 3. We denote by  $\Gamma_{1,2} := \{x = \beta\} \times (0, 1)$  the

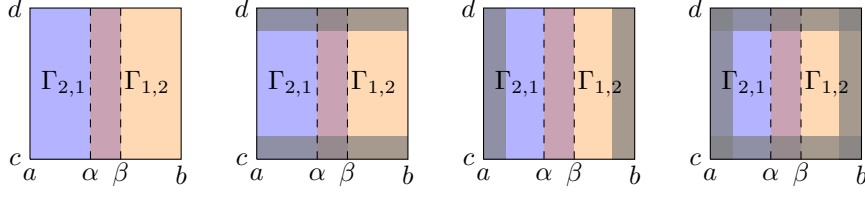


FIG. 3. Domain decomposition for the model problem, from left to right: *D-D*, *D-PML*, *PML-D* and *PML-PML*. In blue we show the domain  $\Omega_1$ , in orange the domain  $\Omega_2$  and in gray the PML region. The overlapping area is delimited by the boundaries  $\Gamma_{1,2}$  and  $\Gamma_{2,1}$ .

200 interface of  $\Omega_1$  (within  $\Omega_2$ ) and  $\Gamma_{2,1} := \{x = \alpha\} \times (0, 1)$  the interface of  $\Omega_2$  (within  $\Omega_1$ ).  
 201 Then, a general iterative Schwarz algorithm computes for iteration index  $n = 1, 2, \dots$   
 202 the subdomain solutions

$$\begin{aligned}
 (3.2) \quad & -\operatorname{div}(A\nabla u_1^n) + \mathbf{i}\mathbf{a} \cdot \nabla u_1^n + \mu u_1^n = f_1 && \text{in } \Omega_1, \\
 & u_1^n = 0 && \text{on } \partial\Omega_1 \setminus \Gamma_{1,2}, \\
 & (a_{11}\partial_x + p_{1,2} - \mathbf{i}\frac{a_1}{2})u_1^n = (a_{11}\partial_x + p_{1,2} - \mathbf{i}\frac{a_1}{2})u_2^{n-1} && \text{on } \Gamma_{1,2}, \\
 & -\operatorname{div}(A\nabla u_2^n) + \mathbf{i}\mathbf{a} \cdot \nabla u_2^n + \mu u_2^n = f_2 && \text{in } \Omega_2, \\
 & u_2^n = 0 && \text{on } \partial\Omega_2 \setminus \Gamma_{2,1}, \\
 & (-a_{11}\partial_x + p_{2,1} + \mathbf{i}\frac{a_1}{2})u_2^n = (-a_{11}\partial_x + p_{2,1} + \mathbf{i}\frac{a_1}{2})u_1^n && \text{on } \Gamma_{2,1},
 \end{aligned}$$

204 where  $f^i$  is the restriction of  $f$  to  $\Omega_i$ ,  $i \in \{1, 2\}$ , and  $p_{1,2}$ ,  $p_{2,1}$  are complex constants.  
 205 We emphasize that when using PML, the algorithm should be written with complex  
 206 stretched coordinates, or equivalently with the PML formulation as described in sub-  
 207 section 2.3. Also, note that we consider here a particular Robin type transmission  
 208 condition to get a condition similar to the ABC (2.6) at the interfaces. Moreover, in  
 209 the PML formulation it is interesting to note that the boundary term coming from  
 210 the integration by parts of  $\operatorname{div}(u\mathbf{a}_{PML})$  is canceled by this choice of transmission  
 211 condition, see Remark 3.3 for more details.

212 To study the convergence of the Schwarz algorithm (3.2) as  $n$  goes to infinity,  
 213 we consider the error  $u - u_i^n|_{\Omega_i}$ ,  $i \in \{1, 2\}$ , which amounts to consider the algorithm  
 214 (3.2) with zero source terms. Using the equivalence with the Helmholtz equation, the  
 215 iterative algorithm (3.2) for the error becomes

$$\begin{aligned}
 (3.3) \quad & -\Delta'(v')_1^n - \tilde{\omega}^2(v')_1^n = 0 && \text{in } \Omega'_1, \\
 & (v')_1^n = 0 && \text{on } \partial\Omega'_1 \cap \Omega', \\
 & (\partial_{x'} + p'_{1,2})(v')_1^n = (\partial_{x'} + p'_{1,2})(v')_2^{n-1} && \text{on } \Gamma'_{1,2}, \\
 & -\Delta'(v')_2^n - \tilde{\omega}^2(v')_2^n = 0 && \text{in } \Omega'_2, \\
 & (v')_2^n = 0 && \text{on } \partial\Omega'_2 \cap \Omega', \\
 & (-\partial_{x'} + p'_{2,1})(v')_2^n = (-\partial_{x'} + p'_{2,1})(v')_1^n && \text{on } \Gamma'_{2,1},
 \end{aligned}$$

217 where  $\Gamma'_{1,2} := \{x' = \beta'\} \times (c', d')$  and  $\Gamma'_{2,1} := \{x' = \alpha'\} \times (c', d')$ , and

$$(3.4) \quad p'_{2,1} = \frac{p_{2,1}}{g_{11}} \quad \text{and} \quad p'_{1,2} = \frac{p_{1,2}}{g_{11}},$$

219 and we recall that  $(g_{ij})_{ij}$  are the coefficients of the lower triangular matrix  $G$  from the  
 220 Cholesky decomposition  $A = GG^T$ . As a consequence, to study the convergence of  
 221 the Schwarz algorithm (3.2), we will study the convergence of the algorithm rewritten  
 222 for the Helmholtz equation (3.3). A similar idea of using an equivalent algorithm to  
 223 remove the anisotropy and advection term can be found in [18].



224 *Remark 3.1.* The choice of a diagonal matrix  $A$  ensures that in the reformulation  
 225 (3.3) the domain  $\Omega'$  is still a square, which is important for the analytical solution we  
 226 use below.

227 *Remark 3.2.* From the reformulation (3.3) of the Schwarz algorithm (3.2), we can  
 228 obtain optimized transmission conditions, using the optimized parameters  $p'_{1,2}$  and  
 229  $p'_{2,1}$  from [23] and relation (3.4).

230 *Remark 3.3.* For the implementation, in the PML context, the transmission con-  
 231 ditions on  $\Gamma_{1,2}$  and  $\Gamma_{2,1}$  should be equivalently rewritten as

$$232 \quad \frac{s_y}{s_x} a_{11} \partial_x u_1^n + \frac{s_y}{s_x} (p_{1,2} - i \frac{a_1}{2}) u_1^n = \frac{s_y}{s_x} a_{11} \partial_x u_2^{n-1} + \frac{s_y}{s_x} (p_{1,2} - i \frac{a_1}{2}) u_2^{n-1} \quad \text{on } \Gamma_{1,2},$$

233 and

$$234 \quad -\frac{s_y}{s_x} a_{11} \partial_x u_2^n + \frac{s_y}{s_x} (p_{2,1} + i \frac{a_1}{2}) u_2^n = -\frac{s_y}{s_x} a_{11} \partial_x u_1^n + \frac{s_y}{s_x} (p_{2,1} + i \frac{a_1}{2}) u_1^n \quad \text{on } \Gamma_{2,1},$$

235 to get natural variational conditions. This is different from implementing the Després  
 236 like transmission conditions

$$237 \quad \frac{s_y}{s_x} a_{11} \partial_x u_1^n + (p_{1,2} - i \frac{a_1}{2}) u_1^n = \frac{s_y}{s_x} a_{11} \partial_x u_2^{n-1} + (p_{1,2} - i \frac{a_1}{2}) u_2^{n-1} \quad \text{on } \Gamma_{1,2},$$

238 and

$$239 \quad -\frac{s_y}{s_x} a_{11} \partial_x u_2^n + (p_{2,1} + i \frac{a_1}{2}) u_2^n = -\frac{s_y}{s_x} a_{11} \partial_x u_1^n + (p_{2,1} + i \frac{a_1}{2}) u_1^n \quad \text{on } \Gamma_{2,1},$$

240 which can lead to a divergent algorithm when algorithm (3.2) is convergent!

### 241 **3.1. Computation of the convergence factor in the two subdomain case.**

242 We show the computations for the *PML-PML* case, the other cases can be deduced  
 243 by simply taking  $\sigma_x = 0$  or  $\sigma_y = 0$ , see equation (2.10) for the definition of  $\sigma_x$  and  $\sigma_y$ .  
 244 We introduce the complex stretched coordinates in the modified coordinate system  
 245  $(x', y')$ ,

$$246 \quad (3.5) \quad \tilde{x}'(x') = \begin{cases} x' + i\sigma(x' - (a' + \ell')) & \text{if } x' \in (a', a' + \ell'), \\ x' & \text{if } x' \in (a' + \ell', b' - \ell'), \\ x' + i\sigma(x' - (b' - \ell')) & \text{if } (b' - \ell', b'), \end{cases}$$

247 and

$$248 \quad (3.6) \quad \tilde{y}'(y') = \begin{cases} y' + i\sigma_y(y' - (c' + \ell')) & \text{if } y' \in (c', c' + \ell'), \\ y' & \text{if } y' \in (c' + \ell', d' - \ell'), \\ y' + i\sigma_y(y' - (d' - \ell')) & \text{if } y' \in (d' - \ell', d'). \end{cases}$$

249 Due to the rectangular geometry of the domain  $\Omega'$ , and since  $A$  is assumed to be  
 250 diagonal, we can use separation of variables to analytically obtain the errors in the  
 251 Schwarz algorithm (3.3),

$$252 \quad (3.7) \quad (v')_i^n = \sum_{k \in \mathbb{N}^*} \psi_k(y') \left( A_i^n(k) e^{i\lambda(\xi_k) \tilde{x}'(x')} + B_i^n(k) e^{-i\lambda(\xi_k) \tilde{x}'(x')} \right), \quad i \in \{1, 2\},$$

253 where  $\lambda(\xi_k) = \sqrt{\tilde{\omega}^2 - \xi_k^2}$ . The functions  $\psi_k$  and the complex numbers  $\xi_k$  are the  
 254 eigenfunctions and eigenvalues of the eigenvalue problem

$$255 \quad (3.8) \quad \begin{cases} -\partial_{y'}^2 \psi_k = \xi_k^2 \psi_k & \text{for } y' \in (c', d'), \\ \psi_k = 0 & \text{on } y' \in \{c', d'\}, \end{cases}$$

256 and we have, up to normalization

$$257 \quad (3.9) \quad \psi_k \propto \sin(\xi_k(y' - c')) \quad \text{and} \quad \xi_k = \frac{k\pi}{d' - c'} \quad \text{if } \sigma_y = 0,$$

258 and

$$259 \quad (3.10) \quad \psi_k \propto \sin(\xi_k(\tilde{y}'(y') - \tilde{y}'(0))) \quad \text{and} \quad \xi_k = \frac{k\pi}{d' - c' + 2i\ell'\sigma_y} \quad \text{if } \sigma_y > 0.$$

260

261 *Remark 3.4.* If there is no horizontal PML ( $\sigma_y = 0$ ), then the family  $(\psi_k)_k$  is an  
 262 orthonormal basis of  $L^2((c', d'))$ . This does not hold any more when considering PML  
 263 ( $\sigma_y > 0$ ). In fact, although one can show that the family is a complete basis [31], it  
 264 is neither an orthonormal basis nor a Riesz basis. A consequence of this result is that  
 265 the decomposition (3.7) is still justified, but cannot be computed in practice given an  
 266 arbitrary Robin data on  $\Gamma_{1,2}$  or  $\Gamma_{2,1}$ .

267 In the expressions (3.7), the amplitudes  $A_i^n(k)$  and  $B_i^n(k)$  should be chosen to satisfy  
 268 the vertical BCs, namely

- 269 • on  $\Gamma'_0 = \{x' = a'\} \times (c', d')$  and  $\Gamma_{1,2}$  for  $i = 1$ ,
- 270 • and on  $\Gamma'_2 = \{x' = b'\} \times (c', d')$  and  $\Gamma_{2,1}$  for  $i = 2$ ,

271 the horizontal BCs on  $(a', b') \times \{c', d'\}$  being already satisfied. To ensure these BCs,  
 272 we must impose

$$273 \quad B_1^n(k) = -A_1^n(k)e^{2i\lambda(\xi_k)(a' - i\sigma_x\ell')} \quad \text{and} \quad B_2^n(k) = -A_2^n(k)e^{2i\lambda(\xi_k)(b' + i\sigma_x\ell')}.$$

274 Now, the BC on  $\Gamma'_{1,2}$  (on  $x' = \beta'$ ) implies

$$\begin{aligned} & A_1^n(k) \left[ i\lambda(\xi_k) \left( e^{i\lambda(\xi_k)\beta'} + e^{2i\lambda(\xi_k)(a' - i\sigma_x\ell')} e^{-i\lambda(\xi_k)\beta'} \right) \right. \\ & \quad \left. + p'_{1,2} \left( e^{i\lambda(\xi_k)\beta'} - e^{2i\lambda(\xi_k)(a' - i\sigma_x\ell')} e^{-i\lambda(\xi_k)\beta'} \right) \right] \\ 275 \quad & = A_2^{n-1}(k) \left[ i\lambda(\xi_k) \left( e^{i\lambda(\xi_k)\beta'} + e^{2i\lambda(\xi_k)(b' + i\sigma_x\ell')} e^{-i\lambda(\xi_k)\beta'} \right) \right. \\ & \quad \left. + p'_{1,2} \left( e^{i\lambda(\xi_k)\beta'} - e^{2i\lambda(\xi_k)(b' + i\sigma_x\ell')} e^{-i\lambda(\xi_k)\beta'} \right) \right], \end{aligned}$$

276 so that

$$277 \quad A_1^n(k) = \rho_1(k) A_2^{n-1}(k),$$

278 with the first convergence factor component

$$279 \quad (3.11) \quad \rho_1(k) = \frac{(i\lambda(\xi_k) + p'_{1,2}) + e^{2i\lambda(\xi_k)(b' + i\sigma_x\ell' - 2\beta')}}{(i\lambda(\xi_k) + p'_{1,2}) + e^{2i\lambda(\xi_k)(a' - i\sigma_x\ell' - 2\beta')}} (i\lambda(\xi_k) - p'_{1,2}).$$

280 In the same way, we get using the BC on  $\Gamma'_{2,1}$  (on  $x' = \alpha'$ ) that

$$\begin{aligned} & A_2^n(k) \left[ -i\lambda(\xi) \left( e^{i\lambda(\xi_k)\alpha'} + e^{2i\lambda(\xi_k)(b' + i\sigma_x\ell')} e^{-i\lambda(\xi_k)\alpha'} \right) \right. \\ & \quad \left. + p'_{2,1} \left( e^{i\lambda(\xi_k)\alpha'} - e^{2i\lambda(\xi_k)(b' + i\sigma_x\ell')} e^{-i\lambda(\xi_k)\alpha'} \right) \right] \\ 281 \quad & = A_1^n(k) \left[ -i\lambda(\xi_k) \left( e^{i\lambda(\xi_k)\alpha'} + e^{2i\lambda(\xi_k)(a' - i\sigma_x\ell')} e^{-i\lambda(\xi_k)\alpha'} \right) \right. \\ & \quad \left. + p'_{2,1} \left( e^{i\lambda(\xi_k)\alpha'} - e^{2i\lambda(\xi_k)(a' - i\sigma_x\ell')} e^{-i\lambda(\xi_k)\alpha'} \right) \right], \end{aligned}$$

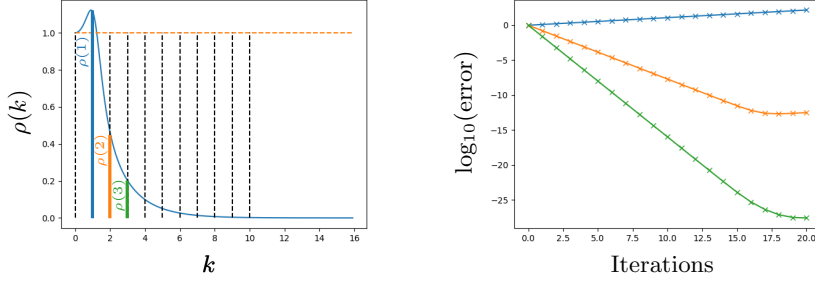


FIG. 4. Convergence factor  $\rho^{DD}$  (on the left) with  $\rho^{DD}(1) \simeq 1.11$  (in blue),  $\rho^{DD}(2) \simeq 0.463$  (in orange) and  $\rho^{DD}(3) \simeq 0.202$  (in green). Error evolution versus iterations (on the right) initializing the error system with mode 1, 2 and 3 (curves in blue, orange and green respectively).

282 so that

$$283 \quad A_2^n(k) = \rho_2(k)A_1^n(k),$$

284 with the second convergence factor component

$$285 \quad (3.12) \quad \rho_2(k) = \frac{(-i\lambda(\xi_k) + p'_{2,1}) - e^{2i\lambda(\xi_k)(a' - i\sigma_x \ell' - \alpha')}}{(-i\lambda(\xi_k) + p'_{2,1}) - e^{2i\lambda(\xi_k)(b' + i\sigma_x \ell' - \alpha')}} \frac{(i\lambda(\xi_k) + p'_{2,1})}{(i\lambda(\xi_k) + p'_{2,1})}.$$

286 Thus, the convergence factor of the Schwarz method is  $\rho(k) = \rho_1(k)\rho_2(k)$ . Let us  
 287 emphasize once again that the convergence factor depends on the case considered  
 288 ( $D$ - $D$ ,  $D$ - $PML$ ,  $PML$ - $D$  or  $PML$ - $PML$ ) through  $\sigma_x$  but also through the eigenvalue  
 289 problem (3.8).

290 As a numerical illustration, let us consider the wave-ray equation,

$$291 \quad (3.13) \quad \begin{aligned} -\Delta u + i\mathbf{a} \cdot \nabla u &= 0 & \text{in } \Omega = (0, 1)^2, \\ u &= 0 & \text{on } \partial\Omega, \end{aligned}$$

292 with  $\mathbf{a} := (10, 0)$ . To solve this problem, we implemented algorithm (3.2) (not its  
 293 equivalent Helmholtz formulation (3.3)) with the parameters

$$294 \quad (3.14) \quad p_{12} = -i\tilde{\omega}g_{11} \quad \text{and} \quad p_{21} = -i\tilde{\omega}g_{11},$$

295 which corresponds to  $p'_{12} = p'_{21} = -i\tilde{\omega}$  (and to the ABC (2.6)). We took  $\alpha = 0.45$   
 296 and  $\beta = 0.55$  so that the overlap is of size 0.1. In the  $D$ - $D$  case, the convergence  
 297 factor is shown in Figure 4 (where the variable  $k$  is “continuified” in the abscissa).  
 298 Since we have  $\rho(-k) = \rho(k)$ , we show the convergence factor only for  $k \geq 0$ . The  
 299 vertical dotted lines correspond to integer values of  $k$ . We see that the algorithm  
 300 is not convergent, since for the first (non-zero) mode, we have  $\rho^{DD}(1) \simeq 1.115$ . In  
 301 the same Figure, we also show the error evolution versus the iterations initializing  
 302 the error equations with mode  $k = 1, 2$  and 3. We see that for modes 2 and 3, the  
 303 algorithm is convergent (as expected from the convergence factor) up to a point where  
 304 the round-off error makes the first mode appear, like in power iterations. Computing  
 305 the slopes of the three lines on the right gives the convergence factors  $\rho^{DD}(1) \simeq 1.115$ ,  
 306  $\rho^{DD}(2) \simeq 0.463$  and  $\rho^{DD}(3) \simeq 0.202$ , matching well the theoretical prediction on the  
 307 left.

308 For the same example, we show the convergence factors  $\rho^{DD}$ ,  $\rho^{DPML}$ ,  $\rho^{PMLD}$   
 309 and  $\rho^{PMLPML}$  in Figure 5. For the PML parameters, we took  $\sigma_x = \sigma_y = 10$  and

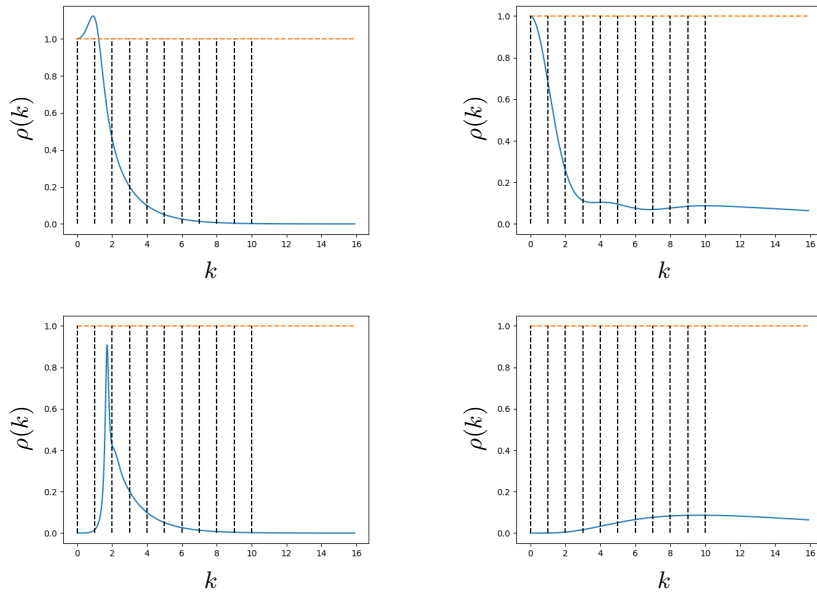


FIG. 5. Convergence factor  $\rho^{DD}$  (top left),  $\rho^{DPML}$  (top right),  $\rho^{PMLD}$  (bottom left) and  $\rho^{PMLPML}$  (bottom right).

310  $\ell' = 0.1$ . We see that the convergence factor is highly influenced by using a PML on  
 311 the outer boundary. In particular, the more we open the domain by adding PMLs,  
 312 the better the convergence factor becomes. This can be understood for wave like  
 313 problems in the sense that when the domain is open, error components can leave the  
 314 domain to infinity, or equivalently they are damped by the PML which emulates the  
 315 unbounded domain. Other boundary conditions reflect these error components and  
 316 inject them back into the iteration, leading to worse convergence, or even divergence.

317 A second remark we can make for waveguide problems, corresponding to the  $D$ -  
 318  $PML$  and  $PML$ - $D$  cases, is that cutting the waveguide in the infinite direction or in  
 319 the transverse direction is very different. Indeed, in the  $PML$ - $D$  case, the convergence  
 320 factor is good for small  $k$ , whereas in the the  $D$ - $PML$  case, the convergence factor is  
 321 better for larger  $k$ ,

322 A last remark is the fact that computing the convergence factor in a vertical  
 323 waveguide  $\Omega' = (a', b') \times \mathbb{R}$  using a Fourier transform in the  $y'$ -direction would lead  
 324 exactly to the same convergence factor as in the  $D$ - $D$  case, since the only change  
 325 is the continuous summation with eigenfunctions  $e^{i\xi y}$  which replaces the discrete  
 326 summation, but the computed solution and performance of the Schwarz method is  
 327 very different in an open wave guide or a closed cavity. In contrast, using a horizontal  
 328 PML as in the  $D$ - $PML$  case leads to a very different convergence factor, whereas the  
 329 computed Schwarz iterates in the physical domain correspond to the solution in the  
 330 unbounded domain! This shows that the two-subdomain analysis is very different if  
 331 we consider the PML or not.

332 *Remark 3.5.* Note that if the PML parameters  $\sigma_x$  and  $\sigma_y$  are too large, then the  
 333 convergence factor deteriorates. In particular, if  $\sigma_x = \sigma_y = \sigma \rightarrow +\infty$ , we do not  
 334 recover the convergence factor one would get in the full space  $\mathbb{R}^2$ , as in [23]. However,

335 we recover this convergence factor if the length of the PML tends to  $+\infty$ .

336 **3.2. Generalization to more subdomains.** The Fourier analysis above can  
 337 be generalized to more subdomains if we still consider vertical slicing of the domain  
 338 to allow us to use separation of variables. Let us consider  $N_s$  subdomains  $\Omega_i =$   
 339  $(\alpha_i, \beta_i) \times (c, d)$ ,  $i \in \{1, \dots, N_s\}$ , where

$$340 \quad a = \alpha_1 < \alpha_2 < \beta_1 < \alpha_2 < \dots < \alpha_{N_s} < \beta_{N_s-1} < \beta_{N_s} = b.$$

341 In that case, for simplicity we will consider the parallel version of algorithm (3.2),  
 342 which gives for its equivalent Helmholtz formulation the error equations

$$343 \quad (3.15) \quad \begin{aligned} -\Delta'(v')_i^n - \tilde{\omega}^2(v')_i^n &= 0 && \text{in } \Omega'_i, \\ (v')_i^n &= 0 && \text{on } \partial\Omega'_i \cap \Omega', \\ (\partial_{x'} + p'_{i,i+1})(v')_i^n &= (\partial_{x'} + p_{i,i+1})(v')_{i+1}^{n-1} && \text{on } \Gamma'_{i,i+1}, \\ (-\partial_{x'} + p'_{i,i-1})(v')_i^n &= (-\partial_{x'} + p_{i,i-1})(v')_{i-1}^{n-1} && \text{on } \Gamma'_{i,i-1}, \end{aligned}$$

344 where  $\Gamma'_{i,i-1} := \{x' = \alpha'_i\} \times (c', d')$  and  $\Gamma'_{i,i+1} := \{x' = \beta'_i\} \times (c', d')$ . Then, with  
 345 separation of variables, we still have as in (3.7), for all  $i \in \{1, \dots, N_s\}$

$$346 \quad (3.16) \quad (v')_i^n = \sum_{k \in \mathbb{N}^*} \psi_k(y') \left( A_i^n(k) e^{\imath\lambda(\xi_k)\tilde{x}'(x')} + B_i^n(k) e^{-\imath\lambda(\xi_k)\tilde{x}'(x')} \right).$$

347 Here again, the horizontal BCs are satisfied by definition of  $\psi_k$ . For each mode, the  
 348 BC on  $\{x' = a'\} \times (c', d')$  imposes that

$$349 \quad (3.17) \quad B_1^n(k) = -A_1^n(k) e^{2\imath\lambda(\xi_k)(a' - i\sigma_x \ell')},$$

350 whereas the BC on  $\{x' = b'\} \times (c', d')$  imposes that

$$351 \quad (3.18) \quad B^{N_s, n}(k) = -A^{N_s, n}(k) e^{2\imath\lambda(\xi_k)(b' + i\sigma_x \ell')}.$$

352 The transmission conditions on  $\Gamma'_{i,i+1}$  give

$$353 \quad \begin{aligned} & A_i^n(k) (\imath\lambda(\xi_k) + p'_{i,i+1}) e^{\imath\lambda(\xi_k)\tilde{x}'(\beta_i)} \\ & + B_i^n(k) (-\imath\lambda(\xi_k) + p'_{i,i+1}) e^{-\imath\lambda(\xi_k)\tilde{x}'(\beta_i)} \\ & = A_{i+1}^{n-1}(k) (\imath\lambda(\xi_k) + p'_{i,i+1}) e^{\imath\lambda(\xi_k)\tilde{x}'(\beta_i)} \\ & + B_{i+1}^{n-1}(k) (-\imath\lambda(\xi_k) + p'_{i,i+1}) e^{-\imath\lambda(\xi_k)\tilde{x}'(\beta_i)}, \end{aligned}$$

354 and similarly the transmission conditions on  $\Gamma'_{i,i-1}$  give

$$355 \quad \begin{aligned} & A_i^n(k) (-\imath\lambda(\xi_k) + p'_{i,i-1}) e^{\imath\lambda(\xi_k)\tilde{x}'(\alpha_i)} \\ & + B_i^n(k) (\imath\lambda(\xi_k) + p'_{i,i-1}) e^{-\imath\lambda(\xi_k)\tilde{x}'(\alpha_i)} \\ & = A_{i-1}^{n-1}(k) (-\imath\lambda(\xi_k) + p'_{i,i-1}) e^{\imath\lambda(\xi_k)\tilde{x}'(\alpha_i)} \\ & + B_{i-1}^{n-1}(k) (\imath\lambda(\xi_k) + p'_{i,i-1}) e^{-\imath\lambda(\xi_k)\tilde{x}'(\alpha_i)}. \end{aligned}$$

356 Combining these relations, we get for the mode  $k$  the iteration relation

$$357 \quad \mathbf{c}^n(k) = I(k) \mathbf{c}^{n-1}(k), \quad \text{where } I(k) = D^{-1}(k)K(k),$$

358 where  $\mathbf{c}^n(k) := [A_1^n(k) \ B_1^n(k) \ \cdots \ A^{N_s, n}(k) \ B^{N_s, n}(k)]^T$ . The matrix  $D$  is block  
 359 diagonal

$$360 \quad D = \begin{bmatrix} D_1 & 0 & \cdots & \cdots & 0 \\ 0 & D_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & D_{N_s} \end{bmatrix},$$

361 where the matrices  $D_i$  are  $2 \times 2$  matrices s.t. for all  $i \in \{2, \dots, N_s - 1\}$

$$362 \quad D_i = \begin{bmatrix} (-\imath\lambda(\xi_k) + p'_{i,i-1})e^{2\imath\lambda(\xi_k)\tilde{x}'(\alpha_i)} & \imath\lambda(\xi_k) + p'_{i,i-1} \\ (\imath\lambda(\xi_k) + p'_{i,i+1})e^{2\imath\lambda(\xi_k)\tilde{x}'(\beta_i)} & -\imath\lambda(\xi_k) + p'_{i,i+1} \end{bmatrix},$$

363 and, to take into account the Dirichlet BCs (3.17) and (3.18), we have

$$364 \quad D_1 = \begin{bmatrix} e^{2\imath\lambda(\xi_k)\tilde{x}'(\alpha_1)} & 1 \\ (\imath\lambda(\xi_k) + p'_{1,2})e^{2\imath\lambda(\xi_k)\tilde{x}'(\beta_1)} & -\imath\lambda(\xi_k) + p'_{1,2} \end{bmatrix}$$

365 and

$$366 \quad D_{N_s} = \begin{bmatrix} (-\imath\lambda(\xi_k) + p'_{N_s, N_s-1})e^{2\imath\lambda(\xi_k)\tilde{x}'(\alpha_{N_s})} & \imath\lambda(\xi_k) + p'_{N_s, N_s-1} \\ e^{2\imath\lambda(\xi_k)\tilde{x}'(\beta_{N_s})} & 1 \end{bmatrix}.$$

367 Similarly, the matrix  $K$  is given by

$$368 \quad K = \begin{bmatrix} 0 & K_{1,2} & 0 & \cdots & 0 \\ K_{2,1} & 0 & K_{2,3} & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & K_{N_s-1, N_s} \\ 0 & \cdots & \cdots & K_{N_s, N_s-1} & 0 \end{bmatrix},$$

369 where

$$370 \quad K_{i,i-1} = \begin{bmatrix} (-\imath\lambda(\xi_k) + p'_{i,i-1})e^{2\imath\lambda(\xi_k)\tilde{x}'(\alpha_i)} & \imath\lambda(\xi_k) + p'_{i,i-1} \\ 0 & 0 \end{bmatrix}$$

371 and

$$372 \quad K_{i,i+1} = \begin{bmatrix} 0 & 0 \\ (\imath\lambda(\xi_k) + p'_{i,i+1})e^{2\imath\lambda(\xi_k)\tilde{x}'(\beta_i)} & -\imath\lambda(\xi_k) + p'_{i,i+1} \end{bmatrix}.$$

373 Thus the convergence factor is the spectral radius of the matrix  $I(k)$ ,

$$374 \quad (3.19) \quad \rho(k) = \rho(I(k)).$$

375

376 *Remark 3.6.* In the two subdomain case,  $N_s = 2$ , if we eliminate  $B_1^n(k)$  and  
 377  $B_2^n(k)$  using the outer Dirichlet BCs, the iteration matrix becomes

$$378 \quad \begin{bmatrix} 0 & \rho_1(k) \\ \rho_2(k) & 0 \end{bmatrix}$$

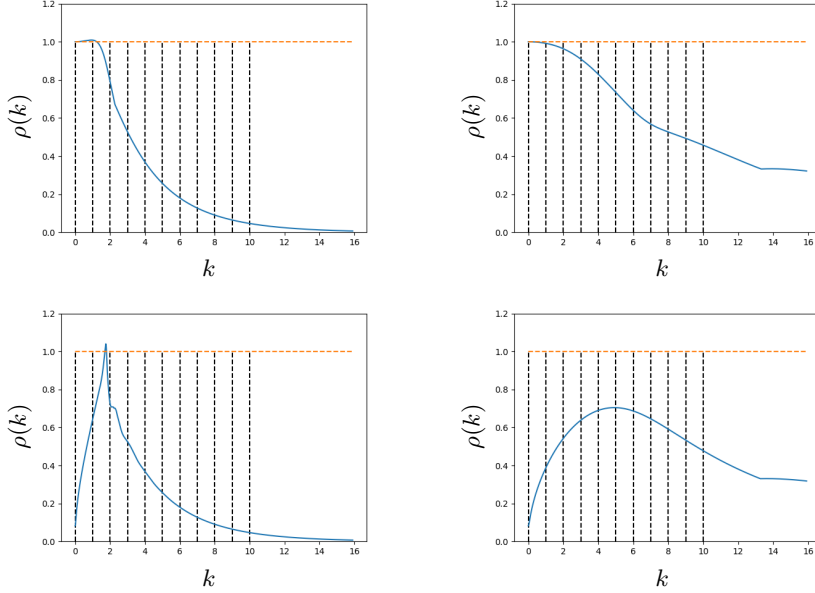


FIG. 6. Convergence factor  $\rho^{DD}$  (top left),  $\rho^{DPML}$  (top right),  $\rho^{PMLD}$  (bottom left) and  $\rho^{PMLPML}$  (bottom right) in the case of 5 subdomains.

379 where  $\rho_1(k)$  and  $\rho_2(k)$  are defined in (3.11) and (3.12). In particular, the convergence  
 380 factor is in that case the square root of the convergence factor defined in the previous  
 381 section. Note that this is simply linked to the fact that in this section, we have  
 382 considered the parallel version of the Schwarz algorithm, whereas before we studied  
 383 the alternating version for two subdomains.

384 *Remark 3.7.* Let us also note that the matrix  $D(k)$  is not invertible if (and only  
 385 if)  $p'_{i,i-1} = -p'_{i,i+1} = \pm i\lambda(\xi_k)$ , which corresponds to the case where the subproblem in  
 386  $\Omega_i$  is not well-posed: the mode  $k$  is a non zero solution of the homogeneous problem.  
 387 Moreover, if  $p'_{i,i-1} = p'_{i,i+1} = -i\lambda(\xi_k)$ , then  $D(k)$  is diagonal and one can show that  
 388  $\rho(k) \rightarrow 0$  as the length  $\ell$  of the PML tends to  $+\infty$ .

389 As a numerical illustration, let us consider again example (3.13) of the previous  
 390 section with the same parameters, except that this time the domain is split into 5  
 391 subdomains. The subdomains are defined by

$$392 \quad (3.20) \quad \begin{aligned} \alpha_1 &= 0, & \alpha_i &= 0.2(i-1) - 0.05 & \text{for } i &\in \{2, \dots, 5\}, \\ \beta_5 &= 1, & \beta_i &= 0.2i + 0.05 & \text{for } i &\in \{1, \dots, 4\}. \end{aligned}$$

393 For the parameters  $p_{i,i+1}$  and  $p_{i,i-1}$ , we chose them s.t.  $p'_{i,i+1} = p'_{i,i-1} = -i\tilde{\omega}$ , as  
 394 before. In Figure 6, we show the convergence factor  $\rho(k)$  for the  $D$ - $D$ ,  $D$ - $PML$ ,  $PML$ -  
 395  $D$  and  $PML$ - $PML$  configurations. As one could expect, the convergence factor is less  
 396 good than for the two subdomain case, but the remarks for the two subdomain case  
 397 still hold. In particular, the more we open up the domain with PML outer boundary  
 398 conditions, the better the convergence becomes.

399 **3.3. Optimized transmission conditions.** Now that we have obtained the  
 400 convergence factor, we can look for optimized parameters  $p'_{i,i+1}$  and  $p'_{i,i-1}$ , and deduce

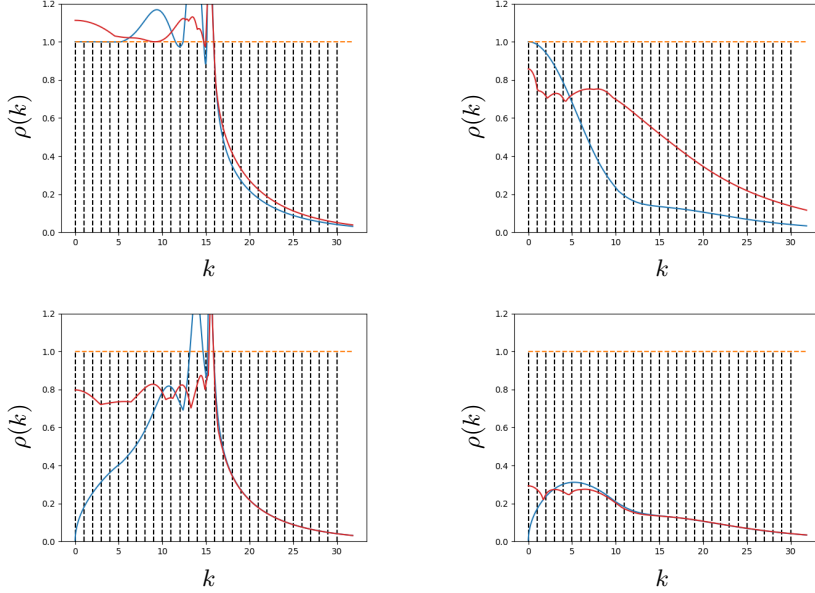


FIG. 7. Convergence factor  $\rho^{DD}$  (top left),  $\rho^{DPMML}$  (top right),  $\rho^{PMLD}$  (bottom left) and  $\rho^{PMLPML}$  (bottom right) in the case of 5 subdomains for the Helmholtz equation. Comparison between the classical transmission conditions  $-\omega$  (in blue) and optimized parameters (in red).

401  $p_{i,i+1}$  and  $p_{i+1,i}$  from equation (3.4). More precisely, we have to solve the classical  
 402 min-max problem

$$403 \quad (3.21) \quad \min_{(p'_{1,2}, p'_{2,1}, p'_{2,3}, \dots, p'_{N_s, N_s-1}) \in \mathbb{C}^{2N_s}} \max_{k \in \mathbb{N}^+} |\rho(k)|.$$

404 In particular, the optimized parameters will be different depending on the outer BCs.  
 405 Also, in contrast to the usual convergence factor in free space, see [23] for instance, here  
 406 the convergence factor can be smaller than 1 at the cut-off frequency  $\xi_k = \tilde{\omega}$ . This is  
 407 due to the fact that we consider a bounded domain for the analysis. As a consequence,  
 408 the optimization can be done for all  $k$ , as in [8], where the equation contained damping.  
 409 Finally, note that solving analytically this min-max problem is difficult, so we use a  
 410 simple optimization process and the function `fmin` of `scipy.optimize`. For first  
 411 results on a many subdomain optimization for a diffusive problem, see [14].

412 As a first numerical example, we consider the case of the Helmholtz equation, so  
 413 that  $u = v'$ ,

$$414 \quad (3.22) \quad -\Delta u - \omega^2 u = 0 \quad \text{in } \Omega = (0, 1)^2.$$

415 We split the domain into 5 subdomains defined as before in equation (3.20). Taking  
 416  $\omega = 50$ , we show in Figure 7 the convergence factor with  $p_{i,i+1} = p_{i,i-1} = -i\omega$  and  
 417 with optimized parameters in the different cases (for the PML cases, we take  $\ell = 0.02$   
 418 and  $\sigma_x = \sigma_y = 10$ ). In particular for the *PML-D* case, one can see that optimized  
 419 parameters allow us to get a convergent algorithm.

420 In Figure 8, we show the error evolution versus the iterations using either the  
 421 Schwarz algorithm as iterative solver, or as preconditioner for GMRES. As we can



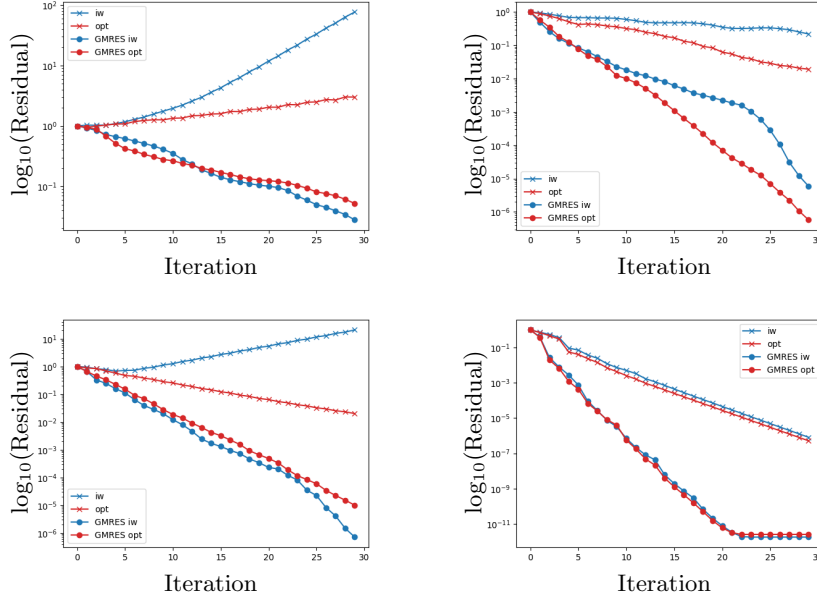


FIG. 8. Relative residual versus iterations in the D-D case (top left), D-PML case (top right), PML-D (bottom left) and in the PML-PML case (bottom right).

422 see, in each case the optimized parameters improve the convergence for the Schwarz  
 423 algorithm. Yet, this is no more true for GMRES. This can be explained since the  
 424 optimization problem (3.21) optimizes the convergence of the iterative Schwarz algo-  
 425 rithm. Therefore, when using it as a preconditioner for GMRES, a priori, we are not  
 426 ensured that the optimized parameters are optimized parameters for GMRES.

427 *Remark 3.8.* Let us emphasize that for the mesh discretization, one must consider  
 428 a sufficiently fine mesh to get accurate results that match the theoretical convergence  
 429 properties. In particular, if the mesh in the PML is too coarse, then the Schwarz  
 430 algorithm can be divergent even if the continuous convergence factor is less than one.

431 As a second, more realistic example, let us consider the case of the convected  
 432 Helmholtz equation

$$433 \quad (3.23) \quad \begin{aligned} -\operatorname{div}(A\nabla u) - 2i\omega \mathbf{v} \cdot \nabla u - \omega^2 u &= \delta \quad \text{in } \Omega = (0, 1)^2 \setminus \mathcal{O}, \\ A\nabla u \cdot \mathbf{n} &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

434 where the obstacle  $\mathcal{O}$  has the rough shape of a submarine, see Figure 9. We consider  
 435 a potential flow  $\mathbf{v} = \nabla\varphi$  coming from the left, which we compute solving the Laplace  
 436 problem

$$437 \quad \begin{aligned} -\Delta\varphi &= 0 \quad \text{in } \Omega, \\ \nabla\varphi \cdot \mathbf{n} &= 0 \quad \text{on } \partial\mathcal{O} \cup (0, 1) \times \{0, 1\}, \\ \nabla\varphi \cdot \mathbf{n} &= -1 \quad \text{on } \{0\} \times (0, 1), \\ \nabla\varphi \cdot \mathbf{n} &= 1 \quad \text{on } \{1\} \times (0, 1), \end{aligned}$$

438 with the same mesh. Note that to get a well-posed problem, we simply impose a value  
 439 of  $\varphi$  inside  $\Omega$ . We compute the gradient of  $\varphi$  inside each cell of the mesh to get  $\mathbf{v}$ .

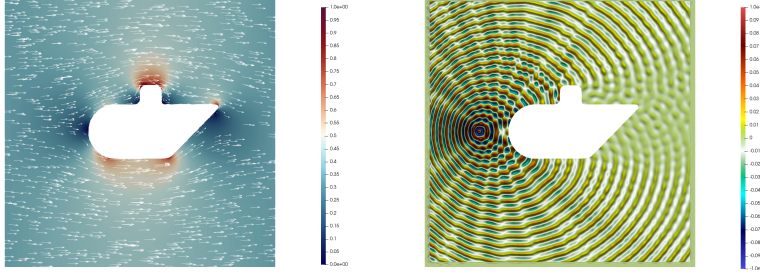


FIG. 9. Potential flow around a submarine on the left (the background color corresponds to the norm of  $\mathbf{v}/\|\mathbf{v}\|_\infty$ ). Diffracted field from a Dirac point source on the right.

440 Then, the velocity is normalized

$$441 \quad \tilde{\mathbf{v}} = \text{Ma} \frac{\mathbf{v}}{\|\mathbf{v}\|_\infty}, \quad \text{where} \quad \|\mathbf{v}\|_\infty = \sup_{(x,y) \in \Omega} \|\mathbf{v}(x,y)\|_2,$$

442 where Ma is the mach number, i.e. the ratio between the velocity of the fluid and the  
443 sound speed in the medium. Thus, the matrix  $A$  in (3.23) is given by

$$444 \quad A = \text{Id} - \tilde{\mathbf{v}}\tilde{\mathbf{v}}^T.$$

445 For this example, we took<sup>2</sup> Ma = 0.7 and  $\omega = 200$ . For the PML, we took  $\ell = 0.02$   
446 and  $\sigma_x = \sigma_y = 15$ . Also, we assume the flow to be constant and horizontal in the  
447 PML region  $\tilde{\mathbf{v}} = (\tilde{v}_{ext}, 0)^T$ , although this is not exact: the flow is almost horizontal  
448 and constant. The domain is decomposed into 5 subdomains defined by

$$449 \quad (3.24) \quad \begin{aligned} \alpha_1 &= 0, & \alpha_i &= 0.2(i-1) - 0.015 & \text{for } i \in \{2, \dots, 5\}, \\ \beta_5 &= 1, & \beta_i &= 0.2i + 0.015 & \text{for } i \in \{1, \dots, 4\}, \end{aligned}$$

450 which corresponds to an overlap of size 0.03. The mesh we use is unstructured, so that  
451 the interfaces between the subdomains are not perfectly straight any more. Moreover,  
452 since  $\mathbf{a}$  is no more constant in the physical domain, a natural generalization of the  
453 transmission conditions, similar to the ABC (2.6), is

$$454 \quad A_{PML} \nabla u \cdot \mathbf{n} - i \frac{1}{2} \mathbf{a}_{PML} \cdot \mathbf{n} + p_{i,i\pm 1} u.$$

455 In particular, we compare in this example the following choices of the parameters  
456  $p_{i,i\pm 1}$ :

- 457 • First, a classical ABC condition as in (2.6),

$$458 \quad (\text{ABC}) \quad p_{i,i\pm 1} = -i\tilde{\omega}\|\mathbf{n}\|_A \quad \text{where} \quad \tilde{\omega} = \sqrt{-\mu + \frac{1}{4}\|\mathbf{a}\|_{A^{-1}}^2}.$$

459 Note that  $\tilde{\omega}$  and  $\|\mathbf{n}\|_A$  are variable.

<sup>2</sup>This value is in fact not realistic since in water the sound speed is much larger than in air. As far as we know, the fastest speed one can reach in water is around mach 0.075 with supercavitation. Nevertheless, this very high speed is only obtained close to corners of the submarine, where the velocity increases a lot. In the rest of the domain, the speed is more realistic.

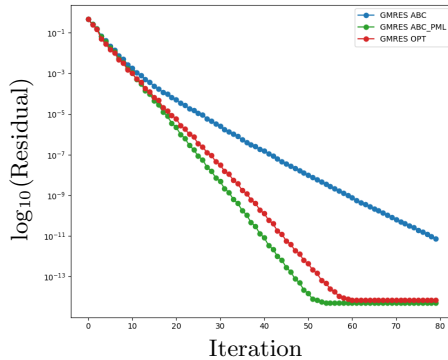


FIG. 10. *GMRES* residual versus iterations for the three types of transmission conditions (ABC), (ABC PML) and (OPT).

- 460       • Second, a similar one but taking the PML approximately into account, see  
 461       Remark 3.3,

462       (ABC PML)      $p_{i,i\pm 1} = -\frac{s_y}{s_x}i\tilde{\omega}\|\mathbf{n}\|_A$    where    $\tilde{\omega} = \sqrt{-\mu + \frac{1}{4}\|\mathbf{a}\|_{A^{-1}}^2}$ ,

463       where we recall that  $s_y$  and  $s_x$  are the PML parameters.

- 464       • Third, a condition that takes the PML into account,

465       (OPT)                  $p_{i,i\pm 1} = \frac{s_y}{s_x}q_{i,i\pm 1}$ ,

466       where  $q_{i,i\pm 1}$  are (numerical) solutions of the min-max problem (3.21) consid-  
 467       ering the medium with no obstacle and with  $\mathbf{a} = -2\omega(\tilde{v}_{ext}, 0)^T$  constant.

468       In Figure 10, we show the evolution of the residual considering these three transmis-  
 469       sion conditions. The best results are obtained with the condition ABC PML and OPT.  
 470       We see that clearly, taking into account the PML coefficient in the parameter is very  
 471       important, as already mentioned in Remark 3.3.

472       **4. Concluding remarks.** We studied Schwarz domain decomposition methods  
 473       for a general diffusion problem with complex advection, which appears in several im-  
 474       portant applications. The complex advection term changes fundamentally the nature  
 475       of the diffusion problem and makes it Helmholtz like. We have shown that for such  
 476       problems the outer boundary conditions imposed on the global domain have a strong  
 477       influence on the convergence of the Schwarz method, and on how one should choose  
 478       optimized parameters. Not taking into account the PML coefficients in the transmis-  
 479       sion conditions deteriorates the convergence of the Schwarz algorithm, both when  
 480       used as iterative solver and as preconditioner for GMRES. Our analysis covers both  
 481       two subdomain and many subdomain situations for decompositions into strips, and  
 482       allowed us to formulate the min-max problem one has to solve to compute optimized  
 483       parameters, which turns out to be difficult to treat theoretically. Furthermore, com-  
 484       puting optimized parameters for GMRES is currently out of reach, for a special case  
 485       in a splitting method, see [6].

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