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# SEMICLASSICAL SPECTRUM OF THE MAGNETIC DIRAC OPERATOR ON AN ANNULUS

E. LAVIGNE BON

ABSTRACT. We consider the magnetic Dirac operator with infinite mass boundary condition on an annulus and prove an explicit asymptotic expansion at the first order for the low-lying positive spectrum. An heuristic of proof is given in the last section for the case of the first negative eigenvalue.

## CONTENTS

1. Introduction	2
1.1. Setting and magnetic Dirac operator	3
1.1.1. Magnetic Dirac operator	3
1.1.2. Scalar and vector potential	3
1.2. Results and discussions	4
2. Characterization of the spectrum	6
2.1. The Hardy space on an annulus	6
2.2. The min-max formula	9
2.3. A similar characterization for fixed angular momentum	10
3. Semiclassical Analysis of the positive eigenvalues	11
3.1. Case of projected states	12
3.1.1. Uniform convergence and upper bound	12
3.1.2. Proof of Proposition 3.3	14
3.2. From the non-linear min-max formula to the projected states formula	16
3.2.1. Rayleigh quotient approximation	16
3.2.2. Proof of Proposition 3.2	17
Appendix A. Sketch of proof for Theorem 1.6	18
Acknowledgements	20

## 1. INTRODUCTION

We consider the magnetic Dirac operator with infinite-mass boundary condition on  $\Omega$  an annulus. We associate to the magnetic field  $B$ , which is assumed to be smooth, a magnetic vector potential  $A = (A_1, A_2)$  satisfying  $B = \partial_x A_2 - \partial_y A_1$ . For  $h > 0$ , this operator acts as  $\mathcal{D}_{h,A} = \sigma \cdot (-ih\nabla - A)$  on

$$\text{Dom}(\mathcal{D}_{h,A}) = \{\varphi = (\varphi_1, \varphi_2) \in H^1(\Omega, \mathbb{C}^2), \varphi_2 = i\mathbf{n}\varphi_1\},$$

where  $\mathbf{n} = (n_1, n_2)$  is the outward pointing normal to the boundary  $\partial\Omega$  we also note  $\mathbf{n} = n_1 + in_2$ ,  $\sigma = (\sigma_1, \sigma_2)$  the matrix vector made of the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and  $\sigma \cdot \mathbf{v} = \sigma_1 v_1 + \sigma_2 v_2$  for  $\mathbf{v} = (v_1, v_2) \in \mathbb{C}^2$ .

In the physical literature this type of model was initially introduced in 1987 by Berry and Mondragon to study two-dimensional neutrino billiards, see [BM97]. In recent years, the study of graphene has stimulated the study of the Dirac equation in two dimensions. In fact at low energies the electrons moving in this type of material are accurately described by this equation (see [GN07], [Cas+09] and references therein).

A detailed study of the asymptotic behavior of the low-lying (positive and negative) spectrum in the limit of strong magnetic field ( $h \rightarrow 0$ ) has been given by Barbaroux et al. (see [Bar+21b]) in the context of an open, smooth and *simply connected domain*. After that, the question of the influence of the topological type of the domain on the semiclassical spectrum is raised. The study of graphene rings has motivated many articles in the physical literature. The Aharonov-Bohm effect has been observed experimentally in graphene rings [Rus+08]. These observations have been confirmed numerically (see [Rec+07] and [TP17]).

The purpose of this article is to discuss the case where the magnetic Dirac operator is defined on an annulus with a radial magnetic field. It is straightforward to check that the spectrum of this operator can be written as a double sequence (cf. Section 1.1.1). We have for  $h > 0$

$$\dots < -\lambda_k^-(h) < \dots < -\lambda_1^-(h) < 0 < \lambda_1^+(h) < \dots < \lambda_k^+(h) < \dots$$

Here we give explicitly the first term of the asymptotic expansion for the first positive eigenvalues (which is our main result). To the best of the author's knowledge, such accurate results have not been established yet for such a basic model. For all  $k \in \mathbb{N}^*$ , we have

$$\lambda_k^+(h) = \min_{\substack{V \subset \mathbb{Z} \\ \#V = k}} \max_{m \in V} \mathfrak{f} \left( m - \frac{c_0}{h} \right) \sqrt{h} e^{\frac{2\phi_{\min}}{h}} (1 + o_{h \rightarrow 0}(1)), \quad (1.1)$$

with  $c_0 \in \mathbb{R}$ ,  $\phi_{\min}$  a strictly negative constant and  $\mathfrak{f} : \mathbb{R} \rightarrow \mathbb{R}$  a coercive function explicitly defined through the magnetic field and  $\Omega$ . The prefactor in  $\lambda_k(h)$  is very close to the one established in [Lav22, Theorem 1.1] where we studied the Dirichlet-Pauli operator. It still encodes Aharonov-Bohm-type oscillations in the semiclassical limit through the 1-periodicity of

$$d \mapsto \min_{\substack{V \subset \mathbb{Z} \\ \#V = k}} \max_{m \in V} \mathfrak{f}(m + d).$$

For more details on this topic we invite the reader to consult [Lav22, Section 1] and its references.

Moreover, the asymptotic expansion at the first order of the first negative eigenvalue is given by

$$\lambda_1^-(h) = Ch^{1/2}(1 + o(1)),$$

where  $C$  is a positive constant, precisely the one exhibited in [Bar+21b]. As it was observed for simply connected domains, the behavior of the negative spectrum is radically different from (1.1). Since the proof of this fact is quite similar to [Bar+21b], we will only sketch it in Appendix A.

**1.1. Setting and magnetic Dirac operator.** Let  $\Omega \subset \mathbb{R}^2$  be an annulus centered at the origin with radii  $0 < \rho_1 < \rho_2$ . Consider a magnetic field  $B \in \mathcal{C}^\infty(\bar{\Omega}, \mathbb{R})$ . Despite the presence of a hole, we have the existence of a vector potential denoted by  $A = (A_1, A_2)$ , which satisfies

$$B = \text{curl}(A) = \partial_x A_2 - \partial_y A_1.$$

Note however that the spectrum will depend on the choice of the vector potential (see Section 1.1.2).

**Assumption 1.1.** *The magnetic field  $B$  is radial and positive.*

1.1.1. *Magnetic Dirac operator.* We are interested in the magnetic Dirac operator  $(\mathcal{D}_{h,A}, \text{Dom}(\mathcal{D}_{h,A}))$  defined for all  $h > 0$  as

$$\mathcal{D}_{h,A} = \sigma \cdot (\mathbf{p} - A) = \begin{pmatrix} 0 & d_{h,A} \\ d_{h,A}^\times & 0 \end{pmatrix},$$

with

$$\mathbf{p} = -ih\nabla, \quad d_{h,A} = -2ih\partial_z - A_1 + iA_2 \quad \text{and} \quad d_{h,A}^\times = -2ih\partial_{\bar{z}} - A_1 - iA_2,$$

acting on the domain

$$\text{Dom}(\mathcal{D}_{h,A}) = \{\varphi = (\varphi_1, \varphi_2) \in H^1(\Omega, \mathbb{C}^2), \quad \varphi_2 = i\mathbf{n}\varphi_1\}.$$

Under these assumptions this operator is selfadjoint with compact resolvent, see [Ben+17]. By the spectral theorem, the spectrum of  $(\mathcal{D}_{h,A}, \text{Dom}(\mathcal{D}_{h,A}))$  is real, discrete and can be written as a sequence tending to  $+\infty$  in absolute value. Moreover, similarly to [Bar+21b, Proposition 1.5.],  $\mathcal{D}_{h,A}$  has no zero modes. Thus,

$$\text{sp}(\mathcal{D}_{h,A}) \cap \mathbb{R}_\pm = \left\{ \pm \lambda_j^\pm(h) \mid j \in \mathbb{N}^* \right\}.$$

In this paper we estimate the low-lying positive spectrum of  $\mathcal{D}_{h,A}$  in the semiclassical limit. Then, we also give the main ideas to obtain the first term of the asymptotic expansion of  $\lambda_1^-(h)$ .

1.1.2. *Scalar and vector potential.* The choice of  $A$  will play an important role. A particular choice is associated with the scalar potential, unique solution in  $H_0^1(\Omega, \mathbb{R})$  of the Poisson equation

$$\begin{cases} \Delta\phi = B & \text{on } \Omega \\ \phi = 0 & \text{in } \partial\Omega. \end{cases} \quad (1.2)$$

Since  $B$  is positive,  $\phi$  is a subharmonic function, which satisfies

$$\max_{x \in \bar{\Omega}} \phi = \max_{x \in \partial\Omega} \phi = 0, \quad (1.3)$$

by the maximum principle.

**Remark 1.2.** Assumption 1.1 on the magnetic field and the uniqueness of  $\phi$  ensures that  $\phi$  is radial and admits a unique circle of minimum centered at the origin and of radius  $r_{\min} \in (\rho_1, \rho_2)$ . We will denote by  $\phi_{\min}$  the value of  $\phi$  in this minimum. According to (1.3), the minima of  $\phi$  is negative. In polar coordinates,  $\phi$  is the solution of

$$\begin{cases} \phi''(r) + \frac{1}{r}\phi'(r) = B(r) & \text{on } ]\rho_1, \rho_2[ \\ \phi(\rho_1) = \phi(\rho_2) = 0. \end{cases}$$

so that  $\phi''(r_{\min}) = B(r_{\min}) \geq b_0 = \inf\{B(x) : x \in \Omega\}$ .

To deal with the hole, let us also consider  $\theta$  the unique solution of

$$\begin{cases} \Delta\theta = 0 & \text{on } \Omega \\ \theta = 1 & \text{in } \partial\Omega_{int} \quad \text{and} \quad \theta = 0 & \text{in } \partial\Omega_{ext}, \end{cases}$$

where  $\partial\Omega_{int} = \mathcal{C}(0, \rho_1)$  and  $\partial\Omega_{ext} = \mathcal{C}(0, \rho_2)$ . In polar coordinates, we have  $\theta(r) = \ln(r/\rho_2)/\ln(\rho_1/\rho_2)$ . Thanks to  $\phi$  and  $\theta$ , we have a family of vector potentials that give rise to unitarily equivalent operators.

**Proposition 1.3** (Proposition II.3. [Lav22]). *For all  $p \in \mathbb{Z}$  and  $h > 0$ , consider*

$$\mathbf{A}_{h,p} = \nabla^\perp \phi + h \gamma_{h,p} \ln \left( \frac{\rho_1}{\rho_2} \right) \nabla^\perp \theta,$$

with  $\gamma_{h,p} = p + c_0/h$  and  $c_0 = \rho_1 \partial_r \phi(\rho_1) - \frac{1}{2\pi} \int_{\partial\Omega_{int}} \mathbf{A}$ .

Then,  $\text{curl } \mathbf{A}_{h,p} = B$  and we have the following identity

$$\sigma \cdot (\mathbf{p} - \mathbf{A}_{h,p}) = e^{ip \arg} \sigma \cdot (\mathbf{p} - \mathbf{A}) e^{-ip \arg},$$

where  $\arg(\cdot)$  is the principal value of the argument that lies within the interval  $[0, 2\pi)$ .

**1.2. Results and discussions.** Before stating our results, let us point out to two recent papers which are the starting points of the work conducted here.

- The asymptotic behavior of the low-lying (positive and negative) spectrum of the magnetic Dirac operator in the limit of strong magnetic field have been the subject of a recent paper, see [Bar+21b]. Here, the authors assume that the domain is simply connected and consider generalized MIT bag boundary conditions. For infinite-mass boundary condition they show that for  $k \in \mathbb{N}^*$ , there exists  $C_k^+ > 0$  such that, as  $h \rightarrow 0$ ,

$$\lambda_k^+(h) = C_k^+ h^{1-k} e^{-2\phi_{\min}/h} (1 + o(1)),$$

where  $\phi_{\min} < 0$  is the unique minimum of the scalar potential  $\phi$  (which is assumed to be non-degenerate). They also show that there exists  $C^- > 0$  such that

$$\lambda_1^-(h) = C^- h^{1/2} (1 + o(1)).$$

A thorough study of the the low-lying negative spectrum is exposed in [Bar+21b, Section 1.2.4.]. Here, Agmon estimates ensure that eigenfunctions associated to negative energies are exponentially localized near the boundary.

The techniques employed in the proof are mainly based on a non-linear min-max characterization (see [Bar+21b, Section 2]), the ellipticity of Cauchy–Riemann operators (see [Bar+21a, Section 4]) and a microlocal dimensional reduction to the boundary for the accurate description of the negative spectrum (see [Bar+21b, Section 7 & 8]).

- The Dirichlet-Pauli operator is the (formal) square of the magnetic Dirac operator. Recent works has been devoted to the study of the behavior of its lowest eigenvalues in the semiclassical limit, for example [HP17; Bar+21a; Lav22]. In [Lav22], an explicit asymptotic expansion at the first order of the lowest eigenvalues is given in the context of an annulus with a radial magnetic field. For all  $k \in \mathbb{N}^*$ , we have

$$\Lambda_k(h) = F(h) \sqrt{h} e^{2\phi_{\min}/h} (1 + o(1)), \tag{1.4}$$

where  $\phi_{\min} < 0$  is defined in Remark 1.2, and  $F(h)$  is a very similar function to the one highlighted in Theorem 1.4.

The square of the magnetic Dirac operator with the so-called "zigzag boundary condition" is precisely the Dirichlet-Pauli operator, see [Bar+21b, Section 1.4.]. One deduces directly its spectrum by taking the square root of the equation (1.4). In the present paper, taking the square of the Dirac operator will not be particularly helpful. Despite the statements look similar, they don't really follow from the same ideas. Here, one will more rely on the strategies of [Bar+21b, Section 2], the non-linear min-max formula.

The main result of this article is the following:

**Theorem 1.4.** *Let  $B \in C^\infty(\overline{\Omega}, \mathbb{R})$  be radial such that*

$$b_0 = \inf\{B(x), x \in \Omega\} > 0,$$

*and  $A \in C^\infty(\overline{\Omega}, \mathbb{R}^2)$  be an associated vector potential.*

*Then, for all fixed  $k \in \mathbb{N}^*$ , we have*

$$\lambda_k^+(h) = \alpha_k(h) \sqrt{h} e^{2\phi_{\min}/h} (1 + o_{h \rightarrow 0}(1)),$$

where

$$\alpha_k(h) = \min_{\substack{V \subset \mathbb{Z} \\ \#V = k}} \max_{m \in V} \mathfrak{f}(m - \frac{c_0}{h}),$$

with  $c_0 = \rho_1 \partial_r \phi(\rho_1) - \frac{1}{2\pi} \int_{\partial\Omega_{int}} A$  and  $\mathfrak{f}: \mathbb{R} \rightarrow \mathbb{R}$  given by

$$\mathfrak{f}(m) = \sqrt{\frac{B(r_{\min})}{\pi}} \left( \left( \frac{\rho_1}{r_{\min}} \right)^{2m+1} + \left( \frac{\rho_2}{r_{\min}} \right)^{2m+1} \right).$$

Let us give a brief outline of the ideas to establish the main theorem.

- i. In Section 2, we follow [Bar+21b, Section 2] to obtain a non-linear min-max characterization in our context. Then, we make a fibration of the magnetic Dirac operator. We give the non-linear min-max formula corresponding to each fiber.
- ii. By using elliptic inequalities established in [Lav22], we show (cf. Section 2.3) that the excited states of the fibered operator can be discarded for the study of the low-lying positive spectrum. Thus, it is now enough to order the set  $\{\lambda_{1,m}(h) \mid m \in \mathbb{Z}\}$  and to study the behavior (in  $h$ ) of each ground state  $\lambda_{1,m}(h)$  of each fibered operator with Fourier mode  $m$ .
- iii. The proof of Theorem 1.4 relies on Propositions 3.2 and 3.3:
  1. We consider a simplified min-max formula and the ground state of the associated operator (denoted by  $\nu_m(h)$  where  $m \in \mathbb{Z}$  is the fixed angular momentum). In Proposition 3.3, we give the asymptotic behavior of each  $\nu_m(h)$ .
  2. We finally show in Proposition 2.14 that, in the semiclassical limit,  $\nu_m(h)$  is a good approximation of  $\lambda_{1,m}(h)$ .

Let us now say a few words about  $\lambda_1^-(h)$ , the first negative eigenvalue, which remarkably is not influenced by the geometry of the annulus contrary to the positive eigenvalues.

Let us give some notations before stating our result.

**Remark 1.5.** As in [Bar+21b, Theorem 1.15.] the universal constant  $a_0 \in (0, \sqrt{2})$  appears in the prefactor of  $\lambda_1^-(h)$ . Recall that  $a_0$  corresponds to the spectral gap of  $\mathcal{D}_{\mathbb{R}_+^2}$ , the homogenous Dirac operators on  $\mathbb{R}_+^2 = \mathbb{R} \times \mathbb{R}_+$ . Consider  $B = 1$  and  $A_0 = (-y, 0)$ , this operator act as  $\sigma \cdot (-i\nabla - A_0)$  on

$$\text{Dom}(\mathcal{D}_{\mathbb{R}_+^2}) = \{u \in H^1(\mathbb{R}_+^2, \mathbb{C}^2), y\phi \in L^2(\mathbb{R}_+^2, \mathbb{C}^2), \sigma_1 u = u \text{ on } \partial\mathbb{R}_+^2\}.$$

According to [Bar+21b, Theorem 1.24.], the spectrum of  $\mathcal{D}_{\mathbb{R}_+^2}$  is purely absolutely continuous and we have

$$\text{sp}(\mathcal{D}_{\mathbb{R}_+^2}) = (-\infty, -a_0] \cup [0, +\infty).$$

**Theorem 1.6.** *We have*

$$\lambda_1^-(h) = \min(\sqrt{2b_0}, a_0 \sqrt{b'_0}) h^{1/2} + o(h^{1/2}),$$

with  $b_0 = \min_{x \in \Omega} B(x)$ ,  $b'_0 = \min_{x \in \partial\Omega} B(x)$  and  $a_0 \in (0, \sqrt{2})$  defined in Remark 1.5.

## 2. CHARACTERIZATION OF THE SPECTRUM

The main result of this section is Proposition 2.10 and its counterpart for fixed angular momentum, Proposition 2.14.

We come back here to a min-max formula which has recently been established in [Bar+21b, Theorem 1.9.] for bounded and simply connected domains. This characterization plays a central role in the asymptotic analysis of the positive spectrum. Furthermore, its proof does not depend on the topological nature of the domain. We just have to be careful with the definition of the Hardy space on the annulus and make sure that its injection is compact in  $L^2(\Omega)$ .

We invite the reader to read [Bar+21b, Section 2] for the detailed proof of the min-max formula. To explore the topics of Hardy spaces the reader can look at [Rud20] for the case of the disk, [Sar65] for the case of the annulus and [Dur70] for general domains.

**2.1. The Hardy space on an annulus.** The Hardy space is defined as

$$\mathcal{H}^2(\Omega) = \left\{ f \in \mathcal{O}(\Omega) \text{ s.t. } \sup_{r \in (\rho_1, \rho_2)} M_2(f, r) < +\infty \right\},$$

with  $M_2(f, r) = \int_0^{2\pi} |f(re^{it})|^2 r dt$  for  $f \in \mathcal{H}^2(\Omega)$ ,  $r \in (\rho_1, \rho_2)$ .

According to Minkowski's inequality, the Hardy space is a vector space. Let  $f$  be a function in  $\mathcal{O}(\Omega)$  with Laurent expansion  $f(z) = \sum_{n \in \mathbb{Z}} a_n z^n$ , we define

$$\frac{1}{2\pi} \|f\|_{\partial\Omega}^2 = \sum_{n \in \mathbb{Z}} |a_n|^2 \left( \rho_1^{2n+1} + \rho_2^{2n+1} \right) \in [0, +\infty],$$

and

$$\frac{1}{2\pi} \|f\|_2^2 = \sum_{n \in \mathbb{Z} \setminus \{-1\}} |a_n|^2 \left( \frac{\rho_2^{2n+2} - \rho_1^{2n+2}}{2n+2} \right) + |a_{-1}|^2 \ln \left( \frac{\rho_2}{\rho_1} \right) \in [0, +\infty].$$

For  $f \in \mathcal{H}^2(\Omega)$  they are finite. In fact, by Fatou's lemma and Plancherel theorem

$$\begin{aligned} \|f\|_{\partial\Omega}^2 &\leq 2\pi \left( \liminf_{r \rightarrow \rho_1} \sum_{n \in \mathbb{Z}} |a_n|^2 r^{2n+1} + \liminf_{r \rightarrow \rho_2} \sum_{n \in \mathbb{Z}} |a_n|^2 r^{2n+1} \right) \\ &= \liminf_{r \rightarrow \rho_1} M_2(f, r) + \liminf_{r \rightarrow \rho_2} M_2(f, r) \\ &\lesssim \sup_{r \in (\rho_1, \rho_2)} M_2(f, r) < +\infty, \end{aligned}$$

and

$$\|f\|_2^2 = \int_{\rho_1}^{\rho_2} M_2(f, r) dr \lesssim \sup_{r \in (\rho_1, \rho_2)} M_2(f, r) < +\infty.$$

Let  $f \in \mathcal{O}(\Omega)$  such that  $\|f\|_{\partial\Omega} < +\infty$  then  $f \in \mathcal{H}^2(\Omega)$ . In fact, let  $f$  an holomorphic function on  $\Omega$  such that for all  $z \in \Omega$ ,  $f(z) = \sum_{n \in \mathbb{Z}} a_n z^n$  and  $\|f\|_{\partial\Omega} < +\infty$ . For  $r \geq 0$ ,  $r^n$  is an increasing (resp. decreasing) function of  $r$  for  $n \geq 0$  (resp.  $n < 0$ ). So that, for  $r \in (\rho_1, \rho_2)$

$$\frac{1}{2\pi} M_2(f, r) = \sum_{n \in \mathbb{Z}} |a_n|^2 r^{2n+1} \leq \sum_{n \geq 0} |a_n|^2 \rho_2^{2n+1} + \sum_{n < 0} |a_n|^2 \rho_1^{2n+1} \leq \frac{1}{2\pi} \|f\|_{\partial\Omega}^2.$$

**Notation 1.** For convenience, we also use  $\|\cdot\|_{\partial\Omega}$  and  $\|\cdot\|_2$  as norms in spaces of sequences. More precisely, for  $\mathbf{a} = (a_n)_{n \in \mathbb{Z}}$  being the coefficients of the Laurent expansion of  $f \in \mathcal{H}^2(\Omega)$ , we set

$$\|\mathbf{a}\|_2 = \|f\|_2 \text{ and } \|\mathbf{a}\|_{\partial\Omega} = \|f\|_{\partial\Omega}.$$

**Lemma 2.1.** *The space  $(\mathcal{H}^2(\Omega), \|\cdot\|_{\partial\Omega})$  is isomorphic to  $(\ell^2(\mathbb{Z}, \mathbb{C}), \|\cdot\|_{\partial\Omega})$  and  $(L^2(\Omega), \|\cdot\|_2)$  is isomorphic to  $(\ell^2(\mathbb{Z}, \mathbb{C}), \|\cdot\|_2)$ . In particular, the Hardy space is a Hilbert space.*

The next proposition plays an important role in order to show that infima are minima in the characterization of eigenvalues that we will state in Definition 2.8 and use in Proposition 2.9.

**Proposition 2.2.** *The embedding  $(\mathcal{H}^2(\Omega), \|\cdot\|_{\partial\Omega}) \hookrightarrow (L^2(\Omega), \|\cdot\|_2)$  is compact.*

We use the following lemma in the proof of Proposition 2.2.

**Lemma 2.3.** *Let us consider, for  $n \in \mathbb{Z}$ , the sequence defined by*

$$\varrho_n = \frac{\rho_2^{2n+2} - \rho_1^{2n+2}}{2(\rho_2^{2n+1} + \rho_1^{2n+1})}.$$

We have  $(\rho_n)_{n \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z}, \mathbb{R})$ .

*Proof of Proposition 2.2.* By Lemma 2.1 we just have to prove that the embedding  $(\ell^2(\mathbb{Z}, \mathbb{C}), \|\cdot\|_{\partial\Omega}) \hookrightarrow (\ell^2(\mathbb{Z}, \mathbb{C}), \|\cdot\|_2)$  is compact.

Let  $(\mathbf{u}^{(\mathbf{m})})_{\mathbf{m} \in \mathbb{N}}$  be a bounded sequence of  $(\ell^2(\mathbb{Z}, \mathbb{C}), \|\cdot\|_{\partial\Omega})$ . For  $m \in \mathbb{N}$  we denote  $\mathbf{u}^{(\mathbf{m})} = (u_n^{(m)})_{n \in \mathbb{Z}}$ . We define the sequence  $\mathbf{v}^{(\mathbf{m})} = (v_n^{(m)})_{n \in \mathbb{Z}}$  with

$$v_n^{(m)} = \sqrt{2\pi} u_n^{(m)} \left( \rho_2^{2n} + \rho_1^{2n} \right)^{1/2}.$$

Let us remark that

$$\|\mathbf{u}^{(\mathbf{m})}\|_{\partial\Omega}^2 = \sum_{n \in \mathbb{Z}} |v_n^{(m)}|^2 \quad \text{and} \quad \|\mathbf{u}^{(\mathbf{m})}\|_2^2 = |v_{-1}^{(m)}|^2 \frac{(\rho_1 \rho_2)^2 \ln(\rho_2/\rho_1)}{\rho_1^2 + \rho_2^2} + \sum_{n \in \mathbb{Z} \setminus \{-1\}} |v_n^{(m)}|^2 \frac{\varrho_n}{n+1}.$$

1. The proof relies on a diagonal argument : there exists  $\psi : \mathbb{N} \rightarrow \mathbb{N}$  such that for all  $n \in \mathbb{Z}$ ,  $(v_n^{(\psi(m))})_{m \in \mathbb{N}}$  converges.
2. According to the Lemma 2.3, for any  $m \in \mathbb{N}$ , we have

$$\sum_{n \in \mathbb{Z} \setminus \{-1\}} |v_n^{(m)}|^2 \frac{\varrho_n}{n+1} = \sum_{n \in \mathbb{Z} \setminus \{-1\}} |v_n^{(m)}|^2 \left| \frac{\varrho_n}{n+1} \right| \leq \|(\rho_n)_n\|_\infty \sum_{n \in \mathbb{Z} \setminus \{-1\}} \frac{|v_n^{(m)}|^2}{|n+1|}.$$

Thus, for all  $\varepsilon > 0$ , there exists  $N_0 \in \mathbb{N}$  such that

$$\sum_{|n| \geq N_0+1} |v_n^{(m)}|^2 \frac{\varrho_n}{n+1} \leq \frac{\|(\rho_n)_n\|_\infty \|\mathbf{u}^{(\mathbf{m})}\|_{\partial\Omega}^2}{|N_0+1|} \leq \frac{\varepsilon^2}{4}.$$

Then, for all  $p, m \in \mathbb{N}$ , we have by Minkowski's inequality

$$\sum_{|n| \geq N_0+1} |v_n^{(\psi(m))} - v_n^{(\psi(p))}|^2 \frac{\varrho_n}{n+1} \leq \varepsilon.$$

It follows from 1. and 2. that  $(\mathbf{u}^{(\mathbf{m})})_{\mathbf{m} \in \mathbb{N}}$  admits a Cauchy sub-sequence for  $\|\cdot\|_2$ . □

**Remark 2.4.** Let us remark that  $\mathcal{H}^2(\Omega) \not\subseteq H^1(\Omega)$ , in fact

$$\tau : z \mapsto \ln(1-z) \in \mathcal{H}^2(\Omega) \quad \text{but} \quad (\partial_z \tau)|_{\partial\Omega} \notin L^2(\partial\Omega).$$

Therefore, the trace operator on Sobolev spaces is not defined on Hardy space. In order to consider the boundary values of functions in  $\mathcal{H}^2(\Omega)$  we use the following fact: a function in  $\mathcal{H}^2(\Omega)$  has non-tangential limits at almost every point of the boundary (see [Hof07, 1st Corollary p. 38]). This

limit function on  $\partial\Omega$  belong to  $\mathcal{H}^2(\partial\Omega)$ : the closure (in  $L^2(\partial\Omega)$ ) of rational functions having no poles in the closure of  $\Omega$  (see [Sar65, page 6]).

Finally, let us introduce the functional spaces adapted to our problem.

**Definition 2.5.** For  $h > 0$ , let us consider the functional spaces

$$\mathcal{H}_{h,A}^2(\Omega) = e^{-\phi/h}\mathcal{H}^2(\Omega) \quad \text{and} \quad \mathfrak{H}_{h,A} = e^{-\phi/h}\mathfrak{H}_0,$$

where  $\phi$  satisfies (1.2) and

$$\mathfrak{H}_0 = H^1(\Omega) + \mathcal{H}^2(\Omega).$$

We equip  $\mathfrak{H}_{h,A}$  with the scalar product

$$\langle \cdot, \cdot \rangle_{\mathfrak{H}_{h,A}} = \langle \cdot, \cdot \rangle_{L^2(e^{-2\phi/h})} + \langle -2ih\partial_{\bar{z}}\cdot, -2ih\partial_{\bar{z}}\cdot \rangle_{L^2(e^{-2\phi/h})} + \langle \cdot, \cdot \rangle_{\partial\Omega}.$$

In the same way as in the simply connected case (see [Bar+21b, Lemma 2.3.]) the space  $\mathfrak{H}_{h,A}$  satisfies the following properties.

**Proposition 2.6.** *For all  $h > 0$ , we have*

- (i)  $(\mathfrak{H}_{h,A}, \langle \cdot, \cdot \rangle_{\mathfrak{H}_{h,A}})$  is an Hilbert space.
- (ii) The embedding  $(\mathfrak{H}_{h,A}, \langle \cdot, \cdot \rangle_{\mathfrak{H}_{h,A}}) \hookrightarrow (L^2(\Omega), \|\cdot\|_2)$  is compact.
- (iii)  $H^1(\Omega)$  is dense in  $(\mathfrak{H}_{h,A}, \langle \cdot, \cdot \rangle_{\mathfrak{H}_{h,A}})$ .

Let us recall some elliptic estimates for the magnetic Cauchy-Riemann operator.

**Lemma 2.7** (Theorem 4.6. [Bar+21a]). *There exists  $c > 0$  such that, for all  $h > 0$ , and for all  $u \in \{v \in L^2(\Omega), d_{h,A}^\times \in L^2(\Omega)\}$ ,*

$$\|d_{h,A}^\times u\|_2 \geq \sqrt{2hb_0} \|\Pi_{h,A}^\perp u\|_2, \quad \|d_{h,A}^\times u\|_2 \geq ch^2(\|\Pi_{h,A}^\perp u\|_{\partial\Omega} + \|\nabla \Pi_{h,A}^\perp u\|_2),$$

where  $\Pi_{h,A}$  is the projection on the kernel of the adjoint of the operator  $d_{h,A}$  (defined in Section 1.1.1) with Dirichlet boundary conditions, i.e.  $(d_{h,A}, H_0^1(\Omega))^*$ , and

$$\text{Id} = \Pi_{h,A} + \Pi_{h,A}^\perp.$$

*Proof.* (i) Let  $(u_n)$  be a Cauchy sequence in  $(\mathfrak{H}_{h,A}, \langle \cdot, \cdot \rangle_{\mathfrak{H}_{h,A}})$ . In particular it is also a Cauchy sequence for  $\|\cdot\|_2$  and  $\|\cdot\|_{\partial\Omega}$ . According to Lemma 2.7, we have  $u_n = \Pi_{h,A}u_n + \Pi_{h,A}^\perp u_n$  and  $(\Pi_{h,A}^\perp u_n)$  is a Cauchy sequence in  $H^1(\Omega)$ . So that,  $(\Pi_{h,A}^\perp u_n)$  converges to some  $u^\perp$  in  $H^1(\Omega)$ . Moreover, by Lemma 2.7,  $(\Pi_{h,A}u_n)$  is a Cauchy sequence in  $\mathcal{H}_{h,A}^2(\Omega)$ . By Proposition 2.2, it converges to  $u \in \mathcal{H}_{h,A}^2(\Omega)$ . It is now clear that  $(u_n)$  converges to  $u + u^\perp$  in  $\mathfrak{H}_{h,A}$ .

(ii) By using Lemma 2.7 and the compact embedding of  $H^1(\Omega)$  and  $\mathcal{H}_{h,A}^2(\Omega)$  in  $L^2(\Omega)$  we have the result.

(iii) It is sufficient to show that for  $\mathcal{H}^2(\Omega)$ . Let  $f \in \mathcal{H}^2(\Omega)$ , according to [Dur70, Theorem 10.12. p. 181], there exists  $h \in \mathcal{H}^2(D(0, \rho_2))$  and  $g \in \mathcal{H}^2(\mathbb{C} \setminus D(0, \rho_1))$  such that  $f = h + g$ . Suppose that, for  $z \in \Omega$ ,  $f(z) = \sum_{n \in \mathbb{Z}} a_n z^n$ , so that  $h(z) = \sum_{n \geq 0} a_n z^n$  and  $g(z) = \sum_{n \leq -1} a_n z^n$ . In the same way of [Bar+21a, Lemma C.1.], we make a rescaling for  $\varepsilon > 0$ . For  $z \in \Omega$ , we define  $f_\varepsilon = h_\varepsilon + g_\varepsilon$  with

$$h_\varepsilon(z) = h((1 - \varepsilon)z), \quad g_\varepsilon(z) = g((1 - \varepsilon)^{-1}z).$$

Let us remark that  $f_\varepsilon \in \mathcal{H}^2(\Omega_\varepsilon)$ , where

$$\Omega_\varepsilon = \{z \in \mathbb{C} \text{ s.t. } (1 - \varepsilon)\rho_1 < |z| < \rho_2/(1 - \varepsilon)\}.$$

In particular,  $f_\varepsilon \in H^1(\Omega)$ . We have

$$\frac{1}{2\pi} \|f - (h_\varepsilon + g_\varepsilon)\|_{\partial\Omega}^2 = \sum_{n \in \mathbb{Z}} |a_n|^2 (\rho_1^{2n+1} + \rho_2^{2n+1}) |1 - (1 - \varepsilon)^{|n|}|^2.$$

Thus, by Lebesgue's theorem  $(f_\varepsilon)_{\varepsilon \in (0,1)}$  converges to  $f$  in  $\mathcal{H}^2(\Omega)$ . □

**2.2. The min-max formula.** The results of this section are straightforward adaptations of [Bar+21b, Section 2], we just give here a digest of it. The main idea is to establish an isomorphism between the eigenspace  $\ker(\mathcal{D}_{h,A} - \lambda)$  (for  $\lambda \geq 0$ ) and the kernel of an auxiliary operator  $\mathcal{L}_\lambda$ .

Let  $\lambda \geq 0$  and consider the quadratic form defined for all  $u \in \mathfrak{H}_{h,A}$

$$Q_\lambda(u) = \|d_{h,A}^\times u\|^2 + h\lambda \|u\|_{\partial\Omega}^2 - \lambda^2 \|u\|^2,$$

and for all  $k \geq 0$

$$l_k(\lambda) = \inf_{\substack{V \subset H^1(\Omega) \\ \dim V = k}} \sup_{v \in V \setminus \{0\}} \frac{Q_\lambda(v)}{\|v\|^2}.$$

**Definition 2.8.** For  $h > 0$  and  $k \in \mathbb{N}^*$ , we define

$$\mu_k(h) = \inf_{\substack{V \subset H^1(\Omega) \\ \dim V = k}} \sup_{v \in V \setminus \{0\}} \rho_+(v),$$

with

$$\rho_+(v) = \frac{h \|v\|_{\partial\Omega}^2 + \sqrt{h^2 \|v\|_{\partial\Omega}^4 + 4 \|v\|^2 \|d_{h,A}^\times v\|^2}}{2 \|v\|^2},$$

In the next proposition, we use Proposition 2.6 to deduce that infima of Definition 2.8 are minima. Other results are from [Bar+21b, Lemma 2.9].

**Proposition 2.9.** For  $\lambda > 0$ , the quadratic form  $Q_\lambda$  is closed. The associated self-adjoint operator  $\mathcal{L}_\lambda$  has compact resolvent, and its eigenvalues are characterized by the usual min-max formulas

$$l_k(\lambda) = \inf_{\substack{V \subset H^1(\Omega) \\ \dim V = k}} \sup_{v \in V \setminus \{0\}} \frac{Q_\lambda(v)}{\|v\|^2} = \min_{\substack{V \subset \mathfrak{H}_{h,A} \\ \dim V = k}} \max_{v \in V \setminus \{0\}} \frac{Q_\lambda(v)}{\|v\|^2}.$$

Moreover, we have

i. The unique positive solution of  $l_k(\lambda) = 0$  is  $\mu_k(h)$ .

ii. For all  $\lambda > 0$ ,

$$|l_k(\lambda)| \geq \lambda |\mu_k(h) - \lambda|.$$

The next proposition provides the connection between  $\lambda_k^+(h)$  and  $\mu_k(h)$ .

**Proposition 2.10.** We have for all  $h > 0$  and  $k \geq 0$

$$\lambda_k^+(h) = \mu_k(h) = \min_{\substack{V \subset \mathfrak{H}_{h,A} \\ \dim V = k}} \max_{v \in V \setminus \{0\}} \rho_+(v). \quad (2.1)$$

**2.3. A similar characterization for fixed angular momentum.** After a change of variables in polar coordinates, we decompose the Dirac operator, defined with the potential  $\mathbf{A}_{\mathbf{h},\mathbf{p}} = \nabla^\perp \psi_h$ , into Fourier series. Then, we reduce to a flat metric with the change of function  $g(r) = \sqrt{r} f(r)$ . By fixing the angular momentum, we will be able to reduce to a one-dimensional problem. Now we can define the fibered operator.

**Notation 2.** We let:  $\mathcal{I} = (\rho_1, \rho_2)$ .

**Definition 2.11.** Let  $h \in (0, 1)$  and  $m, p \in \mathbb{Z}$ . We define the operator  $(\mathcal{D}_{h,m}, \text{Dom}(\mathcal{D}_{h,m}))$  on  $L^2(\mathcal{I}, \mathbb{C}^2)$  as the operator acting as

$$\mathcal{D}_{h,m} = \begin{pmatrix} 0 & d_{h,m} \\ d_{h,m}^\times & 0 \end{pmatrix},$$

on  $\text{Dom}(\mathcal{D}_{h,m}) = H^1(\mathcal{I}, \mathbb{C}^2)$  with

$$d_{h,m} = -ih \left( \partial_r + \frac{m - \gamma_{h,p} + 1/2}{r} - \frac{\partial_r \phi}{h} \right),$$

and  $d_{h,m}^\times$  its formal adjoint, moreover  $\gamma_{h,p}$  is defined in Proposition 1.3.

**Notation 3.** As in Section 1.1.1,  $(\mathcal{D}_{h,m}, \text{Dom}(\mathcal{D}_{h,m}))$  has no zero modes. We note

$$\text{sp}(\mathcal{D}_{h,m}) \cap \mathbb{R}_\pm = \left\{ \pm \lambda_{j,m}^\pm(h) \mid j \in \mathbb{N}^* \right\}.$$

**Remark 2.12.** Let us recall some properties of the Dirac operator (see [Lav22, Proposition 3.8]). Let  $h > 0$  and  $m \in \mathbb{R}$ ,

- i. The operator  $(d_{h,m}, H_0^1(\mathcal{I}, \mathbb{C}))$  is closed with closed range.
- ii. The adjoint  $(d_{h,m}^*, \text{Dom}(d_{h,m}^*))$  acts as  $d_{h,m}^\times$  on  $\text{Dom}(d_{h,m}^*) = H^1(\mathcal{I}, \mathbb{C})$  and

$$\mathcal{H}_{h,m,\phi}^2 := \text{Ker}(d_{h,m}^*) = \text{Vect}(r \mapsto r^{m+1/2} e^{-\phi(r)/h}).$$

We define  $\Pi_{h,m}$  the orthogonal projection on  $\mathcal{H}_{h,m,\phi}^2$ .

We give in the next lemma the connection between the spectrum of  $\mathcal{D}_{h,A}$  and that of  $\mathcal{D}_{h,m}$  for  $m \in \mathbb{Z}$ .

**Lemma 2.13.** *Under the assumptions of Section 1, we have for all  $h \in (0, 1)$  :*

$$\text{Sp}(\mathcal{D}_{h,A}) = \bigcup_{m \in \mathbb{Z}} \text{Sp}(\mathcal{D}_{h,m}),$$

with  $(\mathcal{D}_{h,m}, \text{Dom}(\mathcal{D}_{h,m}))$  given in Definition 2.11.

In the same way as in the Section 2.2, we can prove a nonlinear min-max formula for fixed angular momentum.

**Notation 4.** For  $h > 0$ ,  $k \in \mathbb{N}^*$  and  $m \in \mathbb{Z}$ , we define

$$\mu_{k,m}(h) = \inf_{\substack{V \subset H^1(\mathcal{I}) \\ \dim V = k}} \sup_{v \in V \setminus \{0\}} \rho_m(v),$$

with

$$\rho_m(v) = \frac{h \|v\|_{\partial \mathcal{I}}^2 + \sqrt{h^2 \|v\|_{\partial \mathcal{I}}^4 + 4 \|v\|^2 \|d_{h,m}^\times v\|^2}}{2 \|v\|^2},$$

and for  $v \in H^1(\mathcal{I}, \mathbb{C})$

$$\|v\|_{\partial \mathcal{I}}^2 = |v(\rho_1)|^2 + |v(\rho_2)|^2.$$

**Proposition 2.14.** *For  $h > 0$ ,  $k \in \mathbb{N}^*$  and  $m \in \mathbb{Z}$ , we have*

$$\lambda_{k,m}^+(h) = \mu_{k,m}(h).$$

We give a lower bound for positive excited states. This result allows us to focus in the following on the ground states associated with the fibered operators, in the semiclassical limit.

Before that, let us recall elliptic estimates related to the Dirac operator with real fixed angular momentum.

**Proposition 2.15.** *Let  $h > 0$  and  $m \in \mathbb{R}$ , there exists  $c > 0$  such that for all  $u \in H^1(\mathcal{I})$*

$$\|d_{h,m}^\times u\| \geq \sqrt{2hb_0} \|\Pi_{h,m}^\perp u\| \quad \text{and} \quad \|d_{h,m}^\times u\| \geq ch^{3/4} \|\Pi_{h,m}^\perp u\|_{\partial\mathcal{I}},$$

with  $b_0 = \min_{x \in \Omega} B(x)$  and  $\Pi_{h,m}$  defined in Remark 2.12.

We get the following lower bound for the excited states.

**Proposition 2.16.** *Let  $h > 0$ ,  $k \geq 2$ . For all  $m \in \mathbb{R}$ , we have*

$$\lambda_{k,m}^+(h) \geq \sqrt{2hb_0},$$

with  $b_0 = \min_{x \in \Omega} B(x)$ .

*Proof.* Let  $h > 0$ ,  $k \in \mathbb{N}^*$  and  $m \in \mathbb{R}$ . Using Proposition 2.14, we have

$$\begin{aligned} \lambda_{k,m}^+(h) &= \inf_{\substack{V \subset H^1(\mathcal{I}) \\ \dim V = k}} \sup_{v \in V \setminus \{0\}} \frac{h \|u\|_{\partial\mathcal{I}}^2 + \sqrt{h^2 \|u\|_{\partial\mathcal{I}}^4 + 4 \|u\|^2 \|d_{h,m}^\times u\|^2}}{2 \|u\|^2} \\ &\geq \inf_{\substack{V \subset H^1(\mathcal{I}) \\ \dim V = k}} \sup_{u \in V \setminus \{0\}} \frac{\|d_{h,m}^\times u\|}{\|u\|} \geq \left( \Lambda_{k,m}^N(h) \right)^{1/2}, \end{aligned}$$

where

$$\Lambda_{k,m}^N(h) = \inf_{\substack{V \subset H^1(\mathcal{I}) \\ \dim V = k}} \sup_{u \in V \setminus \{0\}} \frac{\|d_{h,m}^\times u\|^2}{\|u\|^2},$$

being the  $k$ -th eigenvalue of  $\mathcal{L}_{h,m}^N$ , the fibred Pauli operator with Neumann boundary conditions (it acts as  $d_{h,m} d_{h,m}^\times$  on  $H^2(\mathcal{I}, \mathbb{C})$ ).

The first eigenvalue of  $(\mathcal{L}_{h,m}^N, H^2(\mathcal{I}, \mathbb{C}))$  is zero, indeed, the function  $r \mapsto r^{m+1/2}$  cancels the quadratic form. This implies that for  $k \geq 2$ ,

$$\ker \left( \mathcal{L}_{h,m}^N - \Lambda_{k,m}^N(h) \right) \subset \ker \left( \mathcal{L}_{h,m}^N \right)^\perp = \ker \left( d_{h,m}^\times \right)^\perp.$$

However, according to Proposition 2.15, we have for all  $u \in \text{Dom}(d_{h,m}) \cap \ker(d_{h,m}^\times)^\perp$ ,

$$\|d_{h,m}^\times u\|^2 \geq 2hb_0 \|u\|^2.$$

Thus, for  $k \geq 2$ ,  $\Lambda_{k,m}^N(h) \geq 2hb_0$  and we get the result.  $\square$

### 3. SEMICLASSICAL ANALYSIS OF THE POSITIVE EIGENVALUES

In this section we establish Theorem 1.4. To do so, let us consider the following family of ground states energy.

$$\nu_m(h) = \inf_{v \in \mathcal{H}_{h,m,\phi}^2} \frac{h \|v\|_{\partial\mathcal{I}}^2}{\|v\|^2}, \quad (3.1)$$

where  $m \in \mathbb{Z}$  and  $\mathcal{H}_{h,m,\phi}^2$  is defined in Remark 2.12.

The nonlinear min-max formula of Proposition 2.14 (see Notation 4) naturally leads us to consider this quantity and we have clearly:

**Lemma 3.1.** *For all  $h > 0$  and  $m \in \mathbb{R}$ , we have*

$$\mu_{1,m}(h) \leq \nu_m(h).$$

Now we can use Proposition 1.3 and choose, for each  $h > 0$ , a good magnetic potential. For all  $h > 0$ , we consider the Dirac operator associated to the vector potential

$$\mathbf{A}_h = \nabla^\perp \phi + h\gamma(h) \ln \left( \frac{\rho_1}{\rho_2} \right) \nabla^\perp \theta,$$

with  $\gamma(h) = \frac{c_0}{h} - \lfloor \frac{c_0}{h} \rfloor$  in such a way that  $\gamma(h) \in [0, 1]$ .

Definition 2.11 and Lemma 2.13 ensure that

$$Sp(\mathcal{D}_{h,A}) = \left\{ \lambda_{j, m-\gamma(h)}^+(h), m \in \mathbb{Z}, j \in \mathbb{N}^* \right\}. \quad (3.2)$$

**Notation 5.** We let for all  $h > 0$ ,

$$\mathbb{Z}(h) = \mathbb{Z} + \gamma(h).$$

Our main result is a consequence of the following two propositions.

**Proposition 3.2.** *For all  $k \in \mathbb{N}^*$  and  $h > 0$*

$$\lambda_k^+(h) = \left[ \min_{\substack{V \subset \mathbb{Z}(h) \\ \#V = k}} \max_{m \in V} \nu_m(h) \right] (1 + \mathcal{O}(h^\infty)).$$

**Proposition 3.3.** *For all fixed  $k \in \mathbb{N}^*$ , there exists  $h_0 > 0$  such that for all  $h \in (0, h_0)$ ,*

$$\min_{\substack{V \subset \mathbb{Z}(h) \\ \#V = k}} \max_{m \in V} \nu_m(h) = \alpha_k(h) \sqrt{h} e^{2\phi_{\min}/h} (1 + o_{h \rightarrow 0}(1)),$$

where  $\alpha_k(h)$  is defined in Lemma 3.8.

**3.1. Case of projected states.** In this section, we are interested in the behavior of the ground state energy  $\nu_m(h)$ . We give a proof of Proposition 3.3.

3.1.1. *Uniform convergence and upper bound.*

**Notation 6.** Let  $h > 0$  and  $m \in \mathbb{R}$ ,

$$\mathfrak{f}_h(m) = \frac{\nu_m(h)}{\sqrt{h} e^{2\phi_{\min}/h}},$$

and

$$\mathfrak{f}(m) = \sqrt{\frac{\phi''_{\min}}{\pi}} \left( \left( \frac{\rho_1}{r_{\min}} \right)^{2m+1} + \left( \frac{\rho_2}{r_{\min}} \right)^{2m+1} \right).$$

**Remark 3.4.** For all  $m \in \mathbb{R}$ ,

$$\mathfrak{f}(m) \geq \sqrt{\frac{\phi''_{\min}}{\pi}} > 0.$$

Section 2.3 gives us an explicit expression for the eigenvalue  $\nu_m(h)$ .

**Lemma 3.5.** *For all  $h > 0$  and  $m \in \mathbb{R}$ , we have*

$$\nu_m(h) = h \left( \rho_1^{2m+1} + \rho_2^{2m+1} \right) \left( \int_{\rho_1}^{\rho_2} e^{-2\phi/h} r^{2m+1} dr \right)^{-1},$$

where  $\phi$  is the solution of (1.2).

Let us now show that if the angular momenta are bounded then the sequence  $(\mathfrak{f}_h)_h$  is uniformly convergent.

**Proposition 3.6.** *The function sequence  $(\mathfrak{f}_h)_{h \in ]0,1]}$  is uniformly convergent on any compact to  $\mathfrak{f}: \mathbb{R} \rightarrow \mathbb{R}^+$  when  $h$  tends to 0.*

*Proof.* Let  $K$  a compact of  $\mathbb{R}$ . Laplace's method gives us

$$\int_{\rho_1}^{\rho_2} e^{-2(\phi - \phi_{\min})/h} r^{2m+1} \frac{dr}{\sqrt{h}} = \sqrt{\frac{\pi}{\phi''_{\min}}} r_{\min}^{2m+1} (1 + o(1)),$$

where  $o(1)$  depends only on  $K$ . □

**Corollary 3.7.** *Let  $\mathcal{M} \subset \mathbb{R}$  a compact. There exists  $h_0 > 0$  such that for all  $h \in (0, h_0)$  and  $m \in \mathcal{M}$ ,*

$$\nu_m(h) = \mathfrak{f}(m) \sqrt{h} e^{2\phi_{\min}/h} (1 + o(1)),$$

where  $\mathfrak{f}(\cdot)$  is defined in Notation 6 and  $o(1)$  only depends on  $\mathcal{M}$  and  $h$ .

The following lemma justifies the existence of the prefactor given in Theorem 1.4.

**Lemma 3.8.** *Let  $k \in \mathbb{N}^*$  and  $h > 0$ , we consider the non-decreasing sequence*

$$\alpha_k(h) = \inf_{\substack{V \subset \mathbb{Z} \\ \#V = k}} \sup_{m \in V} \mathfrak{f}(m - \gamma(h)). \quad (3.3)$$

*Then, there exists a  $k$ -tuple*

$$V_k(h) = \{\xi_1(h), \dots, \xi_k(h)\}$$

that realizes the infimum. Moreover, the functions  $h \mapsto \alpha_k(h)$  and  $h \mapsto \xi_j(h)$  are bounded for  $j \in \{1, \dots, k\}$ .

*Proof.* The proof relies on the coercivity of  $\mathfrak{f}$ , see [Lav22, Lemma 4.3] for a similar proof. □

Next Proposition establishes the upper bound of Proposition 3.3.

**Proposition 3.9.** *Let  $k \in \mathbb{N}^*$ , there exists  $h_0 > 0$  such that for all  $h \in (0, h_0)$ , we have*

$$\min_{\substack{V \subset \mathbb{Z}(h) \\ \#V = k}} \max_{m \in V} \nu_m(h) \leq \alpha_k(h) \sqrt{h} e^{2\phi_{\min}/h} (1 + o_{h \rightarrow 0}(1)).$$

*Proof.* Let  $k \geq 1$ . Proposition 3.6 and the boundedness of  $V_k(h)$  (Lemma 3.8) give the existence of  $h_0 > 0$  such that for all  $h \in (0, h_0)$ ,

$$\min_{\substack{V \subset \mathbb{Z}(h) \\ \#V = k}} \max_{m \in V} \nu_m(h) \leq \max_{\xi \in V_k(h)} \nu_{\xi - \gamma(h)}(h) = \max_{\xi \in V_k(h)} \mathfrak{f}(\xi - \gamma(h)) \sqrt{h} e^{2\phi_{\min}/h} (1 + o(1)).$$

□

3.1.2. *Proof of Proposition 3.3.* Here, we prove by a weak coercivity argument the lower bound of Proposition 3.3.

**Notation 7.** For  $K > 0$  and  $h_0 > 0$ , let us denote

$$\mathcal{M}_{K,h_0}^\nu = \{m \in \mathbb{R} \mid \forall h \in (0, h_0) \text{ f}_h(m) \leq K\}.$$

From Proposition 3.9, we know that for  $K > \|\alpha_1(\cdot)\|_\infty$  the set  $\mathcal{M}_{K,h_0}^\nu$  is non empty. Throughout this section, we consider such a constant  $K$ . Let us show that under these assumptions,  $\mathcal{M}_{K,h_0}^\nu$  is bounded.

**Lemma 3.10.** *There exists  $h_0 > 0$  such that  $\mathcal{M}_{K,h_0}^\nu$  is bounded.*

*Proof.* Let  $K > 0$ . By contradiction, let us assume that for all  $h_0 > 0$ ,  $\mathcal{M}_{K,h_0}^\nu$  is not bounded. Fix  $h_0 > 0$ , we have for  $m \in \mathcal{M}_{K,h_0}^\nu$ ,

$$\text{f}_h(m) \leq \frac{\rho_1^{2m+1} + \rho_2^{2m+1}}{\int_{\rho_1}^{\rho_2} e^{-2(\phi - \phi_{\min})/h} r^{2m+1} \frac{d}{\sqrt{h}} \leq K. \quad (3.4)$$

Recall that  $\phi$  is the unique solution of 1.2, then for sufficiently small  $\delta > 0$ ,

$$\phi(r) \geq \phi_{\min} + \frac{(r - r_{\min})^2}{2} \phi''_{\min} (1 + \mathcal{O}(r - r_{\min})), \text{ with } r \in (\rho_1, r_{\min} - \delta) \cup (r_{\min} + \delta, \rho_2).$$

Thus,

$$\int_{\rho_1}^{\rho_2} e^{-2(\phi - \phi_{\min})/h} r^{2m+1} \frac{d}{\sqrt{h}} \geq \int_{\rho_1}^{\rho_2} e^{-\phi''_{\min} \frac{(r - r_{\min})^2}{h}} r^{2m+1} \frac{dr}{\sqrt{h}} (1 + o(1)).$$

Let us start by assuming that  $m \in \mathcal{M}_{K,h_0}^\nu \cap \mathbb{R}_+$ . We have  $\rho_2^{2m+1} \leq \rho_1^{2m+1} + \rho_2^{2m+1}$ , so that,

$$\begin{aligned} \frac{1}{K} &\leq \int_{\rho_1}^{\rho_2} e^{-\phi''_{\min} \frac{(r - r_{\min})^2}{h}} \left(\frac{r}{\rho_2}\right)^{2m+1} \frac{dr}{\sqrt{h}} \\ &\leq \left[ \int_{\rho_1}^{\rho_2 - \eta} + \int_{\rho_2 - \eta}^{\rho_2} \right] e^{-\phi''_{\min} \frac{(r - r_{\min})^2}{h}} \left(\frac{r}{\rho_2}\right)^{2m+1} \frac{dr}{\sqrt{h}}. \end{aligned}$$

with  $0 < \eta < (\rho_2 - r_{\min})/2$ .

Then on the one hand, there exists  $C > 0$  (independent of  $h$  and  $m$ ) such that

$$\int_{\rho_1}^{\rho_2 - \eta} e^{-\phi''_{\min} \frac{(r - r_{\min})^2}{h}} \left(\frac{r}{\rho_2}\right)^{2m+1} \frac{dr}{\sqrt{h}} \leq C \left(\frac{\rho_2 - \eta}{\rho_2}\right)^{2m+1}.$$

On the other hand, for all  $\varepsilon > 0$  there exists  $h_0 > 0$  such that for all  $h \in (0, h_0)$

$$\int_{\rho_2 - \eta}^{\rho_2} e^{-\phi''_{\min} \frac{(r - r_{\min})^2}{h}} \left(\frac{r}{\rho_2}\right)^{2m+1} \frac{dr}{\sqrt{h}} \leq \frac{\eta}{\sqrt{h}} e^{-\phi''_{\min} \frac{(r - r_{\min} - \eta)^2}{h}} < \varepsilon.$$

Taking  $\varepsilon = 1/2K$ , there exist  $M_K \in \mathbb{R}_+$  and  $h_0 > 0$  such that for all  $m \geq M_K$  and  $0 < h < h_0$ , we have

$$C \left(\frac{\rho_2 - \eta}{\rho_2}\right)^{2m+1} + \frac{\eta}{\sqrt{h}} e^{-\phi''_{\min} \frac{(r - r_{\min} - \eta)^2}{h}} < \frac{1}{2K},$$

which is in contradiction with (3.4).

We proceed in a similar way for  $m \in \mathcal{M}_{K,h_0}^\nu \cap \mathbb{R}_-$ . □

Let us consider a similar formula to the one of Lemma 3.8 but for the renormalized eigenvalue.

**Lemma 3.11.** *Let  $k \in \mathbb{N}^*$  and  $h > 0$ , consider the non-decreasing sequence*

$$\beta_k(h) = \inf_{\substack{V \subset \mathbb{Z} \\ \#V = k}} \sup_{m \in V} f_h(m - \gamma(h)). \quad (3.5)$$

*Assume that there exists  $h_0 > 0$  such that*

$$\sup_{h \in (0, h_0)} \beta_k(h) < +\infty.$$

*Then, for all  $h \in (0, h_0)$ , there exists a  $k$ -tuple*

$$W_k(h) = \{m_1(h), \dots, m_k(h)\} \subset \mathbb{Z}$$

*that realizes the infimum. Moreover, the functions  $h \mapsto m_j(h)$  are bounded for  $j \in \{1, \dots, k\}$ .*

*Proof.* Let  $k \in \mathbb{N}^*$  fixed. There exists  $h_0 > 0$  such that

$$K = \sup_{h \in (0, h_0)} \beta_k(h) < +\infty$$

i) For all  $\varepsilon \in (0, 1)$  and for all  $h \in (0, h_0)$ , there exists  $X_\varepsilon(h) \subset \mathbb{Z}$  such that  $\#X_\varepsilon(h) = k$  and

$$\beta_k(h) \leq \max_{m \in X_\varepsilon(h)} f_h(m - \gamma(h)) \leq \beta_k(h) + \varepsilon \leq K + 1.$$

Thus, for all  $h \in (0, h_0)$  and for all  $m \in \bigcup_{(h, \varepsilon) \in (0, h_0) \times (0, 1)} X_\varepsilon(h)$ , we have

$$f_h(m - \gamma(h)) \leq K + 1.$$

Lemma 3.10 ensures that there exist  $M_k \in \mathbb{R}_+^*$  independent of  $h$  and  $\varepsilon$  such that for any  $\varepsilon \in (0, 1)$  and for any  $h \in (0, h_0)$ ,  $X_\varepsilon(h) \subset [-M_k, M_k]$ .

ii) Let us consider for all  $h \in (0, h_0)$ ,  $(X_{1/n}(h))_{n \in \mathbb{N}^*}$  the minimizing sequence of (3.5). From the point i), the  $k$ -tuple  $X_{1/n}(h)$  are uniformly bounded with respect to  $n$  and  $h$ . Consequently there exists a  $k$ -tuple, bounded in  $h$ , which realizes the infimum. □

**Lemma 3.12.** *Let  $k \in \mathbb{N}^*$ , there exists  $h_0 > 0$ , such that*

$$\sup_{h \in ]0, h_0]} \beta_k(h) < +\infty,$$

*and for all  $h \in ]0, h_0]$ ,*

$$\min_{\substack{V \subset \mathbb{Z}(h) \\ \#V = k}} \max_{m \in V} \nu_m(h) = \max_{m \in W_k(h)} \nu_{m - \gamma(h)}(h),$$

*where  $\mathbb{Z}(h)$  is defined in Notation 5 and  $W_k(h)$  in Lemma 3.11.*

*Proof.* Let  $k \in \mathbb{N}^*$ . Recall that according to (3.5) and Notation 6, we have

$$\min_{\substack{V \subset \mathbb{Z}(h) \\ \#V = k}} \max_{m \in V} \nu_m(h) = \beta_k(h) \sqrt{h} e^{2\phi_{\min}/h}.$$

By using Proposition 3.9 and Lemma 3.11, we have

$$\sup_{h \in ]0, h_0]} \beta_k(h) < +\infty \quad \text{and} \quad \min_{\substack{V \subset \mathbb{Z}(h) \\ \#V = k}} \max_{m \in V} \nu_m(h) = \max_{m \in W_k(h)} \nu_{m - \gamma(h)}(h).$$

□

*Proof of Proposition 3.3.* Let  $k \in \mathbb{N}^*$ . From Lemma 3.12, we have for all  $h \in ]0, h_0]$ ,

$$\min_{\substack{V \subset \mathbb{Z}(h) \\ \#V = k}} \max_{m \in V} \nu_m(h) = \max_{m \in W_k(h)} \nu_{m-\gamma(h)}(h) = \max_{m \in W_k(h)} \mathfrak{f}_h(m - \gamma(h)) \sqrt{h} e^{2\phi_{\min}/h}. \quad (3.6)$$

Moreover, by Lemma 3.11, for any  $j \in \llbracket 1, k \rrbracket$ , the functions  $h \mapsto m_j(h) - \gamma(h)$  are uniformly bounded. Proposition 3.9 and Remark 3.4 ensure that

$$\begin{aligned} \max_{m \in W_k(h)} \mathfrak{f}_h(m - \gamma(h)) &= (1 + o_{h \rightarrow 0}(1)) \max_{m \in W_k(h)} \mathfrak{f}(m - \gamma(h)) \\ &\geq (1 + o_{h \rightarrow 0}(1)) \alpha_k(h). \end{aligned} \quad (3.7)$$

Using (3.6) and (3.7), we have the following lower bound

$$\min_{\substack{V \subset \mathbb{Z}(h) \\ \#V = k}} \max_{m \in V} \nu_m(h) \geq \alpha_k(h) \sqrt{h} e^{2\phi_{\min}/h} (1 + o_{h \rightarrow 0}(1)).$$

The upper bound of Proposition 3.9 completes the proof.  $\square$

**3.2. From the non-linear min-max formula to the projected states formula.** Now, let us explain why  $\mu_{1,h}(m)$  is close to  $\nu_m(m)$  (uniformly in  $m$ ).

The upper bound is a straightforward consequence of the min-max principle.

**Proposition 3.13.** *Let  $k \in \mathbb{N}^*$ , for all  $h > 0$ , we have*

$$\lambda_k(h) \leq \min_{\substack{V \subset \mathbb{Z}(h) \\ \#V = k}} \max_{m \in V} \nu_m(h),$$

where  $\mathbb{Z}(h)$  is defined in Notation 5.

*Proof.* Lemma 3.1 combined with considerations on min-max formulas give the statement.  $\square$

**3.2.1. Rayleigh quotient approximation.** The next two lemmas are the counterparts (at fixed angular momentum) of those in [Bar+21b, Section 3.2].

**Lemma 3.14.** *Let  $h > 0$  and  $m \in \mathbb{R}$ . Let us consider  $u_m \in H^1(I)$  an eigenfunction associated to  $\mu_{1,m}(h)$ , we have*

$$\begin{aligned} h \|u_m\|_{\partial\mathcal{I}}^2 &\leq \mu_{1,m}(h) \|u_m\|^2 \\ \|d_{h,m}^\times u_m\| &\leq \mu_{1,m}(h) \|u_m\| \end{aligned} \quad (3.8)$$

*Proof.* The proof is a straightforward consequence of the formula

$$\mu_{1,m}(h) = \frac{h \|u_m\|_{\partial\Omega}^2 + \sqrt{h^2 \|u_m\|_{\partial\Omega}^4 + 4 \|u_m\|^2 \|d_{h,m}^\times u_m\|^2}}{2 \|u_m\|^2},$$

where  $u_m \in H^1(I)$  being an eigenfunction associated to  $\mu_{1,m}$ .  $\square$

The following lemma is a straightforward applications of Proposition 2.15.

**Lemma 3.15.** *Let  $h > 0$  and  $m \in \mathbb{R}$ . Let us consider  $u_m \in H^1(I)$  an eigenfunction associated with  $\mu_{1,m}(h)$ , we have*

$$c h^{3/4} \|\Pi_{h,m}^\perp u_m\|_{\partial\mathcal{I}} \leq \mu_{1,m}(h) \|u_m\| \quad (3.9)$$

$$\sqrt{2hb_0} \|\Pi_{h,m}^\perp u_m\| \leq \mu_{1,m}(h) \|u_m\|, \quad (3.10)$$

with  $c > 0$  the constant in Proposition 2.15.

Moreover for  $h$  small enough (and fixed  $m$ )  $\Pi_{h,m}$  is injective on the associated eigenspace.

**Proposition 3.16.** *Let  $h > 0$  and  $m \in \mathbb{R}$ . Let us consider  $u_m \in H^1(I)$  an eigenfunction associated to  $\mu_{1,m}(h)$ . There exists  $h_0 > 0$  such that for all  $h \in (0, h_0)$ ,*

$$0 < \nu_m(h) \mathcal{R}_h(\mu_{1,m}(h)) \leq \mu_{1,m}(h),$$

where for  $\mu \neq \sqrt{2hb_0}$ ,

$$\mathcal{R}_h(\mu) = \left[ \frac{1 - \frac{\mu}{\sqrt{2hb_0}}}{1 + \frac{\sqrt{\mu}}{ch^{1/4}}} \right]^2.$$

with  $c > 0$  defined in Lemma 3.15.

*Proof.* Let us begin with (3.8), by using the reverse triangle inequality, we have

$$\sqrt{h} \left( \|\Pi_{h,m} u_m\|_{\partial\Omega} - \|\Pi_{h,m}^\perp u_m\|_{\partial\Omega} \right) \leq \|u_m\| (\mu_{1,m}(h))^{1/2},$$

so that, according to (3.9),

$$\begin{aligned} \sqrt{h} \|\Pi_{h,m} u_m\|_{\partial\Omega} &\leq \|u_m\| (\mu_{1,m}(h))^{1/2} + \sqrt{h} \|\Pi_{h,m}^\perp u_m\|_{\partial\Omega} \\ &\leq \|u_m\| (\mu_{1,m}(h))^{1/2} \left( 1 + \frac{\sqrt{\mu_{1,m}(h)}}{ch^{1/4}} \right). \end{aligned} \quad (3.11)$$

In other hand, using (3.10),

$$\|u_m\| \leq \|\Pi_{h,m} u_m\| + \|\Pi_{h,m}^\perp u_m\| \leq \|\Pi_{h,m} u_m\| + \frac{\mu_{1,m}(h)}{\sqrt{2hb_0}} \|u_m\|,$$

hence,

$$\left( 1 - \frac{\mu_{1,m}(h)}{\sqrt{2hb_0}} \right) \|u_m\| \leq \|\Pi_{h,m} u_m\|. \quad (3.12)$$

Propositions 3.3 & 3.13 ensure that there exists  $h_0 > 0$  such that for all  $h \in (0, h_0)$ ,  $1 - \frac{\mu_{1,m}(h)}{\sqrt{2hb_0}} > 0$ . Then (3.11) & (3.12) give

$$0 < \frac{\sqrt{h} \|\Pi_{h,m} u_m\|_{\partial\Omega}^2}{\|\Pi_{h,m} u_m\|^2} \mathcal{R}_h(\mu_{1,m}(h)) \leq \mu_{1,m}(h),$$

and by definition of  $\nu_m(h)$  (3.1), we get the result.  $\square$

3.2.2. *Proof of Proposition 3.2.* Proposition 3.2 is a consequence of the following three lemmas.

**Lemma 3.17.** *There exists  $h_0 > 0$  such that for all  $h \in ]0, h_0]$ ,*

$$\lambda_k(h) = \inf_{\substack{V \subset \mathbb{Z}(h) \\ \#V = k}} \sup_{m \in V} \mu_{1,m}(h),$$

with  $\mathbb{Z}(h)$  defined in 5.

*Proof.* Let  $k \in \mathbb{N}^*$ , according to Proposition 2.14 and (3.2) there exist  $n_k(h) \in \mathbb{Z}$  and  $j_k(h) \in \mathbb{N}^*$  such that

$$\lambda_k(h) = \lambda_{j_k(h), n_k(h) - \gamma(h)}^+(h) = \mu_{j_k(h), n_k(h) - \gamma(h)}(h).$$

Moreover, Proposition 2.16 ensures that for all  $h > 0$ ,  $j \geq 2$  and  $m \in \mathbb{R}$ ,

$$\lambda_{j,m}^+(h) \geq 2hb_0.$$

Then, using Propositions 3.3 & 3.13 and the boundedness of  $h \mapsto \alpha_k(h)$  (Lemma 3.8), we have the existence of  $h_0 > 0$  such that for all  $h \in (0, h_0)$ ,  $j_k(h) = 1$ . Hence,

$$\lambda_k(h) = \inf_{\substack{V \subset \mathbb{Z}(h) \\ \#V = k}} \sup_{m \in V} \mu_{1,m}(h).$$

□

**Notation 8.** Let  $h > 0$  and  $m \in \mathbb{R}$ ,

$$\mathfrak{g}_h(m) = \frac{\mu_{1,m}(h)}{\sqrt{h} e^{2\phi_{\min}/h}},$$

let us also note for  $K > 0$  and  $h_0 > 0$ ,

$$\mathcal{M}_{K,h_0}^\mu = \{m \in \mathbb{R} \mid \forall h \in (0, h_0) \ \mathfrak{g}_h(m) \leq K\}.$$

**Lemma 3.18.** Let  $k \in \mathbb{N}^*$  and  $h > 0$ , consider the non-decreasing sequence

$$\omega_k(h) = \inf_{\substack{V \subset \mathbb{Z} \\ \#V = k}} \sup_{m \in V} \mathfrak{g}_h(m - \gamma(h)). \quad (3.13)$$

There exists  $h_0 > 0$  such that

$$\sup_{h \in ]0, h_0]} \omega_k(h) < +\infty.$$

*Proof.* Using Lemma 3.1 and Notations 6 & 8, we have for all  $h > 0$ ,

$$\omega_k(h) \leq \beta_k(h).$$

We conclude by using Lemma 3.12. □

**Lemma 3.19.** Let  $K > 0$  and  $h_0 > 0$ . For all  $h \in (0, h_0)$  and  $m \in \mathcal{M}_{K,h_0}^\mu$ , we have

$$\mu_{1,m}(h) = \nu_m(h) (1 + \mathcal{O}(h^\infty)),$$

with  $\mathcal{O}(h^\infty)$  independent of  $m$ .

*Proof.* First of all, recall that for all  $h > 0$  and  $m \in \mathbb{R}$ ,

$$\mu_{1,m}(h) \leq \nu_m(h).$$

On the other hand, by using Proposition 3.16, we have for  $h$  small enough and  $m \in \mathbb{R}$ ,

$$0 < \nu_m(h) \mathcal{R}_h(\mu_{1,m}(h)) \leq \mu_{1,m}(h).$$

Then, for  $m \in \mathcal{M}_{K,h_0}^\mu$ ,

$$\mathcal{R}_h(\mu_{1,m}(h)) \geq \frac{1 - \frac{K e^{2\phi_{\min}/h}}{\sqrt{2b_0}}}{1 + \frac{K h^{1/4} e^{2\phi_{\min}/h}}{c}} = 1 + \mathcal{O}(h^\infty),$$

with  $\mathcal{O}(h^\infty)$  is independent of  $m$ . □

## APPENDIX A. SKETCH OF PROOF FOR THEOREM 1.6

This section is devoted to the study of the negative spectrum, we only sketch the proof of Theorem 1.6 which essentially follows [Bar+21b, Section 6]. We check that the topology type of the domain does not affect the first term of the asymptotic expansion of  $\lambda_1^-(h)$ .

The min-max formula (2.1) (in Section 2.2) can be also used to describe the negative part of the spectrum of  $\mathcal{D}_{h,A}$ . According to [Bar+21b, Remark 1.10.], it is enough to change the sign of the magnetic field.

**Lemma A.1.** *For all  $h > 0$ , we have*

$$\mathrm{Sp}(\mathcal{D}_{h,A}) = -\mathrm{Sp}(\mathcal{D}_{h,-A}).$$

This lead us to consider, for  $\lambda \geq 0$

$$\tilde{Q}_\lambda(u) = q_{\lambda,h}(u) - \lambda^2 \|u\|^2 \quad \text{and} \quad q_{\lambda,h}(u) = \|d_{h,-A}^\times u\|^2 + h\lambda \|u\|_{\partial\Omega},$$

the quadratic form for the new magnetic field.

We note  $\tilde{l}_1(\lambda)$  and  $\gamma_1(\lambda, h)$  the ground states energy of the operators associated to  $\tilde{Q}_\lambda(\cdot)$  and  $q_{\lambda,h}(\cdot)$ . According to point (i) of Proposition 2.9 and Proposition 2.10, the unique positive solution of the equation  $l_1(\lambda) = 0$  is  $\lambda_1^-(h)$ . Proposition 2.9 ensures that

$$|\gamma_1(\lambda, h) - \lambda^2| \geq \lambda |\lambda_1^-(h) - \lambda|.$$

Let us pose, for  $a > 0$ ,  $\lambda = ah^{1/2}$  and  $\lambda_1^-(h) = e_1(h)h^{1/2}$ , we have

$$|h^{-1}\gamma_1(ah^{1/2}) - a^2| \geq a|a - e_1(h)| \tag{A.1}$$

Theorem 1.6 is a consequence of the following proposition.

**Proposition A.2.** *We have for all  $a > 0$*

$$\gamma_1(ah^{1/2}) = \Lambda(a)h + o(1),$$

with  $\Lambda(a) = \min(2b_0, b'_0\nu(a(b'_0)^{-1/2}))$  where  $b_0, b'_0$  are defined in Theorem 1.6 and  $\nu(\cdot)$  corresponds to the ground state of the magnetic Schrödinger operator on a half-plane with a constant magnetic field (equaling 1) and equipped by a Robin-like boundary condition (see [Bar+21b, Section 4.7]).

By using (A.1) and Proposition A.2, we have

$$|\Lambda(a) - a^2 + o(1)| \geq a|a - e_1(h)|.$$

We choose  $a > 0$  such that  $\Lambda(a) = a^2$ , so that

$$e_1(h) = a + o(1).$$

There is a unique solution of

$$\min(2b_0, b'_0\nu(a(b'_0)^{-1/2})) = a^2$$

which is  $a = \min(\sqrt{2b_0}, a_0\sqrt{b'_0})$ , where  $a_0$  is the unique positive solution of  $\nu(\alpha) = \alpha^2$  (a complete analysis is given in [Bar+21b, Section 4.7]). This completes the proof.

*Sketch of proof for Proposition A.2.* Let  $h, a > 0$ . Recall that,

$$\gamma_1(ah^{1/2}, h) = \inf_{\substack{u \in \mathfrak{H}_{h,-A}(\Omega) \\ \|u\|=1}} q_{ah^{1/2},h}(u),$$

with  $q_{ah^{1/2},h}(u) = \|d_{h,-A}^\times u\|^2 + ah^{3/2}\|u\|_{\partial\Omega}$ .

In a same way of [Bar+21b, Section 6], the sketch of proof is divided into three parts.

- For  $\delta \in (0, 1/2)$ , we consider a semiclassical radial partition of the unity  $(\chi_1, \chi_2, \chi_3)$  such that (in polar coordinates):
  - (1) we have  $\mathrm{supp}(\chi_1) \subset [\rho_1, \rho_1 + h^\delta] \times [-\pi, \pi[$ ,  $\mathrm{supp}(\chi_2) \subset [\rho_2 - h^\delta, \rho_2] \times [-\pi, \pi[$  and  $\mathrm{supp}(\chi_3) \subset [\rho_1 + h^\delta, \rho_2 - h^\delta] \times [-\pi, \pi[$ .
  - (2) for all  $(r, s) \in [\rho_1, \rho_2] \times [-\pi, \pi[$ ,

$$\chi_1^2(r, s) + \chi_2^2(r, s) + \chi_3^2(r, s) = 1.$$

(3) for all  $(r, s) \in [\rho_1, \rho_2] \times [-\pi, \pi[$ , there exists  $C > 0$  such that,

$$\partial_r \chi_1^2(r, s) + \partial_r \chi_2^2(r, s) + \partial_r \chi_3^2(r, s) \leq Ch^{-2\delta}.$$

- Then we look for a good lower bound. Up to some remainders we get three localized quadratic forms. First, the Dirichlet realization of

$$d_{h,-A} d_{h,-A}^\times = (-ih\nabla + A)^2 + hB,$$

is bounded from below by  $2hb_0$ . So that

$$q_{ah^{1/2},h}(\chi_3 u) = \|d_{h,-A}^\times(\chi_3 u)\|^2 \geq 2hb_0 \|\chi_3 u\|^2.$$

For the quadratic forms localized near the boundary we can always linearize the magnetic field and straighten the boundary in order to get back to the study of a Dirac operator on  $\mathbb{R}_+^2$ . We can show that

$$q_{ah^{1/2},h}(\chi_1 u) \geq hb'_0 \min_{\xi \in \mathbb{R}} \nu_{\mathbb{R}_+,1}^-(a, -\xi - \mathbf{m}_1(h)) \|\chi_1 u\|^2,$$

and

$$q_{ah^{1/2},h}(\chi_2 u) \geq hb'_0 \min_{\xi \in \mathbb{R}} \nu_{\mathbb{R}_+,1}^-(a, \xi + \mathbf{m}_2(h)) \|\chi_2 u\|^2,$$

where  $\nu_{\mathbb{R}_+,1}^-(a, \cdot)$  is defined in [Bar+21b, Section 4.4] and  $\mathbf{m}_j(h) = h^{-1/2} \rho_j^{-1} B(\rho_j)^{-1/2} c_j$  with

$$c_1 = \frac{1}{2\pi} \int_{\partial\Omega_{int}} A \quad \text{and} \quad c_2 = c_1 + \rho_2 \partial_r \phi(\rho_2) - \rho_1 \partial_r \phi(\rho_1).$$

By min-max principle, we get

$$\gamma_1(ah^{1/2}, h) \geq \Lambda(a)h + o(h).$$

- The same strategy of [Bar+21b, Section 6.2.3.] can be used to obtain the upper bound of Proposition A.2. □

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