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Paul-Emile Paradan

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# Eigenvalues, singular values, and the O’Shea-Sjamaar Theorem

Paul-Emile Paradan\*

March 20, 2023

## Abstract

The main focus of this work is the study of several cones relating the eigenvalues or singular values of a matrix to those of its off-diagonal blocks. Some of these cones were studied by Fomin-Fulton-Li-Poon in [15, 6], but we will explain why their description is incomplete and how we can remedy it using the O’Shea-Sjamaar theorem.

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## 1 Introduction

Let  $p \geq q \geq 1$  and  $n = p + q$ . Let us denote by  $Herm(n)$  the vector space of Hermitian  $n$ -square matrices. The spectrum of  $X \in Herm(n)$  is denoted by  $e(X) = (e_1 \geq \dots \geq e_n)$  and the singular spectrum of a matrix  $Y \in M_{p,q}(\mathbb{C})$  is denoted by  $s(Y) = (s_1 \geq \dots \geq s_q \geq 0)$ .

The main purpose of this article is the description of the following cones:

$$\begin{aligned} \mathcal{A}(p, q) &= \left\{ (e(X), s(X_{12})), X \in Herm(n) \right\}, \\ \mathcal{S}(p, q) &= \left\{ (s(X), s(X_{12}), s(X_{21})), X \in M_{n,n}(\mathbb{C}) \right\}, \\ \mathcal{T}(p, q) &= \left\{ (s(X), s(X_{11}), s(X_{22})), X \in M_{n,n}(\mathbb{C}) \right\}. \end{aligned}$$

Here, a  $n$ -square complex matrix  $X$  is written by blocs  $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$  where  $X_{12} \in M_{p,q}(\mathbb{C})$  and  $X_{21} \in M_{q,p}(\mathbb{C})$ .

In the 1970s, Thompson gave some inequalities satisfied by the elements of  $\mathcal{T}(p, q)$  [24], and more recently Fomin, Fulton, Li and Poon obtained sets of inequalities that (should) describe the cones  $\mathcal{A}(p, q)$  and  $\mathcal{E}(p, q) := \mathcal{S}(p, q) \cap \{s(X_{12}) = s(X_{21})\}$  [15, 6].

Indeed, in [15, 6] the authors state that an element  $(\lambda, s)$  belongs to  $\mathcal{A}(p, q)$  if and only if

$$(1) \quad \sum_{i \in I} \lambda_i - \sum_{j \in J^o} \lambda_j \geq 2 \sum_{k \in K} s_k$$

holds for all triple  $(I, J, K)$  of subsets of  $[q] = \{1, \dots, q\}$  that belongs to  $\bigcup_{r \leq q} LR_r^q$ . Here  $LR_r^q$  denotes the list of triples of cardinal  $r$  defined inductively by Horn [8], and we have denoted  $J^o = \{n + 1 - \ell, \ell \in J\}$ .

The main purpose of this work is to explain how a direct application of O'Shea-Sjamaar's theorem [18] gives complete sets of inequalities for the cones  $\mathcal{A}(p, q)$ ,  $\mathcal{S}(p, q)$ ,  $\mathcal{T}(p, q)$  and  $\mathcal{E}(p, q)$ . In addition, we will explain why the descriptions of the cones  $\mathcal{A}(p, q)$  and  $\mathcal{E}(p, q)$  given in [15, 6] are incomplete.

Nevertheless, this application of the O'Shea-Sjamaar theorem does not allow to describe these cones in an optimal way, because it leads to a large number of redundancies in the list of inequalities. In a future work [20], we give a more efficient method to describe these inequality sets, using the main result of [19].

In §3, we show that  $\mathcal{A}(p, q)$  is described by the following inequalities:

$$(2) \quad \sum_{i \in I} \lambda_i - \sum_{j \in J^o} \lambda_j \geq 2 \sum_{k \in K \cap [q]} s_k - 2 \sum_{k \in K^o \cap [q]} s_k$$

where the triplets  $(I, J, K)$  belongs to  $\bigcup_{r < n} \text{LR}_r^n$ . In [20], we improve this result by showing that this system can be restricted to the triplets  $(I, J, K)$  satisfying the following additional conditions:  $I, J, K$  are of cardinal  $r \leq q$ ,  $I \cap J^o = \emptyset$  and  $K = K_+ \cup (K_-)^o$  where  $K_+, K_-$  are disjoint subsets of  $[q]$ .

The example  $\mathcal{A}(3, 3)$  that we detail in §3.4.2 permits to see that inequalities (1) given by Fomin, Fulton, Li and Poon in [15, 6] are not sufficient to define  $\mathcal{A}(p, q)$ . We find here two inequalities of the type (2) which are independent of those of the type (1):

$$(3) \quad \begin{aligned} \lambda_1 + \lambda_3 + \lambda_4 - \lambda_2 - \lambda_5 - \lambda_6 &\geq 2(s_1 - s_2 - s_3) \\ \lambda_1 + \lambda_2 + \lambda_5 - \lambda_3 - \lambda_4 - \lambda_6 &\geq 2(s_1 - s_2 - s_3). \end{aligned}$$

The independence of the last inequalities is ensured by the example  $s_0 = (1, 0, 0)$  and  $\lambda_0 = (1, 1, 1, 1, -1, -1)$ . The element  $(\lambda_0, s_0)$  does not verify the inequality (3), and then  $(\lambda_0, s_0) \notin \mathcal{A}(3, 3)$ , whereas it verifies all the inequalities of the type (1).

In [6], the authors pose the question of finding the collection of linear inequalities that describes  $\mathcal{S}(p, q)$  (Problem 1.15). We answer this problem in §4. Here, the list of inequalities is parametrized by some Littlewood-Richardson coefficients. Recall that one associates to an increasing sequence  $A = \{a_1 < \dots < a_p\}$  of positive integers the partition  $\mu(A) = (a_p - p \geq \dots \geq a_1 - 1 \geq 0)$ .

Our main result of §4 states that an element  $(\gamma, s, t) \in \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^q$  belongs to  $\mathcal{S}(p, q)$  if and only the following set of inequalities are satisfied:

$$(4) \quad \gamma = (\gamma_1 \geq \dots \geq \gamma_n \geq 0), \quad s = (s_1 \geq \dots \geq s_q \geq 0), \quad t = (t_1 \geq \dots \geq t_q \geq 0),$$

$$(5) \quad \gamma_\ell \geq s_\ell, \quad \gamma_\ell \geq t_\ell, \quad \forall \ell \in [q],$$

$$(6) \quad \sum_{i \in A \cap [n]} \gamma_i - \sum_{i \in A^o \cap [n]} \gamma_i \geq \sum_{j \in B \cap [q]} s_j - \sum_{j \in B^o \cap [q]} s_j + \sum_{k \in C \cap [q]} t_k - \sum_{k \in C^o \cap [q]} t_k,$$

for any triplets  $(A, B, C)$  satisfying the following requirements:

- $B, C$  are strict subsets of  $[n]$ ,
- $A \subset [2n]$  and  $\#A = \#B + \#C \leq n$ ,

- the Littlewood-Richardson coefficient  $c_{\mu(B), \mu(C)}^{\mu(A)}$  is non-zero.

In §5, we study the subcone  $\mathcal{E}(p, q) = \{(\gamma, s) \in \mathbb{R}^n \times \mathbb{R}^q, (\gamma, s, s) \in \mathcal{S}(p, q)\}$ . Our previous result allows us to assert that  $(\gamma, s)$  belongs to  $\mathcal{E}(p, q)$  if and only if  $(\gamma, s, s)$  satisfies (4), (5) and (6). We will show how to refine this result: on the one hand the conditions (5) are not necessary, and on the other hand, for the inequalities (6), it is sufficient to restrict ourselves to those where  $B = C$ . At the end of §5, we compare our result on  $\mathcal{E}(p, q)$  with that of Fomin, Fulton, Li and Poon in [6]. We will then explain why their description requires further explanation since they use their erroneous description of  $\mathcal{A}(p, q)$  to get it.

In the last section, we give a set of inequalities describing the cone  $\mathcal{T}(p, q)$ . In the case  $\mathcal{T}(p, 1)$ , we recover the interlacing inequalities of the singular values obtained by Thompson [23].

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## Notations

Throughout the paper :

- We fix  $p \geq q \geq 1$  and  $n = p + q$ .
- We write  $0_{ab}$  for the zero matrix of size  $a \times b$ .
- Let  $M_{a,b}(\mathbb{C})$  be the vector space of complex  $a \times b$  matrices.
- $\mathbb{R}_+^\ell$  is the set of sequences  $x = (x_1 \geq \dots \geq x_\ell)$  of real numbers.
- $\mathbb{R}_{++}^\ell$  is the set of sequences  $x = (x_1 \geq \dots \geq x_\ell \geq 0)$  of non-negative real numbers.
- For any positive integer  $\ell$ , let  $[\ell]$  be the set  $\{1, \dots, \ell\}$ .
- If  $x \in \mathbb{R}^\ell$  and  $A \subset [\ell]$ , we write  $|x|_A = \sum_{a \in A} x_a$  and  $|x| = \sum_{i=1}^\ell x_i$ .
- For  $A \subset [\ell]$ , we define  $A^o := \{\ell + 1 - a, a \in A\}$  and  $A^c := [\ell] \setminus A$ .
- If  $x \in \mathbb{R}^\ell$ , let  $\text{Diag}(x)$  be the diagonal  $\ell \times \ell$  matrix with diagonal entries equal to  $x_1, \dots, x_\ell$ .

## 2 Reminder of some classical results

We recall some classical facts that we will need later.

### 2.1 Singular values

Let  $X$  be a rectangular matrix, say  $m \times n$ , with complex entries, and denote by  $X^*$  the complex conjugate transpose of  $X$ . Let  $\mu_1(X) \geq \dots \geq \mu_m(X) \geq 0$  denote the eigenvalues of the positive semidefinite matrix  $XX^*$ . Notice that  $\mu_k(X) = 0$  when  $k > p := \inf\{m, n\}$ .

The singular values of the matrix  $X$  are the coordinates of the vector

$$s(X) := \left( \mu_1(X)^{\frac{1}{2}}, \dots, \mu_p(X)^{\frac{1}{2}} \right) \in \mathbb{R}_{++}^p.$$

Consider the canonical action of the unitary group  $U_m \times U_n$  on  $M_{m,n}(\mathbb{C})$ :  $(g, h) \cdot X = gXh^{-1}$ ,  $\forall (g, h) \in U_m \times U_n$ . The singular values map  $s : M_{m,n}(\mathbb{C}) \rightarrow \mathbb{R}_{++}^p$  induces a bijective application

$$s : M_{m,n}(\mathbb{C})/U_m \times U_n \xrightarrow{\sim} \mathbb{R}_{++}^p.$$

### 2.2 Augmented matrices

If  $X$  is a  $n$ -square matrix, we denote by  $\widehat{X}$  the  $2n$ -square Hermitian matrix  $\begin{pmatrix} 0_{nn} & X \\ X^* & 0_{nn} \end{pmatrix}$ . It is easy to check that the spectrum of  $\widehat{X}$  is equal to  $\widehat{\mu} := (\mu_1, \dots, \mu_n, -\mu_n, \dots, -\mu_1) \in \mathbb{R}_+^{2n}$ , where  $\mu \in \mathbb{R}_{++}^n$  is the singular spectrum of  $X$ .

If  $Y$  is a  $p \times q$  matrix, we denote by  $\widehat{Y}^{p,q}$  the  $n$ -square Hermitian matrix  $\begin{pmatrix} 0_{pp} & Y \\ Y^* & 0_{qq} \end{pmatrix}$ . Here the spectrum of  $\widehat{Y}^{p,q}$  is equal to  $\widehat{s}^{p,q} := (s_1, \dots, s_q, 0, \dots, 0, -s_q, \dots, -s_1) \in \mathbb{R}_+^n$ , where  $s \in \mathbb{R}_{++}^q$  is the singular spectrum of  $Y$ .

If  $Z$  is a  $q \times p$  matrix, we denote by  $\widehat{Y}^{q,p}$  the  $n$ -square Hermitian matrix  $\begin{pmatrix} 0_{qq} & Z \\ Z^* & 0_{pp} \end{pmatrix}$ . Here also the spectrum of  $\widehat{Z}^{q,p}$  is equal to  $\widehat{t}^{q,p}$ , where  $t \in \mathbb{R}_{++}^q$  is the singular spectrum of  $Z$ .

### 2.3 Horn inequalities

Recall that  $e(X) \in \mathbb{R}_+^n$  denotes the vector of eigenvalues of a  $n$ -square Hermitian matrix  $X$ . We consider the Horn cone

$$\text{Horn}(n) := \left\{ (e(X), e(Y), e(X + Y)); X, Y \in \text{Herm}(n) \right\}.$$

Denote the set of cardinality  $r$ -subsets  $I = \{i_1 < i_2 < \dots < i_r\}$  of  $[n]$  by  $\mathcal{P}_r^n$ .

**Definition 2.1** To any  $1 \leq r < n$ ,  $\text{LR}_r^n$  refers to the set of triplet  $(I, J, K) \in (\mathcal{P}_r^n)^3$  such that  $(\mu(I), \mu(J), \mu(K)) \in \text{Horn}(r)$ .

The following theorem was conjectured by Horn [8] and proved by a combination of the works of Klyachko [10] and Knutson-Tao [13].

**Theorem 2.2** *The triplet  $(x, y, z) \in (\mathbb{R}_+^n)^3$  belongs to  $\text{Horn}(n)$  if and only if the following conditions holds:*

- $|x| + |y| = |z|$ ,
- $|x|_I + |y|_J \geq |z|_K$ , for any  $r < n$  and any  $(I, J, K) \in \text{LR}_r^n$ .

In the following sections, we will use the Littlewood Richardson coefficients to parameterize some inequalities. Let us recall their definition. Let  $\lambda, \mu$ , and  $\nu$  be three partitions of length less than  $n \geq 1$ . We associate to them the irreducible representations  $V_\lambda, V_\mu$  and  $V_\nu$  of the unitary group  $U_n$ . The Littlewood Richardson coefficient  $c_{\mu, \nu}^\lambda$  can be characterized by the relation  $c_{\mu, \nu}^\lambda = \dim [V_\lambda^* \otimes V_\mu \otimes V_\nu]^{U_n}$ . Thanks to the saturation Theorem of Knutson and Tao [13], we know that  $c_{\mu, \nu}^\lambda \neq 0 \iff (\mu, \nu, \lambda) \in \text{Horn}(n)$ .

The following kind of duality is used in the next sections: for all  $(I, J, K) \in (\mathcal{P}_r^n)^3$ , we have

$$(7) \quad c_{\mu(I), \mu(J)}^{\mu(K)} \neq 0 \iff c_{\mu((I^o)^c), \mu((J^o)^c)}^{\mu((K^o)^c)} \neq 0.$$

Since the relation  $|x|_{(I^o)^c} + |y|_{(J^o)^c} \geq |z|_{(K^o)^c}$  is equivalent to  $|x|_{I^o} + |y|_{J^o} \leq |z|_{K^o}$ , in Theorem 2.2, we can rewrite the last condition by requiring that

$$|x|_I + |y|_J \geq |z|_K \quad \text{and} \quad |x|_{I^o} + |y|_{J^o} \leq |z|_{K^o}$$

for any  $r \leq \frac{n}{2}$  and any  $(I, J, K) \in \text{LR}_r^n$ .

## 2.4 The cone $LR(U, \tilde{U})$

Let  $\iota : U \hookrightarrow \tilde{U}$  be two connected compact Lie groups. We choose an invariant scalar product  $(-, -)$  on the Lie algebra  $\tilde{\mathfrak{u}}$  of  $\tilde{U}$ , and we denote by  $\pi : \tilde{\mathfrak{u}} \rightarrow \mathfrak{u}$  the orthogonal projection.

Select maximal tori  $T$  in  $U$  and  $\tilde{T}$  in  $\tilde{U}$  such that  $T \subset \tilde{T}$ , and Weyl chambers  $\mathfrak{t}_+ \subset \mathfrak{t}$  and  $\tilde{\mathfrak{t}}_+ \subset \tilde{\mathfrak{t}}$ , where  $\mathfrak{t}$  and  $\tilde{\mathfrak{t}}$  denote the Lie algebras of  $T$ , resp.  $\tilde{T}$ . The aim of this section is to recall the description of the following cone obtained in [4, 21]:

$$LR(U, \tilde{U}) = \left\{ (\xi, \tilde{\xi}) \in \mathfrak{t}_+ \times \tilde{\mathfrak{t}}_+, U\xi \subset \pi(\tilde{U}\tilde{\xi}) \right\}.$$

Consider the lattice  $\wedge := \frac{1}{2\pi} \ker(\exp : \mathfrak{t} \rightarrow T)$  and the Weyl groups  $\tilde{W} = N_{\tilde{U}}(\tilde{T})/\tilde{T}$  and  $W = N_U(T)/T$ . We denote by  $w_o \in W$  the longest element. A vector  $\gamma \in \mathfrak{t}$  is called *rational* if it belongs to the  $\mathbb{Q}$ -vector space  $\mathfrak{t}_{\mathbb{Q}}$  generated by  $\wedge$ . We will see that the cone  $LR(U, \tilde{U})$  is completely described by inequalities of the form

$$(\tilde{\xi}, \tilde{w}\gamma) \geq (\xi, w_o w\gamma)$$

with  $\gamma$  rational anti-dominant and  $(w, \tilde{w}) \in W \times \tilde{W}$ .

### 2.4.1 Admissible elements

We let  $\Sigma(\tilde{\mathfrak{u}}/\mathfrak{u}) \subset \mathfrak{t}^*$  denote the set of weights relative to the  $T$ -action on  $(\tilde{\mathfrak{u}}/\mathfrak{u}) \otimes \mathbb{C}$ . If  $\gamma \in \mathfrak{t}$ , we denote by  $\Sigma(\tilde{\mathfrak{u}}/\mathfrak{u}) \cap \gamma^\perp$  the subset of weights vanishing against  $\gamma$ .

**Definition 2.3** *A rational element  $\gamma \in \mathfrak{t}$  is said admissible when*

$$(8) \quad \text{Vect}(\Sigma(\tilde{\mathfrak{u}}/\mathfrak{u}) \cap \gamma^\perp) = \text{Vect}(\Sigma(\tilde{\mathfrak{u}}/\mathfrak{u})) \cap \gamma^\perp.$$

We work under the following assumption.

**Assumption 2.4** *The subspace  $\mathfrak{z} := \{X \in \mathfrak{t}, \alpha(X) = 0, \forall \alpha \in \Sigma(\tilde{\mathfrak{u}}/\mathfrak{u})\}$  is contained in the center  $Z_{\tilde{\mathfrak{u}}}$  of  $\tilde{\mathfrak{u}}$ .*

This assumption means that any ideal of  $\tilde{\mathfrak{u}}$  contained in  $\mathfrak{u}$  is a subspace of  $Z_{\tilde{\mathfrak{u}}} \cap \mathfrak{u}$ .

Let  $\mathfrak{t} = \mathfrak{z} \oplus \mathfrak{t}_1$  be a rational decomposition. Let us denote by  $\Sigma(\tilde{\mathfrak{u}}/\mathfrak{u})'$  the image of  $\Sigma(\tilde{\mathfrak{u}}/\mathfrak{u})$  through the projection  $\mathfrak{t}^* \rightarrow (\mathfrak{t}_1)^*$ : thanks to our assumption,  $\Sigma(\tilde{\mathfrak{u}}/\mathfrak{u})'$  generates  $(\mathfrak{t}_1)^*$ . Any rational element  $\gamma \in \mathfrak{t}$  can be written  $\gamma = \gamma_0 + \gamma_1$  where  $\gamma_0$  is a rational element of  $\mathfrak{z}$  and  $\gamma_1$  is a rational element of  $\mathfrak{t}_1$ . We see then that a rational element  $\gamma$  is admissible if and only if  $\gamma_1$  is admissible. The later condition is equivalent to asking that the hyperplane  $(\gamma_1)^\perp \subset (\mathfrak{t}_1)^*$  is generated by a finite subset of  $\Sigma(\tilde{\mathfrak{u}}/\mathfrak{u})'$ . Thus, there are a finite number of choices for  $\gamma_1$  (up to multiplication by  $\mathbb{Q}^{>0}$ ).

### 2.4.2 Polarized trace

Let  $\mathfrak{R}(\mathfrak{u})$  and  $\mathfrak{R}(\tilde{\mathfrak{u}})$  be the set of roots associated to the Lie algebras  $\mathfrak{u}$  and  $\tilde{\mathfrak{u}}$ . The choice of the Weyl chambers  $\mathfrak{t}_+$  and  $\tilde{\mathfrak{t}}_+$  define subsets of positive roots  $\mathfrak{R}^+ \subset \mathfrak{R}(\mathfrak{u})$  and  $\tilde{\mathfrak{R}}^+ \subset \mathfrak{R}(\tilde{\mathfrak{u}})$ .

For a rational element  $\gamma \in \mathfrak{t}$  and  $(w, \tilde{w}) \in W \times \tilde{W}$ , we will use the following condition to parameterize the inequalities of  $LR(U, \tilde{U})$ :

$$(9) \quad \sum_{\substack{\alpha \in \mathfrak{R}^+ \\ \langle \alpha, w\gamma \rangle > 0}} \langle \alpha, w\gamma \rangle + \sum_{\substack{\tilde{\alpha} \in \tilde{\mathfrak{R}}^+ \\ \langle \tilde{\alpha}, \tilde{w}\gamma \rangle < 0}} \langle \tilde{\alpha}, \tilde{w}\gamma \rangle = 0$$

### 2.4.3 Schubert calculus

Let  $\iota : U_{\mathbb{C}} \hookrightarrow \tilde{U}_{\mathbb{C}}$  be the complexification of  $\iota : U \hookrightarrow \tilde{U}$ . To any non-zero rational element  $\gamma \in \mathfrak{t}$ , we associate the parabolic subgroups

$$(10) \quad \tilde{P}_\gamma = \{g \in \tilde{U}_{\mathbb{C}}, \lim_{t \rightarrow \infty} \exp(-it\gamma)g \exp(it\gamma) \text{ exists}\} \quad \text{and} \quad P_\gamma = \tilde{P}_\gamma \cap U_{\mathbb{C}}.$$

We consider the projective varieties  $\mathcal{F}_\gamma := U_{\mathbb{C}}/P_\gamma$  and  $\tilde{\mathcal{F}}_\gamma := \tilde{U}_{\mathbb{C}}/\tilde{P}_\gamma$ , with the canonical embedding  $\iota : \mathcal{F}_\gamma \hookrightarrow \tilde{\mathcal{F}}_\gamma$ . Let  $B \subset U_{\mathbb{C}}$  (resp.  $\tilde{B} \subset \tilde{U}_{\mathbb{C}}$ ) be the Borel subgroup associated to the choice of the Weyl chamber  $\mathfrak{t}_+$  (resp.  $\tilde{\mathfrak{t}}_+$ ).

We associate to  $(w, \tilde{w}) \in W \times \widetilde{W}$ , the Schubert cells

$$\tilde{\mathfrak{X}}_{\tilde{w}, \gamma}^o := \tilde{B}[\tilde{w}] \subset \tilde{\mathcal{F}}_\gamma \quad \text{and} \quad \mathfrak{X}_{w, \gamma}^o := B[w] \subset \mathcal{F}_\gamma.$$

The corresponding Schubert varieties are  $\tilde{\mathfrak{X}}_{\tilde{w}, \gamma} := \overline{\tilde{\mathfrak{X}}_{\tilde{w}, \gamma}^o}$  and  $\mathfrak{X}_{w, \gamma} := \overline{\mathfrak{X}_{w, \gamma}^o}$ .

We consider the cohomology<sup>1</sup> rings  $H^*(\tilde{\mathcal{F}}_\gamma, \mathbb{Z})$  and  $H^*(\mathcal{F}_\gamma, \mathbb{Z})$ . Let

$$\iota^* : H^*(\tilde{\mathcal{F}}_\gamma, \mathbb{Z}) \rightarrow H^*(\mathcal{F}_\gamma, \mathbb{Z})$$

be the pull-back map in cohomology. If  $Y$  is an irreducible closed subvariety of  $\tilde{\mathcal{F}}_\gamma$ , we denote by  $[Y] \in H^{2n_Y}(\tilde{\mathcal{F}}_\gamma, \mathbb{Z})$  its cycle class in cohomology : here  $n_Y = \text{codim}_{\mathbb{C}}(Y)$ . Recall that the cohomology class  $[pt]$  associated to a singleton  $Y = \{pt\} \subset \mathcal{F}_\gamma$  is a basis of  $H^{\max}(\mathcal{F}_\gamma, \mathbb{Z})$ .

In the next section we will consider a rational element  $\gamma \in \mathfrak{t}$  and  $(w, \tilde{w}) \in W \times \widetilde{W}$  satisfying the relation  $[\mathfrak{X}_{w, \gamma}] \cdot \iota^*([\tilde{\mathfrak{X}}_{\tilde{w}, \gamma}]) = k[pt]$  in  $H^*(\mathcal{F}_\gamma, \mathbb{Z})$ , with  $k \geq 1$ . This cohomological condition implies in particular that  $\dim_{\mathbb{C}}(\mathfrak{X}_{w, \gamma}) = \text{codim}_{\mathbb{C}}(\tilde{\mathfrak{X}}_{\tilde{w}, \gamma})$  which is equivalent to the relation

$$(11) \quad \# \{ \alpha \in \mathfrak{A}^+, \langle \alpha, w\gamma \rangle > 0 \} = \# \{ \tilde{\alpha} \in \tilde{\mathfrak{A}}^+, \langle \tilde{\alpha}, \tilde{w}\gamma \rangle < 0 \}.$$

We finish this section by considering the particular case where  $U_{\mathbb{C}} = GL_n(\mathbb{C})$  is embedded diagonally in  $\tilde{U}_{\mathbb{C}} = GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ . For  $1 \leq r < n$ , the vector  $\gamma_r = \underbrace{(-1, \dots, -1, 0, \dots, 0)}_{r \text{ times}} \in \mathbb{R}^n \simeq \mathfrak{t}$  is admissible and the flag manifolds  $\mathcal{F}_{\gamma_r}$  and  $\tilde{\mathcal{F}}_{\gamma_r}$  admits a canonical identifications respectively with the Grassmanians  $\mathbb{G}(r, n)$  and  $\mathbb{G}(r, n) \times \mathbb{G}(r, n)$ . The map  $\iota : \mathcal{F}_\gamma \hookrightarrow \tilde{\mathcal{F}}_\gamma$  corresponds to the diagonal embedding  $\iota : \mathbb{G}(r, n) \rightarrow \mathbb{G}(r, n) \times \mathbb{G}(r, n)$ .

For  $w \in W \simeq \mathfrak{S}_n$ , the Schubert variety  $\mathfrak{X}_{w, \gamma_r} \subset \mathbb{G}(r, n)$ , which depends only of the subset  $K = w([r]) \subset [n]$ , is denoted  $\mathfrak{X}_K$ . Similarly, for  $w = (w_1, w_2) \in \widetilde{W} \simeq \mathfrak{S}_n \times \mathfrak{S}_n$ , the Schubert variety  $\tilde{\mathfrak{X}}_{\tilde{w}, \gamma_r} \subset \mathbb{G}(r, n) \times \mathbb{G}(r, n)$  is equal to  $\mathfrak{X}_I \times \mathfrak{X}_J$ , where  $I = w_1([r])$  and  $J = w_2([r])$ .

In this setting, we have the following classical result.

**Lemma 2.5** *The following statements are equivalent:*

- $[\mathfrak{X}_{w, \gamma}] \cdot \iota^*([\tilde{\mathfrak{X}}_{\tilde{w}, \gamma}]) = \ell[pt]$  in  $H^*(\mathcal{F}_\gamma, \mathbb{Z})$ , with  $\ell \geq 1$ .
- $[\mathfrak{X}_I] \cdot [\mathfrak{X}_J] \cdot [\mathfrak{X}_K] = \ell[pt]$  in  $H^*(\mathbb{G}(r, n), \mathbb{Z})$ , with  $\ell \geq 1$ .
- *The Littlewood-Richardson coefficient  $c_{\mu(I^o), \mu(J^o)}^{\mu(K)}$  is non-zero.*

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<sup>1</sup>Here, we use singular cohomology with integer coefficients.

#### 2.4.4 Description of $LR(U, \tilde{U})$

We can finally describe the cone  $LR(U, \tilde{U})$ .

**Theorem 2.6** *Let  $(\xi, \tilde{\xi}) \in \mathfrak{t}_+ \times \tilde{\mathfrak{t}}_+$ . We have  $U\xi \in \pi(\tilde{U}\tilde{\xi})$  if and only if*

$$(12) \quad \langle \tilde{\xi}, \tilde{w}\gamma \rangle \geq \langle \xi, w_0 w \gamma \rangle$$

for any  $(\gamma, w, \tilde{w}) \in \mathfrak{t} \times W \times \tilde{W}$  satisfying the following properties:

- a)  $\gamma$  is admissible antidominant.
- b)  $[\mathfrak{X}_{w, \gamma}] \cdot \iota^*([\tilde{\mathfrak{X}}_{\tilde{w}, \gamma}]) = [pt]$  in  $H^*(\mathcal{F}_\gamma, \mathbb{Z})$ .
- c) Identity (9) holds.

The result still holds if we replace b) by the weaker condition

$$b') \quad [\mathfrak{X}_\gamma] \cdot \iota^*([\tilde{\mathfrak{X}}_{\tilde{w}, \gamma}]) = \ell[pt], \ell \geq 1 \quad \text{in} \quad H^*(\mathcal{F}_\gamma, \mathbb{Z}).$$

**Remark 2.7** *Suppose that there exists  $c_\gamma > 0$  such that  $|\langle \alpha, w\gamma \rangle|$  and  $|\langle \tilde{\alpha}, \tilde{w}\gamma \rangle|$  belongs to  $\{0, c_\gamma\}$ ,  $\forall (w, \tilde{w}) \in W \times \tilde{W}$ ,  $\forall (\alpha, \tilde{\alpha}) \in \mathfrak{R}(\mathfrak{u}) \times \mathfrak{R}(\tilde{\mathfrak{u}})$ . Then condition c) follows from condition b) (see (11)).*

When the closed connected subgroups  $\iota : U \subset \tilde{U}$  satisfy Assumption 2.4 the subspace  $\mathfrak{z} := \{X \in \mathfrak{t}, \alpha(X) = 0, \forall \alpha \in \Sigma(\tilde{\mathfrak{u}}/\mathfrak{u})\}$  is equal to  $Z_{\tilde{\mathfrak{u}}} \cap \mathfrak{u}$ . Let  $\mathfrak{t} = Z_{\tilde{\mathfrak{u}}} \cap \mathfrak{u} \oplus \mathfrak{t}_1$  be a rational decomposition. Any rational element  $\gamma \in \mathfrak{t}$  can be written  $\gamma = \gamma_0 + \gamma_1$  where  $\gamma_0 \in Z_{\tilde{\mathfrak{u}}} \cap \mathfrak{u}$  and  $\gamma_1 \in \mathfrak{t}_1$  are rational. Two cases occur :

- If  $\gamma_1 = 0$ , then  $\gamma$  satisfies conditions a), b) and c). The inequalities (12) given by these central elements shows that  $\tilde{\xi} - \xi$  is orthogonal to  $Z_{\tilde{\mathfrak{u}}} \cap \mathfrak{u}$ .
- If  $\gamma_1 \neq 0$  then it is immediate to see that  $\gamma$  satisfies a), b) and c) if and only if  $\gamma_1$  does also. Moreover, as  $\tilde{\xi} - \xi$  is orthogonal to  $Z_{\tilde{\mathfrak{u}}} \cap \mathfrak{u}$ ,  $\langle \tilde{\xi}, \tilde{w}\gamma \rangle \geq \langle \xi, w_0 w \gamma \rangle$  if and only if  $\langle \tilde{\xi}, \tilde{w}\gamma_1 \rangle \geq \langle \xi, w_0 w \gamma_1 \rangle$ .

Many people contribute to Theorem 2.6. The first input was given by Klyachko [10] with a refinement by Belkale [1], in the case of  $SL(n) \hookrightarrow (SL(n))^s$ . The case  $U_{\mathbb{C}} \hookrightarrow (U_{\mathbb{C}})^s$  was treated by Kapovich-Leeb-Millson [9] following an analogous slightly weaker result proved by Berenstein-Sjamaar [4]. Condition c) is related to the notion of Levi-movability introduced by Belkale-Kumar [2]. Finally Ressayre [21, 22] considered the general case and proved furthermore the irredundancy of the list of inequalities.

We refer the reader to the survey articles [7, 5, 11] for details.

## 2.5 The cone $LR(m, n)$

Let  $m, n \geq 1$ . Let us write an  $m+n$ -square matrix  $X$  by blocs  $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$  where  $X_{11} \in M_{m,m}(\mathbb{C})$  and  $X_{22} \in M_{n,n}(\mathbb{C})$ . In this section, we are interested in the following cone:

$$LR(m, n) := \{(e(X), e(X_{11}), e(X_{12})); X \in Herm(m+n)\}$$

Thanks to the theorem 2.6, we obtain the following description of  $LR(m, n)$ . The details are given in the next section.

**Theorem 2.8** *The triplet  $(x, y, z) \in \mathbb{R}_+^{m+n} \times \mathbb{R}_+^m \times \mathbb{R}_+^n$  belongs to  $LR(m, n)$  if and only if the following conditions holds:*

- $|x| = |y| + |z|$ ,
- $x_{n+k} \leq y_k \leq x_k, \forall k \in [m]$ ,
- $x_{m+\ell} \leq z_\ell \leq x_\ell, \forall \ell \in [n]$ ,
- $|x|_A \geq |y|_B + |z|_C$ , for any triplet  $A, B, C$  satisfying:
  1.  $B \subset [m]$  and  $C \subset [n]$  are strict subsets,
  2.  $A \subset [m+n]$  and  $\sharp A = \sharp B + \sharp C$ ,
  3. the Littlewood-Richardson coefficient  $c_{\mu(B), \mu(C)}^{\mu(A)}$  is non-zero.

Moreover, the condition  $c_{\mu(B), \mu(C)}^{\mu(A)} \neq 0$  is equivalent to  $(\mu(A), \mu(B), \mu(C)) \in LR(u, v)$ , where  $u = \sharp B$  and  $v = \sharp C$ .

**Remark 2.9** *In Theorem 2.8, we can strenghten the condition by requiring that  $c_{\mu(B), \mu(C)}^{\mu(A)} = 1$ .*

We will see in the next section that  $c_{\mu(B), \mu(C)}^{\mu(A)} \neq 0$  if and only  $c_{\mu((B^o)^c), \mu((C^o)^c)}^{\mu((A^o)^c)} \neq 0$ . Since the relation  $|x|_{(A^o)^c} \geq |y|_{(C^o)^c} + |z|_{(C^o)^c}$  is equivalent to  $|x|_{A^o} \leq |y|_{C^o} + |z|_{C^o}$ , in Theorem 2.8, we can rewrite the last condition by requiring that

$$(13) \quad |x|_A \geq |y|_B + |z|_C \quad \text{and} \quad |x|_{A^o} \leq |y|_{C^o} + |z|_{C^o}$$

for all strict subsets  $A \subset [m+n]$ ,  $B \subset [m]$ ,  $C \subset [n]$  that satisfy  $\sharp A = \sharp B + \sharp C \leq \frac{1}{2}(m+n)$  and  $c_{\mu(B), \mu(C)}^{\mu(A)} \neq 0$ .

## 2.6 Proof of Theorem 2.8

We work with the unitary group  $\tilde{U} = U_{m+n}$  and the subgroup  $U = U_m \times U_n$  embedded diagonally. We consider the orthogonal projection  $\pi_0 : \text{Herm}(m+n) \rightarrow \text{Herm}(m) \times \text{Herm}(n)$  that sends  $X$  to  $\pi_0(X) = (X_{11}, X_{22})$ . The cone  $LR(m, n)$  is formed by the triplet  $(x, y, z) \in \mathbb{R}_+^{m+n} \times \mathbb{R}_+^m \times \mathbb{R}_+^n$  satisfying

$$U_m \cdot \text{Diag}(y) \times U_n \cdot \text{Diag}(z) \subset \pi_0(U_{m+n} \cdot \text{Diag}(x)).$$

Thus  $LR(m, n) = LR(U_m \times U_n, U_{m+n})$ .

### 2.6.1 Admissible elements

We work with the maximal torus  $T \subset U$  of diagonal matrices. The set of roots relatively to the action of  $T$  on  $\tilde{\mathfrak{u}}/\mathfrak{u} \simeq M_{m,n}(\mathbb{C})$  is  $\Sigma := \{e_i^* - f_j^*; i \in [m], j \in [n]\}$ .

The center of  $\tilde{\mathfrak{u}}$  is generated by  $\gamma_o := (1, \dots, 1) \in \mathbb{R}^{m+n} \simeq \mathfrak{t}$ . For any  $(r, s) \in \{0, \dots, m\} \times \{0, \dots, n\}$ , we define

$$\gamma_{r,s} = \underbrace{(-1, \dots, -1)}_{r \text{ times}}, 0, \dots, 0 \oplus \underbrace{(-1, \dots, -1)}_{s \text{ times}}, 0, \dots, 0 \in \mathbb{R}^m \times \mathbb{R}^n \simeq \mathfrak{t}.$$

**Lemma 2.10** *Let  $\gamma \in \mathfrak{t}$  be an admissible element. There exists  $(a, b) \in \mathbb{Q} \times \mathbb{Q}^{\geq 0}$ ,  $(w, w') \in \mathfrak{S}_m \times \mathfrak{S}_n$ , and  $(r, s)$  such that  $\gamma = a\gamma_o + b(w, w')\gamma_{r,s}$ . The couple  $(r, s)$  must satisfy the auxiliary conditions: either  $0 < r < m$  and  $0 < s < n$  or  $(r, s) \in \{(1, 0), (0, 1), (m-1, n), (n, m-1)\}$ .*

*Proof :* Consider an admissible vector  $\gamma = (\gamma_1, \dots, \gamma_m; \gamma'_1, \dots, \gamma'_n)$  that is linearly independent to  $\gamma_o$ . The relation  $\text{Vect}(\Sigma \cap \gamma^\perp) = \text{Vect}(\Sigma) \cap \gamma^\perp$  means that  $(\Sigma \cap \gamma^\perp)^\perp$  is a subspace of dimension 2. Here  $\Sigma \cap \gamma^\perp$  is the set of vectors  $e_i^* - f_j^*$  such that  $\gamma_i = \gamma'_j$ . For  $\alpha \in \mathbb{R}$ , we define  $[m]_\alpha := \{i \in [m], \gamma_i = \alpha\}$  and  $[n]_\alpha := \{j \in [n], \gamma'_j = \alpha\}$ . Hence  $\Sigma \cap \gamma^\perp$  is parameterized by  $\coprod_{\alpha \in L} [m]_\alpha \times [n]_\alpha$  where  $L = \{\alpha \in \mathbb{R}, [m]_\alpha \neq \emptyset \text{ and } [n]_\alpha \neq \emptyset\}$  is a finite set.

Consider first the case where  $\cup_{\alpha \in L} [m]_\alpha \neq [m]$ . Let  $k \notin \cup_{\alpha \in L} [m]_\alpha$ . Then  $(\Sigma \cap \gamma^\perp)^\perp$ , which is of dimension 2, contains the vectors  $\gamma_o, \gamma$  and  $e_k$ . Hence,  $\gamma$  is a linear combination of  $\gamma_o$  and  $e_k$ : we check easily that there exists  $(a, b) \in \mathbb{Q} \times \mathbb{Q}^{>0}$ ,  $w \in \mathfrak{S}_m$ , and  $(r, s) \in \{(1, 0), (m-1, n)\}$  such that  $\gamma = a\gamma_o + bw\gamma_{r,s}$ .

If  $\cup_{\alpha \in L} [n]_\alpha \neq [n]$ , we prove similarly that  $\gamma = a\gamma_o + bw'\gamma_{r,s}$  for  $(a, b) \in \mathbb{Q} \times \mathbb{Q}^{>0}$ ,  $w' \in \mathfrak{S}_n$ , and  $(r, s) \in \{(0, 1), (m, n-1)\}$ .

Let us consider the last case where  $\cup_{\alpha \in L} [m]_\alpha = [m]$  and  $\cup_{\alpha \in L} [n]_\alpha = [n]$ . Then  $\gamma = \sum_{\alpha \in L} \alpha V_\alpha$  with  $V_\alpha = \sum_{(i,j) \in [m]_\alpha \times [n]_\alpha} e_i + f_j$ . The vectors  $\{V_\alpha, \alpha \in L\}$  defines an independent family of the subspace  $(\Sigma \cap \gamma^\perp)^\perp$  which is of dimension 2, hence  $\#L \leq 2$ . As  $\gamma$  is linearly independent to  $\gamma_o$ , the set  $L$  is of cardinal 2. Now we see that there exists  $(a, b) \in \mathbb{Q} \times \mathbb{Q}^{>0}$ ,  $(w, w') \in \mathfrak{S}_m \times \mathfrak{S}_n$ , and  $r < m, s < n$  such that  $\gamma = a\gamma_o + b(w, w')\gamma_{r,s}$ .  $\square$

Here, the remark 2.7 applies, so condition c) will follow from condition b).

## 2.6.2 Cohomological conditions and inequalities

The Lie algebra  $\mathfrak{t}$  is identified with  $\mathbb{R}^{m+n} \simeq \mathbb{R}^m \times \mathbb{R}^n$ .

**First case:** The two vectors  $\pm\gamma_o$  are admissible elements which satisfy conditions a), b) and c) of Theorem 2.6 in an obvious way. In this cases, the corresponding inequalities  $\pm(x, \gamma_o) \geq \pm((y, z), w_o\gamma_o)$  are equivalent to  $|x| = |y| + |z|$ .

**Second case:** We work now with the admissible element  $\gamma_{r,s}$  in the situation where  $r \in [m-1]$  and  $s \in [n-1]$ . The flag manifold  $GL_m(\mathbb{C}) \times GL_n(\mathbb{C})/P_{\gamma_{r,s}}$  admits a natural identification with the product of Grassmannians  $\mathbb{G}(r, m) \times \mathbb{G}(s, n)$ . Similarly, the flag manifold  $GL_{m+n}(\mathbb{C})/\tilde{P}_{\gamma_{r,s}}$  is isomorphic to the Grassmannian  $\mathbb{G}(r+s, m+n)$ . The map  $\iota_{\mathbb{C}} : GL_m(\mathbb{C}) \times GL_n(\mathbb{C}) \rightarrow GL_{m+n}(\mathbb{C})$  factorises to a smooth map  $\iota_{r,s} : \mathbb{G}(r, m) \times \mathbb{G}(s, n) \rightarrow \mathbb{G}(r+s, m+n)$  defined by  $\iota_{r,s}(V_1, V_2) = V_1 \oplus V_2$ .

Let  $w = (w_1, w_2) \in W \simeq \mathfrak{S}_m \times \mathfrak{S}_n$  and let  $B = w_1([r]) \subset [m]$  and  $C = w_2([s]) \subset [n]$  be the corresponding subsets. The associated Schubert variety is  $\mathfrak{X}_{w, \gamma_{r,s}} = \mathfrak{X}_B \times \mathfrak{X}_C \subset \mathbb{G}(r, m) \times \mathbb{G}(s, n)$ .

In the same way, to  $\tilde{w} \in \tilde{W} \simeq \mathfrak{S}_{m+n}$ , we associate the subset  $A = \tilde{w}([r]) \cup \tilde{w}([s] + m) \subset [m+n]$  and the Schubert variety  $\tilde{\mathfrak{X}}_{\tilde{w}, \gamma_{r,s}} = \tilde{\mathfrak{X}}_A \subset \mathbb{G}(r+s, m+n)$ .

**Lemma 2.11** *The following identities are equivalent:*

1.  $[\mathfrak{X}_{w, \gamma}] \cdot \iota^*([\tilde{\mathfrak{X}}_{\tilde{w}, \gamma}]) = \ell[pt], \ell \geq 1,$
2.  $c_{\mu(B), \mu(C)}^{\mu(A^o)} = \ell \geq 1,$
3.  $c_{\mu((B^o)^c), \mu((C^o)^c)}^{\mu(A^c)} = \ell \geq 1.$

*Proof:* Recall that we associate a partition  $\lambda(A) = (\lambda_1 \geq \dots \geq \lambda_r)$  with a subset  $A = \{a_1 < a_2 < \dots < a_r\} \subset [n]$  of cardinal  $r$ , by posing  $\lambda_k = n - r + k - a_k, \forall k \in [r]$ .

Let  $\Lambda_r[x] = \mathbb{Z}[x_1, \dots, x_r]^{\mathfrak{S}_r}$  be the rings of symmetric polynomials, with integral coefficients, in  $r$  variables. For any partition  $\nu$  of length  $r$ , we associate its Schur polynomial  $\mathbf{s}_\nu(x) \in \Lambda_r[x]$ . The family  $(\mathbf{s}_\nu)$  determine a  $\mathbb{Z}$ -basis of  $\Lambda_r[x]$ .

Let us recall the following classical fact (see §3.2.2 in [17]). The map  $\phi_r : \Lambda_r[x] \rightarrow H^*(\mathbb{G}(r, m))$  defined by the relations

$$\phi_r(\mathbf{s}_\nu) = \begin{cases} \sigma_\nu & \text{if } \nu_1 \leq m - r, \\ 0 & \text{if } \nu_1 > m - r. \end{cases}$$

is a ring morphism. Here  $\sigma_\nu$  denotes the cohomology class  $[\mathfrak{X}_D]$  defined by a subset  $D \subset [m]$  of cardinal  $r$  such that  $\nu = \lambda(D)$ . In the same way we consider the ring  $\Lambda_{r+s}[x, y] = \mathbb{Z}[x_1, \dots, x_r, y_1, \dots, y_s]^{\mathfrak{S}_{r+s}}$  and the morphism  $\phi_{r+s} : \Lambda_{r+s}[x, y] \rightarrow H^*(\mathbb{G}(r+s, m+n))$ . Let us denote by  $R : \Lambda_{r+s}[x, y] \rightarrow \Lambda_r[x] \otimes \Lambda_s[y]$  the restriction morphism. It is not hard

to check that the following diagram is commutative:

$$\begin{array}{ccc}
\bigwedge_r[x] & \otimes & \bigwedge_s[y] \xleftarrow{R} \bigwedge_{r+s}[x, y] \\
\downarrow \phi_r & & \downarrow \phi_s \qquad \qquad \downarrow \phi_{r+s} \\
H^*(\mathbb{G}(r, m)) & \otimes & H^*(\mathbb{G}(s, n)) \xleftarrow{j^*} H^*(\mathbb{G}(r+s, m+n)).
\end{array}$$

As  $[\mathfrak{X}_{w,r}] = \sigma_{\lambda(B)} \otimes \sigma_{\lambda(C)}$  and  $[\tilde{\mathfrak{X}}_{\tilde{w},r}] = \sigma_{\lambda(A)}$ , the previous diagram tell us that the integer  $\ell$  such that  $[\mathfrak{X}_{w,r}] \cdot \iota_r^*([\tilde{\mathfrak{X}}_{\tilde{w},r}]) = \ell[pt]$  is equal to the coefficient of  $R(\mathfrak{s}_{\lambda(A)}(x, y))$  relatively to  $\mathfrak{s}_{\lambda(B^o)}(x) \otimes \mathfrak{s}_{\lambda(C^o)}(y)$ : in other words  $\ell$  is equal to the Littlewood-Richardson coefficient  $c_{\lambda(B^o), \lambda(C^o)}^{\lambda(A)} = c_{\mu(B), \mu(C)}^{\mu(A^o)}$  (see [16], §I.5). The equivalence between 1. and 2. is proved.

Let us consider  $r', s'$  such that  $r + r' = m$  and  $s + s' = n$ . The canonical bilinear form on  $\mathbb{C}^{m+n}$  permits to define the map  $\delta : \mathbb{G}(r' + s', m + n) \rightarrow \mathbb{G}(r + s, m + n)$  that sends a subspace  $F \subset \mathbb{C}^{m+n}$  to its orthogonal  $F^\perp$ . Let  $\delta^* : H^*(\mathbb{G}(r + s, m + n)) \rightarrow H^*(\mathbb{G}(r' + s', m + n))$  denotes the pullback map in cohomology. If we consider the similar maps  $\delta^* : H^*(\mathbb{G}(r, m)) \rightarrow H^*(\mathbb{G}(r', m))$  and  $\delta^* : H^*(\mathbb{G}(s, n)) \rightarrow H^*(\mathbb{G}(s', n))$ , we have a commutative diagram:

$$\begin{array}{ccc}
H^*(\mathbb{G}(r, m)) & \otimes & H^*(\mathbb{G}(s, n)) \xleftarrow{j^*} H^*(\mathbb{G}(r+s, m+n)) \\
\downarrow \delta^* & & \downarrow \delta^* \qquad \qquad \downarrow \delta^* \\
H^*(\mathbb{G}(r', m)) & \otimes & H^*(\mathbb{G}(s', n)) \xleftarrow{j^*} H^*(\mathbb{G}(r'+s', m+n)).
\end{array}$$

This allows us to see that  $\sigma_{\lambda(B)} \otimes \sigma_{\lambda(C)} \cdot j^*(\sigma_{\lambda(A)}) = \ell[pt]$ ,  $k \geq 1$  if and only if  $\delta^*(\sigma_{\lambda(B)}) \otimes \delta^*(\sigma_{\lambda(C)}) \cdot j^*(\delta^*(\sigma_{\lambda(A)})) = \ell[pt]$ ,  $k \geq 1$ . As in general  $\delta^*(\sigma_{\lambda(X)}) = \sigma_{\mu(X^c)}$ , we see that the previous relation is equivalent to  $c_{\mu((B^o)^c), \mu((C^o)^c)}^{\mu(A^c)} = \ell \geq 1$ . The equivalence between 2. and 3. is proved.  $\square$

The inequalities associated to  $\gamma_{r,s}$  are

$$-|x|_A = \langle x, \tilde{w}\gamma_{r,s} \rangle \geq \langle (y, z), w_o w \gamma \rangle = -|y|_{B^o} - |z|_{C^o}$$

Using  $|x| = |y| + |z|$ , we obtain  $|x|_{A^c} \geq |y|_{(B^o)^c} + |z|_{(C^o)^c}$  for any strict subsets  $A \subset [m+n]$ ,  $B \subset [m]$  and  $C \subset [n]$  satisfying  $\#A = \#B + \#C$  and  $c_{\mu((B^o)^c), \mu((C^o)^c)}^{\mu(A^c)} = \ell \geq 1$ .

**Third case:**  $(r, s) \in \{(1, 0), (0, 1), (m-1, n), (n, m-1)\}$ . We use here the same type of argument as before.

$(r, s) = (1, 0)$ : we obtain the inequalities  $x_{n+k} \leq y_k, \forall k \in [m]$ .

$(r, s) = (0, 1)$ : we obtain the inequalities  $x_{m+\ell} \leq z_\ell, \forall \ell \in [n]$ .

$(r, s) = (m-1, n)$ : we obtain the inequalities  $y_k \leq x_k, \forall k \in [m]$ .

$(r, s) = (m, n-1)$ : we obtain the inequalities  $z_\ell \leq x_\ell, \forall \ell \in [n]$ .

The proof of Theorem 2.8 is completed.  $\square$

## 2.7 A consequence of the O'Shea-Sjamaar Theorem

### 2.7.1 First setting: compact Lie groups with involution

Let  $\tilde{U}$  be a compact connected Lie group equipped with an involution  $\sigma$ . The Lie algebra of  $\tilde{U}$  admit the decomposition  $\tilde{\mathfrak{u}} = \tilde{\mathfrak{u}}^\sigma \oplus \tilde{\mathfrak{u}}^{-\sigma}$  that is invariant under the action of the subgroup  $\tilde{U}^\sigma$ . We start with a basic but important fact (see [18], Example 2.9).

**Lemma 2.12** *For any adjoint orbit  $\tilde{\mathcal{O}} \subset \tilde{\mathfrak{u}}$ , the intersection  $\tilde{\mathcal{O}} \cap \tilde{\mathfrak{u}}^{-\sigma}$  is either empty or an orbit of the connected subgroup  $\tilde{K} := (\tilde{U}^\sigma)_0$ .*

Let  $U \subset \tilde{U}$  be a subgroup invariant under  $\sigma$ . Let us choose an invariant scalar product  $(-, -)$  on the Lie algebra  $\tilde{\mathfrak{u}}$  of  $\tilde{U}$  such that  $\sigma \in O(\tilde{\mathfrak{u}})$ . At the level of Lie algebras, we consider the orthogonal projection  $\pi : \tilde{\mathfrak{u}} \rightarrow \mathfrak{u}$  relatively to the scalar product  $(-, -)$ .

One of the main tool used in this paper is the following result, which is a consequence of the O'Shea-Sjamaar Theorem (see [18], Section 3). Let  $K$  be the connected component of  $U^\sigma$ .

**Proposition 2.13** *Let  $\xi \in \mathfrak{u}^{-\sigma}$  and  $\tilde{\xi} \in \tilde{\mathfrak{u}}^{-\sigma}$ . The following conditions are equivalent:*

1.  $U\xi \subset \pi(\tilde{U}\tilde{\xi})$ ,
2.  $K\xi \subset \pi(\tilde{K}\tilde{\xi})$ .

### 2.7.2 Second setting: real reductive Lie groups

Let  $\iota : G \hookrightarrow \tilde{G} \subset GL_N(\mathbb{R})$  be two connected real reductive Lie groups admitting a complexification  $\iota_{\mathbb{C}} : G_{\mathbb{C}} \hookrightarrow \tilde{G}_{\mathbb{C}} \subset GL_N(\mathbb{C})$ . It is for example the case when  $G$  and  $\tilde{G}$  are semisimple (see [12], §VII.1). Let us denote by

- $K = G \cap SO_N(\mathbb{R})$  and  $\tilde{K} = \tilde{G} \cap SO_N(\mathbb{R})$  the maximal compact subgroups of  $G$  and  $\tilde{G}$ . Their Lie algebras are denoted by  $\iota : \mathfrak{k} \hookrightarrow \tilde{\mathfrak{k}}$ .
- $U = G_{\mathbb{C}} \cap U_N$  and  $\tilde{U} = \tilde{G}_{\mathbb{C}} \cap U_N$  the maximal compact subgroups of  $G_{\mathbb{C}}$  and  $\tilde{G}_{\mathbb{C}}$ . Their Lie algebras are denoted by  $\iota : \mathfrak{u} \hookrightarrow \tilde{\mathfrak{u}}$ .

Consider the Cartan decompositions,  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  and  $\tilde{\mathfrak{g}} = \tilde{\mathfrak{k}} \oplus \tilde{\mathfrak{p}}$ , of  $G$  and  $\tilde{G}$ . At the level of Lie algebras, we have  $\tilde{\mathfrak{u}} = \tilde{\mathfrak{k}} \oplus i\tilde{\mathfrak{p}}$  and  $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ . The complex conjugation on  $GL_N(\mathbb{C})$  defines an involution  $\sigma$  on  $U \hookrightarrow \tilde{U}$  such that  $K \hookrightarrow \tilde{K}$  are respectively equal to the connected components of  $U^\sigma$  and  $\tilde{U}^\sigma$ . We see also that  $\mathfrak{u}^{-\sigma} = i\mathfrak{p}$  and  $\tilde{\mathfrak{u}}^{-\sigma} = i\tilde{\mathfrak{p}}$ .

Let  $\pi : \tilde{\mathfrak{g}}_{\mathbb{C}} \rightarrow \mathfrak{g}_{\mathbb{C}}$  be the orthogonal projection relatively to the Hermitian norm  $\text{Tr}(X^*X)^{1/2}$  on  $\mathfrak{gl}_N(\mathbb{C})$ .

**Proposition 2.14** *Let  $X \in \mathfrak{p}$  and  $\tilde{X} \in \tilde{\mathfrak{p}}$ . The following conditions are equivalent:*

1.  $UX \subset \pi(\tilde{U}\tilde{X})$ ,
2.  $KX \subset \pi(\tilde{K}\tilde{X})$ .

*Proof:* It follows from Proposition 2.13 and the fact that the projection  $\pi : \tilde{\mathfrak{g}}_{\mathbb{C}} \rightarrow \mathfrak{g}_{\mathbb{C}}$  is complex linear.  $\square$

### 3 The cone $\mathcal{A}(p, q)$

We work with the reductive real Lie groups  $G := U(p, q)$  and  $\tilde{G} := GL_n(\mathbb{C})$ . Let us denote by  $\iota : G \rightarrow \tilde{G}$  the canonical embedding. The unitary group  $\tilde{K} := U_n$  is a maximal compact subgroup of  $\tilde{G}$ . Let  $\tilde{\mathfrak{p}} := \text{Herm}(n) \subset \mathfrak{gl}_n(\mathbb{C})$  be the subspace of Hermitian matrices.

The subgroup  $K := \tilde{K} \cap U(p, q) \simeq U_p \times U_q$  is a maximal compact subgroup of  $G$ , and the map  $Y \mapsto \hat{Y}^{p,q} = \begin{pmatrix} 0_{pp} & Y \\ Y^* & 0_{qq} \end{pmatrix}$  defines an identification between  $M_{p,q}(\mathbb{C})$  and the subspace  $\mathfrak{p} := \tilde{\mathfrak{p}} \cap \mathfrak{g}$ .

#### 3.1 Complexification and antiholomorphic involution

The complexification of the group  $G$  is  $G_{\mathbb{C}} := GL_n(\mathbb{C})$ . We consider the antiholomorphic involution  $\sigma$  on  $G_{\mathbb{C}}$  defined by  $\sigma(g) = I_{p,q}(g^*)^{-1}I_{p,q}$ , where  $I_{p,q} = \text{Diag}(I_p, -I_q)$ . The subgroup  $G$  is the fixed point set of  $\sigma$ .

The complexification of the group  $\tilde{G}$  is  $\tilde{G}_{\mathbb{C}} := GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ . The inclusion  $\tilde{G} \hookrightarrow \tilde{G}_{\mathbb{C}}$  is given by the map  $g \mapsto (g, \bar{g})$ . We consider the antiholomorphic involution  $\tilde{\sigma}$  on  $\tilde{G}_{\mathbb{C}}$  defined by  $\tilde{\sigma}(g_1, g_2) = (\overline{g_2}, \overline{g_1})$ . The subgroup  $\tilde{G}$  corresponds to the fixed point set of  $\tilde{\sigma}$ .

The embedding  $\iota : G \hookrightarrow \tilde{G}$  admits a complexification  $\iota_{\mathbb{C}} : G_{\mathbb{C}} \hookrightarrow \tilde{G}_{\mathbb{C}}$  defined by  $\iota_{\mathbb{C}}(g) = (g, \overline{\sigma(g)})$ : notice that  $\iota_{\mathbb{C}} \circ \sigma = \tilde{\sigma} \circ \iota_{\mathbb{C}}$ .

The groups  $U = U_n$  and  $\tilde{U} = U_n \times U_n$  are respectively maximal compact subgroups of  $G_{\mathbb{C}}$  and  $\tilde{G}_{\mathbb{C}}$ . The embedding  $\iota_{\mathbb{C}} : U \hookrightarrow \tilde{U}$  is defined by  $\iota_{\mathbb{C}}(k) = (k, I_{p,q}\bar{k}I_{p,q})$ . The fixed point subgroups of the involutions are  $U^{\sigma} = K$  and  $\tilde{U}^{\tilde{\sigma}} = \tilde{K}$ .

At the level of Lie algebra, we have a morphism  $\iota_{\mathbb{C}} : \mathfrak{gl}_n(\mathbb{C}) \hookrightarrow \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})$  defined by  $\iota_{\mathbb{C}}(X) = (X, \overline{\sigma(X)})$ , where  $\sigma(X) = -I_{p,q}X^*I_{p,q}$ .

#### 3.2 Orthogonal projection of orbits

We use on  $\mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})$  the euclidean norm  $\|(X, Y)\|^2 = \text{Tr}(XX^*) + \text{Tr}(YY^*)$ . The subspace orthogonal to the image of  $\iota_{\mathbb{C}}$  is  $\{(X, -\overline{\sigma(X)}), X \in \mathfrak{gl}_n(\mathbb{C})\}$ . Hence the orthogonal projection

$$\pi : \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C}) \longrightarrow \mathfrak{gl}_n(\mathbb{C}),$$

is defined by the relations  $\pi(X, Y) = \frac{1}{2}(X + \overline{\sigma(Y)})$ . Note that  $\pi$  commutes with the involutions :  $\pi \circ \tilde{\sigma} = \sigma \circ \pi$ .

If  $X \in Herm(n)$ , the corresponding adjoint orbit  $U_n \cdot X$ , which is entirely determined by the spectrum  $e(X)$ , is denoted by  $\mathcal{O}_{e(X)}$ . If  $\lambda = (\lambda_1, \dots, \lambda_n)$ , we denote by  $\lambda^*$  the vector  $(-\lambda_n, \dots, -\lambda_1)$ : we see that  $e(-X) = e(X)^*$  for any  $X \in Herm(n)$ .

The subspace  $\tilde{\mathfrak{p}} \subset \tilde{\mathfrak{g}}$  is identified with  $\{(X, \overline{X}), X \in Herm(n)\} \subset \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})$ . For any  $X \in Herm(n)$ , the image by the projection  $\pi$  of the orbit  $\tilde{U} \cdot (X, \overline{X})$  is equal to

$$\frac{1}{2}(U_n \cdot X + U_n \cdot \sigma(X)) = \frac{1}{2}(U_n \cdot X + U_n \cdot (-X)) = \frac{1}{2}(\mathcal{O}_\lambda + \mathcal{O}_{\lambda^*}),$$

where  $\lambda = e(X)$ .

If  $Y \in M_{p,q}(\mathbb{C})$  has singular spectrum  $s \in \mathbb{R}_{++}^q$ , the spectrum of the Hermitian matrix  $\widehat{Y}^{p,q}$  is equal to  $\widehat{s}^{p,q}$ , hence  $U \cdot \widehat{Y}^{p,q}$  is equal to  $\mathcal{O}_{\widehat{s}^{p,q}}$ . At this stage, we have proved that for any  $(X, Y) \in Herm(n) \times M_{p,q}(\mathbb{C})$  the following statements are equivalents:

- $U \cdot \widehat{Y}^{p,q} \subset \pi(\tilde{U} \cdot (X, \overline{X}))$ ,
- $2\mathcal{O}_{\widehat{s}^{p,q}} \subset \mathcal{O}_\lambda + \mathcal{O}_{\lambda^*}$ ,
- $(\lambda, \lambda^*, 2\widehat{s}^{p,q}) \in \text{Horn}(n)$ ,

where  $\lambda = e(X)$  and  $s = s(Y)$ .

The group  $K \simeq U_p \times U_q$  acts canonically  $M_{p,q}(\mathbb{C}) \simeq \mathfrak{p}$ . For any  $Y \in M_{p,q}(\mathbb{C})$ , the orbit  $K \cdot \widehat{Y}^{p,q} \subset \mathfrak{p}$  is entirely determined by the singular spectrum  $s(Y)$ . If one restricts the projection  $\pi : \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C}) \rightarrow \mathfrak{gl}_n(\mathbb{C})$  to the subspace  $\tilde{\mathfrak{p}} = Herm(n)$ , we obtain the map  $\pi : \tilde{\mathfrak{p}} \rightarrow \mathfrak{p}$  that sends an Hermitian matrix  $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{pmatrix}$  to  $\widehat{X}_{12}^{p,q} = \begin{pmatrix} 0_{pp} & X_{12} \\ X_{12}^* & 0_{qq} \end{pmatrix}$ .

Since the orbit  $\tilde{K} \cdot X$  is equal to  $\mathcal{O}_{e(X)}$ , we see then that  $\mathcal{A}(p, q)$  can be defined as follows:  $(\lambda, s) \in \mathbb{R}_+^n \times \mathbb{R}_{++}^q$  belongs to the cone  $\mathcal{A}(p, q)$  if and only if for any  $(X, Y) \in Herm(n) \times M_{p,q}(\mathbb{C})$  satisfying  $\lambda = e(X)$  and  $s = s(Y)$ , we have  $K \cdot \widehat{Y}^{p,q} \subset \pi(\tilde{K} \cdot X)$ .

### 3.3 Inequalities determining $\mathcal{A}(p, q)$

The computations done in the previous section, together with Proposition 2.14, gives us the following result.

**Proposition 3.1** *Let  $(\lambda, s) \in \mathbb{R}_+^n \times \mathbb{R}_{++}^q$ , and let  $(X, Y) \in Herm(n) \times M_{p,q}(\mathbb{C})$  such that  $\lambda = e(X)$  and  $s = s(Y)$ . The following statements are equivalent:*

- $(\lambda, s) \in \mathcal{A}(p, q)$ ,
- $K \cdot \widehat{Y}^{p,q} \subset \pi(\tilde{K} \cdot X)$ ,
- $U \cdot \widehat{Y}^{p,q} \subset \pi(\tilde{U} \cdot (X, \overline{X}))$ ,

- $(\lambda, \lambda^*, 2\widehat{s}^{p,q}) \in \text{Horn}(n)$ .

Thanks to the description of the Horn cone  $\text{Horn}(n)$  given in Theorem 2.2, we can conclude with the following description of  $\mathcal{A}(p, q)$ .

**Theorem 3.2** *An element  $(\lambda, s) \in \mathbb{R}_+^n \times \mathbb{R}_{++}^q$  belongs to  $\mathcal{A}(p, q)$  if and only if*

$$(\star)_{I,J,K} \quad |\lambda|_I - |\lambda|_{J^c} \geq 2|s|_{K \cap [q]} - 2|s|_{K^c \cap [q]}$$

for any  $r < n$  and any  $(I, J, K) \in \text{LR}_r^n$ .

**Remark 3.3** *In the formulation of the previous theorem we have used that  $|\lambda^*|_J = -|\lambda|_{J^c}$  and  $|\widehat{s}^{p,q}|_K = |s|_{K \cap [q]} - |s|_{K^c \cap [q]}$ .*

**Remark 3.4** *As we have said in the introduction, we can restrict the system of inequalities in Theorem 3.2 by considering uniquely triplets  $(I, J, K) \in \text{LR}_r^n$  with  $r \leq q$  (see [20]).*

## 3.4 Examples

### 3.4.1 Computation of $\mathcal{A}(2, 2)$

The set  $\text{LR}_1^4$  corresponds to the set of triplets  $(i, j, k)$  of elements of  $[4]$  such that  $i + j = k + 1$ : the corresponding (non-trivial) inequalities are

$$\lambda_1 - \lambda_4 \geq 2s_1, \quad \lambda_2 - \lambda_4 \geq 2s_2, \quad \lambda_1 - \lambda_3 \geq 2s_2.$$

The set  $\text{LR}_2^4$  corresponds to the set of triplets  $(I = \{i_1 < i_2\}, J = \{j_1 < j_2\}, K = \{k_1 < k_2\})$  of subsets of  $[4]$  satisfying Horn's conditions:

1.  $i_1 + i_2 + j_1 + j_2 = k_1 + k_2 + 3$ ,
2.  $i_1 + j_1 \leq k_1 + 1, \quad i_1 + j_2 \leq k_2 + 1, \quad i_2 + j_1 \leq k_2 + 1$ .

Here the inequality  $(\star)_{I,J,K}$  is non trivial only in one case: when  $I = J = K = \{1, 2\}$  we obtain  $\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 \geq 2(s_1 + s_2)$ .

We summarize our computations as follows.

**Proposition 3.5** *An element  $(\lambda, s) \in \mathbb{R}_+^4 \times \mathbb{R}_{++}^2$  belongs to  $\mathcal{A}(2, 2)$  if and only if the following conditions holds*

- $\lambda_1 - \lambda_4 \geq 2s_1, \quad \lambda_2 - \lambda_4 \geq 2s_2, \quad \lambda_1 - \lambda_3 \geq 2s_2$ .
- $\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 \geq 2(s_1 + s_2)$ .

### 3.4.2 Computation of $\mathcal{A}(3, 3)$

The non-trivial inequalities associated to  $\text{LR}_1^6$  are

$$(14) \quad \begin{array}{ll} \lambda_1 - \lambda_6 \geq 2s_1 & \lambda_1 - \lambda_5 \geq 2s_2 \\ \lambda_2 - \lambda_6 \geq 2s_2 & \lambda_1 - \lambda_4 \geq 2s_3 \\ \lambda_2 - \lambda_5 \geq 2s_3 & \lambda_3 - \lambda_6 \geq 2s_3. \end{array}$$

The non-trivial inequalities associated to  $\text{LR}_2^6$  are

$$(15) \quad \begin{array}{l} \lambda_1 + \lambda_2 - \lambda_5 - \lambda_6 \geq 2(s_1 + s_2) \\ \lambda_1 + \lambda_2 - \lambda_4 - \lambda_6 \geq 2(s_1 + s_3) \\ \lambda_1 + \lambda_3 - \lambda_5 - \lambda_6 \geq 2(s_1 + s_3) \\ \lambda_1 + \lambda_2 - \lambda_4 - \lambda_5 \geq 2(s_2 + s_3) \\ \lambda_1 + \lambda_3 - \lambda_4 - \lambda_6 \geq 2(s_2 + s_3) \\ \lambda_2 + \lambda_3 - \lambda_5 - \lambda_6 \geq 2(s_2 + s_3). \end{array}$$

Note that the inequality  $\lambda_1 + \lambda_4 - \lambda_5 - \lambda_6 \geq 2(s_2 + s_3)$  is not valid, even if it looks like the previous ones, since the triplet  $(\{1, 4\}, \{1, 2\}, \{2, 3\})$  does not belong to  $\text{LR}_2^6$ .

The non-trivial inequalities associated to  $\text{LR}_3^6$  are

$$(16) \quad \begin{array}{l} \lambda_1 + \lambda_3 + \lambda_4 - \lambda_2 - \lambda_5 - \lambda_6 \geq 2(s_1 - s_2 - s_3) \\ \lambda_1 + \lambda_2 + \lambda_5 - \lambda_3 - \lambda_4 - \lambda_6 \geq 2(s_1 - s_2 - s_3) \\ \lambda_1 + \lambda_2 + \lambda_3 - \lambda_4 - \lambda_5 - \lambda_6 \geq 2(s_1 + s_2 + s_3). \end{array}$$

In the case of  $\text{LR}_3^6$ , the *trivial* inequalities are those induced by inequalities obtained with  $\text{LR}_1^6$  and  $\text{LR}_2^6$ . For example, the inequalities corresponding to the triplets  $(\{1, 2, 5\}, \{2, 3, 4\}, \{2, 3, 6\})$  and  $(\{1, 2, 4\}, \{1, 2, 3\}, \{1, 2, 4\})$  of  $\text{LR}_2^6$  are respectively

$$\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 \geq 2(-s_1 + s_2 + s_3) \quad \text{and} \quad \lambda_1 + \lambda_2 - \lambda_5 - \lambda_6 \geq 2(s_1 + s_2 - s_3).$$

The former is induced by  $\lambda_1 - \lambda_4 \geq 2s_3$  obtained with  $\text{LR}_1^6$  and  $\lambda_2 - \lambda_3 \geq 0 \geq s_2 - s_1$  while the latter is induced by  $\lambda_1 + \lambda_2 - \lambda_5 - \lambda_6 \geq 2(s_1 + s_2)$  obtained with  $\text{LR}_2^6$  and  $s_3 \geq 0$ .

**Proposition 3.6** *An element  $(\lambda, s) \in \mathbb{R}_+^6 \times \mathbb{R}_{++}^3$  belongs to  $\mathcal{A}(3, 3)$  if and only if the inequalities listed in (14), (15) and (16) are satisfied.*

**Remark 3.7** *The cone  $\mathcal{A}(3, 3) \subset \mathbb{R}^6 \times \mathbb{R}^3$  corresponds to the intersection of the Horn cone  $\text{Horn}(6) \subset \mathbb{R}^{18}$  with the subspace  $\{(\lambda, \lambda^*, 2\widehat{\mathcal{S}}^{p,q}), (\lambda, s) \in \mathbb{R}^6 \times \mathbb{R}^3\}$ . It is striking that  $\mathcal{A}(3, 3)$  is determined by 23 inequalities while  $\text{Horn}(6)$  is described with a minimal list of 536 inequalities.*

## 4 The cone $\mathcal{S}(p, q)$

We work with the real reductive Lie groups  $G := U(p, q) \times U(q, p)$  and  $\tilde{G} := U(n, n)$ . The embedding  $\iota : G \rightarrow \tilde{G}$  is defined as follows:

$$(17) \quad \iota(g, h) = \begin{pmatrix} g_{11} & 0_{pn} & g_{12} \\ 0_{np} & h & 0_{nq} \\ g_{21} & 0_{qn} & g_{22} \end{pmatrix}, \quad \text{when} \quad g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}.$$

Here  $g_{11} \in M_{p,p}(\mathbb{C})$ ,  $g_{12} \in M_{p,q}(\mathbb{C})$ ,  $g_{2,1} \in M_{q,p}(\mathbb{C})$  and  $g_{22} \in M_{q,q}(\mathbb{C})$ .

The unitary group  $\tilde{K} := U_n \times U_n$  is a maximal compact subgroup of  $\tilde{G}$ . The subspace  $\tilde{\mathfrak{p}} := \{\hat{X}, X \in M_{n,n}(\mathbb{C})\} \subset \tilde{\mathfrak{g}}$  admits a canonical action of  $\tilde{K}$ . The subgroup  $K = K_1 \times K_2$ , with  $K_1 \simeq U_p \times U_q$  and  $K_2 \simeq U_q \times U_p$ , is a maximal compact subgroup of  $G$ , and the subspace  $\mathfrak{p} = \mathfrak{p}_1 \times \mathfrak{p}_2$  admits a natural identification with  $M_{p,q}(\mathbb{C}) \times M_{q,p}(\mathbb{C})$ :

$$(Y, Z) \in M_{p,q}(\mathbb{C}) \times M_{q,p}(\mathbb{C}) \longmapsto (\hat{Y}^{p,q}, \hat{Z}^{q,p}) \in \mathfrak{p}_1 \times \mathfrak{p}_2.$$

See §2.2 for the notations  $\hat{X}$ ,  $\hat{Y}^{p,q}$ , and  $\hat{Z}^{q,p}$ .

### 4.1 Complexification, antiholomorphic involution and orthogonal projection

The complexification of the group  $G$  is  $G_{\mathbb{C}} := GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ . We consider the antiholomorphic involution  $\sigma$  on  $G_{\mathbb{C}}$  defined by  $\sigma(g, h) = (I_{p,q}(g^*)^{-1}I_{p,q}, I_{q,p}(h^*)^{-1}I_{q,p})$ , where  $I_{p,q} = \text{Diag}(Id_p, -Id_q)$ . The subgroup  $\tilde{G}$  is the fixed point set of  $\sigma$ .

The complexification of the group  $\tilde{G}$  is  $\tilde{G}_{\mathbb{C}} := GL_{2n}(\mathbb{C})$ . We consider the antiholomorphic involution  $\tilde{\sigma}$  on  $\tilde{G}_{\mathbb{C}}$  defined by  $\tilde{\sigma}(g) = I_{n,n}(g^*)^{-1}I_{n,n}$ . The subgroup  $\tilde{G}$  corresponds to the fixed point set of  $\tilde{\sigma}$ .

The groups  $U = U_n \times U_n$  and  $\tilde{U} = U_{2n}$  are respectively maximal compact subgroups of  $G_{\mathbb{C}}$  and  $\tilde{G}_{\mathbb{C}}$ . The fixed point subgroups of the involutions are  $U^{\sigma} = K$  and  $\tilde{U}^{\tilde{\sigma}} = \tilde{K}$ .

The embedding  $\iota : G \hookrightarrow \tilde{G}$  admits a complexification  $\iota_{\mathbb{C}} : G_{\mathbb{C}} \hookrightarrow \tilde{G}_{\mathbb{C}}$ . At the level of Lie algebra, we have a morphism  $\iota_{\mathbb{C}} : \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C}) \hookrightarrow \mathfrak{gl}_{2n}(\mathbb{C})$ , still defined by (17).

The orthogonal projection  $\pi : \mathfrak{gl}_{2n}(\mathbb{C}) \rightarrow \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})$  dual to the morphism  $\iota_{\mathbb{C}}$  is defined by the relations :

$$(18) \quad A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \longmapsto \pi(A) = \left( \begin{pmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{pmatrix}, A_{22} \right).$$

Here the matrix  $A \in \mathfrak{gl}_{2n}(\mathbb{C})$  is written by blocs relatively to the decomposition  $2n = p + n + q$ .

We work also we another projection  $\pi_0 : \mathfrak{gl}_{2n}(\mathbb{C}) \rightarrow \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})$  defined by the relations :

$$(19) \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \longmapsto \pi_0(B) = (B_{11}, B_{22}).$$

Here each matrix  $B_{ij}$  belongs to  $\mathfrak{gl}_n(\mathbb{C})$ .

**Lemma 4.1** *For any  $U_{2n}$ -orbit  $\mathcal{O} \subset \mathfrak{gl}_{2n}(\mathbb{C})$ , we have  $\pi(\mathcal{O}) = \pi_0(\mathcal{O})$ .*

*Proof:* Consider the permutation  $w_0$  of  $[2n]$  defined as follows:  $w_0(k) = k$  if  $1 \leq k \leq p$ ,  $w_0(k) = k + q$  if  $p + 1 \leq k \leq n + p$  and  $w_0(k) = k - n$  if  $n + p + 1 \leq k \leq 2n$ . We check now that  $\pi = \pi_0 \circ \text{Ad}(w_0)$ . Our lemma follows from this relation.  $\square$

## 4.2 Description of $\mathcal{S}(p, q)$ through $LR(n, n)$

For  $t \in \mathbb{R}^q$ , we consider the  $n$ -square Hermitian matrix

$$(20) \quad Y(t) := \begin{pmatrix} 0_{qq} & 0_{q,p-q} & \text{Diag}(t) \\ 0_{p-q,q} & 0_{p-q,p-q} & 0_{p-q,q} \\ \text{Diag}(t) & 0_{q,p-q} & 0_{qq} \end{pmatrix} \subset \mathfrak{p}_1 \cap \mathfrak{p}_2.$$

Notice that for any  $t \in \mathbb{R}_{++}^q$ , we have:

- the orbit  $U(p) \times U(q) \cdot Y(t)$  is equal to  $\{\widehat{Y}^{p,q} \in \mathfrak{p}_1, s(Y) = t\}$ ,
- the orbit  $U(q) \times U(p) \cdot Y(t)$  is equal to  $\{\widehat{Y}^{q,p} \in \mathfrak{p}_2, s(Y) = t\}$ ,
- the orbit  $U(n) \cdot Y(t)$  is equal to  $\mathcal{O}_{\widehat{t}^{p,q}} := \{Z \in \text{Herm}(n), e(Z) = \widehat{t}^{p,q}\}$ .

Here is the main application of the Proposition 2.14.

**Proposition 4.2** *Let  $(\gamma, s, t) \in \mathbb{R}_{++}^n \times \mathbb{R}_{++}^q \times \mathbb{R}_{++}^q$ . The following statements are equivalent:*

1.  $(\gamma, s, t) \in \mathcal{S}(p, q)$ ,
2.  $\exists A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in M_{n,n}(\mathbb{C})$ , such that  $s(A) = \gamma$ ,  $s(A_{12}) = s$ , and  $s(A_{21}) = t$ ,
3.  $\exists X \in \widehat{\mathfrak{p}}$ , with  $\pi(X) = (X_1, X_2)$ , s.t.  $e(X) = \widehat{\gamma}$ ,  $e(X_1) = \widehat{s}^{p,q}$  and  $e(X_2) = \widehat{t}^{p,q}$ ,
4.  $\pi \left( U(n) \times U(n) \cdot \widehat{\text{Diag}}(\gamma) \right)$  contains  $(U(p) \times U(q) \cdot Y(s)) \times (U(q) \times U(p) \cdot Y(t))$ .
5.  $\pi \left( U(2n) \cdot \widehat{\text{Diag}}(\gamma) \right)$  contains  $U(n) \cdot Y(s) \times U(n) \cdot Y(t)$ .
6.  $\pi_0(U(2n) \cdot \widehat{\text{Diag}}(\widehat{\gamma}))$  contains  $\mathcal{O}_{\widehat{t}^{p,q}} \times \mathcal{O}_{\widehat{s}^{p,q}}$ .
7.  $(\widehat{\gamma}, \widehat{s}^{p,q}, \widehat{t}^{p,q}) \in LR(n, n)$ .

*Proof:* 4.  $\iff$  5. follows from Proposition 2.14, and 5.  $\iff$  6. is a consequence of Lemma 4.1 and the fact that the orbit  $U(2n) \cdot \widehat{\text{Diag}}(\gamma)$  is equal to  $U(2n) \cdot \widehat{\text{Diag}}(\widehat{\gamma})$ . The other equivalences are left to the reader.  $\square$

### 4.3 Inequalities determining $\mathcal{S}(p, q)$

Thanks to Proposition 4.2 and Theorem 2.8, we obtain the following description of the cone  $\mathcal{S}(p, q)$ .

**Theorem 4.3** *An element  $(\gamma, s, t) \in \mathbb{R}_{++}^n \times \mathbb{R}_{++}^q \times \mathbb{R}_{++}^q$  belongs to  $\mathcal{S}(p, q)$  if and only if,  $\gamma_k \geq s_k$  and  $\gamma_k \geq t_k$ ,  $\forall k \in [q]$ , and*

$$(21) \quad |\gamma|_{A \cap [n]} - |\gamma|_{A^\circ \cap [n]} \geq |s|_{B \cap [q]} - |s|_{B^\circ \cap [q]} + |t|_{C \cap [q]} - |t|_{C^\circ \cap [q]},$$

holds for any triplets  $(A, B, C)$  satisfying the following conditions :

- $B, C$  are strict subsets of  $[n]$ ,
- $A \subset [2n]$  and  $\sharp A = \sharp B + \sharp C$ ,
- the Littlewood-Richardson coefficient  $c_{\mu(B), \mu(C)}^{\mu(A)}$  is non-zero.

Let us use some duality to minimize the number of equations (see (13)). The equation (21) means that  $(\widehat{\gamma}, \widehat{s}^{p,q}, \widehat{t}^{p,q})$  satisfies  $|x|_A \geq |y|_B + |z|_C$ , that is  $|\widehat{\gamma}|_A \geq |\widehat{s}^{p,q}|_B + |\widehat{t}^{p,q}|_C$ . If we apply  $(\widehat{\gamma}, \widehat{s}^{p,q}, \widehat{t}^{p,q})$  to the relation  $|x|_{A^\circ} \leq |y|_{C^\circ} + |z|_{C^\circ}$ , we get  $|\widehat{\gamma}|_{A^\circ} \leq |\widehat{s}^{p,q}|_{B^\circ} + |\widehat{t}^{p,q}|_{C^\circ}$  which is equivalent to (21) since  $|\widehat{\gamma}|_{A^\circ} = -|\widehat{\gamma}|_A$ ,  $|\widehat{s}^{p,q}|_{B^\circ} = -|\widehat{s}^{p,q}|_B$  and  $|\widehat{t}^{p,q}|_{C^\circ} = -|\widehat{t}^{p,q}|_C$ .

We can therefore rewrite Theorem 4.3, by requiring that (21) holds for all strict subsets  $A \subset [2n]$ ,  $B, C \subset [n]$ , which satisfy  $\sharp A = \sharp B + \sharp C \leq n$  and  $c_{\mu(B), \mu(C)}^{\mu(A)} \neq 0$ .

## 5 The cone $\mathcal{E}(p, q)$

Recall that  $\mathcal{E}(p, q) = \{(\gamma, s) \in \mathbb{R}_{++}^n \times \mathbb{R}_{++}^q, (\gamma, s, s) \in \mathcal{S}(p, q)\}$ . Theorem 5.2 gives a description of the cone  $\mathcal{E}(p, q)$  in terms of a finite set of inequalities:  $\mathcal{E}(p, q) = \{(\gamma, s) \in \mathbb{R}_{++}^n \times \mathbb{R}_{++}^q, (\widehat{\gamma}, \widehat{s}^{p,q}, \widehat{s}^{p,q}) \in LR(n, n)\}$ . In the next section, we explain how to improve this result.

### 5.1 A subcone of $LR(n, n)$

We start by defining the following subcone of  $LR(n, n)$ :

$$\mathcal{E}(n) := \{(x, y) \in \mathbb{R}_+^{2n} \times \mathbb{R}_+^n, (x, y, y) \in LR(n, n)\}.$$

By definition,  $(x, y) \in \mathcal{E}(n)$  if there exists an  $2n$ -square Hermitian matrix of the form

$$X = \begin{pmatrix} X_{11} & Z \\ Z^* & X_{22} \end{pmatrix} \quad \text{with} \quad e(X) = x \quad \text{and} \quad e(X_{11}) = e(X_{22}) = y.$$

Theorem 2.8 gives the following description:  $(x, y) \in \mathbb{R}_+^{2n} \times \mathbb{R}_+^n$  belongs to  $\mathcal{E}(n)$  if and only if the following conditions holds:

1.  $|x| = 2|y|$ ,
2.  $x_{n+k} \leq y_k \leq x_k, \forall k \in [n]$ ,
3.  $|x|_A \geq 2|y|_B + 2|y|_C$  and  $|x|_{A^o} \leq 2|y|_{B^o} + 2|y|_{C^o}$ , for any triplet  $A, B, C$  satisfying:  $B, C \subset [n]$  and  $A \subset [2n]$  are strict subsets such that  $\#A = \#B + \#C \leq n$ , and  $c_{\mu(B), \mu(C)}^{\mu(A)}$  is non-zero.

In fact, condition 2. is not necessary and for the inequalities  $|x|_A \geq |y|_B + |y|_C$  it is enough to restrict to those where  $B = C$ . The following refinement is proved in [20].

**Theorem 5.1**  $(x, y) \in \mathbb{R}_+^{2n} \times \mathbb{R}_+^n$  belongs to  $\mathcal{E}(n)$  if and only if  $|x| = 2|y|$  and if the following relations holds

$$|x|_A \geq 2|y|_B \quad \text{and} \quad |x|_{A^o} \leq 2|y|_{B^o}$$

for any couple  $A, B$  satisfying:

- $A \subset [2n]$  and  $B \subset [n]$  satisfy  $\#A = 2\#B \leq n$
- $(\mu(A), \mu(B)) \in \mathcal{E}(r)$  for  $r = \#B \leq \frac{n}{2}$ .

## 5.2 Description of $\mathcal{E}(p, q)$

Thanks to Theorem 5.1, we obtain the following description of the cone  $\mathcal{E}(p, q)$ .

**Theorem 5.2** An element  $(\gamma, s) \in \mathbb{R}_{++}^n \times \mathbb{R}_{++}^q$  belongs to  $\mathcal{E}(p, q)$  if and only if

$$|\gamma|_{A \cap [n]} - |\gamma|_{A^o \cap [n]} \geq 2(|s|_{B \cap [q]} - |s|_{B^o \cap [q]})$$

holds for any couple  $(A, B)$  satisfying the following conditions :

- $A \subset [2n]$  and  $B \subset [n]$  satisfy  $\#A = 2\#B \leq n$
- the Littlewood-Richardson coefficient  $c_{\mu(B), \mu(B)}^{\mu(A)}$  is non-zero.

## 5.3 Description of $\mathcal{E}(p, q)$ proposed by Fomin-Fulton-Li-Poon

In [6], the authors work with a cone FFLP( $p, q$ ) that is defined as follows. If  $I = \{i_1, \dots, i_r\}$  is an increasing sequence of positive integers we define  $2I = \{2i_1, \dots, 2i_r\}$  and  $2I - 1 = \{2i_1 - 1, \dots, 2i_r - 1\}$ .

**Definition 5.3** FFLP( $p, q$ ) is the set of couple  $(\gamma, s) \in \mathbb{R}_{++}^n \times \mathbb{R}_{++}^q$  satisfying

$$|\gamma|_{2I} + |\gamma|_{2J-1} \geq 2|s|_K,$$

for any  $r \leq q$  and any  $(I, J, K) \in \text{LR}_r^q$ .

The following property is proved in [6].

**Proposition 5.4** *We have  $\mathcal{E}(p, q) \subset \text{FFLP}(p, q)$ .*

*Proof:* We recall the arguments given in [6]. If  $I$  and  $J$  are two subsets of  $\mathbb{N} \setminus \{0\}$  of the same cardinal  $r \geq 1$ , we define  $\tau(I, J) = 2I \cup 2J - 1$ . We will use the following property (see [6], Proposition 1.19):

$$(22) \quad (I, J, K) \in \text{LR}_r^n \implies (\tau(I, J), \tau(I, J), \tau(K, K)) \in \text{LR}_{2r}^{2n}.$$

If  $s \in \mathbb{R}_{++}^q$ , let us write  $d(s) = (s_1, s_1, \dots, s_q, s_q, \underbrace{0, \dots, 0}_{p-q \text{ times}}) \in \mathbb{R}_{++}^n$ . We consider the following subsets of  $\mathbb{R}_{++}^n \times \mathbb{R}_{++}^q$ :

- $\mathfrak{A}_1 := \left\{ (\gamma, s); (\widehat{\gamma}, \widehat{\gamma}, 2\widehat{d}(s)) \in \text{Horn}(2n) \right\}$
- $\mathfrak{A}_2 := \left\{ (\gamma, s); |\widehat{\gamma}|_A \geq |\widehat{d}(s)|_B, \forall (A, A, B) \in \text{LR}_r^{2n}, \forall r < 2n \right\}$
- $\mathfrak{A}_3 := \left\{ (\gamma, s); |\widehat{\gamma}|_A \geq |\widehat{d}(s)|_B, \forall (A, A, B) \in \text{LR}_{2r}^{2n}, \forall r < n \right\}$
- $\mathfrak{A}_4 := \left\{ (\gamma, s); |\widehat{\gamma}|_{\tau(I, J)} \geq |\widehat{d}(s)|_{\tau(K, K)}, \forall (I, J, K) \in \text{LR}_r^n, \forall r < n \right\}$
- $\mathfrak{A}_5 := \left\{ (\gamma, s); |\widehat{\gamma}|_{\tau(I, J)} \geq |\widehat{d}(s)|_{\tau(K, K)}, \forall (I, J, K) \in \text{LR}_r^q, \forall r \leq q \right\}$

The proof of the proposition follows from the next lemma.

**Lemma 5.5** *The following relations hold*

$$\mathcal{E}(p, q) \subset \mathfrak{A}_1 \subset \mathfrak{A}_2 \subset \mathfrak{A}_3 \subset \mathfrak{A}_4 \subset \mathfrak{A}_5 = \text{FFLP}(p, q).$$

*Proof of the Lemma :* The inclusions  $\mathfrak{A}_2 \subset \mathfrak{A}_3$  and  $\mathfrak{A}_4 \subset \mathfrak{A}_5$  are immediate,  $\mathfrak{A}_1 \subset \mathfrak{A}_2$  follows from Theorem 2.2,  $\mathfrak{A}_3 \subset \mathfrak{A}_4$  is due to (22), and  $\mathfrak{A}_5 = \text{FFLP}(p, q)$  follows from the fact that the relation  $|\widehat{\gamma}|_{\tau(I, J)} \geq |\widehat{d}(s)|_{\tau(K, K)}$  is equivalent to  $|\gamma|_{2I} + |\gamma|_{2J-1} \geq 2|s|_K$ , when  $I, J, K \subset [q]$ .

The inclusion  $\mathcal{E}(p, q) \subset \mathfrak{A}_1$  is obtained through the following construction. Let  $(\gamma, s) \in \mathcal{E}(p, q)$ : there exists a  $n$ -square complex matrix of the form  $M = \begin{pmatrix} Z & X \\ Y & T \end{pmatrix}$ , with  $X \in M_{p,q}(\mathbb{C})$ ,  $Y \in M_{p,q}(\mathbb{C})$  satisfying the relations  $s(M) = \gamma$  and  $s(X) = s(Y) = s$ .

Let us define the matrices  $N = \begin{pmatrix} -Z & X \\ Y & -T \end{pmatrix}$  and  $P = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}$ : we see that  $M + N = 2P$  with  $s(N) = \gamma$  and  $s(P) = d(s)$ . Finally, the relation  $\widehat{M} + \widehat{N} = 2\widehat{P}$  of  $2n$ -square hermitians matrices shows that  $(\widehat{\gamma}, \widehat{\gamma}, 2\widehat{d}(s)) \in \text{Horn}(2n)$  since  $e(\widehat{M}) = e(\widehat{N}) = \widehat{\gamma}$  and  $e(\widehat{P}) = \widehat{d}(s)$ .  $\square$

In [6], the authors claim that  $\mathcal{E}(p, q) = \text{FFLP}(p, q)$  by showing that the inverse inclusion  $\text{FFLP}(p, q) \subset \mathcal{E}(p, q)$  is always true. But their proof of the latter inclusion is not correct because they use their erroneous description of  $\mathcal{A}(p, q)$  to obtain it (see [6], §1.5). Note also that the equality  $\mathcal{E}(p, q) = \text{FFLP}(p, q)$  would imply the identities  $\mathcal{E}(p, q) = \mathfrak{A}_1 = \mathfrak{A}_2 = \mathfrak{A}_3 = \mathfrak{A}_4 = \mathfrak{A}_5$ , which are far from obvious.

In the previous sections, we have proven that  $\mathcal{E}(p, q)$  is equal to

$$\text{PEP}(p, q) := \{(\gamma, s) \in \mathbb{R}_{++}^n \times \mathbb{R}_{++}^q; (\widehat{\gamma}, \widehat{s}^{p,q}, \widehat{s}^{p,q}) \in LR(n, n)\}.$$

Let us finish this section with the following remark.

**Lemma 5.6** *We have  $\text{PEP}(p, q) \subset \mathfrak{A}_1$ .*

*Proof:* We follow the same strategy as for the inclusion proof  $\mathcal{E}(p, q) \subset \mathfrak{A}_1$ . Let  $(\gamma, s) \in \text{PEP}(p, q)$ : there exists a  $2n$ -square Hermitian matrix of the form  $M = \begin{pmatrix} X & Z \\ Z^* & Y \end{pmatrix}$ , with  $X, Y \in \text{Herm}(n)$  satisfying the relations  $e(M) = \widehat{\gamma}$  and  $e(X) = e(Y) = \widehat{s}^{p,q}$ .

Now, we define the matrices  $N = \begin{pmatrix} X & -Z \\ -Z^* & Y \end{pmatrix}$  and  $P = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$ : we see that  $N + M = 2P$  with  $e(N) = e(M) = \widehat{\gamma}$  and  $e(P) = \widehat{d}(s)$ . Finally, we have shown that  $(\widehat{\gamma}, \widehat{\gamma}, 2\widehat{d}(s)) \in \text{Horn}(2n)$ . Hence,  $(\gamma, s) \in \mathfrak{A}_1$ .  $\square$

By this way, we can conclude that  $\text{PEP}(p, q) \subset \text{FFLP}(p, q)$ , but the question of whether these cones are equal remains open.

## 6 The cone $\mathcal{T}(p, q)$

We work with the real reductive Lie groups  $G := U(p, p) \times U(q, q)$  and  $\widetilde{G} := U(n, n)$ . The embedding  $\iota : G \rightarrow \widetilde{G}$  is defined as follows:

$$(23) \quad \iota(g, h) = \begin{pmatrix} g_{11} & 0_{p,2q} & g_{12} \\ 0_{2q,p} & h & 0_{2q,p} \\ g_{21} & 0_{p,2q} & g_{22} \end{pmatrix}, \quad \text{when} \quad g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}.$$

Here  $g_{ij} \in M_{p,p}(\mathbb{C})$  and  $h \in U(q, q) \subset M_{2q,2q}(\mathbb{C})$ .

The unitary group  $\widetilde{K} := U_n \times U_n$  is a maximal compact subgroup of  $\widetilde{G}$ . The subspace  $\widetilde{\mathfrak{p}} := \{\widehat{X}, X \in M_{n,n}(\mathbb{C})\} \subset \widetilde{\mathfrak{g}}$  admits a canonical action of  $\widetilde{K}$ . The subgroup  $K = K_1 \times K_2$ , with  $K_1 \simeq U_p \times U_p$  and  $K_2 \simeq U_q \times U_q$ , is a maximal compact subgroup of  $G$ , and the subspace  $\mathfrak{p} = \mathfrak{p}_1 \times \mathfrak{p}_2$  admits a natural identification with  $M_{p,p}(\mathbb{C}) \times M_{q,q}(\mathbb{C})$ :

$$(Y, Z) \in M_{p,p}(\mathbb{C}) \times M_{q,q}(\mathbb{C}) \longmapsto (\widehat{Y}, \widehat{Z}) \in \mathfrak{p}_1 \times \mathfrak{p}_2.$$

See §2.2 for the notations  $\widehat{X}$ .

## 6.1 Complexification, antiholomorphic involution and orthogonal projection

The complexification of the group  $G$  is  $G_{\mathbb{C}} := GL_{2p}(\mathbb{C}) \times GL_{2q}(\mathbb{C})$ . We consider the antiholomorphic involution  $\sigma$  on  $G_{\mathbb{C}}$  defined by  $\sigma(g, h) = (I_{p,p}(g^*)^{-1}I_{p,p}, I_{q,q}(h^*)^{-1}I_{q,q})$ , where  $I_{\ell,\ell} = \text{Diag}(Id_{\ell}, -Id_{\ell})$ . The subgroup  $G$  is the fixed point set of  $\sigma$ .

The complexification of the group  $\tilde{G}$  is  $\tilde{G}_{\mathbb{C}} := GL_{2n}(\mathbb{C})$ . We consider the antiholomorphic involution  $\tilde{\sigma}$  on  $\tilde{G}_{\mathbb{C}}$  defined by  $\tilde{\sigma}(g) = I_{n,n}(g^*)^{-1}I_{n,n}$ . The subgroup  $\tilde{G}$  corresponds to the fixed point set of  $\tilde{\sigma}$ .

The groups  $U = U_{2p} \times U_{2q}$  and  $\tilde{U} = U_{2n}$  are respectively maximal compact subgroups of  $G_{\mathbb{C}}$  and  $\tilde{G}_{\mathbb{C}}$ . The fixed point subgroups of the involutions are  $U^{\sigma} = K$  and  $\tilde{U}^{\tilde{\sigma}} = \tilde{K}$ .

The embedding  $\iota : G \hookrightarrow \tilde{G}$  admits a complexification  $\iota_{\mathbb{C}} : G_{\mathbb{C}} \hookrightarrow \tilde{G}_{\mathbb{C}}$ . At the level of Lie algebra, we have a morphism  $\iota_{\mathbb{C}} : \mathfrak{gl}_{2p}(\mathbb{C}) \times \mathfrak{gl}_{2q}(\mathbb{C}) \hookrightarrow \mathfrak{gl}_{2n}(\mathbb{C})$ , still defined by (23).

The orthogonal projection  $\pi : \mathfrak{gl}_{2n}(\mathbb{C}) \longrightarrow \mathfrak{gl}_{2p}(\mathbb{C}) \times \mathfrak{gl}_{2q}(\mathbb{C})$  dual to the morphism  $\iota_{\mathbb{C}}$  is defined by the relations :

$$(24) \quad A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \quad \longmapsto \quad \pi(A) = \left( \begin{pmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{pmatrix}, A_{22} \right).$$

Here the matrix  $A \in \mathfrak{gl}_{2n}(\mathbb{C})$  is written by blocs relatively to the decomposition  $2n = p + 2q + p$ .

Let us consider another projection  $\pi_0 : \mathfrak{gl}_{2n}(\mathbb{C}) \longrightarrow \mathfrak{gl}_{2p}(\mathbb{C}) \times \mathfrak{gl}_{2q}(\mathbb{C})$  defined by the relations :

$$(25) \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \quad \longmapsto \quad \pi_0(B) = (B_{11}, B_{22}).$$

Here  $B_{11} \in \mathfrak{gl}_{2p,2p}(\mathbb{C})$  and  $B_{22} \in \mathfrak{gl}_{2q,2q}(\mathbb{C})$

**Lemma 6.1** *For any  $U_{2n}$ -orbit  $\mathcal{O} \subset \mathfrak{gl}_{2n}(\mathbb{C})$ , we have  $\pi(\mathcal{O}) = \pi_0(\mathcal{O})$ .*

*Proof:* Same type of proof as for the lemma 4.1.  $\square$

## 6.2 Description of $\mathcal{T}(p, q)$ through $LR(2p, 2q)$

Here is the main application of the Proposition 2.14. Recall that for  $(\gamma, s, t) \in \mathbb{R}_{++}^n \times \mathbb{R}_{++}^p \times \mathbb{R}_{++}^q$ , we define  $\hat{\gamma} := (\gamma_1, \dots, \gamma_n, -\gamma_n, \dots, -\gamma_1)$ ,  $\hat{s} := (s_1, \dots, s_p, -s_p, \dots, -s_1)$  and  $\hat{t} := (t_1, \dots, t_q, -t_q, \dots, -t_1)$

**Proposition 6.2** *Let  $(\gamma, s, t) \in \mathbb{R}_{++}^n \times \mathbb{R}_{++}^p \times \mathbb{R}_{++}^q$ . The following statements are equivalent:*

1.  $(\gamma, s, t) \in \mathcal{T}(p, q)$ ,

2.  $\exists A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in M_{n,n}(\mathbb{C})$ , such that  $s(A) = \gamma$ ,  $s(A_{11}) = s$ , and  $s(A_{22}) = t$ ,
3.  $\exists X \in \tilde{\mathfrak{p}}$ , with  $\pi(X) = (X_1, X_2)$ , s.t.  $e(X) = \hat{\gamma}$ ,  $e(X_1) = \hat{s}$  and  $e(X_2) = \hat{t}$ ,
4.  $\pi \left( U(n) \times U(n) \cdot \widehat{\text{Diag}}(\gamma) \right)$  contains  $\left( U(p) \times U(p) \cdot \widehat{\text{Diag}}(s) \right) \times \left( U(q) \times U(q) \cdot \widehat{\text{Diag}}(t) \right)$ .
5.  $\pi \left( U(2n) \cdot \widehat{\text{Diag}}(\gamma) \right)$  contains  $U(2p) \cdot \widehat{\text{Diag}}(s) \times U(2q) \cdot \widehat{\text{Diag}}(t)$ .
6.  $\pi_0 \left( U(2n) \cdot \widehat{\text{Diag}}(\hat{\gamma}) \right)$  contains  $U(2p) \cdot \widehat{\text{Diag}}(\hat{s}) \times U(2q) \cdot \widehat{\text{Diag}}(\hat{t})$ .
7.  $(\hat{\gamma}, \hat{s}, \hat{t}) \in LR(2p, 2q)$ .

*Proof:* 4.  $\iff$  5. follows from Proposition 2.14, and 5.  $\iff$  6. is a consequence of Lemma 4.1. The other equivalences are left to the reader.  $\square$

### 6.3 Inequalities determining $\mathcal{T}(p, q)$

Thanks to Proposition 6.2 and Theorem 2.8, we obtain the following description of the cone  $\mathcal{T}(p, q)$ .

**Theorem 6.3** *An element  $(\gamma, s, t) \in \mathbb{R}_{++}^n \times \mathbb{R}_{++}^p \times \mathbb{R}_{++}^q$  belongs to  $\mathcal{T}(p, q)$  if and only if the following inequalities hold:*

1.  $\gamma_k \geq s_k, \forall k \leq p$ ,
2.  $\gamma_j \geq t_j, \forall j \leq q$ ,
3.  $\gamma_{2q+\ell} \leq s_\ell, \forall \ell \leq p - q$ ,
4. we have  $|\gamma|_{A \cap [n]} - |\gamma|_{A^\circ \cap [n]} \geq |s|_{B \cap [p]} - |s|_{B^\circ \cap [p]} + |t|_{C \cap [q]} - |t|_{C^\circ \cap [q]}$

for any triplets  $(A, B, C)$  satisfying the following conditions :

- $B \subset [2p]$  and  $C \subset [2q]$  are strict subsets,
- $A \subset [2n]$  and  $\sharp A = \sharp B + \sharp C \leq n$ ,
- the Littlewood-Richardson coefficient  $c_{\mu(B), \mu(C)}^{\mu(A)}$  is non-zero.

### 6.4 Interlacing inequalities for singular values

Let us consider the case where  $p \geq q = 1$ .

Let  $\gamma_1 \geq \dots \geq \gamma_{p+1} \geq 0$  be the singular values of a  $p + 1$ -square complex matrix  $X$ . Let  $X'$  be the  $p$ -square submatrix of  $X$  obtained by deleting a row and a column: we denote by  $s_1 \geq \dots \geq s_p \geq 0$  its singular spectrum.

Points 1. and 3. of Theorem 6.3 yields interlacing inequalities which were first observed by Thompson [23]:

$$\begin{aligned}
& \gamma_3 \leq s_1 \leq \gamma_1, \\
& \gamma_4 \leq s_2 \leq \gamma_2, \\
& \dots \\
& \gamma_{j+2} \leq s_j \leq \gamma_j, \quad 1 \leq j \leq p-2, \\
& \dots \\
& \gamma_p \leq s_{p-2} \leq \gamma_{p-2}, \\
& \quad s_{p-1} \leq \gamma_{p-1}, \\
& \quad s_p \leq \gamma_p.
\end{aligned}$$

## References

- [1] P. Belkale : Geometric proofs of Horn and saturation conjectures. *J. Algebraic Geom.* **15**, 133–173 (2006).
- [2] P. Belkale and S. Kumar : Eigenvalue problem and a new product in cohomology of flag varieties. *Invent. Math.* **166**, 185–228 (2006).
- [3] P. Belkale and S. Kumar : Eigencone, saturation and Horn problems for symplectic and odd orthogonal groups. *Journal of Algebraic Geometry* **19**, 199–242 (2010).
- [4] A. Berenstein and R. Sjamaar : Coadjoint orbits, moment polytopes, and the Hilbert-Mumford criterion. *Journal of the American Mathematical Society* **13**, 433–466 (2000).
- [5] M. Brion : Restriction de représentations et projections d’orbites coadjointes. *Séminaire Bourbaki*, (2012).
- [6] S. Fomin, W. Fulton, C.-K. Li and Y.-T. Poon : Eigenvalues, singular values, and Littlewood-Richardson coefficients. *American Journal of Mathematics* **127**, 101–127 (2005).
- [7] W. Fulton : Eigenvalues, invariant factors, highest weights, and Schubert calculus. *Bulletin of the American Mathematical Society* **37**, 209–249 (2000).
- [8] A. Horn : Eigenvalues of sums of Hermitian matrices. *Pacific J. Math.* **12**, 225–241 (1962).
- [9] M. Kapovich B. Leeb and J. Millson : The generalized triangle inequalities in symmetric spaces and buildings with applications to algebra, *Mem. Amer. Math. Soc.* **192** , no. 896, (2008).
- [10] A. Klyachko : Stable bundles, representation theory and Hermitian operators, *Selecta Mathematica* **4**, 419–445 (1998).
- [11] S. Kumar : A survey of the additive eigenvalue problem (with appendix by M. Kapovich), *Transformation Groups* **19**, 1051–1148 (2014).

- [12] A.W. Knapp : Lie groups beyond an introduction, Progress in Math. **140**, Birkhäuser, Springer (2004).
- [13] A. Knutson and T. Tao : The honeycomb model of  $GL_n(\mathbb{C})$  tensor products I : Proof of the saturation conjecture. Journal of the A.M.S. **12**, 1055–1090 (1999).
- [14] A. Knutson, T. Tao and C. Woodward : The honeycomb model of  $GL_n(\mathbb{C})$  tensor products II: Puzzles determine facets of the Littlewood-Richardson cone. Journal of the A.M.S. **17**, 19–48 (2004).
- [15] C.-K. Li and Y.-T. Poon : Off-diagonal submatrices of a Hermitian matrix. Proceedings of the American Mathematical Society **132**, 2849–2856 (2004).
- [16] I. G. Macdonald : Symmetric functions and Hall polynomials. Oxford university press, 1998.
- [17] L. Manivel : Fonctions symétriques, polynômes de Schubert et lieux de dégénérescence. Cours spécialisés **3**, S.M.F. (1998).
- [18] L. O’Shea and R. Sjamaar : Moment maps and Riemannian symmetric pairs. Mathematische Annalen **317**, 415–457 (2000).
- [19] P.-E. Paradan : Moment polytopes in real symplectic geometry II: applications to singular value inequalities. arXiv preprint arXiv:2111.13399.
- [20] P.-E. Paradan : Moment polytopes in real symplectic geometry III: examples. *in preparation*
- [21] N. Ressayre : Geometric invariant theory and the generalized eigenvalue problem. Inventiones mathematicae **180**, 389–441 (2010).
- [22] N. Ressayre : Geometric invariant theory and generalized eigenvalue problem II. Annales de L’institut Fourier **61**, 1467–1491 (2011).
- [23] R. C. Thompson : Principal submatrices IX: Interlacing inequalities for singular values of submatrices. Linear Algebra and its Applications **5**, 1–12 (1972).
- [24] R. C. Thompson : Singular value inequalities for matrix sums and minors, Linear Algebra and its Applications **11**, 251–269 (1975).