



Eigenvalues, singular values, and the O'Shea-Sjamaar Theorem

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► To cite this version:

| Paul-Emile Paradan. Eigenvalues, singular values, and the O'Shea-Sjamaar Theorem. 2023. <hal-04037272v2>

HAL Id: hal-04037272

<https://hal.science/hal-04037272v2>

Preprint submitted on 13 Nov 2023

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Eigenvalues, singular values, and the O'Shea-Sjamaar Theorem

Paul-Emile Paradan*

November 13, 2023

Abstract

The main focus of this work is the study of several cones relating the eigenvalues or singular values of a matrix to those of its off-diagonal blocks.

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1 Introduction

Let $p \geq q \geq 1$ and $n = p + q$. Let $\text{Herm}(n)$ denote the vector space of n -square Hermitian matrices. The spectrum of $X \in \text{Herm}(n)$ is denoted by $\text{e}(X) = (\text{e}_1 \geq \cdots \geq \text{e}_n)$ and the singular spectrum of a matrix $Y \in M_{p,q}(\mathbb{C})$ is denoted by $\text{s}(Y) = (\text{s}_1 \geq \cdots \geq \text{s}_q \geq 0)$.

The main purpose of this article is to describe the following cones:

$$\begin{aligned}\mathcal{A}(p, q) &= \left\{ (\text{e}(X), \text{s}(X_{12})), X \in \text{Herm}(n) \right\}, \\ \mathcal{S}(p, q) &= \left\{ (\text{s}(X), \text{s}(X_{12}), \text{s}(X_{21})), X \in M_{n,n}(\mathbb{C}) \right\}, \\ \mathcal{T}(p, q) &= \left\{ (\text{s}(X), \text{s}(X_{11}), \text{s}(X_{22})), X \in M_{n,n}(\mathbb{C}) \right\}.\end{aligned}$$

Here, a n -square complex matrix X is written by blocks $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ where $X_{12} \in M_{p,q}(\mathbb{C})$ and $X_{21} \in M_{q,p}(\mathbb{C})$.

In the 1970s, Thompson gave some inequalities satisfied by the elements of $\mathcal{T}(p, q)$ [22, 23], and more recently Fomin, Fulton, Li and Poon obtained sets of inequalities that describe the cone $\mathcal{A}(p, q)$ [14, 5].

The main objective of this work is to explain how a direct application of O'Shea-Sjamaar's theorem [17] yields complete sets of inequalities for the cones $\mathcal{A}(p, q)$, $\mathcal{S}(p, q)$, and $\mathcal{T}(p, q)$. However, this method does not provide an optimal description of these cones, as it leads to a large number of redundancies in the list of inequalities. We'll see, for example, that the Fomin-Fulton-Li-Poon description of $\mathcal{A}(p, q)$ is more accurate than ours. In a future work [19], we will propose a more precise method to describe these inequality sets, using the main result of [18].

Throughout this article, we make extensive use of Horn cones $\text{Horn}(n)$ and Littlewood-Richardson cones $\text{LR}(m, n)$. Let's recall their definition. To any integers $n, m \geq 1$, we associate

$$\begin{aligned}\text{Horn}(n) &= \left\{ (e(X), e(Y), e(X+Y)), X, Y \in \text{Herm}(n) \right\} \\ \text{LR}(m, n) &= \left\{ (e(M), e(M_{\mathbf{I}}), e(M_{\mathbf{II}})), M \in \text{Herm}(m+n) \right\},\end{aligned}$$

where $M_{\mathbf{I}} \in \text{Herm}(m)$ and $M_{\mathbf{II}} \in \text{Herm}(n)$ are the extracted matrices such that $M = \begin{pmatrix} M_{\mathbf{I}} & * \\ * & M_{\mathbf{II}} \end{pmatrix}$.

In §2, we recall the description obtained by Klyachko [9] and Knutson-Tao [12] for $\text{Horn}(n)$ cones, and that obtained by Berenstein-Sjamaar [3] and Ressayre [20] of $\text{LR}(m, n)$ cones. In both cases, the inequalities are parameterized using Littlewood-Richardson coefficients.

In §3, we show that $\mathcal{A}(p, q)$ can be characterized as a sub-cone of¹ $\text{Horn}(n)$. To any $\lambda = (\lambda_1 \geq \dots \geq \lambda_n)$ and $s = (s_1 \geq \dots \geq s_q \geq 0)$, we associate $\lambda^* = (-\lambda_n \geq \dots \geq -\lambda_1)$ and

$$\widehat{s}^{p,q} := (s_1 \geq \dots \geq s_q \geq \underbrace{0, \dots, 0}_{p-q} \geq -s_q \geq \dots \geq -s_1).$$

We then show that $(\lambda, s) \in \mathcal{A}(p, q)$ if and only if $(\lambda, \lambda^*, 2\widehat{s}^{p,q}) \in \text{Horn}(n)$. This makes it possible to describe $\mathcal{A}(p, q)$ by means of the inequalities defining $\text{Horn}(n)$, but we'll see that the resulting description is less precise than that given by Fomin-Fulton-Li-Poon in [5].

In [5], the authors pose the question of finding a collection of linear inequalities that describes $\mathcal{S}(p, q)$ (Problem 1.15). We answer this problem in §4 by showing that $\mathcal{S}(p, q)$ can be characterized as the intersection of $\text{LR}(n, n)$ with the subspace of $\mathbb{R}^{2n} \times \mathbb{R}^n \times \mathbb{R}^n$ formed by the elements $(\widehat{\gamma}^{n,n}, \widehat{s}^{p,q}, \widehat{t}^{p,q})$ where $(\gamma, s, t) \in \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^q$.

In the last section, we give a set of inequalities describing the cone $\mathcal{T}(p, q)$, showing that $\mathcal{T}(p, q)$ is characterized as the intersection of $\text{LR}(2p, 2q)$ with the subspace of $\mathbb{R}^{2n} \times \mathbb{R}^{2p} \times \mathbb{R}^{2q}$ formed by the elements $(\widehat{\gamma}^{n,n}, \widehat{s}^{p,p}, \widehat{t}^{q,q})$ where $(\gamma, s, t) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q$.

In the case $\mathcal{T}(p, 1)$, we recover the interleaving inequalities of singular values obtained by Thompson [22].

Acknowledgement

I would like to thank Nicolas Ressayre for the interesting discussions we had on this subject. I'm also grateful to a group of young students, Martina Agüera Sanchez and Ariane & Constantin Paradan, who implemented A.S. Buch's "Littlewood-Richardson calculator" in various programs, enabling me to calculate a few examples. I would also like to thank the referees for their comments, which helped me improve this text.

¹Here $n = p + q$.

Notations

Throughout the paper :

- We fix $p \geq q \geq 1$ and $n = p + q$.
- We write 0_{ab} for the zero matrix of size $a \times b$.
- Let $M_{a,b}(\mathbb{C})$ be the vector space of complex $a \times b$ matrices.
- \mathbb{R}_+^ℓ is the set of sequences $x = (x_1 \geq \dots \geq x_\ell)$ of real numbers.
- \mathbb{R}_{++}^ℓ is the set of sequences $x = (x_1 \geq \dots \geq x_\ell \geq 0)$ of non-negative real numbers.
- For any positive integer ℓ , let $[\ell]$ be the set $\{1, \dots, \ell\}$.
- If $x \in \mathbb{R}^\ell$ and $A \subset [\ell]$, we write $|x|_A = \sum_{a \in A} x_a$ and $|x| = \sum_{i=1}^\ell x_i$.
- For $A \subset [\ell]$, we define $A^o := \{\ell + 1 - a, a \in A\}$ and $A^c := [\ell] \setminus A$.
- If $x \in \mathbb{R}^\ell$, let $\text{Diag}(x)$ be the diagonal $\ell \times \ell$ matrix with diagonal entries equal to x_1, \dots, x_ℓ .
- If $A = \{a_1 < \dots < a_p\}$ is an increasing sequence of positive integers, let $\mu(A) = (a_p - p \geq \dots \geq a_1 - 1 \geq 0)$.

2 Reminder of some classical results

We recall some classical facts that we'll be needing later on.

2.1 Singular values

Let X be a rectangular matrix, say $m \times n$, with complex entries, and let X^* denote the complex conjugate transpose of X . Let $\eta_1(X) \geq \dots \geq \eta_m(X) \geq 0$ be the eigenvalues of the positive semidefinite matrix XX^* . Notice that $\eta_k(X) = 0$ when $k > \ell := \inf\{m, n\}$.

The singular values of the matrix X are the coordinates of the vector

$$s(X) := \left(\sqrt{\eta_1(X)}, \dots, \sqrt{\eta_\ell(X)} \right) \in \mathbb{R}_{++}^\ell.$$

Consider the canonical action of the unitary group $U_m \times U_n$ on $M_{m,n}(\mathbb{C})$: $(g, h) \cdot X = gXh^{-1}$, $\forall (g, h) \in U_m \times U_n$. The singular values map $s : M_{m,n}(\mathbb{C}) \rightarrow \mathbb{R}_{++}^\ell$ induces a bijective application $M_{m,n}(\mathbb{C})/U_m \times U_n \xrightarrow{\sim} \mathbb{R}_{++}^\ell$.

2.2 Augmented matrices

Let $p \geq q \geq 1$.

If Y is a $p \times q$ matrix, we denote by $\hat{Y}^{p,q}$ the n -square Hermitian matrix $\begin{pmatrix} 0_{pp} & Y \\ Y^* & 0_{qq} \end{pmatrix}$.

Here the spectrum of $\hat{Y}^{p,q}$ is equal to $\hat{s}^{p,q} := (s_1, \dots, s_q, 0, \dots, 0, -s_q, \dots, -s_1) \in \mathbb{R}_+^n$, where $s \in \mathbb{R}_{++}^q$ is the singular spectrum of Y .

If Z is a $q \times p$ matrix, we denote by $\hat{Y}^{q,p}$ the n -square Hermitian matrix $\begin{pmatrix} 0_{qq} & Z \\ Z^* & 0_{pp} \end{pmatrix}$.

Here the spectrum of $\hat{Z}^{q,p}$ is also equal to $\hat{t}^{p,q}$, where $t = s(Z) \in \mathbb{R}_{++}^q$.

If X is a k -square matrix, we simply denote $\hat{X}^{k,k}$ by \hat{X} . The spectrum of \hat{X} is also simply denoted $\hat{\mu} := (\mu_1, \dots, \mu_k, -\mu_k, \dots, -\mu_1) \in \mathbb{R}_+^{2k}$, where $\mu = s(X)$.

2.3 Horn inequalities

Denote the set of cardinality r -subsets $I = \{i_1 < i_2 < \dots < i_r\}$ of $[n]$ by \mathcal{P}_r^n .

Definition 2.1 For any $1 \leq r < n$, LR_r^n refers to the set of triplets $(I, J, K) \in (\mathcal{P}_r^n)^3$ such that $(\mu(I), \mu(J), \mu(K)) \in \text{Horn}(r)$.

The following theorem was conjectured by Horn [7] and proved by a combination of the works of Klyachko [9] and Knutson-Tao [12].

Theorem 2.2 The triplet $(x, y, z) \in (\mathbb{R}_+^n)^3$ belongs to $\text{Horn}(n)$ if and only if the following conditions hold:

- $|x| + |y| = |z|$,
- $|x|_I + |y|_J \geq |z|_K$, for any $r < n$ and any $(I, J, K) \in \text{LR}_r^n$.

In the following sections, we'll use Littlewood-Richardson coefficients to parameterize certain inequalities. Let's recall their definition. Let λ, μ , and ν be three partitions of length less than $n \geq 1$. We associate them with the irreducible representations V_λ , V_μ and V_ν of the unitary group U_n . The Littlewood-Richardson coefficient $c_{\mu, \nu}^\lambda$ can be characterized by the relation $c_{\mu, \nu}^\lambda = \dim [V_\lambda^* \otimes V_\mu \otimes V_\nu]^{U_n}$. Thanks to the saturation Theorem of Knutson and Tao [12], we know that $c_{\mu, \nu}^\lambda \neq 0 \iff (\mu, \nu, \lambda) \in \text{Horn}(n)$.

The following kind of duality is used in the next sections: for all $(I, J, K) \in (\mathcal{P}_r^n)^3$, we have

$$(1) \quad c_{\mu(I), \mu(J)}^{\mu(K)} \neq 0 \iff c_{\mu((I^o)^c), \mu((J^o)^c)}^{\mu((K^o)^c)} \neq 0.$$

Since the relation $|x|_{(I^o)^c} + |y|_{(J^o)^c} \geq |z|_{(K^o)^c}$ is equivalent to $|x|_{I^o} + |y|_{J^o} \leq |z|_{K^o}$, in Theorem 2.2, we can rewrite the last condition by requiring that

$$|x|_I + |y|_J \geq |z|_K \quad \text{and} \quad |x|_{I^o} + |y|_{J^o} \leq |z|_{K^o}$$

for any $r \leq \frac{n}{2}$ and any $(I, J, K) \in \text{LR}_r^n$.

2.4 The cone $\text{LR}(U, \tilde{U})$

Let $\iota : U \hookrightarrow \tilde{U}$ be two connected compact Lie groups. We choose an invariant scalar product $(-, -)$ on the Lie algebra $\tilde{\mathfrak{u}}$ of \tilde{U} , and we denote by $\pi : \tilde{\mathfrak{u}} \rightarrow \mathfrak{u}$ the orthogonal projection.

Select maximal tori T in U and \tilde{T} in \tilde{U} such that $T \subset \tilde{T}$, and Weyl chambers $\mathfrak{t}_+ \subset \mathfrak{t}$ and $\tilde{\mathfrak{t}}_+ \subset \tilde{\mathfrak{t}}$, where \mathfrak{t} and $\tilde{\mathfrak{t}}$ denote the Lie algebras of T , resp. \tilde{T} . The aim of this section is to recall the description of the following cone given in [3, 20]:

$$\text{LR}(U, \tilde{U}) = \left\{ (\xi, \tilde{\xi}) \in \mathfrak{t}_+ \times \tilde{\mathfrak{t}}_+, U\xi \subset \pi(\tilde{U}\tilde{\xi}) \right\}.$$

Consider the lattice $\Lambda := \frac{1}{2\pi} \ker(\exp : \mathfrak{t} \rightarrow T)$ and the Weyl groups $\tilde{W} = N_{\tilde{U}}(\tilde{T})/\tilde{T}$ and $W = N_U(T)/T$. We denote by $w_o \in W$ the longest element. A vector $\gamma \in \mathfrak{t}$ is called *rational* if it belongs to the \mathbb{Q} -vector space $\mathfrak{t}_{\mathbb{Q}}$ generated by Λ . We will see that the cone $\text{LR}(U, \tilde{U})$ is completely described by inequalities of the form

$$(\tilde{\xi}, \tilde{w}\gamma) \geq (\xi, w_o w\gamma)$$

with γ rational anti-dominant and $(w, \tilde{w}) \in W \times \tilde{W}$.

2.4.1 Admissible elements

We let $\Sigma(\tilde{\mathfrak{u}}/\mathfrak{u}) \subset \mathfrak{t}^*$ denote the set of weights relative to the T -action on $(\tilde{\mathfrak{u}}/\mathfrak{u}) \otimes \mathbb{C}$. If $\gamma \in \mathfrak{t}$, we denote by $\Sigma(\tilde{\mathfrak{u}}/\mathfrak{u}) \cap \gamma^\perp$ the subset of weights vanishing against γ .

Definition 2.3 *A rational element $\gamma \in \mathfrak{t}$ is said admissible when*

$$(2) \quad \text{Vect}(\Sigma(\tilde{\mathfrak{u}}/\mathfrak{u}) \cap \gamma^\perp) = \text{Vect}(\Sigma(\tilde{\mathfrak{u}}/\mathfrak{u})) \cap \gamma^\perp.$$

When the following assumption is satisfied, we'll see in Section §2.4.4 that $\text{LR}(U, \tilde{U})$ is described by inequalities parameterized by a finite number of admissible elements.

Assumption 2.4 *The subspace $\mathfrak{z} := \{X \in \mathfrak{t}, \alpha(X) = 0, \forall \alpha \in \Sigma(\tilde{\mathfrak{u}}/\mathfrak{u})\}$ is contained in the center $Z_{\tilde{\mathfrak{u}}}$ of $\tilde{\mathfrak{u}}$.*

This assumption means that any ideal of $\tilde{\mathfrak{u}}$ contained in \mathfrak{u} is a subspace of $Z_{\tilde{\mathfrak{u}}} \cap \mathfrak{u}$.

Let $\mathfrak{t} = \mathfrak{z} \oplus \mathfrak{t}_1$ be a rational decomposition. Let us denote by $\Sigma(\tilde{\mathfrak{u}}/\mathfrak{u})'$ the image of $\Sigma(\tilde{\mathfrak{u}}/\mathfrak{u})$ through the projection $\mathfrak{t}^* \rightarrow (\mathfrak{t}_1)^*$. If Assumption 2.4 holds, $\Sigma(\tilde{\mathfrak{u}}/\mathfrak{u})'$ generates $(\mathfrak{t}_1)^*$. Any rational element $\gamma \in \mathfrak{t}$ can be written $\gamma = \gamma_0 + \gamma_1$ where γ_0 is a rational element of \mathfrak{z} and γ_1 is a rational element of \mathfrak{t}_1 . We see then that a rational element γ is admissible if and only if γ_1 is admissible. The later condition is equivalent to asking that the hyperplane $(\gamma_1)^\perp \subset (\mathfrak{t}_1)^*$ is generated by a finite subset of $\Sigma(\tilde{\mathfrak{u}}/\mathfrak{u})'$. Thus, there are a finite number of choices for γ_1 (up to multiplication by $\mathbb{Q}^{>0}$).

2.4.2 Polarized trace

Let $\mathfrak{R}(\mathfrak{u})$ and $\mathfrak{R}(\tilde{\mathfrak{u}})$ be the set of roots associated to the Lie algebras \mathfrak{u} and $\tilde{\mathfrak{u}}$. The choice of the Weyl chambers \mathfrak{t}_+ and $\tilde{\mathfrak{t}}_+$ define subsets of positive roots $\mathfrak{R}^+ \subset \mathfrak{R}(\mathfrak{u})$ and $\tilde{\mathfrak{R}}^+ \subset \mathfrak{R}(\tilde{\mathfrak{u}})$.

For a rational element $\gamma \in \mathfrak{t}$ and $(w, \tilde{w}) \in W \times \tilde{W}$, we will use the following condition to parameterize the inequalities of $\text{LR}(U, \tilde{U})$:

$$(3) \quad \sum_{\substack{\alpha \in \mathfrak{R}^+ \\ \langle \alpha, w\gamma \rangle > 0}} \langle \alpha, w\gamma \rangle + \sum_{\substack{\tilde{\alpha} \in \tilde{\mathfrak{R}}^+ \\ \langle \tilde{\alpha}, \tilde{w}\gamma \rangle < 0}} \langle \tilde{\alpha}, \tilde{w}\gamma \rangle = 0$$

2.4.3 Schubert calculus

Let $\iota : U_{\mathbb{C}} \hookrightarrow \tilde{U}_{\mathbb{C}}$ be the complexification of $\iota : U \hookrightarrow \tilde{U}$. To any non-zero rational element $\gamma \in \mathfrak{t}$, we associate the parabolic subgroups

$$(4) \quad \tilde{P}_{\gamma} = \{g \in \tilde{U}_{\mathbb{C}}, \lim_{t \rightarrow \infty} \exp(-it\gamma)g \exp(it\gamma) \text{ exists}\} \quad \text{and} \quad P_{\gamma} = \tilde{P}_{\gamma} \cap U_{\mathbb{C}}.$$

We consider the projective varieties $\mathcal{F}_{\gamma} := U_{\mathbb{C}}/P_{\gamma}$ and $\tilde{\mathcal{F}}_{\gamma} := \tilde{U}_{\mathbb{C}}/\tilde{P}_{\gamma}$, with the canonical embedding $\iota : \mathcal{F}_{\gamma} \hookrightarrow \tilde{\mathcal{F}}_{\gamma}$. Let $B \subset U_{\mathbb{C}}$ (resp. $\tilde{B} \subset \tilde{U}_{\mathbb{C}}$) be the Borel subgroup associated to the choice of the Weyl chamber \mathfrak{t}_+ (resp. $\tilde{\mathfrak{t}}_+$).

We associate to $(w, \tilde{w}) \in W \times \tilde{W}$, the Schubert cells

$$\tilde{\mathfrak{X}}_{\tilde{w}, \gamma}^o := \tilde{B}[\tilde{w}] \subset \tilde{\mathcal{F}}_{\gamma} \quad \text{and} \quad \mathfrak{X}_{w, \gamma}^o := B[w] \subset \mathcal{F}_{\gamma}.$$

The corresponding Schubert varieties are $\tilde{\mathfrak{X}}_{\tilde{w}, \gamma} := \overline{\tilde{\mathfrak{X}}_{\tilde{w}, \gamma}^o}$ and $\mathfrak{X}_{w, \gamma} := \overline{\mathfrak{X}_{w, \gamma}^o}$.

We consider the cohomology² rings $H^*(\tilde{\mathcal{F}}_{\gamma}, \mathbb{Z})$ and $H^*(\mathcal{F}_{\gamma}, \mathbb{Z})$. Let

$$\iota^* : H^*(\tilde{\mathcal{F}}_{\gamma}, \mathbb{Z}) \rightarrow H^*(\mathcal{F}_{\gamma}, \mathbb{Z})$$

be the pull-back map in cohomology. If Y is an irreducible closed subvariety of $\tilde{\mathcal{F}}_{\gamma}$, we denote by $[Y] \in H^{2n_Y}(\tilde{\mathcal{F}}_{\gamma}, \mathbb{Z})$ its cycle class in cohomology : here $n_Y = \text{codim}_{\mathbb{C}}(Y)$. Recall that the cohomology class $[pt]$ associated to a singleton $Y = \{pt\} \subset \mathcal{F}_{\gamma}$ is a basis of $H^{\max}(\mathcal{F}_{\gamma}, \mathbb{Z})$.

In the next section we will consider a rational element $\gamma \in \mathfrak{t}$ and $(w, \tilde{w}) \in W \times \tilde{W}$ satisfying the relation $[\mathfrak{X}_{w, \gamma}] \cdot \iota^*([\tilde{\mathfrak{X}}_{\tilde{w}, \gamma}]) = k[pt]$ in $H^*(\mathcal{F}_{\gamma}, \mathbb{Z})$, with $k \geq 1$. This cohomological condition implies in particular that $\dim_{\mathbb{C}}(\mathfrak{X}_{w, \gamma}) = \text{codim}_{\mathbb{C}}(\tilde{\mathfrak{X}}_{\tilde{w}, \gamma})$ which is equivalent to the relation

$$(5) \quad \#\{\alpha \in \mathfrak{R}^+, \langle \alpha, w\gamma \rangle > 0\} = \#\{\tilde{\alpha} \in \tilde{\mathfrak{R}}^+, \langle \tilde{\alpha}, \tilde{w}\gamma \rangle < 0\}.$$

²Here, we use singular cohomology with integer coefficients.

We finish this section by considering the particular case where $U_{\mathbb{C}} = GL_n(\mathbb{C})$ is embedded diagonally in $\tilde{U}_{\mathbb{C}} = GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$. For $1 \leq r < n$, the vector $\gamma_r = \underbrace{(-1, \dots, -1)}_{r \text{ times}}, 0, \dots, 0) \in \mathbb{R}^n \simeq \mathfrak{t}$ is admissible and the flag manifolds \mathcal{F}_{γ_r} and $\tilde{\mathcal{F}}_{\gamma_r}$ admits a canonical identifications respectively with the Grassmanians $\mathbb{G}(r, n)$ and $\mathbb{G}(r, n) \times \mathbb{G}(r, n)$. The map $\iota : \mathcal{F}_{\gamma} \hookrightarrow \tilde{\mathcal{F}}_{\gamma}$ corresponds to the diagonal embedding $\iota : \mathbb{G}(r, n) \rightarrow \mathbb{G}(r, n) \times \mathbb{G}(r, n)$.

For $w \in W \simeq \mathfrak{S}_n$, the Schubert variety $\mathfrak{X}_{w, \gamma_r} \subset \mathbb{G}(r, n)$, which depends only of the subset $K = w([r]) \subset [n]$, is denoted \mathfrak{X}_K . Similarly, for $w = (w_1, w_2) \in \tilde{W} \simeq \mathfrak{S}_n \times \mathfrak{S}_n$, the Schubert variety $\tilde{\mathfrak{X}}_{\tilde{w}, \gamma_r} \subset \mathbb{G}(r, n) \times \mathbb{G}(r, n)$ is equal to $\mathfrak{X}_I \times \mathfrak{X}_J$, where $I = w_1([r])$ and $J = w_2([r])$.

In this setting, we have the following classical result.

Lemma 2.5 *The following statements are equivalent:*

- $[\mathfrak{X}_{w, \gamma}] \cdot \iota^*([\tilde{\mathfrak{X}}_{\tilde{w}, \gamma}]) = \ell[pt]$ in $H^*(\mathcal{F}_{\gamma}, \mathbb{Z})$, with $\ell \geq 1$.
- $[\mathfrak{X}_I] \cdot [\mathfrak{X}_J] \cdot [\mathfrak{X}_K] = \ell[pt]$ in $H^*(\mathbb{G}(r, n), \mathbb{Z})$, with $\ell \geq 1$.
- The Littlewood-Richardson coefficient $c_{\mu(I^o), \mu(J^o)}^{\mu(K)}$ is non-zero.

2.4.4 Description of $\text{LR}(U, \tilde{U})$

We can finally describe the cone $\text{LR}(U, \tilde{U})$.

Theorem 2.6 *Let $(\xi, \tilde{\xi}) \in \mathfrak{t}_+ \times \tilde{\mathfrak{t}}_+$. We have $U\xi \subset \pi(\tilde{U}\tilde{\xi})$ if and only if*

$$(6) \quad \langle \tilde{\xi}, \tilde{w}\gamma \rangle \geq \langle \xi, w_o w \gamma \rangle$$

for any $(\gamma, w, \tilde{w}) \in \mathfrak{t} \times W \times \tilde{W}$ satisfying the following properties:

- γ is admissible antidominant.
- $[\mathfrak{X}_{w, \gamma}] \cdot \iota^*([\tilde{\mathfrak{X}}_{\tilde{w}, \gamma}]) = [pt]$ in $H^*(\mathcal{F}_{\gamma}, \mathbb{Z})$.
- Identity (3) holds.

The result still holds if we replace b) by the weaker condition

$$b') \quad [\mathfrak{X}_{\gamma}] \cdot \iota^*([\tilde{\mathfrak{X}}_{\tilde{w}, \gamma}]) = \ell[pt], \ell \geq 1 \quad \text{in } H^*(\mathcal{F}_{\gamma}, \mathbb{Z}).$$

Remark 2.7 *Suppose that there exists $c_{\gamma} > 0$ such that $|\langle \alpha, w\gamma \rangle|$ and $|\langle \tilde{\alpha}, \tilde{w}\gamma \rangle|$ belongs to $\{0, c_{\gamma}\}$, $\forall (w, \tilde{w}) \in W \times \tilde{W}$, $\forall (\alpha, \tilde{\alpha}) \in \mathfrak{R}(\mathfrak{u}) \times \mathfrak{R}(\tilde{\mathfrak{u}})$. Then condition c) follows from condition b) (see (5)).*

When the closed connected subgroups $\iota : U \hookrightarrow \tilde{U}$ satisfy Assumption 2.4 the subspace $\mathfrak{z} := \{X \in \mathfrak{t}, \alpha(X) = 0, \forall \alpha \in \Sigma(\tilde{\mathfrak{u}}/\mathfrak{u})\}$ is equal to $Z_{\tilde{\mathfrak{u}}} \cap \mathfrak{u}$. Let $\mathfrak{t} = Z_{\tilde{\mathfrak{u}}} \cap \mathfrak{u} \oplus \mathfrak{t}_1$ be a rational decomposition. Any rational element $\gamma \in \mathfrak{t}$ can be written $\gamma = \gamma_0 + \gamma_1$ where $\gamma_0 \in Z_{\tilde{\mathfrak{u}}} \cap \mathfrak{u}$ and $\gamma_1 \in \mathfrak{t}_1$ are rational. Two cases occur :

- If $\gamma_1 = 0$, then γ satisfies conditions a), b) and c). The inequalities (6) given by these central elements shows that $\tilde{\xi} - \xi$ is orthogonal to $Z_{\tilde{\mathfrak{u}}} \cap \mathfrak{u}$.
- If $\gamma_1 \neq 0$ then it is immediate to see that γ satisfies a), b) and c) if and only if γ_1 does also. Moreover, as $\tilde{\xi} - \xi$ is orthogonal to $Z_{\tilde{\mathfrak{u}}} \cap \mathfrak{u}$, $\langle \tilde{\xi}, \tilde{w}\gamma \rangle \geq \langle \xi, w_o w \gamma \rangle$ if and only if $\langle \tilde{\xi}, \tilde{w}\gamma_1 \rangle \geq \langle \xi, w_o w \gamma_1 \rangle$.

Thus, when Assumption 2.4 is satisfied, $\text{LR}(U, \tilde{U})$ is described by the condition $\tilde{\xi} - \xi \in (Z_{\tilde{\mathfrak{u}}} \cap \mathfrak{u})^\perp$ and a finite number of inequalities of the form $\langle \tilde{\xi}, \tilde{w}\gamma_1 \rangle \geq \langle \xi, w_o w \gamma_1 \rangle$.

Many people have contributed to Theorem 2.6. The first input was given by Klyachko [9] with a refinement by Belkale [1], in the case of $SL_n \hookrightarrow (SL_n)^s$. The case $U_{\mathbb{C}} \hookrightarrow (U_{\mathbb{C}})^s$ has been treated by Belkale-Kumar [2] and by Kapovich-Leeb-Millson [8]. Recall that *Condition c)* is related to the notion of Levi-movability introduced by Belkale-Kumar [2]. Finally, Berenstein-Sjamaar [3] and Ressayre [20, 21] have studied the general case. Ressayre [20] also proved the irredundancy of the list of inequalities.

We refer the reader to the survey articles [6, 4, 10] for details.

2.5 The cone $\text{LR}(m, n)$

Let $m, n \geq 1$. Let us write an Hermitian matrix $X \in \text{Herm}(m+n)$ by blocks $X = \begin{pmatrix} X_{\mathbf{I}} & * \\ * & X_{\mathbf{II}} \end{pmatrix}$ where $X_{\mathbf{I}} \in \text{Herm}(m)$ and $X_{\mathbf{II}} \in \text{Herm}(n)$. In this section, we are interested in the cone $\text{LR}(m, n) := \{(e(X), e(X_{\mathbf{I}}), e(X_{\mathbf{II}})) ; X \in \text{Herm}(m+n)\}$. Thanks to Theorem 2.6, we obtain the following description of $\text{LR}(m, n)$. The details of the proof are given in the next section.

Theorem 2.8 *The triplet $(x, y, z) \in \mathbb{R}_+^{m+n} \times \mathbb{R}_+^m \times \mathbb{R}_+^n$ belongs to $\text{LR}(m, n)$ if and only if the following conditions hold:*

- $|x| = |y| + |z|$,
- $x_{n+k} \leq y_k \leq x_k, \forall k \in [m]$,
- $x_{m+\ell} \leq z_\ell \leq x_\ell, \forall \ell \in [n]$,
- $|x|_A \geq |y|_B + |z|_C$, for any triplet A, B, C satisfying:
 1. $B \subset [m]$ and $C \subset [n]$ are strict subsets,
 2. $A \subset [m+n]$ and $\sharp A = \sharp B + \sharp C$,

3. the Littlewood-Richardson coefficient $c_{\mu(B), \mu(C)}^{\mu(A)}$ is non-zero.

Moreover, the condition $c_{\mu(B), \mu(C)}^{\mu(A)} \neq 0$ is equivalent to $(\mu(A), \mu(B), \mu(C)) \in \text{LR}(u, v)$, where $u = \sharp B$ and $v = \sharp C$.

Remark 2.9 In Theorem 2.8, we can strengthen condition 3. by requiring that $c_{\mu(B), \mu(C)}^{\mu(A)} = 1$.

Remark 2.10 In [13], Li and Poon also obtained a characterization of the cone $\text{LR}(m, n)$ by means of the following inequalities: $|x|_I \leq |y|_{J \cap [m]} + |z|_{K \cap [n]}$, $\forall (I, J, K) \in \text{LR}_r^{n+m}$, $\forall r < n + m$.

We will see in the next section that $c_{\mu(B), \mu(C)}^{\mu(A)} \neq 0$ if and only $c_{\mu((B^o)^c), \mu((C^o)^c)}^{\mu((A^o)^c)} \neq 0$. Since the relation $|x|_{(A^o)^c} \geq |y|_{(C^o)^c} + |z|_{(C^o)^c}$ is equivalent to $|x|_{A^o} \leq |y|_{C^o} + |z|_{C^o}$, in Theorem 2.8, we can rewrite the last condition by requiring that

$$(7) \quad |x|_A \geq |y|_B + |z|_C \quad \text{and} \quad |x|_{A^o} \leq |y|_{C^o} + |z|_{C^o}$$

for all strict subsets $A \subset [m+n]$, $B \subset [m]$, $C \subset [n]$ that satisfy $\sharp A = \sharp B + \sharp C \leq \frac{1}{2}(m+n)$ and $c_{\mu(B), \mu(C)}^{\mu(A)} \neq 0$.

2.6 Proof of Theorem 2.8

We work with the unitary group $\tilde{U} = U_{m+n}$ and the subgroup $U = U_m \times U_n$ embedded diagonally. We consider the orthogonal projection $\pi_0 : \text{Herm}(m+n) \rightarrow \text{Herm}(m) \times \text{Herm}(n)$ that sends X to $\pi_0(X) = (X_{\mathbf{I}}, X_{\mathbf{II}})$. The cone $\text{LR}(m, n)$ is formed by the triplets $(x, y, z) \in \mathbb{R}_+^{m+n} \times \mathbb{R}_+^m \times \mathbb{R}_+^n$ satisfying

$$U_m \cdot \text{Diag}(y) \times U_n \cdot \text{Diag}(z) \subset \pi_0(U_{m+n} \cdot \text{Diag}(x)).$$

Thus $\text{LR}(m, n) = \text{LR}(U_m \times U_n, U_{m+n})$.

2.6.1 Admissible elements

We work with the maximal torus $T \subset U$ of diagonal matrices. The set of roots relatively to the action of T on $\tilde{\mathfrak{u}}/\mathfrak{u} \simeq M_{m,n}(\mathbb{C})$ is $\Sigma := \{e_i^* - f_j^*; i \in [m], j \in [n]\}$.

The center of $\tilde{\mathfrak{u}}$ is generated by $\gamma_o := (1, \dots, 1) \in \mathbb{R}^{m+n} \simeq \mathfrak{t}$. For any $(r, s) \in \{0, \dots, m\} \times \{0, \dots, n\}$, we define

$$\gamma_{r,s} = \underbrace{(-1, \dots, -1, 0, \dots, 0)}_{r \text{ times}} \oplus \underbrace{(-1, \dots, -1, 0, \dots, 0)}_{s \text{ times}} \in \mathbb{R}^m \times \mathbb{R}^n \simeq \mathfrak{t}.$$

Lemma 2.11 Let $\gamma \in \mathfrak{t}$ be an admissible element. There exists $(a, b) \in \mathbb{Q} \times \mathbb{Q}^{\geq 0}$, $(w, w') \in \mathfrak{S}_m \times \mathfrak{S}_n$, and (r, s) such that $\gamma = a\gamma_o + b(w, w')\gamma_{r,s}$. The couple (r, s) must satisfy the auxiliary conditions: either $0 < r < m$ and $0 < s < n$ or $(r, s) \in \{(1, 0), (0, 1), (m-1, n), (n, m-1)\}$.

Proof : Consider an admissible vector $\gamma = (\gamma_1, \dots, \gamma_m; \gamma'_1, \dots, \gamma'_n)$ that is linearly independent to γ_o . The relation $\text{Vect}(\Sigma \cap \gamma^\perp) = \text{Vect}(\Sigma) \cap \gamma^\perp$ means that $(\Sigma \cap \gamma^\perp)^\perp$ is a subspace of dimension 2. Here $\Sigma \cap \gamma^\perp$ is the set of vectors $e_i^* - f_j^*$ such that $\gamma_i = \gamma'_j$. For $\alpha \in \mathbb{R}$, we define $[m]_\alpha := \{i \in [m], \gamma_i = \alpha\}$ and $[n]_\alpha := \{j \in [n], \gamma'_j = \alpha\}$. Hence $\Sigma \cap \gamma^\perp$ is parameterized by $\coprod_{\alpha \in L} [m]_\alpha \times [n]_\alpha$ where $L = \{\alpha \in \mathbb{R}, [m]_\alpha \neq \emptyset \text{ and } [n]_\alpha \neq \emptyset\}$ is a finite set.

Consider first the case where $\cup_{\alpha \in L} [m]_\alpha \neq [m]$. Let $k \notin \cup_{\alpha \in L} [m]_\alpha$. Then $(\Sigma \cap \gamma^\perp)^\perp$, which is of dimension 2, contains the vectors γ_o, γ and e_k . Hence, γ is a linear combinaison of γ_o and e_k : we check easily that there exists $(a, b) \in \mathbb{Q} \times \mathbb{Q}^{>0}$, $w \in \mathfrak{S}_m$, and $(r, s) \in \{(1, 0), (m-1, n)\}$ such that $\gamma = a\gamma_o + bw\gamma_{r,s}$.

If $\cup_{\alpha \in L} [n]_\alpha \neq [n]$, we prove similarly that $\gamma = a\gamma_o + bw'\gamma_{r,s}$ for $(a, b) \in \mathbb{Q} \times \mathbb{Q}^{>0}$, $w' \in \mathfrak{S}_n$, and $(r, s) \in \{(0, 1), (m, n-1)\}$.

Let us consider the last case where $\cup_{\alpha \in L} [m]_\alpha = [m]$ and $\cup_{\alpha \in L} [n]_\alpha = [n]$. Then $\gamma = \sum_{\alpha \in L} \alpha V_\alpha$ with $V_\alpha = \sum_{i \in [m]_\alpha, j \in [n]_\alpha} e_i + f_j$. The vectors $\{V_\alpha, \alpha \in L\}$ define an independent family of the subspace $(\Sigma \cap \gamma^\perp)^\perp$ which is of dimension 2, so $\#L \leq 2$. Since γ is linearly independent to γ_o , the set L has cardinal 2. Now we see that there exists $(a, b) \in \mathbb{Q} \times \mathbb{Q}^{>0}$, $(w, w') \in \mathfrak{S}_m \times \mathfrak{S}_n$, and $r < m, s < n$ such that $\gamma = a\gamma_o + b(w, w')\gamma_{r,s}$. \square

Here, the remark 2.7 applies, so condition c) will follow from condition b).

2.6.2 Cohomological conditions and inequalities

The Lie algebra \mathfrak{t} is identified with $\mathbb{R}^{m+n} \simeq \mathbb{R}^m \times \mathbb{R}^n$.

First case: The two vectors $\pm\gamma_o$ are admissible elements, and satisfy conditions a), b) and c) of Theorem 2.6 in an obvious way. In this cases, the corresponding inequalities $\pm(x, \gamma_o) \geq \pm((y, z), w_o\gamma_o)$ are equivalent to $|x| = |y| + |z|$.

Second case: We work now with the admissible element $\gamma_{r,s}$ in the situation where $r \in [m-1]$ and $s \in [n-1]$. The flag manifold $GL_m(\mathbb{C}) \times GL_n(\mathbb{C}) / P_{\gamma_{r,s}}$ admits a natural identification with the product of Grassmannians $\mathbb{G}(r, m) \times \mathbb{G}(s, n)$. Similarly, the flag manifold $GL_{m+n}(\mathbb{C}) / \tilde{P}_{\gamma_{r,s}}$ is isomorphic to the Grassmannian $\mathbb{G}(r+s, m+n)$. The map $\iota_{\mathbb{C}} : GL_m(\mathbb{C}) \times GL_n(\mathbb{C}) \rightarrow GL_{m+n}(\mathbb{C})$ factorises to a smooth map $\iota_{r,s} : \mathbb{G}(r, m) \times \mathbb{G}(s, n) \rightarrow \mathbb{G}(r+s, m+n)$ defined by $\iota_{r,s}(V_1, V_2) = V_1 \oplus V_2$.

Let $w = (w_1, w_2) \in W \simeq \mathfrak{S}_m \times \mathfrak{S}_n$ and let $B = w_1([r]) \subset [m]$ and $C = w_2([s]) \subset [n]$ be the corresponding subsets. The associated Schubert variety is $\mathfrak{X}_{w, \gamma_{r,s}} = \mathfrak{X}_B \times \mathfrak{X}_C \subset \mathbb{G}(r, m) \times \mathbb{G}(s, n)$.

In the same way, to $\tilde{w} \in \tilde{W} \simeq \mathfrak{S}_{m+n}$, we associate the subset $A = \tilde{w}([r]) \cup \tilde{w}([s] + m) \subset [m+n]$ and the Schubert variety $\tilde{\mathfrak{X}}_{\tilde{w}, \gamma_{r,s}} = \tilde{\mathfrak{X}}_A \subset \mathbb{G}(r+s, m+n)$.

Lemma 2.12 *The following identities are equivalent:*

1. $[\mathfrak{X}_{w, \gamma}] \cdot \iota^*([\tilde{\mathfrak{X}}_{\tilde{w}, \gamma}]) = \ell[pt], \ell \geq 1,$

$$2. \ c_{\mu(B), \mu(C)}^{\mu(A^o)} = \ell \geq 1,$$

$$3. \ c_{\mu((B^o)^c), \mu((C^o)^c)}^{\mu(A^c)} = \ell \geq 1.$$

Proof: Recall that we associate a partition $\lambda(A) = (\lambda_1 \geq \dots \geq \lambda_r)$ with a subset $A = \{a_1 < i_2 < \dots < a_r\} \subset [n]$ of cardinality r , by posing $\lambda_k = n - r + k - a_k$, $\forall k \in [r]$.

Let $\bigwedge_r[x] = \mathbb{Z}[x_1, \dots, x_r]^{\mathfrak{S}_r}$ be the ring of symmetric polynomials, with integral coefficients, in r variables. For any partition ν of length r , we associate its Schur polynomial $\mathbf{s}_\nu(x) \in \bigwedge_r[x]$. The family (\mathbf{s}_ν) determine a \mathbb{Z} -basis of $\bigwedge_r[x]$.

Let us recall the following classical fact (see §3.2.2 in [16]). The map $\phi_r : \bigwedge_r[x] \rightarrow H^*(\mathbb{G}(r, m))$ defined by the relations

$$\phi_r(\mathbf{s}_\nu) = \begin{cases} \sigma_\nu & \text{if } \nu_1 \leq m - r, \\ 0 & \text{if } \nu_1 > m - r. \end{cases}$$

is a ring morphism. Here σ_ν denotes the cohomology class $[\mathfrak{X}_D]$ defined by a subset $D \subset [m]$ of cardinality r such that $\nu = \lambda(D)$. In the same way we consider the ring $\bigwedge_{r+s}[x, y] = \mathbb{Z}[x_1, \dots, x_r, y_1, \dots, y_s]^{\mathfrak{S}_{r+s}}$ and the morphism $\phi_{r+s} : \bigwedge_{r+s}[x, y] \rightarrow H^*(\mathbb{G}(r+s, m+n))$. Let us denote by $R : \bigwedge_{r+s}[x, y] \rightarrow \bigwedge_r[x] \otimes \bigwedge_s[y]$ the restriction morphism. It is not hard to check that the following diagram is commutative:

$$\begin{array}{ccc} \bigwedge_r[x] & \otimes & \bigwedge_s[y] \xleftarrow{R} \bigwedge_{r+s}[x, y] \\ \downarrow \phi_r & & \downarrow \phi_s \quad \downarrow \phi_{r+s} \\ H^*(\mathbb{G}(r, m)) & \otimes & H^*(\mathbb{G}(s, n)) \xleftarrow{j^*} H^*(\mathbb{G}(r+s, m+n)). \end{array}$$

As $[\mathfrak{X}_{w,r}] = \sigma_{\lambda(B)} \otimes \sigma_{\lambda(C)}$ and $[\tilde{\mathfrak{X}}_{\bar{w},r}] = \sigma_{\lambda(A)}$, the previous diagram tell us that the integer ℓ such that $[\mathfrak{X}_{w,r}] \cdot \iota_r^*([\tilde{\mathfrak{X}}_{\bar{w},r}]) = \ell[pt]$ is equal to the coefficient of $R(\mathbf{s}_{\lambda(A)}(x, y))$ relatively to $\mathbf{s}_{\lambda(B^o)}(x) \otimes \mathbf{s}_{\lambda(C^o)}(y)$: in other words ℓ is equal to the Littlewood-Richardson coefficient $c_{\lambda(B^o), \lambda(C^o)}^{\lambda(A)} = c_{\mu(B), \mu(C)}^{\mu(A^o)}$ (see [15], §I.5). The equivalence between 1. and 2. is proved.

Let us consider r', s' such that $r + r' = m$ and $s + s' = n$. The canonical bilinear form on \mathbb{C}^{m+n} permits to define the map $\delta : \mathbb{G}(r' + s', m + n) \rightarrow \mathbb{G}(r + s, m + n)$ that sends a subspace $F \subset \mathbb{C}^{m+n}$ to its orthogonal F^\perp . Let $\delta^* : H^*(\mathbb{G}(r + s, m + n)) \rightarrow H^*(\mathbb{G}(r' + s', m + n))$ denote the pullback map in cohomology. If we consider similar maps $\delta^* : H^*(\mathbb{G}(r, m)) \rightarrow H^*(\mathbb{G}(r', m))$ and $\delta^* : H^*(\mathbb{G}(s, n)) \rightarrow H^*(\mathbb{G}(s', n))$, we have a commutative diagram:

$$\begin{array}{ccc} H^*(\mathbb{G}(r, m)) & \otimes & H^*(\mathbb{G}(s, n)) \xleftarrow{j^*} H^*(\mathbb{G}(r + s, m + n)) \\ \downarrow \delta^* & & \downarrow \delta^* \quad \downarrow \delta^* \\ H^*(\mathbb{G}(r', m)) & \otimes & H^*(\mathbb{G}(s', n)) \xleftarrow{j^*} H^*(\mathbb{G}(r' + s', m + n)). \end{array}$$

This allows us to see that $\sigma_{\lambda(B)} \otimes \sigma_{\lambda(C)} \cdot j^*(\sigma_{\lambda(A)}) = \ell[pt], k \geq 1$ if and only if $\delta^*(\sigma_{\lambda(B)}) \otimes \delta^*(\sigma_{\lambda(C)}) \cdot j^*(\delta^*(\sigma_{\lambda(A)})) = \ell[pt], k \geq 1$. Since we have $\delta^*(\sigma_{\lambda(X)}) = \sigma_{\mu(X^c)}$ as a general rule, the previous relation is equivalent to $c_{\mu((B^o)^c), \mu((C^o)^c)}^{\mu(A^c)} = \ell \geq 1$. The equivalence between 2. and 3. is proved. \square

The inequalities associated to $\gamma_{r,s}$ are

$$-|x|_A = \langle x, \tilde{w}\gamma_{r,s} \rangle \geq \langle (y, z), w_o w \gamma \rangle = -|y|_{B^o} - |z|_{C^o}$$

Using $|x| = |y| + |z|$, we obtain $|x|_{A^c} \geq |y|_{(B^o)^c} + |z|_{(C^o)^c}$ for any strict subsets $A \subset [m+n]$, $B \subset [m]$ and $C \subset [n]$ satisfying $\sharp A = \sharp B + \sharp C$ and $c_{\mu((B^o)^c), \mu((C^o)^c)}^{\mu(A^c)} = \ell \geq 1$.

Third case: $(r, s) \in \{(1, 0), (0, 1), (m-1, n), (n, m-1)\}$. Here we use the same type of argument as before.

$(r, s) = (1, 0)$: we obtain the inequalities $x_{n+k} \leq y_k, \forall k \in [m]$.

$(r, s) = (0, 1)$: we obtain the inequalities $x_{m+\ell} \leq z_\ell, \forall \ell \in [n]$.

$(r, s) = (m-1, n)$: we obtain the inequalities $y_k \leq x_k, \forall k \in [m]$.

$(r, s) = (m, n-1)$: we obtain the inequalities $z_\ell \leq x_\ell, \forall \ell \in [n]$.

The proof of Theorem 2.8 is completed. \square

2.7 A consequence of the O'Shea-Sjamaar Theorem

2.7.1 First setting: compact Lie groups with involution

Let \tilde{U} be a compact connected Lie group equipped with an involution σ . The Lie algebra of \tilde{U} admit the decomposition $\tilde{\mathfrak{u}} = \tilde{\mathfrak{u}}^\sigma \oplus \tilde{\mathfrak{u}}^{-\sigma}$ that is invariant under the action of the subgroup \tilde{U}^σ . We start with a basic but important fact (see [17], Example 2.9).

Lemma 2.13 *For any adjoint orbit $\tilde{\mathcal{O}} \subset \tilde{\mathfrak{u}}$, the intersection $\tilde{\mathcal{O}} \cap \tilde{\mathfrak{u}}^{-\sigma}$ is either empty or an orbit of the connected subgroup $\tilde{K} := (\tilde{U}^\sigma)_0$.*

Let $U \subset \tilde{U}$ be a subgroup invariant under σ . Let us choose an invariant scalar product $(-, -)$ on the Lie algebra $\tilde{\mathfrak{u}}$ of \tilde{U} such that $\sigma \in O(\tilde{\mathfrak{u}})$. At the level of Lie algebras, we consider the orthogonal projection $\pi : \tilde{\mathfrak{u}} \rightarrow \mathfrak{u}$ relatively to the scalar product $(-, -)$.

One of the main tool used in this paper is the following result, which is a consequence of the O'Shea-Sjamaar Theorem (see [17], Section 3). Let K be the connected component of U^σ .

Proposition 2.14 *Let $\xi \in \mathfrak{u}^{-\sigma}$ and $\tilde{\xi} \in \tilde{\mathfrak{u}}^{-\sigma}$. The following conditions are equivalent:*

1. $U\xi \subset \pi(\tilde{U}\tilde{\xi})$,
2. $K\xi \subset \pi(\tilde{K}\tilde{\xi})$.

2.7.2 Second setting: real reductive Lie groups

Let $\iota : G \hookrightarrow \tilde{G} \subset GL_N(\mathbb{R})$ be two connected real reductive Lie groups admitting a complexification $\iota_{\mathbb{C}} : G_{\mathbb{C}} \hookrightarrow \tilde{G}_{\mathbb{C}} \subset GL_N(\mathbb{C})$. It is for example the case when G and \tilde{G} are semisimple (see [11], §VII.1). Let us denote by

- $K = G \cap SO_N(\mathbb{R})$ and $\tilde{K} = \tilde{G} \cap SO_N(\mathbb{R})$ the maximal compact subgroups of G and \tilde{G} . Their Lie algebras are denoted by $\iota : \mathfrak{k} \hookrightarrow \tilde{\mathfrak{k}}$.
- $U = G_{\mathbb{C}} \cap U_N$ and $\tilde{U} = \tilde{G}_{\mathbb{C}} \cap U_N$ the maximal compact subgroups of $G_{\mathbb{C}}$ and $\tilde{G}_{\mathbb{C}}$. Their Lie algebras are denoted by $\iota : \mathfrak{u} \hookrightarrow \tilde{\mathfrak{u}}$.

Consider the Cartan decompositions, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $\tilde{\mathfrak{g}} = \tilde{\mathfrak{k}} \oplus \tilde{\mathfrak{p}}$, of G and \tilde{G} . At the level of Lie algebras, we have $\tilde{\mathfrak{u}} = \tilde{\mathfrak{k}} \oplus i\tilde{\mathfrak{p}}$ and $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$. The antilinear conjugation on $GL_N(\mathbb{C})$ defines an involution σ on $U \hookrightarrow \tilde{U}$ such that $K \hookrightarrow \tilde{K}$ are respectively equal to the connected components of U^{σ} and \tilde{U}^{σ} . We see also that $\mathfrak{u}^{-\sigma} = i\mathfrak{p}$ and $\tilde{\mathfrak{u}}^{-\sigma} = i\tilde{\mathfrak{p}}$.

Let $\pi : \tilde{\mathfrak{g}}_{\mathbb{C}} \rightarrow \mathfrak{g}_{\mathbb{C}}$ be the orthogonal projection relatively to the Hermitian norm $\text{Tr}(X^*X)^{1/2}$ on $\mathfrak{gl}_N(\mathbb{C})$.

Proposition 2.15 *Let $X \in \mathfrak{p}$ and $\tilde{X} \in \tilde{\mathfrak{p}}$. The following conditions are equivalent:*

1. $UX \subset \pi(\tilde{U}\tilde{X})$,
2. $KX \subset \pi(\tilde{K}\tilde{X})$.

Proof: It follows from Proposition 2.14 and the fact that the projection $\pi : \tilde{\mathfrak{g}}_{\mathbb{C}} \rightarrow \mathfrak{g}_{\mathbb{C}}$ is complex linear. \square

3 The cone $\mathcal{A}(p, q)$

Here, we work with the reductive real Lie group $U(p, q) = \{g \in GL_n(\mathbb{C}), g^* I_{p,q} g = I_{p,q}\}$, where $I_{p,q} = \text{Diag}(I_p, -I_q)$.

3.1 Matrix identities

Let us decompose a n -square hermitian matrix $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{pmatrix}$ by blocks, where $X_{12} \in M_{p,q}(\mathbb{C})$. Recall that $(\lambda, s) \in \mathcal{A}(p, q)$ if and only if there exists an hermitian matrix X such that $\lambda = e(X)$ and $s = s(X_{12})$. Let us consider

$$\tilde{X} = -I_{p,q} X I_{p,q} = \begin{pmatrix} -X_{11} & X_{12} \\ X_{12}^* & -X_{22} \end{pmatrix} \quad \text{and} \quad X + \tilde{X} = 2 \begin{pmatrix} 0 & X_{12} \\ X_{12}^* & 0 \end{pmatrix}.$$

If we look at the eigenvalues of this three Hermitian matrices, we obtain, following the notations of Section 2.2,

$$e(\tilde{X}) = e(X)^* \quad \text{and} \quad e(X + \tilde{X}) = 2s(\widehat{X_{12}})^{p,q},$$

i.e. $(e(X), e(X)^*, 2s(\widehat{X_{12}})^{p,q}) \in \text{Horn}(n)$.

From the above identities, we see that any $(\lambda, s) \in \mathcal{A}(p, q)$ satisfies the relation

$$(8) \quad (\lambda, \lambda^*, 2\widehat{s}^{p,q}) \in \text{Horn}(n).$$

In the following sections, we explain how the O'Shea-Sjamaar theorem (see Proposition 2.15) allows us to see that relation (8) characterizes the cone $\mathcal{A}(p, q)$.

3.2 Complexification and antiholomorphic involution

We work with the reductive real Lie groups $G := U(p, q)$ and $\tilde{G} := GL_n(\mathbb{C})$. Let us denote by $\iota : G \rightarrow \tilde{G}$ the canonical embedding. The unitary group $\tilde{K} := U_n$ is a maximal compact subgroup of \tilde{G} . Let $\tilde{\mathfrak{p}} := \text{Herm}(n) \subset \mathfrak{gl}_n(\mathbb{C})$ be the subspace of Hermitian matrices.

The subgroup $K := \tilde{K} \cap U(p, q) \simeq U_p \times U_q$ is a maximal compact subgroup of G , and the map $Y \mapsto \hat{Y}^{p,q} = \begin{pmatrix} 0_{pp} & Y \\ Y^* & 0_{qq} \end{pmatrix}$ defines an identification between $M_{p,q}(\mathbb{C})$ and the subspace $\mathfrak{p} := \tilde{\mathfrak{p}} \cap \mathfrak{g}$.

The complexification of the group G is $G_{\mathbb{C}} := GL_n(\mathbb{C})$. We consider the antiholomorphic involution σ on $G_{\mathbb{C}}$ defined by $\sigma(g) = I_{p,q}(g^*)^{-1}I_{p,q}$. The subgroup G is the fixed point set of σ .

The complexification of the group \tilde{G} is $\tilde{G}_{\mathbb{C}} := GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$. The inclusion $\tilde{G} \hookrightarrow \tilde{G}_{\mathbb{C}}$ is given by the map $g \mapsto (g, \bar{g})$. We consider the antiholomorphic involution $\tilde{\sigma}$ on $\tilde{G}_{\mathbb{C}}$ defined by $\tilde{\sigma}(g_1, g_2) = (\bar{g}_2, \bar{g}_1)$. The subgroup \tilde{G} corresponds to the fixed point set of $\tilde{\sigma}$. The embedding $\iota : G \hookrightarrow \tilde{G}$ admits a complexification $\iota_{\mathbb{C}} : G_{\mathbb{C}} \hookrightarrow \tilde{G}_{\mathbb{C}}$ defined by $\iota_{\mathbb{C}}(g) = (g, \overline{\sigma(g)})$: notice that $\iota_{\mathbb{C}} \circ \sigma = \tilde{\sigma} \circ \iota_{\mathbb{C}}$.

The groups $U = U_n$ and $\tilde{U} = U_n \times U_n$ are respectively maximal compact subgroups of $G_{\mathbb{C}}$ and $\tilde{G}_{\mathbb{C}}$. The embedding $\iota_{\mathbb{C}} : U \hookrightarrow \tilde{U}$ is defined by $\iota_{\mathbb{C}}(k) = (k, I_{p,q}\bar{k}I_{p,q})$. The fixed point subgroups of the involutions are $U^{\sigma} = K$ and $\tilde{U}^{\tilde{\sigma}} = \tilde{K}$.

At the level of Lie algebra, we have a morphism $\iota_{\mathbb{C}} : \mathfrak{gl}_n(\mathbb{C}) \hookrightarrow \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})$ defined by $\iota_{\mathbb{C}}(X) = (X, \overline{\sigma(X)})$, where $\sigma(X) = -I_{p,q}X^*I_{p,q}$.

3.3 Orthogonal projection of orbits

We use on $\mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})$ the euclidean norm $\|(X, Y)\|^2 = \text{Tr}(XX^*) + \text{Tr}(YY^*)$. The subspace orthogonal to the image of $\iota_{\mathbb{C}}$ is $\{(X, -\overline{\sigma(X)}), X \in \mathfrak{gl}_n(\mathbb{C})\}$. Hence the orthogonal projection

$$\pi : \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C}) \longrightarrow \mathfrak{gl}_n(\mathbb{C}),$$

is defined by the relations $\pi(X, Y) = \frac{1}{2}(X + \overline{\sigma(Y)})$. Note that π commutes with the involutions : $\pi \circ \tilde{\sigma} = \sigma \circ \pi$.

If $X \in \text{Herm}(n)$, the corresponding adjoint orbit $U_n \cdot X$, which is entirely determined by the spectrum $e(X)$, is denoted by $\mathcal{O}_{e(X)}$. If $\lambda = (\lambda_1, \dots, \lambda_n)$, we denote by λ^* the vector $(-\lambda_n, \dots, -\lambda_1)$: we see that $e(-X) = e(X)^*$ for any $X \in \text{Herm}(n)$.

The subspace $\tilde{\mathfrak{p}} \subset \tilde{\mathfrak{g}}$ is identified with $\{(X, \overline{X}), X \in \text{Herm}(n)\} \subset \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})$. For any $X \in \text{Herm}(n)$, the image by the projection π of the orbit $\tilde{U} \cdot (X, \overline{X})$ is equal to

$$\frac{1}{2}(U_n \cdot X + U_n \cdot \sigma(X)) = \frac{1}{2}(U_n \cdot X + U_n \cdot (-X)) = \frac{1}{2}(\mathcal{O}_\lambda + \mathcal{O}_{\lambda^*}),$$

where $\lambda = e(X)$.

If $Y \in M_{p,q}(\mathbb{C})$ has singular spectrum $s \in \mathbb{R}_{++}^q$, the spectrum of the Hermitian matrix $\widehat{Y}^{p,q}$ is equal to $\widehat{s}^{p,q}$, hence $U \cdot \widehat{Y}^{p,q}$ is equal to $\mathcal{O}_{\widehat{s}^{p,q}}$. At this stage, we have proved that for any $(X, Y) \in \text{Herm}(n) \times M_{p,q}(\mathbb{C})$ the following statements are equivalents:

- $U \cdot \widehat{Y}^{p,q} \subset \pi(\tilde{U} \cdot (X, \overline{X}))$,
- $2\mathcal{O}_{\widehat{s}^{p,q}} \subset \mathcal{O}_\lambda + \mathcal{O}_{\lambda^*}$,
- $(\lambda, \lambda^*, 2\widehat{s}^{p,q}) \in \text{Horn}(n)$,

where $\lambda = e(X)$ and $s = s(Y)$.

The group $K \simeq U_p \times U_q$ acts canonically $M_{p,q}(\mathbb{C}) \simeq \mathfrak{p}$. For any $Y \in M_{p,q}(\mathbb{C})$, the orbit $K \cdot \widehat{Y}^{p,q} \subset \mathfrak{p}$ is equal to the set of matrices with singular spectrum equal to $s(Y)$. If one restricts the projection $\pi : \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C}) \rightarrow \mathfrak{gl}_n(\mathbb{C})$ to the subspace $\tilde{\mathfrak{p}} \simeq \text{Herm}(n)$, we obtain the map $\pi : \tilde{\mathfrak{p}} \rightarrow \mathfrak{p}$ that sends an Hermitian matrix $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{pmatrix}$ to

$$\widehat{X}_{12}^{p,q} = \begin{pmatrix} 0_{pp} & X_{12} \\ X_{12}^* & 0_{qq} \end{pmatrix}.$$

Since the orbit $\tilde{K} \cdot X$ is equal to $\mathcal{O}_{e(X)}$, we see then that $\mathcal{A}(p, q)$ can be defined as follows: $(\lambda, s) \in \mathbb{R}_+^n \times \mathbb{R}_{++}^q$ belongs to the cone $\mathcal{A}(p, q)$ if and only if for any $(X, Y) \in \text{Herm}(n) \times M_{p,q}(\mathbb{C})$ satisfying $\lambda = e(X)$ and $s = s(Y)$, we have $K \cdot \widehat{Y}^{p,q} \subset \pi(\tilde{K} \cdot X)$.

3.4 Inequalities determining $\mathcal{A}(p, q)$

The computations done in the previous section, together with Proposition 2.15, gives us the following result.

Proposition 3.1 *Let $(\lambda, s) \in \mathbb{R}_+^n \times \mathbb{R}_{++}^q$, and let $(X, Y) \in \text{Herm}(n) \times M_{p,q}(\mathbb{C})$ such that $\lambda = e(X)$ and $s = s(Y)$. The following statements are equivalent:*

- $(\lambda, s) \in \mathcal{A}(p, q)$,
- $K \cdot \widehat{Y}^{p,q} \subset \pi(\tilde{K} \cdot X)$,

- $U \cdot \widehat{Y}^{p,q} \subset \pi \left(\widetilde{U} \cdot (X, \overline{X}) \right),$
- $(\lambda, \lambda^*, 2\widehat{s}^{p,q}) \in \text{Horn}(n).$

Thanks to the description of the $\text{Horn}(n)$ cone given in Theorem 2.2, we can conclude with the following description of $\mathcal{A}(p, q)$. Note that $|\lambda^*|_J = -|\lambda|_{J^o}$ and $|\widehat{s}^{p,q}|_K = |s|_{K \cap [q]} - |s|_{K^o \cap [q]}.$

Proposition 3.2 *An element $(\lambda, s) \in \mathbb{R}_+^n \times \mathbb{R}_{++}^q$ belongs to $\mathcal{A}(p, q)$ if and only if*

$$(\star)_{I,J,K} \quad |\lambda|_I - |\lambda|_{J^o} \geq 2|s|_{K \cap [q]} - 2|s|_{K^o \cap [q]}$$

for any $r \leq \frac{n}{2}$ and any $(I, J, K) \in \text{LR}_r^n$.

However, our description is less precise than that obtained by Fomin-Fulton-Li-Poon [5]. They show the remarkable fact that it suffices to consider inequalities $(\star)_{I,J,K}$ when I, J, K are subsets of $[q]$.

Theorem 3.3 ([5]) *An element $(\lambda, s) \in \mathbb{R}_+^n \times \mathbb{R}_{++}^q$ belongs to $\mathcal{A}(p, q)$ if and only if*

$$|\lambda|_I - |\lambda|_{J^o} \geq 2|s|_K$$

for any $r \leq q$ and any $(I, J, K) \in \text{LR}_r^q$.

3.5 Examples

Computation of $\mathcal{A}(2, 2)$

The inequalities associated to $(I, J, K) \in \text{LR}_1^2$ are

$$(9) \quad \lambda_1 - \lambda_4 \geq 2s_1, \quad \lambda_2 - \lambda_4 \geq 2s_2, \quad \lambda_1 - \lambda_3 \geq 2s_2.$$

The inequality associated to $I = J = K = \{1, 2\}$ is

$$(10) \quad \lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 \geq 2(s_1 + s_2).$$

Theorem 3.3 give us the following description.

Corollary 3.4 *An element $(\lambda, s) \in \mathbb{R}_+^4 \times \mathbb{R}_{++}^2$ belongs to $\mathcal{A}(2, 2)$ if and only if the conditions (9) and (10) hold.*

Computation of $\mathcal{A}(3, 3)$

The inequalities associated to LR_1^3 are

$$(11) \quad \begin{array}{ll} \lambda_1 - \lambda_6 \geq 2s_1 & \lambda_1 - \lambda_4 \geq 2s_3 \\ \lambda_2 - \lambda_6 \geq 2s_2 & \lambda_2 - \lambda_5 \geq 2s_3 \\ \lambda_1 - \lambda_5 \geq 2s_2 & \lambda_3 - \lambda_6 \geq 2s_3. \end{array}$$

The inequalities associated to LR_2^3 are

$$(12) \quad \begin{array}{ll} \lambda_1 + \lambda_2 - \lambda_5 - \lambda_6 & \geq 2(s_1 + s_2) \\ \lambda_1 + \lambda_2 - \lambda_4 - \lambda_6 & \geq 2(s_1 + s_3) \\ \lambda_1 + \lambda_3 - \lambda_5 - \lambda_6 & \geq 2(s_1 + s_3) \\ \lambda_1 + \lambda_2 - \lambda_4 - \lambda_5 & \geq 2(s_2 + s_3) \\ \lambda_1 + \lambda_3 - \lambda_4 - \lambda_6 & \geq 2(s_2 + s_3) \\ \lambda_2 + \lambda_3 - \lambda_5 - \lambda_6 & \geq 2(s_2 + s_3). \end{array}$$

The inequality associated to $I = J = K = \{1, 2, 3\}$ is

$$(13) \quad \lambda_1 + \lambda_2 + \lambda_3 - \lambda_4 - \lambda_5 - \lambda_6 \geq 2(s_1 + s_2 + s_3).$$

The result of Fulton-Fomin-Li-Poon (Theorem 3.3) gives the following description of $\mathcal{A}(3, 3)$.

Proposition 3.5 *An element $(\lambda, s) \in \mathbb{R}_+^6 \times \mathbb{R}_{++}^3$ belongs to $\mathcal{A}(3, 3)$ if and only if the inequalities listed in (11), (12) and (13) are satisfied.*

Remark 3.6 *The cone $\mathcal{A}(3, 3) \subset \mathbb{R}^6 \times \mathbb{R}^3$ corresponds to the intersection of the Horn cone $\text{Horn}(6) \subset \mathbb{R}^{18}$ with the subspace $\{(\lambda, \lambda^*, 2\hat{s}^{p,q}), (\lambda, s) \in \mathbb{R}^6 \times \mathbb{R}^3\}$. Strikingly, $\mathcal{A}(3, 3)$ is determined by 21 inequalities, while $\text{Horn}(6)$ is described with a minimal list of 536 inequalities.*

4 The cone $\mathcal{S}(p, q)$

We work with the projection $\pi_0 : \mathfrak{gl}_{2n}(\mathbb{C}) \longrightarrow \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})$ defined by the relations:

$$(14) \quad B = \begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix} \longmapsto \pi_0(B) = (B_{00}, B_{11}).$$

Here each matrix B_{ij} belongs to $\mathfrak{gl}_n(\mathbb{C})$.

Recall that $(\lambda, \mu, \nu) \in \text{LR}(n, n)$ if and only if there exists a $2n$ -square Hermitian matrix B such that $\lambda = e(B)$, $\mu = e(B_{00})$ and $\nu = e(B_{11})$.

4.1 Matrix identities

Here we use the notations $\widehat{Y}^{p,q}, \widehat{\mu}^{p,q}$ introduced in the §2.2.

Let us decompose a n -square complex matrix $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ by blocks where $X_{12} \in M_{p,q}(\mathbb{C})$. Let $\widehat{X} = \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}$ be the associated $2n$ -square Hermitian matrix.

Recall that $e(\widehat{X}) = \widehat{s(X)}$ (see Section 2.2).

Let $P_\tau \in O_{2n}(\mathbb{R})$ be the orthogonal matrix associated with the permutation $\tau : [2n] \rightarrow [2n]$ which is defined as follows: $\tau(k) = k$ if $1 \leq k \leq p$, $\tau(k) = k + q$ if $p + 1 \leq k \leq n + p$ and $\tau(k) = k - n$ if $n + p + 1 \leq k \leq 2n$.

We see then that $P_\tau \widehat{X} P_\tau^{-1}$ is a $2n$ -square hermitian matrix such that

$$(P_\tau \widehat{X} P_\tau^{-1})_{00} = \begin{pmatrix} 0 & X_{12} \\ X_{12}^* & 0 \end{pmatrix} \quad \text{and} \quad (P_\tau \widehat{X} P_\tau^{-1})_{11} = \begin{pmatrix} 0 & X_{21} \\ X_{21}^* & 0 \end{pmatrix}$$

Finally we obtain the relations $\widehat{s(X)} = e(\widehat{X}) = e(P_\tau \widehat{X} P_\tau^{-1})$,

$$e((P_\tau \widehat{X} P_\tau^{-1})_{00}) = \widehat{s(X_{12})}^{p,q} \quad \text{and} \quad e((P_\tau \widehat{X} P_\tau^{-1})_{11}) = \widehat{s(X_{21})}^{p,q}.$$

In other words, $(\widehat{s(X)}, \widehat{s(X_{12})}^{p,q}, \widehat{s(X_{21})}^{p,q}) \in \text{LR}(n, n)$ for any n -square complex matrix X . At this point, we have shown that any $(\gamma, s, t) \in \mathcal{S}(p, q)$ satisfies the relation

$$(15) \quad (\widehat{\gamma}, \widehat{s}^{p,q}, \widehat{t}^{p,q}) \in \text{LR}(n, n).$$

In the next sections, we explain how the O'Shea-Sjamaar theorem (see Proposition 2.15) allows us to show that (15) characterizes the cone $\mathcal{S}(p, q)$.

4.2 Antiholomorphic involution and orthogonal projection

We work with the real reductive Lie groups $G := U(p, q) \times U(q, p)$ and $\widetilde{G} := U(n, n)$. The embedding $\iota : G \rightarrow \widetilde{G}$ is defined as follows:

$$(16) \quad \iota(g, h) = \begin{pmatrix} g_{11} & 0_{pn} & g_{12} \\ 0_{np} & h & 0_{nq} \\ g_{21} & 0_{qn} & g_{22} \end{pmatrix}, \quad \text{when} \quad g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}.$$

Here $g_{11} \in M_{p,p}(\mathbb{C})$, $g_{12} \in M_{p,q}(\mathbb{C})$, $g_{21} \in M_{q,p}(\mathbb{C})$ and $g_{22} \in M_{q,q}(\mathbb{C})$.

The unitary group $\widetilde{K} := U_n \times U_n$ is a maximal compact subgroup of \widetilde{G} . The subspace $\widetilde{\mathfrak{p}} := \{\widehat{X}, X \in \mathfrak{gl}_n(\mathbb{C})\} \subset \widetilde{\mathfrak{g}}$ admits a canonical action of \widetilde{K} . The subgroup $K = K_1 \times K_2$, with $K_1 \simeq U_p \times U_q$ and $K_2 \simeq U_q \times U_p$, is a maximal compact subgroup of G , and the subspace $\mathfrak{p} = \widetilde{\mathfrak{p}} \cap \mathfrak{g}$ admits a natural identification with $M_{p,q}(\mathbb{C}) \times M_{q,p}(\mathbb{C})$:

$$(Y, Z) \in M_{p,q}(\mathbb{C}) \times M_{q,p}(\mathbb{C}) \longmapsto (\widehat{Y}^{p,q}, \widehat{Z}^{q,p}) \in \mathfrak{p}.$$

The complexification of the group G is $G_{\mathbb{C}} := GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$. We consider the antiholomorphic involution σ on $G_{\mathbb{C}}$ defined by $\sigma(g, h) = (I_{p,q}(g^*)^{-1}I_{p,q}, I_{q,p}(h^*)^{-1}I_{q,p})$. The subgroup G is the fixed point set of σ .

The complexification of the group \tilde{G} is $\tilde{G}_{\mathbb{C}} := GL_{2n}(\mathbb{C})$. We consider the antiholomorphic involution $\tilde{\sigma}$ on $\tilde{G}_{\mathbb{C}}$ defined by $\tilde{\sigma}(g) = I_{n,n}(g^*)^{-1}I_{n,n}$. The subgroup \tilde{G} corresponds to the fixed point set of $\tilde{\sigma}$.

The groups $U = U_n \times U_n$ and $\tilde{U} = U_{2n}$ are respectively maximal compact subgroups of $G_{\mathbb{C}}$ and $\tilde{G}_{\mathbb{C}}$. The fixed point subgroups of the involutions are $U^{\sigma} = K$ and $\tilde{U}^{\tilde{\sigma}} = \tilde{K}$.

The embedding $\iota : G \hookrightarrow \tilde{G}$ admits a complexification $\iota_{\mathbb{C}} : G_{\mathbb{C}} \hookrightarrow \tilde{G}_{\mathbb{C}}$. At the level of Lie algebra, we have a morphism $\iota_{\mathbb{C}} : \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C}) \hookrightarrow \mathfrak{gl}_{2n}(\mathbb{C})$, still defined by (16).

The orthogonal projection $\pi_1 : \mathfrak{gl}_{2n}(\mathbb{C}) \longrightarrow \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})$ dual to the morphism $\iota_{\mathbb{C}}$ is defined by the relations :

$$(17) \quad A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \quad \longmapsto \quad \pi_1(A) = \left(\begin{pmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{pmatrix}, A_{22} \right).$$

Here the matrix $A \in \mathfrak{gl}_{2n}(\mathbb{C})$ is written by blocks relatively to the decomposition $2n = p + n + q$.

In the beginning of §4, we have consider another projection π_0 (see (14)).

Lemma 4.1 *For any U_{2n} -orbit $\mathcal{O} \subset \mathfrak{gl}_{2n}(\mathbb{C})$, we have $\pi_1(\mathcal{O}) = \pi_0(\mathcal{O})$.*

Proof: Let $P_{\tau} \in O_{2n}(\mathbb{R})$ the orthogonal matrix defined in Section 4.1. We check that $\pi_1(M) = \pi_0(P_{\tau}MP_{\tau}^{-1})$, $\forall M \in \mathfrak{gl}_{2n}(\mathbb{C})$. Our lemma follows from this relation. \square

4.3 Description of $\mathcal{S}(p, q)$ through $\text{LR}(n, n)$

For $t \in \mathbb{R}^q$, we consider the n -square Hermitian matrix

$$(18) \quad Y(t) := \begin{pmatrix} 0_{qq} & 0_{q,p-q} & \text{Diag}(t) \\ 0_{p-q,q} & 0_{p-q,p-q} & 0_{p-q,q} \\ \text{Diag}(t) & 0_{q,p-q} & 0_{qq} \end{pmatrix}.$$

Here is the main application of Proposition 2.15.

Proposition 4.2 *Let $(\gamma, s, t) \in \mathbb{R}_{++}^n \times \mathbb{R}_{++}^q \times \mathbb{R}_{++}^q$. The following statements are equivalent:*

1. $(\gamma, s, t) \in \mathcal{S}(p, q)$,
2. $\exists A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathfrak{gl}_n(\mathbb{C})$, such that $s(A) = \gamma$, $s(A_{12}) = s$, and $s(A_{21}) = t$,

3. $\exists M \in \tilde{\mathfrak{p}}$, with $\pi_1(M) = (M_1, M_2)$, s.t. $e(M) = \hat{\gamma}$, $e(M_1) = \hat{s}^{p,q}$ and $e(M_2) = \hat{t}^{p,q}$,
4. $\pi_1 \left(U_n \times U_n \cdot \widehat{\text{Diag}(\gamma)} \right)$ contains $\left(U_p \times U_q \cdot Y(s) \right) \times \left(U_q \times U_p \cdot Y(t) \right)$.
5. $\pi_1 \left(U_{2n} \cdot \widehat{\text{Diag}(\gamma)} \right)$ contains $U_n \cdot Y(s) \times U_n \cdot Y(t)$.
6. $\pi_0 \left(U_{2n} \cdot \text{Diag}(\hat{\gamma}) \right)$ contains $\mathcal{O}_{\hat{s}^{p,q}} \times \mathcal{O}_{\hat{t}^{p,q}}$.
7. $(\hat{\gamma}, \hat{s}^{p,q}, \hat{t}^{p,q}) \in \text{LR}(n, n)$.

Proof: Equivalences “1. \iff 2.” and “6. \iff 7.” are true by definition. Equivalence “2. \iff 3.” is proved by taking $M = \hat{A}$ (see §4.1). Equivalence “3. \iff 4.” is obtained by noting the following relations

$$\left\{ M \in \tilde{\mathfrak{p}}; e(M) = \hat{\gamma} \right\} = U_n \times U_n \cdot \widehat{\text{Diag}(\gamma)},$$

and $\{(X, Y) \in \mathfrak{p}; e(X) = \hat{s} \text{ and } e(Y) = \hat{t}\} = (U_p \times U_q \cdot Y(s)) \times (U_q \times U_p \cdot Y(t))$. Equivalence “4. \iff 5.” follows from Proposition 2.15, and “5. \iff 6.” is a consequence of Lemma 4.1 and the fact that the orbit $U_{2n} \cdot \widehat{\text{Diag}(\gamma)}$ is equal to $U_{2n} \cdot \text{Diag}(\hat{\gamma})$. \square

4.4 Inequalities determining $\mathcal{S}(p, q)$

Thanks to Proposition 4.2 and Theorem 2.8, we obtain the following description of the cone $\mathcal{S}(p, q)$.

Theorem 4.3 *An element $(\gamma, s, t) \in \mathbb{R}_{++}^n \times \mathbb{R}_{++}^q \times \mathbb{R}_{++}^q$ belongs to $\mathcal{S}(p, q)$ if and only if,*

$$(19) \quad \gamma_k \geq s_k \quad \text{and} \quad \gamma_k \geq t_k, \quad \forall k \in [q],$$

and

$$(20) \quad |\gamma|_{A \cap [n]} - |\gamma|_{A^\circ \cap [n]} \geq |s|_{B \cap [q]} - |s|_{B^\circ \cap [q]} + |t|_{C \cap [q]} - |t|_{C^\circ \cap [q]},$$

holds for any triplets (A, B, C) satisfying the following conditions :

- B, C are strict subsets of $[n]$,
- $A \subset [2n]$ and $\sharp A = \sharp B + \sharp C$,
- the Littlewood-Richardson coefficient $c_{\mu(B), \mu(C)}^{\mu(A)}$ is non-zero.

Let us use some duality to minimize the number of equations (see (7)). The equation (20) means that $(\hat{\gamma}, \hat{s}^{p,q}, \hat{t}^{p,q})$ satisfies $|x|_A \geq |y|_B + |z|_C$, that is $|\hat{\gamma}|_A \geq |\hat{s}^{p,q}|_B + |\hat{t}^{p,q}|_C$. If we apply $(\hat{\gamma}, \hat{s}^{p,q}, \hat{t}^{p,q})$ to the relation $|x|_{A^\circ} \leq |y|_{C^\circ} + |z|_{C^\circ}$, we get $|\hat{\gamma}|_{A^\circ} \leq |\hat{s}^{p,q}|_{B^\circ} + |\hat{t}^{p,q}|_{C^\circ}$ which is equivalent to (20) since $|\hat{\gamma}|_{A^\circ} = -|\hat{\gamma}|_A$, $|\hat{s}^{p,q}|_{B^\circ} = -|\hat{s}^{p,q}|_B$ and $|\hat{t}^{p,q}|_{C^\circ} = -|\hat{t}^{p,q}|_C$.

We can therefore rewrite Theorem 4.3, by requiring that (20) holds for all strict subsets $A \subset [2n]$, $B, C \subset [n]$, which satisfy $\sharp A = \sharp B + \sharp C \leq n$ and $c_{\mu(B), \mu(C)}^{\mu(A)} \neq 0$.

4.5 Examples

The cone $\mathcal{S}(1, 1)$

We have to look to subsets $B = \{b\} \subset [2]$, $C = \{c\} \subset [2]$, and $A = \{a_2 > a_1\} \subset [4]$ such that the Littlewood-Richardson coefficient $c_{\mu(B), \mu(C)}^{\mu(A)}$ is non-zero. Here's the list and the corresponding inequalities:

- i) $B = C = \{2\}$ and $A = \{4, 1\}$ or $\{3, 2\}$: $0 \geq -s - t$.
- ii) $B = \{1\}$, $C = \{2\}$ and $A = \{3, 1\}$: $\lambda_1 - \lambda_2 \geq s - t$.
- iii) $B = \{2\}$, $C = \{1\}$ and $A = \{3, 1\}$: $\lambda_1 - \lambda_2 \geq -s + t$.
- iv) $B = C = \{1\}$ and $A = \{2, 1\}$: $\lambda_1 + \lambda_2 \geq s + t$.

Note that the inequalities (19) are here a consequence of ii), iii) and iv). Thus, an element $(\gamma, s, t) \in \mathbb{R}_{++}^2 \times \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0}$ belongs to $\mathcal{S}(1, 1)$ if and only if

$$\lambda_1 - \lambda_2 \geq |s - t| \quad \text{and} \quad \lambda_1 + \lambda_2 \geq s + t.$$

We recover the computation done in [5] (see Example 1.17).

The cone $\mathcal{S}(2, 1)$

First, we look to subsets $B = \{b\} \subset [3]$, $C = \{c\} \subset [3]$, and $A = \{a_2 > a_1\} \subset [6]$ such that the Littlewood-Richardson coefficient $c_{\mu(B), \mu(C)}^{\mu(A)}$ is non-zero. The corresponding inequality (20) is called trivial when it is a consequence of the following relations

$$(21) \quad \gamma_1 \geq \gamma_2 \geq \gamma_3 \geq 0, \quad \gamma_1 \geq t \geq 0, \quad \gamma_1 \geq s \geq 0.$$

Here's the list of the non-trivial inequalities:

- $B = \{1\}$, $C = \{1\}$ and $A = \{2, 1\}$: $\gamma_1 + \gamma_2 \geq s + t$.
- $B = \{1\}$, $C = \{3\}$ and $A = \{4, 1\}$: $\gamma_1 - \gamma_3 \geq s - t$.
- $B = \{3\}$, $C = \{1\}$ and $A = \{4, 1\}$: $\gamma_1 - \gamma_3 \geq -s + t$.

Next, we examine subsets $B = \{b_1 > b_2\} \subset [3]$, $C = \{c\} \subset [3]$, and $A = \{a_3 > a_2 > a_1\} \subset [6]$ such that $c_{\mu(B), \mu(C)}^{\mu(A)} \neq 0$. An easy check shows that all inequalities obtained here are a consequence of (21) and the inequalities

$$(22) \quad \lambda_1 + \lambda_2 \geq s + t, \quad \lambda_1 - \lambda_3 \geq |s - t|,$$

which we have just proved above.

Corollary 4.4 *An element $(\gamma, s, t) \in \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}$ belongs to $\mathcal{S}(2, 1)$ if and only if the inequalities (21) and (22) hold.*

5 The cone $\mathcal{T}(p, q)$

We consider here the projections $\pi_0, \pi_1 : \mathfrak{gl}_{2n}(\mathbb{C}) \longrightarrow \mathfrak{gl}_{2p}(\mathbb{C}) \times \mathfrak{gl}_{2q}(\mathbb{C})$:

- π_1 is defined by the relations:

$$(23) \quad A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \quad \longmapsto \quad \pi(A) = \left(\begin{pmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{pmatrix}, A_{22} \right),$$

where the matrix $A \in \mathfrak{gl}_{2n}(\mathbb{C})$ is written by blocks relatively to the decomposition $2n = p + 2q + p$.

- π_0 is defined by the relations:

$$(24) \quad B = \begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix} \quad \longmapsto \quad \pi_0(B) = (B_{00}, B_{11}),$$

where $B_{00} \in \mathfrak{gl}_{2p}(\mathbb{C})$ and $B_{11} \in \mathfrak{gl}_{2q}(\mathbb{C})$

Lemma 5.1 *For any U_{2n} -orbit $\mathcal{O} \subset \mathfrak{gl}_{2n}(\mathbb{C})$, we have $\pi_1(\mathcal{O}) = \pi_0(\mathcal{O})$.*

Proof: Same proof as for Lemma 4.1. \square

5.1 Matrix identities

Let us decompose a n -square complex matrix $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ by blocks where $X_{12} \in M_{p,q}(\mathbb{C})$. We want to find a link between the singular eigenvalues of X , X_{11} and X_{22} .

The matrix $Q = \begin{pmatrix} 0_{pq} & Id_p \\ Id_q & 0_{qp} \end{pmatrix}$ is orthogonal and the matrix $X' := XQ = \begin{pmatrix} X_{12} & X_{11} \\ X_{22} & X_{21} \end{pmatrix}$ has the same singular values as X . The image of the $2n$ -square Hermitian matrix $\widehat{X'} := \begin{pmatrix} 0_{nn} & X' \\ (X')^* & 0_{nn} \end{pmatrix}$ through the projection π_1 is equal to

$$\pi_1(\widehat{X'}) = (\widehat{X_{11}}, \widehat{X_{22}}).$$

The orbit $\mathcal{O} := U_{2n} \cdot \widehat{X'}$ is equal to the subset of $2n$ -square Hermitian matrices Y satisfying $e(Y) = e(\widehat{X'}) = \widehat{s(X')} = \widehat{s(X)}$, and the projection $\pi_0(\mathcal{O}) = \pi_1(\mathcal{O})$ contains $(\widehat{X_{11}}, \widehat{X_{22}})$, so $(\widehat{s(X)}, \widehat{s(X_{11})}, \widehat{s(X_{22})}) \in \text{LR}(2p, 2q)$.

We have just shown that any $(\gamma, s, t) \in \mathcal{T}(p, q)$ satisfies the relation

$$(25) \quad (\widehat{\gamma}, \widehat{s}, \widehat{t}) \in \text{LR}(2p, 2q).$$

In the next sections, we explain how the O'Shea-Sjamaar theorem (see Proposition 2.15) allows us to see that relation (25) characterizes the cone $\mathcal{T}(p, q)$.

5.2 Antiholomorphic involution and orthogonal projection

We work with the real reductive Lie groups $G := U(p, p) \times U(q, q)$ and $\tilde{G} := U(n, n)$. The embedding $\iota : G \rightarrow \tilde{G}$ is defined as follows:

$$(26) \quad \iota(g, h) = \begin{pmatrix} g_{11} & 0_{p,2q} & g_{12} \\ 0_{2q,p} & h & 0_{2q,p} \\ g_{21} & 0_{p,2q} & g_{22} \end{pmatrix}, \quad \text{when} \quad g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}.$$

Here $g_{ij} \in \mathfrak{gl}_p(\mathbb{C})$ and $h \in U(q, q) \subset \mathfrak{gl}_{2q}(\mathbb{C})$.

The unitary group $\tilde{K} := U_n \times U_n$ is a maximal compact subgroup of \tilde{G} . The subspace $\tilde{\mathfrak{p}} := \{\hat{X}, X \in M_{n,n}(\mathbb{C})\} \subset \tilde{\mathfrak{g}}$ admits a canonical action of \tilde{K} . The subgroup $K = K_1 \times K_2$, with $K_1 \simeq U_p \times U_p$ and $K_2 \simeq U_q \times U_q$, is a maximal compact subgroup of G , and the subspace $\mathfrak{p} = \tilde{\mathfrak{p}} \cap \mathfrak{g}$ admits a natural identification with $\mathfrak{gl}_p(\mathbb{C}) \times \mathfrak{gl}_q(\mathbb{C})$:

$$(Y, Z) \in \mathfrak{gl}_p(\mathbb{C}) \times \mathfrak{gl}_q(\mathbb{C}) \mapsto (\hat{Y}, \hat{Z}) \in \mathfrak{p}.$$

The complexification of the group G is $G_{\mathbb{C}} := GL_{2p}(\mathbb{C}) \times GL_{2q}(\mathbb{C})$. We consider the antiholomorphic involution σ on $G_{\mathbb{C}}$ defined by $\sigma(g, h) = (I_{p,p}(g^*)^{-1} I_{p,p}, I_{q,q}(h^*)^{-1} I_{q,q})$. The subgroup G is the fixed point set of σ .

The complexification of the group \tilde{G} is $\tilde{G}_{\mathbb{C}} := GL_{2n}(\mathbb{C})$. We consider the antiholomorphic involution $\tilde{\sigma}$ on $\tilde{G}_{\mathbb{C}}$ defined by $\tilde{\sigma}(g) = I_{n,n}(g^*)^{-1} I_{n,n}$. The subgroup \tilde{G} corresponds to the fixed point set of $\tilde{\sigma}$.

The groups $U = U_{2p} \times U_{2q}$ and $\tilde{U} = U_{2n}$ are respectively maximal compact subgroups of $G_{\mathbb{C}}$ and $\tilde{G}_{\mathbb{C}}$. The fixed point subgroups of the involutions are $U^{\sigma} = K$ and $\tilde{U}^{\tilde{\sigma}} = \tilde{K}$.

The embedding $\iota : G \hookrightarrow \tilde{G}$ admits a complexification $\iota_{\mathbb{C}} : G_{\mathbb{C}} \hookrightarrow \tilde{G}_{\mathbb{C}}$. At the level of Lie algebra, we have a morphism $\iota_{\mathbb{C}} : \mathfrak{gl}_{2p}(\mathbb{C}) \times \mathfrak{gl}_{2q}(\mathbb{C}) \hookrightarrow \mathfrak{gl}_{2n}(\mathbb{C})$, still defined by (26).

The orthogonal projection $\mathfrak{gl}_{2n}(\mathbb{C}) \rightarrow \mathfrak{gl}_{2p}(\mathbb{C}) \times \mathfrak{gl}_{2q}(\mathbb{C})$ dual to the morphism $\iota_{\mathbb{C}}$ is the map π_1 defined at the start of Section 5.

5.3 Description of $\mathcal{T}(p, q)$ through $\text{LR}(2p, 2q)$

Here is the main application of the Proposition 2.15. Recall that for $(\gamma, s, t) \in \mathbb{R}_{++}^n \times \mathbb{R}_{++}^p \times \mathbb{R}_{++}^q$, we define $\hat{\gamma} := (\gamma_1, \dots, \gamma_n, -\gamma_n, \dots, -\gamma_1)$, $\hat{s} := (s_1, \dots, s_p, -s_p, \dots, -s_1)$ and $\hat{t} := (t_1, \dots, t_q, -t_q, \dots, -t_1)$.

Proposition 5.2 *Let $(\gamma, s, t) \in \mathbb{R}_{++}^n \times \mathbb{R}_{++}^p \times \mathbb{R}_{++}^q$. The following statements are equivalent:*

1. $(\gamma, s, t) \in \mathcal{T}(p, q)$,
2. $\exists A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathfrak{gl}_n(\mathbb{C})$, such that $s(A) = \gamma$, $s(A_{11}) = s$, and $s(A_{22}) = t$,
3. $\exists M \in \tilde{\mathfrak{p}}$, with $\pi_1(M) = (M_1, M_2)$, s.t. $e(M) = \hat{\gamma}$, $e(M_1) = \hat{s}$ and $e(M_2) = \hat{t}$,

4. $\pi_1 \left(U_n \times U_n \cdot \widehat{\text{Diag}(\gamma)} \right)$ contains $\left(U_p \times U_p \cdot \widehat{\text{Diag}(s)} \right) \times \left(U_q \times U_q \cdot \widehat{\text{Diag}(t)} \right)$.
5. $\pi_1 \left(U_{2n} \cdot \widehat{\text{Diag}(\gamma)} \right)$ contains $U_{2p} \cdot \widehat{\text{Diag}(s)} \times U_{2q} \cdot \widehat{\text{Diag}(t)}$.
6. $\pi_0 (U_{2n} \cdot \widehat{\text{Diag}(\gamma)})$ contains $U_{2p} \cdot \widehat{\text{Diag}(\hat{s})} \times U_{2q} \cdot \widehat{\text{Diag}(\hat{t})}$.
7. $(\hat{\gamma}, \hat{s}, \hat{t}) \in \text{LR}(2p, 2q)$.

Proof: Equivalences “2. \iff 3.” and “6. \iff 7.” are true by definition. Equivalence “2. \iff 3.” is proved by taking $M = \widehat{A'}$ (see §5.1). Equivalence “3. \iff 4.” is obtained by noting the following relations

$$\left\{ M \in \tilde{\mathfrak{p}}; e(M) = \hat{\gamma} \right\} = U_n \times U_n \cdot \widehat{\text{Diag}(\gamma)},$$

and $\left\{ (X, Y) \in \mathfrak{p}; e(X) = \hat{s} \text{ and } e(Y) = \hat{t} \right\} = \left(U_p \times U_p \cdot \widehat{\text{Diag}(s)} \right) \times \left(U_q \times U_q \cdot \widehat{\text{Diag}(t)} \right)$. Equivalence “4. \iff 5.” follows from Proposition 2.15, and “5. \iff 6.” is a consequence of Lemma 4.1. \square

5.4 Inequalities determining $\mathcal{T}(p, q)$

Thanks to Proposition 5.2 and Theorem 2.8, we obtain the following description of the cone $\mathcal{T}(p, q)$.

Theorem 5.3 *An element $(\gamma, s, t) \in \mathbb{R}_{++}^n \times \mathbb{R}_{++}^p \times \mathbb{R}_{++}^q$ belongs to $\mathcal{T}(p, q)$ if and only if the following inequalities hold:*

1. $\gamma_k \geq s_k, \forall k \leq p$,
2. $\gamma_j \geq t_j, \forall j \leq q$,
3. $\gamma_{2q+\ell} \leq s_\ell, \forall \ell \leq p - q$,
4. we have $|\gamma|_{A \cap [n]} - |\gamma|_{A^\circ \cap [n]} \geq |s|_{B \cap [p]} - |s|_{B^\circ \cap [p]} + |t|_{C \cap [q]} - |t|_{C^\circ \cap [q]}$

for any triplets (A, B, C) satisfying the following conditions :

- $B \subset [2p]$ and $C \subset [2q]$ are strict subsets,
- $A \subset [2n]$ and $\sharp A = \sharp B + \sharp C \leq n$,
- the Littlewood-Richardson coefficient $c_{\mu(B), \mu(C)}^{\mu(A)}$ is non-zero.

5.5 Examples

The cone $\mathcal{T}(1, 1)$

We need to find subsets $B = \{b\} \subset [2]$, $C = \{c\} \subset [2]$, and $A = \{a_2 > a_1\} \subset [4]$ such that the Littlewood-Richardson coefficient $c_{\mu(B), \mu(C)}^{\mu(A)}$ is non-zero. This work has been done for the cone $\mathcal{S}(1, 1)$. Thus, an element $(\gamma, s, t) \in \mathbb{R}_{++}^2 \times \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0}$ belongs to $\mathcal{T}(1, 1)$ if and only if

$$(27) \quad \gamma_1 - \gamma_2 \geq |s - t| \quad \text{and} \quad \gamma_1 + \gamma_2 \geq s + t.$$

Note that inequalities $\gamma_1 \geq s$ and $\gamma_1 \geq t$ follow from (27).

The cone $\mathcal{T}(2, 1)$

We work with $(\gamma, s, t) \in \mathbb{R}_{++}^3 \times \mathbb{R}_{++}^2 \times \mathbb{R}^{\geq 0}$ satisfying

$$(28) \quad \gamma_1 \geq s_1, \quad \gamma_1 \geq t, \quad \gamma_2 \geq s_2, \quad \gamma_3 \leq s_1.$$

We're now interested in the inequalities associated with triplets (A, B, C) such that $c_{\mu(B), \mu(C)}^{\mu(A)}$ is non-zero and $\sharp A = \sharp B + \sharp C \leq 3$. We obtain the following inequalities when $\sharp A = 2$:

$$(29) \quad \begin{aligned} \gamma_1 + \gamma_2 &\geq s_1 + t \\ \gamma_1 + \gamma_3 &\geq s_2 + t \\ \gamma_1 - \gamma_3 &\geq |s_2 - t| \\ \gamma_1 - \gamma_2 &\geq -s_1 + t. \end{aligned}$$

In the list (29), I haven't included inequality $\gamma_2 + \gamma_3 \geq s_2 - t$, which is associated with $A = \{2, 3\}$, $B = \{2\}$, $C = \{2\}$, since it follows from (28) and the fact that $\gamma_3, t \geq 0$. When $\sharp A = 3$ we get:

$$(30) \quad \begin{aligned} \gamma_1 + \gamma_2 + \gamma_3 &\geq s_1 + s_2 + t \\ \gamma_1 + \gamma_2 - \gamma_3 &\geq s_1 - s_2 + t \\ \gamma_1 + \gamma_2 - \gamma_3 &\geq s_1 + s_2 - t \\ \gamma_1 - \gamma_2 + \gamma_3 &\geq -s_1 + s_2 + t \\ \gamma_1 - \gamma_2 - \gamma_3 &\geq -s_1 + s_2 - t \\ \gamma_1 - \gamma_2 - \gamma_3 &\geq -s_1 - s_2 + t. \end{aligned}$$

Thus, an element $(\gamma, s, t) \in \mathbb{R}_{++}^3 \times \mathbb{R}_{++}^2 \times \mathbb{R}^{\geq 0}$ belongs to $\mathcal{T}(2, 1)$ if and only if the inequalities (28), (29) and (30) are satisfied.

5.6 Interlacing inequalities for singular values

Let us consider the case where $p \geq q = 1$.

Let $\gamma_1 \geq \cdots \geq \gamma_{p+1} \geq 0$ be the singular values of a $p+1$ -square complex matrix X . Let X' be the p -square submatrix of X obtained by deleting a row and a column: we denote by $s_1 \geq \cdots \geq s_p \geq 0$ its singular spectrum.

Points 1. and 3. of Theorem 5.3 yields interlacing inequalities which were first observed by Thompson [22]:

$$\begin{aligned} \gamma_3 &\leq s_1 \leq \gamma_1, \\ \gamma_4 &\leq s_2 \leq \gamma_2, \\ &\dots \\ \gamma_{j+2} &\leq s_j \leq \gamma_j, \quad 1 \leq j \leq p-2, \\ &\dots \\ \gamma_p &\leq s_{p-2} \leq \gamma_{p-2}, \\ s_{p-1} &\leq \gamma_{p-1}, \\ s_p &\leq \gamma_p. \end{aligned}$$

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