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On the parameterized complexity of non-hereditary relaxations of clique

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Abstract

We investigate the parameterized complexity of several problems formalizing cluster identification in graphs. In other words we ask whether a graph contains a large enough and sufficiently connected subgraph. We study here three relaxations of CLIQUE: s -CLUB and s -CLIQUE, in which the relaxation is focused on the distances in respectively the cluster and the original graph, and γ -COMPLETE SUBGRAPH in which the relaxation is made on the minimal degree in the cluster. As these three problems are known to be NP-hard, we study here their parameterized complexities. We prove that s -CLUB and s -CLIQUE are NP-hard even restricted to graphs of degeneracy ≤ 3 whenever $s \geq 3$, and to graphs of degeneracy ≤ 2 whenever $s \geq 5$, which is a strictly stronger result than its W[1]-hardness parameterized by the degeneracy. Concerning γ -COMPLETE SUBGRAPH, we prove that it is W[1]-hard parameterized both by the degeneracy, implying the W[1]-hardness parameterized by the number of vertices in the γ -complete-subgraph, and by the number of elements outside the γ -complete subgraph.

1 Introduction

Finding large clusters in graphs is a fundamental problem in graph theory. Among the applications, one could think of friends recommendation in social networks as it might help to find users with similar interests, as well as the applications to biological networks, where two elements in the same clusters can reasonably be considered to share a common behavior [1, 2, 20].

A very natural way to formalize the concept of clusters could be to define them as induced cliques in the given graph. The problem CLIQUE of finding a

large enough clique in the input graph is of paramount importance in computational complexity theory, and is even one of Karp's 21 NP-complete problem [9]. Not only is CLIQUE NP-complete, it is also known that the size of the largest clique is NP-hard to approximate with a factor $O(n^{1-\varepsilon})$ with n the number of vertices and $\varepsilon > 0$ is fixed [22].

However, asking for a clique is often too restrictive in practice, as it is reasonable to consider a subgraph to be a cluster even if a few edges are missing between some pairs of elements. Therefore, relaxations of CLIQUE are considered: one can think of a relaxation by degree, connectivity, density, or distance [11].

Unsurprisingly, most of these relaxations have quickly been proven to be NP-hard [4, 14, 15, 16, 17, 21]. In order to get a better understanding of these problems, it is then natural to tackle them under the paradigm of parameterized complexity. Among the relevant parameters to consider, one could think of the number of vertices of the sought cluster k , the number of vertices left out of the cluster $\ell = n - k$ (with n the number of vertices of the input graph), the degree of the graph Δ , the h -index of the graph h , and its degeneracy d .

The case of CLIQUE parameterized by k has been treated successfully thanks to powerful meta-theorems [10] relying on the heredity of the class of cliques, i.e. stable by vertex deletion, which leads to its W[1]-hardness. Concerning the parameterization by d , it is sufficient to observe that a clique is always contained in the neighborhood of each of its vertices to get an FPT algorithm [11]. As an immediate consequence, we get that CLIQUE is also FPT when parameterized by Δ and h , since $d \leq h \leq \Delta$. The case of ℓ can be treated with the very well known FPT-reduction from VERTEX-COVER parameterized by k which leads to an FPT algorithm running in time $O(1.28^\ell + n^2)$ with n the number of vertices, making use of the $O(1.28^k + n^2)$ algorithm for VERTEX-COVER [5].

Unfortunately, treating the relaxations of CLIQUE is usually a lot more difficult. A major technical obstruction to the study of the parameterized complexity of the relaxation of CLIQUE is often that the class of clusters defined in these problems are not hereditary. Therefore, the most commonly useful meta-theorems [10] cannot be applied, and both algorithms and reductions become more difficult to find. Despite this difficulty, the parameterized complexity of the γ -QUASI-CLIQUE problem, (with $\gamma \in]0, 1[$ a rational) -that looks for a subgraph of size at least k and of density at least γ - has been completely classified regarding the parameters k, ℓ, Δ, h and d evoked earlier [13]. This success can be attributed to the quasi-hereditary nature of the class of γ -quasi-clique (one vertex can always be removed from a γ -quasi-clique of size ≥ 2 to get a smaller γ -quasi-clique). However, the problems tackled here are even harder to handle as it presents no hereditary properties. Therefore, the techniques used here to circumvent the non-heredity are quite novel and might be applied to other non-hereditary problems.

Two different distance based relaxations of CLIQUE, have been introduced so far: given an integer $s \geq 2$, they both ask for a subgraph of size k where every pair of vertices is at distance at most s . If the distance considered is the distance in the induced subgraph chosen, the problem defined thereby is known as the

s -CLUB problem, whose multivariate complexity analysis is now completed for $s = 2$ and almost completed for other values of s [4, 8, 11, 18] (despite not being quasi-hereditary). If the distance is taken in the original graph instead, the problem is called the s -CLIQUE problem, which received significantly less attention than s -CLUB. Surprisingly enough, s -CLUB and s -CLIQUE with $s \geq 2$ are FPT parameterized by k (and ℓ) [11, 18]. This differs from the 1 -CLUB = 1 -CLIQUE = CLIQUE problem which is known to be W[1]-hard parameterized by k . Some additional results are known in the particular case of 2 -CLUB: it is W[1]-hard when parameterized by the h -index, and it is NP-hard even for graphs of degeneracy at most 6 [8]. However, it was still open to determine if s -CLUB for $s \geq 3$ shares this common behavior with 2 -CLUB.

Our first contribution is to complete the study of s -CLUB parameterized by the degeneracy d . We prove in Section 3 that, for $s \geq 3$, it is NP-hard even on graphs of degeneracy ≤ 3 . This implies that s -CLUB is para-NP-hard when parameterized by d . This result does not only prove the W[1]-hardness of s -CLUB when parameterized by d , but also rule out the existence of any XP algorithm (i.e. algorithm running in time $O(F(d) \times n^{F(d)})$ with n the size of the graph and F a computable function), under the assumption that $P \neq NP$. Moreover, our reduction leads to a bipartite graph if s is odd, and to a graph of degeneracy 2 if $s \geq 5$, leading to the NP-hardness on these respective classes of graphs.

In addition, we show that our reduction to s -CLUB with $s \geq 3$ can also be used to derive the exact same complexity results for s -CLIQUE. As it is known that s -CLUB and s -CLIQUE can be solved in polynomial time on graphs of degeneracy 1 [19], proving that the NP-hardness on graphs of degeneracy 2 obtained when $s \geq 5$ is essentially optimal.

Concerning the degree based relaxations of CLIQUE, the γ -COMPLETE SUBGRAPH problem (with $\gamma \in]0, 1[$ a rational), where one looks for a subgraph of size at least k where every vertex has a proportion at least γ of neighbors, has received very little attention, possibly because the class of γ -complete-graphs is not even quasi-hereditary. So far, it is only known that γ -COMPLETE SUBGRAPH is FPT when parameterized by the h -index (which leads to it being FPT when parameterized by Δ) and W[1]-hard when parameterized by k , but only in the case where $\gamma \geq 0.5$ [3].

Our second and third contributions in this paper consist in the completion of the multivariate complexity analysis of γ -COMPLETE SUBGRAPH: we first prove in Section 4 that it is W[1]-hard when parameterized by d , and as an immediate corollary, we obtain that γ -COMPLETE SUBGRAPH is also W[1]-hard when parameterized by k , extending the result of Baril *et al.* [3] from rational $\gamma \in [0.5, 1[$ to any rational $\gamma \in]0, 1[$ (note that the case $\gamma = 0$ corresponds to a trivial problem, and that the case $\gamma=1$ corresponds exactly to the CLIQUE problem and is thus already known). Second, in Section 5, we prove that γ -COMPLETE SUBGRAPH is W[1]-hard when parameterized by ℓ . Note that trivial XP algorithms exist for both of these parameterizations and thus the W[1]-hardness results obtained cannot be extended to para-NP-hardness results.

In order to circumvent the non-heredity, for γ -COMPLETE SUBGRAPH, our

reduction is made with constraints strong enough so that if there is a solution, it has size exactly k , whereas for s -club our reduction guarantees that if a solution exists, a lot of vertices are forced to be in it.

Our contributions are summarized in Figure 1. Notice that the only computational problem of this list that were already completely classified were 2-CLUB, CLIQUE and γ -QUASI CLIQUE (the latter two being respectively hereditary and quasi-hereditary), whereas the parameterized complexity of the non-hereditary problems s -CLUB and γ -COMPLETE SUBGRAPH remained unknown.

Problem	k	ℓ	h	d
CLIQUE	<u>W[1]-h [6]</u>	FPT [11]	FPT [11]	FPT [11]
2-CLUB	FPT [11]	FPT [11]	W[1]-h[8]	NP-h for $d = 6$ [8]
s -CLUB with $s \in \{3, 4\}$	FPT [11]	FPT [11]	?	<u>NP-h for $d = 3$</u>
s -CLUB with $s \geq 5$	FPT [11]	FPT [11]	?	<u>NP-h for $d = 2$</u>
2-CLIQUE	FPT [11]	FPT [11]	?	?
s -CLIQUE with $s \in \{3, 4\}$	FPT [11]	FPT [11]	?	<u>NP-h for $d = 3$</u>
s -CLIQUE with $s \geq 5$	FPT [11]	FPT [11]	?	<u>NP-h for $d = 2$</u>
γ -QUASI CLIQUE	W[1]-h [13]	W[1]-h [13]	FPT [13]	W[1]-h [13]
γ -COMPLETE SUBGRAPH	<u>W[1]-h for any γ</u>	<u>W[1]-h</u>	FPT [3]	<u>W[1]-h</u>

Figure 1: Complexities of relaxations of CLIQUE, our contributions underlined.

2 Preliminaries

In this paper, all graphs will be simple, loopless and non-oriented. In any context where a graph is called $G = (V_G, E_G)$, we will denote by $n := |V_G|$ its number of vertices, $m := |E_G|$ its number of edges and $\bar{m} := |\bar{E}_G| = \binom{n}{2} - m$ its number of non-edges. Let $G = (V_G, E_G)$ be a graph for the rest of this section.

Given a set V , we denote by $\binom{V}{2}$ the set of pairs of V . Formally, we have $\binom{V}{2} := \{\{u, v\} \mid (u, v) \in V^2, u \neq v\}$. If V is a set of vertices, $\binom{V}{2}$ is the set of edges in the complete graph whose vertex set is V .

Notation 1. For $(u, v) \in (V_G)^2$, denote by $\deg_G(u)$ the degree of u in G , and $\text{dist}_G(u, v)$ the distance between u and v in G . For $r \geq 1$ and $u \in V_G$, denote by $\bar{B}(u, r) := \{v \in V_G \mid \text{dist}_G(u, v) \leq r\}$ the ball of center u and radius r .

For $S \subseteq V_G$, denote by $G[S]$ the graph induced on G by S , i.e. the graph $(S, E_G \cap \binom{S}{2})$. Also, for any $u \in V_G$ (respectively $(u, v) \in (V_G)^2$), we denote by $\deg_S(u)$ (respectively $\text{dist}_S(u, v)$) the number of neighbors of u in S , i.e. degree of u in the graph $G[S \cup \{u\}]$ (respectively the distance between u and v in $G[S \cup \{u, v\}]$). For $w \in V_G$, denote by $G - w$ the graph $G[V_G \setminus \{w\}]$: this is the graph obtained from G after the deletion of w .

Definition 2. The *diameter* of a connected graph G denoted by $\text{diam}(G)$ is the maximal distance between two vertices in G .

By an easy induction on m , we obtain Lemma 3 (also known as the "hand-shaking lemma").

Lemma 3. For any graph G , we have $\sum_{u \in V_G} \deg_G(u) = 2m$.

2.1 Parameterized complexity

Given an computational problem Π on a language Σ^* , a *parameter* of Π is a function $\lambda : \Sigma^* \mapsto \mathbb{N}$ computable in polynomial time (where \mathbb{N} is the set of positive integers). A parameterized problem is a couple of the form (Π, λ) where Π is an computational problem, and λ is a parameter of Π .

We say that an algorithm is FPT (Fixed Parameter Tractable) parameterized by a parameter λ if it runs in time $O(F(\lambda(x)) \times \|x\|^{O(1)})$ on any instance x of size $\|x\|$. Here, F can be any computable function (that can be assumed to be monotonous). A parameterized problem (Π, λ) is said to be FPT if it can be solved by an FPT algorithm. Note that a polynomial time is in particular an FPT time for any parameter.

A *FPT-reduction* from a parameterized problem (Π_1, λ_1) to a parameterized problem (Π_2, λ_2) is a function R that maps any instance x of Π_1 to an instance $R(x)$ of Π_2 that satisfies the following properties:

- For all instance x of Π_1 , $x \in \Pi_1 \iff R(x) \in \Pi_2$.
- R is computable in FPT time parameterized by λ_1 .
- For all instance x of Π_1 , $\lambda_2(R(x)) \leq G(\lambda_1(x))$ (here, G can be any computable function).

The interest of this definition is that if there exists a FPT-reduction from (Π_1, λ_1) to (Π_2, λ_2) and if (Π_2, λ_2) is FPT, then so is (Π_1, λ_1) .

In this paper, we say that a parameterized problem (Π, λ) is $W[1]$ -hard if it can be FPT-reduced from (CLIQUE, k) (where k is the size of the demanded clique). Note that it is not the rigorous definition of the $W[1]$ -hardness in the literature, but this definition is equivalent, and sufficiently precise for the purpose of this paper.

Remark 4. Take two parameters λ_1 and λ_2 on the same non-trivial (i.e., neither always yes nor always no) computational problem Π , such that (Π, λ_1) is $W[1]$ -hard, and $\lambda_2 \leq \lambda_1$. Then, (Π, λ_2) is also $W[1]$ -hard. More generally, if we only assume that there exists a poly-time computable function G such that Π is solvable in polynomial time on any instance x that does not satisfy $\lambda_2(x) \leq G(\lambda_1(x))$, then (Π, λ_2) is also $W[1]$ -hard.

Proof. In the case where $\lambda_2 \leq \lambda_1$, the reduction that maps any instance to itself is a FPT-reduction.

In the more general case, the reduction R consists on an instance x , to test in polynomial time if $\lambda_2(x) \leq G(\lambda_1(x))$ holds. In case it does not, the problem is solvable in polynomial time: solve it and return any equivalent small instance (of size $O(1)$), and in case it does, set $R(x) = x$, we indeed have $\lambda_2(R(x)) \leq G(\lambda_1(x))$. □

Analogously to the $P \neq NP$ conjecture in polynomial complexity, the central conjecture in the field of parameterized complexity is that $FPT \neq W[1]$ (the definition of the class $W[1]$ is omitted here). This conjecture is widely believed to be true, and implies that any $W[1]$ -hard problem can not be FPT (similarly to how under $P \neq NP$, any NP -hard problem can not be polynomial).

A parameterized problem (Π, λ) is said to be XP if it can be solved in polynomial time on any class of instance of bounded parameter, or equivalently, if it can be solved in time $O(\|x\|^{F(\lambda(x))})$ on any instance x of size $\|x\|$ (here, F can be any computable function). Notice that $FPT \subseteq XP$, but the reverse is false under $FPT \neq W[1]$. For instance, (CLIQUE, k) is $W[1]$ -hard, but belongs to XP , since testing every set of vertices of size k of the input graph and checking if it is a clique can be done in time $O(k^2 \times \binom{n}{k}) = O(k^2 \times n^k) = O(n^{k+2})$ (as $k \leq n$).

Remark 5. *Take two parameters λ_1 and λ_2 on the same computational problem Π such that (Π, λ_2) is XP , and $\lambda_2 \leq \lambda_1$. Then, (Π, λ_1) is also XP . More generally, if we only assume that there exists a poly-time computable function G such that Π is solvable in polynomial time on any instance x that does not satisfy $\lambda_2(x) \leq G(\lambda_1(x))$, then (Π, λ_1) is also XP .*

Proof. In the case where $\lambda_2 \leq \lambda_1$ any algorithm that is XP parameterized by λ_2 is also XP parameterized by λ_1 .

In the more general case, on an instance x , one can test in polynomial time if $\lambda_2(x) \leq G(\lambda_1(x))$ holds. In case it does not, the problem is solvable in polynomial time, and in case it does, (Π, λ_2) being XP leads to an algorithm running in time $O(\|x\|^{F(\lambda_2(x))}) = O(\|x\|^{H(\lambda_1(x))})$ with $H = F \circ G$. □

A parameterized problem (Π, λ) is said to be para- NP -hard if it is NP -complete on a class of instances of Π bounded for λ . Note that under $P \neq NP$, a para- NP -hard problem can not be in XP . Note also that any para- NP -hard problem is automatically $W[1]$ -hard (the reduction ensuring the para- NP -hardness is indeed also an FPT -reduction), even though the reverse is not true. For instance, (CLIQUE, k) is $W[1]$ -hard, but is not para- NP -hard, since it is in XP .

2.2 Problems studied

To prove the $W[1]$ -hardness and para- NP -hardness of relaxations of CLIQUE , it seems natural to start our reduction from the CLIQUE problem, which is known to be $W[1]$ -hard when parameterized by k , the size of the demanded clique.

Problem (CLIQUE).

Input: A graph G and an integer k .

Question: Does there exist a subset $K \subseteq V_G$ with $|K| = k$, and such that $\forall (u, v) \in K^2, u \neq v \implies \{u, v\} \in E_G$?

This problem is known to be $W[1]$ -hard parameterized by k . However, for some technical reasons, it might be useful to assume additional properties on k . In fact, one can notice that for any subset $I = \{ar + b \mid r \in \mathbb{N}, r \geq r_0\}$ with three constants a, b and r_0 with $a \in \mathbb{N}^*$ and $(b, r_0) \in \mathbb{N}^2$, the problem:

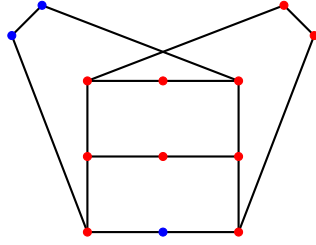


Figure 2: The above graph does not contain any 3-club of size ≥ 10 . However, the red vertices form a 3-clique of size 10.

Problem (CLIQUE(I)).

Input: A graph G and an integer $k \in I$.

Question: Does there exist a subset $K \subseteq V_G$ with $|K| = k$, and such that $\forall(u, v) \in K^2, u \neq v \implies \{u, v\} \in E_G$?

is still $W[1]$ -hard when parameterized by k . Indeed, any instance (G, k) of CLIQUE is equivalent to the instance $(G + q, k + q)$ of CLIQUE for any $q \geq 0$ (denoting by $G + q$ the graph G where q universal vertices have been added). By choosing q such that $k + q \in I$, we can always reach an instance of CLIQUE(I). Notice also that the number q of vertices to add depend only on k and not on the graph G , which proves that CLIQUE(I) is also $W[1]$ -hard.

In the first part of this paper, we will study for $s \geq 2$ the s -CLUB and s -CLIQUE problem, formally defined as:

Problem (s -CLUB).

Input: A graph G and an integer k .

Question: Does there exist a subset $S \subseteq V_G$ with $|S| \geq k$, and such that $\forall(u, v) \in S^2, \text{dist}_S(u, v) \leq s$?

Problem (s -CLIQUE).

Input: A graph G and an integer k .

Question: Does there exist a subset $S \subseteq V_G$ with $|S| \geq k$, and such that $\forall(u, v) \in S^2, \text{dist}_G(u, v) \leq s$?

Note that the only difference between s -CLUB and s -CLIQUE is that the distance is taken in $G[S]$ in the case of s -CLUB, and in G in the case of s -CLIQUE. A subset of vertices $S \subseteq V_G$ satisfying $\forall(u, v) \in S^2, \text{dist}_S(u, v) \leq s$ (respectively $\text{dist}_G(u, v) \leq s$) is called a s -club of G (respectively a s -clique of G). Notice that for $s = 1$, the three problems s -CLUB, s -CLIQUE, and CLIQUE are the same.

Also, even if these concepts seem highly similar, it is important to note that they are different. Figure 2 presents a graph with a 3-clique of size 10 and no 3-club of size ≥ 10 .

Then, given a rational $\gamma \in]0, 1[$, we will focus on the γ -COMPLETE SUBGRAPH problem, formally defined as:

Problem (γ -COMPLETE SUBGRAPH).

Input: A graph G and an integer k .

Question: Does there exist a subset $S \subseteq V_G$ with $|S| \geq k$, and such that $\forall u \in S, \deg_S(u) \geq \gamma(|S| - 1)$?

A subset of vertices $S \subseteq V_G$ satisfying $\forall u \in S, \deg_S(u) \geq \gamma(|S| - 1)$ is called a γ -complete subgraph of G . The γ -COMPLETE SUBGRAPH problem is said to be a relative relaxation of clique by degree. Note that indeed, if $\gamma = 1$, γ -COMPLETE SUBGRAPH corresponds to CLIQUE.

Note that when studying the multivariate complexity of s -CLUB or s -CLIQUE (respectively γ -COMPLETE SUBGRAPH), the value of s (respectively γ) will always be considered to be a constant and will never be considered to be a part of the input. Thus, in the FPT-reduction we will perform, the size of the outputs and the values of the considered parameters are allowed to depend on s (respectively γ), but we won't be able to choose a particular value of s (respectively γ) depending on the input.

Note also that not authorizing s (respectively γ) to be part of the input strengthen our results of hardness, since it proves that the problem with s (respectively γ) as part of the input stays hard even if a particular value of s (respectively γ) is fixed.

2.3 Notations for parameters

In this subsection, we introduce the notations and the definitions of the parameter used to study the parameterized complexities of s -CLUB, s -CLIQUE and γ -COMPLETE SUBGRAPH.

- The parameter k , consistently with the notation used in the definition of CLIQUE, s -CLUB, s -CLIQUE and γ -COMPLETE SUBGRAPH, will always refer to the (minimal) size of an aimed subgraph (either a clique, an s -club, an s -clique or a γ -complete-subgraph).
- The parameter $\ell := n - k = |V_G| - k$ designs the (maximal) number of vertices outside the aimed subgraph.
- The parameter $\Delta := \Delta(G)$ is the maximal degree in the input graph.
- The parameter $h := h(G)$ is the h -index of the input graph G , i.e. the largest integer h such that G has at least h vertices of degree at least h .
- The parameter $d := d(G)$ designs the degeneracy of the input graph G , i.e. the largest integer d such that every subgraph of G has a vertex of degree at most d . A graph G is d -degenerated (i.e. its degeneracy is upper-bounded by d) if, and only if, there is an order over the vertices of G in which each vertex has at most d inferior neighbors. Such an order is called a d -elimination order, or simply an elimination order. Note that an order \leq over the vertices of G is a d -elimination order if, and only if, when eliminating sequentially the vertices by decreasing order relatively to \leq , we eliminate a vertex of degree at most d at each step.

Problem	k	ℓ	h	d
CLIQUE	W[1]-h[6]	FPT [11]	FPT [11]	FPT [11]
2-CLUB	FPT [11]	FPT [11]	W[1]-h [8]	NP-h for $d = 6$ [8]
s -CLUB with $s \geq 3$	FPT [11]	FPT [11]	?	?
s -CLIQUE	FPT [11]	FPT [11]	?	?
γ -COMPLETE SUBGRAPH	W[1]-h if $\gamma \geq \frac{1}{2}$ [3]	?	FPT [3]	?

Figure 3: State of the art of parameterized complexities of relaxations of CLIQUE.

One can remark that on any graph G , we have $d \leq h \leq \Delta$. As a consequence, every W[1]-hard problem parameterized by Δ is also W[1]-hard parameterized by both h or d , and every FPT problem parameterized by d is automatically FPT when parameterized by both h or Δ .

2.4 Contributions

Let $s \geq 2$ an integer and $\gamma \in]0, 1[$ a rational. The table in Figure 3 sums up the state of the art on the parameterized complexities of CLIQUE, s -CLUB, s -CLIQUE and γ -COMPLETE SUBGRAPH parameterized by k , ℓ , Δ , h and d . We omit the column representing Δ since each of these problems is FPT when parameterized by Δ [11].

In this paper, we give five contributions:

- We prove in Section 3 that if $s \geq 3$, s -CLUB is NP-hard even on graphs of degeneracy at most 3. This implies in particular that for $s \geq 3$, s -CLUB is para-NP-hard when parameterized by d . Moreover, this result of NP-hardness holds even on bipartite graphs if $s \geq 5$ is odd
- Also in Section 3, we prove that the results listed in the previous point also hold for s -CLIQUE.
- We prove in Section 4 that γ -COMPLETE SUBGRAPH is W[1]-hard when parameterized by the degeneracy d of the input graph.
- As an immediate corollary, we obtain that γ -COMPLETE SUBGRAPH is W[1]-hard when parameterized by k regardless of the value of γ , extending the result by Baril *et al.* [3] even for $\gamma < \frac{1}{2}$.
- Finally, we prove in Section 5 that γ -COMPLETE SUBGRAPH is W[1]-hard when parameterized by ℓ : the minimal number of vertices not taken in the γ -complete subgraph.

3 s -club and s -clique NP-hard for graphs of bounded degeneracy

We first study the s -CLUB and s -CLIQUE problems for $s \geq 3$. We establish that they are para-NP-hard when parameterized by the degeneracy. More precisely, if $s \geq 3$ is odd, s -CLUB and s -CLIQUE are NP-hard even on bipartite graphs of degeneracy 3 (even of degeneracy 2 if $s \geq 5$), and if $s \geq 4$ is even, s -CLUB and s -CLIQUE are NP-hard even on graphs of degeneracy 3 (even of degeneracy 2 if $s \geq 6$).

When performing our reductions, we will need to distinguish whether s is odd or even. We prove four distinct theorems in order to treat these two cases separately (and depending on whether we study s -CLUB or s -CLIQUE) even though the conclusion is essentially the same (despite the graph obtained for s even not being bipartite). We treat the odd case in Theorem 6 for s -CLUB and in Theorem 8 for s -CLIQUE, and the even case in Theorem 9 for s -CLUB and in Theorem 11 for s -CLIQUE.

Theorem 6. *For any $s \geq 3$ odd, s -CLUB is NP-hard, even on bipartite 3-degenerate graphs. Moreover, if $s \geq 5$, s -CLUB is NP-hard, even on bipartite 2-degenerate graphs.*

Proof. Let $s \geq 3$ be an odd integer.

We provide a NP-hardness reduction from CLIQUE. Let (G, k) be an instance of CLIQUE. Without loss of generality, we assume that G has no isolated vertices. We build an instance (G', k') of s -CLUB as in Figure 4 and Figure 5.

Construction of G' :

First, in order to decrease the degeneracy of the input graph and to increase the distances between the original vertices we subdivide $s - 2$ times each edge by replacing them with a path of red vertices of length $s - 1$ (i.e. with $s - 1$ edges) as done in Figure 4.

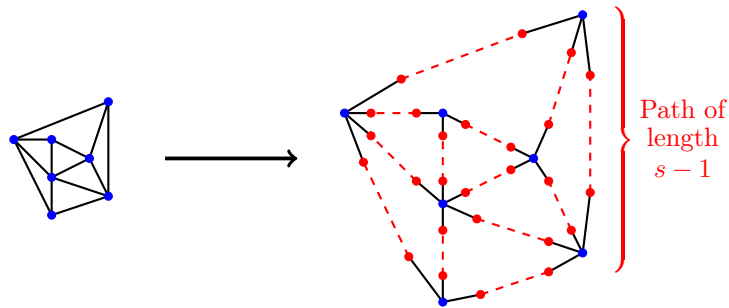


Figure 4: 2-degeneracy transformation of a graph.

The rest of the construction of G' is done as follows. $V_{G'}$ contains three types of vertices:

- The original vertices of V_G which we call the blue vertices.
- $s - 2$ red vertices for each original edge $\{u, v\} \in E_G$ added during the subdivision of the edge. The set formed by the red vertices is denoted V_R .
- Finally, we add a new vertex y which we call yellow.

The vertices of $V_{G'}$ are connected by the following edges:

- For each original edge $\{u, v\} \in E_G$, the edges of the subdivision exist in G' .
- The yellow vertex y is linked to the middle vertex of each red path.

The graph obtained by this construction is illustrated in Figure 5. Note that the graph G' can be constructed in time $n^{O(1)}$ where $n = |V_G|$.

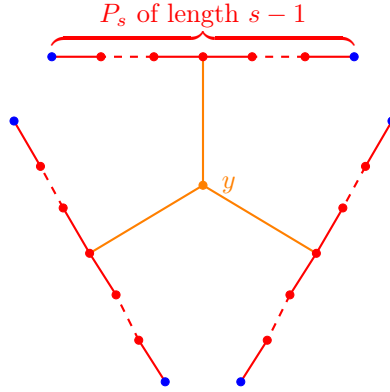


Figure 5: Construction of G' on G being three disjoint edges for s odd.

Soundness of the reduction:

Let $k' := k + |V_R| + 1 = k + (s - 2)m + 1$. We prove that G has a clique of order k if, and only if, G' has an s -club of order k' .

Claim 7. For every $(u, v) \in (V_{G'})^2$ with $u \neq v$, we have:

1. $\text{dist}_{G'}(u, v) \leq s$ if either u or v is not in V_G
2. else, $\text{dist}_{G'}(u, v) = \begin{cases} s - 1 & \text{if } \{u, v\} \in E_G \\ s + 1 & \text{if } \{u, v\} \notin E_G \end{cases}$

3. For any $(u, v) \in (V_{G'})^2$ with $\text{dist}_{G'}(u, v) \leq s$, there exists a path P from u to v of length $\leq s$ such that $P \setminus \{u, v\}$ does not intersect V_G : i.e. $\text{dist}_{V_{G'} \setminus V_G}(u, v) \leq s$.

Proof of the claim.

1. We notice that $(V_{G'} \setminus V_G)$ is included in $\overline{B}_{G'}\left(y, \frac{s-1}{2}\right)$, the closed ball of center y and radius $\frac{s-1}{2}$. Thus, every pair of non-blue vertices of G' (i.e. in $V_{G'} \setminus V_G$) is at distance at most $s-1$. Now, since every blue vertex $u \in V_G$ has a neighbor in V_R , it is at a distance lower than s from any non-blue vertex.
2. By construction, for all pair of blue vertices that forms an edge in G : $\{u, v\} \in E_G$, we have $\text{dist}_{G'}(u, v) = s-1$ (taking the red path of length $s-1$ that comes from the subdivision of the original edge $\{u, v\}$).
Conversely, any shortest path between two blue vertices u and v in V_G such that $\{u, v\} \notin E_G$ is, starting from u , first getting to y in $\frac{s-1}{2} + 1$ steps, and symmetrically getting to v in $1 + \frac{s-1}{2}$ steps, which gives a path of length $s+1$. Indeed, not using y would imply to take the red vertices that come from the subdivisions of at least two edges in G , which is a path of length at least $2(s-1) \geq s+1$, since $s \geq 3$.
3. The paths given by what precedes in the cases $\text{dist}_{G'}(u, v) \leq s$ satisfy the requirement. \diamond

We now prove that G has a clique of order k if and only if G' has a s -club of order at least k' .

- \implies Suppose G has a clique $K \subseteq V_G$ of order k . Consider in G' the set $S := K \cup V_R \cup \{y\}$. Note that $|S| = k'$. According to Claim 7.1 and 7.2, since S does not contain any pair $\{u, v\} \notin E_G$ (because K is a clique), every pair of vertices is at distance $\leq s$ in G' . Note that since $V_R \cup \{y\} \subseteq S$, we get by Claim 7.3 that every pair of vertices in S is at distance $\leq s$ in $G[S]$. Therefore, S is an s -club of G' .
- \impliedby Reversely, if G' contains an s -club S of order at least k' : let $K := S \cap V_G$. As $|V_{G'} \setminus V_G| = k' - k$, we have $|K| \geq k$. Using that S is an s -club and that the distance between any two vertices in S is necessarily greater than their distance in G' , we get by Claim 7.1 and 7.2 that $\forall (u, v) \in K^2$ with $u \neq v$, $\{u, v\} \in E_G$, i.e. K is a clique of G .

Degeneracy of G' and bipartition:

We prove that G' has degeneracy at most 3:

- First, remove the red vertices in any order: they all have degree 3 or less.
- Then, we are left with an independent set, remove them in an arbitrary order.

Moreover, if $s \geq 5$, by beginning with the elimination of red vertices that are non-adjacent to y , we start with degree 2 vertices. Then, we can remove the red vertices adjacent to y that now have degree 1. Then, we are left with an independent set. This proves that the degeneracy of G' is bounded by 2 if $s \geq 5$.

We prove that G' is bipartite by showing that it does not contain any odd cycle. Let C be a cycle of G' . First it is easy to see that the subgraph induced by the red and yellow vertices is a tree. Hence, C contains at least a blue vertex. Second, let P be a simple path (or cycle) linking two consecutive blue vertices (possibly the same) in this cycle (possibly, $P = C$) using thus only red and yellow vertices. Either P does not use y : it is an edge subdivided, i.e. a path of length $s - 1$. Or P uses y and is constituted of: half of a subdivided edge, i.e. a path of length $\frac{s-1}{2}$, two edges containing y and another half of another subdivided edge. Thus, the total length of P is $s+1$. In both cases the length of P is even. Since C contains a blue vertex, it is a concatenation of paths between blue vertices. Hence, the length of C is a sum of even integers and thus is even. \square

We now adapt the proof to also extend the result of Theorem 6 to s -CLIQUE, in the form of Theorem 8.

Theorem 8. *For any $s \geq 3$ odd, s -CLIQUE is NP-hard, even on bipartite 3-degenerate graphs. Moreover, if $s \geq 5$, s -CLIQUE is NP-hard, even on bipartite 2-degenerate graphs.*

Proof. We provide a NP-hardness reduction from CLIQUE. Let (G, k) be an instance of CLIQUE.

Consider the graph G' and the integer k' defined in the proof of Theorem 6. We already know that G' is bipartite, and we already have studied the degeneracy of G' .

There only remains to prove that G has a clique of size k if, and only if, G' has an s -clique of size k' . We will consider exactly the same sets as in the proof of Theorem 6.

\implies Suppose G has a clique $K \subseteq V_G$ of order k . Consider in G' the set $S = K \cup V_R \cup \{y\}$. Note that $|S| = k'$. According to Claim 7.1 and 7.2, since S does not contain any pair $\{u, v\} \notin E_G$ (because K is a clique), every pair of vertices is at distance $\leq s$ in G' : S is an s -clique.

\impliedby Reversely, if G' contains an s -clique S of order at least k' : let $K = S \cap V_G$. As $|V_{G'} \setminus V_G| = k' - k$, we have $|K| \geq k$. Using that S is an s -clique, we get by Claim 7.1 and 7.2 that $\forall (u, v) \in K^2$ with $u \neq v$, $\{u, v\} \in E_G$, i.e. K is a clique of G .

□

The reduction for s even is slightly more complicated, but still leads to a graph of degeneracy ≤ 3 (even of degeneracy ≤ 2 if $s \geq 6$), even though the graph obtained is not bipartite. Note that the case $s = 2$ is covered neither by Theorem 9 nor is it by Theorem 11.

Theorem 9. *For any $s \geq 4$ even, s -CLUB is NP-hard even on 3-degenerate graphs. Moreover, if $s \geq 6$, s -CLUB is NP-hard even on 2-degenerate graphs.*

Proof. Let $s \geq 4$ be an even integer.

We provide a NP-hardness reduction from CLIQUE. Let (G, k) be an instance of CLIQUE. Without loss of generality, we assume that G has no isolated vertices and at least 2 edges. We build an instance (G', k') of s -CLUB, as in Figure 4 and Figure 6.

Construction of G' :

We start again by subdividing $s - 2$ times each original edge of G as done in Figure 4.

The rest of the construction of G' is done as follows. $V_{G'}$ contains four types of vertices:

- The original vertices of V_G which we call the blue vertices.
- $s - 2$ red vertices for each original edge $(u, v) \in E_G$ added during the subdivision of the edge we note these vertices respectively $e_{u,v}^1, \dots, e_{u,v}^{s-2}$. The set formed by the red vertices is denoted V_R .
- For any pair of red vertices $\{e_{u,v}^i, e_{u',v'}^j\}$ with $1 \leq i, j \leq s - 2$ and $\{u, v\} \neq \{u', v'\}$, we add $s - 2$ new vertices which we call the green vertices.
- Finally, we add a new yellow vertex y .

The vertices of $V_{G'}$ are linked by the following edges:

- For each original edge $(u, v) \in E_G$, the edges of the subdivision exist in G' . i.e. u, v and the $s - 2$ corresponding red vertices $e_{u,v}^1, \dots, e_{u,v}^{s-2}$ form a path $u, e_{u,v}^1, \dots, e_{u,v}^{s-2}, v$ of length $s - 1$ between u and v .
- For each pair of red vertices $\{e_{u,v}^i, e_{u',v'}^j\}$ with $1 \leq i, j \leq s - 2$ and $\{u, v\} \neq \{u', v'\}$, $e_{u,v}^i, e_{u',v'}^j$ and the $s - 2$ corresponding green vertices form a path of length $s - 1$ between $e_{u,v}^i$ and $e_{u',v'}^j$.
- The yellow vertex y is linked to the two middle vertices of each green path.

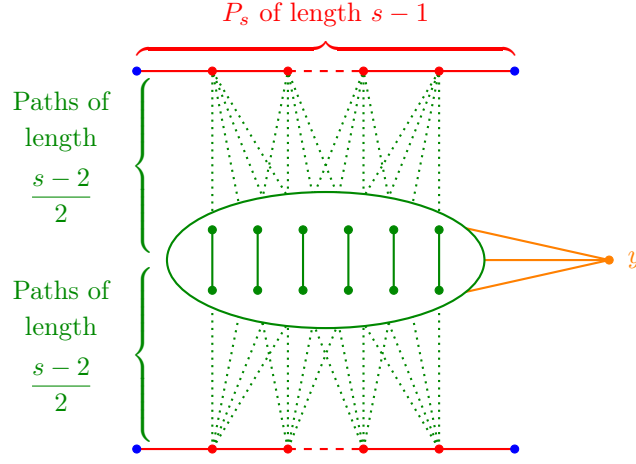


Figure 6: Construction of G' on G being two distinct edges, for s even.

See Figure 6 for an illustration of the construction of G' . Note that G' can be constructed in time $n^{O(1)}$, where $n = |V_G|$.

Soundness of the reduction:

Let $k' := k + (s-2)m + (s-2)\binom{(s-2)m}{2} + 1$, note that there are $(s-2)m$ red vertices and $(s-2)\binom{(s-2)m}{2}$ green vertices. We prove that G has a clique of order k if, and only if, G' has an s -club of order k' .

Claim 10. For every $(u, v) \in (V_{G'})^2$ with $u \neq v$, we have:

1. $\text{dist}_{G'}(u, v) \leq s$ if either u or v is not in V_G
2. else, $\text{dist}_{G'}(u, v) = \begin{cases} s-1 & \text{if } \{u, v\} \in E_G \\ s+1 & \text{if } \{u, v\} \notin E_G \end{cases}$
3. For any $(u, v) \in (V_{G'})^2$ with $\text{dist}_{G'}(u, v) \leq s$, there exists a path P from u to v of length $\leq s$ such that $P \setminus \{u, v\}$ does not intersect V_G : i.e. $\text{dist}_{V_{G'} \setminus V_G}(u, v) \leq s$.

Proof of the claim.

1. We notice the green vertices are all contained in $\overline{B}_{G'}\left(y, \frac{s-2}{2}\right)$ (the closed ball of center y and radius $\frac{s-2}{2}$). Thus, every pair of green or yellow vertices is at distance $\leq s-2$. Now, since every vertex $u \in V_{G'}$ is at distance at most 2 of a green vertex, it is at distance at most s of any green or yellow vertex. Moreover, two red vertices are at distance $\leq s-1$ by construction. Since any blue vertex $u \in V_G$ is adjacent to a

red vertex, it is at distance at most s from any red vertex. All cases have been treated.

2. By construction, for all $\{u, v\} \in E_G$, $\text{dist}_{G'}(u, v) = s - 1$ (taking the red path of length $s - 1$ that comes from the subdivision of the original edge $\{u, v\}$).

Conversely, since $s > 2$, any shortest path between two blue vertices u and v in V_G such that $\{u, v\} \notin E_G$ is, starting from u , first to get to a red neighbor, then to go to a red neighbor of v (in $s - 1$ steps), and then to get to v . This leads to a path of length $s + 1$.

3. The paths given by what precedes in the cases $\text{dist}_{G'}(u, v) \leq s$ satisfy the requirement. \diamond

\implies Suppose that G has a clique $K \subseteq V_G$ of order k . Consider in G' the set $S = V_{G'} \setminus (V_G \setminus K)$, i.e. all vertices except the blue ones that are not in K . Note that $|S| = k'$. According to Claim 10.1 and 10.2, since S does not contain any pair $\{u, v\} \notin E_G$ (because K is a clique), every pair of vertices is at distance $\leq s$ in G' . Note that since S contain every non-blue vertices, we get by Claim 10.3 that every pair of vertices in S is at distance $\leq s$ in $G[S]$: S is an s -club of G' .

\impliedby Reversely, if G' contains an s -club S of order at least k' : let $K = S \cap V_G$. As $|V_{G'} \setminus V_G| = k' - k$, we have $|K| \geq k$. Using that S is an s -club and that the distance between any two vertices in S is necessarily greater than their distance in G' , we get by Claim 10.1 and 10.2 that $\forall (u, v) \in K^2$ with $u \neq v$, $\{u, v\} \in E_G$, i.e. K is a clique of G .

Degeneracy of G' :

We prove that G' has degeneracy at most 3:

- First, remove the green vertices: they all have degree 3 or less.
- Then, remove the red vertices: they now all have degree 2.
- Finally, we are left with an independent set (because $s \geq 4$), remove the remaining vertices arbitrarily.

Moreover, if $s \geq 6$, one can begin with the elimination of green vertices that are not adjacent to y , and therefore have degree 2. Then, we can remove the green vertices adjacent to y and the red vertices that now have degree 2. Finally, we are left with an independent set. This proves that the degeneracy of G' is bounded by 2 if $s \geq 6$. \square

We now adapt the proof to also extend the result Theorem 9 to s -CLIQUE, in the form of Theorem 11.

Theorem 11. *For any $s \geq 4$ even, s -CLIQUE is NP-hard even on 3-degenerate graphs. Moreover, if $s \geq 6$, s -CLIQUE is NP-hard even on 2-degenerate graphs.*

Proof. We provide a NP-hardness reduction from CLIQUE. Let (G, k) be an instance of CLIQUE.

Consider the graph G' and the integer k' defined in the proof of Theorem 9. We have already studied the degeneracy of G' .

There only remains to prove that G has a clique of size k if, and only if, G' has an s -clique of size k' . We will consider exactly the same sets as in the proof of Theorem 9.

\implies Suppose G has a clique $K \subseteq V_G$ of order k . Consider in G' the set $S = V_{G'} \setminus (V_G \setminus K)$. Note that $|S| = k'$. According to Claim 10.1 and 10.2, since S does not contain any pair $\{u, v\} \notin E_G$ (because K is a clique), every pair of vertices is at distance $\leq s$ in G' : S is an s -clique.

\impliedby Reversely, if G' contains an s -clique S of order at least k' : let $K = S \cap V_G$. As $|V_{G'} \setminus V_G| = k' - k$, we have $|K| \geq k$. Using that S is an s -clique, we get by Claim 10.1 and 10.2 that $\forall (u, v) \in K^2$ with $u \neq v$, $\{u, v\} \in E_G$, i.e. K is a clique of G .

□

As 1-clubs are exactly cliques, and as CLIQUE is FPT parameterized by the degeneracy [7], and as 2-CLUB is known to be NP-hard even on graphs of degeneracy 6 [18] Theorem 6 and Theorem 9 ends the study of s -CLUB parameterized by d . For s -CLIQUE, the complexity of 2-CLIQUE parameterized by the degeneracy still remains open.

Note that, as 1-degenerated graphs are exactly forests, and it is already known that a largest s -club can be founded in polynomial time in forests [19], this results ends the study of the complexity of s -club parameterized by the degeneracy of the graph.

4 γ -complete subgraph W[1]-hard for the degeneracy

In this section, we establish the W[1]-hardness of the γ -COMPLETE SUBGRAPH problem when parameterized by the degeneracy d , for any rational $\gamma \in]0, 1[$. As a corollary, we obtain that it is also W[1]-hard when parameterized by k .

Theorem 12. *γ -COMPLETE SUBGRAPH is W[1]-hard parameterized by d (the degeneracy of the input graph).*

Proof. We take two integers $a, b > 0$ such that $\gamma = \frac{a}{b}$.

Let (G, k) be an instance of CLIQUE. We can assume that k is of the form $2r(b - a) + 2$ with $r > 0$ an integer (indeed, $b - a > 0$ since $\gamma < 1$).

Technicalities:

Recall that we have assumed that k is of the form $2r(b-a) + 2$ with $r > 0$. Now, let $R := r(k+3) > 0$ and $p := Ra - k + 1 > 0$ (recall that since $\gamma \neq 0$, it holds that $a \neq 0$).

We choose these values because, in our reduction, we will ask for a γ -complete subgraph of size $k' := 2k - 3 + \binom{k}{2} + p$, and the minimal degree in such a subgraph will necessarily be exactly $d_\gamma := k - 1 + p$. Thus, we want to ensure that such a subgraph is (tightly) γ -complete, i.e. that $\gamma = \frac{d_\gamma}{k' - 1}$.

Let us check that indeed, $\gamma = \frac{d_\gamma}{k' - 1}$. We will prove first that $d_\gamma = Ra$ and then that $k' - 1 = Rb$, which will imply that $\frac{d_\gamma}{k' - 1} = \frac{a}{b} = \gamma$.

First, clearly, $d_\gamma = k - 1 + p = k - 1 + Ra - k + 1 = Ra$ by definitions of d_γ and p .

Second:

$$\begin{aligned}
k' - 1 &= 2k - 4 + \binom{k}{2} + p \\
&= 2k - 4 + \binom{k}{2} + Ra - k + 1 \\
&= k - 3 + \frac{k(k-1)}{2} + Ra \\
&= \frac{k(k+1)}{2} - 3 + Ra \\
&= \frac{k}{2}(k+1) - 3 + Ra \\
&= (r(b-a) + 1)(2r(b-a) + 3) - 3 + Ra \\
&= 2r^2(b-a)^2 + 2r(b-a) + 3r(b-a) + 3 - 3 + Ra \\
&= r(b-a)(2r(b-a) + 2 + 3) + Ra \\
&= r(b-a)(k+3) + Ra \\
&= R(b-a) + Ra \\
&= Rb
\end{aligned}$$

This proves that $\gamma = \frac{Ra}{Rb} = \frac{d_\gamma}{k' - 1}$ (note that both R and b are strictly positive).

Construction of G' :

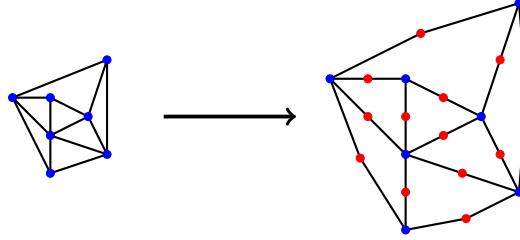


Figure 7: 2-degeneracy transformation of a graph.

First, in order to decrease the degeneracy of the input graph we subdivide each edge by adding a red vertex as done in Figure 7.

The rest of the construction of G' is done as follows. $V_{G'}$ contains four types of vertices:

- The original vertices of V_G , which we will call the blue vertices.
- A red vertex $e_{u,v}$ for each original edge $\{u, v\} \in E_G$ added during the subdivision of the edges. We denote V_R the set formed by the red vertices.
- $k - 3$ new vertices, which we will call the purple vertices. We denote V_P the set formed by the purple vertices.
- p new vertices, which we will call the yellow vertices. We denote V_Y the set formed by the yellow vertices.

We will refer to every vertex that is either yellow or purple as a *special* vertex. The edges of G' are the following:

- For every $\{u, v\} \in E_G$, the edges of the subdivision, i.e. $\{e_{u,v}, u\}$ and $\{e_{u,v}, v\}$, exist in G' .
- Every yellow vertex is a universal vertex of G' .
- Every purple vertex is a neighbor of every red vertex.
- The set of special vertices (yellow and purple) forms a clique (of size $p + k - 3$).

The graph obtained by this construction is illustrated in Figure 8.

Note that G' can be constructed in time $n^{O(1)}$, where $n = |V_G|$.

Soundness of the reduction:

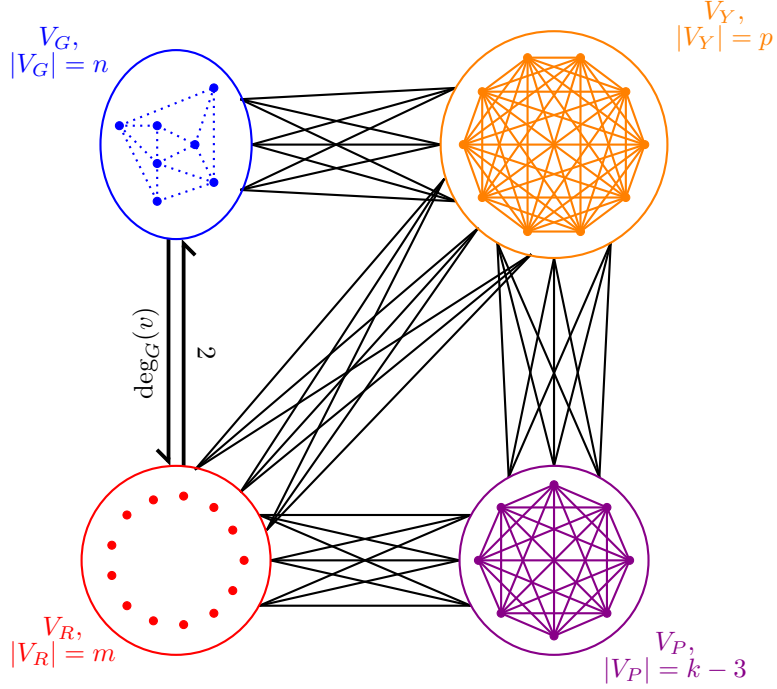


Figure 8: Construction of G' . 9 edges between sets of vertices represent a complete linkage. The double arrow represent that each blue vertex v has $\deg_G(v)$ red neighbors and each red vertex has 2 blue neighbors.

Recall that we defined $k' := 2k - 3 + \binom{k}{2} + p = k + \binom{k}{2} + |V_Y| + |V_P|$. We prove that (G, k) is a yes-instance of CLIQUE if, and only if, (G', k') is a yes-instance of γ -COMPLETE SUBGRAPH.

But first, the following claim proves useful properties on any γ -complete-subgraph of G' of size $\geq k'$.

Claim 13. *Let S be a γ -complete subgraph of size $\geq k'$ of G' . Then:*

1. S contains a red vertex.
2. $|S| = k'$.
3. If a red vertex $e_{u,v}$ is in S , then the two blue vertices u and v are also in S .
4. S contains every special vertices
5. If a blue vertex u is in S , then it has at least $k - 1$ red neighbors in S .

Proof of the claim.

1. Assume by contradiction that S contains no red vertex. Since there are exactly p special vertices, and that $p < k' \leq |S|$, S contains at least one non-special vertex u , i.e.: u is either blue or red: u is therefore blue. On the one hand, since S is a γ -complete graph, $\deg_S(u) \geq \gamma(|S| - 1) \geq \gamma(k' - 1) = d_\gamma$. On the other hand, the neighbors of u in S are neither blue, nor purple (because a blue vertex is never adjacent to a blue/purple vertex) nor red (because we assumed that S does not contain red vertices): so $\deg_S(u) \leq p$ (the only possible neighbors of u in S are the p yellow vertices), which is absurd because $p < d_\gamma$ (recall that $d_\gamma = k - 1 + p$).
2. Now that we have proven that S contains thus a red vertex $e_{u,v}$, assume by contradiction that $|S| > k'$. On the one hand, since S is a γ -complete subgraph, we have $\deg_S(e_{u,v}) \geq \gamma(|S| - 1) > \gamma(k' - 1) = d_\gamma$. On the other hand, $\deg_{G'}(e_{u,v}) = 2 + (k - 3) + p = d_\gamma$ (a red vertex has exactly 2 blue neighbors, $k - 3$ purple neighbors, and p yellow neighbors). We conclude that $\deg_S(e_{u,v}) > \deg_{G'}(e_{u,v})$, which is absurd (the degree of a vertex in an induced subgraph can not be strictly greater than its degree in the original graph). This proves that $|S| = k'$.
3. For every red vertex $e_{u,v}$ of S , the similar remark shows that we have $\deg_S(e_{u,v}) \geq \gamma(k' - 1) = d_\gamma = \deg_{G'}(e_{u,v}) \geq \deg_S(e_{u,v})$ holds, and therefore, $\deg_S(e_{u,v}) = \deg_{G'}(e_{u,v})$, i.e. every neighbor of $e_{u,v}$ in G' is in S : in particular, u and v are in S .
4. Since we know from 1. that S has at least one red vertex $e_{u,v}$, the reasoning of 3. above also shows that every special vertex is in S (because the special vertices are all adjacent to $e_{u,v}$).
5. Every blue vertex in S must have at least $d_\gamma = k - 1 + p$ neighbors in S . Since it has already p neighbors within the special vertices, it must have at least $k - 1$ other neighbors, which must be red since any two blue vertices are not adjacent in G' .

◇

We are ready to prove that (G, k) is a yes-instance of CLIQUE if, and only if, (G', k') is a yes-instance of γ -COMPLETE SUBGRAPH:

⇒ Assume that there is a clique $K \subseteq V_G$ of size k in G .

Take in S :

- The k blue vertices of K .
- The $\binom{k}{2}$ red vertices of the form $e_{u,v}$ for $(u, v) \in K^2$ with $u \neq v$ (which all exists since K is a clique in G).
- All $p + k - 3$ special vertices.

We have indeed $S \subseteq V_{G'}$ and $|S| = k + \binom{k}{2} + p + k - 3 = k' \geq k'$.

We verify that S is a γ -complete subgraph of G' :

- Every blue vertex $u \in S$ (i.e. $u \in K$) is adjacent to the $k - 1$ red vertices of the form $e_{u,v}$ for $v \in K \setminus \{u\}$. These red vertices are indeed in S . In addition, u is also adjacent to each of the p yellow vertices. This proves that $\deg_S(u) \geq k - 1 + p = d_\gamma = \gamma(k' - 1) = \gamma(|S| - 1)$.
- Every red vertex $e_{u,v}$ in S is adjacent to the 2 blue vertices u and v . These 2 vertices are indeed in S . In addition, $e_{u,v}$ is adjacent to the $k - 3$ purple vertices, and to the p yellow vertices. Thus proves that $\deg_S(e_{u,v}) \geq 2 + (k - 3) + p = d_\gamma = \gamma(|S| - 1)$.
- Every special vertex is adjacent to every other special vertex, and to every red vertex in S : thus, as $\binom{k}{2} \geq 3$, their degree is at least $p + (k - 3) - 1 + \binom{k}{2} \geq d_\gamma$.

We have proven that S is a γ -complete subgraph of size $\geq k'$ of G' .

\Leftarrow Assume that there is a γ -complete subgraph S of size $\geq k'$ of G' .

Let V_K be the set of blue vertices in S (i.e. $V_K = S \cap V_G$), and E_K be the set of edges $\{u, v\}$ of G that are such that $e_{u,v} \in S$ (formally, we have $E_K = \{\{u, v\} \in E_G \mid e_{u,v} \in S\}$). By Claim 13.3, for any red vertex $e_{u,v} \in S$, it holds that $\{u, v\} \in (V_K)^2$. In other words, we have that $K = (V_K, E_K)$ is a subgraph (not necessarily induced) of G . We will prove that K is an induced clique of size k in G .

By Claim 13.2 and 13.4, we have $|S| = k' = 2k - 3 + \binom{k}{2} + p$, and that S contains all $p + k - 3$ special vertices. It follows that $|V_K| + |E_K| = k + \binom{k}{2}$. We will prove that $|V_K| = k$ and that $|E_K| = \binom{k}{2}$.

Assume by contradiction that $|V_K| < k$. Then the graph K which has $|V_K|$ vertices has at most $\binom{|V_K|}{2} < \binom{k}{2}$ edges, i.e. $|E_K| < \binom{k}{2}$. We have a contradiction with $|V_K| + |E_K| = k + \binom{k}{2}$.

Assume by contradiction that $|V_K| > k$. Then, by Claim 13.5 the degree of any vertex in the graph K is at least $k - 1$. Thus, we have by Lemma 3 that $2|E_K| = \sum_{u \in V_K} \deg_K(u) \geq \sum_{u \in V_K} (k - 1) = |V_K|(k - 1) > k(k - 1)$. It

follows that $|E_K| > \frac{k(k - 1)}{2} = \binom{k}{2}$. We have again a contradiction with $|V_K| + |E_K| = k + \binom{k}{2}$.

This proves that $|V_K| = k$, and thus that $|E_K| = \binom{k}{2}$. Therefore, K is a subgraph of G with k vertices and $\binom{k}{2}$ edges: it is an induced clique of size k in G .

Degeneracy of G' :

Finally, we prove that the degeneracy of G' is at most $k - 1 + p$, and thus, depends only on k , by giving an elimination order:

- First, remove the red vertices: they have degree $2 + k - 3 + p = k - 1 + p$.
- Second, remove the blue vertices: they are now only adjacent to the p yellow vertices.
- Finally, remove the $p + k - 3$ special vertices.

Since CLIQUE is W[1]-hard parameterized by k , this completes the proof that γ -COMPLETE SUBGRAPH is W[1]-hard parameterized by d , the degeneracy of the input graph. □

As an immediate corollary of Theorem 12, by comparing the parameters k and d on graphs that have a γ -complete-subgraphs, we obtained the W[1]-hardness of γ -COMPLETE SUBGRAPH when parameterized by k .

Corollary 14. *γ -COMPLETE SUBGRAPH is W[1]-hard parameterized by k .*

Proof. This is due to Remark 4 and the fact that γ -COMPLETE SUBGRAPH is trivial (there are only no-instances) as soon as $d < \gamma(k - 1)$ (where d is the degeneracy of the input graph). Indeed, if S is a subset of at least k vertices of a graph G that forms a γ -complete-subgraph of G , it has a vertex of degree at most d by definition of d . Using that S is a γ -complete-subgraph, we get $d \geq \gamma(|S| - 1) \geq \gamma(k - 1)$. Alternatively, one could notice that in the reduction in the proof Theorem 12, k' depends only on k , and is thus also a reduction to γ -COMPLETE SUBGRAPH parameterized by k . □

Notice also that unless P=NP the result of W[1]-hardness of γ -COMPLETE SUBGRAPH, presented in Theorem 12, cannot be improved in a result of para-NP-hardness. Indeed, γ -COMPLETE SUBGRAPH is clearly XP when parameterized by d . For (G, k) an instance of γ -COMPLETE SUBGRAPH and d the degeneracy of G , if $d < \gamma(k - 1)$ then (G, k) is a no-instance. Otherwise, one just has to test, for every $S \subseteq V_G$ such that $k \leq |S| \leq \frac{d}{\gamma} + 1$, if S is a γ -complete-subgraph of G . This process can be performed in XP time, more precisely in time $O(n^3 \times n^{\frac{d}{\gamma} + 1})$.

5 γ -complete subgraph W[1]-hard for $\ell = n - k$

In this section, we establish that γ -COMPLETE SUBGRAPH is also W[1]-hard when parameterized by $\ell := n - k$, the number of vertices outside the γ -complete-subgraph.

This result comes as a surprise, since not only is CLIQUE FPT when parameterized by ℓ [11], but so is the s -PLEX problem [12]: that asks, given a graph G and an integer k , if there exists $S \subseteq V_G$ with $|S| \geq k$ and $\forall u \in S, \deg_S(u) \geq |S| - s$. The only difference between s -PLEX and γ -COMPLETE SUBGRAPH is that the constraint on the number of non-neighbors depends on the size of S in the latter.

Theorem 15. γ -COMPLETE SUBGRAPH is $W[1]$ -hard parameterized by $\ell = n - k$ where k is the size of the γ -complete subgraph.

Proof. Let (G, k) be an instance of CLIQUE. We assume without loss of generality that $k > \frac{2}{\gamma}$. We construct (G', k') an equivalent instance of γ -COMPLETE SUBGRAPH as follows.

Technicalities:

For the sake of our reduction, we will need an integer N that satisfies some useful properties.

Claim 16. *There exists N an integer such that:*

1. $N > \frac{1}{1-\gamma}(\gamma n + \gamma \bar{m} + 1)$,
2. $\lceil \gamma(N + n + \bar{m}) \rceil < \lceil \gamma(N + n + \bar{m} + 1) \rceil$,
3. $N > (k + 1)(k + \bar{m})$.
4. N is polynomial in n .

Proof. (Claim 16)

Note that 2 is true at least once every $\lceil \frac{1}{\gamma} \rceil$ integers. Recall that k is bounded by n , hence, an N verifying points 1 to 3 always exists. Moreover, the smallest possible N verifies:

$$N < \max \left(\frac{1}{1-\gamma}(\gamma n + \gamma \bar{m} + 1), (k + 1)(k + \bar{m}) \right) + \left\lceil \frac{1}{\gamma} \right\rceil + 1.$$

□

Let $d_\gamma := \lceil \gamma(N + n + \bar{m}) \rceil$. Note that $N \geq d_\gamma$: it is indeed a direct consequence of Claim 16.1.

Construction of G' :

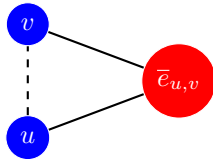


Figure 9: Widget performed on the non-edges of G .

We construct $V_{G'}$ with the following vertices.

- The original vertices of V_G which we call the blue vertices.
- For each $\{u, v\} \in \overline{E_G}$ the vertex $\bar{e}_{u,v}$ which we call a red vertex. We denote V_R the set formed by the red vertices.
- $k + 1$ new purple vertices forming the set V_P .
- N new yellow vertices forming the set V_Y .

The vertices of $V_{G'}$ are connected by the following edges:

- The original edges of E_G .
- For each $\{u, v\} \in \overline{E_G}$, the edges $\{u, \bar{e}_{u,v}\}, \{v, \bar{e}_{u,v}\}$ as done in Figure 9.
- All possible edges between the red and purple vertices, i.e. the bipartite graph induced by (V_R, V_P) is complete.
- All possible edges between the purple vertices, i.e. V_P is a clique of G' .
- All possible edges between yellow vertices, i.e. V_Y is a clique of G' .
- All possible edges between the blue and yellow vertices, i.e. the bipartite graph induced by (V_G, V_Y) is complete.
- For each $\bar{e}_{u,v} \in V_R$: $d_\gamma - k - 2$ neighbors in the yellow vertices V_Y ($d_\gamma - k - 2$ is indeed positive).
- For each $v \in V_P$: $d_\gamma - \bar{m} - k$ neighbors in the yellow vertices V_Y ($d_\gamma - \bar{m} - k$ is indeed positive). The first purple with the $d_\gamma - \bar{m} - k$ first yellow vertices, the second purple with the next $d_\gamma - \bar{m} - k$ yellow vertices and so on. More formally the i -th purple vertex is linked to the yellow vertices with indices from $i \times (d_\gamma - \bar{m} - k) \bmod N$ to $(i + 1) \times (d_\gamma - \bar{m} - k) - 1 \bmod N$.
Note that by definition of $d_\gamma := \lceil \gamma(N + n + \bar{m}) \rceil$, we have that $d_\gamma \geq \gamma N$. Since $k > \frac{2}{\gamma}$ it holds that $(k + 1)d_\gamma > 2N$, then using Claim 16.3 we obtain that $(k + 1)(d_\gamma - \bar{m} - k) > 2N - (k + 1)(\bar{m} + k) > N$. This proves that each yellow vertex has at least a purple neighbor.

The graph obtained by this construction is illustrated in Figure 10.

Let $k' := N + n + \bar{m} + 1$. Notice that $d_\gamma = \lceil \gamma(k' - 1) \rceil$.

Note that G' can be constructed in time $n^{O(1)}$.

Soundness of the reduction:

First, the following claim proves useful properties on any γ -complete-subgraph of G' of size $\geq k'$.

Claim 17. *Let S be a γ -complete subgraph of G' of size $\geq k'$. Then:*

1. S contains every non-blue vertex.

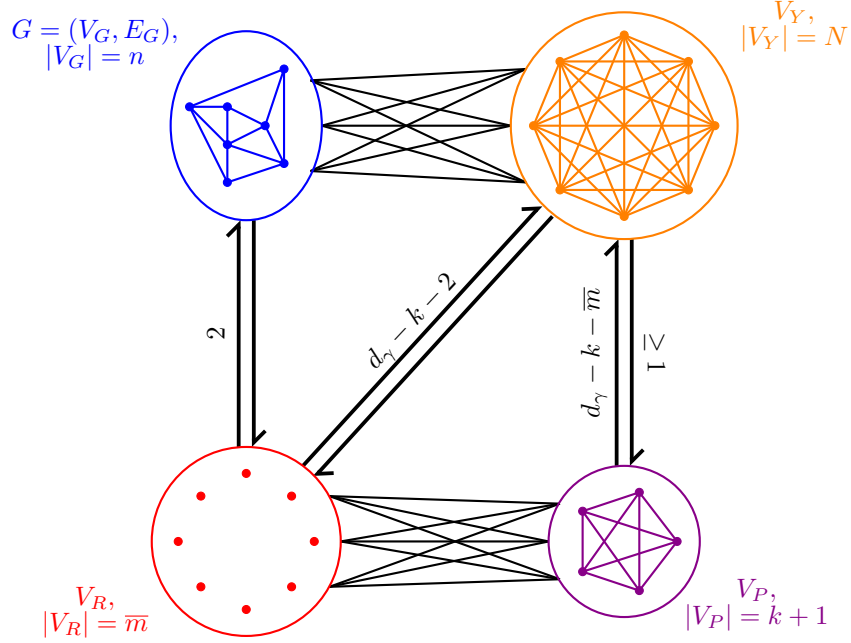


Figure 10: Construction of G' . 9 edges between sets of vertices represent a complete linkage. A double arrow $A \overset{a}{\underset{b}{\rightleftarrows}} B$ represents that the vertices in A have a neighbors in B and the vertices in B have b neighbors in A , if either a or b is not mentioned it means that this number is irrelevant to the proof.

2. $|S| = k'$.
3. For every $\{u, v\} \in \overline{E_G}$, either u or v is in S .

Proof of the claim.

1. Since $|V_{G'} \setminus S| \leq k$ and $|V_P| \geq k + 1$, it holds that S contains at least one purple vertex. Let $v \in S \cap V_P$, since $\deg_S(v) \geq \lceil \gamma(|S| - 1) \rceil \geq \lceil \gamma(k' - 1) \rceil = d_\gamma = \deg_{G'}(v) \geq \deg_S(v)$, it holds that $\deg_{G'}(v) = \deg_S(v)$, i.e. all neighbors of v are in S : which implies that V_R and V_P are contained in S . Using the same argument on each other purple vertex and since each yellow vertex has a purple neighbor, it holds that $V_Y \subseteq S$. This proves that every non-blue vertex is in S .
2. From what precedes, we obtain that $\lceil \gamma(k' - 1) \rceil = \lceil \gamma(|S| - 1) \rceil$. By Claim 16.2 it holds that $\lceil \gamma(k' - 1) \rceil < \lceil \gamma k' \rceil$. Using $|S| \geq k'$, we get $|S| = k'$.
3. Let $\{u, v\} \in \overline{E_G}$. Assume by contradiction that neither u nor v is in S . By what precedes, $\bar{e}_{u,v} \in S$ and since u and v are two distinct neighbors of $\bar{e}_{u,v}$

outside S , we have $\deg_S(\bar{e}_{u,v}) \leq \deg_{G'}(\bar{e}_{u,v}) - 2 = d_\gamma - 1 < \lceil \gamma(|S| - 1) \rceil$, contradicting that S is a γ -complete-subgraph. This proves that either u or v belongs to S .

◇

⇒ Let $K \subseteq V_G$ be a clique of G of size k .

Let $S = V_{G'} \setminus K$. It holds that $|S| = n + \bar{m} + k + 1 + N - k = k'$. Also:

- For any blue vertex in S (i.e. $v \in V_G \cap S$), let's recall that v is linked to all V_Y , so: $\deg_S(v) \geq |V_Y| = N \geq d_\gamma$.
- For any red vertex $\bar{e}_{u,v} \in V_R$, let's recall that $\bar{e}_{u,v}$ is linked in G to u, v , and in S to all $k + 1$ vertices of V_P and $d_\gamma - k - 2$ vertices of V_Y . Since u and v are not adjacent in G it holds that at least one of them is not in K and thus is in S . So $\deg_S(\bar{e}_{u,v}) \geq d_\gamma - 1 + 1 = d_\gamma$.
- For any purple vertex $v \in V_P$, let's recall that v is linked to all \bar{m} vertices of V_R , all k other vertices of V_P and $d_\gamma - k - \bar{m}$ yellow vertices. All of its neighbors are in S , thus $\deg_S(v) = d_\gamma$.
- For any yellow vertex $v \in V_Y$, let's recall that v is linked to all yellow vertices and at least one purple vertex. Hence, $\deg_S(v) \geq N - 1 + 1 \geq d_\gamma$.

Hence, S is a γ -complete subgraph of G' of size k' .

⇐ Let S be a γ -complete subgraph of G' of size $\geq k'$. Let $K = V_{G'} \setminus S$.

We have $|S| = k'$ by Claim 17.2, and thus $|K| = |V_{G'}| - |S| = k$. Moreover, S contains every non-blue vertex by Claim 17.1: it follows that K contains only blue vertices (i.e. vertices of V_G). We will prove that K is a clique.

Let $(u, v) \in K^2$ with $u \neq v$, and assume by contradiction that $\{u, v\} \notin E_G$, i.e. $\{u, v\} \in \overline{E_G}$. By Claim 17.3, either u or v is in S , contradicting that $(u, v) \in K^2$. This proves that $\{u, v\} \in E_G$.

Thus K is a clique of G of size k .

Preservation of the parameter:

Since we reduce the instance (G, k) of CLIQUE parameterized by k to the instance (G', k') of γ -COMPLETE SUBGRAPH parameterized by ℓ , we need to check that $\ell' := |V_{G'}| - k'$ depends only on k . It is indeed true since $\ell' = k$. □

Note that, even if the concept of s -plex with $s \geq 2$ (a subset $S \subseteq V_G$ such that $\forall u \in S, \deg_S(u) \geq |S| - s$) is similar to γ -complete subgraph, this proof cannot be adapted to s -plexes. Indeed, we use the fact that with a large enough number of vertices, the vertices can also have a large enough number of non-neighbors in the γ -complete subgraph. Something similar cannot be said for s -plexes where the number of non-neighbors is always constant.

Also, as discussed in the end of Section 4, unless $P=NP$, the result of $W[1]$ -hardness obtained in Theorem 15 cannot be improved to a result of para- NP -hardness unless $P=NP$. Indeed, testing if every subset $S \subseteq V_G$ with $|S| \geq n - \ell$ is a γ -complete-subgraph takes time $O(n^2 \times \ell \times \binom{n}{\ell}) = O(n^3 \times n^\ell)$.

6 Discussion

In this paper, we have completed the study of the parameterized complexity of s - $CLUB$ when parameterized by the degeneracy d , as we established its para- NP -hardness for all value of $s \geq 3$. The para- NP -hardness of 2 - $CLUB$ was indeed already known [18], and noticing that 1 - $CLUB$, which corresponds to $CLIQUE$, is FPT when parameterized by d , all cases are now covered. More precisely, if $s \geq 3$ is odd, we proved that s - $CLUB$ is NP -hard even on bipartite graphs of degeneracy 3, and if $s \geq 4$ is even, s - $CLUB$ is NP -hard even on graphs of degeneracy 3^1 . We also proved that the exact same complexity results hold for s - $CLIQUE$.

Then, we have established the $W[1]$ -hardness of the γ - $COMPLETE$ $SUBGRAPH$ problem parameterized by 3 different parameters: k , ℓ and d , which ends the study of the parameterized complexity of γ - $COMPLETE$ $SUBGRAPH$ when parameterized by the most relevant parameters, as proposed by Komusiewicz [11].

Most of the complexity results are now known about s - $CLUB$, s - $CLIQUE$ and γ - $COMPLETE$ $SUBGRAPH$, our contribution is summarized in Figure 11. Note that, unless $P = NP$, most of our $W[1]$ -hardness results can not be improved to para- NP -hardness results since XP algorithm exist to solve these problems, as discussed at the end of Section 4 and Section 5.

Problem	k	ℓ	h	d
$CLIQUE$	$W[1]$ -h [6]	FPT [11]	FPT [11]	FPT [11]
2 - $CLUB$	FPT [11]	FPT [11]	$W[1]$ -h[8]	NP -h for $d = 6$ [8]
s - $CLUB$ with $s \geq 3$	FPT [11]	FPT [11]	?	NP-h for $d = 3^1$
2 - $CLIQUE$	FPT [11]	FPT [11]	?	?
s - $CLIQUE$ with $s \geq 3$	FPT [11]	FPT [11]	?	NP-h for $d = 3^1$
γ - $COMPLETE$ $SUBGRAPH$	<u>$W[1]$-h for any γ</u>	<u>$W[1]$-h</u>	FPT [3]	<u>$W[1]$-h</u>

Figure 11: Complexities of relaxations of $CLIQUE$, our contributions underlined.

Nevertheless, the complexity of s - $CLUB$ and s - $CLIQUE$ parameterized by the h -index (even though the particular case of 2 - $CLUB$ is known [18]), and the complexity of 2 - $CLIQUE$ when parameterized by the degeneracy d remain open. Note that as 1 -clique is FPT parameterized by d and as s -clique with $s \geq 3$ is para- NP -hard for the same parameter, this case is intermediate. Moreover, for $s \in \{3, 4\}$, the graph obtained by our reduction is only 3-degenerated and for $s = 2$, the graph obtained [8] is only 6-degenerated. Thus, the complexity for 2-degenerated graphs for $s \in \{3, 4\}$, and the complexity on the class of d -degenerated graphs, for $d \in \{2, \dots, 5\}$ and $s = 2$ are also still open.

¹We even get the NP -hardness for $d = 2$ if $s \geq 5$, and for bipartite graphs if s is odd

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