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# Systèmes entrée-sortie non linéaires et applications en audio-acoustique

Thomas Hélie

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Thomas Hélie. Systèmes entrée-sortie non linéaires et applications en audio-acoustique. Doctorat. 8ième École d'été en Mécanique Théorique : Théorie du Contrôle en Mécanique, Quiberon, France. 2019. hal-04036312

**HAL Id: hal-04036312**

**<https://hal.science/hal-04036312v1>**

Submitted on 19 Mar 2023

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*Systèmes entrée-sortie non linéaires  
et applications en audio-acoustique*

**Séries de Volterra**

Thomas Hélie, CNRS

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Laboratoire des Sciences et Technologies de la Musique et du Son  
IRCAM - CNRS - SU, Paris, France

Ecole Thématique "Théorie du Contrôle en Mécanique"  
2019

Vito Volterra [1860 (Ancona) - 1940 (Roma)]



**Mathématicien et physicien italien** connu pour ses travaux sur les **équations intégro-différentielles**, **la dynamique des populations**, ... Prédécesseur de Fréchet et Banach, il est considéré comme l'un des fondateurs de l'analyse fonctionnelle. Il s'oppose au régime fasciste (1922) et refuse les honneurs académiques par conviction politique. Il vit alors en grande partie à l'étranger et revient à Rome peu avant sa mort.

Royal Society (1910) - Royal Society of Edinburgh (1913)  
Un cratère de lune porte son nom.

## Avant de commencer, quelques références bibliographiques



**V. Volterra.** *Theory of Functionals and of Integral and Integro-Differential Equations.* (Dover Pub., 1959).



**R. W. Brockett.** Volterra series and geometric control theory. *Automatica*, 12:167–176, 1976).



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**M. Fliess et al.** An algebraic approach to nonlinear functional expansions. (*IEEE-TCAS*, 30(8):554–570, 1983).



**A. Isidori.** *Nonlinear control systems (3rd ed).* (Springer, 3rd ed., 1995).



**W. J. Rugh.** *Nonlinear System Theory, The Volterra/Wiener approach.* (The Johns Hopkins University Press, Baltimore, 1981).



**M. Schetzen.** *The Volterra and Wiener theories of nonlinear systems.* (Wiley-Interscience, 1989).



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**S. Boyd and L. Chua.** Fading memory and the problem of approximating nonlinear operators with Volterra series. (*IEEE-TCAS*, 32(11):1150–1161, 1985).



**M. Hasler.** Phénomènes non linéaires. (*Ed. Ecole Polytechnique Fédérale de Lausanne*, 1999).



**F. Bullo.** Series expansions for analytic systems linear in control. (*Automatica*, 38:1425–1432, 2002).



**Doyle et al.** Identification and Control Using Volterra Models. (*Springer*, 2002).



**Hélie & collaborators.** Quelques articles personnels joints (2003-2019).

# Plan

- 1 Préambule
- 2 Séries de Volterra : généralités
- 3 Calcul des noyaux de Volterra d'un système différentiel
- 4 Exercices et applications en audio-acoustique
- 5 Convergence
- 6 Extension en dimension infinie et application
- 7 Conclusion

# Plan

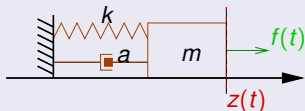
- 1 **Préambule**
  - Systèmes linéaires invariants dans le temps (/exemple)
  - Perturbations non linéaires régulières
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# 1. AN EXAMPLE: Mass-Spring-Damper system

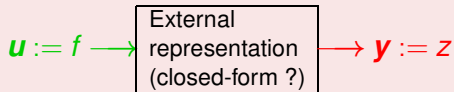
Problem (at rest for  $t < 0$ )



$$mz''(t) + az'(t) + kz(t) = f(t)$$

**Find the trajectory  $z(t)$  with respect to the force  $f(t)$**

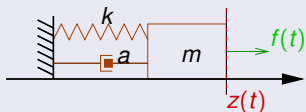
System with input ( $u$ ) / output ( $y$ )





# 1. AN EXAMPLE: Mass-Spring-Damper system

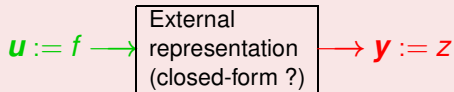
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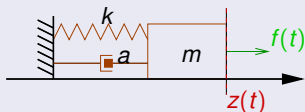


State-space representation:  $\mathbf{x}(t) = [z(t), z'(t)]^T$ ,  $\mathbf{x}(0) = [0, 0]^T$

$$\underbrace{\begin{bmatrix} z'(t) \\ z''(t) \end{bmatrix}}_{\mathbf{x}'(t)} = \underbrace{\begin{bmatrix} 0 & 1 \\ -k/m & -a/m \end{bmatrix}}_A \underbrace{\begin{bmatrix} z(t) \\ z'(t) \end{bmatrix}}_{\mathbf{x}(t)} + \underbrace{\begin{bmatrix} 0 \\ 1/m \end{bmatrix}}_B u(t)$$

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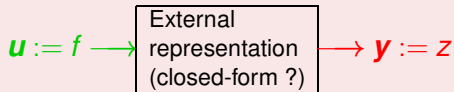
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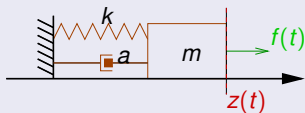
$$\text{Eq.: } \mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\text{Sol.: } \mathbf{x}(t) = \int_0^t e^{\mathbf{A}\tau} \mathbf{B}\mathbf{u}(t-\tau) d\tau$$

$$\begin{aligned} \mathbf{y}(t) &= [1, 0] \mathbf{x}(t) \\ &= \mathbf{C} \mathbf{x}(t) \end{aligned}$$

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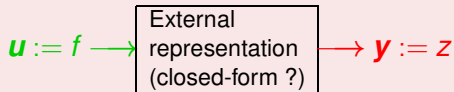
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Find the trajectory  $z(t)$  with respect to the force  $f(t)$

System with input ( $u$ ) / output ( $y$ )



$$y(t) = \int_0^t h(\tau) u(t-\tau) d\tau$$

Convolution(/filtering) with the impulse response

$$h(\tau) = \mathbf{C} e^{\mathbf{A}\tau} \mathbf{B} \mathbf{1}_{\tau \geq 0}$$

State-space representation:  $\mathbf{x}(t) = [z(t), z'(t)]^T$ ,  $\mathbf{x}(0) = [0, 0]^T$

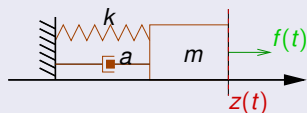
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## 2. AN EXAMPLE: in the LAPLACE domain

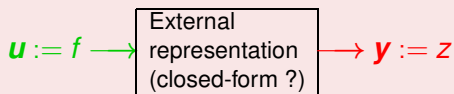
Problem (at rest for  $t < 0$ )



$$mz''(t) + az'(t) + kz(t) = f(t)$$

Find the trajectory  $z(t)$  with respect to the force  $f(t)$

System with input ( $u$ ) / output ( $y$ )



$$Y(s) = H(s) U(s)$$

Transfer function (/filter)

$$H(s) = \mathbf{C}(s\mathbf{I}_2 - \mathbf{A})^{-1}\mathbf{B}$$

State-space representation:  $\mathbf{x}(t) = [z(t), z'(t)]^T$ ,  $\mathbf{x}(0) = [0, 0]^T$

$$\text{Eq.: } s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$

$$\text{Sol.: } \mathbf{X}(s) = (s\mathbf{I}_2 - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$

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$$= \mathbf{C} \mathbf{X}(s)$$

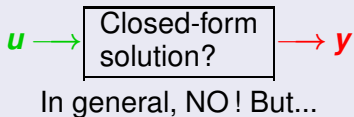
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### 3. What about nonlinear systems ?

#### Input/Output nonlinear differential system (state $x$ )

$$\begin{aligned} \mathbf{x}'(t) &= \mathbf{F}(\mathbf{x}(t), \mathbf{u}(t)) \\ \mathbf{y}(t) &= \mathbf{G}(\mathbf{x}(t), \mathbf{u}(t)) \end{aligned}$$



#### Linear case

$$\mathbf{F}(\mathbf{x}, \mathbf{u}) = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{G}(\mathbf{x}, \mathbf{u}) = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

#### I/O relation: linear filter

Kernel:  $h(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{B} + \mathbf{D}$

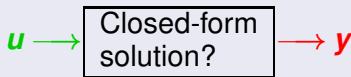
Trsf. fct:  $H(s) = \mathbf{C}(s\mathbf{I} + \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$

**Interests:** control, spectral analysis, identification, simulation, etc.

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In general, NO ! But...

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#### Weakly nonlinear case

$\mathbf{F}$ ,  $\mathbf{G}$ : power series expansions around equilibrium point 0 (nonzero coeff. at order 1)

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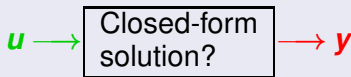
#### Example: a nonlinear spring

$$mz''(t) + az'(t) + \kappa(z(t)) = f(t)$$

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#### I/O relation: Volterra series

Interests: idem!



## 4. From a qualitative point of view...

### A few comparisons:

Case	closed-form sol. w.r.t. input	distortions	self-oscillations bifucations, chaos
General	no	yes	yes
Volterra	yes	yes	no
Linear	yes	no	no

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Case	closed-form sol. w.r.t. input	distortions	self-oscillations bifucations, chaos
General	no	yes	yes
Volterra	yes	yes	no
Linear	yes	no	no

### Interest of Volterra series:

- **Natural distortions** for high amplitudes
- Possible extensions to **partial differential equations**
- Audio-acoustics: **large dynamics (/fortissimo)**

## 5. What is the idea ?

(regular perturbation method)

### For a Weakly Nonlinear System ...

$$\begin{aligned} \mathbf{x}'(t) &= \mathbf{F}(\mathbf{x}(t), \mathbf{u}(t)) & \mathbf{F}(X, U) &= \sum_{m,n} \frac{D_{m,n} \mathbf{F}(0,0)}{m!n!} (X, \dots, X, U, \dots, U) \\ \mathbf{y}(t) &= \mathbf{G}(\mathbf{x}(t), \mathbf{u}(t)) & \mathbf{G}(X, U) &= \sum_{m,n} \frac{D_{m,n} \mathbf{G}(0,0)}{m!n!} (X, \dots, X, U, \dots, U) \end{aligned}$$

Consider the input as a **perturbation** of the system.  
Mark it with  $\eta \in (0, 1)$ :  $\mathbf{u}(t) = \eta \mathbf{v}(t)$

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- (i) Introduce  $\mathbf{x}(t) = \sum_n \eta^n \mathbf{x}_n(t)$  and  $\mathbf{y}(t) = \sum_n \eta^n \mathbf{y}_n(t)$
- (ii) Inject these series expansions in the system equations
- (iii) Sort equations w.r.t.  $\eta^n$

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**We obtain a sequence of linear ODEs, indexed w.r.t.  $n$**

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**We obtain a sequence of linear ODEs, indexed w.r.t.  $n$**

- (iv) Solution: Each  $x_n$  is a **multiple convolution** of  $n$  repeated versions of the input and a **computable multivariate kernel**  
→ **Volterra kernel**

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# Time domain

- Definition and examples
- A convergence criterion
- Non-uniqueness of kernels
- Remark on time-varying systems

# Volterra series: definition

## Definition

A system  $u \rightarrow \boxed{\{h_n\}} \rightarrow y$  is defined by the **Volterra series**  $\{h_n\}_{n \geq 1}$  if

$$y(t) = \underbrace{\sum_{n=1}^{+\infty}}_{\text{Sum}} \underbrace{\int_{\mathbb{R}^n} h_n(\tau_1, \dots, \tau_n) u(t - \tau_1) \dots u(t - \tau_n) d\tau_1 \dots d\tau_n}_{\text{of multiple convolutions}}$$

*(with several possible functional settings)*

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## Example

- Linear filters:  $h_n = 0$ , if  $n \geq 2$ .

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## Example

- Linear filters:  $h_n = 0$ , if  $n \geq 2$ .
- Memoryless fct:  $h_n(\tau_1, \dots, \tau_n) = \alpha_n \delta(\tau_1, \dots, \tau_n)$ , ( $\delta$ : Dirac).

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- General case:  $n=1$  (linear contrib.),  $n=2$  (quadratic), etc.

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- General case:  $n=1$  (linear contrib.),  $n=2$  (quadratic), etc.

## A system is causal if

$$\begin{aligned} \tau < 0 &\Rightarrow h(\tau) = 0 && \text{(linear)} \\ \exists k \text{ s.t. } \tau_k < 0 &\Rightarrow h_n(\tau_1, \dots, \tau_k, \dots, \tau_n) = 0 && \text{(Volterra)} \end{aligned}$$

# A convergence criterion (1/2)

(see e.g. [Boyd,1984])

RECALL: definition of a Volterra series

$$x(t) = \sum_{n=1}^{+\infty} x_n(t) \text{ with } x_n(t) = \int_{\mathbb{R}^n} h_n(\tau_1, \dots, \tau_n) u(t - \tau_1) \dots u(t - \tau_n) d\tau_1 \dots d\tau_n$$

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$$\text{Hence, } \|x\|_{\infty} \leq \sum_{n=1}^{+\infty} \|x_n\|_{\infty} \leq \sum_{n=1}^{+\infty} \|h_n\|_1 (\|u\|_{\infty})^n.$$

## A convergence criterion (2/2)

(see e.g. [Boyd,1984])

### Gain bound function $\varphi$

Define  $\varphi(z) = \sum_{n \geq 1} \|h_n\|_1 z^n$  with convergence radius  $\rho$  at  $z = 0$ .

### Theorem (BIBO result)

**If**  $\|u\|_\infty < \rho$ , **then** the Volterra series expansion of  $x$  is normally convergent and

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Moreover, the **truncation error is bounded**:

$$\left\| \sum_{n=N+1}^{+\infty} x_n \right\|_\infty \leq \sum_{n=N+1}^{+\infty} \|h_n\|_1 (\|u\|_\infty)^n$$

# Non-uniqueness of Volterra kernels

## Remark:

Permuting variables  $\tau_k$  in  $y(t) = \sum_{n=1}^{+\infty} \int_{\mathbb{R}^n} h_n(\tau_1, \dots, \tau_n) u(t - \tau_1) \dots u(t - \tau_n) d\tau_1 \dots d\tau_n$  leaves the output  $y$  unchanged.

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$h_2(\tau_1, \tau_2)$ ,  $h_2(\tau_2, \tau_1)$ , but also  $\alpha h_1(\tau_1, \tau_2) + (1 - \alpha) h_2(\tau_2, \tau_1)$  define the same Input-Output system.



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Symmetrical versions of Volterra kernels  $\text{SYM}[h_n]$  are unique

$$\text{SYM}[h_n](\tau_1, \dots, \tau_n) = \frac{1}{n!} \sum_{\pi} h_n(\tau_{\pi(1)}, \dots, \tau_{\pi(n)})$$

Other unique versions (triangular kernels, regular kernels) are also available.

## Remark on time varying systems

A definition is also available for time-varying systems:

$$y(t) = \sum_{n=1}^{+\infty} \int_{\mathbb{R}^n} g_n(t, \theta_1, \dots, \theta_n) u(\theta_1) \dots u(\theta_n) d\theta_1 \dots d\theta_n$$

Time-Invariant (TI) case and link with kernels  $h_n$

$$\text{TI case: } y(t) = \sum_{n=1}^{+\infty} \int_{\mathbb{R}^n} h_n(\tau_1, \dots, \tau_n) u(t - \tau_1) \dots u(t - \tau_n) d\tau_1 \dots d\tau_n$$

Kernels  $g_n$  of a TI system are such that ( $\theta_k = t - \tau_k$ )

$$g_n(t, t - \tau_1, \dots, t - \tau_n) = h_n(\tau_1, \dots, \tau_n)$$

does not depend on  $t$

Causal kernels  $g_n$

$$\exists k \text{ s.t. } \theta_k > t \Rightarrow g_n(t, \theta_1, \dots, \theta_k, \dots, \theta_n) = 0$$

# Outline

- 1 Préambule
- 2 **Séries de Volterra : généralités**
  - Domaine temporel
  - **Domaine fréquentiel et de Laplace**
- 3 Calcul des noyaux de Volterra d'un système différentiel
- 4 Exercices et applications en audio-acoustique
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# Laplace(/Fourier) domain and analogies with linear systems

## Laplace domain (or Fourier domain with $s = 2i\pi f$ )

Transfer function:  $H(s) = \int_{\mathbb{R}} h(\tau) e^{-s\tau} d\tau$  (lin.)

Transfer kernel:  $H_n(s_{1:n}) = \int_{\mathbb{R}^n} h_n(\tau_{1:n}) e^{-(s_1\tau_1 + \dots + s_n\tau_n)} d\tau_1 \dots d\tau_n$  (Volt.)

denoting  $(s_{1:n}) = (s_1, \dots, s_n)$  and  $(\tau_{1:n}) = (\tau_1, \dots, \tau_n)$ .

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## Causal stable system: **NO** poles (and **NO** singularities)

of  $H(s)$  for  $\Re(s) > 0$  (linear)

of  $H_n(s_{1:n})$  for  $\Re(s_k) > 0$  (Volterra)

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## Input/Output relation

$u \rightarrow$  system  $\rightarrow y$

Transfer function:  $Y(s) = H(s) U(s)$  (lin.)

Transfer kernel: more complex (Volt.)

(next part: use interconnection laws)

## A result on periodic signals

Analytic input signal  $u(t) = a e^{i\omega t}$

$$u(t) = a e^{i\omega t} \longrightarrow \boxed{\{h_n\}} \longrightarrow y(t) = \sum_{n=1}^{+\infty} a^n H_n(i\omega, \dots, i\omega) e^{in\omega t}$$

Periodic input signals / Fourier series

$$u(t) = \sum_{k=-\infty}^{+\infty} u_k e^{ik\omega t} \longrightarrow \boxed{\{h_n\}} \longrightarrow y(t) = \sum_{k=-\infty}^{+\infty} y_k e^{ik\omega t}$$

$$\text{with } y_k = \sum_{n=1}^{+\infty} \sum_{\substack{k_1, \dots, k_n = -\infty \\ k_1 + \dots + k_n = k}}^{+\infty} u_{k_1} \dots u_{k_n} H_n(ik_1\omega, \dots, ik_n\omega)$$

Distortion coefficient for  $u(t) = a \cos(\omega t)$

$$D(a, \omega) = \sum_{n=2}^{+\infty} |y_n|^2 / |y_1|^2 : \text{closed-form solution w.r.t. } a, \omega, H_n.$$



## In summary:

### A Volterra series ...

- catches distortions (memory combined with nonlinearities)
- sorts the nonlinear responses w.r.t. the degree  $n$  of homogenous contributions of  $u$
- generalizes the convolution principle
- can be described by transfer kernels in the frequency domain (as filters).
- is usually non unique but uniquely defined versions are available (useful for identification purposes)

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