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*Systèmes entrée-sortie non linéaires
et applications en audio-acoustique*

Séries de Volterra

Thomas Hélie, CNRS

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Laboratoire des Sciences et Technologies de la Musique et du Son
IRCAM - CNRS - SU, Paris, France

Ecole Thématique "Théorie du Contrôle en Mécanique"
2019

Avant de commencer, quelques références bibliographiques

-  **V. Volterra.** *Theory of Functionals and of Integral and Integro-Differential Equations.* (Dover Pub., 1959).
-  **R. W. Brockett.** Volterra series and geometric control theory. *Automatica*, 12:167–176, 1976).
-  **E. G. Gilbert.** Functional expansions for the response of nonlinear differential systems. (*IEEE-TCAS*, 22:909–921, 1977).
-  **M. Fliess *et al.*** An algebraic approach to nonlinear functional expansions. (*IEEE-TCAS*, 30(8):554–570, 1983).
-  **A. Isidori.** *Nonlinear control systems (3rd ed).* (Springer, 3rd ed., 1995).
-  **W. J. Rugh.** *Nonlinear System Theory, The Volterra/Wiener approach.* (The Johns Hopkins University Press, Baltimore, 1981).
-  **M. Schetzen.** *The Volterra and Wiener theories of nonlinear systems.* (Wiley-Interscience, 1989).
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-  **S. Boyd and L. Chua.** Fading memory and the problem of approximating nonlinear operators with Volterra series. (*IEEE-TCAS*, 32(11):1150–1161, 1985).
-  **M. Hasler.** Phénomènes non linéaires. (*Ed. Ecole Polytechnique Fédérale de Lausanne*, 1999).
-  **F. Bullo.** Series expansions for analytic systems linear in control. (*Automatica*, 38:1425–1432, 2002).
-  **Doyle *et al.*** Identification and Control Using Volterra Models. (*Springer*, 2002).
-  **Hélie & collaborators.** Quelques articles personnels joints (2003-2019).

Plan

- 1 Préambule
- 2 Séries de Volterra : généralités
- 3 Calcul des noyaux de Volterra d'un système différentiel
- 4 Exercices et applications en audio-acoustique
- 5 Convergence
- 6 Extension en dimension infinie et application
- 7 Conclusion

Plan

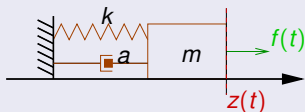
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 - Systèmes linéaires invariants dans le temps (/exemple)
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Outline

- 1 **Préambule**
 - **Systèmes linéaires invariants dans le temps (/exemple)**
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1. AN EXAMPLE: Mass-Spring-Damper system

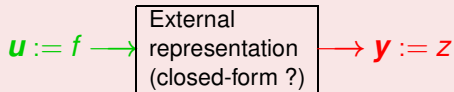
Problem (at rest for $t < 0$)



$$mz''(t) + az'(t) + kz(t) = f(t)$$

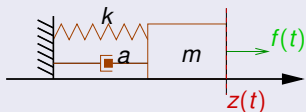
Find the trajectory $z(t)$ with respect to the force $f(t)$

System with input (u) / output (y)



1. AN EXAMPLE: Mass-Spring-Damper system

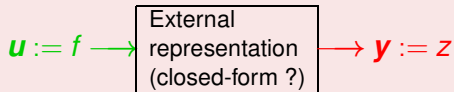
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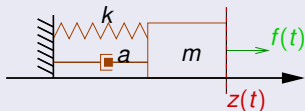


State-space representation: $\mathbf{x}(t) = [z(t), z'(t)]^T$, $\mathbf{x}(0) = [0, 0]^T$

$$\underbrace{\begin{bmatrix} z'(t) \\ z''(t) \end{bmatrix}}_{\mathbf{x}'(t)} = \underbrace{\begin{bmatrix} 0 & 1 \\ -k/m & -a/m \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} z(t) \\ z'(t) \end{bmatrix}}_{\mathbf{x}(t)} + \underbrace{\begin{bmatrix} 0 \\ 1/m \end{bmatrix}}_{\mathbf{B}} u(t)$$

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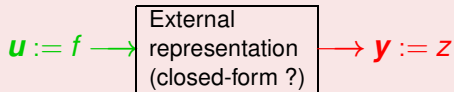
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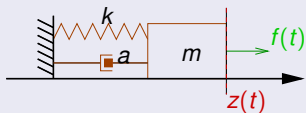
$$\text{Eq.: } \mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\text{Sol.: } \mathbf{x}(t) = \int_0^t e^{\mathbf{A}\tau} \mathbf{B}\mathbf{u}(t-\tau) d\tau$$

$$\begin{aligned} \mathbf{y}(t) &= [1, 0] \mathbf{x}(t) \\ &= \mathbf{C} \mathbf{x}(t) \end{aligned}$$

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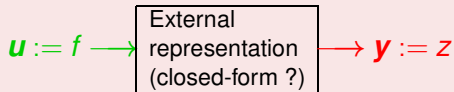
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Find the trajectory $z(t)$ with respect to the force $f(t)$

System with input (u) / output (y)



$$y(t) = \int_0^t h(\tau) u(t-\tau) d\tau$$

Convolution(/filtering) with the impulse response

$$h(\tau) = \mathbf{C} e^{\mathbf{A}\tau} \mathbf{B} \mathbf{1}_{\tau \geq 0}$$

State-space representation: $\mathbf{x}(t) = [z(t), z'(t)]^T$, $\mathbf{x}(0) = [0, 0]^T$

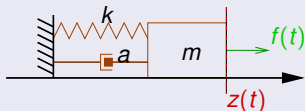
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2. AN EXAMPLE: in the LAPLACE domain

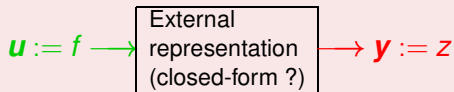
Problem (at rest for $t < 0$)



$$mz''(t) + az'(t) + kz(t) = f(t)$$

Find the trajectory $z(t)$ with respect to the force $f(t)$

System with input (u) / output (y)



$$Y(s) = H(s) U(s)$$

Transfer function (/filter)

$$H(s) = \mathbf{C}(s\mathbf{I}_2 - \mathbf{A})^{-1}\mathbf{B}$$

State-space representation: $\mathbf{x}(t) = [z(t), z'(t)]^T$, $\mathbf{x}(0) = [0, 0]^T$

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$$\text{Sol.: } \mathbf{X}(s) = (s\mathbf{I}_2 - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$

$$\mathbf{Y}(s) = \begin{bmatrix} 1, & 0 \end{bmatrix} \mathbf{X}(s)$$

$$= \mathbf{C} \mathbf{X}(s)$$

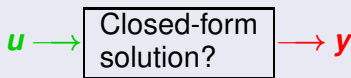
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3. What about nonlinear systems ?

Input/Output nonlinear differential system (state x)

$$\begin{aligned} \mathbf{x}'(t) &= \mathbf{F}(\mathbf{x}(t), \mathbf{u}(t)) \\ \mathbf{y}(t) &= \mathbf{G}(\mathbf{x}(t), \mathbf{u}(t)) \end{aligned}$$



In general, NO ! But...

Linear case

$$\mathbf{F}(\mathbf{x}, \mathbf{u}) = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{G}(\mathbf{x}, \mathbf{u}) = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

I/O relation: linear filter

$$\text{Kernel: } h(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{B} + \mathbf{D}$$

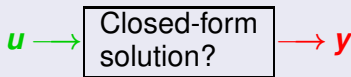
$$\text{Trsf. fct: } H(s) = \mathbf{C}(s\mathbf{I} + \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

Interests: control, spectral analysis, identification, simulation, etc.

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Weakly nonlinear case

\mathbf{F} , \mathbf{G} : power series expansions around equilibrium point 0 (nonzero coeff. at order 1)

I/O relation: linear filter

Kernel: $h(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{B} + \mathbf{D}$

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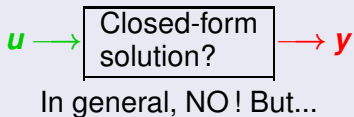
Example: a nonlinear spring

$$mz''(t) + az'(t) + \kappa(z(t)) = f(t)$$

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Linear case

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I/O relation: Volterra series

Interests: idem!

4. From a qualitative point of view...

A few comparisons:

Case	closed-form sol. w.r.t. input	distortions	self-oscillations bifucations, chaos
General	no	yes	yes
Volterra	yes	yes	no
Linear	yes	no	no

4. From a qualitative point of view...

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Volterra	yes	yes	no
Linear	yes	no	no

Interest of Volterra series:

- **Natural distortions** for high amplitudes
- Possible extensions to **partial differential equations**
- Audio-acoustics: **large dynamics (/fortissimo)**

5. What is the idea ?

(regular perturbation method)

For a Weakly Nonlinear System ...

$$\begin{aligned} \mathbf{x}'(t) &= \mathbf{F}(\mathbf{x}(t), \mathbf{u}(t)) & \mathbf{F}(X, U) &= \sum_{m,n} \frac{D_{m,n} \mathbf{F}(0,0)}{m!n!} (X, \dots, X, U, \dots, U) \\ \mathbf{y}(t) &= \mathbf{G}(\mathbf{x}(t), \mathbf{u}(t)) & \mathbf{G}(X, U) &= \sum_{m,n} \frac{D_{m,n} \mathbf{G}(0,0)}{m!n!} (X, \dots, X, U, \dots, U) \end{aligned}$$

Consider the input as a **perturbation** of the system.
Mark it with $\eta \in (0, 1)$: $\mathbf{u}(t) = \eta \mathbf{v}(t)$

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- (i) Introduce $\mathbf{x}(t) = \sum_n \eta^n \mathbf{x}_n(t)$ and $\mathbf{y}(t) = \sum_n \eta^n \mathbf{y}_n(t)$
- (ii) Inject these series expansions in the system equations
- (iii) Sort equations w.r.t. η^n

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We obtain a sequence of linear ODEs, indexed w.r.t. n

- (iv) Solution: Each x_n is a **multiple convolution** of n repeated versions of the input and a **computable multivariate kernel**
→ **Volterra kernel**

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Time domain

- Definition and examples
- A convergence criterion
- Non-uniqueness of kernels
- Remark on time-varying systems

Volterra series: definition

Definition

A system $u \rightarrow \boxed{\{h_n\}} \rightarrow y$ is defined by the Volterra series $\{h_n\}_{n \geq 1}$ if

$$y(t) = \underbrace{\sum_{n=1}^{+\infty}}_{\text{Sum}} \underbrace{\int_{\mathbb{R}^n} h_n(\tau_1, \dots, \tau_n) u(t - \tau_1) \dots u(t - \tau_n) d\tau_1 \dots d\tau_n}_{\text{of multiple convolutions}}$$

(with several possible functional settings)

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Example

- Linear filters: $h_n = 0$, if $n \geq 2$.

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Example

- Linear filters: $h_n = 0$, if $n \geq 2$.
- Memoryless fct: $h_n(\tau_1, \dots, \tau_n) = \alpha_n \delta(\tau_1, \dots, \tau_n)$, (δ : Dirac).

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- General case: $n=1$ (linear contrib.), $n=2$ (quadratic), etc.

A system is **causal** if

$$\begin{aligned} \tau < 0 &\Rightarrow h(\tau) = 0 && \text{(linear)} \\ \exists k \text{ s.t. } \tau_k < 0 &\Rightarrow h_n(\tau_1, \dots, \tau_k, \dots, \tau_n) = 0 && \text{(Volterra)} \end{aligned}$$

A convergence criterion (1/2)

(see e.g. [Boyd,1984])

RECALL: definition of a Volterra series

$$x(t) = \sum_{n=1}^{+\infty} x_n(t) \text{ with } x_n(t) = \int_{\mathbb{R}^n} h_n(\tau_1, \dots, \tau_n) u(t - \tau_1) \dots u(t - \tau_n) d\tau_1 \dots d\tau_n$$

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Bounded Input Bounded Output (BIBO) result

$$(\|u\|_{\infty} = \sup_{t \in \mathbb{R}} |u(t)|)$$

$$|x_n(t)| = \left| \int_{\mathbb{R}^n} h_n(\tau_1, \dots, \tau_n) u(t - \tau_1) \dots u(t - \tau_n) d\tau_1 \dots d\tau_n \right|$$

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$$\begin{aligned} |x_n(t)| &= \left| \int_{\mathbb{R}^n} h_n(\tau_1, \dots, \tau_n) u(t - \tau_1) \dots u(t - \tau_n) d\tau_1 \dots d\tau_n \right| \\ &\leq \int_{\mathbb{R}^n} |h_n(\tau_1, \dots, \tau_n)| \underbrace{|u(t - \tau_1)| \dots |u(t - \tau_n)|}_{\leq \|u\|_{\infty}^n} d\tau_1 \dots d\tau_n \end{aligned}$$

A convergence criterion (1/2)

(see e.g. [Boyd,1984])

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$$\text{Hence, } \|x\|_\infty \leq \sum_{n=1}^{+\infty} \|x_n\|_\infty \leq \sum_{n=1}^{+\infty} \|h_n\|_1 (\|u\|_\infty)^n.$$

A convergence criterion (2/2)

(see e.g. [Boyd,1984])

Gain bound function φ

Define $\varphi(z) = \sum_{n \geq 1} \|h_n\|_1 z^n$ with convergence radius ρ at $z = 0$.

Theorem (BIBO result)

If $\|u\|_\infty < \rho$, **then** the Volterra series expansion of x is normally convergent and

$$\|x\|_\infty \leq \varphi(\|u\|_\infty) < +\infty.$$

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Moreover, the **truncation error is bounded**:

$$\left\| \sum_{n=N+1}^{+\infty} x_n \right\|_\infty \leq \sum_{n=N+1}^{+\infty} \|h_n\|_1 (\|u\|_\infty)^n$$

Non-uniqueness of Volterra kernels

Remark:

Permuting variables τ_k in $y(t) = \sum_{n=1}^{+\infty} \int_{\mathbb{R}^n} h_n(\tau_1, \dots, \tau_n) u(t - \tau_1) \dots u(t - \tau_n) d\tau_1 \dots d\tau_n$ leaves the output y unchanged.

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Example

$h_2(\tau_1, \tau_2)$, $h_2(\tau_2, \tau_1)$, but also $\alpha h_1(\tau_1, \tau_2) + (1 - \alpha) h_2(\tau_2, \tau_1)$ define the same Input-Output system.

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Permuting variables τ_k in $y(t) = \sum_{n=1}^{+\infty} \int_{\mathbb{R}^n} h_n(\tau_1, \dots, \tau_n) u(t-\tau_1) \dots u(t-\tau_n) d\tau_1 \dots d\tau_n$ leaves the output y unchanged.

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Symmetrical versions of Volterra kernels $\text{SYM}[h_n]$ are unique

$$\text{SYM}[h_n](\tau_1, \dots, \tau_n) = \frac{1}{n!} \sum_{\pi} h_n(\tau_{\pi(1)}, \dots, \tau_{\pi(n)})$$

Other unique versions (triangular kernels, regular kernels) are also available.

Remark on time varying systems

A definition is also available for time-varying systems:

$$y(t) = \sum_{n=1}^{+\infty} \int_{\mathbb{R}^n} g_n(t, \theta_1, \dots, \theta_n) u(\theta_1) \dots u(\theta_n) d\theta_1 \dots d\theta_n$$

Time-Invariant (TI) case and link with kernels h_n

$$\text{TI case: } y(t) = \sum_{n=1}^{+\infty} \int_{\mathbb{R}^n} h_n(\tau_1, \dots, \tau_n) u(t - \tau_1) \dots u(t - \tau_n) d\tau_1 \dots d\tau_n$$

Kernels g_n of a TI system are such that ($\theta_k = t - \tau_k$)

$$g_n(t, t - \tau_1, \dots, t - \tau_n) = h_n(\tau_1, \dots, \tau_n)$$

does not depend on t

Causal kernels g_n

$$\exists k \text{ s.t. } \theta_k > t \Rightarrow g_n(t, \theta_1, \dots, \theta_k, \dots, \theta_n) = 0$$

Outline

- 1 Préambule
- 2 **Séries de Volterra : généralités**
 - Domaine temporel
 - **Domaine fréquentiel et de Laplace**
- 3 Calcul des noyaux de Volterra d'un système différentiel
- 4 Exercices et applications en audio-acoustique
- 5 Convergence
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- 7 Conclusion

Laplace(/Fourier) domain and analogies with linear systems

Laplace domain (or Fourier domain with $s = 2i\pi f$)

Transfer function: $H(s) = \int_{\mathbb{R}} h(\tau) e^{-s\tau} d\tau$ (lin.)

Transfer kernel: $H_n(s_{1:n}) = \int_{\mathbb{R}^n} h_n(\tau_{1:n}) e^{-(s_1\tau_1 + \dots + s_n\tau_n)} d\tau_1 \dots d\tau_n$ (Volt.)

denoting $(s_{1:n}) = (s_1, \dots, s_n)$ and $(\tau_{1:n}) = (\tau_1, \dots, \tau_n)$.

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Causal stable system: **NO** poles (and **NO** singularities)

of $H(s)$ for $\Re(s) > 0$ (linear)

of $H_n(s_{1:n})$ for $\Re(s_k) > 0$ (Volterra)

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Input/Output relation

$u \rightarrow$ system $\rightarrow y$

Transfer function: $Y(s) = H(s) U(s)$ (lin.)

Transfer kernel: more complex (Volt.)

(next part: use interconnection laws)

A result on periodic signals

Analytic input signal $u(t) = ae^{i\omega t}$

$$u(t) = ae^{i\omega t} \longrightarrow \boxed{\{h_n\}} \longrightarrow y(t) = \sum_{n=1}^{+\infty} a^n H_n(i\omega, \dots, i\omega) e^{in\omega t}$$

Periodic input signals / Fourier series

$$u(t) = \sum_{k=-\infty}^{+\infty} u_k e^{ik\omega t} \longrightarrow \boxed{\{h_n\}} \longrightarrow y(t) = \sum_{k=-\infty}^{+\infty} y_k e^{ik\omega t}$$

$$\text{with } y_k = \sum_{n=1}^{+\infty} \sum_{\substack{k_1, \dots, k_n = -\infty \\ k_1 + \dots + k_n = k}}^{+\infty} u_{k_1} \dots u_{k_n} H_n(ik_1\omega, \dots, ik_n\omega)$$

Distortion coefficient for $u(t) = a \cos(\omega t)$

$$D(a, \omega) = \sum_{n=2}^{+\infty} |y_n|^2 / |y_1|^2 : \text{closed-form solution w.r.t. } a, \omega, H_n.$$

In summary:

A Volterra series ...

- catches distortions (memory combined with nonlinearities)
- sorts the nonlinear responses w.r.t. the degree n of homogenous contributions of u
- generalizes the convolution principle
- can be described by transfer kernels in the frequency domain (as filters).
- is usually non unique but uniquely defined versions are available (useful for identification purposes)

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