De Rham logarithmic classes and Tate conjecture

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July 18, 2023

Abstract

We introduce the notion of De Rham logarithmic classes. We show that the De Rham class of an algebraic cycle of a smooth algebraic variety over a field of characteristic zero is logarithmic and conversely that a logarithmic class of bidegree (d,d) is the De Rham class of an algebraic cycle (of codimension d). We deduce the Tate conjecture for smooth projective varieties over fields of finite type over $\mathbb{Q}$ and over p-adic fields for $\mathbb{Q}_p$ coefficients, under good reduction hypothesis.

1 Introduction

1.1 De Rham logarithmic classes

In this work, we introduce for X a noetherian scheme, the definition of logarithmic De Rham cohomology classes which are for each $j \in \mathbb{Z}$, the subgroup $H^jOL_X(\mathbb{H}^j_{et}(X,\Omega^\bullet_{X,\log})) \subset \mathbb{H}^j_{et}(X,\Omega^\bullet_X) =: H^j_{DR}(X)$ of the De Rham cohomology abelian group. The subcomplex $OL_X : \Omega^\bullet_{X,\log} \hookrightarrow \Omega^\bullet_X$ of abelian sheaves on X consisting of logarithmic forms is introduced in definition 1.

Let X be a smooth algebraic variety over a field k of characteristic zero. To an algebraic cycle $Z \in \mathcal{Z}^d(X)$, we associate its De Rham cohomology class $[Z] \in H^{2d}_{DR}(X)$, which is by definition, as for any Weil cohomology theory, the image of the fundamental class $[Z] \in H^{2d}_{DR,Z}(X)$ by the canonical morphism $H^{2d}_{DR,Z}(X) \to H^{2d}(X)$. In section 3, we show (c.f. theorem 1(i)) that $[Z] \in H^{2d}_{DR}(X)$ is logarithmic of bidegree (d,d), that is

$$[Z] = H^{2d}OL_X([Z]_L) \in H^{2d}_{DR}(X),\ [Z]_L \in H^d_{et}(X,\Omega^d_{X,\log})$$

where $H^d_{et}(X,\Omega^d_{X,\log}) \subset \mathbb{H}^{2d}_{et}(X,\Omega^\bullet_{X,\log})$ is the canonical subspace, as all the differentials of $\Omega^\bullet_{X,\log}$ vanishes since by definition a logarithmic form is closed. This fact is a consequence of two properties :

- the Euler class of a vector bundle of rank d over an algebraic variety is logarithmic of bidegree (d,d) (c.f. proposition 2)
- the motivic isomorphisms applied to De Rham cohomology preserve logarithmic classes (c.f. proposition 1). The key point is that logarithmic De Rham forms on algebraic varieties over k are closed, (trivially) $\mathbb{A}^1_k$ invariant and compatible with the transfers maps induced by finite morphisms of algebraic varieties and in particular finite correspondences. Since $\Omega^l_{\log} \subset \Omega^l$ are (trivially) $\mathbb{A}^1_k$ invariant presheaves with transfers on the category of smooth algebraic varieties over k, by a theorem of Voevodsky the cohomology presheaves of $\mathbb{A}^1_k$ invariant presheaves with transfers are $\mathbb{A}^1_k$ invariant.

Conversely, we show (c.f. theorem 1(ii)), that a logarithmic class $w \in H^{2d}_{DR}(X)$ of bidegree (d,d) is the De Rham class of an algebraic cycle (of codimension d).
The proof works as follows : A logarithmic class of bidegree (p,q) is locally acyclic for the Zariski topology of X since it is the etale cohomology of a single sheaf. This allows us to proceed by a finite induction using the crucial fact that the purity isomorphism for De Rham cohomology preserve logarithmic classes (c.f. proposition 3). The proof of proposition 3 follows from the fact that the purity isomorphism is

motivic (see [5]), that the Euler class of a vector bundle of rank d over an algebraic variety is logarithmic of bidegree (d,d) (c.f. proposition 2) and that the motivic isomorphisms applied to De Rham cohomology preserve logarithmic classes (c.f. proposition 1). At the final step, we use, for simplicity, the fact that for a scheme Y, $H^1_{et}(Y,\Omega^1_{Y,\log}) = H^1(Y,O_Y^*)$ is the Picard group of Y. In the case X projective, we also get a vanishing result (c.f. theorem 1(iii)') : $H^{2d}OL_X(H^{d-k}_{et}(X,\Omega^{d+k}_{X,\log})) = 0$ for $k > 0$.

1.2 Tate conjecture

Let X be a smooth projective variety over a field k of finite type over $\mathbb{Q}$. Let p be a prime number unramified over k such that X has good reduction at p, $\sigma_p : k \hookrightarrow \mathbb{C}_p$ a fixed embedding and $\bar{k}$ the algebraic closure of k inside $\mathbb{C}_p$. Denote by $\hat{k}_{\sigma_p} \subset \mathbb{C}_p$ the p adic completion of k with respect to σ_p. We show (c.f. proposition 7) that Tate classes $\alpha \in H^{2d}_{et}(X_{\bar{k}},\mathbb{Z}_p)(d)^G$ of X gives by the p-adic crystalline comparison isomorphism

$$H^{2d}R\alpha(X) : H^{2d}_{et}(X_{\mathbb{C}_p},\mathbb{Z}_p)\otimes_{\mathbb{Z}_p}\mathbb{B}_{cris,\hat{k}_{\sigma_p}} \xrightarrow{\sim} H^{2d}_{DR}(X)\otimes_k \mathbb{B}_{cris,\hat{k}_{\sigma_p}},$$

a $\mathbb{Q}_p$ linear combination of logarithmic classes

$$w(\alpha) := H^{2d}R\alpha(X)(\alpha) := \sum_{i=1}^{s}\lambda_i w(\alpha)_{L,i} \in H^{2d}_{DR}(X_{\mathbb{C}_p}),\ \lambda_i \in \mathbb{Q}_p$$

with for each $1 \leq i \leq s$, $\lambda_i \in \mathbb{Q}_p$ and $w(\alpha)_{L,i} \in H^{2d}OL_X\mathbb{H}^{2d}_{et}(X_{\mathbb{C}_p},\Omega^{\geq d}_{X_{\mathbb{C}_p},\log})$, using a result of [6] together with results of [8] on the pro-etale topology, and a GAGA result on logarithmic De Rham classes (c.f. proposition 5) whose proof is motivic and similar to the proof of theorem 1. We deduce, by theorem 1(ii) and (iii)', the Tate conjecture for X with $\mathbb{Q}_p$ coefficient (c.f. theorem 2).

Note that for a p-adic field $K \subset \mathbb{C}_p$, $\dim H^j_{et}(X_{\mathbb{C}_p},\mathbb{Z}_l)^G > \dim H^j_{et}(X_{\mathbb{C}_p},\mathbb{Z}_p)^G$ for $l \neq p$ in general. Hence it is not clear whether Tate conjecture holds for smooth projective varieties over finite fields k_0, since the base change for etale cohomology for $f : X \to O_K$, $O_K = W(k_0)$ only holds for $\mathbb{Z}_l$ coefficients, $l \neq p$.

Let X a smooth projective variety over $\mathbb{C}$. Then X is defined over a subfield $k \subset \mathbb{C}$ of finite type over $\mathbb{Q}$, that is $X = X_k \otimes_k \mathbb{C}$. Take an isomorphism $\mathbb{C} \simeq \mathbb{C}_p$ with $p \in \mathbb{N}$ a prime number such that $X_{\mathbb{C}_p}$ has good reduction at p. As Tate conjecture holds for X_k with $\mathbb{Q}_p$ coefficients, the standards conjectures holds for X and any absolute Hodge class of X is the class of an algebraic cycle (corollary 2).

I am grateful for professor F.Mokrane for help and support during this work.

2 Preliminaries and Notations

- Denote by Top the category of topological spaces and RTop the category of ringed spaces.
- Denote by Cat the category of small categories and RCat the category of ringed topos.
- For $\mathcal{S} \in$ Cat and $X \in \mathcal{S}$, we denote $\mathcal{S}/X \in$ Cat the category whose objects are $Y/X := (Y,f)$ with $Y \in \mathcal{S}$ and $f : Y \to X$ is a morphism in $\mathcal{S}$, and whose morphisms $\mathrm{Hom}((Y',f'),(Y,f))$ consists of $g : Y' \to Y$ in $\mathcal{S}$ such that $f \circ g = f'$.
- For $(\mathcal{S},O_S) \in$ RCat a ringed topos, we denote by
  - $\mathrm{PSh}(\mathcal{S})$ the category of presheaves of O_S modules on $\mathcal{S}$ and $\mathrm{PSh}_{O_S}(\mathcal{S})$ the category of presheaves of O_S modules on $\mathcal{S}$, whose objects are $\mathrm{PSh}_{O_S}(\mathcal{S})^0 := \{(M,m), M \in \mathrm{PSh}(\mathcal{S}), m : M \otimes O_S \to M\}$, together with the forgetful functor $o : \mathrm{PSh}(\mathcal{S}) \to \mathrm{PSh}_{O_S}(\mathcal{S})$,
  - $C(\mathcal{S}) = C(\mathrm{PSh}(\mathcal{S}))$ and $C_{O_S}(\mathcal{S}) = C(\mathrm{PSh}_{O_S}(\mathcal{S}))$ the big abelian category of complexes of presheaves of O_S modules on $\mathcal{S}$,

- $C_{O_S(2)fil}(\mathcal{S}) := C_{(2)fil}(\mathrm{PSh}_{O_S}(\mathcal{S})) \subset C(\mathrm{PSh}_{O_S}(\mathcal{S}), F, W)$, the big abelian category of (bi)filtered complexes of presheaves of O_S modules on $\mathcal{S}$ such that the filtration is biregular and $\mathrm{PSh}_{O_S(2)fil}(\mathcal{S}) := (\mathrm{PSh}_{O_S}(\mathcal{S}), F, W)$.

- Let $(\mathcal{S}, O_S) \in \mathrm{RCat}$ a ringed topos with topology τ. For $F \in C_{O_S}(\mathcal{S})$, we denote by $k : F \to E_\tau(F)$ the canonical flasque resolution in $C_{O_S}(\mathcal{S})$ (see [3]). In particular for $X \in \mathcal{S}$, $H^*(X, E_\tau(F)) \xrightarrow{\sim} \mathbb{H}^*_\tau(X, F)$.

- For $f : \mathcal{S}' \to \mathcal{S}$ a morphism with $\mathcal{S}, \mathcal{S}' \in \mathrm{RCat}$, endowed with topology τ and τ' respectively, we denote for $F \in C_{O_S}(\mathcal{S})$ and each $j \in \mathbb{Z}$,

  - $f^* := H^j\Gamma(\mathcal{S}, k \circ \mathrm{ad}(f^*, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \to \mathbb{H}^j(\mathcal{S}', f^*F)$,
  - $f^* := H^j\Gamma(\mathcal{S}, k \circ \mathrm{ad}(f^{*mod}, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \to \mathbb{H}^j(\mathcal{S}', f^{*mod}F)$,

  the canonical maps.

- For $\mathcal{X} \in \mathrm{Cat}$ a (pre)site and p a prime number, we consider the full subcategory

$$\mathrm{PSh}_{\mathbb{Z}_p}(\mathcal{X}) \subset \mathrm{PSh}(\mathbb{N} \times \mathcal{X}), \;\; F = (F_n)_{n\in\mathbb{N}},\; p^n F_n = 0,\; F_{n+1}/p^n \xrightarrow{\sim} F_n$$

  $C_{\mathbb{Z}_p}(\mathcal{X}) := C(\mathrm{PSh}_{\mathbb{Z}_p}(\mathcal{X})) \subset C(\mathbb{N} \times \mathcal{X})$ and

$$\mathbb{Z}_p := \mathbb{Z}_{p,\mathcal{X}} := ((\mathbb{Z}/p^*\mathbb{Z})_{\mathcal{X}}) \in \mathrm{PSh}_{\mathbb{Z}_p}(\mathcal{X})$$

  the diagram of constant presheaves on $\mathcal{X}$.

- Denote by $\mathrm{Sch} \subset \mathrm{RTop}$ the subcategory of schemes (the morphisms are the morphisms of locally ringed spaces). We denote by $\mathrm{PSch} \subset \mathrm{Sch}$ the full subcategory of proper schemes. For a field k, we consider $\mathrm{Sch}\,/k := \mathrm{Sch}\,/\operatorname{Spec} k$ the category of schemes over $\operatorname{Spec} k$, that is whose object are $X := (X, a_X)$ with $X \in \mathrm{Sch}$ and $a_X : X \to \operatorname{Spec} k$ a morphism and whose objects are morphism of schemes $f : X' \to X$ such that $f \circ a_{X'} = a_X$. We then denote by

  - $\mathrm{Var}(k) = \mathrm{Sch}^{ft}\,/k \subset \mathrm{Sch}\,/k$ the full subcategory consisting of algebraic varieties over k, i.e. schemes of finite type over k,
  - $\mathrm{PVar}(k) \subset \mathrm{QPVar}(k) \subset \mathrm{Var}(k)$ the full subcategories consisting of quasi-projective varieties and projective varieties respectively,
  - $\mathrm{PSmVar}(k) \subset \mathrm{SmVar}(k) \subset \mathrm{Var}(k)$, $\mathrm{PSmVar}(k) := \mathrm{PVar}(k) \cap \mathrm{SmVar}(k)$, the full subcategories consisting of smooth varieties and smooth projective varieties respectively.

  For a morphism of field $\sigma : k \hookrightarrow K$, we have the extention of scalar functor

$$\otimes_k K : \mathrm{Sch}\,/k \to \mathrm{Sch}\,/K,\; X \mapsto X_K := X_{K,\sigma} := X \otimes_k K,\; (f : X' \to X) \mapsto (f_K := f \otimes I : X'_K \to X_K).$$

  which is left ajoint to the restriction of scalar

$$Res_{k/K} : \mathrm{Sch}\,/K \to \mathrm{Sch}\,/k,\; X = (X, a_X) \mapsto X = (X, \sigma \circ a_X),\; (f : X' \to X) \mapsto (f : X' \to X)$$

  The extention of scalar functor restrict to a functor

$$\otimes_k K : \mathrm{Var}(k) \to \mathrm{Var}(K),\; X \mapsto X_K := X_{K,\sigma} := X \otimes_k K,\; (f : X' \to X) \mapsto (f_K := f \otimes I : X'_K \to X_K).$$

  and for $X \in \mathrm{Var}(k)$ we have $\pi_{k/K}(X) : X_K \to X$ the projection in $\mathrm{Sch}\,/k$.

- For $X \in \mathrm{Sch}$, we denote $X^{pet} \subset \mathrm{Sch}/X$ the etale site and $X^{pet} \subset \mathrm{Sch}/X$ the pro etale site (see [2]) which is the full subcategory of Sch/X whose object consists of weakly etale maps $U \to X$ (that is flat maps $U \to X$ such that $\Delta_U : U \to U \times_X U$ is also flat) and whose topology is generated by fpqc covers. We then have the canonical morphism of site

$$\nu_X : X^{pet} \to X^{et}, (U \to X) \mapsto (U \to X)$$

For $F \in C(X^{et})$,

$$\mathrm{ad}(\nu_X^*, R\nu_{X*})(F) : F \to R\nu_{X*}\nu_X^* F$$

is an isomorphism in $D(X^{et})$, in particular, for each $n \in \mathbb{Z}$

$$\nu_X^* : \mathbb{H}^n_{et}(X, F) \xrightarrow{\sim} \mathbb{H}^n_{pet}(X, \nu_X^* F)$$

are isomorphisms, For $X \in \mathrm{Sch}$, we denote

  - $\underline{\mathbb{Z}_p}_X := \varprojlim_{n\in\mathbb{N}} \nu_X^*(\mathbb{Z}/p^n\mathbb{Z})_{X^{et}} \in \mathrm{PSh}(X^{pet})$ the constant presheaf on $\mathcal{X}$,
  - $l_{p,\mathcal{X}} := (p(*)) : \underline{\mathbb{Z}_p}_X \to \nu_X^*(\mathbb{Z}/p\mathbb{Z})_{X^{et}}$ the projection map in $\mathrm{PSh}(\mathbb{N} \times \mathcal{X}^{pet})$.

- An affine scheme $U \in \mathrm{Sch}$ is said to be w-contractible if any faithfully flat weakly etale map $V \to U$, $V \in \mathrm{Sch}$, admits a section. We will use the facts that (see [2]):

  - Any scheme $X \in \mathrm{Sch}$ admits a pro-etale affine cover $(r_i : X_i \to X)_{i\in I}$ with for each $i \in I$, X_i a w-contractile affine scheme and $r_i : X_i \to X$ a weakly etale map. For $X \in \mathrm{Var}(k)$ with k a field, we may assume I finite since the topological space X is then quasi-compact.
  - If $U \in \mathrm{Sch}$ is a w-contractible affine scheme, then for any sheaf $F \in \mathrm{Shv}(U^{pet})$, $H^i_{pet}(U, F) = 0$ for $i \neq 0$ since $\Gamma(U, -)$ is an exact functor.

- Let $X \in \mathrm{Sch}$. Considering its De Rham complex $\Omega^\bullet_X := DR(X)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H^j_{DR}(X) := \mathbb{H}^j(X, \Omega^\bullet_X)$. For $X \in \mathrm{Var}(k)$, the differentials of $\Omega^\bullet_X := \Omega^\bullet_{X/k}$ are by definition k-linear, thus $H^j_{DR}(X)$ has a structure of a k vector space. If $X \in \mathrm{SmVar}(k)$, then $H^j_{DR}(X) = \mathbb{H}^j_{et}(X, \Omega^\bullet_X)$ since $\Omega^\bullet_{/k} \in C(\mathrm{SmVar}(k))$ is $\mathbb{A}^1$ local and admits transfers where

$$\Omega^\bullet_{/k} \in C(\mathrm{Var}(k)), X \mapsto \Omega^\bullet_{/k}(X) := \Gamma(X, \Omega^\bullet_X),$$
$$(f : X' \to X) \mapsto \Omega^\bullet_{/k}(f) := f^* : \Gamma(X, \Omega^\bullet_X) \to \Gamma(X', \Omega^\bullet_{X'})$$

(see [3]).

- Let $X \in \mathrm{Var}(k)$. Let $X = \cup_{i=1}^s X_i$ an open affine cover. For $I \subset [1, \ldots, s]$, we denote $X_I := \cap_{i\in I} X_i$. We get $X_\bullet \in \mathrm{Fun}(P([1, \ldots, s]), \mathrm{Var}(k))$. Since quasi-coherent sheaves on affine noetherian schemes are acyclic, we have for each $j \in \mathbb{Z}$, $H^j_{DR}(X) = \Gamma(X_\bullet, \Omega^\bullet_{X_\bullet})$.

- For $X \in \mathrm{Sch}$ noetherian irreducible and $d \in \mathbb{N}$, we denote by $\mathcal{Z}^d(X)$ the group of algebraic cycles of codimension d, which is the free abelian group generated by irreducible closed subsets of codimension d.

- For K a field which is complete with respect to a p-adic norm, we consider $O_K \subset K$ the subring of K consisting of integral elements, that is $x \in K$ such that $|x| \leq 1$.

  - For $X \in \mathrm{PVar}(K)$, we will consider $X^{\mathcal{O}} \in \mathrm{PSch}/O_K$ a (non canonical) integral model of X, i.e. satisfying $X^{\mathcal{O}} \otimes_{O_K} K = X$
  - For $X \in \mathrm{Var}(K)$, we will consider $X^{\mathcal{O}} \in \mathrm{Sch}/O_K$ a (non canonical) integral model of X, i.e. $X^{\mathcal{O}} = \bar{X}^{\mathcal{O}} \backslash Z^{\mathcal{O}}$ for $\bar{X} \in \mathrm{PVar}(K)$ a compactification of X, $Z := \bar{X} \backslash X$, where $\bar{X}^{\mathcal{O}} \in \mathrm{PSch}/O_K$ is an integral model of $\bar{X}$ and $Z^{\mathcal{O}} \in \mathrm{PSch}/O_K$ is an integral model of Z.

For $X \in \mathrm{Var}(K)$, we will consider $X^{\mathcal{O}} \in \mathrm{Sch}\,/O_K$ a (non canonical) integral model of X, we then have the commutative diagram of sites

$$\begin{array}{ccc} X^{pet} & \xrightarrow{r} & X^{\mathcal{O},pet} \\ \downarrow{\scriptstyle \nu_X} & & \downarrow{\scriptstyle \nu_{X^{\mathcal{O}}}} \\ X^{et} & \xrightarrow{r} & X^{\mathcal{O},et} \end{array} \quad , \quad r(t : U \to X^{\mathcal{O}}) := (t \otimes_{O_K} K : U \otimes_{O_K} K \to X^{\mathcal{O}} \otimes_{O_K} K = X)$$

- For $X \in \mathrm{Sch}$ and p a prime number, we denote by $c : \hat{X}^{(p)} \to X$ the morphism in RTop which is the completion along the ideal generated by p.
- Let K a field which is complete with respect to a p-adic norm and $X \in \mathrm{PVar}(K)$ projective. For $X^{\mathcal{O}} \in \mathrm{PSch}\,/O_K$ an integral model of X, i.e. satisfying $X^{\mathcal{O}} \otimes_{O_K} K = X$, we consider the morphism in RTop
$$c := (c \otimes I) : \hat{X}^{(p)} := \hat{X}^{\mathcal{O},(p)} \otimes_{O_K} K \to X^{\mathcal{O}} \otimes_{O_K} K = X.$$
We have then, by GAGA (c.f. EGA 3), for $F \in \mathrm{Coh}_{O_X}(X)$ a coherent sheaf of O_X module, $c^* : H^k(X^{\mathcal{O}}, F) \xrightarrow{\sim} H^k(\hat{X}^{\mathcal{O},(p)}, c^{*mod} F)$ for all $k \in \mathbb{Z}$, in particular
$$c^* : \mathbb{H}^k(X, \Omega_X^{\bullet \geq l}) \xrightarrow{\sim} \mathbb{H}^k(\hat{X}^{(p)}, \Omega_{\hat{X}^{(p)}}^{\bullet \geq l})$$
for all $k, l \in \mathbb{Z}$.
- Denote by $\mathrm{AnSp}(\mathbb{C}) \subset \mathrm{RTop}$ the full subcategory of analytic spaces over $\mathbb{C}$, and by $\mathrm{AnSm}(\mathbb{C}) \subset \mathrm{AnSp}(\mathbb{C})$ the full subcategory of smooth analytic spaces (i.e. complex analytic manifold). Denote by $\mathrm{CW} \subset \mathrm{Top}$ the full subcategory of CW complexes. Denote by $\mathrm{Diff}(\mathbb{R}) \subset \mathrm{RTop}$ the full subcategory of differentiable (real) manifold.
- Denote $\mathrm{Sch}^2 \subset \mathrm{RTop}^2$ the subcategory whose objects are couples (X, Z) with $X = (X, O_X) \in \mathrm{Sch}$ and $Z \subset X$ a closed subset and whose set of morphisms $\mathrm{Hom}((X', Z'), (X, Z))$ consists of $f : X' \to X$ of locally ringed spaces such that $f^{-1}(Z) \subset Z'$.
- Let k a field of characteristic zero. Denote $\mathrm{SmVar}^2(k) \subset \mathrm{Var}^2(k) \subset \mathrm{Sch}^2\,/k$ the full subcategories whose objects are (X, Z) with $X \in \mathrm{Var}(k)$, resp. $X \in \mathrm{SmVar}(k)$, and $Z \subset X$ is a closed subset, and whose morphisms $\mathrm{Hom}((X', Z') \to (X, Z))$ consists of $f : X' \to X$ of schemes over k such that $f^{-1}(Z) \subset Z'$.
- Let k a field of characteristic zero. For $X, X' \in \mathrm{Var}(k)$, with X' irreducible, we denote $\mathcal{Z}^{fs/X'}(X' \times X) \subset \mathcal{Z}_{d_{X'}}(X' \times X)$ which consist of algebraic cycles $\alpha = \sum_i n_i \alpha_i \in \mathcal{Z}_{d_{X'}}(X' \times X)$ such that, denoting $\mathrm{supp}(\alpha) = \cup_i \alpha_i \subset X' \times X$ its support and $p' : X' \times X \to X'$ the projection, $p'_{|\mathrm{supp}(\alpha)} : \mathrm{supp}(\alpha) \to X'$ is finite surjective. We denote by $\mathrm{Cor}\,\mathrm{SmVar}(k)$ the category such that the objects are $\{X \in \mathrm{SmVar}(k)\}$ and such that $\mathrm{Hom}_{\mathrm{Cor}\,\mathrm{SmVar}(k)}(X', X) := \mathcal{Z}^{fs/X'}(X' \times X)$.
- For $X \in \mathrm{Top}$ and $Z \subset X$ a closed subset, denoting $j : X \backslash Z \hookrightarrow X$ the open complementary, we will consider
$$\Gamma_Z^{\vee} \mathbb{Z}_X := \mathrm{Cone}(\mathrm{ad}(j_!, j^*)(\mathbb{Z}_X) : j_! j^* \mathbb{Z}_X \hookrightarrow \mathbb{Z}_X) \in C(X)$$
and denote for short $\gamma_Z^{\vee} := \gamma_X^{\vee}(\mathbb{Z}_X) : \mathbb{Z}_X \to \Gamma_Z^{\vee} \mathbb{Z}_X$ the canonical map in $C(X)$.
- For $X \in \mathrm{Var}(k)$ and $Z \subset X$ a closed subset, denoting $j : X \backslash Z \hookrightarrow X$ the open complementary, we will consider
$$\Gamma_Z^{\vee} \mathbb{Z}_X := \mathrm{Cone}(\mathrm{ad}(j_\sharp, j^*)(\mathbb{Z}_X) : \mathbb{Z}_X \hookrightarrow \mathbb{Z}_X) \in C(\mathrm{Var}(k)^{sm}/X)$$
and denote for short $\gamma_Z^{\vee} := \gamma_Z^{\vee}(\mathbb{Z}_X) : \mathbb{Z}_X \to \Gamma_Z^{\vee} \mathbb{Z}_X$ the canonical map in $C(\mathrm{Var}(k)^{sm}/X)$. Denote $a_X : X \to \mathrm{Spec}\, k$ the structural map. For $X \in \mathrm{Var}(k)$ and $Z \subset X$ a closed subset, we have the motive of X with support in Z defined as
$$M_Z(X) := a_{X!} \Gamma_Z^{\vee} a_X^! \mathbb{Z} \in \mathrm{DA}(k).$$

If $X \in \mathrm{SmVar}(k)$, we will also consider

$$a_{X\sharp}\Gamma_Z^\vee \mathbb{Z}_X := \mathrm{Cone}(a_{X\sharp} \circ \mathrm{ad}(j_\sharp, j^*)(\mathbb{Z}_X) : \mathbb{Z}(U) \hookrightarrow \mathbb{Z}(X)) =: \mathbb{Z}(X, X\backslash Z) \in C(\mathrm{SmVar}(k)).$$

Then for $X \in \mathrm{SmVar}(k)$ and $Z \subset X$ a closed subset

$$M_Z(X) := a_{X!}\Gamma_Z^\vee a_X^! \mathbb{Z} = a_{X\sharp}\Gamma_Z^\vee \mathbb{Z}_X =: \mathbb{Z}(X, X\backslash Z) \in \mathrm{DA}(k).$$

- We denote $\mathbb{I}^n := [0,1]^n \in \mathrm{Diff}(\mathbb{R})$ (with boundary). For $X \in \mathrm{Top}$ and R a ring, we consider its singular cochain complex

$$C^*_{\mathrm{sing}}(X, R) := (\mathbb{Z}\,\mathrm{Hom}_{\mathrm{Top}}(\mathbb{I}^*, X)^\vee) \otimes R$$

and for $l \in \mathbb{Z}$ its singular cohomology $H^l_{\mathrm{sing}}(X, R) := H^n C^*_{\mathrm{sing}}(X, R)$. In particular, we get by functoriality the complex

$$C^*_{X, R\,\mathrm{sing}} \in C_R(X),\ (U \subset X) \mapsto C^*_{\mathrm{sing}}(U, R)$$

We will consider the canonical embedding

$$C^*\iota_{2i\pi\mathbb{Z}/\mathbb{C}}(X) : C^*_{\mathrm{sing}}(X, 2i\pi\mathbb{Z}) \hookrightarrow C^*_{\mathrm{sing}}(X, \mathbb{C}),\ \alpha \mapsto \alpha \otimes 1$$

whose image consists of cochains $\alpha \in C^j_{\mathrm{sing}}(X, \mathbb{C})$ such that $\alpha(\gamma) \in 2i\pi\mathbb{Z}$ for all $\gamma \in \mathbb{Z}\,\mathrm{Hom}_{\mathrm{Top}}(\mathbb{I}^*, X)$. We get by functoriality the embedding in $C(X)$

$$C^*\iota_{2i\pi\mathbb{Z}/\mathbb{C},X} : C^*_{X,2i\pi\mathbb{Z},\mathrm{sing}} \hookrightarrow C^*_{X,\mathbb{C},\mathrm{sing}},$$
$$(U \subset X) \mapsto (C^*\iota_{2i\pi\mathbb{Z}/\mathbb{C}}(U) : C^*_{\mathrm{sing}}(U, 2i\pi\mathbb{Z}) \hookrightarrow C^*_{\mathrm{sing}}(U, \mathbb{C}))$$

We recall we have

  - For $X \in \mathrm{Top}$ locally contractile, e.g. $X \in \mathrm{CW}$, and R a ring, the inclusion in $C_R(X)$ $c_X : R_X \to C^*_{X, R\,\mathrm{sing}}$ is by definition an equivalence top local and that we get by the small chain theorem, for all $l \in \mathbb{Z}$, an isomorphism $H^l c_X : H^l(X, R_X) \xrightarrow{\sim} H^l_{\mathrm{sing}}(X, R)$.
  - For $X \in \mathrm{Diff}(\mathbb{R})$, the restriction map

$$r_X : \mathbb{Z}\,\mathrm{Hom}_{\mathrm{Diff}(\mathbb{R})}(\mathbb{I}^*, X)^\vee \to C^*_{\mathrm{sing}}(X, R),\ w \mapsto w : (\phi \mapsto w(\phi))$$

    is a quasi-isomorphism by Whitney approximation theorem.

- We will use the following theorem of Voevodsky (see e.g. [5]): Let k a perfect field (e.g. k a field of characteristic zero). Let $F \in \mathrm{PSh}(\mathrm{SmVar}(k))$. If F is $\mathbb{A}^1$ invariant and admits transfers, then for all $j \in \mathbb{Z}$, $H^j E^\bullet_{et}(F) \in \mathrm{PSh}(\mathrm{SmVar}(k))$ are $\mathbb{A}^1$ invariant. That is, if F is $\mathbb{A}^1$ invariant and admits transfers then F is $\mathbb{A}^1$ local.

We introduce the logarithmic De Rham complexes

Definition 1. *(i) Let $X = (X, O_X) \in \mathrm{RCat}$ a ringed topos, we have in $C(X)$ the subcomplex of presheaves of abelian groups*

$$OL_X : \Omega^\bullet_{X,\log} \hookrightarrow \Omega^\bullet_X,\ \textit{s.t. for } X^o \in X \textit{ and } p \in \mathbb{N},\ p \geq 1,$$
$$\Omega^p_{X,\log}(X^0) :=< df_{\alpha_1}/f_{\alpha_1} \wedge \cdots \wedge df_{\alpha_p}/f_{\alpha_p}, f_{\alpha_k} \in \Gamma(X^o, O_X)^* >_{\subset} \Omega^p_X(X^0),$$

where $\Omega^\bullet_X := DR(X)(O_X) \in C(X)$ is the De Rham complex and $\Gamma(X^o, O_X)^* \subset \Gamma(X^o, O_X)$ is the multiplicative subgroup consisting of invertible elements for the multiplication, here $<,>$ stand for

the sub-abelian group generated by. By definition, for $w \in \Omega^p_X(X^o)$, $w \in \Omega^p_{X,\log}(X^o)$ *if and only if there exists* $(n_i)_{1\leq i\leq s} \in \mathbb{Z}$ *and* $(f_{i,\alpha_k})_{1\leq i\leq s, 1\leq k\leq p} \in \Gamma(X^o, O_X)^*$ *such that*

$$w = \sum_{1\leq i\leq s} n_i df_{i,\alpha_1}/f_{i,\alpha_1} \wedge \cdots \wedge df_{i,\alpha_p}/f_{i,\alpha_p} \in \Omega^p_X(X^o).$$

For $p = 0$, *we set* $\Omega^0_{X,\log} := \mathbb{Z}$. *Let* $f : X' = (X', O_{X'}) \to X = (X, O_X)$ *a morphism with* $X, X' \in$ RCat. *Consider the morphism* $\Omega(f) : \Omega^\bullet_X \to f_*\Omega^\bullet_{X'}$ *in* $C(X)$. *Then,* $\Omega(f)(\Omega^\bullet_{X,\log}) \subset f_*\Omega^\bullet_{X',\log}$.

(ii) For k *a field, we get from (i), for* $X \in \mathrm{Var}(k)$, *the embedding in* $C(X)$

$$OL_X : \Omega^\bullet_{X,\log} \hookrightarrow \Omega^\bullet_X := \Omega^\bullet_{X/k},$$

such that, for $X^o \subset X$ *an open subset and* $w \in \Omega^p_X(X^o)$, $w \in \Omega^p_{X,\log}(X^o)$ *if and only if there exists* $(n_i)_{1\leq i\leq s} \in \mathbb{Z}$ *and* $(f_{i,\alpha_k})_{1\leq i\leq s, 1\leq k\leq p} \in \Gamma(X^o, O_X)^*$ *such that*

$$w = \sum_{1\leq i\leq s} n_i df_{i,\alpha_1}/f_{i,\alpha_1} \wedge \cdots \wedge df_{i,\alpha_p}/f_{i,\alpha_p} \in \Omega^p_X(X^o),$$

and for $p = 0$, $\Omega^0_{X,\log} := \mathbb{Z}$. *Let* k *a field. We get an embedding in* $C(\mathrm{Var}(k))$

$$OL : \Omega^\bullet_{/k,\log} \hookrightarrow \Omega^\bullet_{/k}, \text{ given by, for } X \in \mathrm{Var}(k),$$
$$OL(X) := OL_X : \Omega^\bullet_{/k,\log}(X) := \Gamma(X, \Omega^\bullet_{X,\log}) \hookrightarrow \Gamma(X, \Omega^\bullet_X) =: \Omega^\bullet_{/k}(X)$$

and its restriction to $\mathrm{SmVar}(k) \subset \mathrm{Var}(k)$.

(iii) Let K *a field of characteristic zero which is complete for a p-adic norm. Let* $X \in \mathrm{Var}(K)$. *Let* $X^{\mathcal{O}} \in \mathrm{Sch}/O_K$ *an integral model of* X, *in particular* $X^{\mathcal{O}} \otimes_{O_K} K = X$. *We have then the morphisms of sites* $r : X^{et} \to X^{\mathcal{O},et}$ *and* $r : X^{pet} \to X^{\mathcal{O},pet}$ *such that* $\nu_{X^{\mathcal{O}}} \circ r = r \circ \nu_X$. *We then consider the embedding of* $C(X^{et})$

$$OL_X := OL_X \circ \iota : \Omega^\bullet_{X^{et},\log,\mathcal{O}} := r^*\Omega^\bullet_{X^{\mathcal{O},et},\log} \hookrightarrow \Omega^\bullet_{X^{et},\log} \hookrightarrow \Omega^\bullet_{X^{et}}$$

consisting of integral logarithimc De Rham forms, with $\iota : r^*\Omega^\bullet_{X^{\mathcal{O},et},\log} \hookrightarrow \Omega^\bullet_{X^{et},\log}$. *We will also consider the embedding of* $C(X^{\mathcal{O},pet})$

$$OL_{\hat{X}^{\mathcal{O},(p)}} := (OL_{X^{\mathcal{O}}/p^n})_{n\in\mathbb{N}} : \Omega^\bullet_{\hat{X}^{(p)},\log,\mathcal{O}} := \varprojlim_{n\in\mathbb{N}} \nu^*_{X^{\mathcal{O}}}\Omega^\bullet_{X^{\mathcal{O},et}/p^n,\log}$$
$$\hookrightarrow \Omega^\bullet_{\hat{X}^{\mathcal{O},(p)}} := \varprojlim_{n\in\mathbb{N}} \nu^*_{X^{\mathcal{O}}}\Omega^\bullet_{X^{\mathcal{O},et}/p^n/(O_K/p^n)}$$

where we recall $c : \hat{X}^{\mathcal{O},(p)} \to X^{\mathcal{O}}$ *the morphism in* RTop *is given by the completion along the ideal generated by* p. *We then get the embeddings of* $C(X^{pet})$

$$m \circ (OL_X \otimes I) : \Omega^\bullet_{X^{pet},\log,\mathcal{O}} \otimes \mathbb{Z}_p := \Omega^\bullet_{X^{pet},\log,\mathcal{O}} \otimes \underline{\mathbb{Z}_p}_X \hookrightarrow \Omega^\bullet_{X^{pet}},\ (w \otimes \lambda_n)_{n\in\mathbb{N}} \mapsto (\lambda_n)_{n\in\mathbb{N}} \cdot w$$

and

$$OL_{\hat{X}^{(p)}} := r^*OL_{\hat{X}^{\mathcal{O},(p)}} : r^*\Omega^\bullet_{\hat{X}^{(p)},\log,\mathcal{O}} \hookrightarrow r^*\Omega^\bullet_{\hat{X}^{\mathcal{O},(p)}} \hookrightarrow \Omega^\bullet_{\hat{X}^{(p)}} := \Omega^\bullet_{\hat{X}^{(p)}/K},$$

where we recall $c := (c \otimes_{O_K} K) : \hat{X}^{(p)} \to X$ *the morphism in* RTop. *Note that the inclusion* $\Omega^l_{X^{\mathcal{O},et},\log}/p^n \subset \Omega^\bullet_{X^{\mathcal{O},et}/p^n,\log}$ *is strict in general. Note that*

$$\Omega^\bullet_{X^{pet},\log,\mathcal{O}} := \nu^*_X\Omega^\bullet_{X^{et},\log,\mathcal{O}} \in C(X^{pet}),$$

but

$$\underline{\mathbb{Z}_p}_X := \varprojlim_{n\in\mathbb{N}} \nu_X^*(\mathbb{Z}/p^n\mathbb{Z})_{X^{et}} \in C(X^{pet}),\ \Omega^\bullet_{\hat{X}^{(p)}} \in C(X^{pet}),\ and$$
$$r^*\Omega^\bullet_{\hat{X}^{(p)},\log,\mathcal{O}} := r^* \varprojlim_{n\in\mathbb{N}} \nu_{X^{\mathcal{O}}}^* \Omega_{X^{\mathcal{O},et}/p^n,\log} \in C(X^{pet})$$

are NOT the pullback of etale sheaves by ν_X. We consider $\mathrm{Sch}^{int}/O_K := O(\mathrm{PSch}^2/O_K) \subset \mathrm{Sch}^{ft}/O_K$ *the full subcategory consisting of integrable models of algebraic varieties over K, where* $O : \mathrm{PSch}^2/O_K \to \mathrm{Sch}^{ft}/O_K$, $O(X,Z) = X\backslash Z$ *is the canonical functor. We denote by* $\mathrm{Var}(K)^{pet}$ *and* $(\mathrm{Sch}^{int}/O_K)^{pet}$ *the big pro-etale sites. We have then the morphism of sites* $r : \mathrm{Var}(K)^{pet} \to (\mathrm{Sch}^{int}/O_K)^{pet}$. *We will consider the embedding of* $C((\mathrm{Sch}^{int}/O_K)^{pet})$

$$OL_{/O_K,an} : \Omega^{\bullet,an}_{/K,\log,\mathcal{O}} \hookrightarrow \Omega^{\bullet,an}_{/O_K},\ for\, X^{\mathcal{O}} \in (\mathrm{Sch}^{int}/O_K)^{pet},$$
$$OL_{/O_K,an}(X^{\mathcal{O}}) := OL_{\hat{X}^{\mathcal{O},(p)}}(X^{\mathcal{O}}) : \Omega^\bullet_{\hat{X}^{(p)},\log,\mathcal{O}}(\hat{X}^{\mathcal{O},(p)}) \hookrightarrow \Omega^\bullet_{\hat{X}^{\mathcal{O},(p)}}(\hat{X}^{\mathcal{O},(p)})$$

We get the embedding of $C(\mathrm{Var}(K)^{pet})$

$$OL_{/K,an} := r^*OL_{/O_K,an} : r^*\Omega^{\bullet,an}_{/K,\log,\mathcal{O}} \hookrightarrow r^*\Omega^{\bullet,an}_{/O_K} \hookrightarrow \Omega^{\bullet,an}_{/K}$$

and its restriction to $\mathrm{SmVar}(K)^{pet} \subset \mathrm{Var}(K)^{pet}$.

3 De Rham logarithmic classes

3.1 The De Rham classes of algebraic cycles vs De Rham logarithmic classes

Let $X \in \mathrm{Sch}$. Recall we have the canonical sub-complex $OL_X : \Omega^\bullet_{X,\log} \hookrightarrow \Omega^\bullet_X$ in $C(X^{et})$ (c.f. definition 1). All the differential of $\Omega^\bullet_{X,\log}$ vanishes since by definition the logarithmic forms are closed. For $j \in \mathbb{Z}$, the De Rham logarithmic classes consist of the image

$$H^jOL_X(\mathbb{H}^j_{et}(X,\Omega^\bullet_{X,\log})) \subset \mathbb{H}^j_{et}(X,\Omega^\bullet_X) = H^j_{DR}(X).$$

The filtered complex $\Gamma(X, E_{et}(\Omega^\bullet_{X^{et},\log}, F_b)) \in C_{fil}(\mathbb{Z})$ has its differentials at the E_1 level vanishes since the logarithmic forms are closed, hence we have a canonical splitting

$$\mathbb{H}^j_{et}(X,\Omega^\bullet_{X,\log}) = \oplus_{0\le l\le j} H^{j-l}_{et}(X,\Omega^l_{X,\log}).$$

Let $X \in \mathrm{Sch}$ a noetherian scheme. We have by definition the exact sequence in $C(X^{et})$

$$0 \to F^* \to O_X^* \xrightarrow{dlog} \Omega_{X,\log} \to 0$$

where F is a prime field (i.e. $F = \mathbb{Q}$ or $F = \mathbb{Z}/p\mathbb{Z}$ for p a prime number). Hence $H^1_{et}(X,\Omega_{X,\log}) = H^1_{et}(X,O_X^*)$ and $H^q_{et}(X,\Omega_{X,\log}) = H^q_{et}(X,O_X^*) = 0$ for $q \le 2$. Let $X \in \mathrm{Sch}$ a noetherian proper scheme. We have $H^0(X,\Omega^l_{X,\log}) = 0$.

Let k a field of characteristic zero. We get (see definition 1) the embedding in $C(\mathrm{SmVar}(k))$

$$OL : \Omega^\bullet_{/k,\log} \hookrightarrow \Omega^\bullet_{/k},\ \text{given by, for}\, X \in \mathrm{SmVar}(k),$$
$$OL(X) := OL_X : \Omega^\bullet_{/k,\log}(X) := \Gamma(X,\Omega^\bullet_{X,\log}) \hookrightarrow \Gamma(X,\Omega^\bullet_X) =: \Omega^\bullet_{/k}(X).$$

We have also the sheaves

$$O_k, O_k^* \in \mathrm{PSh}(\mathrm{SmVar}(k)),\ X \in \mathrm{SmVar}(k) \mapsto O_k(X) := O(X),\ O_k^*(X) := O(X)^*,$$
$$(g : Y \to X) \mapsto a_g(X) : O_X(X) \to O_Y(Y),\ O_X(X)^* \to O_Y(Y)^*$$

The sheaf $O_k^* \in \mathrm{PSh}(\mathrm{SmVar}(k))$ admits transfers : for $W \subset X' \times X$ with $X, X' \in \mathrm{SmVar}(k)$ and W finite over X' and $f \in O(X)^*$, $W^*f := N_{W/X'}(p_X^* f)$ where $p_X : W \hookrightarrow X' \times X \to X$ is the projection and $N_{W/X'} : k(W)^* \to k(X')^*$ is the norm map. This gives transfers on $\Omega^1_{/k,\log} \in \mathrm{PSh}(\mathrm{SmVar}(k))$ compatible with transfers on $\Omega^1_{/k} \in \mathrm{PSh}(\mathrm{SmVar}(k))$: for $W \subset X' \times X$ with $X, X' \in \mathrm{SmVar}(k)$ and W finite over X' and $f \in O(X)^*$,

$$W^* df/f := dW^* f / W^* f = Tr_{W/X'}(p_X^*(df/f)),$$

where where $p_X : W \hookrightarrow X' \times X \to X$ is the projection and $Tr_{W/X'} : O_W \to O_X$ is the trace map. Note that $d(fg)/fg = df/f + dg/g$. We get transfers on

$$\otimes^l_{\mathbb{Q}} \Omega^1_{/k,\log},\ \otimes^l_{O_k} \Omega^1_{/k} \in \mathrm{PSh}(\mathrm{SmVar}(k))$$

since $\otimes^l_{\mathbb{Q}} \Omega^1_{/k,\log} = H^0(\otimes^{L,l}_{\mathbb{Q}} \Omega^1_{/k,\log})$ and $\otimes^l_{O_k} \Omega^1_{/k} = H^0(\otimes^{L,l}_{O_k} \Omega^1_{/k})$. This induces transfers on

$$\wedge^l_{\mathbb{Q}} \Omega^1_{/k,\log} := \mathrm{coker}(\oplus_{I_2 \subset [1,\dots,l]} \otimes^{l-1}_{\mathbb{Q}} \Omega^1_{/k,\log} \xrightarrow{\oplus_{I_2 \subset [1,\dots,l]} \Delta_{I_2} := (w \otimes w' \mapsto w \otimes w \otimes w')} \otimes^l_{\mathbb{Q}} \Omega^1_{/k,\log}) \in \mathrm{PSh}(\mathrm{SmVar}(k)).$$

and

$$\wedge^l_{O_k} \Omega^1_{/k} := \mathrm{coker}(\oplus_{I_2 \subset [1,\dots,l]} \otimes^{l-1}_{O_k} \Omega^1_{/k} \xrightarrow{\oplus_{I_2 \subset [1,\dots,l]} \Delta_{I_2} := (w \otimes w' \mapsto w \otimes w \otimes w')}) \otimes^l_{O_k} \Omega^1_{/k} \in \mathrm{PSh}(\mathrm{SmVar}(k)).$$

Let $X \in \mathrm{Var}(k)$. We have by definition the exact sequence in $C(X^{et})$

$$0 \to k^* \to O_X^* \xrightarrow{dlog} \Omega_{X,\log} \to 0$$

Hence $H^1_{et}(X, \Omega_{X,\log}) = H^1_{et}(X, O_X^*)$ and $H^q_{et}(X, \Omega_{X,\log}) = H^q_{et}(X, O_X^*) = 0$ for $q \leq 2$. For $X \in \mathrm{PVar}(k)$, we have $H^0(X, \Omega^l_{X,\log}) = 0$.

- Let $(X, Z) \in \mathrm{Sch}^2$ with $X \in \mathrm{Sch}$ a noetherian scheme and $Z \subset X$ a closed subset. We have the deformation $(D_Z X, \mathbb{A}^1_Z) \to \mathbb{A}^1$, $(D_Z X, \mathbb{A}^1_Z) \in \mathrm{Sch}^2$ of (X, Z) by the normal cone $C_{Z/X} \to Z$, i.e. such that
$$(D_Z X, \mathbb{A}^1_Z)_s = (X, Z),\ s \in \mathbb{A}^1 \backslash 0,\ (D_Z X, \mathbb{A}^1_Z)_0 = (C_{Z/X}, Z).$$
We denote by $i_1 : (X, Z) \hookrightarrow (D_Z X, \mathbb{A}^1_Z)$ and $i_0 : (C_{Z/X}, Z) \hookrightarrow (D_Z X, \mathbb{A}^1_Z)$ the closed embeddings in Sch^2.

- Let k a field of characteristic zero. Let $X \in \mathrm{SmVar}(k)$. For $Z \subset X$ a closed subset of pure codimension c, consider a desingularisation $\epsilon : \tilde{Z} \to Z$ of Z and denote $n : \tilde{Z} \xrightarrow{\epsilon} Z \subset X$. We have then the morphism in $\mathrm{DA}(k)$
$$G_{Z,X} : M(X) \xrightarrow{D(\mathbb{Z}(n)_{BM})} M(\tilde{Z})(c)[2c] \xrightarrow{\mathbb{Z}(\epsilon)} M(Z)(c)[2c]$$
where $D : \mathrm{Hom}_{\mathrm{DA}(k)}(M_c(\tilde{Z}), M_c(X)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{DA}(k)}(M(X), M(\tilde{Z})(c)[2c])$ is the duality isomorphism from the six functors formalism (moving lemma of Suzlin and Voevodsky) and $\mathbb{Z}(n) := \mathrm{ad}(n_!, n^!)(a_X^! \mathbb{Z})$, noting that $n_! = n_*$ since n is proper and that $a_X^! = a_X^*[d_X]$ and $a_{\tilde{Z}}^! = a_{\tilde{Z}}^*[d_Z]$ since X, resp. $\tilde{Z}$, are smooth (considering the connected components, we may assume X and $\tilde{Z}$ of pure dimension).

We recall the following facts (see [5] and [3]):

Proposition 1. *Let k a field a characteristic zero. Let $X \in \mathrm{SmVar}(k)$ and $i : Z \subset X$ a smooth subvariety of pure codimension d. Then $C_{Z/X} = N_{Z/X} \to Z$ is a vector bundle of rank d.*

(i) The closed embeddings $i_1 : (X, Z) \hookrightarrow (D_Z X, \mathbb{A}^1_Z)$ and $i_0 : (C_{Z/X}, Z) \hookrightarrow (D_Z X, \mathbb{A}^1_Z)$ in $\mathrm{SmVar}^2(k)$ induces isomorphisms of motives $\mathbb{Z}(i_1) : M_Z(X) \xrightarrow{\sim} M_{\mathbb{A}^1_Z}(D_Z X)$ and $\mathbb{Z}(i_0) : M_Z(N_{Z/X}) \xrightarrow{\sim} M_{\mathbb{A}^1_Z}(D_Z X)$ in $\mathrm{DA}(k)$. We get the excision isomorphism in $\mathrm{DA}(k)$

$$P_{Z,X} := \mathbb{Z}(i_0)^{-1} \circ \mathbb{Z}(i_1) : M_Z(X) \xrightarrow{\sim} M_Z(N_{Z/X}).$$

We have

$$Th(N_{Z/X}) \circ P_{Z,X} \circ \gamma_Z^{\vee}(\mathbb{Z}_X) = G_{Z,X} := D(\mathbb{Z}(i)) : M(X) \to M(Z)(d)[2d].$$

(ii) For each $j \in \mathbb{Z}$, the excision isomorphism induced by (i)

$$H^j\Omega(P_{Z,X}) := H^j E_{et}(\Omega^{\bullet}_{/k})(\mathbb{Z}(i_0)) \circ H^j E_{et}(\Omega^{\bullet}_{/k})(\mathbb{Z}(i_1))^{-1} :$$
$$H^j_{DR,Z}(X) := H^j\Gamma_Z(X, E_{et}(\Omega^{\bullet}_X)) \xrightarrow{\sim} H^j_{DR,Z}(N_{Z/X}) := H^j\Gamma_Z(N_{Z/X}, E_{et}(\Omega^{\bullet}_{N_{Z/X}}))$$

preserve logarithmic De Rham classes, that is for each $l \in \mathbb{Z}$,

$$H^j\Omega(P_{Z,X})(H^j OL_X(H^{j-l}_{et,Z}(X, \Omega^l_{X,\log}))) = H^j OL_{N_{Z/X}}(H^{j-l}_{et,Z}(N_{Z/X}, \Omega^l_{N_{Z/X},\log}))$$

Proof. (i): See [5].
(ii): The de Rham complex $\Omega^{\bullet}_{/k} \in C(\mathrm{SmVar}(k))$ admits transfers using trace map for commutative rings, that is $\Omega^{\bullet} \in C(\mathrm{Cor}\,\mathrm{SmVar}(k))$ (see [7]) and the inclusion $OL : \Omega^l_{/k,\log}[-l] \hookrightarrow \Omega^{\bullet}_{/k}$ in $C(\mathrm{SmVar}(k))$ is compatible with transfers. Since $\Omega^l_{/k,\log} \in \mathrm{PSh}(\mathrm{SmVar}(k))$ is $\mathbb{A}^1_k$ invariant ($O^*_{\mathbb{A}^1_k} = k^*$ and for $\lambda \in k^*$ we have $d\lambda = 0$) and admits transfers, for each $j \in \mathbb{Z}$, the presheaves $H^{j-l}_{et}\Omega^l_{/k,\log} \in \mathrm{PSh}(\mathrm{SmVar}(k))$ are $\mathbb{A}^1_k$ invariant by a theorem of Voevodsky (c.f. [5] for example). It then follows from (i) (see [5] or [3]) that

$$H^j\Omega_{\log}(P_{Z,X}) := H^j E_{et}(\Omega^{\bullet}_{/k,\log})(\mathbb{Z}(i_0)) \circ H^j E_{et}(\Omega^{\bullet}_{/k,\log})(\mathbb{Z}(i_1))^{-1} :$$
$$H^{j-l}_{et,Z}(X, \Omega^l_{X,\log}) \xrightarrow{\sim} H^{j-l}_{et,Z}(N_{Z/X}, \Omega^l_{N_{Z/X},\log}).$$

Hence,

$$\begin{aligned} H^j\Omega(P_{Z,X})(H^j OL_X(H^{j-l}_{et,Z}(X, \Omega^l_{X,\log}))) &= H^j OL_{N_{Z/X}}(\Omega_{\log}(P_{Z,X})(H^{j-l}_{et,Z}(X, \Omega^l_{X,\log}))) \\ &= H^j OL_{N_{Z/X}}(H^{j-l}_{et,Z}(N_{Z,X}, \Omega^l_{N_{Z/X},\log})). \end{aligned}$$

□

Remark 1. *Let k a field a characteristic zero.*

(i) The statement of proposition 1(ii) also holds for the subcomplex $\Omega^{\bullet,\partial=0}_{/k} \hookrightarrow \Omega^{\bullet}_{/k}$ of closed forms by the same argument since the inclusion $\Omega^{l,\partial=0}_{/k}[-l] \hookrightarrow \Omega^{\bullet}_{/k}$ in $C(\mathrm{SmVar}(k))$ is compatible with transfers.

(ii) The statement of proposition 1(ii) does NOT hold for presheaves which do not admits transfers (they are $\mathbb{A}^1$ invariant but not $\mathbb{A}^1$ local in general). Note that the result of proposition 1(ii) does NOT hold for the embedding in $C(\mathrm{SmVar}(k))$, associated to an embedding $\sigma : k \hookrightarrow \mathbb{C}$,

$$\mathcal{H}^{l,j-l}[-l] \hookrightarrow \mathcal{A}^{\bullet},\ X \in \mathrm{SmVar}(k), \Gamma(X, \mathcal{H}^{l,j-l}_{X^{an}_{\mathbb{C}}})[-l] \hookrightarrow \Gamma(X, \mathcal{A}^{\bullet}_{X^{an}_{\mathbb{C}}})$$

of the subsheaf of harmonic differential forms, the sheaves of differential forms $\mathcal{A} \in C(\mathrm{SmVar}(k))$ does NOT admits transfers (finite algebraic correspondences are not smooth and for $f : X' \to X$ a morphism with $X', X \in \mathrm{Diff}(\mathbb{R})$ vector fields on X only lift if f is a smooth morphism i.e. if the differential of df is surjective), transfers maps are only defined on cohomology.

Let k a field of characteristic zero. Let $X \in \mathrm{SmVar}(k)$ and $Z \subset X$ a smooth subvariety of pure codimension d. We have for each $j \in \mathbb{Z}$, the purity isomorphism given by $H^j\Omega(P_{Z,X})$ (see proposition 1) and the cup product with the Euler class of of the normal tangent bundle $N_{Z/X} \to Z$:

$$H^jDR(P_{Z,X}) : H^j_{DR,Z}(X) \xrightarrow{H^j\Omega(P_{Z,X})} H^j_{DR,Z}(N_{Z/X}) \xrightarrow{((-)\cdot e(N_{Z/X}))^{-1}} H^{j-2d}_{DR}(Z).$$

Now, we have the following :

Proposition 2. *Let k a field of characteristic zero. Let $p : E \to X$ a vector bundle of rank $d \in \mathbb{N}$ with $X, E \in \mathrm{SmVar}(k)$ connected. Then*

(i) the Euler class $e(E) \in H^{2d}_{DR,X}(E)$ is logarithmic of type (d,d), that is $e(E) = H^{2d}OL_E(e(E)) \in H^{2d}_{DR,X}(E)$ with $e(E) \in H^d_{X,et}(E, \Omega^d_{E,\log})$,

(ii) the Euler class $e(E) \in H^{2d}_{DR,X}(E)$ induces for each $i, j \in \mathbb{Z}$ an isomorphism

$$((-)\cdot e(E)) : H^{i+j}OL_X(H^j_{et}(X, \Omega^i_{X,\log})) \xrightarrow{\sim} H^{2d+i+j}OL_E(H^{j+d}_{et}(E, \Omega^{i+d}_{E,\log})).$$

Proof. (i):Let $X = \cup_i X_i$ an open affine cover such that $E_i := E_{|X_i}$ is trivial : $(s_1, \cdots, s_d) : E_{|X_i} \xrightarrow{\sim} X_i \times \mathbb{A}^d$ with $s_j \in \Gamma(X_i, E)$. Then

$$e(E)_{|X_i} = \partial(ds_1/s_1 \wedge \cdots \wedge ds_d/s_d) \in H^{2d}OL_{E_i}(H^d_{X_i}(E_i, \Omega^d_{E,\log}))$$

where $\partial : H^{d-1}(E_i \backslash X_i, \Omega^d_{E,\log}) \to H^d_{X_i}(E_i, \Omega^d_{E,\log})$ is the boundary map. Hence

$$e(E) \in H^{2d}OL_E(H^0(E, H^d_X\Omega^d_{E,\log})) = H^d_{et,X}(E, \Omega^d_{E,\log}) \subset H^0(E, H^{2d}_X\Omega^{\bullet \geq d}_E) \subset H^{2d}_{DR,X}(E).$$

(ii):Follows from (i) and Kunneth formula for De Rham cohomology : let $X = \cup_i X_i$ an open affine cover such that $E_i := E_{|X_i}$ is trivial and consider the morphism of bi-complexes of abelian groups

$$((-)\cdot e(E)) : OL_X(\Gamma(X_\bullet, E_{et}(\Omega^i_{X,\log}))) \to OL_E(\Gamma(E_\bullet, E_{et}(\Omega^{i+d}_{E,\log})))[2d].$$

By (i) an Kunneth formula for De Rham cohomology, $((-)\cdot e(E))$ induces an isomorphism on the E_1 level of the spectral sequences for the filtrations associated to bi-complex structures on the total complexes. Hence, $((-)\cdot e(E))$ is a quasi-isomorphism. This proves (ii). □

We deduce from proposition 1 and proposition 2, the following key proposition

Proposition 3. *Let k a field of characteristic zero. Let $X \in \mathrm{SmVar}(k)$ and $Z \subset X$ a smooth subvariety of codimension d. For each $j \in \mathbb{Z}$, the purity isomorphism*

$$H^jDR(P_{Z,X}) : H^j_{DR,Z}(X) \xrightarrow{H^j\Omega(P_{Z,X})} H^j_{DR,Z}(N_{Z/X}) \xrightarrow{((-)\cdot e(N_{Z/X}))^{-1}} H^{j-2d}_{DR}(Z)$$

preserve logarithmic De Rham classes, that is for each $l \in \mathbb{Z}$,

$$H^jDR(P_{Z,X})(H^jOL_X(H^{j-l}_{et,Z}(X, \Omega^l_{X,\log}))) = H^jOL_Z(H^{j-l}_{et}(Z, \Omega^l_{Z,\log})).$$

Proof. Follows from proposition 1(ii) and proposition 2(ii). □

Let k a field of characteristic zero. For $X \in \mathrm{SmVar}(k)$, we have the cycle class map for De Rham cohomology

$$Z \in \mathcal{Z}^d(X) \mapsto [Z] := H^{2d}\Omega(\gamma_Z^\vee)([Z]) \subset H^{2d}_{DR}(X),\ [Z] \in H^{2d}_{DR,|Z|}(X),$$
$$H^{2d}\Omega(\gamma_Z^\vee) : H^{2d}_{DR,|Z|}(X) \to H^{2d}_{DR}(X)$$

where

$$\Omega(\gamma_Z^\vee) := \mathrm{Hom}(\gamma_Z^\vee(\mathbb{Z}_X), E_{et}(\Omega_X^\bullet)) : \Gamma_Z(X, E_{et}(\Omega_X^\bullet)) \hookrightarrow \Gamma(X, E_{et}(\Omega_X^\bullet))$$

and, as for any Weil cohomology theory, we have the isomorphism given by purity : if $X^o \subset X$ is an open subset such that $Z^o := X^o \cap Z$ has smooth components we have the isomorphism

$$H^{2d}DR(P_{Z^o,X^o}) : H^{2d}_{DR,|Z|}(X) = H^{2d}_{DR,|Z^o|}(X^o) = H^{2d}_{DR,|Z^o|}(N_{Z^o/X^o}) \xrightarrow{\sim} H^0_{DR}(|Z^o|).$$

where the first equality follows from dimension reason : for $X \in \mathrm{SmVar}(k)$ and $Z' \subset X$ smooth, we have $H^i_{DR,Z'}(X) = 0$ for $i < 2\,\mathrm{codim}(Z', X)$ by the purity isomorphism $H^iDR(P_{Z',X})$.

The main result of this section is the following :

Theorem 1. *Let k a field of characteristic zero. Let $X \in \mathrm{SmVar}(k)$. Let $d \in \mathbb{N}$.*

(i) The De Rham cohomology class of an algebraic cycle is logarithmic and is of type (d,d), that is, for $Z \in \mathcal{Z}^d(X)$

$$[Z] := H^{2d}\Omega(\gamma_Z^\vee)([Z]) \subset H^{2d}OL_X(H^d_{et}(X, \Omega^d_{X,\log})) \subset H^{2d}_{DR}(X).$$

(ii) Conversely, any $w \in H^{2d}OL_X(H^d_{et}(X, \Omega^d_{X,\log}))$ is the class of an algebraic cycle. Note that it implies $w \in W_0H^{2d}_{DR}(X)$ as it is easily seen for $d = 1$ by the Chow moving lemma since $j^* : \mathrm{Pic}(\bar{X}) \to \mathrm{Pic}(X)$ is surjective if $j : X \hookrightarrow \bar{X}$ is a smooth compactification of X, $\bar{X} \in \mathrm{PSmVar}(k)$ and since $H^1_{et}(Y, \Omega^1_{Y,\log}) = \mathrm{Pic}(Y)$ for $Y \in \mathrm{Var}(k)$.

(iii) We have $H^jOL_X(H^{j-l}_{et}(X, \Omega^l_{X,\log})) = 0$ for $j, l \in \mathbb{Z}$ such that $2l < j$.

(iii)' If $X \in \mathrm{PSmVar}(k)$, we also have $H^jOL_X(H^{j-l}_{et}(X, \Omega^l_{X,\log})) = 0$ for $j, l \in \mathbb{Z}$ such that $2l > j$. That is, if $X \in \mathrm{PSmVar}(k)$, we only have logarithmic classes in bidegree (d,d) for $d \in \mathbb{N}$, in particular there is no non trivial logarithmic classes for odd degree De Rham cohomology $H^{2d+1}_{DR}(X)$.

Proof. (i):We have

$$[Z] := H^{2d}\Omega(\gamma_Z^\vee)([Z]) = H^{2d}OL_X \circ H^d\Omega^d_{\log}(\gamma_Z^\vee)([Z]) \subset H^{2d}_{DR}(X),\ [Z] \in H^d_{et,|Z|}(X, \Omega^d_{X,\log})$$

where

$$\Omega^d_{\log}(\gamma_Z^\vee) := \mathrm{Hom}(\gamma_Z^\vee(\mathbb{Z}_X), E_{et}(\Omega^d_{X,\log})) : \Gamma_Z(X, E_{et}(\Omega^d_{X,\log})) \hookrightarrow \Gamma(X, E_{et}(\Omega^d_{X,\log})),$$

since if $X^o \subset X$ is a Zariski open subset such that $Z^o := X^o \cap Z$ has smooth components and $N_{Z^o/X^o} \to Z^o$ is the normal tangent bundle, we have by proposition 3 the isomorphism

$$\begin{aligned} H^{2d}DR(P_{Z^o,X^o}) : H^{2d}OL_X(H^d_{et,|Z|}(X, \Omega^d_{X,\log})) &= H^{2d}OL_X(H^d_{et,|Z^o|}(X^o, \Omega^d_{X^o,\log})) \\ &= H^{2d}OL_X(H^d_{et,|Z^o|}(N_{Z^o/X^o}, \Omega^d_{N_{Z^o/X^o},\log})). \end{aligned}$$

(ii):By assumption we have

$$w = H^{2d}OL_X(w) \in H^{2d}_{DR}(X),\ w \in H^d_{et}(X, \Omega^d_{X,\log})$$

As $\Omega^d_{X,\log} \in \mathrm{PSh}(X^{et})$ consist of a single presheaf, that is a complex of presheaves concentrated one degree, there exist an etale open cover $r = r(w) = (r_i : X_i \to X)_{1 \le i \le s}$, depending on w, such that $r_i^* w = 0 \in H^d_{et}(X_i, \Omega^d_{X,\log})$ for each i. Choose $i = 1$ and denote $j : U := r_1(X_1) \hookrightarrow X$ the corresponding (Zariski) open subset. As $\Omega^d_{X,\log}$ has no torsion and admits transfers, we have $j^*w = 0 \in H^d_{et}(U, \Omega^d_{X,\log})$. Hence, denoting $D := X \backslash U$, we have

$$w = H^{2d}OL_X(w), w \in H^d\Omega^d_{\log}(\gamma_D^\vee)(H^d_{et,D}(X, \Omega^d_{X,\log})).$$

We may assume, up to shrinking U, that $D \subset X$ is a divisor. Denote $D^o \subset D$ its smooth locus and $l : X^o \hookrightarrow X$ a Zariski open subset such that $X^o \cap D = D^o$. We then have by proposition 3

$$l^*w \in H^{2d}DR(P_{D^o,X^o})(H^{2d-2}OL_{D^o}(H^{d-1}_{et}(D^o, \Omega^{d-1}_{D^o,\log}))).$$

We repeat this procedure with each connected components of D^o instead of X. By a finite induction of d steps, we get

$$w = H^{2d}OL_X(w),\ w \in H^d\Omega^d_{\log}(\gamma_Z^\vee)(H^d_{et,Z}(X, \Omega^d_{X,\log})),$$

with $Z := D_d \subset \cdots \subset D \subset X$ a pure codimension d (Zariski) closed subset and $w = \sum_i n_i[Z_i] \in H^{2d}_{DR}(X)$, where $n_i \in \mathbb{Z}$ and $(Z_i)_{1\leq i\leq t} \subset Z$ are the irreducible components of Z, using in the final step the isomorphism

$$H^1_{et}(D^o_{d-1}, \Omega_{D^o_{d-1},\log}) = H^1_{et}(D^o_{d-1}, O^*_{D_{d-1}}) = Pic(D^o_{d-1}).$$

(iii):Let $j > 2l$. Let $w \in H^jOL_X(H^{j-l}_{et}(X, \Omega^l_{X,\log}))$. By the proof of (ii) there exists $Z \subset X$ a closed subset of pure codimension l such that

$$w = H^jOL_X(w),\ w \in H^{j-l}\Omega^l_{\log}(\gamma_Z^\vee)(H^{j-l}_{et,Z}(X, \Omega^l_{X,\log})),$$

By a finite induction of $d_X - l$ steps, restricting to the smooth locus of closed subsets of Z, $w = 0$ since by proposition 1(i), $H^jOL_X(H^{j-l}_{et,Z'}(X, \Omega^l_{X,\log})) = 0$ for all $Z' \in \mathrm{SmVar}(k)$ such that $\mathrm{codim}(Z', X) \geq l$.

(iii)':Let $j < 2l$. Let $w \in H^jOL_X(H^{j-l}_{et}(X, \Omega^l_{X,\log}))$. By the proof of (ii) there exists $Z \subset X$ a closed subset of pure codimension $j - l$ such that

$$w = (H^jOL_X \circ H^{j-l}\Omega^l_{\log}(\gamma_Z^\vee))(w),\ w \in H^{j-l}_{et,Z}(X, \Omega^l_{X,\log}),$$

For $Z' \subset X$ a closed subset of pure codimension c, consider a desingularisation $\epsilon : \tilde{Z}' \to Z'$ of Z' and denote $n : \tilde{Z}' \xrightarrow{\epsilon} Z' \subset X$. The morphism in $\mathrm{DA}(k)$

$$G_{Z',X} : M(X) \xrightarrow{D(\mathbb{Z}(n))} M(\tilde{Z}')(c)[2c] \xrightarrow{\mathbb{Z}(\epsilon)} M(Z')(c)[2c]$$

where $D : \mathrm{Hom}_{\mathrm{DA}(k)}(M_c(\tilde{Z}'), M_c(X)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{DA}(k)}(M(X), M(\tilde{Z}')(c)[2c])$ is the duality isomorphism from the six functor formalism (moving lemma of Suzlin and Voevodsky) and $\mathbb{Z}(n) := \mathrm{ad}(n_!, n^!)(a_X^!\mathbb{Z})$, is given by a morphism in $C(\mathrm{SmVar}(k))$

$$\hat{G}_{Z',X} : \mathbb{Z}_{tr}(X) \to E_{et}(C_*\mathbb{Z}_{tr}(Z'))(c)[2c].$$

Let $l : X^o \hookrightarrow X$ be an open embedding such that $Z^o := Z \cap X^o$ is the smooth locus of Z. We then have the following commutative diagram of abelian groups

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{j-l}_{et,Z}(X, \Omega^l_{X,\log}) & \xrightarrow{l^*} & H^{j-l}_{et,Z^o}(X^o, \Omega^l_{X,\log}) & \xrightarrow{\partial} & H^{j-l+1}_{et,Z\setminus Z^o}(X, \Omega^l_{X,\log}) & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle \Omega(\hat{G}_{Z,X})} & & \uparrow{\scriptstyle DR(P_{Z^o,X^o})} & & \uparrow{\scriptstyle \Omega(\hat{G}_{Z,X})} & & \\
0 & \longrightarrow & H^0_{et}(Z, \Omega^{2l-j}_{Z,\log}) & \xrightarrow{l*} & H^0_{et}(Z^o, \Omega^{2l-j}_{Z^o,\log}) & \xrightarrow{\partial} & H^1_{et,Z\setminus Z^o}(Z, \Omega^{2l-j}_{Z,\log}) & \longrightarrow & \cdots
\end{array}$$

whose rows are exact sequences. Consider

$$l^*w = DR(P_{Z^o,X^o})(w^o) \in H^{j-l}_{et,Z^o}(X^o, \Omega^l_{X,\log}),\ w^o \in H^0_{et}(Z^o, \Omega^{2l-j}_{Z^o,\log}).$$

Since $\partial l^*w = 0 \in H^{j-l+1}_{et,Z\setminus Z^o}(X, \Omega^l_{X,\log})$, we get using proposition 3 applied to $((Z\setminus Z^o)^o, X^{oo})$, where $(Z\setminus Z^o)^o \subset Z\setminus Z^o$ is the smooth locus and $X^{oo} \subset X$ is an open subset such that $X \cap (Z\setminus Z^o)^o = X^{oo} \cap (Z\setminus Z^o)$,

$$\partial w^o = 0 \in H^1_{et,Z\setminus Z^o}(Z, \Omega^{2l-j}_{Z,\log}),$$

since $H^1_{et,(Z\setminus Z^o)\setminus(Z\setminus Z^o)^o}(Z, \Omega^{2l-j}_{Z,\log}) = 0$ for dimension reasons, that is

$$w = \Omega(\hat{G}_{Z,X})(w), \text{ with } w \in H^0_{et}(Z, \Omega^{2l-j}_{Z,\log}).$$

Hence $w = 0$ since $H^0(Z', \Omega^k_{Z',\log}) = 0$ for all $k > 0$ and all $Z' \in \mathrm{PVar}(k)$. □

3.2 Complex integral periods

Let k a field of characteristic zero.

Let $X \in \mathrm{SmVar}(k)$ a smooth variety. Let $X = \cup_{i=1}^{s} X_i$ an open affine cover. We have for $\sigma : k \hookrightarrow \mathbb{C}$ an embedding, the evaluation period embedding map which is the morphism of bi-complexes

$$ev(X)^{\bullet}_{\bullet} : \Gamma(X_{\bullet}, \Omega^{\bullet}_{X_{\bullet}}) \to \mathbb{Z}\,\mathrm{Hom}_{\mathrm{Diff}}(\mathbb{I}^{\bullet}, X^{an}_{\mathbb{C},\bullet})^{\vee} \otimes \mathbb{C},$$

$$w^l_I \in \Gamma(X_I, \Omega^l_{X_I}) \mapsto (ev(X)^l_I(w^l_I) : \phi^l_I \in \mathbb{Z}\,\mathrm{Hom}_{\mathrm{Diff}}(\mathbb{I}^l, X^{an}_{\mathbb{C},I})^{\vee} \otimes \mathbb{C} \mapsto ev^l_I(w^l_I)(\phi^l_I) := \int_{\mathbb{I}^l} \phi^{l*}_I w^l_I)$$

given by integration. By taking all the affine open cover $(j_i : X_i \hookrightarrow X)$ of X, we get for $\sigma : k \hookrightarrow \mathbb{C}$, the evaluation period embedding map

$$ev(X) := \varinjlim_{(j_i : X_i \hookrightarrow X)} ev(X)^{\bullet}_{\bullet} : \varinjlim_{(j_i : X_i \hookrightarrow X)} \Gamma(X_{\bullet}, \Omega^{\bullet}_{X_{\bullet}}) \to \varinjlim_{(j_i : X_i \hookrightarrow X)} \mathbb{Z}\,\mathrm{Hom}_{\mathrm{Diff}(\mathbb{R})}(\mathbb{I}^{\bullet}, X^{an}_{\mathbb{C},\bullet})^{\vee} \otimes \mathbb{C}$$

It induces in cohomology, for $j \in \mathbb{Z}$, the evaluation period map

$$H^j ev(X) = H^j ev(X)^{\bullet}_{\bullet} : H^j_{DR}(X) = H^j \Gamma(X_{\bullet}, \Omega^{\bullet}_{X_{\bullet}}) \to H^j_{\mathrm{sing}}(X^{an}_{\mathbb{C}}, \mathbb{C}) = H^j(\mathrm{Hom}_{\mathrm{Diff}(\mathbb{R})}(\mathbb{I}^{\bullet}, X^{an}_{\mathbb{C},\bullet})^{\vee} \otimes \mathbb{C}).$$

which does NOT depend on the choice of the affine open cover by acyclicity of quasi-coherent sheaves on affine noetherian schemes for the left hand side and from Mayer-Vietoris quasi-isomorphism for singular cohomology of topological spaces and Whitney approximation theorem for differential manifolds for the right hand side.

Proposition 4. *Let k a field of characteristic zero. Let $X \in \mathrm{SmVar}(k)$. Let $\sigma : k \hookrightarrow \mathbb{C}$ an embedding.*

(i) Let $w \in H^j_{DR}(X) := \mathbb{H}^j(X, \Omega^{\bullet}_X) = \mathbb{H}^j_{pet}(X, \Omega^{\bullet}_X)$. If

$$w \in H^j OL_X(\mathbb{H}^j_{pet}(X_{\mathbb{C}}, \Omega^{\bullet}_{X^{et}_{\mathbb{C}},\log}))$$

then $H^j ev(X)(w) \in H^j_{\mathrm{sing}}(X^{an}_{\mathbb{C}}, 2i\pi\mathbb{Q})$.

(ii) Let $p \in \mathbb{N}$ a prime number and $\sigma_p : k \hookrightarrow \mathbb{C}_p$ an embedding. Let $j \in \mathbb{Z}$. Let $w \in H^j_{DR}(X) := \mathbb{H}^j(X, \Omega^{\bullet}_X) = \mathbb{H}^j_{pet}(X, \Omega^{\bullet}_X)$. If

$$w := \pi_{k/\mathbb{C}_p}(X)^* w \in H^j OL_X(\mathbb{H}^j_{et}(X_{\mathbb{C}_p}, \Omega^{\bullet}_{X^{et}_{\mathbb{C}_p},\log,\mathcal{O}}))$$

then $H^j ev(X)(w) \in H^j_{\mathrm{sing}}(X^{an}_{\mathbb{C}}, 2i\pi\mathbb{Q})$. Recall that $\mathbb{H}^j_{pet}(X_{\mathbb{C}}, \Omega^{\bullet}_{X^{pet}_{\mathbb{C}_p},\log,\mathcal{O}}) = \mathbb{H}^j_{et}(X_{\mathbb{C}}, \Omega^{\bullet}_{X^{et}_{\mathbb{C}_p},\log,\mathcal{O}})$.

Proof. (i): Let

$$w \in H^j_{DR}(X) := \mathbb{H}^j(X, \Omega^{\bullet}_X) = H^j \Gamma(X_{\bullet}, \Omega^{\bullet}_X).$$

where $(r_i : X_i \to X)_{1 \leq i \leq s}$ is an affine etale cover. Let $X^{an}_{\mathbb{C}} = \cup_{i=1}^{r} \mathbb{D}_i$ an open cover with $\mathbb{D}_i \simeq D(0,1)^d$. Denote $j_{IJ} : X_I \cap \mathbb{D}_J \hookrightarrow X_I$ the open embeddings. Then by definition $H^j ev(X)(w) = H^j ev(X^{an}_{\mathbb{C}})(j^*_{\bullet} \circ \mathrm{an}^*_{X_{\bullet}} w)$ with

$$j^*_{\bullet} \circ \mathrm{an}^*_{X_{\bullet}} w \in H^j \Gamma(X^{an}_{\bullet,\mathbb{C}} \cap \mathbb{D}_{\bullet}, \Omega^{\bullet}_{X^{an}_{\mathbb{C}}}).$$

Now, if $w = H^j OL_X(\mathbb{H}^j_{et}(X_{\mathbb{C}}, \Omega^{\bullet}_{X^{et}_{\mathbb{C}},\log}))$, we have a canonical splitting

$$w = \sum_{l=0}^{j} w^{l,j-l}_L = \sum_{l=d}^{j} w^{l,j-l} \in H^j_{DR}(X_{\mathbb{C}}),\ w^{l,j-l}_L \in H^{j-l}(X_{\mathbb{C}}, \Omega^l_{X^{et}_{\mathbb{C}},\log}),\ w^{l,j-l} := H^j OL_{X^{et}_{\mathbb{C}}}(w^{l,j-l}_L).$$

Let $0 \leq l \leq j$. Using an affine w-contractile pro-etale cover of X, we see that there exists an affine etale cover $r = r(w^{l,j-l}) = (r_i : X_i \to X)_{1 \leq i \leq n}$ of X (depending on $w^{l,j-l}$) such that

$$w^{l,j-l} = [(w^{l,j-l}_{L,I})_I] \in H^j OL_{X^{et}_{\mathbb{C}}}(H^{j-l}\Gamma(X_{\mathbb{C},\bullet}, \Omega^l_{X_{\mathbb{C}},\log})) \subset \mathbb{H}^j\Gamma(X_{\mathbb{C},\bullet}, \Omega^\bullet_{X_{\mathbb{C}}}).$$

Note that since X is an algebraic variety, this also follows from a comparison theorem between Chech cohomology of etale covers and etale cohomology. By [4] lemma 2, we may assume, up to take a desingularization $\pi : X' \to X$ of $(X, \cup_i(r_i(X \backslash X_i)))$ and replace w with $\pi^* w$, that $r_i(X_i) = r_i(X_i(w)) = X \backslash D_i$ with $D_i \subset X$ smooth divisors with normal crossing For $1 \leq l \leq j$, we get

$$w^{l,j-l}_{L,I} = \sum_\nu df_{\nu_1}/f_{\nu_1} \wedge \cdots \wedge df_{\nu_l}/f_{\nu_l} \in \Gamma(X_{\mathbb{C},I}, \Omega^l_{X_{\mathbb{C}}}).$$

For $l = 0$, we get

$$w^{0,j} = [(\lambda_I)] \in H^j\Gamma(X_{\mathbb{C},I}, O_{X_{\mathbb{C}_p,I}}),\ \lambda_I \in \Gamma(X_{\mathbb{C},I}, \mathbb{Z}_{X^{et}_{\mathbb{C},I}})$$

There exists $k' \subset \mathbb{C}$ containing k such that $w^{l,j-l}_{L,I} \in \Gamma(X_{k',I}, \Omega^l_{X_{k'}})$ for all $0 \leq l \leq j$. Taking an embedding $\sigma' : k' \hookrightarrow \mathbb{C}$ such that $\sigma'_{|k} = \sigma$, we then have

$$j^*_\bullet \circ \mathrm{an}^*_{X_\bullet} w = j^*_\bullet((m_l \cdot w^{l,j-l}_L)_{0 \leq l \leq j}) = (w^{l,j-l}_{L,I,J})_{l,I,J} \in H^j\Gamma(X^{an}_{\bullet,\mathbb{C}} \cap \mathbb{D}_\bullet, \Omega^\bullet_{X^{an}_{\mathbb{C}}}).$$

where for each (I, J, l) with $cardI + cardJ + l = j$,

$$w^{l,j-l}_{L,I,J} := j^*_{IJ} w^{l,j-l}_{L,I} \in \Gamma(X^{an}_{I,\mathbb{C}} \cap \mathbb{D}_J, \Omega^l_{X^{an}_{\mathbb{C}}}).$$

We have by a standard computation, for each (I, J, l) with $cardI + cardJ + l = j$,

$$H^*_{\mathrm{sing}}(X^{an}_{I,\mathbb{C}} \cap \mathbb{D}_J, \mathbb{Z}) = < \gamma_1, \cdots, \gamma_{cardI} >,$$

where for $1 \leq i \leq cardI$, $\gamma_i \in \mathrm{Hom}(\Delta^*, X^{an}_{I,\mathbb{C}} \cap \mathbb{D}_J)$ are products of loops around the origin inside the pointed disc $\mathbb{D}^1 \backslash 0$. On the other hand,

- $w^{l,j-l}_{L,I,J} = j^*_J(\sum_\nu df_{\nu_1}/f_{\nu_1} \wedge \cdots \wedge df_{\nu_l}/f_{\nu_l}) \in \Gamma(X^{an}_{I,\mathbb{C}} \cap \mathbb{D}_J, \Omega^l_{X^{an}_{\mathbb{C}}})$ for $1 \leq l \leq j$,
- $w^{0,j}_{L,I,J} = \lambda_I$ is a constant.

Hence, for $\mu \in P([1, \cdots, s])$ with $card\mu = l$, we get, for $l = 0$ $H^l ev(X^{an}_{\mathbb{C}})_{I,J}(w^{0,j}_{L,I,J}) = 0$ and, for $1 \leq l \leq j$,

$$H^l ev(X^{an}_{\mathbb{C}})_{I,J}(w^{l,j-l}_{L,I,J})(\gamma_\mu) = \sum_k \delta_{\nu,\mu} 2i\pi \in 2i\pi\mathbb{Z}.$$

where $\gamma_\mu := \gamma_{\mu_1} \cdots \gamma_{\mu_l}$. We conclude by [4] lemma 1.
(ii):It is a particular case of (i). See [4] proposition 1. □

Let k a field of characteristic zero. Let $X \in \mathrm{SmVar}(k)$. Let $X = \cup_{i=1}^s X_i$ an open affine cover with $X_i := X \backslash D_i$ with $D_i \subset X$ smooth divisors with normal crossing. Let $\sigma : k \hookrightarrow \mathbb{C}$ an embedding. By proposition 4, we have a commutative diagram of graded algebras

$$\begin{array}{ccc} H^*_{DR}(X) & \xrightarrow{H^* ev(X)} & H^*_{\mathrm{sing}}(X^{an}_{\mathbb{C}}, \mathbb{C}) \\ \subset \uparrow & & \uparrow H^* C^* \iota_{2i\pi\mathbb{Q}/\mathbb{C}}(X^{an}_{\mathbb{C}}) \\ H^* OL_X(\mathbb{H}^*_{et}(X, \Omega^\bullet_{X^{et},\log})) \cap H^*_{DR}(X) & \xrightarrow{H^* ev(X)} & H^*_{\mathrm{sing}}(X^{an}_{\mathbb{C}}, 2i\pi\mathbb{Q}) \end{array}$$

where

$$C^*\iota_{2i\pi\mathbb{Q}/\mathbb{C}}(X^{an}_{\mathbb{C}}) : C^\bullet_{\text{sing}}(X^{an}_{\mathbb{C}}, 2i\pi\mathbb{Q}) \hookrightarrow C^\bullet_{\text{sing}}(X^{an}_{\mathbb{C}}, \mathbb{C})$$

is the subcomplex consiting of $\alpha \in C^j_{\text{sing}}(X^{an}_{\mathbb{C}}, \mathbb{C})$ such that $\alpha(\gamma) \in 2i\pi\mathbb{Q}$ for all $\gamma \in C^{\text{sing}}_j(X^{an}_{\mathbb{C}}, \mathbb{Q})$. Recall that

$$H^*ev(X_{\mathbb{C}}) = H^*R\Gamma(X^{an}_{\mathbb{C}}, \alpha(X)) \circ \Gamma(X^{an}_{\mathbb{C}}, E_{zar}(\Omega(\text{an}_X))) : H^*_{DR}(X_{\mathbb{C}}) \xrightarrow{\sim} H^*_{\text{sing}}(X^{an}_{\mathbb{C}}, \mathbb{C})$$

is the canonical isomorphism induced by the analytical functor and $\alpha(X) : \mathbb{C}_{X^{an}_{\mathbb{C}}} \hookrightarrow \Omega^\bullet_{X^{an}_{\mathbb{C}}}$, which gives the periods elements $H^*ev(X)(H^*_{DR}(X)) \subset H^*_{\text{sing}}(X^{an}_{\mathbb{C}}, \mathbb{C})$. On the other side the induced map

$$H^*ev(X_{\mathbb{C}}) : H^*OL_X(\mathbb{H}^*_{et}(X_{\mathbb{C}}, \Omega^\bullet_{X^{et}_{\mathbb{C}},\log})) \hookrightarrow H^*\iota_{2i\pi\mathbb{Q}/\mathbb{C}}H^*_{\text{sing}}(X^{an}_{\mathbb{C}}, 2i\pi\mathbb{Q})$$

is NOT surjective in general since the left hand side is invariant by the action of the group $Aut(\mathbb{C})$ (the group of field automorphism of $\mathbb{C}$) whereas the right hand side is not. The fact for a de Rham cohomology class of being logarithmic is algebraic and invariant under isomorphism of (abstract) schemes.

Corollary 1. *Let $X \in \text{PSmVar}(\mathbb{C})$. Then the Hodge conjecture holds for X if and only if the Hodge classes are given by logarithmic De Rham classes.*

Proof. Follows from theorem 1. □

3.3 Rigid GAGA for logarithmic de Rham classes

Let K a field of characteristic zero which is complete for a p-adic norm. We consider $\text{Sch}^{int}/O_K := O(\text{PSch}^2/O_K) \subset \text{Sch}^{ft}/O_K$ the full subcategory consisting of integrable models of algebraic varieties over K, where $O : \text{PSch}^2/O_K \to \text{Sch}^{ft}/O_K,\ O(X, Z) = X\backslash Z$ is the canonical functor. Denote by $\text{Var}(K)^{pet}$ and $(\text{Sch}^{int}/O_K)^{pet}$ the big pro-etale sites. We then have the morphsim of sites $r : \text{Var}(K)^{pet} \to (\text{Sch}^{int}/O_K)^{pet}$ We will consider

$$\Omega^{\bullet,an}_{/K,\log,\mathcal{O}} \in C((\text{Sch}^{int}/O_K)^{pet}),\ X^{\mathcal{O}} \mapsto \Omega^\bullet_{\hat{X}^{(p)},\log,\mathcal{O}}(\hat{X}^{\mathcal{O},(p)}),\ (f : X' \to X) \mapsto \Omega(f) := f^*$$

and the embedding of $C(\text{Var}(K)^{pet})$ (see definition 1(iii))

$$OL := OL_{/K,an} : r^*\Omega^{\bullet,an}_{/K,\log,\mathcal{O}} \hookrightarrow \Omega^{\bullet,an}_{/K},\ \text{for } X \in \text{Var}(K)^{pet},$$

$$OL_{/K,an}(X) := OL_{\hat{X}^{(p)}}(X) : \varinjlim_{X^{\mathcal{O}} \in (\text{Sch}^{int}/O_K)^{pet}, X^{\mathcal{O}} \times_{O_K} K = X} \Omega^\bullet_{\hat{X}^{(p)},\log,\mathcal{O}}(\hat{X}^{(p)}) \to \Omega^\bullet_{\hat{X}^{(p)}}(\hat{X}^{(p)}).$$

and their restrictions to $\text{SmVar}(K)^{pet} \subset \text{Var}(K)^{pet}$. Denote FSch/O_K is the category of formal schemes over O_K. The morphism of site in RCat given by completion with respect to (p)

$$c : \text{FSch}/O_K \to \text{Var}(K),\ X \mapsto \hat{X} := \hat{X}^{\mathcal{O},(p)} \otimes_{O_K} K,\ X^{\mathcal{O}} \in \text{Sch}^{int}/O_K,\ \text{s.t}\, X^{\mathcal{O}} \otimes_{O_K} K = X$$

induces the map in $C(\text{Var}(K)^{pet})$

$$c^* : \Omega^\bullet_{/K} \to \Omega^{\bullet,an}_{/K} := c_*\Omega^{\bullet,an}_{/K},\ \text{for } X \in \text{Var}(K),\ c^* : \Omega^\bullet_X(X) \to \Omega^\bullet_{\hat{X}^{(p)}}(\hat{X}^{(p)}),$$

which induces for each $j \in \mathbb{Z}$ and $X \in \text{Var}(K)$ the morphism

$$c^* : H^j_{DR}(X) = \mathbb{H}^j_{pet}(X, \Omega^\bullet_X) \to \mathbb{H}^j_{pet}(X, \Omega^\bullet_{\hat{X}^{(p)}}) = \mathbb{H}^j_{pet}(\hat{X}^{(p)}, \Omega^\bullet_{\hat{X}^{(p)}})$$

which is an isomorphism for $X \in \text{SmVar}(K)$ or $X \in \text{PVar}(K)$ by GAGA (c.f. EGA 3) and the properties of w-contractile rings (for the pro-etale toplogy $c_* = Rc_*$). On the on the other hand, it is well-known (see [8]) that for $X \in \text{SmVar}(K)$

$$\mathbb{H}^j_{pet}(\hat{X}^{(p)}, \Omega^\bullet_{\hat{X}^{(p)}}) = \mathbb{H}^j_{et}(\hat{X}^{(p)}, \Omega^\bullet_{\hat{X}^{(p)}}),\ \text{that is, } \mathbb{H}^j_{pet}(X, \Omega^\bullet_{\hat{X}^{(p)}}) = \mathbb{H}^j_{et}(X, \Omega^\bullet_{\hat{X}^{(p)}}).$$

We have also the sheaves

$$\hat{O}, \hat{O}^* \in \mathrm{PSh}(\mathrm{Var}(K)),\ X \in \mathrm{Var}(K) \mapsto \hat{O}(X) := \varprojlim_{n\in\mathbb{N}} O(X^{\mathcal{O}}_{/p^n}),\ \hat{O}^*(X) := \varprojlim_{n\in\mathbb{N}} O(X^{\mathcal{O}}_{/p^n})^*,$$
$$(g : Y \to X) \mapsto (a_g(X^{\mathcal{O}}_{/p^n}))_{n\in\mathbb{N}} : \hat{O}(X) \to \hat{O}(Y),\ \hat{O}(X)^* \to \hat{O}(Y)^*$$

The sheaf $\hat{O}^* \in \mathrm{PSh}(\mathrm{SmVar}(k))$ admits transfers : for $W \subset X' \times X$ with $X, X' \in \mathrm{SmVar}(K)$ and W finite over X' and $f \in O(X^{\mathcal{O}}_{/p^n})^*$, $W^*f := N_{W/X'}(p_X^* f)$ where $p_X : W \hookrightarrow X' \times X \to X$ is the projection and $N_{W/X'} : k(W)^* \to k(X')^*$ is the norm map. This gives transfers on $\Omega^{1,an}_{/K,\log,\mathcal{O}} \in \mathrm{PSh}(\mathrm{SmVar}(K))$ compatible with transfers on $\Omega^{1,an}_{/K} \in \mathrm{PSh}(\mathrm{SmVar}(K))$: for $W \subset X' \times X$ with $X, X' \in \mathrm{SmVar}(k)$ and W finite over X' and $f \in O(X^{\mathcal{O}}_{/p^n})^*$,

$$W^* df/f := dW^* f/W^* f = Tr_{W/X'}(p_X^*(df/f)),$$

where where $p_X : W \hookrightarrow X' \times X \to X$ is the projection and $Tr_{W/X'} : O_W \to O_X$ is the trace map. Note that $d(fg)/fg = df/f + dg/g$. We get transfers on

$$\otimes^l_{\mathbb{Q}_p} \Omega^{1,an}_{/K,\log,\mathcal{O}},\ \otimes^l_{\hat{O}} \Omega^{1,an}_{/K} \in \mathrm{PSh}(\mathrm{SmVar}(K))$$

since $\otimes^l_{\mathbb{Q}_p} \Omega^{1,an}_{/K,\log,\mathcal{O}} = H^0(\otimes^{L,l}_{\mathbb{Q}_p} \Omega^{1,an}_{/K,\log,\mathcal{O}})$ and $\otimes^l_{\hat{O}} \Omega^{1,an}_{/k} = H^0(\otimes^{L,l}_{\hat{O}} \Omega^{1,an}_{/K})$. This induces transfers on

$$\wedge^l_{\mathbb{Q}_p} \Omega^{1,an}_{/K,\log,\mathcal{O}} := \mathrm{coker}(\oplus_{I_2\subset[1,\dots,l]} \otimes^{l-1}_{\mathbb{Q}_p} \Omega^{1,an}_{/K,\log,\mathcal{O}} \xrightarrow{\oplus_{I_2\subset[1,\dots,l]}\Delta_{I_2} := (w\otimes w' \mapsto w\otimes w\otimes w')} \otimes^l_{\mathbb{Q}_p} \Omega^{1,an}_{/K,\log,\mathcal{O}})$$
$$\in \mathrm{PSh}(\mathrm{SmVar}(K)).$$

and

$$\wedge^l_{\hat{O}} \Omega^{1,an}_{/K} := \mathrm{coker}(\oplus_{I_2\subset[1,\dots,l]} \otimes^{l-1}_{\hat{O}} \Omega^{1,an}_{/K} \xrightarrow{\oplus_{I_2\subset[1,\dots,l]}\Delta_{I_2} := (w\otimes w' \mapsto w\otimes w\otimes w')}) \otimes^l_{\hat{O}} \Omega^{1,an}_{/K} \in \mathrm{PSh}(\mathrm{SmVar}(K)).$$

The result of this section is the following ([4] proposition 2 (i)), we recall and give some precisions to the proof :

Proposition 5. *Let K a field of characteristic zero which is complete for a p-adic norm. Let $X \in \mathrm{PSmVar}(K)$. Let $X^{\mathcal{O}} \in \mathrm{PSch}/O_K$ such that $X^{\mathcal{O}} \otimes_{O_K} K = X$. We have then the morphism $c : \hat{X}^{(p)} \to X$ in* RTop *given by completion with respect to (p). For each $j \in \mathbb{Z}$, the isomorphism $c^* : \mathbb{H}^j_{pet}(X, \Omega^\bullet_X) \xrightarrow{\sim} \mathbb{H}^j_{pet}(X, \Omega^\bullet_{\hat{X}^{(p)}})$ and its inverse preserve logarithmic classes, that is for each $j, l \in \mathbb{Z}$,*

$$c^*(H^j(m \circ (OL_X \otimes I))(H^{j-l}_{pet}(X, \Omega^l_{X^{pet},\log,\mathcal{O}} \otimes \mathbb{Z}_p))) = H^j OL_{\hat{X}^{(p)}}(H^{j-l}_{pet}(X, \Omega^l_{\hat{X}^{(p)},\log,\mathcal{O}}))$$

is an isomorphism.

Proof. Consider, for $j, l \in \mathbb{Z}$, the presheaf

$$L^{l,j-l} := H^j OL_{/K,an}(H^{j-l} E^\bullet_{pet} \Omega^{l,an}_{/K,\log,\mathcal{O}}) \in \mathrm{PSh}(\mathrm{SmVar}(K)).$$

By definition, for $X \in \mathrm{SmVar}(K)$,

$$L^{l,j-l}(X) := H^j OL_{/K,an}(H^{j-l} \mathrm{Hom}(\mathbb{Z}(X), E^\bullet_{pet}(r^* \Omega^{l,an}_{/K,\log,\mathcal{O}}))) = H^j OL_{\hat{X}^{(p)}}(H^{j-l}_{pet}(X, \Omega^l_{\hat{X}^{(p)},\log,\mathcal{O}})).$$

Consider also for $X \in \mathrm{SmVar}(K)$ and $Z \subset X$ a closed subset,

$$L^{l,j-l}_Z(X) := H^j OL_{/K,an}(H^{j-l} \mathrm{Hom}(\mathbb{Z}(X, X\backslash Z), E^\bullet_{pet}(r^* \Omega^{l,an}_{/K,\log,\mathcal{O}}))) = H^j OL_{\hat{X}^{(p)}}(H^{j-l}_{pet,Z}(X, \Omega^l_{\hat{X}^{(p)},\log,\mathcal{O}})).$$

For the second equalities of $L^{l,j-l}(X)$ and $L^{l,j-l}_Z(X)$, note that

$$a_{et}OL_{/K,an}(r^*\Omega^{l,an}_{/K,\log,\mathcal{O}})_{|X^{pet}} = a_{et}OL_{\hat{X}^{(p)}}(r^*\Omega_{\hat{X}^{(p)},\log,\mathcal{O}}),$$

where $X^{pet} \subset \mathrm{Var}(K)^{pet}$ is the small pro etale site and $a_{et} : \mathrm{PSh}(X^{pet}) \to \mathrm{Shv}_{et}(X^{pet})$ is the sheaftification functor. For $l, n \in \mathbb{N}$, consider the presheaves in $(\mathrm{Sch}^{int}/O_K)^{pet}$

$$OL_{/O_K,p^n} : \Omega^l_{/(O_K/p^n),\log} \hookrightarrow \Omega^l_{/(O_K/p^n)}, \text{ for } X^{\mathcal{O}} \in (\mathrm{Sch}^{int}/O_K)^{pet},$$
$$\Omega^l_{/(O_K/p^n),\log}(X^{\mathcal{O}}) := \Omega_{X^{\mathcal{O}}/p^n,\log}(X^{\mathcal{O}}) \hookrightarrow \Omega^l_{/(O_K/p^n)}(X^{\mathcal{O}}) := \Omega_{X^{\mathcal{O}}/p^n}(X^{\mathcal{O}})$$

which induces the presheaves in $\mathrm{SmVar}(K)^{pet}$

$$OL_{/(O_K/p^n)} := r^*OL_{/O_K,p^n} : r^*\Omega^l_{/(O_K/p^n),\log} \hookrightarrow r^*\Omega^l_{/(O_K/p^n)}.$$

Note that by definition $\Omega^l_{/(O_K/p^n),\log} = \nu^*\nu_*\Omega^l_{/(O_K/p^n),\log}$ and $\Omega^l_{/(O_K/p^n)} = \nu^*\nu_*\Omega^l_{/(O_K/p^n)}$, where $\nu : (\mathrm{Sch}^{int}/O_K)^{pet} \to \mathrm{Sch}^{int}/O_K$. We have then for $X \in \mathrm{SmVar}(K)$

$$\begin{aligned} L^{l,j-l}(X) &= H^jOL_{\hat{X}^{(p)}}(\varprojlim_{n\in\mathbb{N}} H^{j-l}_{et}(X, \Omega^l_{X^{\mathcal{O}}/p^n,\log,\mathcal{O}})) \\ &:= H^jOL_{/K,an}(\varprojlim_{n\in\mathbb{N}} H^{j-l}\,\mathrm{Hom}(\mathbb{Z}(X, X\backslash Z), E^\bullet_{et}(r^*\Omega^l_{/(O_K/p^n),\log}))). \end{aligned}$$

We have for $X \in \mathrm{SmVar}(K)$ and $Z \subset X$ a closed subset,

$$\begin{aligned} L^{l,j-l}_Z(X) &= H^jOL_{\hat{X}^{(p)}}(\varprojlim_{n\in\mathbb{N}} H^{j-l}_{et,Z}(X, \Omega^l_{X^{\mathcal{O}}/p^n,\log,\mathcal{O}})) \\ &:= H^jOL_{/K,an}(\varprojlim_{n\in\mathbb{N}} H^{j-l}\,\mathrm{Hom}(\mathbb{Z}(X, X\backslash Z), E^\bullet_{et}(r^*\Omega^l_{/(O_K/p^n),\log}))). \end{aligned}$$

The presheaves $\Omega^l_{/(O_K/p^n),\log} \in \mathrm{PSh}(\mathrm{SmVar}(K))$, $l, n \in \mathbb{N}$, are $\mathbb{A}^1$ invariant and admit transfers. Hence by a theorem of Voevodsky (c.f. [5] for example), $\Omega^l_{/(O_K/p^n),\log} \in \mathrm{PSh}(\mathrm{SmVar}(K))$, $l, n \in \mathbb{N}$, are $\mathbb{A}^1$ local since they are $\mathbb{A}^1$ invariant and admit transfers. This gives in particular, for $Z \subset X$ a smooth subvariety of (pure) codimension d, by proposition 1(i) an isomorphism

$$(\Omega^l_{/(O_K/p^n),\log}(P_{Z,X}))_{n\in\mathbb{N}} : L^{l,j-l}_Z(X) \xrightarrow{\sim} L^{l,j-l}_Z(N_{Z/X}) \xrightarrow{\sim} L^{l-d,j-l-d}(Z).$$

Let

$$\alpha = OL_{\hat{X}^{(p)}}(\alpha) \in L^{l,j-l}(X) := H^jOL_{\hat{X}^{(p)}}(H^{j-l}_{pet}(X, \Omega^l_{\hat{X}^{(p)},\log,\mathcal{O}})) = H^{j-l}\,\mathrm{Hom}(\mathbb{Z}(X), E^\bullet_{pet}(r^*\Omega^{l,an}_{/K,\log,\mathcal{O}})).$$

Let $r = (r_i) : Y := \varprojlim_{i\in I} U_i \to U \hookrightarrow X$ a pro-etale map where U is an affine open subset such that there exists an etale map $e : U \to T^{d_X} \subset \mathbb{A}^{d_X}$ and Y is w-contractile such that $\bar{r}_i(Y^{(0)}) = U$, where $Y^{(0)} \subset Y$ are the closed points. As $\Omega^{l,an}_{/K,\log,\mathcal{O}}$ consists of single presheaf, we have

$$r^*\alpha = 0 \in H^j_{DR}(\hat{Y}^{(p)}) = H^j\Omega^\bullet_{\hat{X}^{(p)}}(Y) = \mathbb{H}^j_{pet}(Y, \Omega^\bullet_{\hat{X}^{(p)}}),$$

that is

$$r^*\alpha = 0 = [\partial(\eta_n)_{n\in\mathbb{N}}] = [(\partial\eta_n)_{n\in\mathbb{N}}] \in H^j_{DR}(\hat{Y}^{(p)}), \text{ with } (\eta_n)_{n\in\mathbb{N}} \in \Omega^{j+1}_{\hat{X}^{(p)}}(Y).$$

Denote $j : U \hookrightarrow X$ the open embedding. Consider

$$j^*\alpha = [(w_n)_{n\in\mathbb{N}}] \in H^j_{DR}(\hat{U}^{(p)}) = H^j\Omega^\bullet_{\hat{X}^{(p)}}(U) = \mathbb{H}^j_{pet}(U, \Omega^\bullet_{\hat{X}^{(p)}}).$$

Let $n \in \mathbb{N}$. There exists $i_n \in I$ (depending on n) such that $\eta_n = r_{i_n}^* \tilde{\eta}_n$, with $\tilde{\eta}_n \in \Omega^{j+1}_{\hat{X}^{(p)}}(U_{i_n})$. Then, there exists $a_{ij} : U_{j_n} \to U_{i_n}$, $j_n, a_{ij} \in I$ such that

$$r_{j_n}^* w_n = a_{ij}^* \partial \tilde{\eta}_n + \partial \beta_n \in \Omega^\bullet_{X^{\mathcal{O}}/p^n}(U_{j_n})$$

Since $e : U \to \mathbb{A}^{d_X}$ is an etale map, the divisors, hence the closed subset of U are given by irreducible components of closed subsets given by global equations, hence

$$w_n = \partial(\tilde{\eta}_n + \beta_n) \in \Omega^\bullet_{X^{\mathcal{O}}/p^n}(U).$$

Hence $j^*\alpha = 0 \in L^{l,j-l}(U)$. Let $r_\bullet : Y_\bullet \to U$ be a pro-etale cover by w-contractile schemes. Then,

$$j^*\alpha = (\theta_I) \in H^{j-l}_{pet}(U, \Omega^l_{\hat{X}^{(p)},\log,\mathcal{O}}) = H^{j-l}\Gamma(Y_\bullet, \Omega^l_{\hat{X}^{(p)},\log,\mathcal{O}}).$$

Since $j^*\alpha = 0 \in L^{l,j-l}(U)$, we have

$$\theta_I = \tilde{\theta}_{I'|Y_I} + \partial \gamma_I \in \Gamma(Y_I, \Omega^l_{\hat{X}^{(p)}}),$$

with $I \subset I'$, $\tilde{\theta}_{I'} \in \Gamma(Y_{I'}, \Omega^l_{\hat{X}^{(p)}})$ and $\gamma_I \in \Gamma(Y_I, \Omega^{l+1}_{\hat{X}^{(p)}})$. Since the closed subset of U are given by irreducible components of closed subsets given by global equations, we get $\theta_I = \tilde{\theta}_{I'|Y_I}$ and

$$j^*\alpha = \partial_\bullet(\tilde{\theta}_{I'}) = 0 \in H^{j-l}_{pet}(U, \Omega^l_{\hat{X}^{(p)},\log,\mathcal{O}}) = H^{j-l}\Gamma(Y_\bullet, \Omega^l_{\hat{X}^{(p)},\log,\mathcal{O}}).$$

Considering a divisor $X \backslash U \subset D \subset X$, we get

$$\alpha = H^{j-l} E^\bullet_{pet}(r^* \Omega^{l,an}_{/K,\log,\mathcal{O}})(\gamma_D^\vee)(\alpha),\ \alpha \in L^{l,j-l}_D(X).$$

Similarly to the proof of theorem 1, we get by induction (restricting to the smooth locus of the divisors) a closed subset $Z \subset X$ of pure codimension $c = min(l, j-l)$ such that

$$\alpha = H^{j-l} E^\bullet_{pet}(r^* \Omega^{l,an}_{/K,\log,\mathcal{O}})(\gamma_Z^\vee)(\alpha),\ \alpha \in L^{l,j-l}_Z(X).$$

Hence, as in the proof of theorem 1,

- if $j \neq 2l$, $\alpha = 0$, we use the fact that X is projective for $j < 2l$,
- if $j = 2l$, $\alpha \in \oplus_{1 \leq t \leq s} \mathbb{Z}_p[Z_i]$, where $(Z_i)_{1 \leq i \leq t} \subset Z$ are the irreducible components of Z.

□

4 Tate conjecture

Let k a field of finite type over $\mathbb{Q}$. Denote $\bar{k}$ the algebraic closure of k and $G = Gal(\bar{k}/k)$ the absolute Galois group of k.

Let $X \in \mathrm{SmVar}(k)$ be a smooth variety. Let $p \in \mathbb{N}$ a prime number. Consider an embedding $\sigma_p : k \hookrightarrow \mathbb{C}_p$. Denote $\hat{k}_{\sigma_p} \subset \mathbb{C}_p$ the p-adic completion of k with respect to σ_p. We have the commutative diagram in $C_{\mathbb{B}_{dr} fil, \hat{G}_{\sigma_p}}(X^{an,pet}_{\mathbb{C}_p})$

$$\begin{array}{ccc} (\mathbb{B}_{dr,X_{\mathbb{C}_p}}, F) & \xrightarrow{\alpha(X)} & (\Omega^\bullet_{X_{\mathbb{C}_p}}, F_b) \otimes_{O_{X_{\mathbb{C}_p}}} (O\mathbb{B}_{dr,X_{\mathbb{C}_p}}, F) \\ {\scriptstyle \iota'_{X^{pet}_{\mathbb{C}_p}} := l \mapsto l.1} \uparrow & & \uparrow {\scriptstyle OL_X \otimes I := (w \otimes \lambda_n)_{n \in \mathbb{N}} \mapsto (w \otimes \lambda_n)_{n \in \mathbb{N}}} \\ \underline{\mathbb{Z}_p}_{X_{\mathbb{C}_p}} & \xrightarrow{\iota_{X^{pet}_{\mathbb{C}_p}} := (I,0)} & (\Omega^\bullet_{X_{\mathbb{C}_p},\log,\mathcal{O}} \otimes \mathbb{Z}_p, F_b) \end{array}.$$

Consider an integral model $X^{\mathcal{O}}_{\hat{k}_{\sigma_p}} \in \mathrm{Sch}^{int}/O_{\hat{k}_{\sigma_p}}$ of $X_{\hat{k}_{\sigma_p}}$. If $X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}$ has good reduction modulo p, we have (see e.g. [1]) the embedding in $C((X^{\mathcal{O}}_{\hat{k}_{\sigma_p}})^{Falt})$

$$\alpha(X) : \mathbb{B}_{cris, X_{\hat{k}_{\sigma_p}}} \hookrightarrow a_{\bullet *}\Omega^{\bullet}_{X^{\mathcal{O},\bullet}_{\hat{k}_{\sigma_p}}} \otimes_{O_{X^{\mathcal{O}}}} O\mathbb{B}_{cris, X^{\bullet}_{\hat{k}_{\sigma_p}}}$$

which is a filtered quasi-isomorphism compatible with the action of $Gal(\mathbb{C}_p/\hat{k}_{\sigma_p})$ and the action of the Frobenius ϕ_p, where $(X^{\mathcal{O}}_{\hat{k}_{\sigma_p}})^{Falt}$ denote the Falting site.

For k a field of finite type over $\mathbb{Q}$ and $X \in \mathrm{SmVar}(k)$, we denote $\delta(k, X) \subset N$ the finite set consisting of prime numbers such that if $p \in \mathbb{N}\backslash\delta(k, X)$ is a prime number, k is unramified at p and there exists an integral model $X^{\mathcal{O}}_{\hat{k}_{\sigma_p}} \in \mathrm{Sch}^{int}/O_{\hat{k}_{\sigma_p}}$ of $X_{\hat{k}_{\sigma_p}}$ with good reduction modulo p for all embeddings $\sigma_p : k \hookrightarrow \mathbb{C}_p$, $\hat{k}_{\sigma_p} \subset \mathbb{C}_p$ being the p-adic completion of k with respect to σ_p.

Remark 2. *To see that $\delta(k, X)$ is a finite set, considering $k = k_0(x_1, \cdots, x_d)(x_{d+1})$ where $x_1, \cdots x_d$ are algebraicaly independent and k_0 is a number field one can take an integral model $Y \to O_{k_0}[x_1, \cdots, x_{d+1}]/f_{d+1}$ of X, a desingularization of $Y' \to Y$ of Y and then see that $\delta(k, X)$ is contained in the discriminant of the family $Y' \to Y \to O_{k_0}[x_1, \cdots, x_{d+1}]/f_{d+1}$.*

Let $X \in \mathrm{PSmVar}(k)$ be a smooth projective variety. Let $p \in \mathbb{N}\backslash\delta(k, X)$ a prime number. Consider an embedding $\sigma_p : k \hookrightarrow \mathbb{C}_p$. Then $X_{\hat{k}_{\sigma_p}}$ has good reduction modulo p and let $X^{\mathcal{O}}_{\hat{k}_{\sigma_p}} \in \mathrm{PSch}/O_{\hat{k}_{\sigma_p}}$ be a smooth model, i.e. $X^{\mathcal{O}}_{\hat{k}_{\sigma_p}} \otimes_{O_{\hat{k}_{\sigma_p}}} \hat{k}_{\sigma_p} = X_{\hat{k}_{\sigma_p}}$ and $X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}$ is smooth with smooth special fiber. The main result of [1] say in this case that the embedding in $C((X^{\mathcal{O}}_{\hat{k}_{\sigma_p}})^{Falt})$

$$\alpha(X) : \mathbb{B}_{cris, X_{\hat{k}_{\sigma_p}}} \hookrightarrow a_{\bullet *}\Omega^{\bullet}_{X^{\mathcal{O},\bullet}_{\hat{k}_{\sigma_p}}} \otimes_{O_{X^{\mathcal{O}}}} O\mathbb{B}_{cris, X^{\bullet}_{\hat{k}_{\sigma_p}}}$$

induces a filtered quasi-isomorphism compatible for each $j \in \mathbb{Z}$, a filtered isomorphism of filtered abelian groups

$$H^j R\alpha(X) : H^j_{et}(X_{\mathbb{C}_p}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{B}_{cris, \hat{k}_{\sigma_p}} \xrightarrow{H^j T(a_X, \mathbb{B}_{cris})^{-1}} H^j_{et}((X)^{Falt})(\mathbb{B}_{cris, X_{\hat{k}_{\sigma_p}}})$$
$$\xrightarrow{H^j R\Gamma(X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}, \alpha(X))} H^j_{DR}(X_{\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}} \mathbb{B}_{cris, \hat{k}_{\sigma_p}}$$

compatible with the action of $Gal(\mathbb{C}_p/\hat{k}_{\sigma_p})$ and of the Frobenius ϕ_p.

Definition 2. *Let k a field of finite type over $\mathbb{Q}$. Denote $\bar{k}$ the algebraic closure of k and $G = Gal(\bar{k}/k)$ the absolute Galois group of k. Let $X \in \mathrm{PSmVar}(k)$. Let $p \in \mathbb{N}\backslash\delta(k, X)$ a prime number. Consider an embedding $\sigma_p : k \hookrightarrow \mathbb{C}_p$. Denote $\hat{k}_{\sigma_p} \subset \mathbb{C}_p$ the p-adic completion of k with respect to σ_p. For $\alpha \in H^j_{et}(X_{\mathbb{C}_p}, \mathbb{Z}_p)$, we consider*

$$w(\alpha) := H^j R\alpha(X)(\alpha \otimes 1) \in H^j_{DR}(X_{\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}} \mathbb{B}_{cris, \hat{k}_{\sigma_p}}.$$

the associated de Rham class by the p-adic periods. We recall

$$H^j R\alpha(X) : H^j_{et}(X_{\mathbb{C}_p}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{B}_{cris, \hat{k}_{\sigma_p}} \xrightarrow{\sim} H^j_{DR}(X_{\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}} \mathbb{B}_{cris, \hat{k}_{\sigma_p}}$$

is the canonical filtered isomorphism compatible with the action of $\hat{G}_{\sigma_p}$ and with the action of the Frobenius ϕ_p.

We recall the following result from Illusie:

Proposition 6. *Let k a field of finite type over $\mathbb{Q}$. Let $X \in \mathrm{SmVar}(k)$. Let $p \in \mathbb{N}\backslash\delta(k,X)$ a prime number. Consider an embedding $\sigma_p : k \hookrightarrow \mathbb{C}_p$. Denote $\hat{k}_{\sigma_p} \subset \mathbb{C}_p$ the p-adic completion of k with respect to σ_p. Consider $X^{\mathcal{O}}_{\hat{k}_{\sigma_p}} \in \mathrm{Sch}^{int}/O_{\hat{k}_{\sigma_p}}$ a smooth model of $X_{\hat{k}_{\sigma_p}}$, in particular $X^{\mathcal{O}}_{\hat{k}_{\sigma_p}} \otimes_{O_{\hat{k}_{\sigma_p}}} \hat{k}_{\sigma_p} = X_{\hat{k}_{\sigma_p}}$ and $X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}$ is smooth with smooth special fiber. Assume there exist lifts $\phi_n : X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}/p^n \to X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}/p^n$ of the Frobenius $\phi : X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}/p \to X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}/p$, such that for $n' > n$ the following diagram commutes*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & O_{X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}/p^{n'-n}} & \xrightarrow{p^n\cdot} & O_{X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}/p^{n'}} & \xrightarrow{/p^{n'-n}} & O_{X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}/p^{n}} & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \phi_{n'-n}} & & \uparrow{\scriptstyle \phi_{n'}} & & \uparrow{\scriptstyle \phi_{n}} & & \\ 0 & \longrightarrow & O_{X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}/p^{n'-n}} & \xrightarrow{p^n\cdot} & O_{X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}/p^{n'}} & \xrightarrow{/p^{n'-n}} & O_{X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}/p^{n}} & \longrightarrow & 0 \end{array}$$

Let $l \in \mathbb{Z}$. For each $n \in \mathbb{N}$, the sequence in $C(X^{\mathcal{O},et}_{\hat{k}_{\sigma_p}})$

$$0 \to \Omega^{\bullet\geq l}_{X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}/p^n,\log} \xrightarrow{OL_{X^{\mathcal{O}}/p^n}} \Omega^{\bullet\geq l}_{X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}/p^n} \xrightarrow{\phi_n - I} \Omega^{\bullet\geq l}_{X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}/p^n} \to 0$$

is exact as a sequence of complexes of etale sheaves (i.e. we only have local surjectivity on the right).

Proof. It follows from [6] for $n = 1$. It then follows for $n \geq 2$ by induction on n by a trivial devissage. □

We have the following key proposition (the projective case of [4]), we state and prove it for smooth projective varieties, the case for smooth varieties is obtained in the same way using a smooth compactification with normal crossing divisors. The projective case suffices for our purpose:

Proposition 7. *Let k a field of finite type over $\mathbb{Q}$. Let $X \in \mathrm{PSmVar}(k)$. Let $p \in \mathbb{N}\backslash\delta(k,X)$ a prime number. Consider an embedding $\sigma_p : k \hookrightarrow \mathbb{C}_p$. Denote $k \subset \hat{k}_{\sigma_p} \subset \mathbb{C}_p$ the p-adic completion of k with respect to σ_p. Consider $X^{\mathcal{O}}_{\hat{k}_{\sigma_p}} \in \mathrm{PSch}/O_{\hat{k}_{\sigma_p}}$ a smooth model of $X_{\hat{k}_{\sigma_p}}$, i.e. $X^{\mathcal{O}}_{\hat{k}_{\sigma_p}} \otimes_{O_{\hat{k}_{\sigma_p}}} \hat{k}_{\sigma_p} = X_{\hat{k}_{\sigma_p}}$ and $X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}$ is smooth with smooth special fiber.*

(i) Let $j, l \in \mathbb{Z}$. We have, see definition 1(iii),

$$F^l H^j_{DR}(X_{\hat{k}_{\sigma_p}}) \cap H^j R\alpha(X)(H^j_{et}(X_{\mathbb{C}_p}, \mathbb{Z}_p)) = (c^*)^{-1}(H^j OL_{\hat{X}^{(p)}_{\hat{k}_{\sigma_p}}}(\mathbb{H}^j_{pet}(X_{\hat{k}_{\sigma_p}}, \Omega^{\bullet\geq l}_{\hat{X}^{(p)}_{\hat{k}_{\sigma_p}},\log,\mathcal{O}})))$$
$$\subset H^j_{DR}(X_{\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}} \mathbb{B}_{cris,\hat{k}_{\sigma_p}},$$

where we recall $c : \hat{X}^{(p)}_{\hat{k}_{\sigma_p}} \to X_{\hat{k}_{\sigma_p}}$ is the completion with respect to (p), and

$$c^* : \mathbb{H}^j_{pet}(X_{\hat{k}_{\sigma_p}}, \Omega^{\bullet}_{X_{\hat{k}_{\sigma_p}}}) \xrightarrow{\sim} \mathbb{H}^j_{pet}(X_{\hat{k}_{\sigma_p}}, \Omega^{\bullet}_{\hat{X}^{(p)}_{\hat{k}_{\sigma_p}}})$$

is an isomorphism by GAGA and the properties of w-contractile open covers.

(ii) Let $j, l \in \mathbb{Z}$. We have, see definition 1(iii),

$$F^l H^j_{DR}(X_{\hat{k}_{\sigma_p}}) \cap H^j R\alpha(X)(H^j_{et}(X_{\mathbb{C}_p}, \mathbb{Z}_p)) = H^j(m \circ (OL_X \otimes I))(\mathbb{H}^j_{pet}(X_{\hat{k}_{\sigma_p}}, \Omega^{\bullet\geq l}_{X_{\hat{k}_{\sigma_p}},\log,\mathcal{O}} \otimes \mathbb{Z}_p))$$
$$\subset H^j_{DR}(X_{\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}} \mathbb{B}_{cris,\hat{k}_{\sigma_p}},$$

(ii)' For $\alpha \in H^j_{et}(X_{\mathbb{C}_p}, \mathbb{Z}_p)$ such that $w(\alpha) \in F^l H^j_{DR}(X_{\hat{k}_{\sigma_p}})$ (see definition 2), there exist

$$(\lambda_i)_{1\leq i\leq n} \in \mathbb{Z}_p \text{ and } (w_{Li})_{1\leq i\leq n} \in \mathbb{H}^j_{pet}(X_{\mathbb{C}_p}, \Omega^{\bullet\geq l}_{X_{\mathbb{C}_p},\log,\mathcal{O}})$$

such that

$$w(\alpha) = \sum_{1\leq i\leq n} \lambda_i \cdot w_{Li} \in \mathbb{H}^j_{pet}(X_{\mathbb{C}_p}, \Omega^{\bullet\geq l}_{X_{\mathbb{C}_p}}) = F^l H^j_{DR}(X_{\mathbb{C}_p}),\ w_{Li} := H^j OL_X(w_{Li}).$$

Proof. (i): Consider $c : \hat{X}^{\mathcal{O},(p)}_{\hat{k}_{\sigma_p}} \to X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}$ the morphism in RTop which is the formal completion along the ideal (p). Take a Zariski or etale cover $r = (r_i : X_i \to X)_{1\leq i\leq r}$ such that for each i there exists an etale map $X_i \to \mathbb{A}^{d_{X_i}}$. Then, by [1], we have for each i explicit lifts of Frobenius $\phi^i_n : X^{\mathcal{O}}_{i,\hat{k}_{\sigma_p}}/p^n \to X^{\mathcal{O}}_{i,\hat{k}_{\sigma_p}}/p^n$ of the Frobenius $\phi^i_1 : X^{\mathcal{O}}_{i,\hat{k}_{\sigma_p}}/p \to X^{\mathcal{O}}_{i,\hat{k}_{\sigma_p}}/p$, such that for $n' > n$ the following diagram commutes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & O_{X^{\mathcal{O}}_{i,\hat{k}_{\sigma_p}}/p^{n'-n}} & \xrightarrow{p^n\cdot} & O_{X^{\mathcal{O}}_{i,\hat{k}_{\sigma_p}}/p^{n'}} & \xrightarrow{/p^{n'-n}} & O_{X^{\mathcal{O}}_{i,\hat{k}_{\sigma_p}}/p^n} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \phi^i_{n'-n}} & & \uparrow{\scriptstyle \phi^i_{n'}} & & \uparrow{\scriptstyle \phi^i_n} & & \\
0 & \longrightarrow & O_{X^{\mathcal{O}}_{i,\hat{k}_{\sigma_p}}/p^{n'-n}} & \xrightarrow{p^n\cdot} & O_{X^{\mathcal{O}}_{i,\hat{k}_{\sigma_p}}/p^{n'}} & \xrightarrow{/p^{n'-n}} & O_{X^{\mathcal{O}}_{i,\hat{k}_{\sigma_p}}/p^n} & \longrightarrow & 0
\end{array}$$

and such that the action of ϕ^i_n on $\Omega^{\bullet}_{X^{\mathcal{O}}_{i,\hat{k}_{\sigma_p}}/p^n}$ is a morphism of complex, i.e. commutes with the differentials. On the other hand, by [6], we have action of the Frobenius on $H^j_{DR}(X_{\hat{k}_{\sigma_p}}) = H^j_{DR}(X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}) \otimes_{O_{\hat{k}_{\sigma_p}}} \hat{k}_{\sigma_p}$ by

$$\phi : H^j_{DR}(X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}) \xrightarrow{\sim} \mathbb{H}^j(X_{\hat{k}_{\sigma_p}}, W\Omega^{\bullet}_{X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}}) \xrightarrow{\phi^*_{W(X_{\hat{k}_{\sigma_p}})}} \mathbb{H}^j(X_{\hat{k}_{\sigma_p}}, W\Omega^{\bullet}_{X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}}) \xrightarrow{\sim} H^j_{DR}(X^{\mathcal{O}}_{\hat{k}_{\sigma_p}})$$

We then have the following commutative diagram, where $R := [1, \ldots, r]$ and $X_R := X_1 \times_X \cdots \times_X X_r$,

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & F^l H^{j-1}_{DR}(X^{\mathcal{O}}_{R,\hat{k}_{\sigma_p}}) & \xrightarrow{\partial} & F^l H^j_{DR}(X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}) & \xrightarrow{r_i^*} & \oplus^r_{i=1} F^l H^j_{DR}(X^{\mathcal{O}}_{i,\hat{k}_{\sigma_p}}) & \xrightarrow{r_I^*} & \cdots \\
 & & \uparrow{\scriptstyle I-\phi^R} & & \uparrow{\scriptstyle I-\phi} & & \uparrow{\scriptstyle I-\phi^i} & & \\
\cdots & \longrightarrow & F^l H^{j-1}_{DR}(X^{\mathcal{O}}_{R,\hat{k}_{\sigma_p}}) & \xrightarrow{\partial} & F^l H^j_{DR}(X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}) & \xrightarrow{r_i^*} & \oplus^r_{i=1} F^l H^j_{DR}(X^{\mathcal{O}}_{i,\hat{k}_{\sigma_p}}) & \xrightarrow{r_I^*} & \cdots \\
 & & \uparrow{\scriptstyle H^{j-1}r^*OL_{\hat{X}^{\mathcal{O},(p)}_{R,\hat{k}_{\sigma_p}}}} & & \uparrow{\scriptstyle H^j r^*OL_{\hat{X}^{\mathcal{O},(p)}_{i,\hat{k}_{\sigma_p}}}} & & \uparrow{\scriptstyle H^j r^*OL_{\hat{X}^{\mathcal{O},(p)}_{i,\hat{k}_{\sigma_p}}}} & & \\
\cdots & \longrightarrow & \mathbb{H}^{j-1}_{pet}(X_{R,\hat{k}_{\sigma_p}}, \Omega^{\bullet\geq l}_{\hat{X}^{(p)}_{\hat{k}_{\sigma_p}},\log,\mathcal{O}}) & \xrightarrow{\partial} & \mathbb{H}^j_{pet}(X_{\hat{k}_{\sigma_p}}, \Omega^{\bullet\geq l}_{\hat{X}^{(p)}_{\hat{k}_{\sigma_p}},\log,\mathcal{O}}) & \xrightarrow{r_i^*} & \oplus^r_{i=1}\mathbb{H}^j_{pet}(X_{i,\hat{k}_{\sigma_p}}, \Omega^{\bullet\geq l}_{\hat{X}^{(p)}_{\hat{k}_{\sigma_p}},\log,\mathcal{O}}) & \xrightarrow{r_I^*} & \cdots
\end{array} \tag{1}$$

whose rows are exact sequences. By [1], $\alpha \in H^j_{et}(X_{\mathbb{C}_p}, \mathbb{Z}_p)$ is such that $w(\alpha) \in F^l H^j_{DR}(X_{\hat{k}_{\sigma_p}})$ if and only if

$$w(\alpha) \in \ker(I - \phi : F^l H^j_{DR}(X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}) \to F^l H^j_{DR}(X^{\mathcal{O}}_{\hat{k}_{\sigma_p}})).$$

On the other hand, for each $I \subset [1, \ldots, r]$, the sequence in $C(X^{\mathcal{O},pet}_I)$

$$0 \to \Omega^{\bullet\geq l}_{\hat{X}^{(p)}_{I,\hat{k}_{\sigma_p}},\log,\mathcal{O}} \xrightarrow{OL_{\hat{X}^{\mathcal{O},(p)}_{\hat{k}_{\sigma_p}}}} \Omega^{\bullet\geq l}_{\hat{X}^{\mathcal{O},(p)}_{I,\hat{k}_{\sigma_p}}} \xrightarrow{\phi^I - I := (\phi^I_n - I)_{n\in\mathbb{N}}} \Omega^{\bullet\geq l}_{\hat{X}^{\mathcal{O},(p)}_{I,\hat{k}_{\sigma_p}}} \to 0$$

is exact for the pro-etale topology by proposition 6 and since for each $l, n \in \mathbb{N}$ the map in $\mathrm{PSh}(X^{pet})$

$$\Omega(/p) : \Omega^l_{X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}/p^n} \to \Omega^l_{X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}/p^{n+1}}$$

are surjective and since the pro-etale site is a replete topos by [2]. Hence, for each $I \subset [1, \dots, r]$, by applying r^* where $r : X_I^{pet} \to X_I^{\mathcal{O},pet}$, the sequence in $C(X_I^{pet})$

$$0 \to r^*\Omega^{\bullet \geq l}_{\hat{X}^{(p)}_{I,\hat{k}_{\sigma_p}},\log,\mathcal{O}} \xrightarrow{r^*OL_{\hat{X}^{\mathcal{O},(p)}_{\hat{k}_{\sigma_p}}}} r^*\Omega^{\bullet \geq l}_{\hat{X}^{\mathcal{O},(p)}_{I,\hat{k}_{\sigma_p}}} \xrightarrow{\phi^I - I := (\phi_n^I - I)_{n \in \mathbb{N}}} r^*\Omega^{\bullet \geq l}_{\hat{X}^{\mathcal{O},(p)}_{I,\hat{k}_{\sigma_p}}} \to 0$$

is exact for the pro-etale topology. Hence the columns of the diagram (1) are exact. This proves (i).
(ii): Follows from (i) and proposition 5.
(ii)': By (ii), if $(r_i : X_i \to X)_{1 \leq i \leq r}$ is a w-contractile affine pro-etale cover

$$w(\alpha) = (\sum_{1 \leq i \leq n_J} \lambda_{iJ} \cdot w_{LiJ})_{J \subset [1,\dots,r], cardJ=j} \in H^j\Gamma(X_{\bullet,\hat{k}_{\sigma_p}}, m \circ (OL_X \otimes I))(H^j\Gamma(X_{\bullet,\hat{k}_{\sigma_p}}, \Omega^{\bullet \geq l}_{X_{\hat{k}_{\sigma_p}},\log,\mathcal{O}} \otimes \mathbb{Z}_p)),$$

with $w_{LiJ} \in \Gamma(X_J, \Omega^{\bullet \geq l'}_{X_{\hat{k}_{\sigma_p}},\log,\mathcal{O}})$, $l' \geq l$ and $\lambda_{iJ} \in \mathbb{Z}_p$. Since $w(\alpha)$ is a Chech etale cycle (i.e. closed for the Chech differential) without torsion and since the topology consists of etale covers of $X \in \mathrm{PSmVar}(k)$ (which we way assume connected), $n_J = n$ and $\lambda_{iJ} = \lambda_i \in \mathbb{Z}_p$ for each J, which gives

$$w(\alpha) = \sum_{1 \leq i \leq n} \lambda_i \cdot w_{Li} \in \mathbb{H}^j_{pet}(X_{\mathbb{C}_p}, \Omega^{\bullet \geq l}_{X_{\mathbb{C}_p}}) = F^l H^j_{DR}(X_{\mathbb{C}_p}),\ w_{Li} := H^j OL_X(w_{Li}).$$

□

We also have

Proposition 8. *Let k a field of finite type over $\mathbb{Q}$. Denote $\bar{k}$ the algebraic closure of k. Denote $G := Gal(\bar{k}/k)$ its absolute Galois group. Let $X \in \mathrm{PSmVar}(k)$ a smooth projective variety. Let $p \in \mathbb{N}$ a prime number. Consider an embedding $\sigma_p : k \hookrightarrow \mathbb{C}_p$. Denote $k \subset \hat{k}_{\sigma_p} \subset \mathbb{C}_p$ being the p-adic completion with respect to the p adic norm induced by σ_p. Then $\hat{G}_{\sigma_p} := Gal(\mathbb{C}_p/\hat{k}_{\sigma_p}) \subset G := Gal(\bar{k}/k)$. Let $\alpha \in H^j_{et}(X_{\mathbb{C}_p}, \mathbb{Z}_p)$. Consider then its associated De Rham class (see definition 2)*

$$w(\alpha) := H^j R\alpha(X)(\alpha \otimes 1) \in H^j_{DR}(X_{\mathbb{C}_p}) \otimes_{\mathbb{C}_p} \mathbb{B}_{dr,\mathbb{C}_p}.$$

Then $\alpha \in H^j_{et}(X_{\mathbb{C}_p}, \mathbb{Z}_p)(l)^{\hat{G}_{\sigma_p}}$ if and only if $w(\alpha) \in F^l H^j_{DR}(X_{\hat{k}_{\sigma_p}}) = F^l H^j_{DR}(X_{\mathbb{C}_p}) \cap H^j_{DR}(X_{\hat{k}_{\sigma_p}})$. That is we have

$$H^j_{et}(X_{\mathbb{C}_p}, \mathbb{Z}_p)(l)^{\hat{G}_{\sigma_p}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p =< F^l H^j_{DR}(X_{\hat{k}_{\sigma_p}}) \cap H^j R\alpha(X)(H^j_{et}(X_{\mathbb{C}_p}, \mathbb{Z}_p)) >_{\mathbb{Q}_p} \subset H^j_{et}(X_{\mathbb{C}_p}, \mathbb{Z}_{p,X^{et}}) \otimes \mathbb{B}_{dr,\mathbb{C}_p},$$

where $< - >_{\mathbb{Q}_p}$ denote the $\mathbb{Q}_p$ vector space generated by $(-)$.

Proof. Follows immediately from the fact that $H^j R\alpha(X)$ is a filtered quasi-isomorphism compatible with the Galois action of $\hat{G}_{\sigma_p}$ by [1]. □

These propositions, together with the results of section 3 implies the Tate conjecture :

Theorem 2. *(i) Let k a field of finite type over $\mathbb{Q}$. Let $X \in \mathrm{PSmVar}(k)$. Let $p \in \mathbb{N} \backslash \delta(k, X)$ be a prime number. Then the Tate conjecture holds for X. That is for $d \in \mathbb{Z}$, the cycle class map*

$$\mathcal{Z}^d(X) \otimes \mathbb{Q}_p \to H^{2d}_{et}(X_{\bar{k}}, \mathbb{Q}_p)^G,\ Z \mapsto [Z]$$

is surjective, where $G = Gal(\bar{k}/k)$.

(ii) Let K a p-adic field which is unramified. Let $X \in \mathrm{PSmVar}(K)$ with good reduction. Then the Tate conjecture holds for X. That is for $d \in \mathbb{Z}$, the cycle class map

$$\mathcal{Z}^d(X) \otimes \mathbb{Q}_p \to H^{2d}_{et}(X_{\mathbb{C}_p}, \mathbb{Q}_p)^G,\ Z \mapsto [Z]$$

is surjective, where $G = Gal(\mathbb{C}_p, K)$.

Proof. (i):Consider an embedding $\sigma_p : k \hookrightarrow \mathbb{C}_p$. Then $k \subset \bar{k} \subset \mathbb{C}_p$ and $k \subset \hat{k}_{\sigma_p} \subset \mathbb{C}_p$, where $\hat{k}_{\sigma_p}$ is the p-adic field which is the completion of k with respect the p adic norm given by σ_p. Consider $X^{\mathcal{O}}_{\hat{k}_{\sigma_p}} \in \mathrm{PSch}/O_{\hat{k}_{\sigma_p}}$ a smooth model of $X_{\hat{k}_{\sigma_p}}$, i.e. $X^{\mathcal{O}}_{\hat{k}_{\sigma_p}} \otimes_{O_{\hat{k}_{\sigma_p}}} \hat{k}_{\sigma_p} = X_{\hat{k}_{\sigma_p}}$ and $X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}$ is smooth with smooth special fiber. Let $\alpha \in H^{2d}_{et}(X_{\bar{k}}, \mathbb{Z}_p)(d)^G$. Using definition 2, by proposition 7(ii)' and proposition 8 there exists $(\lambda_i)_{1\leq i\leq n} \in \mathbb{Z}_p$ and

$$(w_{Li})_{1\leq i\leq n} \in \mathbb{H}^{2d}_{pet}(X_{\mathbb{C}_p}, \Omega^{\bullet\geq d}_{X_{\mathbb{C}_p},\log,\mathcal{O}})$$

such that

$$w(\alpha) = \sum_{1\leq i\leq n} \lambda_i \cdot w_{Li} \in \mathbb{H}^{2d}_{pet}(X_{\mathbb{C}_p}, \Omega^{\bullet\geq d}_{X_{\mathbb{C}_p}}) = F^d H^{2d}_{DR}(X_{\mathbb{C}_p}),\ w_{Li} := H^{2d}OL_X(w_{Li}).$$

By theorem 1(ii) and (iii)', there exists $Z_i \in \mathcal{Z}^d(X_{\mathbb{C}_p})$ such that $w_{Li} = [Z_i] \in H^{2d}_{DR}(X_{\mathbb{C}_p})$. Hence,

$$Z := \sum_{1\leq i\leq n} \lambda_i [Z_i] \in \mathcal{Z}^d(X_{\mathbb{C}_p}) \otimes \mathbb{Q}_p$$

satisfies $w(\alpha) = [Z] \in H^{2d}_{DR}(X_{\mathbb{C}_p})$. By [1], considering the commutative diagram of abelian groups whose rows are exact, where $j : X\backslash|Z| \hookrightarrow X$ denote the open embedding,

$$\begin{array}{ccccc}
H^{2d}_{et,Z}(X_{\bar{k}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{B}_{cris} & \xrightarrow{B_{cris,X}(\gamma_Z^\vee)} & H^{2d}_{et}(X_{\bar{k}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{B}_{cris} & \xrightarrow{j^*} & H^{2d}_{et}((X\backslash Z)_{\bar{k}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{B}_{cris}\ , \\
\downarrow{\scriptstyle Ra_{X*}\Gamma_Z \alpha(X)} & & \downarrow{\scriptstyle R\alpha(X)} & & \downarrow{\scriptstyle R\alpha(X\backslash Z)} \\
H^{2d}_{DR,Z}(X_{\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}} \mathbb{B}_{cris} & \xrightarrow{DR(X)(OB_{cris,X})(\gamma_Z^\vee)} & H^{2d}_{DR}(X_{\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}} \mathbb{B}_{cris} & \xrightarrow{j^*} & H^{2d}_{DR}((X\backslash Z)_{\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}} \mathbb{B}_{cris}
\end{array}$$

we get $\alpha = [Z] \in H^{2d}_{et}(X_{\bar{k}}, \mathbb{Z}_p)$. Since $[Z]$ is G invariant,

$$Z' := 1/card(gZ, g \in G) \sum_{g\in G/G_Z} gZ \in \mathcal{Z}^d(X) \otimes \mathbb{Q}_p$$

satisfy $[Z'] = [Z] = \alpha \in H^{2d}_{et}(X_{\bar{k}}, \mathbb{Z}_p)$.
(ii):Similar to (i).

□

It is well known that theorem 2 implies the following :

Corollary 2. *Let $X \in \mathrm{PSmVar}(\mathbb{C})$.*

(i) The standard conjectures holds for X.

(ii) Let $k \subset \mathbb{C}$ a subfield of finite type over $\mathbb{Q}$ over which X is defined, that is $X \simeq X_k \otimes_k \mathbb{C}$ in $\mathrm{PSmVar}(\mathbb{C})$ with $X_k \in \mathrm{PSmVar}(k)$. For $\theta \in Aut(\mathbb{C}/\bar{k})$, we have the isomorphism $\theta : X \xrightarrow{\sim} X_\theta$ in Sch. Then $\theta \in Aut(\mathbb{C}/\bar{k})$ induces, for each $j \in \mathbb{Z}$, an isomorphism

$$\theta : H^j_{\mathrm{sing}}(X^{an}, \mathbb{C}) \xrightarrow{H^j ev(X)^{-1}} H^j_{DR}(X) \xrightarrow{\theta^*} H^j_{DR}(X_\theta) \xrightarrow{H^j ev(X_\theta)} H^j_{\mathrm{sing}}(X^{an}_\theta, \mathbb{C}).$$

Let $d \in \mathbb{N}$. Let $\alpha \in F^d H^{2d}(X^{an}, \mathbb{Q})$ where $F^d H^{2d}(X^{an}, \mathbb{Q}) := F^d H^{2d}_{DR}(X) \cap H^{2d}_{\mathrm{sing}}(X^{an}, \mathbb{Q}) \subset H^{2d}(X^{an}, \mathbb{C})$. If $\theta(\alpha) \in H^{2d}_{\mathrm{sing}}(X^{an}_\theta, \mathbb{Q})$ for all $\theta \in Aut(\mathbb{C}/\bar{k})$, then $\alpha = [Z]$ with $Z \in \mathcal{Z}^d(X)$.

Proof. Standard. We consider, for $i \in \mathbb{Z}$, $k' \subset \mathbb{C}$ a subfield and $Y \in \mathrm{Var}(k')$, the canonical morphism

$$T(Y) := T^i(Y) : H^i_{\mathrm{sing}}(Y^{an}_{\mathbb{C}}, \mathbb{Q}) \xrightarrow{(/p^n)_{n\in\mathbb{N}}} H^i_{\mathrm{sing}}(Y^{an}_{\mathbb{C}}, \mathbb{Q}_p) \xrightarrow{\sim} H^i_{et}(Y_{\mathbb{C}}, \mathbb{Q}_p) \xrightarrow{\sim} H^i_{et}(Y_{\bar{k}'}, \mathbb{Q}_p).$$

(i): Let $k \subset \mathbb{C}$ of finite type over $\mathbb{Q}$ such that X is defined. Denote $G = Gal(\bar{k}/k)$ the Galois group. Denote $d_X = \dim(X)$ and for short $X^{an} = X$. Let $L^j \in H^{2d_X-2j}_{\mathrm{sing}}(X \times X, \mathbb{Q})$ the class inducing the cup product with of the intersection of j hyperplane sections on $H^{d_X-j}_{\mathrm{sing}}(X, \mathbb{Q})$ and zero on $H^i_{\mathrm{sing}}(X, \mathbb{Q})$, $i \neq d_X - j$. Let $\Lambda^j \in H^{2d_X+2j}_{\mathrm{sing}}(X \times X, \mathbb{Q})$ inducing the inverse of L_j on $H^{d_X+j}_{\mathrm{sing}}(X, \mathbb{Q})$ and zero on $H^i_{\mathrm{sing}}(X, \mathbb{Q})$, $i \neq d_X + j$. Consider an isomorphism $\sigma_p : \mathbb{C} \xrightarrow{\sim} \mathbb{C}_p$ such that $\hat{k}_{\sigma_p} \subset \mathbb{C}_p$ is unramified and X has good reduction. We have $T(X \times X)(L^j) \in H^{2d_X-2j}_{et}((X \times X)_{\bar{k}}, \mathbb{Q}_p)^G$ since it is the class of an algebraic cycle. Hence,

$$T(X \times X)(\Lambda^j) \in H^{2d_X+2j}_{et}((X \times X)_{\bar{k}}, \mathbb{Q}_p)^G$$

since the inverse of a Galois invariant morphism is Galois invariant. Hence, by theorem 2(i),

$$\Lambda^j = [Z], \text{ with } Z \in \mathcal{Z}^{d_X+j}(X \times X) \otimes \mathbb{Q}.$$

(ii): As $\alpha \in H^{2d}\Gamma(X_\bullet, \Omega_X)$, where $X = \cup_{i=1}^s X_i$ is an open affine cover, there exists a subfield $k \subset k' \subset \mathbb{C}$ of finite type over $\mathbb{Q}$ over which α is defined, that is $\alpha \in H^{2d}_{DR}(X_{k'}) \subset H^{2d}_{DR}(X)$, where $X_{k'} := X_k \otimes_k k' \in \mathrm{PSmVar}(k')$ satisfy $X_{k'} \otimes_{k'} \mathbb{C} \simeq X$ in $\mathrm{PSmVar}(\mathbb{C})$. Let $p \in \mathbb{N}\backslash\delta(k', X_{k'})$ be a prime number. Consider

$$T(X_{k'})(\alpha) \in H^{2d}(X_{\bar{k}'}, \mathbb{Q}_p).$$

If $\theta(\alpha) \in H^{2d}_{\mathrm{sing}}(X^{an}_\theta, \mathbb{Q})$ for all $\theta \in Aut(\mathbb{C}/\bar{k}')$, we get

$$T(X_{k'})(\alpha) = 1/card(Aut(\mathbb{C}/\bar{k}')\alpha) \sum_{\theta \in Aut(\mathbb{C}/k)} \theta^{-1,*} T(X_\theta)(\theta(\alpha)) \in H^{2d}_{et}(X_{\bar{k}'}, \mathbb{Q}_p) = H^{2d}_{et}(X, \mathbb{Q}_p),$$

hence, for $g \in Gal(\bar{k}'/k)$, we get

$$\begin{aligned} g \cdot T(X_{k'})(\alpha) &= 1/card(Aut(\mathbb{C}/\bar{k}')\alpha) \sum_{\theta \in Aut(\mathbb{C}/k)} g \cdot \theta^{-1,*} T(X_\theta)(\theta(\alpha)) \\ &= 1/card(Aut(\mathbb{C}/k)\alpha) \sum_{\theta \in Aut(\mathbb{C}/k)} \theta^{-1,*} T(X_\theta)(\theta(\alpha)) = T(X_{k'})(\alpha) \in H^{2d}_{et}(X_{\bar{k}'}, \mathbb{Q}_p), \end{aligned}$$

that is, $T(X_{k'})(\alpha) \in H^{2d}(X_{\bar{k}'}, \mathbb{Q}_p)^G$, with $G := Gal(\bar{k}'/k)$. Hence, if $\theta(\alpha) \in H^{2d}_{\mathrm{sing}}(X^{an}_\theta, \mathbb{Q})$ for all $\theta \in Aut(\mathbb{C}/\bar{k}')$, we get by theorem 2(i),

$$T(X)(\alpha) = T(X_{k'})(\alpha) = [Z], \text{ with } Z \in \mathcal{Z}^d(X) \otimes \mathbb{Q}.$$

□

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