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De Rham logarithmic classes and Tate conjecture

Johann Bouali

June 4, 2023

Abstract

We introduce the definition of De Rham logarithmic classes. We show that the De Rham class of an algebraic cycle of a smooth algebraic variety over a field of characteristic zero is logarithmic and conversely that a logarithmic class of bidegree (d, d) is the De Rham class of an algebraic cycle (of codimension d). We deduce the Tate conjecture for smooth projective varieties over fields of finite type over \mathbb{Q} and over p -adic fields for \mathbb{Q}_p coefficients, under good reduction hypothesis.

1 Introduction

In this work, we introduce for X a noetherian scheme, the definition of logarithmic De Rham cohomology classes which are for each $j \in \mathbb{Z}$, the subgroup $H^j OL_X(\mathbb{H}_{et}^j(X, \Omega_{X, \log}^\bullet)) \subset \mathbb{H}_{et}^j(X, \Omega_X^\bullet) =: H_{DR}^j(X)$ of the De Rham cohomology abelian group. The subcomplex $OL_X : \Omega_{X, \log}^\bullet \hookrightarrow \Omega_X^\bullet$ of abelian sheaves on X consisting of logarithmic forms is introduced in definition 1.

Let X be a smooth algebraic variety over a field of characteristic zero k . To an algebraic cycle $Z \in \mathcal{Z}^d(X)$, we associate its De Rham cohomology class $[Z] \in H_{DR}^{2d}(X)$, which is by definition, as for any Weil cohomology theory, the image of the fundamental class $[Z] \in H_{DR, Z}^{2d}(X)$ by the canonical morphism $H_{DR, Z}^{2d}(X) \rightarrow H^{2d}(X)$. In section 3, we show (c.f. theorem 1(i)) that $[Z] \in H_{DR}^{2d}(X)$ is logarithmic of bidegree (d, d) , that is

$$[Z] = H^{2d} OL_X([Z]_L) \in H_{DR}^{2d}(X), [Z]_L \in H_{et}^d(X, \Omega_{X, \log}^d)$$

where $H_{et}^d(X, \Omega_{X, \log}^d) \subset \mathbb{H}_{et}^{2d}(X, \Omega_{X, \log}^\bullet)$ is the canonical subspace, as all the differentials of $\Omega_{X, \log}^\bullet$ vanishes since by definition a logarithmic form is closed. This fact is a consequence of two properties :

- the Euler class of a vector bundle of rank d over an algebraic variety is logarithmic of bidegree (d, d) (c.f. proposition 2)
- the motivic isomorphisms applied to De Rham cohomology preserve logarithmic classes (c.f. proposition 1). The key point is that logarithmic De Rham forms on algebraic varieties over k are closed, (trivially) \mathbb{A}_k^1 invariant and compatible with the transfers maps induced by finite morphisms of algebraic varieties and in particular finite correspondences. Since $\Omega_{\log}^l \subset \Omega^l$ are (trivially) \mathbb{A}_k^1 invariant presheaves with transfers on the category of smooth algebraic varieties over k , by a theorem of Voevodsky the cohomology presheaves of \mathbb{A}_k^1 invariant presheaves with transfers are \mathbb{A}_k^1 invariant.

Conversely, we show (c.f. theorem 1(ii)), that a logarithmic class $w \in H_{DR}^{2d}(X)$ of bidegree (d, d) is the De Rham class of an algebraic cycle (of codimension d).

The proof works as follows : A logarithmic class of bidegree (p, q) is locally acyclic for the Zariski topology of X since it is the etale cohomology of a single sheaf. This allows us to proceed by a finite induction using the crucial fact that the purity isomorphism for De Rham cohomology preserve logarithmic classes (c.f. proposition 3). The proof of proposition 3 follows from the fact that the purity isomorphism is motivic (see [5]), that the Euler class of a vector bundle of rank d over an algebraic variety is logarithmic of bidegree (d, d) (c.f. proposition 2) and that the motivic isomorphisms applied to De Rham cohomology

preserve logarithmic classes (c.f. proposition 1). At the final step, we use, for simplicity, the fact that for a scheme Y , $H_{et}^1(Y, \Omega_{Y, \log}^1) = H^1(Y, O_Y^*)$ is the Picard group of Y . In the case X projective, we also get a vanishing result (c.f. theorem 1(iii)') : $H^{2d}OL_X(H_{et}^{d-k}(X, \Omega_{X, \log}^{d+k})) = 0$ for $k > 0$.

Let X a smooth projective variety over a field k of finite type over \mathbb{Q} . Let p be a prime number unramified over k such that X has good reduction at p , $\sigma_p : k \hookrightarrow \mathbb{C}_p$ a fixed embedding and \hat{k} the algebraic closure of k inside \mathbb{C}_p . Denote by $\hat{k}_{\sigma_p} \subset \mathbb{C}_p$ the p adic completion of k with respect to σ_p . We show (c.f. proposition 7) that Tate classes $\alpha \in H_{et}^{2d}(X_{\hat{k}}, \mathbb{Z}_p)(d)^G$ of X gives by the p -adic comparison isomorphism

$$H^{2d}R\alpha(X) : H_{et}^{2d}(X_{\mathbb{C}_p}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{B}_{cris, \hat{k}_{\sigma_p}} \xrightarrow{\sim} H_{DR}^{2d}(X) \otimes_k \mathbb{B}_{cris, \hat{k}_{\sigma_p}},$$

a \mathbb{Z}_p linear combination of logarithmic classes

$$w(\alpha) := H^{2d}R\alpha(X)(\alpha) := \sum_{i=1}^s \lambda_i w(\alpha)_{L,i} \in H_{DR}^{2d}(X_{\mathbb{C}_p}), \lambda_i \in \mathbb{Z}_p$$

with for each $1 \leq i \leq s$, $\lambda_i \in \mathbb{Z}_p$ and $w(\alpha)_{L,i} \in H^{2d}OL_X \mathbb{H}_{et}^{2d}(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}, \log}^{\geq d})$, using a result of [6] together with results of [8] on the pro-etale topology, and a GAGA result on logarithmic De Rham classes (c.f. proposition 5) whose proof is motivic and similar to the proof of theorem 1. We deduce, by theorem 1(ii) and (iii)', the Tate conjecture for X with \mathbb{Q}_p coefficient (c.f. theorem 2). Note that for a p -adic field $K \subset \mathbb{C}_p$, $\dim H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_l)^G > \dim H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p)^G$ for $l \neq p$ in general. Hence it is not clear whether Tate conjecture holds for smooth projective varieties over finite fields k_0 , since the base change for etale cohomology for $f : X \rightarrow O_K$, $O_K = W(k_0)$ only holds for \mathbb{Z}_l coefficients, $l \neq p$.

Let X a smooth projective variety over \mathbb{C} . Choose an isomorphism $\mathbb{C} \simeq \mathbb{C}_p$ with $p \in \mathbb{N}$ a prime number such that $X_{\mathbb{C}_p}$ has good reduction at p . As Tate conjecture holds for $X_{\mathbb{C}_p}$ with \mathbb{Q}_p coefficients, the standards conjectures holds for X and any absolute Hodge class of X is the class of an algebraic cycle (corollary 2).

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2 Preliminaries and Notations

- Denote by Top the category of topological spaces and RTop the category of ringed spaces.
- Denote by Cat the category of small categories and RCat the category of ringed topoi.
- For $\mathcal{S} \in \text{Cat}$ and $X \in \mathcal{S}$, we denote $\mathcal{S}/X \in \text{Cat}$ the category whose objects are $Y/X := (Y, f)$ with $Y \in \mathcal{S}$ and $f : Y \rightarrow X$ is a morphism in \mathcal{S} , and whose morphisms $\text{Hom}((Y', f'), (Y, f))$ consists of $g : Y' \rightarrow Y$ in \mathcal{S} such that $f \circ g = f'$.
- For $(\mathcal{S}, O_{\mathcal{S}}) \in \text{RCat}$ a ringed topoi, we denote by
 - $\text{PSh}(\mathcal{S})$ the category of presheaves of $O_{\mathcal{S}}$ modules on \mathcal{S} and $\text{PSh}_{O_{\mathcal{S}}}(\mathcal{S})$ the category of presheaves of $O_{\mathcal{S}}$ modules on \mathcal{S} , whose objects are $\text{PSh}_{O_{\mathcal{S}}}(\mathcal{S})^0 := \{(M, m), M \in \text{PSh}(\mathcal{S}), m : M \otimes O_{\mathcal{S}} \rightarrow M\}$, together with the forgetful functor $o : \text{PSh}(\mathcal{S}) \rightarrow \text{PSh}_{O_{\mathcal{S}}}(\mathcal{S})$,
 - $C(\mathcal{S}) = C(\text{PSh}(\mathcal{S}))$ and $C_{O_{\mathcal{S}}}(\mathcal{S}) = C(\text{PSh}_{O_{\mathcal{S}}}(\mathcal{S}))$ the big abelian category of complexes of presheaves of $O_{\mathcal{S}}$ modules on \mathcal{S} ,
 - $C_{O_{\mathcal{S}}(2)fil}(\mathcal{S}) := C_{(2)fil}(\text{PSh}_{O_{\mathcal{S}}}(\mathcal{S})) \subset C(\text{PSh}_{O_{\mathcal{S}}}(\mathcal{S}), F, W)$, the big abelian category of (bi)filtered complexes of presheaves of $O_{\mathcal{S}}$ modules on \mathcal{S} such that the filtration is biregular and $\text{PSh}_{O_{\mathcal{S}}(2)fil}(\mathcal{S}) := (\text{PSh}_{O_{\mathcal{S}}}(\mathcal{S}), F, W)$.
- Let $(\mathcal{S}, O_{\mathcal{S}}) \in \text{RCat}$ a ringed topoi with topology τ . For $F \in C_{O_{\mathcal{S}}}(\mathcal{S})$, we denote by $k : F \rightarrow E_{\tau}(F)$ the canonical flasque resolution in $C_{O_{\mathcal{S}}}(\mathcal{S})$ (see [3]). In particular for $X \in \mathcal{S}$, $H^*(X, E_{\tau}(F)) \xrightarrow{\sim} \mathbb{H}_{\tau}^*(X, F)$.

- For $f : \mathcal{S}' \rightarrow \mathcal{S}$ a morphism with $\mathcal{S}, \mathcal{S}' \in \text{RCat}$, endowed with topology τ and τ' respectively, we denote for $F \in C_{O_S}(\mathcal{S})$ and each $j \in \mathbb{Z}$,

$$\begin{aligned} - f^* &:= H^j \Gamma(\mathcal{S}, k \circ \text{ad}(f^*, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \rightarrow \mathbb{H}^j(\mathcal{S}', f^*F), \\ - f^* &:= H^j \Gamma(\mathcal{S}, k \circ \text{ad}(f^{*mod}, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \rightarrow \mathbb{H}^j(\mathcal{S}', f^{*mod}F), \end{aligned}$$

the canonical maps.

- For $\mathcal{X} \in \text{Cat}$ a (pre)site and p a prime number, we consider the full subcategory

$$\text{PSh}_{\mathbb{Z}_p}(\mathcal{X}) \subset \text{PSh}(\mathbb{N} \times \mathcal{X}), \quad F = (F_n)_{n \in \mathbb{N}}, \quad p^n F_n = 0, \quad F_{n+1}/p^n \xrightarrow{\sim} F_n$$

$$C_{\mathbb{Z}_p}(\mathcal{X}) := C(\text{PSh}_{\mathbb{Z}_p}(\mathcal{X})) \subset C(\mathbb{N} \times \mathcal{X}) \text{ and}$$

$$\mathbb{Z}_p := \mathbb{Z}_{p, \mathcal{X}} := ((\mathbb{Z}/p^* \mathbb{Z})_{\mathcal{X}}) \in \text{PSh}_{\mathbb{Z}_p}(\mathcal{X})$$

the diagram of constant presheaves on \mathcal{X} .

- Denote by $\text{Sch} \subset \text{RTop}$ the subcategory of schemes (the morphisms are the morphisms of locally ringed spaces). For a field k , we consider $\text{Sch}/k := \text{Sch}/\text{Spec } k$ the category of schemes over $\text{Spec } k$, that is whose object are $X := (X, a_X)$ with $X \in \text{Sch}$ and $a_X : X \rightarrow \text{Spec } k$ a morphism and whose objects are morphism of schemes $f : X' \rightarrow X$ such that $f \circ a_{X'} = a_X$. We then denote by
 - $\text{Var}(k) = \text{Sch}^{ft}/k \subset \text{Sch}/k$ the full subcategory consisting of algebraic varieties over k , i.e. schemes of finite type over k ,
 - $\text{PVar}(k) \subset \text{QPVar}(k) \subset \text{Var}(k)$ the full subcategories consisting of quasi-projective varieties and projective varieties respectively,
 - $\text{PSmVar}(k) \subset \text{SmVar}(k) \subset \text{Var}(k)$, $\text{PSmVar}(k) := \text{PVar}(k) \cap \text{SmVar}(k)$, the full subcategories consisting of smooth varieties and smooth projective varieties respectively.

For a morphism of field $\sigma : k \hookrightarrow K$, we have the extension of scalar functor

$$\otimes_k K : \text{Sch}/k \rightarrow \text{Sch}/K, \quad X \mapsto X_K := X_{K, \sigma} := X \otimes_k K, \quad (f : X' \rightarrow X) \mapsto (f_K := f \otimes I : X'_K \rightarrow X_K).$$

which is left ajoint to the restriction of scalar

$$\text{Res}_{k/K} : \text{Sch}/K \rightarrow \text{Sch}/k, \quad X = (X, a_X) \mapsto X = (X, \sigma \circ a_X), \quad (f : X' \rightarrow X) \mapsto (f : X' \rightarrow X)$$

The extension of scalar functor restrict to a functor

$$\otimes_k K : \text{Var}(k) \rightarrow \text{Var}(K), \quad X \mapsto X_K := X_{K, \sigma} := X \otimes_k K, \quad (f : X' \rightarrow X) \mapsto (f_K := f \otimes I : X'_K \rightarrow X_K).$$

and for $X \in \text{Var}(k)$ we have $\pi_{k/K}(X) : X_K \rightarrow X$ the projection in Sch/k .

- For $X \in \text{Sch}$, we denote $X^{pet} \subset \text{Sch}/X$ the pro etale site (see [2]) which is the full subcategory of Sch/X whose object consists of weakly etale maps $U \rightarrow X$ (that is flat maps $U \rightarrow X$ such that $\Delta_U : U \rightarrow U \times_X U$ is also flat) and whose topology is generated by fpqc covers. We then have the canonical morphism of site

$$\nu_X : X^{pet} \rightarrow X, \quad (U \rightarrow X) \mapsto (U \rightarrow X)$$

For $F \in C(X^{et})$,

$$\text{ad}(\nu_X^*, R\nu_{X*})(F) : F \rightarrow R\nu_{X*} \nu_X^* F$$

is an isomorphism in $D(X^{et})$, in particular, for each $n \in \mathbb{Z}$

$$\nu_X^* : \mathbb{H}_{et}^n(X, F) \xrightarrow{\sim} \mathbb{H}_{pet}^n(X, \nu_X^* F)$$

are isomorphisms, For $X \in \text{Sch}$, we denote

- $\underline{\mathbb{Z}}_{p_X} := \varprojlim_{n \in \mathbb{N}} \nu_X^*(\mathbb{Z}/p^n \mathbb{Z})_{X^{et}} \in \text{PSh}(X^{pet})$ the constant presheaf on \mathcal{X} ,
- $l_{p, \mathcal{X}} := (p^*) : \underline{\mathbb{Z}}_{p_X} \rightarrow \nu_X^*(\mathbb{Z}/p\mathbb{Z})_{X^{et}}$ the projection map in $\text{PSh}(\mathbb{N} \times \mathcal{X}^{pet})$.

- An affine scheme $U \in \text{Sch}$ is said to be w -contractible if any faithfully flat weakly etale map $V \rightarrow U$, $V \in \text{Sch}$, admits a section. We will use the facts that (see [2]):
 - Any scheme $X \in \text{Sch}$ admits a pro-etale affine cover $(r_i : X_i \rightarrow X)_{i \in I}$ with for each $i \in I$, X_i a w -contractible affine scheme and $r_i : X_i \rightarrow X$ a weakly etale map. For $X \in \text{Var}(k)$ with k a field, we may assume I finite since the topological space X is then quasi-compact.
 - If $U \in \text{Sch}$ is a w -contractible affine scheme, then for any sheaf $F \in \text{Shv}(U^{pet})$, $H_{pet}^i(U, F) = 0$ for $i \neq 0$ since $\Gamma(U, -)$ is an exact functor.
- Let $X \in \text{Sch}$. Considering its De Rham complex $\Omega_X^\bullet := DR(X)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^\bullet)$. For $X \in \text{Var}(k)$, the differentials of $\Omega_X^\bullet := \Omega_{X/k}^\bullet$ are by definition k -linear, thus $H_{DR}^j(X)$ has a structure of a k vector space. If $X \in \text{SmVar}(k)$, then $H_{DR}^j(X) = \mathbb{H}_{et}^j(X, \Omega_X^\bullet)$ since $\Omega_{/k}^\bullet \in C(\text{SmVar}(k))$ is \mathbb{A}^1 local and admits transfers where

$$\begin{aligned} \Omega_{/k}^\bullet &\in C(\text{Var}(k)), X \mapsto \Omega_{/k}^\bullet(X) := \Gamma(X, \Omega_X^\bullet), \\ (f : X' \rightarrow X) &\mapsto \Omega_{/k}^\bullet(f) := f^* : \Gamma(X, \Omega_X^\bullet) \rightarrow \Gamma(X', \Omega_{X'}^\bullet) \end{aligned}$$

(see [3]).

- Let $X \in \text{Var}(k)$. Let $X = \cup_{i=1}^s X_i$ an open affine cover. For $I \subset [1, \dots, s]$, we denote $X_I := \cap_{i \in I} X_i$. We get $X_\bullet \in \text{Fun}(P([1, \dots, s]), \text{Var}(k))$. Since quasi-coherent sheaves on affine noetherian schemes are acyclic, we have for each $j \in \mathbb{Z}$, $H_{DR}^j(X) = \Gamma(X_\bullet, \Omega_{X_\bullet}^\bullet)$.
- For $X \in \text{Sch}$ noetherian irreducible and $d \in \mathbb{N}$, we denote by $\mathcal{Z}^d(X)$ the group of algebraic cycles of codimension d , which is the free abelian group generated by irreducible closed subsets of codimension d .
- For K a field which is complete with respect to a p -adic norm, we consider $O_K \subset K$ the subring of K consisting of integral elements, that is $x \in K$ such that $|x| \leq 1$. For $X \in \text{Var}(K)$, we denote by $X^\mathcal{O} \in \text{Sch}/O_K$ a (non canonical) integral model of X , i.e. satisfying $X^\mathcal{O} \otimes_{O_K} K = X$.
- For $X \in \text{Sch}$ and p a prime number, we denote by $c : \hat{X}^{(p)} \rightarrow X$ the morphism in RTop which is the completion along the ideal generated by p .
- Let K a field which is complete with respect to a p -adic norm and $X \in \text{PVar}(K)$ projective. For $X^\mathcal{O} \in \text{Sch}/O_K$ an integral model of X , i.e. satisfying $X^\mathcal{O} \otimes_{O_K} K = X$, we consider the morphism in RTop

$$c := (c \otimes I) : \hat{X}^{(p)} := \hat{X}^{\mathcal{O}, (p)} \otimes_{O_K} K \rightarrow X^\mathcal{O} \otimes_{O_K} K = X.$$

We have then, by GAGA (c.f. EGA 3), for $F \in \text{Coh}_{O_X}(X)$ a coherent sheaf of O_X module, $c^* : H^k(X^\mathcal{O}, F) \xrightarrow{\sim} H^k(\hat{X}^{\mathcal{O}, (p)}, c^{*mod} F)$ for all $k \in \mathbb{Z}$, in particular

$$c^* : \mathbb{H}^k(X, \Omega_X^{\bullet \geq l}) \xrightarrow{\sim} \mathbb{H}^k(\hat{X}^{(p)}, \Omega_{\hat{X}^{(p)}}^{\bullet \geq l})$$

for all $k, l \in \mathbb{Z}$.

- For k a field of finite type over \mathbb{Q} and $X \in \text{SmVar}(k)$, we denote $\delta(k, X) \subset \mathbb{N}$ the finite set consisting of prime numbers such that for $p \in \mathbb{N} \setminus \delta$ a prime number k is unramified at p and such that there exists an integral model $X_{\hat{k}_{\sigma_p}}^\mathcal{O}$ of $X_{\hat{k}_{\sigma_p}}$ with good reduction modulo p for all embeddings $\sigma_p : k \hookrightarrow \mathbb{C}_p$, $\hat{k}_{\sigma_p} \subset \mathbb{C}_p$ being the p -adic completion of k with respect to σ_p . To see that $\delta(k, X)$ is a finite set, considering $k = k_0(x_1, \dots, x_d)(x_{d+1})$ where x_1, \dots, x_d are algebraically independent

and k_0 is a number field one can take an integral model $Y \rightarrow O_{k_0}[x_1, \dots, x_{d+1}]/f_{d+1}$ of X , a desingularization of $Y' \rightarrow Y$ of Y and then see that $\delta(k, X)$ is contained in the discriminant of the family $Y' \rightarrow Y \rightarrow O_{k_0}[x_1, \dots, x_{d+1}]/f_{d+1}$.

- Denote by $\text{AnSp}(\mathbb{C}) \subset \text{RTop}$ the full subcategory of analytic spaces over \mathbb{C} , and by $\text{AnSm}(\mathbb{C}) \subset \text{AnSp}(\mathbb{C})$ the full subcategory of smooth analytic spaces (i.e. complex analytic manifold). Denote by $\text{CW} \subset \text{Top}$ the full subcategory of CW complexes. Denote by $\text{Diff}(\mathbb{R}) \subset \text{RTop}$ the full subcategory of differentiable (real) manifold.
- Denote $\text{Sch}^2 \subset \text{RTop}^2$ the subcategory whose objects are couples (X, Z) with $X = (X, O_X) \in \text{Sch}$ and $Z \subset X$ a closed subset and whose set of morphisms $\text{Hom}((X', Z'), (X, Z))$ consists of $f : X' \rightarrow X$ of locally ringed spaces such that $f^{-1}(Z) \subset Z'$.
- Let k a field of characteristic zero. Denote $\text{SmVar}^2(k) \subset \text{Var}^2(k) \subset \text{Sch}^2/k$ the full subcategories whose objects are (X, Z) with $X \in \text{Var}(k)$, resp. $X \in \text{SmVar}(k)$, and $Z \subset X$ is a closed subset, and whose morphisms $\text{Hom}((X', Z') \rightarrow (X, Z))$ consists of $f : X' \rightarrow X$ of schemes over k such that $f^{-1}(Z) \subset Z'$.
- Let k a field of characteristic zero. For $X, X' \in \text{Var}(k)$, with X' irreducible, we denote $\mathcal{Z}^{fs/X'}(X' \times X) \subset \mathcal{Z}_{d_{X'}}(X' \times X)$ which consist of algebraic cycles $\alpha = \sum_i n_i \alpha_i \in \mathcal{Z}_{d_{X'}}(X' \times X)$ such that, denoting $\text{supp}(\alpha) = \cup_i \alpha_i \subset X' \times X$ its support and $p' : X' \times X \rightarrow X'$ the projection, $p'|_{\text{supp}(\alpha)} : \text{supp}(\alpha) \rightarrow X'$ is finite surjective. We denote by $\text{Cor SmVar}(k)$ the category such that the objects are $\{X \in \text{SmVar}(k)\}$ and such that $\text{Hom}_{\text{Cor SmVar}(k)}(X', X) := \mathcal{Z}^{fs/X'}(X' \times X)$.
- For $X \in \text{Top}$ and $Z \subset X$ a closed subset, denoting $j : X \setminus Z \hookrightarrow X$ the open complementary, we will consider

$$\Gamma_Z^\vee \mathbb{Z}_X := \text{Cone}(\text{ad}(j!, j^*)(\mathbb{Z}_X) : j!j^* \mathbb{Z}_X \hookrightarrow \mathbb{Z}_X) \in C(X)$$

and denote for short $\gamma_Z^\vee := \gamma_X^\vee(\mathbb{Z}_X) : \mathbb{Z}_X \rightarrow \Gamma_Z^\vee \mathbb{Z}_X$ the canonical map in $C(X)$.

- For $X \in \text{Var}(k)$ and $Z \subset X$ a closed subset, denoting $j : X \setminus Z \hookrightarrow X$ the open complementary, we will consider

$$\Gamma_Z^\vee \mathbb{Z}_X := \text{Cone}(\text{ad}(j_\sharp!, j^*)(\mathbb{Z}_X) : \mathbb{Z}_X \hookrightarrow \mathbb{Z}_X) \in C(\text{Var}(k)^{\text{sm}}/X)$$

and denote for short $\gamma_Z^\vee := \gamma_X^\vee(\mathbb{Z}_X) : \mathbb{Z}_X \rightarrow \Gamma_Z^\vee \mathbb{Z}_X$ the canonical map in $C(\text{Var}(k)^{\text{sm}}/X)$. Denote $a_X : X \rightarrow \text{Spec } k$ the structural map. For $X \in \text{Var}(k)$ and $Z \subset X$ a closed subset, we have the motive of X with support in Z defined as

$$M_Z(X) := a_{X!} \Gamma_Z^\vee a_X^! \mathbb{Z} \in \text{DA}(k).$$

If $X \in \text{SmVar}(k)$, we will also consider

$$a_{X\sharp} \Gamma_Z^\vee \mathbb{Z}_X := \text{Cone}(a_{X\sharp} \circ \text{ad}(j_\sharp!, j^*)(\mathbb{Z}_X) : \mathbb{Z}(U) \hookrightarrow \mathbb{Z}(X)) =: \mathbb{Z}(X, X \setminus Z) \in C(\text{SmVar}(k)).$$

Then for $X \in \text{SmVar}(k)$ and $Z \subset X$ a closed subset

$$M_Z(X) := a_{X!} \Gamma_Z^\vee a_X^! \mathbb{Z} \mathbb{Z}_X = a_{X\sharp} \Gamma_Z^\vee \mathbb{Z}_X =: \mathbb{Z}(X, X \setminus Z) \in \text{DA}(k).$$

- We denote $\mathbb{I}^n := [0, 1]^n \in \text{Diff}(\mathbb{R})$ (with boundary). For $X \in \text{Top}$ and R a ring, we consider its singular cochain complex

$$C_{\text{sing}}^*(X, R) := (\mathbb{Z} \text{Hom}_{\text{Top}}(\mathbb{I}^*, X)^\vee) \otimes R$$

and for $l \in \mathbb{Z}$ its singular cohomology $H_{\text{sing}}^l(X, R) := H^n C_{\text{sing}}^*(X, R)$. In particular, we get by functoriality the complex

$$C_{X, R \text{ sing}}^* \in C_R(X), (U \subset X) \mapsto C_{\text{sing}}^*(U, R)$$

We will consider the canonical embedding

$$C^* \iota_{2i\pi\mathbb{Z}/\mathbb{C}}(X) : C_{\text{sing}}^*(X, 2i\pi\mathbb{Z}) \hookrightarrow C_{\text{sing}}^*(X, \mathbb{C}), \alpha \mapsto \alpha \otimes 1$$

whose image consists of cochains $\alpha \in C_{\text{sing}}^j(X, \mathbb{C})$ such that $\alpha(\gamma) \in 2i\pi\mathbb{Z}$ for all $\gamma \in \mathbb{Z} \text{Hom}_{\text{Top}}(\mathbb{I}^*, X)$. We get by functoriality the embedding in $C(X)$

$$\begin{aligned} C^* \iota_{2i\pi\mathbb{Z}/\mathbb{C}, X} : C_{X, 2i\pi\mathbb{Z}, \text{sing}}^* &\hookrightarrow C_{X, \mathbb{C}, \text{sing}}^*, \\ (U \subset X) \mapsto (C^* \iota_{2i\pi\mathbb{Z}/\mathbb{C}}(U) : C_{\text{sing}}^*(U, 2i\pi\mathbb{Z}) &\hookrightarrow C_{\text{sing}}^*(U, \mathbb{C})) \end{aligned}$$

We recall we have

- For $X \in \text{Top}$ locally contractile, e.g. $X \in \text{CW}$, and R a ring, the inclusion in $C_R(X)$ $c_X : R_X \rightarrow C_{X, R, \text{sing}}^*$ is by definition an equivalence top local and that we get by the small chain theorem, for all $l \in \mathbb{Z}$, an isomorphism $H^l c_X : H^l(X, R_X) \xrightarrow{\sim} H_{\text{sing}}^l(X, R)$.
- For $X \in \text{Diff}(\mathbb{R})$, the restriction map

$$r_X : \mathbb{Z} \text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^*, X)^\vee \rightarrow C_{\text{sing}}^*(X, R), w \mapsto w : (\phi \mapsto w(\phi))$$

is a quasi-isomorphism by Whitney approximation theorem.

- We will use the following theorem of Voevodsky (see e.g. [5]): Let k a perfect field (e.g. k a field of characteristic zero). Let $F \in \text{PSh}(\text{SmVar}(k))$. If F is \mathbb{A}^1 invariant and admits transfers, then for all $j \in \mathbb{Z}$, $H^j E_{\text{et}}^\bullet(F) \in \text{PSh}(\text{SmVar}(k))$ are \mathbb{A}^1 invariant. That is, if F is \mathbb{A}^1 invariant and admits transfers then F is \mathbb{A}^1 local.

We introduce the logarithmic De Rham complexes

Definition 1. (i) Let $X = (X, O_X) \in \text{RCat}$ a ringed topos, we have in $C(X)$ the subcomplex of presheaves of abelian groups

$$\begin{aligned} OL_X : \Omega_{X, \log}^\bullet &\hookrightarrow \Omega_X^\bullet, \text{ s.t. for } X^\circ \in X \text{ and } p \in \mathbb{N}, p \geq 1, \\ \Omega_{X, \log}^p(X^\circ) &:= \langle df_{\alpha_1}/f_{\alpha_1} \wedge \cdots \wedge df_{\alpha_p}/f_{\alpha_p}, f_{\alpha_k} \in \Gamma(X^\circ, O_X)^* \rangle \subset \Omega_X^p(X^\circ), \end{aligned}$$

where $\Omega_X^\bullet := DR(X)(O_X) \in C(X)$ is the De Rham complex and $\Gamma(X^\circ, O_X)^* \subset \Gamma(X^\circ, O_X)$ is the multiplicative subgroup consisting of invertible elements for the multiplication, here \langle, \rangle stand for the sub-abelian group generated by. By definition, for $w \in \Omega_X^p(X^\circ)$, $w \in \Omega_{X, \log}^p(X^\circ)$ if and only if there exists $(n_i)_{1 \leq i \leq s} \in \mathbb{Z}$ and $(f_{i, \alpha_k})_{1 \leq i \leq s, 1 \leq k \leq p} \in \Gamma(X^\circ, O_X)^*$ such that

$$w = \sum_{1 \leq i \leq s} n_i df_{i, \alpha_1}/f_{i, \alpha_1} \wedge \cdots \wedge df_{i, \alpha_p}/f_{i, \alpha_p} \in \Omega_X^p(X^\circ).$$

For $p = 0$, we set $\Omega_{X, \log}^0 := \mathbb{Z}$. Let $f : X' = (X', O_{X'}) \rightarrow X = (X, O_X)$ a morphism with $X, X' \in \text{RCat}$. Consider the morphism $\Omega(f) : \Omega_X^\bullet \rightarrow f_* \Omega_{X'}^\bullet$ in $C(X)$. Then, $\Omega(f)(\Omega_{X, \log}^\bullet) \subset f_* \Omega_{X', \log}^\bullet$.

(ii) For k a field, we get from (i), for $X \in \text{Var}(k)$, the embedding in $C(X)$

$$OL_X : \Omega_{X, \log}^\bullet \hookrightarrow \Omega_X^\bullet := \Omega_{X/k}^\bullet,$$

such that, for $X^\circ \subset X$ an open subset and $w \in \Omega_X^p(X^\circ)$, $w \in \Omega_{X, \log}^p(X^\circ)$ if and only if there exists $(n_i)_{1 \leq i \leq s} \in \mathbb{Z}$ and $(f_{i, \alpha_k})_{1 \leq i \leq s, 1 \leq k \leq p} \in \Gamma(X^\circ, O_X)^*$ such that

$$w = \sum_{1 \leq i \leq s} n_i df_{i, \alpha_1}/f_{i, \alpha_1} \wedge \cdots \wedge df_{i, \alpha_p}/f_{i, \alpha_p} \in \Omega_X^p(X^\circ),$$

and for $p = 0$, $\Omega_{X,\log}^0 := \mathbb{Z}$. Let k a field. We get an embedding in $C(\text{Var}(k))$

$$\begin{aligned} OL &: \Omega_{/k,\log}^\bullet \hookrightarrow \Omega_{/k}^\bullet, \text{ given by, for } X \in \text{Var}(k), \\ OL(X) &:= OL_X : \Omega_{/k,\log}^\bullet(X) := \Gamma(X, \Omega_{X,\log}^\bullet) \hookrightarrow \Gamma(X, \Omega_X^\bullet) =: \Omega_{/k}^\bullet(X) \end{aligned}$$

and its restriction to $\text{SmVar}(k) \subset \text{Var}(k)$.

(iii) Let K a field of characteristic zero which is complete for a p -adic norm. Let $X \in \text{Var}(K)$. For $X^\circ \in \text{Sch}/O_K$ an integral model of X , i.e. satisfying $X^\circ \otimes_{O_K} K = X$, we write for short

$$OL_X := OL_X \circ \iota : \Omega_{X^\circ,\log,\mathcal{O}}^\bullet := \Omega_{X^\circ,\text{et},\log}^\bullet \hookrightarrow \Omega_{X^\circ,\log}^\bullet \hookrightarrow \Omega_{X^\circ}^\bullet$$

the embedding of $C(X^\circ)$ consisting of integral logarithmic De Rham forms, with $\iota : \Omega_{X^\circ,\text{et},\log}^\bullet \hookrightarrow \Omega_{X^\circ,\log}^\bullet$. We will also consider the embeddings of $C(X^{\text{pet}})$

$$m \circ (OL_X \otimes I) : \Omega_{X^{\text{pet}},\log,\mathcal{O}}^\bullet \otimes \mathbb{Z}_p := \Omega_{X^{\text{pet}},\log,\mathcal{O}}^\bullet \otimes \underline{\mathbb{Z}}_p \rightarrow \Omega_{X^{\text{pet}}}^\bullet, (w \otimes \lambda_n)_{n \in \mathbb{N}} \mapsto (\lambda_n)_{n \in \mathbb{N}} \cdot w$$

and

$$OL_{\hat{X}^{(p)}} := (OL_{X^\circ/p^n})_{n \in \mathbb{N}} : \Omega_{\hat{X}^{(p)},\log,\mathcal{O}}^\bullet := \varprojlim_{n \in \mathbb{N}} \nu_X^* \Omega_{X^\circ,\text{et}/p^n,\log}^\bullet \hookrightarrow \Omega_{\hat{X}^{(p)}}^\bullet,$$

where we recall $c : \hat{X}^{(p)} \rightarrow X$ the morphism in RTop is given by the completion along the ideal generated by p . Note that the inclusion $\Omega_{X^\circ,\text{et},\log}^\bullet/p^n \subset \Omega_{X^\circ,\text{et}/p^n,\log}^\bullet$ is strict in general. Note that

$$\Omega_{X^{\text{pet}},\log,\mathcal{O}}^\bullet := \nu_X^* \Omega_{X^\circ,\log,\mathcal{O}}^\bullet \in C(X^{\text{pet}}),$$

but

$$\underline{\mathbb{Z}}_p := \varprojlim_{n \in \mathbb{N}} \nu_X^*(\mathbb{Z}/p^n \mathbb{Z})_{X^\circ} \in C(X^{\text{pet}}), \text{ and } \Omega_{\hat{X}^{(p)},\log,\mathcal{O}}^\bullet := \varprojlim_{n \in \mathbb{N}} \nu_X^* \Omega_{X^\circ,\text{et}/p^n,\log}^\bullet \in C(X^{\text{pet}})$$

are NOT the pullback of étale sheaves by ν_X . We denote by $\text{Var}(K)^{\text{pet}}$ the big pro-étale site and consider the embedding of $C(\text{Var}(K)^{\text{pet}})$

$$\begin{aligned} OL_{/K,\text{an}} &: \Omega_{/K,\log,\mathcal{O}}^{\bullet,\text{an}} \hookrightarrow \Omega_{/K}^{\bullet,\text{an}}, \text{ for } X \in \text{Var}(K), \\ OL_{/K,\text{an}}(X) &:= OL_{\hat{X}^{(p)}}(X) : \Omega_{\hat{X}^{(p)},\log,\mathcal{O}}^\bullet(\hat{X}^{(p)}) \rightarrow \Omega_{\hat{X}^{(p)}}^\bullet(\hat{X}^{(p)}) \end{aligned}$$

and its restriction to $\text{SmVar}(K)^{\text{pet}} \subset \text{Var}(K)^{\text{pet}}$.

3 De Rham logarithmic classes

3.1 The De Rham classes of algebraic cycles vs De Rham logarithmic classes

Let $X \in \text{Sch}$. Recall we have the canonical sub-complex $OL_X : \Omega_{X,\log}^\bullet \hookrightarrow \Omega_X^\bullet$ in $C(X^\circ)$ (c.f. definition 1). All the differential of $\Omega_{X,\log}^\bullet$ vanishes since by definition the logarithmic forms are closed. For $j \in \mathbb{Z}$, the De Rham logarithmic classes consist of the image

$$H^j OL_X(\mathbb{H}_{\text{et}}^j(X, \Omega_{X,\log}^\bullet)) \subset \mathbb{H}_{\text{et}}^j(X, \Omega_X^\bullet) = H_{DR}^j(X).$$

The filtered complex $\Gamma(X, E_{\text{et}}(\Omega_{X^\circ,\log}^\bullet, F_b)) \in C_{\text{fil}}(\mathbb{Z})$ has its differentials at the E_1 level vanishes since the logarithmic forms are closed, hence we have a canonical splitting

$$\mathbb{H}_{\text{et}}^j(X, \Omega_{X,\log}^\bullet) = \bigoplus_{0 \leq l \leq j} H_{\text{et}}^{j-l}(X, \Omega_{X,\log}^l).$$

Let $X \in \text{Sch}$ a noetherian scheme. We have by definition the exact sequence in $C(X^{et})$

$$0 \rightarrow F^* \rightarrow O_X^* \xrightarrow{dlog} \Omega_{X,\log} \rightarrow 0$$

where F is a prime field (i.e. $F = \mathbb{Q}$ or $F = \mathbb{Z}/p\mathbb{Z}$ for p a prime number). Hence $H_{et}^1(X, \Omega_{X,\log}) = H_{et}^1(X, O_X^*)$ and $H_{et}^q(X, \Omega_{X,\log}) = H_{et}^q(X, O_X^*) = 0$ for $q \leq 2$. Let $X \in \text{Sch}$ a noetherian proper scheme. We have $H^0(X, \Omega_{X,\log}^l) = 0$.

Let k a field of characteristic zero. We get (see definition 1) the embedding in $C(\text{SmVar}(k))$

$$\begin{aligned} OL : \Omega_{/k,\log}^\bullet &\hookrightarrow \Omega_{/k}^\bullet, \text{ given by, for } X \in \text{SmVar}(k), \\ OL(X) := OL_X : \Omega_{/k,\log}^\bullet(X) &:= \Gamma(X, \Omega_{X,\log}^\bullet) \hookrightarrow \Gamma(X, \Omega_X^\bullet) =: \Omega_{/k}^\bullet(X). \end{aligned}$$

We have also the sheaves

$$\begin{aligned} O_k, O_k^* &\in \text{PSh}(\text{SmVar}(k)), \quad X \in \text{SmVar}(k) \mapsto O_k(X) := O(X), \quad O_k^*(X) := O(X)^*, \\ (g : Y \rightarrow X) &\mapsto a_g(X) : O_X(X) \rightarrow O_Y(Y), \quad O_X(X)^* \rightarrow O_Y(Y)^* \end{aligned}$$

The sheaf $O_k^* \in \text{PSh}(\text{SmVar}(k))$ admits transfers : for $W \subset X' \times X$ with $X, X' \in \text{SmVar}(k)$ and W finite over X' and $f \in O(X)^*$, $W^*f := N_{W/X'}(p_X^*f)$ where $p_X : W \hookrightarrow X' \times X \rightarrow X$ is the projection and $N_{W/X'} : k(W)^* \rightarrow k(X')^*$ is the norm map. This gives transfers on $\Omega_{/k,\log}^1 \in \text{PSh}(\text{SmVar}(k))$ compatible with transfers on $\Omega_{/k}^1 \in \text{PSh}(\text{SmVar}(k))$: for $W \subset X' \times X$ with $X, X' \in \text{SmVar}(k)$ and W finite over X' and $f \in O(X)^*$,

$$W^*df/f := dW^*f/W^*f = Tr_{W/X'}(p_X^*(df/f)),$$

where where $p_X : W \hookrightarrow X' \times X \rightarrow X$ is the projection and $Tr_{W/X'} : O_W \rightarrow O_X$ is the trace map. Note that $d(fg)/fg = df/f + dg/g$. We get transfers on

$$\otimes_{\mathbb{Q}}^l \Omega_{/k,\log}^1, \otimes_{O_k}^l \Omega_{/k}^1 \in \text{PSh}(\text{SmVar}(k))$$

since $\otimes_{\mathbb{Q}}^l \Omega_{/k,\log}^1 = H^0(\otimes_{\mathbb{Q}}^{L,l} \Omega_{/k,\log}^1)$ and $\otimes_{O_k}^l \Omega_{/k}^1 = H^0(\otimes_{O_k}^{L,l} \Omega_{/k}^1)$. This induces transfers on

$$\wedge_{\mathbb{Q}}^l \Omega_{/k,\log}^1 := \text{coker}(\oplus_{I_2 \subset [1,\dots,l]} \otimes_{\mathbb{Q}}^{l-1} \Omega_{/k,\log}^1 \xrightarrow{\oplus_{I_2 \subset [1,\dots,l]} \Delta_{I_2} := (w \otimes w' \mapsto w \otimes w \otimes w')} \otimes_{\mathbb{Q}}^l \Omega_{/k,\log}^1) \in \text{PSh}(\text{SmVar}(k)).$$

and

$$\wedge_{O_k}^l \Omega_{/k}^1 := \text{coker}(\oplus_{I_2 \subset [1,\dots,l]} \otimes_{O_k}^{l-1} \Omega_{/k}^1 \xrightarrow{\oplus_{I_2 \subset [1,\dots,l]} \Delta_{I_2} := (w \otimes w' \mapsto w \otimes w \otimes w')} \otimes_{O_k}^l \Omega_{/k}^1) \in \text{PSh}(\text{SmVar}(k)).$$

Let $X \in \text{Var}(k)$. We have by definition the exact sequence in $C(X^{et})$

$$0 \rightarrow k^* \rightarrow O_X^* \xrightarrow{dlog} \Omega_{X,\log} \rightarrow 0$$

Hence $H_{et}^1(X, \Omega_{X,\log}) = H_{et}^1(X, O_X^*)$ and $H_{et}^q(X, \Omega_{X,\log}) = H_{et}^q(X, O_X^*) = 0$ for $q \leq 2$. For $X \in \text{PVar}(k)$, we have $H^0(X, \Omega_{X,\log}^l) = 0$.

- Let $(X, Z) \in \text{Sch}^2$ with $X \in \text{Sch}$ a noetherian scheme and $Z \subset X$ a closed subset. We have the deformation $(D_Z X, \mathbb{A}_Z^1) \rightarrow \mathbb{A}^1$, $(D_Z X, \mathbb{A}_Z^1) \in \text{Sch}^2$ of (X, Z) by the normal cone $C_{Z/X} \rightarrow Z$, i.e. such that

$$(D_Z X, \mathbb{A}_Z^1)_s = (X, Z), \quad s \in \mathbb{A}^1 \setminus 0, \quad (D_Z X, \mathbb{A}_Z^1)_0 = (C_{Z/X}, Z).$$

We denote by $i_1 : (X, Z) \hookrightarrow (D_Z X, \mathbb{A}_Z^1)$ and $i_0 : (C_{Z/X}, Z) \hookrightarrow (D_Z X, \mathbb{A}_Z^1)$ the closed embeddings in Sch^2 .

- Let k a field of characteristic zero. Let $X \in \text{SmVar}(k)$. For $Z \subset X$ a closed subset of pure codimension c , consider a desingularisation $\epsilon : \tilde{Z} \rightarrow Z$ of Z and denote $n : \tilde{Z} \xrightarrow{\epsilon} Z \subset X$. We have then the morphism in $\text{DA}(k)$

$$G_{Z,X} : M(X) \xrightarrow{D(\mathbb{Z}(n)_{BM})} M(\tilde{Z})(c)[2c] \xrightarrow{\mathbb{Z}(\epsilon)} M(Z)(c)[2c]$$

where $D : \text{Hom}_{\text{DA}(k)}(M_c(\tilde{Z}), M_c(X)) \xrightarrow{\sim} \text{Hom}_{\text{DA}(k)}(M(X), M(\tilde{Z})(c)[2c])$ is the duality isomorphism from the six functors formalism (moving lemma of Suzlin and Voevodsky) and $\mathbb{Z}(n) := \text{ad}(n_!, n^!)(a_X^! \mathbb{Z})$, noting that $n_! = n_*$ since n is proper and that $a_X^! = a_X^*[d_X]$ and $a_{\tilde{Z}}^! = a_{\tilde{Z}}^*[d_{\tilde{Z}}]$ since X , resp. \tilde{Z} , are smooth (considering the connected components, we may assume X and \tilde{Z} of pure dimension).

We recall the following facts (see [5] and [3]):

Proposition 1. *Let k a field a characteristic zero. Let $X \in \text{SmVar}(k)$ and $i : Z \subset X$ a smooth subvariety of pure codimension d . Then $C_{Z/X} = N_{Z/X} \rightarrow Z$ is a vector bundle of rank d .*

- (i) *The closed embeddings $i_1 : (X, Z) \hookrightarrow (D_Z X, \mathbb{A}_Z^1)$ and $i_0 : (C_{Z/X}, Z) \hookrightarrow (D_Z X, \mathbb{A}_Z^1)$ in $\text{SmVar}^2(k)$ induces isomorphisms of motives $\mathbb{Z}(i_1) : M_Z(X) \xrightarrow{\sim} M_{\mathbb{A}_Z^1}(D_Z X)$ and $\mathbb{Z}(i_0) : M_Z(N_{Z/X}) \xrightarrow{\sim} M_{\mathbb{A}_Z^1}(D_Z X)$ in $\text{DA}(k)$. We get the excision isomorphism in $\text{DA}(k)$*

$$P_{Z,X} := \mathbb{Z}(i_0)^{-1} \circ \mathbb{Z}(i_1) : M_Z(X) \xrightarrow{\sim} M_Z(N_{Z/X}).$$

We have

$$\text{Th}(N_{Z/X}) \circ P_{Z,X} \circ \gamma_Z^\vee(\mathbb{Z}_X) = G_{Z,X} := D(\mathbb{Z}(i)) : M(X) \rightarrow M(Z)(d)[2d].$$

- (ii) *For each $j \in \mathbb{Z}$, the excision isomorphism induced by (i)*

$$\begin{aligned} H^j \Omega(P_{Z,X}) &:= H^j E_{\text{et}}(\Omega_{/k}^\bullet)(\mathbb{Z}(i_0)) \circ H^j E_{\text{et}}(\Omega_{/k}^\bullet)(\mathbb{Z}(i_1))^{-1} : \\ H_{\text{DR},Z}^j(X) &:= H^j \Gamma_Z(X, E_{\text{et}}(\Omega_X^\bullet)) \xrightarrow{\sim} H_{\text{DR},Z}^j(N_{Z/X}) := H^j \Gamma_Z(N_{Z/X}, E_{\text{et}}(\Omega_{N_{Z/X}}^\bullet)) \end{aligned}$$

preserve logarithmic De Rham classes, that is for each $l \in \mathbb{Z}$,

$$H^j \Omega(P_{Z,X})(H^j OL_X(H_{\text{et},Z}^{j-l}(X, \Omega_{X,\log}^l))) = H^j OL_{N_{Z/X}}(H_{\text{et},Z}^{j-l}(N_{Z/X}, \Omega_{N_{Z/X},\log}^l))$$

Proof. (i): See [5].

(ii): The de Rham complex $\Omega_{/k}^\bullet \in C(\text{SmVar}(k))$ admits transfers using trace map for commutative rings, that is $\Omega^\bullet \in C(\text{Cor SmVar}(k))$ (see [7]) and the inclusion $OL : \Omega_{/k,\log}^l[-l] \hookrightarrow \Omega_{/k}^\bullet$ in $C(\text{SmVar}(k))$ is compatible with transfers. Since $\Omega_{/k,\log}^l \in \text{PSh}(\text{SmVar}(k))$ is \mathbb{A}_k^1 invariant ($O_{\mathbb{A}_k^1}^* = k^*$ and for $\lambda \in k^*$ we have $d\lambda = 0$) and admits transfers, for each $j \in \mathbb{Z}$, the presheaves $H_{\text{et}}^{j-l} \Omega_{/k,\log}^l \in \text{PSh}(\text{SmVar}(k))$ are \mathbb{A}_k^1 invariant by a theorem of Voevodsky (c.f. [5] for example). It then follows from (i) (see [5] or [3]) that

$$\begin{aligned} H^j \Omega_{\log}(P_{Z,X}) &:= H^j E_{\text{et}}(\Omega_{/k,\log}^\bullet)(\mathbb{Z}(i_0)) \circ H^j E_{\text{et}}(\Omega_{/k,\log}^\bullet)(\mathbb{Z}(i_1))^{-1} : \\ &H_{\text{et},Z}^{j-l}(X, \Omega_{X,\log}^l) \xrightarrow{\sim} H_{\text{et},Z}^{j-l}(N_{Z/X}, \Omega_{N_{Z/X},\log}^l). \end{aligned}$$

Hence,

$$\begin{aligned} H^j \Omega(P_{Z,X})(H^j OL_X(H_{\text{et},Z}^{j-l}(X, \Omega_{X,\log}^l))) &= H^j OL_{N_{Z/X}}(\Omega_{\log}(P_{Z,X})(H_{\text{et},Z}^{j-l}(X, \Omega_{X,\log}^l))) \\ &= H^j OL_{N_{Z/X}}(H_{\text{et},Z}^{j-l}(N_{Z,X}, \Omega_{N_{Z/X},\log}^l)). \end{aligned}$$

□

Remark 1. Let k a field a characteristic zero.

(i) The statement of proposition 1(ii) also holds for the subcomplex $\Omega_{/k}^{\bullet, \partial=0} \hookrightarrow \Omega_{/k}^{\bullet}$ of closed forms by the same argument since the inclusion $\Omega_{/k}^{l, \partial=0}[-l] \hookrightarrow \Omega_{/k}^{\bullet}$ in $C(\text{SmVar}(k))$ is compatible with transfers.

(ii) The statement of proposition 1(ii) does NOT hold for presheaves which do not admits transfers (they are \mathbb{A}^1 invariant but not \mathbb{A}^1 local in general). Note that the result of proposition 1(ii) does NOT hold for the embedding in $C(\text{SmVar}(k))$, associated to an embedding $\sigma : k \hookrightarrow \mathbb{C}$,

$$\mathcal{H}^{l, j-l}[-l] \hookrightarrow \mathcal{A}^{\bullet}, X \in \text{SmVar}(k), \Gamma(X, \mathcal{H}_{X_{\mathbb{C}^n}}^{l, j-l})[-l] \hookrightarrow \Gamma(X, \mathcal{A}_{X_{\mathbb{C}^n}}^{\bullet})$$

of the subsheaf of harmonic differential forms, the sheaves of differential forms $\mathcal{A} \in C(\text{SmVar}(k))$ does NOT admits transfers (finite algebraic correspondences are not smooth and for $f : X' \rightarrow X$ a morphism with $X', X \in \text{Diff}(\mathbb{R})$ vector fields on X only lift if f is a smooth morphism i.e. if the differential of df is surjective), transfers maps are only defined on cohomology.

Let k a field of characteristic zero. Let $X \in \text{SmVar}(k)$ and $Z \subset X$ a smooth subvariety of pure codimension d . We have for each $j \in \mathbb{Z}$, the purity isomorphism given by $H^j \Omega(P_{Z, X})$ (see proposition 1) and the cup product with the Euler class of of the normal tangent bundle $N_{Z/X} \rightarrow Z$:

$$H^j DR(P_{Z, X}) : H_{DR, Z}^j(X) \xrightarrow{H^j \Omega(P_{Z, X})} H_{DR, Z}^j(N_{Z/X}) \xrightarrow{((-) \cdot e(N_{Z/X}))^{-1}} H_{DR}^{j-2d}(Z).$$

Now, we have the following :

Proposition 2. Let k a field of characteristic zero. Let $p : E \rightarrow X$ a vector bundle of rank $d \in \mathbb{N}$ with $X, E \in \text{SmVar}(k)$ connected. Then

(i) the Euler class $e(E) \in H_{DR, X}^{2d}(E)$ is logarithmic of type (d, d) , that is $e(E) = H^{2d} OL_E(e(E)) \in H_{DR, X}^{2d}(E)$ with $e(E) \in H_{X, et}^d(E, \Omega_{E, \log}^d)$,

(ii) the Euler class $e(E) \in H_{DR, X}^{2d}(E)$ induces for each $i, j \in \mathbb{Z}$ an isomorphism

$$((-) \cdot e(E)) : H^{i+j} OL_X(H_{et}^j(X, \Omega_{X, \log}^i)) \xrightarrow{\sim} H^{2d+i+j} OL_E(H_{et}^{j+d}(E, \Omega_{E, \log}^{i+d})).$$

Proof. (i): Let $X = \cup_i X_i$ an open affine cover such that $E_i := E|_{X_i}$ is trivial : $(s_1, \dots, s_d) : E|_{X_i} \xrightarrow{\sim} X_i \times \mathbb{A}^d$ with $s_j \in \Gamma(X_i, E)$. Then

$$e(E)|_{X_i} = \partial(ds_1/s_1 \wedge \dots \wedge ds_d/s_d) \in H^{2d} OL_{E_i}(H_{X_i}^d(E_i, \Omega_{E, \log}^d))$$

where $\partial : H^{d-1}(E_i \setminus X_i, \Omega_{E, \log}^d) \rightarrow H_{X_i}^d(E_i, \Omega_{E, \log}^d)$ is the boundary map. Hence

$$e(E) \in H^{2d} OL_E(H^0(E, H_X^d \Omega_{E, \log}^d)) = H_{et, X}^d(E, \Omega_{E, \log}^d) \subset H^0(E, H_X^{2d} \Omega_E^{\bullet \geq d}) \subset H_{DR, X}^{2d}(E).$$

(ii): Follows from (i) and Kunneth formula for De Rham cohomology : let $X = \cup_i X_i$ an open affine cover such that $E_i := E|_{X_i}$ is trivial and consider the morphism of bi-complexes of abelian groups

$$((-) \cdot e(E)) : OL_X(\Gamma(X_{\bullet}, E_{et}(\Omega_{X, \log}^i))) \rightarrow OL_E(\Gamma(E_{\bullet}, E_{et}(\Omega_{E, \log}^{i+d}))) [2d].$$

By (i) an Kunneth formula for De Rham cohomology, $((-) \cdot e(E))$ induces an isomorphism on the E_1 level of the spectral sequences for the filtrations associated to bi-complex structures on the total complexes. Hence, $((-) \cdot e(E))$ is a quasi-isomorphism. This proves (ii). \square

We deduce from proposition 1 and proposition 2, the following key proposition

Proposition 3. *Let k a field of characteristic zero. Let $X \in \text{SmVar}(k)$ and $Z \subset X$ a smooth subvariety of codimension d . For each $j \in \mathbb{Z}$, the purity isomorphism*

$$H^j DR(P_{Z,X}) : H^j_{DR,Z}(X) \xrightarrow{H^j \Omega(P_{Z,X})} H^j_{DR,Z}(N_{Z/X}) \xrightarrow{((-) \cdot e(N_{Z/X}))^{-1}} H^{j-2d}_{DR}(Z)$$

preserve logarithmic De Rham classes, that is for each $l \in \mathbb{Z}$,

$$H^j DR(P_{Z,X})(H^j OL_X(H^{j-l}_{et,Z}(X, \Omega^l_{X,\log}))) = H^j OL_Z(H^{j-l}_{et}(Z, \Omega^l_{Z,\log})).$$

Proof. Follows from proposition 1(ii) and proposition 2(ii). \square

Let k a field of characteristic zero. For $X \in \text{SmVar}(k)$, we have the cycle class map for De Rham cohomology

$$\begin{aligned} Z \in \mathcal{Z}^d(X) &\mapsto [Z] := H^{2d} \Omega(\gamma_Z^\vee)([Z]) \subset H^{2d}_{DR}(X), [Z] \in H^{2d}_{DR,|Z|}(X), \\ &H^{2d} \Omega(\gamma_Z^\vee) : H^{2d}_{DR,|Z|}(X) \rightarrow H^{2d}_{DR}(X) \end{aligned}$$

where

$$\Omega(\gamma_Z^\vee) := \text{Hom}(\gamma_Z^\vee(\mathbb{Z}_X), E_{et}(\Omega_X^\bullet)) : \Gamma_Z(X, E_{et}(\Omega_X^\bullet)) \hookrightarrow \Gamma(X, E_{et}(\Omega_X^\bullet))$$

and, as for any Weil cohomology theory, we have the isomorphism given by purity : if $X^\circ \subset X$ is an open subset such that $Z^\circ := X^\circ \cap Z$ has smooth components we have the isomorphism

$$H^{2d} DR(P_{Z^\circ, X^\circ}) : H^{2d}_{DR,|Z|}(X) = H^{2d}_{DR,|Z^\circ|}(X^\circ) = H^{2d}_{DR,|Z^\circ|}(N_{Z^\circ/X^\circ}) \xrightarrow{\sim} H^0_{DR}(|Z^\circ|).$$

where the first equality follows from dimension reason : for $X \in \text{SmVar}(k)$ and $Z' \subset X$ smooth, we have $H^i_{DR,Z'}(X) = 0$ for $i < 2 \text{codim}(Z', X)$ by the purity isomorphism $H^i DR(P_{Z',X})$.

The main result of this section is the following :

Theorem 1. *Let k a field of characteristic zero. Let $X \in \text{SmVar}(k)$. Let $d \in \mathbb{N}$.*

(i) *The De Rham cohomology class of an algebraic cycle is logarithmic and is of type (d, d) , that is, for $Z \in \mathcal{Z}^d(X)$*

$$[Z] := H^{2d} \Omega(\gamma_Z^\vee)([Z]) \subset H^{2d} OL_X(H^d_{et}(X, \Omega^d_{X,\log})) \subset H^{2d}_{DR}(X).$$

(ii) *Conversely, any $w \in H^{2d} OL_X(H^d_{et}(X, \Omega^d_{X,\log}))$ is the class of an algebraic cycle. Note that it implies $w \in W_0 H^{2d}_{DR}(X)$ as it is easily seen for $d = 1$ by the Chow moving lemma since $j^* : \text{Pic}(\bar{X}) \rightarrow \text{Pic}(X)$ is surjective if $j : X \hookrightarrow \bar{X}$ is a smooth compactification of X , $\bar{X} \in \text{PSmVar}(k)$ and since $H^1_{et}(Y, \Omega^1_{Y,\log}) = \text{Pic}(Y)$ for $Y \in \text{Var}(k)$.*

(iii) *We have $H^j OL_X(H^{j-l}_{et}(X, \Omega^l_{X,\log})) = 0$ for $j, l \in \mathbb{Z}$ such that $2l < j$.*

(iii)' *If $X \in \text{PSmVar}(k)$, we also have $H^j OL_X(H^{j-l}_{et}(X, \Omega^l_{X,\log})) = 0$ for $j, l \in \mathbb{Z}$ such that $2l > j$. That is, if $X \in \text{PSmVar}(k)$, we only have logarithmic classes in bidegree (d, d) for $d \in \mathbb{N}$, in particular there is no non trivial logarithmic classes for odd degree De Rham cohomology $H^{2d+1}_{DR}(X)$.*

Proof. (i): We have

$$[Z] := H^{2d} \Omega(\gamma_Z^\vee)([Z]) = H^{2d} OL_X \circ H^d \Omega^d_{\log}(\gamma_Z^\vee)([Z]) \subset H^{2d}_{DR}(X), [Z] \in H^d_{et,|Z|}(X, \Omega^d_{X,\log})$$

where

$$\Omega^d_{\log}(\gamma_Z^\vee) := \text{Hom}(\gamma_Z^\vee(\mathbb{Z}_X), E_{et}(\Omega^d_{X,\log})) : \Gamma_Z(X, E_{et}(\Omega^d_{X,\log})) \hookrightarrow \Gamma(X, E_{et}(\Omega^d_{X,\log})),$$

since if $X^\circ \subset X$ is a Zariski open subset such that $Z^\circ := X^\circ \cap Z$ has smooth components and $N_{Z^\circ/X^\circ} \rightarrow Z^\circ$ is the normal tangent bundle, we have by proposition 3 the isomorphism

$$\begin{aligned} H^{2d} DR(P_{Z^\circ, X^\circ}) : H^{2d} OL_X(H^d_{et,|Z|}(X, \Omega^d_{X,\log})) &= H^{2d} OL_X(H^d_{et,|Z^\circ|}(X^\circ, \Omega^d_{X^\circ,\log})) \\ &= H^{2d} OL_X(H^d_{et,|Z^\circ|}(N_{Z^\circ/X^\circ}, \Omega^d_{N_{Z^\circ/X^\circ},\log})). \end{aligned}$$

(ii):By assumption we have

$$w = H^{2d}OL_X(w) \in H_{DR}^{2d}(X), \quad w \in H_{et}^d(X, \Omega_{X, \log}^d)$$

As $\Omega_{X, \log}^d \in \text{PSh}(X^{et})$ consist of a single presheaf, that is a complex of presheaves concentrated one degree, there exist an etale open cover $r = r(w) = (r_i : X_i \rightarrow X)_{1 \leq i \leq s}$, depending on w , such that $r_i^*w = 0 \in H_{et}^d(X_i, \Omega_{X, \log}^d)$ for each i . Choose $i = 1$ and denote $j : U := r_1(X_1) \hookrightarrow X$ the corresponding (Zariski) open subset. As $\Omega_{X, \log}^d$ has no torsion and admits transfers, we have $j^*w = 0 \in H_{et}^d(U, \Omega_{X, \log}^d)$. Hence, denoting $D := X \setminus U$, we have

$$w = H^{2d}OL_X(w), \quad w \in H^d \Omega_{\log}^d(\gamma_D^\vee)(H_{et, D}^d(X, \Omega_{X, \log}^d)).$$

We may assume, up to shrinking U , that $D \subset X$ is a divisor. Denote $D^\circ \subset D$ its smooth locus and $l : X^\circ \hookrightarrow X$ a Zariski open subset such that $X^\circ \cap D = D^\circ$. We then have by proposition 3

$$l^*w \in H^{2d}DR(P_{D^\circ, X^\circ})(H^{2d-2}OL_{D^\circ}(H_{et}^{d-1}(D^\circ, \Omega_{D^\circ, \log}^{d-1}))).$$

We repeat this procedure with each connected components of D° instead of X . By a finite induction of d steps, we get

$$w = H^{2d}OL_X(w), \quad w \in H^d \Omega_{\log}^d(\gamma_Z^\vee)(H_{et, Z}^d(X, \Omega_{X, \log}^d)),$$

with $Z := D_d \subset \dots \subset D \subset X$ a pure codimension d (Zariski) closed subset and $w = \sum_i n_i [Z_i] \in H_{DR}^{2d}(X)$, where $n_i \in \mathbb{Z}$ and $(Z_i)_{1 \leq i \leq t} \subset Z$ are the irreducible components of Z , using in the final step the isomorphism

$$H_{et}^1(D_{d-1}^\circ, \Omega_{D_{d-1}^\circ, \log}) = H_{et}^1(D_{d-1}^\circ, \mathcal{O}_{D_{d-1}^\circ}^*) = \text{Pic}(D_{d-1}^\circ).$$

(iii):Let $j > 2l$. Let $w \in H^j OL_X(H_{et}^{j-l}(X, \Omega_{X, \log}^l))$. By the proof of (ii) there exists $Z \subset X$ a closed subset of pure codimension l such that

$$w = H^j OL_X(w), \quad w \in H^{j-l} \Omega_{\log}^l(\gamma_Z^\vee)(H_{et, Z}^{j-l}(X, \Omega_{X, \log}^l)),$$

By a finite induction of $d_X - l$ steps, restricting to the smooth locus of closed subsets of Z , $w = 0$ since by proposition 1(i), $H^j OL_X(H_{et, Z'}^{j-l}(X, \Omega_{X, \log}^l)) = 0$ for all $Z' \in \text{SmVar}(k)$ such that $\text{codim}(Z', X) \geq l$.

(iii)':Let $j < 2l$. Let $w \in H^j OL_X(H_{et}^{j-l}(X, \Omega_{X, \log}^l))$. By the proof of (ii) there exists $Z \subset X$ a closed subset of pure codimension $j - l$ such that

$$w = (H^j OL_X \circ H^{j-l} \Omega_{\log}^l(\gamma_Z^\vee))(w), \quad w \in H_{et, Z}^{j-l}(X, \Omega_{X, \log}^l),$$

For $Z' \subset X$ a closed subset of pure codimension c , consider a desingularisation $\epsilon : \tilde{Z}' \rightarrow Z'$ of Z' and denote $n : \tilde{Z}' \xrightarrow{\epsilon} Z' \subset X$. The morphism in $\text{DA}(k)$

$$G_{Z', X} : M(X) \xrightarrow{D(\mathbb{Z}(n))} M(\tilde{Z}')(c)[2c] \xrightarrow{\mathbb{Z}(\epsilon)} M(Z')(c)[2c]$$

where $D : \text{Hom}_{\text{DA}(k)}(M_c(\tilde{Z}'), M_c(X)) \xrightarrow{\sim} \text{Hom}_{\text{DA}(k)}(M(X), M(\tilde{Z}')(c)[2c])$ is the duality isomorphism from the six functor formalism (moving lemma of Suzlin and Voevodsky) and $\mathbb{Z}(n) := \text{ad}(n_l, n^l)(a_X^! \mathbb{Z})$, is given by a morphism in $C(\text{SmVar}(k))$

$$\hat{G}_{Z', X} : \mathbb{Z}_{tr}(X) \rightarrow E_{et}(C_* \mathbb{Z}_{tr}(Z'))(c)[2c].$$

Let $l : X^\circ \hookrightarrow X$ be an open embedding such that $Z^\circ := Z \cap X^\circ$ is the smooth locus of Z . We then have the following commutative diagram of abelian groups

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{et, Z}^{j-l}(X, \Omega_{X, \log}^l) & \xrightarrow{l^*} & H_{et, Z^\circ}^{j-l}(X^\circ, \Omega_{X, \log}^l) & \xrightarrow{\partial} & H_{et, Z \setminus Z^\circ}^{j-l+1}(X, \Omega_{X, \log}^l) & \longrightarrow & \dots \\ & & \Omega(\hat{G}_{Z, X}) \uparrow & & DR(P_{Z^\circ, X^\circ}) \uparrow & & \Omega(\hat{G}_{Z, X}) \uparrow & & \\ 0 & \longrightarrow & H_{et}^0(Z, \Omega_{Z, \log}^{2l-j}) & \xrightarrow{l^*} & H_{et}^0(Z^\circ, \Omega_{Z^\circ, \log}^{2l-j}) & \xrightarrow{\partial} & H_{et, Z \setminus Z^\circ}^1(Z, \Omega_{Z, \log}^{2l-j}) & \longrightarrow & \dots \end{array}$$

whose rows are exact sequences. Consider

$$l^*w = DR(P_{Z^o, X^o})(w^o) \in H_{et, Z^o}^{j-l}(X^o, \Omega_{X, \log}^l), \quad w^o \in H_{et}^0(Z^o, \Omega_{Z^o, \log}^{2l-j}).$$

Since $\partial l^*w = 0 \in H_{et, Z \setminus Z^o}^{j-l+1}(X, \Omega_{X, \log}^l)$, we get using proposition 3 applied to $((Z \setminus Z^o)^o, X^{oo})$, where $(Z \setminus Z^o)^o \subset Z \setminus Z^o$ is the smooth locus and $X^{oo} \subset X$ is an open subset such that $X \cap (Z \setminus Z^o)^o = X^{oo} \cap (Z \setminus Z^o)$,

$$\partial w^o = 0 \in H_{et, Z \setminus Z^o}^1(Z, \Omega_{Z, \log}^{2l-j}),$$

since $H_{et, (Z \setminus Z^o) \setminus (Z \setminus Z^o)^o}^1(Z, \Omega_{Z, \log}^{2l-j}) = 0$ for dimension reasons, that is

$$w = \Omega(\hat{G}_{Z, X})(w), \quad \text{with } w \in H_{et}^0(Z, \Omega_{Z, \log}^{2l-j}).$$

Hence $w = 0$ since $H^0(Z', \Omega_{Z', \log}^k) = 0$ for all $k > 0$ and all $Z' \in \text{PVar}(k)$. \square

3.2 Complex integral periods

Let k a field of characteristic zero.

Let $X \in \text{SmVar}(k)$ a smooth variety. Let $X = \cup_{i=1}^s X_i$ an open affine cover. We have for $\sigma : k \hookrightarrow \mathbb{C}$ an embedding, the evaluation period embedding map which is the morphism of bi-complexes

$$ev(X) \bullet : \Gamma(X_\bullet, \Omega_{X_\bullet}^\bullet) \rightarrow \mathbb{Z} \text{Hom}_{\text{Diff}}(\mathbb{I}^\bullet, X_{\mathbb{C}, \bullet}^{an})^\vee \otimes \mathbb{C},$$

$$w_I^l \in \Gamma(X_I, \Omega_{X_I}^l) \mapsto (ev(X)_I^l(w_I^l) : \phi_I^l \in \mathbb{Z} \text{Hom}_{\text{Diff}}(\mathbb{I}^l, X_{\mathbb{C}, I}^{an})^\vee \otimes \mathbb{C} \mapsto ev_I^l(w_I^l)(\phi_I^l) := \int_{\mathbb{I}^l} \phi_I^{l*} w_I^l)$$

given by integration. By taking all the affine open cover $(j_i : X_i \hookrightarrow X)$ of X , we get for $\sigma : k \hookrightarrow \mathbb{C}$, the evaluation period embedding map

$$ev(X) := \varinjlim_{(j_i : X_i \hookrightarrow X)} ev(X) \bullet : \varinjlim_{(j_i : X_i \hookrightarrow X)} \Gamma(X_\bullet, \Omega_{X_\bullet}^\bullet) \rightarrow \varinjlim_{(j_i : X_i \hookrightarrow X)} \mathbb{Z} \text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^\bullet, X_{\mathbb{C}, \bullet}^{an})^\vee \otimes \mathbb{C}$$

It induces in cohomology, for $j \in \mathbb{Z}$, the evaluation period map

$$H^j ev(X) = H^j ev(X) \bullet : H_{DR}^j(X) = H^j \Gamma(X_\bullet, \Omega_{X_\bullet}^\bullet) \rightarrow H_{\text{sing}}^j(X_{\mathbb{C}}^{an}, \mathbb{C}) = H^j(\text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^\bullet, X_{\mathbb{C}, \bullet}^{an})^\vee \otimes \mathbb{C}).$$

which does NOT depend on the choice of the affine open cover by acyclicity of quasi-coherent sheaves on affine noetherian schemes for the left hand side and from Mayer-Vietoris quasi-isomorphism for singular cohomology of topological spaces and Whitney approximation theorem for differential manifolds for the right hand side.

Proposition 4. *Let k a field of characteristic zero. Let $X \in \text{SmVar}(k)$. Let $\sigma : k \hookrightarrow \mathbb{C}$ an embedding.*

(i) *Let $w \in H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_{X_\bullet}^\bullet) = \mathbb{H}_{pet}^j(X, \Omega_{X_\bullet}^\bullet)$. If*

$$w \in H^j OL_X(\mathbb{H}_{pet}^j(X_{\mathbb{C}}, \Omega_{X_{\mathbb{C}}^{et}, \log}^\bullet))$$

then $H^j ev(X)(w) \in H_{\text{sing}}^j(X_{\mathbb{C}}^{an}, 2i\pi\mathbb{Q})$.

(ii) *Let $p \in \mathbb{N}$ a prime number and $\sigma_p : k \hookrightarrow \mathbb{C}_p$ an embedding. Let $j \in \mathbb{Z}$. Let $w \in H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_{X_\bullet}^\bullet) = \mathbb{H}_{pet}^j(X, \Omega_{X_\bullet}^\bullet)$. If*

$$w := \pi_{k/\mathbb{C}_p}(X)^* w \in H^j OL_X(\mathbb{H}_{et}^j(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}^{et}, \log, \mathcal{O}}^\bullet))$$

then $H^j ev(X)(w) \in H_{\text{sing}}^j(X_{\mathbb{C}}^{an}, 2i\pi\mathbb{Q})$. Recall that $\mathbb{H}_{pet}^j(X_{\mathbb{C}}, \Omega_{X_{\mathbb{C}}^{pet}, \log, \mathcal{O}}^\bullet) = \mathbb{H}_{et}^j(X_{\mathbb{C}}, \Omega_{X_{\mathbb{C}}^{et}, \log, \mathcal{O}}^\bullet)$.

Proof. (i): Let

$$w \in H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^\bullet) = H^j\Gamma(X_\bullet, \Omega_X^\bullet).$$

where $(r_i : X_i \rightarrow X)_{1 \leq i \leq s}$ is an affine etale cover. Let $X_{\mathbb{C}}^{an} = \cup_{i=1}^r \mathbb{D}_i$ an open cover with $\mathbb{D}_i \simeq D(0, 1)^d$. Denote $j_{IJ} : X_I \cap \mathbb{D}_J \hookrightarrow X_I$ the open embeddings. Then by definition $H^j ev(X)(w) = H^j ev(X_{\mathbb{C}}^{an})(j_\bullet^* \circ \text{an}_{X_\bullet}^* w)$ with

$$j_\bullet^* \circ \text{an}_{X_\bullet}^* w \in H^j\Gamma(X_{\bullet, \mathbb{C}}^{an} \cap \mathbb{D}_\bullet, \Omega_{X_{\mathbb{C}}^{an}}^\bullet).$$

Now, if $w = H^j OL_X(\mathbb{H}_{et}^j(X_{\mathbb{C}}, \Omega_{X_{\mathbb{C}}^{et, \log}}^\bullet))$, we have a canonical splitting

$$w = \sum_{l=0}^j w_L^{l, j-l} = \sum_{l=d}^j w^{l, j-l} \in H_{DR}^j(X_{\mathbb{C}}), \quad w_L^{l, j-l} \in H^{j-l}(X_{\mathbb{C}}, \Omega_{X_{\mathbb{C}}^{et, \log}}^l), \quad w^{l, j-l} := H^j OL_{X_{\mathbb{C}}^{et}}(w_L^{l, j-l}).$$

Let $0 \leq l \leq j$. Using an affine w-contractile pro-etale cover of X , we see that there exists an affine etale cover $r = r(w^{l, j-l}) = (r_i : X_i \rightarrow X)_{1 \leq i \leq n}$ of X (depending on $w^{l, j-l}$) such that

$$w^{l, j-l} = [(w_{L, I}^{l, j-l})] \in H^j OL_{X_{\mathbb{C}}^{et}}(H^{j-l}\Gamma(X_{\mathbb{C}, \bullet}, \Omega_{X_{\mathbb{C}, \log}}^l)) \subset \mathbb{H}^j\Gamma(X_{\mathbb{C}, \bullet}, \Omega_{X_{\mathbb{C}}}^\bullet).$$

Note that since X is an algebraic variety, this also follows from a comparison theorem between Chech cohomology of etale covers and etale cohomology. By [4] lemma 2, we may assume, up to take a desingularization $\pi : X' \rightarrow X$ of $(X, \cup_i (r_i(X \setminus X_i)))$ and replace w with $\pi^* w$, that $r_i(X_i) = r_i(X_i(w)) = X \setminus D_i$ with $D_i \subset X$ smooth divisors with normal crossing For $1 \leq l \leq j$, we get

$$w_{L, I}^{l, j-l} = \sum_{\nu} df_{\nu_1}/f_{\nu_1} \wedge \cdots \wedge df_{\nu_l}/f_{\nu_l} \in \Gamma(X_{\mathbb{C}, I}, \Omega_{X_{\mathbb{C}}}^l).$$

For $l = 0$, we get

$$w^{0, j} = [(\lambda_I)] \in H^j\Gamma(X_{\mathbb{C}, I}, \mathcal{O}_{X_{\mathbb{C}, I}}), \quad \lambda_I \in \Gamma(X_{\mathbb{C}, I}, \mathbb{Z}_{X_{\mathbb{C}}^{et}})$$

There exists $k' \subset \mathbb{C}$ containing k such that $w_{L, I}^{l, j-l} \in \Gamma(X_{k', I}, \Omega_{X_{k'}}^l)$ for all $0 \leq l \leq j$. Taking an embedding $\sigma' : k' \hookrightarrow \mathbb{C}$ such that $\sigma'|_k = \sigma$, we then have

$$j_\bullet^* \circ \text{an}_{X_\bullet}^* w = j_\bullet^* ((m_l \cdot w_L^{l, j-l})_{0 \leq l \leq j}) = (w_{L, I, J}^{l, j-l})_{l, I, J} \in H^j\Gamma(X_{\bullet, \mathbb{C}}^{an} \cap \mathbb{D}_\bullet, \Omega_{X_{\mathbb{C}}^{an}}^\bullet).$$

where for each (I, J, l) with $\text{card}I + \text{card}J + l = j$,

$$w_{L, I, J}^{l, j-l} := j_{IJ}^* w_{L, I}^{l, j-l} \in \Gamma(X_{I, \mathbb{C}}^{an} \cap \mathbb{D}_J, \Omega_{X_{\mathbb{C}}^{an}}^l).$$

We have by a standard computation, for each (I, J, l) with $\text{card}I + \text{card}J + l = j$,

$$H_{\text{sing}}^*(X_{I, \mathbb{C}}^{an} \cap \mathbb{D}_J, \mathbb{Z}) = \langle \gamma_1, \dots, \gamma_{\text{card}I} \rangle,$$

where for $1 \leq i \leq \text{card}I$, $\gamma_i \in \text{Hom}(\Delta^*, X_{I, \mathbb{C}}^{an} \cap \mathbb{D}_J)$ are products of loops around the origin inside the pointed disc $\mathbb{D}^1 \setminus 0$. On the other hand,

- $w_{L, I, J}^{l, j-l} = j_J^* (\sum_{\nu} df_{\nu_1}/f_{\nu_1} \wedge \cdots \wedge df_{\nu_l}/f_{\nu_l}) \in \Gamma(X_{I, \mathbb{C}}^{an} \cap \mathbb{D}_J, \Omega_{X_{\mathbb{C}}^{an}}^l)$ for $1 \leq l \leq j$,
- $w_{L, I, J}^{0, j} = \lambda_I$ is a constant.

Hence, for $\mu \in P([1, \dots, s])$ with $\text{card}\mu = l$, we get, for $l = 0$ $H^l ev(X_{\mathbb{C}}^{an})_{I, J}(w_{L, I, J}^{0, j}) = 0$ and, for $1 \leq l \leq j$,

$$H^l ev(X_{\mathbb{C}}^{an})_{I, J}(w_{L, I, J}^{l, j-l})(\gamma_\mu) = \sum_k \delta_{\nu, \mu} 2i\pi \in 2i\pi\mathbb{Z}.$$

where $\gamma_\mu := \gamma_{\mu_1} \cdots \gamma_{\mu_l}$. We conclude by [4] lemma 1.

(ii): It is a particular case of (i). See [4] proposition 1. □

Let k a field of characteristic zero. Let $X \in \text{SmVar}(k)$. Let $X = \cup_{i=1}^s X_i$ an open affine cover with $X_i := X \setminus D_i$ with $D_i \subset X$ smooth divisors with normal crossing. Let $\sigma : k \hookrightarrow \mathbb{C}$ an embedding. By proposition 4, we have a commutative diagram of graded algebras

$$\begin{array}{ccc} H_{DR}^*(X) & \xrightarrow{H^* ev(X)} & H_{\text{sing}}^*(X_{\mathbb{C}}^{an}, \mathbb{C}) \\ \uparrow \subset & & \uparrow H^* C^* \iota_{2i\pi\mathbb{Q}/\mathbb{C}}(X_{\mathbb{C}}^{an}) \\ H^* OL_X(\mathbb{H}_{et}^*(X, \Omega_{X^{et}, \log}^\bullet) \cap H_{DR}^*(X)) & \xrightarrow{H^* ev(X)} & H_{\text{sing}}^*(X_{\mathbb{C}}^{an}, 2i\pi\mathbb{Q}) \end{array}$$

where

$$C^* \iota_{2i\pi\mathbb{Q}/\mathbb{C}}(X_{\mathbb{C}}^{an}) : C_{\text{sing}}^\bullet(X_{\mathbb{C}}^{an}, 2i\pi\mathbb{Q}) \hookrightarrow C_{\text{sing}}^\bullet(X_{\mathbb{C}}^{an}, \mathbb{C})$$

is the subcomplex consisting of $\alpha \in C_{\text{sing}}^j(X_{\mathbb{C}}^{an}, \mathbb{C})$ such that $\alpha(\gamma) \in 2i\pi\mathbb{Q}$ for all $\gamma \in C_j^{\text{sing}}(X_{\mathbb{C}}^{an}, \mathbb{Q})$. Recall that

$$H^* ev(X_{\mathbb{C}}) = H^* R\Gamma(X_{\mathbb{C}}^{an}, \alpha(X)) \circ \Gamma(X_{\mathbb{C}}^{an}, E_{zar}(\Omega(\text{an}_X))) : H_{DR}^*(X_{\mathbb{C}}) \xrightarrow{\sim} H_{\text{sing}}^*(X_{\mathbb{C}}^{an}, \mathbb{C})$$

is the canonical isomorphism induced by the analytical functor and $\alpha(X) : \mathbb{C}_{X_{\mathbb{C}}^{an}} \hookrightarrow \Omega_{X_{\mathbb{C}}^{an}}^\bullet$, which gives the periods elements $H^* ev(X)(H_{DR}^*(X)) \subset H_{\text{sing}}^*(X_{\mathbb{C}}^{an}, \mathbb{C})$. On the other side the induced map

$$H^* ev(X_{\mathbb{C}}) : H^* OL_X(\mathbb{H}_{et}^*(X_{\mathbb{C}}, \Omega_{X_{\mathbb{C}}^{et}, \log}^\bullet)) \hookrightarrow H^* \iota_{2i\pi\mathbb{Q}/\mathbb{C}} H_{\text{sing}}^*(X_{\mathbb{C}}^{an}, 2i\pi\mathbb{Q})$$

is NOT surjective in general since the left hand side is invariant by the action of the group $\text{Aut}(\mathbb{C})$ (the group of field automorphism of \mathbb{C}) whereas the right hand side is not. The fact for a de Rham cohomology class of being logarithmic is algebraic and invariant under isomorphism of (abstract) schemes.

Corollary 1. *Let $X \in \text{PSmVar}(\mathbb{C})$. Then the Hodge conjecture holds for X if and only if the Hodge classes are given by logarithmic De Rham classes.*

Proof. Follows from theorem 1. □

3.3 Rigid GAGA for logarithmic de Rham classes

Let K a field of characteristic zero which is complete for a p -adic norm. Denote by $\text{Var}(K)^{pet}$ the big pro-etale sheaf. We will consider

$$\Omega_{/K, \log, \mathcal{O}}^{\bullet, an} \in C(\text{Var}(K)^{pet}), X \mapsto \Omega_{\hat{X}^{(p)}, \log, \mathcal{O}}^\bullet(\hat{X}^{(p)}), (f : X' \rightarrow X) \mapsto \Omega(f) := f^*$$

and the embedding of $C(\text{Var}(K)^{pet})$ (see definition 1(iii))

$$\begin{aligned} OL_{/K, an} : \Omega_{/K, \log, \mathcal{O}}^{\bullet, an} &\hookrightarrow \Omega_{/K}^{\bullet, an}, \text{ for } X \in \text{Var}(K), \\ OL_{/K, an}(X) &:= OL_{\hat{X}^{(p)}}(X) : \Omega_{\hat{X}^{(p)}, \log, \mathcal{O}}^\bullet(\hat{X}^{(p)}) \rightarrow \Omega_{\hat{X}^{(p)}}^\bullet(\hat{X}^{(p)}). \end{aligned}$$

and their restrictions to $\text{SmVar}(K)^{pet} \subset \text{Var}(K)^{pet}$. The morphism of site in RCat given by completion with respect to (p)

$$c : \text{FSch}/O_K \rightarrow \text{Var}(K), X \mapsto \hat{X}^{(p)},$$

where FSch/O_K is the category of formal schemes over O_K , induces the map in $C(\text{Var}(K)^{pet})$

$$c^* : \Omega_{/K}^\bullet \rightarrow \Omega_{/K}^{\bullet, an} := c_* \Omega_{/K}^{\bullet, an}, \text{ for } X \in \text{Var}(K), c^* : \Omega_X^\bullet(X) \rightarrow \Omega_{\hat{X}^{(p)}}^\bullet(\hat{X}^{(p)}),$$

which induces for each $j \in \mathbb{Z}$ the morphism

$$c^* : H_{DR}^j(X) = \mathbb{H}_{pet}^j(X, \Omega_X^\bullet) \xrightarrow{\sim} \mathbb{H}_{pet}^j(X, \Omega_{\hat{X}^{(p)}}^\bullet) = \mathbb{H}_{pet}^j(\hat{X}^{(p)}, \Omega_{\hat{X}^{(p)}}^\bullet)$$

which is an isomorphism by GAGA (c.f. EGA 3) and the properties of w-contractile rings (for the pro-étale topology $c_* = Rc_*$). On the other hand, it is well-known (see [8]) that

$$\mathbb{H}_{pet}^j(\hat{X}^{(p)}, \Omega_{\hat{X}^{(p)}}^\bullet) = \mathbb{H}_{et}^j(\hat{X}^{(p)}, \Omega_{\hat{X}^{(p)}}^\bullet), \text{ that is, } \mathbb{H}_{pet}^j(X, \Omega_{\hat{X}^{(p)}}^\bullet) = \mathbb{H}_{et}^j(X, \Omega_{\hat{X}^{(p)}}^\bullet).$$

We have also the sheaves

$$\begin{aligned} \hat{O}, \hat{O}^* \in \text{PSh}(\text{Var}(K)), \quad X \in \text{Var}(K) \mapsto \hat{O}(X) &:= \varprojlim_{n \in \mathbb{N}} O(X/p^n), \quad \hat{O}^*(X) := \varprojlim_{n \in \mathbb{N}} O(X/p^n)^*, \\ (g : Y \rightarrow X) \mapsto (a_g(X/p^n))_{n \in \mathbb{N}} : \hat{O}(X) &\rightarrow \hat{O}(Y), \quad \hat{O}(X)^* \rightarrow \hat{O}(Y)^* \end{aligned}$$

The sheaf $\hat{O}^* \in \text{PSh}(\text{SmVar}(k))$ admits transfers : for $W \subset X' \times X$ with $X, X' \in \text{SmVar}(K)$ and W finite over X' and $f \in O(X/p^n)^*$, $W^*f := N_{W/X'}(p_X^*f)$ where $p_X : W \hookrightarrow X' \times X \rightarrow X$ is the projection and $N_{W/X'} : k(W)^* \rightarrow k(X')^*$ is the norm map. This gives transfers on $\Omega_{/K, \log, \mathcal{O}}^{1, an} \in \text{PSh}(\text{SmVar}(K))$ compatible with transfers on $\Omega_{/K}^{1, an} \in \text{PSh}(\text{SmVar}(K))$: for $W \subset X' \times X$ with $X, X' \in \text{SmVar}(k)$ and W finite over X' and $f \in O(X/p^n)^*$,

$$W^*df/f := dW^*f/W^*f = Tr_{W/X'}(p_X^*(df/f)),$$

where $p_X : W \hookrightarrow X' \times X \rightarrow X$ is the projection and $Tr_{W/X'} : O_W \rightarrow O_X$ is the trace map. Note that $d(fg)/fg = df/f + dg/g$. We get transfers on

$$\otimes_{\mathbb{Q}_p}^l \Omega_{/K, \log, \mathcal{O}}^{1, an}, \otimes_{\hat{O}}^l \Omega_{/K}^{1, an} \in \text{PSh}(\text{SmVar}(K))$$

since $\otimes_{\mathbb{Q}_p}^l \Omega_{/K, \log, \mathcal{O}}^{1, an} = H^0(\otimes_{\mathbb{Q}_p}^{L, l} \Omega_{/K, \log, \mathcal{O}}^{1, an})$ and $\otimes_{\hat{O}}^l \Omega_{/K}^{1, an} = H^0(\otimes_{\hat{O}}^{L, l} \Omega_{/K}^{1, an})$. This induces transfers on

$$\begin{aligned} \wedge_{\mathbb{Q}_p}^l \Omega_{/K, \log, \mathcal{O}}^{1, an} := \text{coker}(\oplus_{I_2 \subset [1, \dots, l]} \otimes_{\mathbb{Q}_p}^{l-1} \Omega_{/K, \log, \mathcal{O}}^{1, an} \xrightarrow{\oplus_{I_2 \subset [1, \dots, l]} \Delta_{I_2} := (w \otimes w' \mapsto w \otimes w \otimes w')} \otimes_{\mathbb{Q}_p}^l \Omega_{/K, \log, \mathcal{O}}^{1, an}) \\ \in \text{PSh}(\text{SmVar}(K)). \end{aligned}$$

and

$$\wedge_{\hat{O}}^l \Omega_{/K}^{1, an} := \text{coker}(\oplus_{I_2 \subset [1, \dots, l]} \otimes_{\hat{O}}^{l-1} \Omega_{/K}^{1, an} \xrightarrow{\oplus_{I_2 \subset [1, \dots, l]} \Delta_{I_2} := (w \otimes w' \mapsto w \otimes w \otimes w')} \otimes_{\hat{O}}^l \Omega_{/K}^{1, an}) \in \text{PSh}(\text{SmVar}(K)).$$

Lemma 1. *Let K a field of characteristic zero which is complete for a p -adic norm. Consider, for $l \in \mathbb{Z}$, the factorization in $\text{PSh}(\text{SmVar}(K))$*

$$OL_{/K, an}^l : \Omega_{/K, \log, \mathcal{O}}^{l, an} \xrightarrow{OL_{/K, an}^l} L^{l, 0} := OL_{/K, an}^l(\Omega_{/K, \log, \mathcal{O}}^{l, an}) \xrightarrow{\subset, {}^l OL^l} H^l \Omega_{/K}^{\bullet, an}, \quad w \mapsto [w].$$

(i) *Let $X \in \text{SmVar}(K)$. Consider, for $j \in \mathbb{Z}$, the following commutative diagram of abelian groups*

$$\begin{array}{ccc} H_{pet}^{j-l}(X, \Omega_{\hat{X}^{(p)}, \log}^l) & \xrightarrow{H^j OL_X} & \mathbb{H}_{pet}^j(X, \Omega_{\hat{X}}^\bullet) \\ & \searrow^{H^j OL_X^{\geq l}} & \swarrow_{\tau := \tau^{\geq l}(OL_X)} \\ & & \mathbb{H}_{pet}^j(X, \tau^{\geq l} \Omega_{\hat{X}^{(p)}}^\bullet) \\ & \searrow^{H^{j-l} OL_X^l} & \swarrow_h \\ H_{pet}^{j-l}(X, L^{l, 0}) & \xrightarrow{H^{j-l} {}^l OL_X^l} & H_{pet}^{j-l}(X, H^l \Omega_{\hat{X}^{(p)}}^\bullet) \end{array}$$

where for A an abelian category and $G \in C(A)$, $\tau^{\geq l}G := (0 \rightarrow G^{l, \partial=0} \rightarrow G^{l+1} \rightarrow \dots)$. Then, $H^j OL_X^l$ is surjective, and $\tau^l := \tau|_{H_{pet}^{j-l}(X, \Omega_{\hat{X}^{(p)}, \log}^l)}$ and $h := h|_{H_{pet}^{j-l}(X, \Omega_{\hat{X}^{(p)}, \log}^l)}$ are isomorphisms.

(ii) Let $X \in \text{SmVar}(K)$ and $Z \subset X$ a closed subset. Consider, for $j \in \mathbb{Z}$, the following commutative diagram of abelian groups

$$\begin{array}{ccc}
H_{pet,Z}^{j-l}(X, \Omega_{\hat{X}^{(p)}, \log}^l) & \xrightarrow{H^j OL_X} & \mathbb{H}_{pet,Z}^j(X, \Omega_{\hat{X}}^\bullet) \\
\downarrow H^{j-l} OL_X^l & \searrow H^{j-l} OL_X^l & \swarrow H^j OL_X^{\geq l} \\
H_{pet,Z}^{j-l}(X, L^{l,0}) & \xrightarrow{H^{j-l} OL_X^l} & H_{pet,Z}^{j-l}(X, H^l \Omega_{\hat{X}^{(p)}}^\bullet) \\
& & \swarrow h \\
& & \mathbb{H}_{pet,Z}^j(X, \tau^{\geq l} \Omega_{\hat{X}^{(p)}}^\bullet)
\end{array}$$

Then, $H^j OL_X^l$ is surjective, and $\tau' := \tau|_{H_{pet,Z}^{j-l}(X, \Omega_{\hat{X}^{(p)}, \log}^l)}$ and $h := h|_{H_{pet,Z}^{j-l}(X, \Omega_{\hat{X}^{(p)}, \log}^l)}$ are isomorphisms.

Proof. (i): Let $r = (r_i : X_i \rightarrow X)$ a w-contractile etale cover. Let

$$w = ([w_I]) \in \Gamma(X_{\bullet, j-l}, L^{l,0})^{\partial=0}.$$

We have then $[w_I]_{X_I \times_X X_j} = [w_J]_{X_I \times_X X_j}$ for all I, j . But since a logarithmic form is exact if and only if it vanishes, $(w_I)_{X_I \times_X X_j} = (w_J)_{X_I \times_X X_j}$ for all I, j . Hence, $w = OL_X^l(w)$, $w = (w_I) \in \Gamma(X_{\bullet, j-l}, \Omega_{\hat{X}^{(p)}, \log}^l)^{\partial=0}$. This shows that $H^{j-l} OL_X^l$ is surjective. Obviously, τ' and h' are surjective. Let

$$w = (w_I) \in \Gamma(X_{\bullet, j-l}, \Omega_{\hat{X}^{(p)}, \log}^l)^{\partial=0}.$$

If $w = \partial(\beta_J) + (\partial\alpha_I)$, then β_J is of degree l and α_I is of degree $l+1$. Hence τ' is injective. If $([w_I]) = \partial([\beta_J])$, that is $[w_I] = [w_{J|X_I}]$ for all I, J , then $w_I = w_{J|X_I}$ for all I, J , since a logarithmic form is exact if and only if it vanishes.

(ii): Follows from (i) applied to X and $X \setminus Z$ considering the distinguish triangle in $\text{DA}(K)$

$$M(X \setminus Z) \xrightarrow{\mathbb{Z}(j)} M(X) \xrightarrow{\gamma_Z^\vee} M_Z(X) \xrightarrow{\partial} M(X \setminus Z)[1]$$

where $j : X \setminus Z \hookrightarrow X$ is the open embedding. □

The result of this section is the following ([4] proposition 2 (i)), we recall and give some precisions to the proof :

Proposition 5. *Let K a field of characteristic zero which is complete for a p -adic norm. Let $X \in \text{PSmVar}(K)$. We have then the morphism $c : \hat{X}^{(p)} \rightarrow X$ in RTop given by completion with respect to (p) . For each $j \in \mathbb{Z}$, the isomorphism $c^* : \mathbb{H}_{pet}^j(X, \Omega_X^\bullet) \xrightarrow{\sim} \mathbb{H}_{pet}^j(X, \Omega_{\hat{X}^{(p)}}^\bullet)$ and its inverse preserve logarithmic classes, that is for each $j, l \in \mathbb{Z}$,*

$$c^*(H^j(m \circ (OL_X \otimes I))(H_{pet}^{j-l}(X, \Omega_{X^{pet}, \log, \mathcal{O}}^l \otimes \mathbb{Z}_p))) = H^j OL_{\hat{X}^{(p)}}(H_{pet}^{j-l}(X, \Omega_{\hat{X}^{(p)}, \log, \mathcal{O}}^l))$$

is an isomorphism.

Proof. Consider, for $j, l \in \mathbb{Z}$, the presheaf

$$L^{l, j-l} := H^j OL_{/K, an}(H_{pet}^{j-l} \Omega_{/K, \log, \mathcal{O}}^{l, an}) \in \text{PSh}(\text{SmVar}(K)).$$

By definition, for $X \in \text{SmVar}(K)$,

$$L^{l, j-l}(X) := H^j OL_{\hat{X}^{(p)}}(H_{pet}^{j-l}(X, \Omega_{\hat{X}^{(p)}, \log, \mathcal{O}}^l)) =: H^j OL_{/K, an}(H^{j-l} \text{Hom}(\mathbb{Z}(X), E_{pet}^\bullet(\Omega_{/K, \log, \mathcal{O}}^{l, an})))$$

Consider also for $X \in \text{SmVar}(K)$ and $Z \subset X$ a closed subset,

$$L_Z^{l,j-l}(X) := H^j OL_{/K,an}(H^{j-l} \text{Hom}(\mathbb{Z}(X, X \setminus Z), E_{pet}^\bullet(\Omega_{/K,log,\mathcal{O}}^{l,an}))) := H^j OL_{\hat{X}(p)}(H_{pet,Z}^{j-l}(X, \Omega_{\hat{X}(p),log,\mathcal{O}}^l)).$$

Consider the factorization in $C(\text{SmVar}(K))$

$$OL_{/K,an} : \Omega_{/K,log,\mathcal{O}}^{l,an} \xrightarrow{OL_{/K,an}^l} L^{l,0} := OL_{/K,an}(\Omega_{/K,log,\mathcal{O}}^{l,an}) \xrightarrow{C, {}^t OL^l} H^l \Omega_{/K}^{\bullet,an}, \quad w \mapsto [w]$$

and the exact sequence in $\text{PSh}(\text{SmVar}(K))$

$$0 \rightarrow L^{l,0} \xrightarrow{{}^t OL^l} H^l \Omega_{/K}^{\bullet,an} \rightarrow Q^{l,0} \rightarrow 0.$$

By lemma 1, we have

$$\begin{aligned} L^{l,j-l}(X) &= H^{j-l} \text{Hom}(\mathbb{Z}(X), E_{pet}^\bullet(L^{l,0})) / H^{j-l-1} \text{Hom}(\mathbb{Z}(X), E_{pet}^\bullet(Q^{l,0})) \\ &= H^{j-l} \text{Hom}(\mathbb{Z}(X), E_{et}^\bullet(L^{l,0})) / H^{j-l-1} \text{Hom}(\mathbb{Z}(X), E_{et}^\bullet(Q^{l,0})) \end{aligned}$$

and

$$\begin{aligned} L_Z^{l,j-l}(X) &= H^{j-l} \text{Hom}(\mathbb{Z}(X, X \setminus Z), E_{pet}^\bullet(L^{l,0})) / H^{j-l-1} \text{Hom}(\mathbb{Z}(X, X \setminus Z), E_{pet}^\bullet(Q^{l,0})) \\ &= H^{j-l} \text{Hom}(\mathbb{Z}(X, X \setminus Z), E_{et}^\bullet(L^{l,0})) / H^{j-l-1} \text{Hom}(\mathbb{Z}(X, X \setminus Z), E_{et}^\bullet(Q^{l,0})). \end{aligned}$$

where the second equalities follows from the fact that

$$\mathbb{H}_{pet}^k(X, \Omega_{\hat{X}(p)}^\bullet) = \mathbb{H}_{et}^k(X, \Omega_{\hat{X}(p)}^\bullet).$$

The presheaves $L^{l,0}, H^l \Omega_{/K}^{\bullet,an} \in \text{PSh}(\text{SmVar}(K))$ are \mathbb{A}^1 invariant and admit transfers, hence $Q^{l,0} \in \text{PSh}(\text{SmVar}(K))$ is \mathbb{A}^1 invariant and admits transfers. Hence by a theorem of Voevodsky (c.f. [5] for example), $L^{l,0}, Q^{l,0} \in \text{PSh}(\text{SmVar}(K))$ are \mathbb{A}^1 local since they are \mathbb{A}^1 invariant and admit transfers. This gives in particular, for $Z \subset X$ a smooth subvariety of (pure) codimension d , by proposition 1(i) an isomorphism

$$L^{l,0}(P_{Z,X}) : L_Z^{l,j-l}(X) \xrightarrow{\sim} L_Z^{l,j-l}(N_{Z/X}) \xrightarrow{\sim} L^{l-d,j-l-d}(Z).$$

Let

$$\begin{aligned} \alpha \in L^{l,j-l}(X) : &= H^j OL_{\hat{X}(p)}(H_{pet}^{j-l}(X, \Omega_{\hat{X}(p),log,\mathcal{O}}^l)) \\ &= H^{j-l} \text{Hom}(\mathbb{Z}(X), E_{et}^\bullet(L^{l,0})) / H^{j-l-1} \text{Hom}(\mathbb{Z}(X), E_{et}^\bullet(Q^{l,0})). \end{aligned}$$

As $L^{l,0}$ and $Q^{l,0}$ consists of single presheaf, there exists an open subset $U = U(\alpha) \subset X$ depending on α such that $j^* \alpha = 0 \in L^{l,j-l}(U)$, where $j : U \hookrightarrow X$ denote the open embedding. Considering a divisor $X \setminus U \subset D \subset X$, we get

$$\alpha = H^{j-l} E_{et}^\bullet(L^{l,0})(\gamma_D^\vee)(\alpha), \quad \alpha \in L_D^{l,j-l}(X).$$

Similarly to the proof of theorem 1, we get by induction (restricting to the smooth locus of the divisors) a closed subset $Z \subset X$ of pure codimension $c = \min(l, j-l)$ such that

$$\alpha = H^{j-l} E_{et}^\bullet(L^{l,0})(\gamma_Z^\vee)(\alpha), \quad \alpha \in L_Z^{l,j-l}(X).$$

Hence, as in the proof of theorem 1,

- if $j \neq 2l$, $\alpha = 0$, we use the fact that X is projective for $j < 2l$,
- if $j = 2l$, $\alpha \in \bigoplus_{1 \leq t \leq s} \mathbb{Z}_p[Z_i]$, where $(Z_i)_{1 \leq i \leq t} \subset Z$ are the irreducible components of Z .

□

4 Tate conjecture

Let k a field of finite type over \mathbb{Q} . Denote \bar{k} the algebraic closure of k and $G = Gal(\bar{k}/k)$ the absolute Galois group of k . Let $X \in \text{PSmVar}(k)$ be a smooth projective variety. Let $p \in \mathbb{N}$ a prime number. Consider an embedding $\sigma_p : k \hookrightarrow \mathbb{C}_p$. Denote $\hat{k}_{\sigma_p} \subset \mathbb{C}_p$ the p -adic completion of k with respect to σ_p . We consider the commutative diagram in $C_{\mathbb{B}dr, \text{Fil}, \hat{G}_{\sigma_p}}(X_{\mathbb{C}_p}^{an, pet})$

$$\begin{array}{ccc} (\mathbb{B}_{dr, X_{\mathbb{C}_p}}, F) & \xrightarrow{\alpha(X)} & (\Omega_{X_{\mathbb{C}_p}}^\bullet, F_b) \otimes_{O_{X_{\mathbb{C}_p}}} (O\mathbb{B}_{dr, X_{\mathbb{C}_p}}, F) \\ \uparrow \iota'_{X_{\mathbb{C}_p}^{pet} := l \mapsto l.1} & & \uparrow OL_X \otimes I := (w \otimes \lambda_n)_{n \in \mathbb{N}} \rightarrow (w \otimes \lambda_n)_{n \in \mathbb{N}} \\ \mathbb{Z}_p X_{\mathbb{C}_p} & \xrightarrow{\iota_{X_{\mathbb{C}_p}^{pet} := (I, 0)}} & (\Omega_{X_{\mathbb{C}_p}, \log, \mathcal{O}}^\bullet \otimes \mathbb{Z}_p, F_b) \end{array},$$

Let $p \in \mathbb{N} \setminus \delta(k, X)$ a prime number. Consider an embedding $\sigma_p : k \hookrightarrow \mathbb{C}_p$. Then $X_{\hat{k}_{\sigma_p}}$ has good reduction modulo p and let $X_{\hat{k}_{\sigma_p}}^\mathcal{O} \in \text{Sch}/O_{\hat{k}_{\sigma_p}}$ be a smooth model, i.e. $X_{\hat{k}_{\sigma_p}}^\mathcal{O} \otimes_{O_{\hat{k}_{\sigma_p}}} \hat{k}_{\sigma_p} = X_{\hat{k}_{\sigma_p}}$ and $X_{\hat{k}_{\sigma_p}}^\mathcal{O}$ is smooth with smooth special fiber. The main result of [1] say in this case that the embedding in $C((X_{\hat{k}_{\sigma_p}}^\mathcal{O})^{Falt})$

$$\alpha(X) : \mathbb{B}_{cris, X_{\hat{k}_{\sigma_p}}} \hookrightarrow a_{\bullet, \bullet} \Omega_{X_{\hat{k}_{\sigma_p}}^\bullet} \otimes_{O_{X^\mathcal{O}}} O\mathbb{B}_{cris, X_{\hat{k}_{\sigma_p}}^\bullet}$$

is a filtered quasi-isomorphism compatible with the action of $Gal(\mathbb{C}_p/\hat{k}_{\sigma_p})$ and the action of the Frobenius ϕ_p , and gives if $X_{\hat{k}_{\sigma_p}}^\mathcal{O}$ has good reduction modulo p , for each $j \in \mathbb{Z}$, a filtered isomorphism of filtered abelian groups

$$\begin{aligned} H^j R\alpha(X) : H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{B}_{cris, \hat{k}_{\sigma_p}} &\xrightarrow{H^j T(a_X, \mathbb{B}_{cris})^{-1}} H_{et}^j((X)^{Falt})(\mathbb{B}_{cris, X_{\hat{k}_{\sigma_p}}}) \\ &\xrightarrow{H^j R\Gamma(X_{\hat{k}_{\sigma_p}}^\mathcal{O}, \alpha(X))} H_{DR}^j(X_{\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}} \mathbb{B}_{cris, \hat{k}_{\sigma_p}} \end{aligned}$$

compatible with the action of $Gal(\mathbb{C}_p/\hat{k}_{\sigma_p})$ and of the Frobenius ϕ_p .

Definition 2. Let k a field of finite type over \mathbb{Q} . Denote \bar{k} the algebraic closure of k and $G = Gal(\bar{k}/k)$ the absolute Galois group of k . Let $X \in \text{PSmVar}(k)$. Let $p \in \mathbb{N} \setminus \delta(k, X)$ a prime number. Consider an embedding $\sigma_p : k \hookrightarrow \mathbb{C}_p$. Denote $\hat{k}_{\sigma_p} \subset \mathbb{C}_p$ the p -adic completion of k with respect to σ_p . For $\alpha \in H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p)$, we consider

$$w(\alpha) := H^j R\alpha(X)(\alpha \otimes 1) \in H_{DR}^j(X_{\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}} \mathbb{B}_{cris, \hat{k}_{\sigma_p}}.$$

the associated de Rham class by the p -adic periods. We recall

$$H^j R\alpha(X) : H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{B}_{cris, \hat{k}_{\sigma_p}} \xrightarrow{\sim} H_{DR}^j(X_{\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}} \mathbb{B}_{cris, \hat{k}_{\sigma_p}}$$

is the canonical filtered isomorphism compatible with the action of \hat{G}_{σ_p} and with the action of the Frobenius ϕ_p .

We recall the following result from Illusie:

Proposition 6. Let k a field of finite type over \mathbb{Q} . Let $X \in \text{SmVar}(k)$. Let $p \in \mathbb{N} \setminus \delta(k, X)$ a prime number. Consider an embedding $\sigma_p : k \hookrightarrow \mathbb{C}_p$. Denote $\hat{k}_{\sigma_p} \subset \mathbb{C}_p$ the p -adic completion of k with respect to σ_p . Consider $X_{\hat{k}_{\sigma_p}}^\mathcal{O} \in \text{Sch}/O_{\hat{k}_{\sigma_p}}$ a smooth model of $X_{\hat{k}_{\sigma_p}}$, i.e. $X_{\hat{k}_{\sigma_p}}^\mathcal{O} \otimes_{O_{\hat{k}_{\sigma_p}}} \hat{k}_{\sigma_p} = X_{\hat{k}_{\sigma_p}}$ and $X_{\hat{k}_{\sigma_p}}^\mathcal{O}$ is

smooth with smooth special fiber. Assume there exist lifts $\phi_n : X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}/p^n \rightarrow X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}/p^n$ of the Frobenius $\phi : X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}/p \rightarrow X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}/p$, such that for $n' > n$ the following diagram commutes

$$\begin{array}{ccccccc} 0 & \longrightarrow & O_{X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}/p^{n'-n}} & \xrightarrow{p^n} & O_{X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}/p^{n'}} & \xrightarrow{/p^{n'-n}} & O_{X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}/p^n} \longrightarrow 0 \\ & & \uparrow \phi_{n'-n} & & \uparrow \phi_{n'} & & \uparrow \phi_n \\ 0 & \longrightarrow & O_{X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}/p^{n'-n}} & \xrightarrow{p^n} & O_{X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}/p^{n'}} & \xrightarrow{/p^{n'-n}} & O_{X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}/p^n} \longrightarrow 0 \end{array}$$

Let $l \in \mathbb{Z}$. For each $n \in \mathbb{N}$, the sequence in $C(X_{\hat{k}_{\sigma_p}}^{et})$

$$0 \rightarrow \Omega_{X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}/p^n, \log}^{\bullet \geq l} \xrightarrow{OL_{X^{\mathcal{O}}/p^n}} \Omega_{X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}/p^n}^{\bullet \geq l} \xrightarrow{\phi_n - I} \Omega_{X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}/p^n}^{\bullet \geq l} \rightarrow 0$$

is exact as a sequence of complexes of etale sheaves (i.e. we only have local surjectivity on the right).

Proof. It follows from [6] for $n = 1$. It then follows for $n \geq 2$ by induction on n by a trivial devissage. \square

We have the following key proposition (the projective case of [4]), we state and prove it for smooth projective varieties, the case for smooth varieties is obtained in the same way using a smooth compactification with normal crossing divisors. The projective case suffices for our purpose:

Proposition 7. *Let k a field of finite type over \mathbb{Q} . Let $X \in \text{PSmVar}(k)$. Let $p \in \mathbb{N} \setminus \delta(k, X)$ a prime number. Consider an embedding $\sigma_p : k \hookrightarrow \mathbb{C}_p$. Denote $k \subset \hat{k}_{\sigma_p} \subset \mathbb{C}_p$ the p -adic completion of k with respect to σ_p .*

(i) *Let $j, l \in \mathbb{Z}$. We have, see definition 1(iii),*

$$\begin{aligned} F^l H_{DR}^j(X_{\hat{k}_{\sigma_p}}) \cap H^j R\alpha(X)(H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p)) &= (c^*)^{-1}(H^j OL_{\hat{X}_{\hat{k}_{\sigma_p}}^{(p)}}(\mathbb{H}_{pet}^j(X_{\hat{k}_{\sigma_p}}, \Omega_{\hat{X}_{\hat{k}_{\sigma_p}}^{(p)}, \log, \mathcal{O}})^{\bullet \geq l})) \\ &\subset H_{DR}^j(X_{\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}} \mathbb{B}_{cris, \hat{k}_{\sigma_p}}, \end{aligned}$$

where we recall $c : \hat{X}_{\hat{k}_{\sigma_p}}^{(p)} \rightarrow X_{\hat{k}_{\sigma_p}}$ is the completion with respect to (p) , and

$$c^* : \mathbb{H}_{pet}^j(X_{\hat{k}_{\sigma_p}}, \Omega_{X_{\hat{k}_{\sigma_p}}}^{\bullet}) \xrightarrow{\sim} \mathbb{H}_{pet}^j(X_{\hat{k}_{\sigma_p}}, \Omega_{\hat{X}_{\hat{k}_{\sigma_p}}^{(p)}}^{\bullet})$$

is an isomorphism by GAGA and the properties of w -contractile open covers.

(ii) *Let $j, l \in \mathbb{Z}$. We have, see definition 1(iii),*

$$\begin{aligned} F^l H_{DR}^j(X_{\hat{k}_{\sigma_p}}) \cap H^j R\alpha(X)(H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p)) &= H^j(m \circ (OL_X \otimes I))(\mathbb{H}_{pet}^j(X_{\hat{k}_{\sigma_p}}, \Omega_{X_{\hat{k}_{\sigma_p}}^{\bullet \geq l}, \log, \mathcal{O}} \otimes \mathbb{Z}_p)) \\ &\subset H_{DR}^j(X_{\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}} \mathbb{B}_{cris, \hat{k}_{\sigma_p}}, \end{aligned}$$

(ii)' *For $\alpha \in H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p)$ such that $w(\alpha) \in F^l H_{DR}^j(X_{\hat{k}_{\sigma_p}})$ (see definition 2), there exist*

$$(\lambda_i)_{1 \leq i \leq n} \in \mathbb{Z}_p \text{ and } (w_{Li})_{1 \leq i \leq n} \in \mathbb{H}_{pet}^j(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}, \log, \mathcal{O}}^{\bullet \geq l})$$

such that

$$w(\alpha) = \sum_{1 \leq i \leq n} \lambda_i \cdot w_{Li} \in \mathbb{H}_{pet}^j(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}}^{\bullet \geq l}) = F^l H_{DR}^j(X_{\mathbb{C}_p}), \quad w_{Li} := H^j OL_X(w_{Li}).$$

Proof. (i): Consider $X_{\hat{k}_{\sigma_p}}^{\mathcal{O}} \in \text{Sch}/O_{\hat{k}_{\sigma_p}}$ a smooth model of $X_{\hat{k}_{\sigma_p}}$, i.e. $X_{\hat{k}_{\sigma_p}}^{\mathcal{O}} \otimes_{O_{\hat{k}_{\sigma_p}}} \hat{k}_{\sigma_p} = X_{\hat{k}_{\sigma_p}}$ and $X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}$ is smooth. Consider $c : \hat{X}_{\hat{k}_{\sigma_p}}^{\mathcal{O},(p)} \rightarrow X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}$ the morphism in RTop which is the formal completion along the ideal (p) . Take a Zariski or etale cover $r = (r_i : X_i \rightarrow X)_{1 \leq i \leq r}$ such that for each i there exists an etale map $X_i \rightarrow \mathbb{A}^{d_{X_i}}$. Then, by [1], we have for each i explicit lifts of Frobenius $\phi_n^i : X_{i,\hat{k}_{\sigma_p}}^{\mathcal{O}}/p^n \rightarrow X_{i,\hat{k}_{\sigma_p}}^{\mathcal{O}}/p^n$ of the Frobenius $\phi_1^i : X_{i,\hat{k}_{\sigma_p}}^{\mathcal{O}}/p \rightarrow X_{i,\hat{k}_{\sigma_p}}^{\mathcal{O}}/p$, such that for $n' > n$ the following diagram commutes

$$\begin{array}{ccccccc} 0 & \longrightarrow & O_{X_{i,\hat{k}_{\sigma_p}}^{\mathcal{O}}/p^{n'-n}} & \xrightarrow{p^n} & O_{X_{i,\hat{k}_{\sigma_p}}^{\mathcal{O}}/p^{n'}} & \xrightarrow{/p^{n'-n}} & O_{X_{i,\hat{k}_{\sigma_p}}^{\mathcal{O}}/p^n} \longrightarrow 0 \\ & & \uparrow \phi_{n'-n}^i & & \uparrow \phi_{n'}^i & & \uparrow \phi_n^i \\ 0 & \longrightarrow & O_{X_{i,\hat{k}_{\sigma_p}}^{\mathcal{O}}/p^{n'-n}} & \xrightarrow{p^n} & O_{X_{i,\hat{k}_{\sigma_p}}^{\mathcal{O}}/p^{n'}} & \xrightarrow{/p^{n'-n}} & O_{X_{i,\hat{k}_{\sigma_p}}^{\mathcal{O}}/p^n} \longrightarrow 0 \end{array}$$

and such that the action of ϕ_n^i on $\Omega_{X_{i,\hat{k}_{\sigma_p}}^{\mathcal{O}}/p^n}^{\bullet}$ is a morphism of complex, i.e. commutes with the differentials. On the other hand, by [6], we have action of the Frobenius on $H_{DR}^j(X_{\hat{k}_{\sigma_p}})$ by

$$\phi : H_{DR}^j(X_{\hat{k}_{\sigma_p}}) \xrightarrow{\sim} \mathbb{H}^j(X_{\hat{k}_{\sigma_p}}, W\Omega_{X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}}^{\bullet}) \xrightarrow{\phi_W^{*(X_{\hat{k}_{\sigma_p}})}} \mathbb{H}^j(X_{\hat{k}_{\sigma_p}}, W\Omega_{X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}}^{\bullet}) \xrightarrow{\sim} H_{DR}^j(X_{\hat{k}_{\sigma_p}})$$

We then have the following commutative diagram, where $R := [1, \dots, r]$ and $X_R := X_1 \times_X \dots \times_X X_r$,

$$\begin{array}{ccccccc} \dots & \longrightarrow & F^l H_{DR}^{j-1}(X_{R,\hat{k}_{\sigma_p}}) & \xrightarrow{\partial} & F^l H_{DR}^j(X_{\hat{k}_{\sigma_p}}) & \xrightarrow{r_i^*} & \bigoplus_{i=1}^r F^l H_{DR}^j(X_{i,\hat{k}_{\sigma_p}}) \xrightarrow{r_i^*} \dots \\ & & \uparrow I - \phi^R & & \uparrow I - \phi & & \uparrow I - \phi^i \\ \dots & \longrightarrow & F^l H_{DR}^{j-1}(X_{R,\hat{k}_{\sigma_p}}) & \xrightarrow{\partial} & F^l H_{DR}^j(X_{\hat{k}_{\sigma_p}}) & \xrightarrow{r_i^*} & \bigoplus_{i=1}^r F^l H_{DR}^j(X_{i,\hat{k}_{\sigma_p}}) \xrightarrow{r_i^*} \dots \\ & & \uparrow H^{j-1} OL_{\hat{X}_{R,\hat{k}_{\sigma_p}}^{(p)}} & & \uparrow H^j OL_{\hat{X}_{i,\hat{k}_{\sigma_p}}^{(p)}} & & \uparrow H^j OL_{\hat{X}_{i,\hat{k}_{\sigma_p}}^{(p)}} \\ \dots & \longrightarrow & \mathbb{H}_{pet}^{j-1}(X_{R,\hat{k}_{\sigma_p}}; \Omega_{\hat{X}_{R,\hat{k}_{\sigma_p}}^{(p)}, \log, \mathcal{O}}^{\bullet \geq l}) & \xrightarrow{\partial} & \mathbb{H}_{pet}^j(X_{\hat{k}_{\sigma_p}}; \Omega_{\hat{X}_{\hat{k}_{\sigma_p}}^{(p)}, \log, \mathcal{O}}^{\bullet \geq l}) & \xrightarrow{r_i^*} & \bigoplus_{i=1}^r \mathbb{H}_{pet}^j(X_{i,\hat{k}_{\sigma_p}}; \Omega_{\hat{X}_{i,\hat{k}_{\sigma_p}}^{(p)}, \log, \mathcal{O}}^{\bullet \geq l}) \xrightarrow{r_i^*} \dots \end{array} \quad (1)$$

whose rows are exact sequences. By [1], $\alpha \in H_{et}^j(X_{C_p}, \mathbb{Z}_p)$ is such that $w(\alpha) \in F^l H_{DR}^j(X_{\hat{k}_{\sigma_p}})$ if and only if

$$w(\alpha) \in \ker(I - \phi : F^l H_{DR}^j(X_{\hat{k}_{\sigma_p}}) \rightarrow F^l H_{DR}^j(X_{\hat{k}_{\sigma_p}})).$$

On the other hand, for each $I \subset [1, \dots, r]$, the sequence in $C(X_I^{pet})$

$$0 \rightarrow \Omega_{\hat{X}_{I,\hat{k}_{\sigma_p}}^{\mathcal{O},(p)}, \log, \mathcal{O}}^{\bullet \geq l} \xrightarrow{OL_{\hat{X}_{\hat{k}_{\sigma_p}}^{(p)}}} \Omega_{\hat{X}_{I,\hat{k}_{\sigma_p}}^{\mathcal{O},(p)}}^{\bullet \geq l} \xrightarrow{\phi^I - I := (\phi_n^I - I)_{n \in \mathbb{N}}} \Omega_{\hat{X}_{I,\hat{k}_{\sigma_p}}^{\mathcal{O},(p)}}^{\bullet \geq l} \rightarrow 0$$

is exact for the pro-etale topology by proposition 6 and since for each $l, n \in \mathbb{N}$ the map in $\text{PSh}(X^{pet})$

$$\Omega(/p) : \Omega_{X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}/p^n}^l \rightarrow \Omega_{X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}/p^{n+1}}^l$$

are surjective and since the pro-etale site is a replete topos by [2]. Hence the columns of the diagram (1) are exact. This proves (i).

(ii): Follows from (i) and proposition 5.

(ii)': By (ii), if $(r_i : X_i \rightarrow X)_{1 \leq i \leq r}$ is a w-contractile affine pro-etale cover

$$w(\alpha) = \left(\sum_{1 \leq i \leq n_J} \lambda_{iJ} \cdot w_{LiJ} \right)_{J \subset [1, \dots, r], \text{card} J = j} \in H^j \Gamma(X_{\bullet, \hat{k}_{\sigma_p}}, m \circ (OL_X \otimes I)) (H^j \Gamma(X_{\bullet, \hat{k}_{\sigma_p}}, \Omega_{X_{\hat{k}_{\sigma_p}}, \log, \mathcal{O}}^{\bullet \geq l} \otimes \mathbb{Z}_p)),$$

with $w_{LiJ} \in \Gamma(X_J, \Omega_{X_{\hat{k}_{\sigma_p}}, \log, \mathcal{O}}^{\bullet \geq l'})$, $l' \geq l$ and $\lambda_{iJ} \in \mathbb{Z}_p$. Since $w(\alpha)$ is a Chech etale cycle (i.e. closed for the Chech differential) without torsion and since the topology consists of etale covers of $X \in \text{PSmVar}(k)$ (which we way assume connected), $n_J = n$ and $\lambda_{iJ} = \lambda_i \in \mathbb{Z}_p$ for each J , which gives

$$w(\alpha) = \sum_{1 \leq i \leq n} \lambda_i \cdot w_{Li} \in \mathbb{H}_{pet}^j(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}}^{\bullet \geq l}) = F^l H_{DR}^j(X_{\mathbb{C}_p}), \quad w_{Li} := H^j OL_X(w_{Li}).$$

□

We also have

Proposition 8. *Let k a field of finite type over \mathbb{Q} . Denote \bar{k} the algebraic closure of k . Denote $G := \text{Gal}(\bar{k}/k)$ its absolute Galois group. Let $X \in \text{PSmVar}(k)$ a smooth projective variety. Let $p \in \mathbb{N}$ a prime number. Consider an embedding $\sigma_p : k \hookrightarrow \mathbb{C}_p$. Denote $k \subset \hat{k}_{\sigma_p} \subset \mathbb{C}_p$ being the p -adic completion with respect to the p adic norm induced by σ_p . Then $\hat{G}_{\sigma_p} := \text{Gal}(\mathbb{C}_p/\hat{k}_{\sigma_p}) \subset G := \text{Gal}(\bar{k}/k)$. Let $\alpha \in H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p)$. Consider then its associated De Rham class (see definition 2)*

$$w(\alpha) := H^j R\alpha(X)(\alpha \otimes 1) \in H_{DR}^j(X_{\mathbb{C}_p}) \otimes_{\mathbb{C}_p} \mathbb{B}_{dr, \mathbb{C}_p}.$$

Then $\alpha \in H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p)(l)^{\hat{G}_{\sigma_p}}$ if and only if $w(\alpha) \in F^l H_{DR}^j(X_{\hat{k}_{\sigma_p}}) = F^l H_{DR}^j(X_{\mathbb{C}_p}) \cap H_{DR}^j(X_{\hat{k}_{\sigma_p}})$. That is we have

$$H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p)(l)^{\hat{G}_{\sigma_p}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \langle F^l H_{DR}^j(X_{\hat{k}_{\sigma_p}}) \cap H^j R\alpha(X)(H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p)) \rangle_{\mathbb{Q}_p} \subset H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p, X^{et}) \otimes \mathbb{B}_{dr, \mathbb{C}_p},$$

where $\langle - \rangle_{\mathbb{Q}_p}$ denote the \mathbb{Q}_p vector space generated by $(-)$.

Proof. Follows immediately from the fact that $H^j R\alpha(X)$ is a filtered quasi-isomorphism compatible with the Galois action of \hat{G}_{σ_p} by [1]. □

These propositions, together with the results of section 3 implies the Tate conjecture :

Theorem 2. (i) *Let k a field of finite type over \mathbb{Q} . Let $X \in \text{PSmVar}(k)$. Let $p \in \mathbb{N} \setminus \delta(k, X)$ a prime number. Then the Tate conjecture holds for X . That is for $d \in \mathbb{Z}$, the cycle class map*

$$\mathcal{Z}^d(X) \otimes \mathbb{Q}_p \rightarrow H_{et}^{2d}(X_{\bar{k}}, \mathbb{Q}_p)^G, \quad Z \mapsto [Z]$$

is surjective, where $G = \text{Gal}(\bar{k}/k)$.

(ii) *Let K a p -adic field which is unramified. Let $X \in \text{PSmVar}(K)$ with good reduction. Then the Tate conjecture holds for X . That is for $d \in \mathbb{Z}$, the cycle class map*

$$\mathcal{Z}^d(X) \otimes \mathbb{Q}_p \rightarrow H_{et}^{2d}(X_{\mathbb{C}_p}, \mathbb{Q}_p)^G, \quad Z \mapsto [Z]$$

is surjective, where $G = \text{Gal}(\mathbb{C}_p, K)$.

Proof. (i): Consider an embedding $\sigma_p : k \hookrightarrow \mathbb{C}_p$. Then $k \subset \bar{k} \subset \mathbb{C}_p$ and $k \subset \hat{k}_{\sigma_p} \subset \mathbb{C}_p$, where \hat{k}_{σ_p} is the p -adic field which is the completion of k with respect the p adic norm given by σ_p . Let $\alpha \in H_{et}^{2d}(X_{\bar{k}}, \mathbb{Z}_p)(d)^G$. Using definition 2, by proposition 7(ii)' and proposition 8 there exists $(\lambda_i)_{1 \leq i \leq n} \in \mathbb{Z}_p$ and

$$(w_{Li})_{1 \leq i \leq n} \in \mathbb{H}_{pet}^{2d}(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}, \log, \mathcal{O}}^{\bullet \geq d})$$

such that

$$w(\alpha) = \sum_{1 \leq i \leq n} \lambda_i \cdot w_{Li} \in \mathbb{H}_{pet}^{2d}(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}}^{\bullet \geq d}) = F^d H_{DR}^{2d}(X_{\mathbb{C}_p}), \quad w_{Li} := H^{2d} OL_X(w_{Li}).$$

By theorem 1(ii) and (iii)', there exists $Z_i \in \mathcal{Z}^d(X_{\mathbb{C}_p})$ such that $w_{Li} = [Z_i] \in H_{DR}^{2d}(X_{\mathbb{C}_p})$. Hence,

$$Z := \sum_{1 \leq i \leq n} \lambda_i [Z_i] \in \mathcal{Z}^d(X_{\mathbb{C}_p}) \otimes \mathbb{Q}_p$$

satisfies $w(\alpha) = [Z] \in H_{DR}^{2d}(X_{\mathbb{C}_p})$. By [1], considering the commutative diagram of abelian groups whose rows are exact, where $j : X \setminus |Z| \hookrightarrow X$ denote the open embedding,

$$\begin{array}{ccccc} H_{et,Z}^{2d}(X_{\bar{k}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{B}_{cris} & \xrightarrow{B_{cris,X}(\gamma_Z^\vee)} & H_{et}^{2d}(X_{\bar{k}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{B}_{cris} & \xrightarrow{j^*} & H_{et}^{2d}((X \setminus Z)_{\bar{k}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{B}_{cris} , \\ \downarrow Ra_{X^*} \Gamma_Z \alpha(X) & & \downarrow Ra(X) & & \downarrow Ra(X \setminus Z) \\ H_{DR,Z}^{2d}(X_{\hat{k}_{\sigma_p}}, \mathbb{B}_{cris}) & \xrightarrow{DR(X)(OB_{cris,X})(\gamma_Z^\vee)} & H_{DR}^{2d}(X_{\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}} \mathbb{B}_{cris} & \xrightarrow{j^*} & H_{DR}^{2d}((X \setminus Z)_{\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}} \mathbb{B}_{cris} \end{array}$$

we get $\alpha = [Z] \in H_{et}^{2d}(X_{\bar{k}}, \mathbb{Z}_p)$. Since $[Z]$ is G invariant,

$$Z' := 1/\text{card}(gZ, g \in G) \sum_{g \in G/G_Z} gZ \in \mathcal{Z}^d(X) \otimes \mathbb{Q}_p$$

satisfy $[Z'] = [Z] = \alpha \in H_{et}^{2d}(X_{\bar{k}}, \mathbb{Z}_p)$.

(ii): Similar to (i). □

It is well known that theorem 2 implies the following :

Corollary 2. *Let $X \in \text{PSmVar}(\mathbb{C})$.*

(i) *The standard conjectures holds for X .*

(ii) *Let $k \subset \mathbb{C}$ a subfield of finite type over \mathbb{Q} over which X is defined, that is $X \simeq X_k \otimes_k \mathbb{C}$ in $\text{PSmVar}(\mathbb{C})$ with $X_k \in \text{PSmVar}(k)$. For $\theta \in \text{Aut}(\mathbb{C}/\bar{k})$, we have the isomorphism $\theta : X \xrightarrow{\sim} X_\theta$ in Sch. Then $\theta \in \text{Aut}(\mathbb{C}/\bar{k})$ induces, for each $j \in \mathbb{Z}$, an isomorphism*

$$\theta : H_{\text{sing}}^j(X^{an}, \mathbb{C}) \xrightarrow{H^j \text{ev}(X)^{-1}} H_{DR}^j(X) \xrightarrow{\theta^*} H_{DR}^j(X_\theta) \xrightarrow{H^j \text{ev}(X_\theta)} H_{\text{sing}}^j(X_\theta^{an}, \mathbb{C}).$$

Let $d \in \mathbb{N}$. Let $\alpha \in F^d H^{2d}(X^{an}, \mathbb{Q})$ where $F^d H^{2d}(X^{an}, \mathbb{Q}) := F^d H_{DR}^{2d}(X) \cap H_{\text{sing}}^{2d}(X^{an}, \mathbb{Q}) \subset H^{2d}(X^{an}, \mathbb{C})$. If $\theta(\alpha) \in H_{\text{sing}}^{2d}(X_\theta^{an}, \mathbb{Q})$ for all $\theta \in \text{Aut}(\mathbb{C}/\bar{k})$, then $\alpha = [Z]$ with $Z \in \mathcal{Z}^d(X)$.

Proof. Standard. We consider, for $i \in \mathbb{Z}$, $k' \subset \mathbb{C}$ a subfield and $Y \in \text{Var}(k')$, the canonical morphism

$$T(Y) := T^i(Y) : H_{\text{sing}}^i(Y_{\mathbb{C}}^{an}, \mathbb{Q}) \rightarrow H_{\text{sing}}^i(Y_{\mathbb{C}}^{an}, \mathbb{Q}_p) \xrightarrow{\sim} H_{et}^i(Y_{\mathbb{C}}, \mathbb{Q}_p) \xrightarrow{\sim} H_{et}^i(Y_{k'}, \mathbb{Q}_p).$$

(i): Let $k \subset \mathbb{C}$ of finite type over \mathbb{Q} such that X is defined. Denote $G = \text{Gal}(\bar{k}/k)$ the Galois group. Denote $d_X = \dim(X)$ and for short $X^{an} = X$. Let $L^j \in H_{\text{sing}}^{2d_X - 2j}(X \times X, \mathbb{Q})$ the class inducing the cup product with of the intersection of j hyperplane sections on $H_{\text{sing}}^{d_X - j}(X, \mathbb{Q})$ and zero on $H_{\text{sing}}^i(X, \mathbb{Q})$, $i \neq d_X - j$. Let $\Lambda^j \in H_{\text{sing}}^{2d_X + 2j}(X \times X, \mathbb{Q})$ inducing the inverse of L_j on $H_{\text{sing}}^{d_X + j}(X, \mathbb{Q})$ and zero on $H_{\text{sing}}^i(X, \mathbb{Q})$, $i \neq d_X + j$. Consider an isomorphism $\sigma_p : \mathbb{C} \xrightarrow{\sim} \mathbb{C}_p$ such that $\hat{k}_{\sigma_p} \subset \mathbb{C}_p$ is unramified and

X has good reduction. We have $T(X \times X)(L^j) \in H_{et}^{2d_X-2j}((X \times X)_{\bar{k}}, \mathbb{Q}_p)^G$ since it is the class of an algebraic cycle. Hence,

$$T(X \times X)(\Lambda^j) \in H_{et}^{2d_X+2j}((X \times X)_{\bar{k}}, \mathbb{Q}_p)^G$$

since the inverse of a Galois invariant morphism is Galois invariant. Hence, by theorem 2(i),

$$\Lambda^j = [Z], \text{ with } Z \in \mathcal{Z}^{d_X+j}(X \times X) \otimes \mathbb{Q}.$$

(ii): As $\alpha \in H^{2d} \Gamma(X_{\bullet}, \Omega_X)$, where $X = \cup_{i=1}^s X_i$ is an open affine cover, there exists a subfield $k \subset k' \subset \mathbb{C}$ of finite type over \mathbb{Q} over which α is defined, that is $\alpha \in H_{DR}^{2d}(X_{k'}) \subset H_{DR}^{2d}(X)$, where $X_{k'} := X_k \otimes_k k' \in \text{PSmVar}(k')$ satisfy $X_{k'} \otimes_{k'} \mathbb{C} \simeq X$ in $\text{PSmVar}(\mathbb{C})$. Consider

$$T(X_{k'}) (\alpha) \in H^{2d}(X_{\bar{k}'}, \mathbb{Q}_p).$$

If $\theta(\alpha) \in H_{\text{sing}}^{2d}(X_{\theta}^{an}, \mathbb{Q})$ for all $\theta \in \text{Aut}(\mathbb{C}/\bar{k}')$, we get

$$T(X_{k'}) (\alpha) = 1/\text{card}(\text{Aut}(\mathbb{C}/\bar{k}')\alpha) \sum_{\theta \in \text{Aut}(\mathbb{C}/k)} \theta^{-1,*} T(X_{\theta})(\theta(\alpha)) \in H_{et}^{2d}(X_{\bar{k}'}, \mathbb{Q}_p) = H_{et}^{2d}(X, \mathbb{Q}_p),$$

hence, for $g \in \text{Gal}(\bar{k}'/k)$, we get

$$\begin{aligned} g \cdot T(X_{k'}) (\alpha) &= 1/\text{card}(\text{Aut}(\mathbb{C}/\bar{k}')\alpha) \sum_{\theta \in \text{Aut}(\mathbb{C}/k)} g \cdot \theta^{-1,*} T(X_{\theta})(\theta(\alpha)) \\ &= 1/\text{card}(\text{Aut}(\mathbb{C}/k)\alpha) \sum_{\theta \in \text{Aut}(\mathbb{C}/k)} \theta^{-1,*} T(X_{\theta})(\theta(\alpha)) = T(X_{k'}) (\alpha) \in H_{et}^{2d}(X_{\bar{k}'}, \mathbb{Q}_p), \end{aligned}$$

that is, $T(X_{k'}) (\alpha) \in H^{2d}(X_{\bar{k}'}, \mathbb{Q}_p)^G$, with $G := \text{Gal}(\bar{k}'/k)$. Hence, if $\theta(\alpha) \in H_{\text{sing}}^{2d}(X_{\theta}^{an}, \mathbb{Q})$ for all $\theta \in \text{Aut}(\mathbb{C}/\bar{k}')$, we get by theorem 2(i),

$$T(X) (\alpha) = T(X_{k'}) (\alpha) = [Z], \text{ with } Z \in \mathcal{Z}^d(X) \otimes \mathbb{Q}.$$

□

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