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# De Rham logarithmic classes and Tate conjecture

Johann Bouali

April 3, 2023

## Abstract

We introduce the definition of De Rham logarithmic classes. We show that the De Rham class of an algebraic cycle of a smooth algebraic variety over a field of characteristic zero is logarithmic and conversely that a logarithmic class of bidegree  $(d, d)$  is the De Rham class of an algebraic cycle (of codimension  $d$ ). We deduce the Tate conjecture for smooth projective varieties over fields of finite type over  $\mathbb{Q}$  and over  $p$ -adic fields for  $\mathbb{Q}_p$  coefficients, under good reduction hypothesis.

## 1 Introduction

In this work, we introduce for  $X$  a noetherian scheme, the definition of logarithmic De Rham cohomology classes which are for each  $j \in \mathbb{Z}$ , the subgroup  $H^j OL_X(\mathbb{H}_{et}^j(X, \Omega_{X, \log}^\bullet)) \subset \mathbb{H}_{et}^j(X, \Omega_X^\bullet) =: H_{DR}^j(X)$  of the De Rham cohomology abelian group. The subcomplex  $OL_X : \Omega_{X, \log}^\bullet \hookrightarrow \Omega_X^\bullet$  of abelian sheaves on  $X$  consisting of logarithmic forms is introduced in definition 1.

Let  $X$  be a smooth algebraic variety over a field of characteristic zero  $k$ . To an algebraic cycle  $Z \in \mathcal{Z}^d(X)$ , we associate its De Rham cohomology class  $[Z] \in H_{DR}^{2d}(X)$ , which is by definition, as for any Weil cohomology theory, the image of the fundamental class  $[Z] \in H_{DR, Z}^{2d}(X)$  by the canonical morphism  $H_{DR, Z}^{2d}(X) \rightarrow H^{2d}(X)$ . In section 3, we show (c.f. theorem 1(i)) that  $[Z] \in H_{DR}^{2d}(X)$  is logarithmic of bidegree  $(d, d)$ , that is

$$[Z] = H^{2d} OL_X([Z]_L) \in H_{DR}^{2d}(X), [Z]_L \in H_{et}^d(X, \Omega_{X, \log}^d)$$

where  $H_{et}^d(X, \Omega_{X, \log}^d) \subset \mathbb{H}_{et}^{2d}(X, \Omega_{X, \log}^\bullet)$  is the canonical subspace, as all the differentials of  $\Omega_{X, \log}^\bullet$  vanishes since by definition a logarithmic form is closed. This fact is a consequence of two properties :

- the Euler class of a vector bundle of rank  $d$  over an algebraic variety is logarithmic of bidegree  $(d, d)$  (c.f. proposition 2)
- the motivic isomorphisms applied to De Rham cohomology preserve logarithmic classes (c.f. proposition 1). The key point is that logarithmic De Rham forms on algebraic varieties over  $k$  are closed, (trivially)  $\mathbb{A}_k^1$  invariant and compatible with the transfers maps induced by finite morphisms of algebraic varieties and in particular finite correspondences. Since  $\Omega_{\log}^l \subset \Omega^l$  are (trivially)  $\mathbb{A}_k^1$  invariant presheaves with transfers on the category of smooth algebraic varieties over  $k$ , by a theorem of Voevodsky the cohomology presheaves of  $\mathbb{A}_k^1$  invariant presheaves with transfers are  $\mathbb{A}_k^1$  invariant.

Conversely, we show (c.f. theorem 1(ii)), that a logarithmic class  $w \in H_{DR}^{2d}(X)$  of bidegree  $(d, d)$  is the De Rham class of an algebraic cycle (of codimension  $d$ ).

The proof works as follows : A logarithmic class of bidegree  $(p, q)$  is locally acyclic for the Zariski topology of  $X$  since it is the etale cohomology of a single sheaf. This allows us to proceed by a finite induction using the crucial fact that the purity isomorphism for De Rham cohomology preserve logarithmic classes (c.f. proposition 3). The proof of proposition 3 follows from the fact that the purity isomorphism is motivic (see [5]), that the Euler class of a vector bundle of rank  $d$  over an algebraic variety is logarithmic of bidegree  $(d, d)$  (c.f. proposition 2) and that the motivic isomorphisms applied to De Rham cohomology

preserve logarithmic classes (c.f. proposition 1). At the final step, we use, for simplicity, the fact that for a scheme  $Y$ ,  $H_{et}^1(Y, \Omega_{Y, \log}^1) = H^1(Y, O_Y^*)$  is the Picard group of  $Y$ . In the case  $X$  projective, we also get a vanishing result (c.f. theorem 1(iii)) :  $H^{2d}OL_X(H_{et}^{d-k}(X, \Omega_{X, \log}^{d+k})) = 0$  for  $k > 0$ .

Let  $X$  a smooth projective variety over a field  $k$  of finite type over  $\mathbb{Q}$ . Let  $p$  be a prime number unramified over  $k$  such that  $X$  has good reduction at  $p$ ,  $\sigma_p : k \hookrightarrow \mathbb{C}_p$  a fixed embedding and  $\hat{k}$  the algebraic closure of  $k$  inside  $\mathbb{C}_p$ . Denote by  $\hat{k}_{\sigma_p} \subset \mathbb{C}_p$  the  $p$  adic completion of  $k$  with respect to  $\sigma_p$ . In a previous work (see [4], prop 2,3), we show that Tate classes  $\alpha \in H_{et}^{2d}(X_{\hat{k}}, \mathbb{Z}_p)(d)^G$  of  $X$  gives by the  $p$ -adic comparison isomorphism

$$H^{2d}R\alpha(X) : H_{et}^{2d}(X_{\mathbb{C}_p}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{B}_{cris, \hat{k}_{\sigma_p}} \xrightarrow{\sim} H_{DR}^{2d}(X) \otimes_k \mathbb{B}_{cris, \hat{k}_{\sigma_p}},$$

a  $\mathbb{Z}_p$  linear combination of logarithmic classes

$$w(\alpha) := H^{2d}R\alpha(X)(\alpha) := \sum_{i=1}^s \lambda_i w(\alpha)_{L,i} \in H_{DR}^{2d}(X_{\mathbb{C}_p}), \lambda_i \in \mathbb{Z}_p$$

with for each  $1 \leq i \leq s$ ,  $w(\alpha)_{L,i} \in H^{2d}OL_X \mathbb{H}_{et}^{2d}(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}, \log}^{\geq d})$ . We deduce, by theorem 1(ii) and (iii), the Tate conjecture for  $X$  with  $\mathbb{Q}_p$  coefficient (c.f. theorem 2). Note that for a  $p$ -adic field  $K \subset \mathbb{C}_p$ ,  $\dim H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_l)^G > \dim H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p)^G$  for  $l \neq p$  in general. Hence it is not clear whether Tate conjecture holds for smooth projective varieties over finite fields  $k_0$ , since the base change for etale cohomology for  $f : X \rightarrow O_K$ ,  $O_K = W(k_0)$  only holds for  $\mathbb{Z}_l$  coefficients,  $l \neq p$ . In particular it is not clear if point (iii) of corollary 1 of [4] holds.

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## 2 Preliminaries and Notations

- Denote by Top the category of topological spaces and RTop the category of ringed spaces.
- Denote by Cat the category of small categories and RCat the category of ringed topoi.
- For  $\mathcal{S} \in \text{Cat}$  and  $X \in \mathcal{S}$ , we denote  $\mathcal{S}/X \in \text{Cat}$  the category whose objects are  $Y/X := (Y, f)$  with  $Y \in \mathcal{S}$  and  $f : Y \rightarrow X$  is a morphism in  $\mathcal{S}$ , and whose morphisms  $\text{Hom}((Y', f'), (Y, f))$  consists of  $g : Y' \rightarrow Y$  in  $\mathcal{S}$  such that  $f \circ g = f'$ .
- For  $(\mathcal{S}, O_{\mathcal{S}}) \in \text{RCat}$  a ringed topos, we denote by
  - $\text{PSh}(\mathcal{S})$  the category of presheaves of  $O_{\mathcal{S}}$  modules on  $\mathcal{S}$  and  $\text{PSh}_{O_{\mathcal{S}}}(\mathcal{S})$  the category of presheaves of  $O_{\mathcal{S}}$  modules on  $\mathcal{S}$ , whose objects are  $\text{PSh}_{O_{\mathcal{S}}}(\mathcal{S})^0 := \{(M, m), M \in \text{PSh}(\mathcal{S}), m : M \otimes O_{\mathcal{S}} \rightarrow M\}$ , together with the forgetful functor  $o : \text{PSh}(\mathcal{S}) \rightarrow \text{PSh}_{O_{\mathcal{S}}}(\mathcal{S})$ ,
  - $C(\mathcal{S}) = C(\text{PSh}(\mathcal{S}))$  and  $C_{O_{\mathcal{S}}}(\mathcal{S}) = C(\text{PSh}_{O_{\mathcal{S}}}(\mathcal{S}))$  the big abelian category of complexes of presheaves of  $O_{\mathcal{S}}$  modules on  $\mathcal{S}$ ,
  - $C_{O_{\mathcal{S}}(2)fil}(\mathcal{S}) := C_{(2)fil}(\text{PSh}_{O_{\mathcal{S}}}(\mathcal{S})) \subset C(\text{PSh}_{O_{\mathcal{S}}}(\mathcal{S}), F, W)$ , the big abelian category of (bi)filtered complexes of presheaves of  $O_{\mathcal{S}}$  modules on  $\mathcal{S}$  such that the filtration is biregular and  $\text{PSh}_{O_{\mathcal{S}}(2)fil}(\mathcal{S}) := (\text{PSh}_{O_{\mathcal{S}}}(\mathcal{S}), F, W)$ .
- Let  $(\mathcal{S}, O_{\mathcal{S}}) \in \text{RCat}$  a ringed topos with topology  $\tau$ . For  $F \in C_{O_{\mathcal{S}}}(\mathcal{S})$ , we denote by  $k : F \rightarrow E_{\tau}(F)$  the canonical flasque resolution in  $C_{O_{\mathcal{S}}}(\mathcal{S})$  (see [3]). In particular for  $X \in \mathcal{S}$ ,  $H^*(X, E_{\tau}(F)) \xrightarrow{\sim} \mathbb{H}_{\tau}^*(X, F)$ .
- For  $f : \mathcal{S}' \rightarrow \mathcal{S}$  a morphism with  $\mathcal{S}, \mathcal{S}' \in \text{RCat}$ , endowed with topology  $\tau$  and  $\tau'$  respectively, we denote for  $F \in C_{O_{\mathcal{S}}}(\mathcal{S})$  and each  $j \in \mathbb{Z}$ ,
  - $f^* := H^j \Gamma(\mathcal{S}, k \circ \text{ad}(f^*, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \rightarrow \mathbb{H}^j(\mathcal{S}', f^*F)$ ,

$$- f^* := H^j \Gamma(\mathcal{S}, k \circ \text{ad}(f^{*mod}, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \rightarrow \mathbb{H}^j(\mathcal{S}', f^{*mod}F),$$

the canonical maps.

- For  $\mathcal{X} \in \text{Cat}$  a (pre)site and  $p$  a prime number, we consider the full subcategory

$$\text{PSh}_{\mathbb{Z}_p}(\mathcal{X}) \subset \text{PSh}(\mathbb{N} \times \mathcal{X}), \quad F = (F_n)_{n \in \mathbb{N}}, \quad p^n F_n = 0, \quad F_{n+1}/p^n \xrightarrow{\sim} F_n$$

$$C_{\mathbb{Z}_p}(\mathcal{X}) := C(\text{PSh}_{\mathbb{Z}_p}(\mathcal{X})) \subset C(\mathbb{N} \times \mathcal{X}) \text{ and}$$

$$\mathbb{Z}_p := \mathbb{Z}_{p,\mathcal{X}} := ((\mathbb{Z}/p^*\mathbb{Z})_{\mathcal{X}}) \in \text{PSh}_{\mathbb{Z}_p}(\mathcal{X})$$

the diagram of constant presheaves on  $\mathcal{X}$ .

- Denote by  $\text{Sch} \subset \text{RTop}$  the subcategory of schemes (the morphisms are the morphisms of locally ringed spaces). For a field  $k$ , we consider  $\text{Sch}/k := \text{Sch}/\text{Spec } k$  the category of schemes over  $\text{Spec } k$ , that is whose object are  $X := (X, a_X)$  with  $X \in \text{Sch}$  and  $a_X : X \rightarrow \text{Spec } k$  a morphism and whose objects are morphism of schemes  $f : X' \rightarrow X$  such that  $f \circ a_{X'} = a_X$ . We then denote by
  - $\text{Var}(k) = \text{Sch}^{ft}/k \subset \text{Sch}/k$  the full subcategory consisting of algebraic varieties over  $k$ , i.e. schemes of finite type over  $k$ ,
  - $\text{PVar}(k) \subset \text{QPVar}(k) \subset \text{Var}(k)$  the full subcategories consisting of quasi-projective varieties and projective varieties respectively,
  - $\text{PSmVar}(k) \subset \text{SmVar}(k) \subset \text{Var}(k)$ ,  $\text{PSmVar}(k) := \text{PVar}(k) \cap \text{SmVar}(k)$ , the full subcategories consisting of smooth varieties and smooth projective varieties respectively.

For a morphism of field  $\sigma : k \hookrightarrow K$ , we have the extension of scalar functor

$$\otimes_k K : \text{Sch}/k \rightarrow \text{Sch}/K, \quad X \mapsto X_K := X_{K,\sigma} := X \otimes_k K, \quad (f : X' \rightarrow X) \mapsto (f_K := f \otimes I : X'_K \rightarrow X_K).$$

which is left adjoint to the restriction of scalar

$$\text{Res}_{k/K} : \text{Sch}/K \rightarrow \text{Sch}/k, \quad X = (X, a_X) \mapsto X = (X, \sigma \circ a_X), \quad (f : X' \rightarrow X) \mapsto (f : X' \rightarrow X)$$

The extension of scalar functor restrict to a functor

$$\otimes_k K : \text{Var}(k) \rightarrow \text{Var}(K), \quad X \mapsto X_K := X_{K,\sigma} := X \otimes_k K, \quad (f : X' \rightarrow X) \mapsto (f_K := f \otimes I : X'_K \rightarrow X_K).$$

and for  $X \in \text{Var}(k)$  we have  $\pi_{k/K}(X) : X_K \rightarrow X$  the projection in  $\text{Sch}/k$ .

- For  $X \in \text{Sch}$ , we denote  $X^{pet} \subset \text{Sch}/X$  the pro etale site (see [2]) which is the full subcategory of  $\text{Sch}/X$  whose object consists of weakly etale maps  $U \rightarrow X$  (that is flat maps  $U \rightarrow X$  such that  $\Delta_U : U \rightarrow U \times_X U$  is also flat) and whose topology is generated by fpqc covers. We then have the canonical morphism of site

$$\nu_X : X^{pet} \rightarrow X, \quad (U \rightarrow X) \mapsto (U \rightarrow X)$$

For  $F \in C(X^{et})$ ,

$$\text{ad}(\nu_X^*, R\nu_{X*})(F) : F \rightarrow R\nu_{X*}\nu_X^*F$$

is an isomorphism in  $D(X^{et})$ , in particular, for each  $n \in \mathbb{Z}$

$$\nu_X^* : \mathbb{H}_{et}^n(X, F) \xrightarrow{\sim} \mathbb{H}_{pet}^n(X, \nu_X^*F)$$

are isomorphisms, For  $X \in \text{Sch}$ , we denote

$$- \underline{\mathbb{Z}}_p := \varprojlim_{n \in \mathbb{N}} \nu_X^*(\mathbb{Z}/p^n\mathbb{Z})_{X^{et}} \in \text{PSh}(X^{pet}) \text{ the constant presheaf on } \mathcal{X},$$

–  $l_{p,\mathcal{X}} := (p(\ast)) : \underline{\mathbb{Z}}_{p_X} \rightarrow \nu_X^\ast(\mathbb{Z}/p\mathbb{Z})_{X^{pet}}$  the projection map in  $\text{PSh}(\mathbb{N} \times \mathcal{X}^{pet})$ .

- An affine scheme  $U \in \text{Sch}$  is said to be  $w$ -contractible if any faithfully flat weakly etale map  $V \rightarrow U$ ,  $V \in \text{Sch}$ , admits a section. We will use the facts that (see [2]):

- Any scheme  $X \in \text{Sch}$  admits a pro-etale affine cover  $(r_i : X_i \rightarrow X)_{i \in I}$  with for each  $i \in I$ ,  $X_i$  a  $w$ -contractible affine scheme and  $r_i : X_i \rightarrow X$  a weakly etale map. For  $X \in \text{Var}(k)$  with  $k$  a field, we may assume  $I$  finite since the topological space  $X$  is then quasi-compact.
- If  $U \in \text{Sch}$  is a  $w$ -contractible affine scheme, then for any sheaf  $F \in \text{Shv}(U^{pet})$ ,  $H_{pet}^i(U, F) = 0$  for  $i \neq 0$  since  $\Gamma(U, -)$  is an exact functor.

- Let  $X \in \text{Sch}$ . Considering its De Rham complex  $\Omega_X^\bullet := DR(X)(O_X)$ , we have for  $j \in \mathbb{Z}$  its De Rham cohomology  $H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^\bullet)$ . For  $X \in \text{Var}(k)$ , the differentials of  $\Omega_X^\bullet := \Omega_{X/k}^\bullet$  are by definition  $k$ -linear, thus  $H_{DR}^j(X)$  has a structure of a  $k$  vector space. If  $X \in \text{SmVar}(k)$ , then  $H_{DR}^j(X) = \mathbb{H}_{et}^j(X, \Omega_X^\bullet)$  since  $\Omega_{/k}^\bullet \in C(\text{SmVar}(k))$  is  $\mathbb{A}^1$  local and admits transfers where

$$\begin{aligned} \Omega_{/k}^\bullet &\in C(\text{SmVar}(k)), X \mapsto \Omega_{/k}^\bullet(X) := \Gamma(X, \Omega_X^\bullet), \\ (f : X' \rightarrow X) &\mapsto \Omega_{/k}^\bullet(f) := f^\ast : \Gamma(X, \Omega_X^\bullet) \rightarrow \Gamma(X', \Omega_{X'}^\bullet) \end{aligned}$$

(see [3]).

- Let  $X \in \text{Var}(k)$ . Let  $X = \cup_{i=1}^s X_i$  an open affine cover. For  $I \subset [1, \dots, s]$ , we denote  $X_I := \cap_{i \in I} X_i$ . We get  $X_\bullet \in \text{Fun}(P([1, \dots, s]), \text{Var}(k))$ . Since quasi-coherent sheaves on affine noetherian schemes are acyclic, we have for each  $j \in \mathbb{Z}$ ,  $H_{DR}^j(X) = \Gamma(X_\bullet, \Omega_{X_\bullet}^\bullet)$ .
- For  $X \in \text{Sch}$  noetherian irreducible and  $d \in \mathbb{N}$ , we denote by  $\mathcal{Z}^d(X)$  the group of algebraic cycles of codimension  $d$ , which is the free abelian group generated by irreducible closed subsets of codimension  $d$ .
- For  $K$  a field which is complete with respect to a  $p$ -adic norm, we consider  $O_K \subset K$  the subring of  $K$  consisting of integral elements, that is  $x \in K$  such that  $|x| \leq 1$ . For  $X \in \text{Var}(K)$ , we denote by  $X^\mathcal{O} \in \text{Sch}/O_K$  a (non canonical) integral model of  $X$ , i.e. satisfying  $X^\mathcal{O} \otimes_{O_K} K = X$ .
- For  $X \in \text{Sch}$  and  $p$  a prime number, we denote by  $c : \hat{X}^{(p)} \rightarrow X$  the morphism in  $\text{RTop}$  which is the completion along the ideal generated by  $p$ .
- Let  $K$  a field which is complete with respect to a  $p$ -adic norm and  $X \in \text{PVar}(K)$  projective. For  $X^\mathcal{O} \in \text{Sch}/O_K$  an integral model of  $X$ , i.e. satisfying  $X^\mathcal{O} \otimes_{O_K} K = X$ , we consider the morphism in  $\text{RTop}$

$$c := (c \otimes I) : \hat{X}^{(p)} := \hat{X}^{\mathcal{O},(p)} \otimes_{O_K} K \rightarrow X^\mathcal{O} \otimes_{O_K} K = X.$$

We have then, by GAGA (c.f. EGA 3), for  $F \in \text{Coh}_{O_X}(X)$  a coherent sheaf of  $O_X$  module,  $c^\ast : H^k(X^\mathcal{O}, F) \xrightarrow{\sim} H^k(\hat{X}^{\mathcal{O},(p)}, c^{\ast mod} F)$  for all  $k \in \mathbb{Z}$ , in particular

$$c^\ast : \mathbb{H}^k(X, \Omega_X^{\bullet \geq l}) \xrightarrow{\sim} \mathbb{H}^k(\hat{X}^{(p)}, \Omega_{\hat{X}^{(p)}}^{\bullet \geq l})$$

for all  $k, l \in \mathbb{Z}$ .

- For  $k$  a field of finite type over  $\mathbb{Q}$  and  $X \in \text{SmVar}(k)$ , we denote  $\delta(k, X) \subset \mathbb{N}$  the finite set consisting of prime numbers such that for  $p \in \mathbb{N} \setminus \delta$  a prime number  $k$  is unramified at  $p$  and such that there exists an integral model  $X_{\hat{k}_{\sigma_p}}^\mathcal{O}$  of  $X_{\hat{k}_{\sigma_p}}$  with good reduction modulo  $p$  for all embeddings  $\sigma_p : k \hookrightarrow \mathbb{C}_p$ ,  $\hat{k}_{\sigma_p} \subset \mathbb{C}_p$  being the  $p$ -adic completion of  $k$  with respect to  $\sigma_p$ . To see that  $\delta(k, X)$  is a finite set, considering  $k = k_0(x_1, \dots, x_d)(x_{d+1})$  where  $x_1, \dots, x_d$  are algebraically independent and  $k_0$  is a number field one can take an integral model  $Y \rightarrow O_{k_0}[x_1, \dots, x_{d+1}]/f_{d+1}$  of  $X$ , a desingularization of  $Y' \rightarrow Y$  of  $Y$  and then see that  $\delta(k, X)$  is contained in the discriminant of the family  $Y' \rightarrow Y \rightarrow O_{k_0}[x_1, \dots, x_{d+1}]/f_{d+1}$ .

- Denote by  $\text{AnSp}(\mathbb{C}) \subset \text{RTop}$  the full subcategory of analytic spaces over  $\mathbb{C}$ , and by  $\text{AnSm}(\mathbb{C}) \subset \text{AnSp}(\mathbb{C})$  the full subcategory of smooth analytic spaces (i.e. complex analytic manifold). Denote by  $\text{CW} \subset \text{Top}$  the full subcategory of  $CW$  complexes. Denote by  $\text{Diff}(\mathbb{R}) \subset \text{RTop}$  the full subcategory of differentiable (real) manifold.
- Denote  $\text{Sch}^2 \subset \text{RTop}^2$  the subcategory whose objects are couples  $(X, Z)$  with  $X = (X, O_X) \in \text{Sch}$  and  $Z \subset X$  a closed subset and whose set of morphisms  $\text{Hom}((X', Z'), (X, Z))$  consists of  $f : X' \rightarrow X$  of locally ringed spaces such that  $f^{-1}(Z) \subset Z'$ .
- Let  $k$  a field of characteristic zero. Denote  $\text{SmVar}^2(k) \subset \text{Var}^2(k) \subset \text{Sch}^2/k$  the full subcategories whose objects are  $(X, Z)$  with  $X \in \text{Var}(k)$ , resp.  $X \in \text{SmVar}(k)$ , and  $Z \subset X$  is a closed subset, and whose morphisms  $\text{Hom}((X', Z') \rightarrow (X, Z))$  consists of  $f : X' \rightarrow X$  of schemes over  $k$  such that  $f^{-1}(Z) \subset Z'$ .
- Let  $k$  a field of characteristic zero. For  $X, X' \in \text{Var}(k)$ , with  $X'$  irreducible, we denote  $\mathcal{Z}^{fs/X'}(X' \times X) \subset \mathcal{Z}_{d_{X'}}(X' \times X)$  which consist of algebraic cycles  $\alpha = \sum_i n_i \alpha_i \in \mathcal{Z}_{d_{X'}}(X' \times X)$  such that, denoting  $\text{supp}(\alpha) = \cup_i \alpha_i \subset X' \times X$  its support and  $p' : X' \times X \rightarrow X'$  the projection,  $p'_{|\text{supp}(\alpha)} : \text{supp}(\alpha) \rightarrow X'$  is finite surjective. We denote by  $\text{Cor SmVar}(k)$  the category such that the objects are  $\{X \in \text{SmVar}(k)\}$  and such that  $\text{Hom}_{\text{Cor SmVar}(k)}(X', X) := \mathcal{Z}^{fs/X'}(X' \times X)$ .
- For  $X \in \text{Top}$  and  $Z \subset X$  a closed subset, denoting  $j : X \setminus Z \hookrightarrow X$  the open complementary, we will consider

$$\Gamma_Z^\vee \mathbb{Z}_X := \text{Cone}(\text{ad}(j!, j^*)(\mathbb{Z}_X) : j!j^* \mathbb{Z}_X \hookrightarrow \mathbb{Z}_X) \in C(X)$$

and denote for short  $\gamma_Z^\vee := \gamma_X^\vee(\mathbb{Z}_X) : \mathbb{Z}_X \rightarrow \Gamma_Z^\vee \mathbb{Z}_X$  the canonical map in  $C(X)$ .

- For  $X \in \text{Var}(k)$  and  $Z \subset X$  a closed subset, denoting  $j : X \setminus Z \hookrightarrow X$  the open complementary, we will consider

$$\Gamma_Z^\vee \mathbb{Z}_X := \text{Cone}(\text{ad}(j_\sharp, j^*)(\mathbb{Z}_X) : \mathbb{Z}_X \hookrightarrow \mathbb{Z}_X) \in C(\text{Var}(k)^{\text{sm}}/X)$$

and denote for short  $\gamma_Z^\vee := \gamma_X^\vee(\mathbb{Z}_X) : \mathbb{Z}_X \rightarrow \Gamma_Z^\vee \mathbb{Z}_X$  the canonical map in  $C(\text{Var}(k)^{\text{sm}}/X)$ . If  $X \in \text{SmVar}(k)$ , it gives

$$a_{X\sharp} \Gamma_Z^\vee \mathbb{Z}_X := \text{Cone}(a_{X\sharp} \circ \text{ad}(j_\sharp, j^*)(\mathbb{Z}_X) : \mathbb{Z}(U) \hookrightarrow \mathbb{Z}(X)) \in C(\text{SmVar}(k)).$$

Then  $M_Z(X) := a_{X\sharp} \Gamma_Z^\vee \mathbb{Z}_X \in \text{DA}(k)$ .

- We denote  $\mathbb{I}^n := [0, 1]^n \in \text{Diff}(\mathbb{R})$  (with boundary). For  $X \in \text{Top}$  and  $R$  a ring, we consider its singular cochain complex

$$C_{\text{sing}}^*(X, R) := (\mathbb{Z} \text{Hom}_{\text{Top}}(\mathbb{I}^*, X)^\vee) \otimes R$$

and for  $l \in \mathbb{Z}$  its singular cohomology  $H_{\text{sing}}^l(X, R) := H^n C_{\text{sing}}^*(X, R)$ . In particular, we get by functoriality the complex

$$C_{X, R \text{sing}}^* \in C_R(X), (U \subset X) \mapsto C_{\text{sing}}^*(U, R)$$

We will consider the canonical embedding

$$C^* \iota_{2i\pi\mathbb{Z}/\mathbb{C}}(X) : C_{\text{sing}}^*(X, 2i\pi\mathbb{Z}) \hookrightarrow C_{\text{sing}}^*(X, \mathbb{C}), \alpha \mapsto \alpha \otimes 1$$

whose image consists of cochains  $\alpha \in C_{\text{sing}}^j(X, \mathbb{C})$  such that  $\alpha(\gamma) \in 2i\pi\mathbb{Z}$  for all  $\gamma \in \mathbb{Z} \text{Hom}_{\text{Top}}(\mathbb{I}^*, X)$ . We get by functoriality the embedding in  $C(X)$

$$\begin{aligned} & C^* \iota_{2i\pi\mathbb{Z}/\mathbb{C}, X} : C_{X, 2i\pi\mathbb{Z}, \text{sing}}^* \hookrightarrow C_{X, \mathbb{C}, \text{sing}}^* \\ (U \subset X) & \mapsto (C^* \iota_{2i\pi\mathbb{Z}/\mathbb{C}}(U) : C_{\text{sing}}^*(U, 2i\pi\mathbb{Z}) \hookrightarrow C_{\text{sing}}^*(U, \mathbb{C})) \end{aligned}$$

We recall we have

- For  $X \in \text{Top}$  locally contractile, e.g.  $X \in \text{CW}$ , and  $R$  a ring, the inclusion in  $C_R(X)$   $c_X : R_X \rightarrow C_{X,R,\text{sing}}^*$  is by definition an equivalence top local and that we get by the small chain theorem, for all  $l \in \mathbb{Z}$ , an isomorphism  $H^l c_X : H^l(X, R_X) \xrightarrow{\sim} H_{\text{sing}}^l(X, R)$ .
- For  $X \in \text{Diff}(\mathbb{R})$ , the restriction map

$$r_X : \mathbb{Z} \text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^*, X)^\vee \rightarrow C_{\text{sing}}^*(X, R), \quad w \mapsto w : (\phi \mapsto w(\phi))$$

is a quasi-isomorphism by Whitney approximation theorem.

In [4], we introduced the logarithmic De Rham complexes

**Definition 1.** (i) Let  $X = (X, O_X) \in \text{RCat}$  a ringed topos, we have in  $C(X)$  the subcomplex of presheaves of abelian groups

$$OL_X : \Omega_{X,\log}^\bullet \hookrightarrow \Omega_X^\bullet, \quad \text{s.t. for } X^\circ \in X \text{ and } p \in \mathbb{N}, p \geq 1,$$

$$\Omega_{X,\log}^p(X^\circ) := \langle df_{\alpha_1}/f_{\alpha_1} \wedge \cdots \wedge df_{\alpha_p}/f_{\alpha_p}, f_{\alpha_k} \in \Gamma(X^\circ, O_X)^* \rangle \subset \Omega_X^p(X^\circ),$$

where  $\Omega_X^\bullet := DR(X)(O_X) \in C(X)$  is the De Rham complex and  $\Gamma(X^\circ, O_X)^* \subset \Gamma(X^\circ, O_X)$  is the multiplicative subgroup consisting of invertible elements for the multiplication, here  $\langle, \rangle$  stand for the sub-abelian group generated by. By definition, for  $w \in \Omega_X^p(X^\circ)$ ,  $w \in \Omega_{X,\log}^p(X^\circ)$  if and only if there exists  $(n_i)_{1 \leq i \leq s} \in \mathbb{Z}$  and  $(f_{i,\alpha_k})_{1 \leq i \leq s, 1 \leq k \leq p} \in \Gamma(X^\circ, O_X)^*$  such that

$$w = \sum_{1 \leq i \leq s} n_i df_{i,\alpha_1}/f_{i,\alpha_1} \wedge \cdots \wedge df_{i,\alpha_p}/f_{i,\alpha_p} \in \Omega_X^p(X^\circ).$$

For  $p = 0$ , we set  $\Omega_{X,\log}^0 := \mathbb{Z}$ .

- (ii) For  $k$  a field of characteristic zero, we get from (i), for  $X \in \text{Var}(k)$ , the embedding  $OL_X : \Omega_{X,\log}^\bullet \hookrightarrow \Omega_X^\bullet$  in  $C(X)$ , where  $\Omega_X^\bullet := \Omega_{X/k}^\bullet \in C(X)$  is the De Rham complex : for  $X^\circ \subset X$  an open subset and  $w \in \Omega_X^p(X^\circ)$   $w \in \Omega_{X,\log}^p(X^\circ)$  if and only if there exists  $(n_i)_{1 \leq i \leq s} \in \mathbb{Z}$  and  $(f_{i,\alpha_k})_{1 \leq i \leq s, 1 \leq k \leq p} \in \Gamma(X^\circ, O_X)^*$  such that

$$w = \sum_{1 \leq i \leq s} n_i df_{i,\alpha_1}/f_{i,\alpha_1} \wedge \cdots \wedge df_{i,\alpha_p}/f_{i,\alpha_p} \in \Omega_X^p(X^\circ).$$

Let  $k$  a field of characteristic zero. We get an embedding in  $C(\text{SmVar}(k))$

$$OL : \Omega_{/k,\log}^\bullet \hookrightarrow \Omega_{/k}^\bullet, \quad \text{given by, for } X \in \text{SmVar}(k),$$

$$OL(X) := OL_X : \Omega_{/k,\log}^\bullet(X) := \Gamma(X, \Omega_{X,\log}^\bullet) \hookrightarrow \Gamma(X, \Omega_X^\bullet) =: \Omega_{/k}^\bullet(X).$$

- (iii) Let  $K$  a field which is complete for a  $p$ -adic norm. Let  $X \in \text{Var}(K)$ . For  $X^\circ \in \text{Sch}/O_K$  an integral model of  $X$ , i.e. satisfying  $X^\circ \otimes_{O_K} K = X$ , we write for short

$$OL_X := OL_X \circ \iota : \Omega_{X^\circ,\log,\mathcal{O}}^\bullet := \Omega_{X^\circ,\text{et},\log}^\bullet \hookrightarrow \Omega_{X^\circ,\text{et},\log}^\bullet \hookrightarrow \Omega_{X^\circ,\text{et}}^\bullet$$

the embedding in  $C(X^{\text{et}})$  consisting of integral logarithmic De Rham forms, with  $\iota : \Omega_{X^\circ,\text{et},\log}^\bullet \hookrightarrow \Omega_{X^\circ,\text{et},\log}^\bullet$ . We will also consider the embeddings of  $C(X^{\text{pet}})$

$$m \circ (OL_X \otimes I) : \Omega_{X^{\text{pet}},\log,\mathcal{O}}^\bullet \otimes \mathbb{Z}_p := \Omega_{X^{\text{pet}},\log,\mathcal{O}}^\bullet \otimes \mathbb{Z}_p \rightarrow \Omega_{X^{\text{pet}}}^\bullet, \quad (w \otimes \lambda_n)_{n \in \mathbb{N}} \mapsto (\lambda_n)_{n \in \mathbb{N}} \cdot w$$

and

$$OL_{\hat{X}^{(p)}} := (OL_{X^\circ/p^n})_{n \in \mathbb{N}} : \Omega_{\hat{X}^{(p)},\log,\mathcal{O}}^\bullet := \varprojlim_{n \in \mathbb{N}} \nu_X^* \Omega_{X^\circ,\text{et}/p^n,\log}^\bullet \hookrightarrow \Omega_{\hat{X}^{(p)}}^\bullet,$$

where we recall  $c : \hat{X}^{(p)} \rightarrow X$  the morphism in  $\mathbf{RTop}$  is given by the completion along the ideal generated by  $p$ . Note that the inclusion  $\Omega_{X^{\circ,et},\log}^l/p^n \subset \Omega_{X^{\circ,et}/p^n,\log}^{\bullet}$  is strict in general. Note that

$$\Omega_{X^{pet},\log,\mathcal{O}}^{\bullet} := \nu_X^* \Omega_{X^{et},\log,\mathcal{O}}^{\bullet} \in C(X^{pet}),$$

but

$$\underline{\mathbb{Z}}_p \nu_X := \varprojlim_{n \in \mathbb{N}} \nu_X^*(\mathbb{Z}/p^n \mathbb{Z})_{X^{et}} \in C(X^{pet}), \text{ and } \Omega_{\hat{X}^{(p)},\log,\mathcal{O}}^{\bullet} := \varprojlim_{n \in \mathbb{N}} \nu_X^* \Omega_{X^{\circ,et}/p^n,\log}^{\bullet} \in C(X^{pet})$$

are NOT the pullback of etale sheaves by  $\nu_X$ .

### 3 De Rham logarithmic classes

#### 3.1 The De Rham classes of algebraic cycles vs De Rham logarithmic classes

Let  $X \in \text{Sch}$ . Recall we have the canonical sub-complex  $OL_X : \Omega_{X,\log}^{\bullet} \hookrightarrow \Omega_X^{\bullet}$  in  $C(X^{et})$  (c.f. definition 1). All the differential of  $\Omega_{X,\log}^{\bullet}$  vanishes since by definition the logarithmic forms are closed. For  $j \in \mathbb{Z}$ , the De Rham logarithmic classes consist of the image

$$H^j OL_X(\mathbb{H}_{et}^j(X, \Omega_{X,\log}^{\bullet})) \subset \mathbb{H}_{et}^j(X, \Omega_X^{\bullet}) = H_{DR}^j(X).$$

The filtered complex  $\Gamma(X, E_{et}(\Omega_{X^{et},\log}^{\bullet}, F_b)) \in C_{fil}(\mathbb{Z})$  has its differentials at the  $E_1$  level vanishes since the logarithmic forms are closed, hence we have a canonical splitting

$$\mathbb{H}_{et}^j(X, \Omega_{X,\log}^{\bullet}) = \bigoplus_{0 \leq l \leq j} H_{et}^{j-l}(X, \Omega_{X,\log}^l).$$

Let  $X \in \text{Sch}$  a noetherian scheme. We have by definition the exact sequence in  $C(X^{et})$

$$0 \rightarrow \mathbb{Z}_X \rightarrow O_X^* \xrightarrow{dlog} \Omega_{X,\log} \rightarrow 0$$

Hence  $H_{et}^1(X, \Omega_{X,\log}) = H_{et}^1(X, O_X^*)$  and  $H_{et}^q(X, \Omega_{X,\log}) = H_{et}^q(X, O_X^*) = 0$  for  $q \leq 2$ . Let  $X \in \text{Sch}$  a noetherian proper scheme. We have  $H^0(X, \Omega_{X,\log}^l) = 0$ .

Let  $k$  a field of characteristic zero. We get (see definition 1) the embedding in  $C(\text{SmVar}(k))$

$$OL : \Omega_{/k,\log}^{\bullet} \hookrightarrow \Omega_{/k}^{\bullet}, \text{ given by, for } X \in \text{SmVar}(k), \\ OL(X) := OL_X : \Omega_{/k,\log}^{\bullet}(X) := \Gamma(X, \Omega_{X,\log}^{\bullet}) \hookrightarrow \Gamma(X, \Omega_X^{\bullet}) =: \Omega_{/k}^{\bullet}(X).$$

We will use the following lemma :

**Lemma 1.** *Let  $X \in \text{Sch}$ . Let  $X = \cup_{i \in I} X_i$  an open cover. Denote  $j_i : X_i \hookrightarrow X$  the open embeddings. Let  $w \in H_{DR}^j(X)$ . Then  $w \in H^j OL_X(H_{et}^{j-l}(X, \Omega_{X,\log}^l))$  if and only if  $j_i^* w \in H^j OL_X(H_{et}^{j-l}(X_i, \Omega_{X_i,\log}^l))$  for all  $i \in I$ .*

*Proof.* The only if part is obvious. Considering the morphism of bicomplexes of abelian groups

$$OL_X : \Gamma(X_{\bullet}, E_{et}^{\bullet}(\Omega_{X,\log}^l))[-l] \hookrightarrow \Gamma(X_{\bullet}, E_{et}^{\bullet}(\Omega_X^{\bullet}))$$

the if part follows from Mayer-Vietoris (i.e. etale hence Zariski descent of flasque etale sheaves) by induction on the filtration of the total complex of a bi-complex, using the spectral sequences associated to these filtrations.  $\square$

Let  $(X, Z) \in \text{Sch}^2$  with  $X \in \text{Sch}$  a noetherian scheme and  $Z \subset X$  a closed subset. We have the deformation  $(D_Z X, \mathbb{A}_Z^1) \rightarrow \mathbb{A}^1$ ,  $(D_Z X, \mathbb{A}_Z^1) \in \text{Sch}^2$  of  $(X, Z)$  by the normal cone  $C_{Z/X} \rightarrow Z$ , i.e. such that

$$(D_Z X, \mathbb{A}_Z^1)_s = (X, Z), s \in \mathbb{A}^1 \setminus 0, (D_Z X, \mathbb{A}_Z^1)_0 = (C_{Z/X}, Z).$$

We denote by  $i_1 : (X, Z) \hookrightarrow (D_Z X, \mathbb{A}_Z^1)$  and  $i_0 : (C_{Z/X}, Z) \hookrightarrow (D_Z X, \mathbb{A}_Z^1)$  the closed embeddings in  $\text{Sch}^2$ .

We recall the following facts (see [5] and [3]):



**Proposition 1.** *Let  $k$  a field a characteristic zero. Let  $X \in \text{SmVar}(k)$  and  $Z \subset X$  a smooth subvariety of codimension  $d$ . Then  $C_{Z/X} = N_{Z/X} \rightarrow Z$  is a vector bundle of rank  $d$ .*

- (i) *The closed embeddings  $i_1 : (X, Z) \hookrightarrow (D_Z X, \mathbb{A}_Z^1)$  and  $i_0 : (C_{Z/X}, Z) \hookrightarrow (D_Z X, \mathbb{A}_Z^1)$  in  $\text{SmVar}^2(k)$  induces isomorphisms of motives  $\mathbb{Z}(i_1) : M_Z(X) \xrightarrow{\sim} M_{\mathbb{A}_Z^1}(D_Z X)$  and  $\mathbb{Z}(i_0) : M_Z(N_{Z/X}) \xrightarrow{\sim} M_{\mathbb{A}_Z^1}(D_Z X)$  in  $\text{DA}(k)$ . We get the excision isomorphism in  $\text{DA}(k)$*

$$P_{Z,X} := \mathbb{Z}(i_0)^{-1} \circ \mathbb{Z}(i_1) : M_Z(X) \xrightarrow{\sim} M_Z(N_{Z/X}).$$

- (ii) *For each  $j \in \mathbb{Z}$ , the excision isomorphism induced by (i)*

$$\begin{aligned} H^j \Omega(P_{Z,X}) &:= H^j E_{\text{et}}(\Omega_{/k}^\bullet)(\mathbb{Z}(i_0)) \circ H^j E_{\text{et}}(\Omega_{/k}^\bullet)(\mathbb{Z}(i_1))^{-1} : \\ H_{DR,Z}^j(X) &:= H^j \Gamma_Z(X, E_{\text{et}}(\Omega_X^\bullet)) \xrightarrow{\sim} H_{DR,Z}^j(N_{Z/X}) := H^j \Gamma_Z(N_{Z/X}, E_{\text{et}}(\Omega_{N_{Z/X}}^\bullet)) \end{aligned}$$

*preserve logarithmic De Rham classes, that is for each  $l \in \mathbb{Z}$ ,*

$$H^j \Omega(P_{Z,X})(H^j OL_X(H_{\text{et},Z}^{j-l}(X, \Omega_{X,\log}^l))) = H^j OL_{N_{Z/X}}(H_{\text{et},Z}^{j-l}(N_{Z/X}, \Omega_{N_{Z/X},\log}^l))$$

*Proof.* (i): See [5].

(ii): The de Rham complex  $\Omega_{/k}^\bullet \in C(\text{SmVar}(k))$  admits transfers using trace map for commutative rings, that is  $\Omega^\bullet \in C(\text{Cor SmVar}(k))$  (see [7]) and that the inclusion  $OL : \Omega_{/k,\log}^l[-l] \hookrightarrow \Omega_{/k}^\bullet$  in  $C(\text{SmVar}(k))$  is compatible with transfers. Since  $\Omega_{/k,\log}^l \in \text{PSh}(\text{SmVar}(k))$  is  $\mathbb{A}_k^1$  invariant ( $O_{\mathbb{A}_k^1}^* = k^*$  and for  $\lambda \in k^*$  we have  $d\lambda = 0$ ) and admits transfers, for each  $j \in \mathbb{Z}$ , the presheaves  $H_{\text{et}}^{j-l} \Omega_{/k,\log}^l \in \text{PSh}(\text{SmVar}(k))$  are  $\mathbb{A}_k^1$  invariant by a theorem of Voevodsky (c.f. [5] for example). It then follows from (i) (see [5] or [3]) that

$$\begin{aligned} H^j \Omega_{\log}(P_{Z,X}) &:= H^j E_{\text{et}}(\Omega_{/k,\log}^\bullet)(\mathbb{Z}(i_0)) \circ H^j E_{\text{et}}(\Omega_{/k,\log}^\bullet)(\mathbb{Z}(i_1))^{-1} : \\ &H_{\text{et},Z}^{j-l}(X, \Omega_{X,\log}^l) \xrightarrow{\sim} H_{\text{et},Z}^{j-l}(N_{Z/X}, \Omega_{N_{Z/X},\log}^l). \end{aligned}$$

Hence,

$$\begin{aligned} H^j \Omega(P_{Z,X})(H^j OL_X(H_{\text{et},Z}^{j-l}(X, \Omega_{X,\log}^l))) &= H^j OL_{N_{Z/X}}(\Omega_{\log}(P_{Z,X})(H_{\text{et},Z}^{j-l}(X, \Omega_{X,\log}^l))) \\ &= H^j OL_{N_{Z/X}}(H_{\text{et},Z}^{j-l}(N_{Z,X}, \Omega_{N_{Z/X},\log}^l)). \end{aligned}$$

□

**Remark 1.** *Let  $k$  a field a characteristic zero. Let  $X \in \text{SmVar}(k)$  and  $Z \subset X$  a smooth subvariety of codimension  $d$ .*

- (i) *The statement of proposition 1(ii) also holds for the subcomplex  $\Omega_{/k}^{\bullet, \partial=0} \hookrightarrow \Omega_{/k}^\bullet$  of closed forms by the same argument since the inclusion  $\Omega_{/k}^{l, \partial=0}[-l] \hookrightarrow \Omega_{/k}^\bullet$  in  $C(\text{SmVar}(k))$  is compatible with transfers.*
- (ii) *The statement of proposition 1(ii) does NOT hold for presheaves which do not admits transfers. Note that the result of proposition 1(ii) does NOT hold for the embedding in  $C(\text{SmVar}(k))$ , associated to an embedding  $\sigma : k \hookrightarrow \mathbb{C}$ ,*

$$\mathcal{H}^{l,j-l}[-l] \hookrightarrow \mathcal{A}^\bullet, X \in \text{SmVar}(k), \Gamma(X, \mathcal{H}_{X_{\mathbb{C}}^{\text{an}}}^{l,j-l})[-l] \hookrightarrow \Gamma(X, \mathcal{A}_{X_{\mathbb{C}}^{\text{an}}}^\bullet)$$

*of the subsheaf of harmonic differential forms, the sheaves of differential forms  $\mathcal{A} \in C(\text{SmVar}(k))$  does NOT admits transfers (finite algebraic correspondences are not smooth and for  $f : X' \rightarrow X$  a morphism with  $X', X \in \text{Diff}(\mathbb{R})$  vector fields on  $X$  only lift if  $f$  is a smooth morphism i.e. if the differential of  $df$  is surjective), transfers maps are only defined on cohomology.*

Let  $k$  a field of characteristic zero. Let  $X \in \text{SmVar}(k)$  and  $Z \subset X$  a smooth subvariety of codimension  $d$ . We have for each  $j \in \mathbb{Z}$ , the purity isomorphism given by  $H^j \Omega(P_{Z,X})$  (see proposition 1) and the cup product with the Euler class of the normal tangent bundle  $N_{Z/X} \rightarrow Z$  :

$$H^j DR(P_{Z,X}) : H_{DR,Z}^j(X) \xrightarrow{H^j \Omega(P_{Z,X})} H_{DR,Z}^j(N_{Z/X}) \xrightarrow{((-) \cdot e(N_{Z/X}))^{-1}} H_{DR}^{j-2d}(Z).$$

Now, we have the following :

**Proposition 2.** *Let  $k$  a field of characteristic zero. Let  $p : E \rightarrow X$  a vector bundle of rank  $d \in \mathbb{N}$  with  $X, E \in \text{SmVar}(k)$  connected. Then*

(i) *the Euler class  $e(E) \in H_{DR,X}^{2d}(E)$  is logarithmic of type  $(d, d)$ , that is  $e(E) = H^{2d} OL_E(e(E)) \in H_{DR,X}^{2d}(E)$  with  $e(E) \in H_{X,et}^d(E, \Omega_{E,\log}^d)$ ,*

(ii) *the Euler class  $e(E) \in H_{DR,X}^{2d}(E)$  induces for each  $i, j \in \mathbb{Z}$  an isomorphism*

$$((-) \cdot e(E)) : H^{i+j} OL_X(H_{et}^j(X, \Omega_{X,\log}^i)) \xrightarrow{\sim} H^{2d+i+j} OL_E(H_{et}^{j+d}(E, \Omega_{E,\log}^{i+d})).$$

*Proof.* (i): Let  $X = \cup_i X_i$  an open affine cover such that  $E_i := E|_{X_i}$  is trivial :  $(s_1, \dots, s_d) : E|_{X_i} \xrightarrow{\sim} X_i \times \mathbb{A}^d$  with  $s_j \in \Gamma(X_i, E)$ . Then

$$e(E)|_{X_i} = \partial(ds_1/s_1 \wedge \dots \wedge ds_d/s_d) \in H^{2d} OL_{E_i}(H_{X_i}^d(E_i, \Omega_{E,\log}^d))$$

where  $\partial : H^{d-1}(E_i \setminus X_i, \Omega_{E,\log}^d) \rightarrow H_{X_i}^d(E_i, \Omega_{E,\log}^d)$  is the boundary map. Hence

$$e(E) \in H^{2d} OL_E(H^0(E, H_X^d \Omega_{E,\log}^d)) = H_{et,X}^d(E, \Omega_{E,\log}^d) \subset H^0(E, H_X^{2d} \Omega_E^{\geq d}) \subset H_{DR,X}^{2d}(E).$$

(ii): Follows from (i) and Kunnetth formula for De Rham cohomology together with lemma 1.  $\square$

We deduce from proposition 1 and proposition 2, the following key proposition

**Proposition 3.** *Let  $k$  a field of characteristic zero. Let  $X \in \text{SmVar}(k)$  and  $Z \subset X$  a smooth subvariety of codimension  $d$ . For each  $j \in \mathbb{Z}$ , the purity isomorphism*

$$H^j DR(P_{Z,X}) : H_{DR,Z}^j(X) \xrightarrow{H^j \Omega(P_{Z,X})} H_{DR,Z}^j(N_{Z/X}) \xrightarrow{((-) \cdot e(N_{Z/X}))^{-1}} H_{DR}^{j-2d}(Z)$$

*preserve logarithmic De Rham classes, that is for each  $l \in \mathbb{Z}$ ,*

$$H^j DR(P_{Z,X})(H^j OL_X(H_{et,Z}^{j-l}(X, \Omega_{X,\log}^l))) = H^j OL_Z(H_{et}^{j-l}(Z, \Omega_{Z,\log}^l)).$$

*Proof.* Follows from proposition 1(ii) and proposition 2(ii).  $\square$

Let  $k$  a field of characteristic zero. For  $X \in \text{SmVar}(k)$  connected, we have the cycle class map for De Rham cohomology

$$Z \in \mathcal{Z}^d(X) \mapsto [Z] := H^{2d} \Omega(\gamma_Z^\vee)([Z]) \subset H_{DR}^{2d}(X), [Z] \in H_{DR,|Z|}^{2d}(X), \\ H^{2d} \Omega(\gamma_Z^\vee) : H_{DR,|Z|}^{2d}(X) \rightarrow H_{DR}^{2d}(X)$$

where

$$\Omega(\gamma_Z^\vee) := \text{Hom}(\gamma_Z^\vee(\mathbb{Z}_X), E_{et}(\Omega_X^\bullet)) : \Gamma_Z(X, E_{et}(\Omega_X^\bullet)) \hookrightarrow \Gamma(X, E_{et}(\Omega_X^\bullet))$$

and, as for any Weil cohomology theory, we have the isomorphism given by purity : if  $X^\circ \subset X$  is an open subset such that  $Z^\circ := X^\circ \cap Z$  has smooth components we have the isomorphism

$$H^{2d} DR(P_{Z^\circ, X^\circ}) : H_{DR,|Z|}^{2d}(X) = H_{DR,|Z^\circ|}^{2d}(X^\circ) = H_{DR,|Z^\circ|}^{2d}(N_{Z^\circ/X^\circ}) \xrightarrow{\sim} H_{DR}^0(|Z^\circ|).$$

where the first equality follows from dimension reason : for  $X \in \text{SmVar}(k)$  and  $Z' \subset X$  smooth, we have  $H_{DR,Z'}^i(X) = 0$  for  $i < 2 \text{codim}(Z', X)$  by the purity isomorphism  $H^i DR(P_{Z',X})$ .

The main result of this section is the following :

**Theorem 1.** *Let  $k$  a field of characteristic zero. Let  $X \in \text{SmVar}(k)$ . Let  $d \in \mathbb{N}$ .*

(i) *The De Rham cohomology class of an algebraic cycle is logarithmic and is of type  $(d, d)$ , that is, for  $Z \in \mathcal{Z}^d(X)$*

$$[Z] := H^{2d}\Omega(\gamma_Z^\vee)([Z]) \subset H^{2d}OL_X(H_{et}^d(X, \Omega_{X, \log}^d)) \subset H_{DR}^{2d}(X).$$

(ii) *Conversely, any  $w \in H^{2d}OL_X(H_{et}^d(X, \Omega_{X, \log}^d))$  is the class of an algebraic cycle. Note that it implies  $w \in W_0 H_{DR}^{2d}(X)$  as it is easily seen for  $d = 1$  by the Chow moving lemma since  $j^* : \text{Pic}(\bar{X}) \rightarrow \text{Pic}(X)$  is surjective if  $j : X \hookrightarrow \bar{X}$  is a smooth compactification of  $X$ ,  $\bar{X} \in \text{PSmVar}(k)$  and since  $H_{et}^1(Y, \Omega_{Y, \log}^1) = \text{Pic}(Y)$  for  $Y \in \text{Var}(k)$ .*

(iii) *We have  $H^j OL_X(H_{et}^{j-1}(X, \Omega_{X, \log}^l)) = 0$  for  $2l < j$ .*

(iii)' *If  $X \in \text{PSmVar}(k)$ , we also have  $H^j OL_X(H_{et}^{j-1}(X, \Omega_{X, \log}^l)) = 0$  for  $2l > j$ .*

*Proof.* (i): We have

$$[Z] := H^{2d}\Omega(\gamma_Z^\vee)([Z]) = H^{2d}OL_X \circ H^d\Omega_{\log}^d(\gamma_Z^\vee)([Z]) \subset H_{DR}^{2d}(X), [Z] \in H_{et, |Z|}^d(X, \Omega_{X, \log}^d)$$

where

$$\Omega_{\log}^d(\gamma_Z^\vee) := \text{Hom}(\gamma_Z^\vee(\mathbb{Z}_X), E_{et}(\Omega_{X, \log}^d)) : \Gamma_Z(X, E_{et}(\Omega_{X, \log}^d)) \hookrightarrow \Gamma(X, E_{et}(\Omega_{X, \log}^d)),$$

since if  $X^\circ \subset X$  is a Zariski open subset such that  $Z^\circ := X^\circ \cap Z$  has smooth components and  $N_{Z^\circ/X^\circ} \rightarrow Z^\circ$  is the normal tangent bundle, we have by proposition 3 the isomorphism

$$\begin{aligned} H^{2d}DR(P_{Z^\circ, X^\circ}) : H^{2d}OL_X(H_{et, |Z|}^d(X, \Omega_{X, \log}^d)) &= H^{2d}OL_X(H_{et, |Z^\circ|}^d(X^\circ, \Omega_{X^\circ, \log}^d)) \\ &= H^{2d}OL_X(H_{et, |Z^\circ|}^d(N_{Z^\circ/X^\circ}, \Omega_{N_{Z^\circ/X^\circ}, \log}^d)). \end{aligned}$$

(ii): By assumption we have

$$w = H^{2d}OL_X(w) \in H_{DR}^{2d}(X), w \in H_{et}^d(X, \Omega_{X, \log}^d)$$

As  $\Omega_{X, \log}^d \in \text{PSh}(X^{et})$  consist of a single presheaf, that is a complex of presheaves concentrated one degree, there exist an etale open cover  $r = r(w) = (r_i : X_i \rightarrow X)_{1 \leq i \leq s}$ , depending on  $w$ , such that  $r_i^* w = 0 \in H_{et}^d(X_i, \Omega_{X, \log}^d)$  for each  $i$ . Choose  $i = 1$  and denote  $j : U := r_1(X_1) \hookrightarrow X$  the corresponding (Zariski) open subset. As  $\Omega_{X, \log}^d$  has no torsion and admits transfers, we have  $j^* w = 0 \in H_{et}^d(U, \Omega_{X, \log}^d)$ . Hence, denoting  $D := X \setminus U$ , we have

$$w = H^{2d}OL_X(w), w \in \Omega_{\log}^d(\gamma_D^\vee)(H_{et, D}^d(X, \Omega_{X, \log}^d)).$$

We may assume, up to shrinking  $U$ , that  $D \subset X$  is a divisor. Denote  $D^\circ \subset D$  its smooth locus and  $l : X^\circ \hookrightarrow X$  a Zariski open subset such that  $X^\circ \cap D = D^\circ$ . We then have by proposition 3

$$l^* w \in H^{2d}DR(P_{D^\circ, X^\circ})(H^{2d-2}OL_{D^\circ}(H_{et}^{d-1}(D^\circ, \Omega_{D^\circ, \log}^{d-1}))).$$

We repeat this procedure with each connected components of  $D^\circ$  instead of  $X$ . By a finite induction of  $d$  steps, we get

$$w = H^{2d}OL_X(w), w \in \Omega_{\log}^d(\gamma_Z^\vee)(H_{et, Z}^d(X, \Omega_{X, \log}^d)),$$

with  $Z := D_d \subset \dots \subset D \subset X$  a codimension  $d$  (Zariski) closed subset and  $w = [Z] \in H_{DR}^{2d}(X)$ , using in the final step the isomorphism

$$H_{et}^1(D_{d-1}^\circ, \Omega_{D_{d-1}^\circ, \log}^0) = H_{et}^1(D_{d-1}^\circ, \mathcal{O}_{D_{d-1}^\circ}^*) = \text{Pic}(D_{d-1}^\circ).$$

(iii): Follows from the proof of (ii) and the fact that  $H^k OL_X(H_{et}^k(X, \mathbb{Z})) = 0$  for all  $k > 0$ .

(iii)': Follows from the proof of (ii) and the fact that for  $X' \in \text{PSmVar}(k)$ ,  $H^0(X', \Omega_{X', \log}^l) = 0$  for all  $l > 0$ .  $\square$

### 3.2 Complex integral periods

Let  $k$  a field of characteristic zero.

Let  $X \in \text{SmVar}(k)$  a smooth variety. Let  $X = \cup_{i=1}^s X_i$  an open affine cover. We have for  $\sigma : k \hookrightarrow \mathbb{C}$  an embedding, the evaluation period embedding map which is the morphism of bi-complexes

$$\begin{aligned} ev(X)_{\bullet} : \Gamma(X_{\bullet}, \Omega_{X_{\bullet}}^{\bullet}) &\rightarrow \mathbb{Z} \text{Hom}_{\text{Diff}}(\mathbb{I}^{\bullet}, X_{\mathbb{C}, \bullet}^{an})^{\vee} \otimes \mathbb{C}, \\ w_I^l \in \Gamma(X_I, \Omega_{X_I}^l) &\mapsto (ev(X)_I^l(w_I^l) : \phi_I^l \in \mathbb{Z} \text{Hom}_{\text{Diff}}(\mathbb{I}^l, X_{\mathbb{C}, I}^{an})^{\vee} \otimes \mathbb{C} \mapsto ev_I^l(w_I^l)(\phi_I^l) := \int_{\mathbb{I}^l} \phi_I^{l*} w_I^l) \end{aligned}$$

given by integration. By taking all the affine open cover  $(j_i : X_i \hookrightarrow X)$  of  $X$ , we get for  $\sigma : k \hookrightarrow \mathbb{C}$ , the evaluation period embedding map

$$ev(X) := \varinjlim_{(j_i : X_i \hookrightarrow X)} ev(X)_{\bullet} : \varinjlim_{(j_i : X_i \hookrightarrow X)} \Gamma(X_{\bullet}, \Omega_{X_{\bullet}}^{\bullet}) \rightarrow \varinjlim_{(j_i : X_i \hookrightarrow X)} \mathbb{Z} \text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^{\bullet}, X_{\mathbb{C}, \bullet}^{an})^{\vee} \otimes \mathbb{C}$$

It induces in cohomology, for  $j \in \mathbb{Z}$ , the evaluation period map

$$H^j ev(X) = H^j ev(X)_{\bullet} : H_{DR}^j(X) = H^j \Gamma(X_{\bullet}, \Omega_{X_{\bullet}}^{\bullet}) \rightarrow H_{\text{sing}}^j(X_{\mathbb{C}}^{an}, \mathbb{C}) = H^j(\text{Hom}_{\text{Diff}(\mathbb{R})}(\mathbb{I}^{\bullet}, X_{\mathbb{C}, \bullet}^{an})^{\vee} \otimes \mathbb{C}).$$

which does NOT depend on the choice of the affine open cover by acyclicity of quasi-coherent sheaves on affine noetherian schemes for the left hand side and from Mayer-Vietoris quasi-isomorphism for singular cohomology of topological spaces and Whitney approximation theorem for differential manifolds for the right hand side.

**Proposition 4.** *Let  $k$  a field of characteristic zero. Let  $X \in \text{SmVar}(k)$ . Let  $\sigma : k \hookrightarrow \mathbb{C}$  an embedding.*

(i) *Let  $w \in H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^{\bullet}) = \mathbb{H}_{\text{pet}}^j(X, \Omega_X^{\bullet})$ . If*

$$w \in H^j OL_X(\mathbb{H}_{\text{pet}}^j(X_{\mathbb{C}}, \Omega_{X_{\mathbb{C}}}^{\bullet, \log}))$$

*then  $H^j ev(X)(w) \in H_{\text{sing}}^j(X_{\mathbb{C}}^{an}, 2i\pi\mathbb{Q})$ .*

(ii) *Let  $p \in \mathbb{N}$  a prime number and  $\sigma_p : k \hookrightarrow \mathbb{C}_p$  an embedding. Let  $j \in \mathbb{Z}$ . Let  $w \in H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^{\bullet}) = \mathbb{H}_{\text{pet}}^j(X, \Omega_X^{\bullet})$ . If*

$$w := \pi_{k/\mathbb{C}_p}(X)^* w \in H^j OL_X(\mathbb{H}_{\text{et}}^j(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}}^{\bullet, \log, \mathcal{O}}))$$

*then  $H^j ev(X)(w) \in H_{\text{sing}}^j(X_{\mathbb{C}}^{an}, 2i\pi\mathbb{Q})$ . Recall that  $\mathbb{H}_{\text{pet}}^j(X_{\mathbb{C}}, \Omega_{X_{\mathbb{C}}}^{\bullet, \log, \mathcal{O}}) = \mathbb{H}_{\text{et}}^j(X_{\mathbb{C}}, \Omega_{X_{\mathbb{C}}}^{\bullet, \log, \mathcal{O}})$ .*

*Proof.* (i): Let

$$w \in H_{DR}^j(X) := \mathbb{H}^j(X, \Omega_X^{\bullet}) = H^j \Gamma(X_{\bullet}, \Omega_{X_{\bullet}}^{\bullet}).$$

where  $(r_i : X_i \rightarrow X)_{1 \leq i \leq s}$  is an affine etale cover. Let  $X_{\mathbb{C}}^{an} = \cup_{i=1}^r \mathbb{D}_i$  an open cover with  $\mathbb{D}_i \simeq D(0, 1)^d$ . Denote  $j_{IJ} : X_I \cap \mathbb{D}_J \hookrightarrow X_I$  the open embeddings. Then by definition  $H^j ev(X)(w) = H^j ev(X_{\mathbb{C}}^{an})(j_{\bullet}^* \circ \text{an}_{X_{\bullet}}^* w)$  with

$$j_{\bullet}^* \circ \text{an}_{X_{\bullet}}^* w \in H^j \Gamma(X_{\bullet, \mathbb{C}}^{an} \cap \mathbb{D}_{\bullet}, \Omega_{X_{\mathbb{C}}^{an}}^{\bullet}).$$

Now, if  $w = H^j OL_X(\mathbb{H}_{\text{et}}^j(X_{\mathbb{C}}, \Omega_{X_{\mathbb{C}}}^{\bullet, \log, \mathcal{O}}))$ , we have a canonical splitting

$$w = \sum_{l=0}^j w_L^{l, j-l} = \sum_{l=d}^j w^{l, j-l} \in H_{DR}^j(X_{\mathbb{C}}), w_L^{l, j-l} \in H^{j-l}(X_{\mathbb{C}}, \Omega_{X_{\mathbb{C}}}^l), w^{l, j-l} := H^j OL_{X_{\mathbb{C}}^{\text{et}}}(w_L^{l, j-l}).$$

Let  $0 \leq l \leq j$ . Using an affine w-contractile pro-étale cover of  $X$ , we see that there exists an affine étale cover  $r = r(w^{l,j-l}) = (r_i : X_i \rightarrow X)_{1 \leq i \leq n}$  of  $X$  (depending on  $w^{l,j-l}$ ) such that

$$w^{l,j-l} = [(w_{L,I}^{l,j-l})_I] \in H^j OL_{X_{\mathbb{C}}^{et}}(H^{j-l} \Gamma(X_{\mathbb{C},\bullet}, \Omega_{X_{\mathbb{C},\log}}^l)) \subset \mathbb{H}^j \Gamma(X_{\mathbb{C},\bullet}, \Omega_{X_{\mathbb{C}}}^{\bullet}).$$

Note that since  $X$  is an algebraic variety, this also follows from a comparison theorem between Čech cohomology of étale covers and étale cohomology. By [4] lemma 2, we may assume, up to take a desingularization  $\pi : X' \rightarrow X$  of  $(X, \cup_i (r_i(X \setminus X_i)))$  and replace  $w$  with  $\pi^* w$ , that  $r_i(X_i) = r_i(X_i(w)) = X \setminus D_i$  with  $D_i \subset X$  smooth divisors with normal crossing. For  $1 \leq l \leq j$ , we get

$$w_{L,I}^{l,j-l} = \sum_{\nu} df_{\nu_1}/f_{\nu_1} \wedge \cdots \wedge df_{\nu_l}/f_{\nu_l} \in \Gamma(X_{\mathbb{C},I}, \Omega_{X_{\mathbb{C}}}^l).$$

For  $l = 0$ , we get

$$w^{0,j} = [(\lambda_I)] \in H^j \Gamma(X_{\mathbb{C},I}, O_{X_{\mathbb{C},p,I}}), \lambda_I \in \Gamma(X_{\mathbb{C},I}, \mathbb{Z}_{X_{\mathbb{C}}^{et}})$$

There exists  $k' \subset \mathbb{C}$  containing  $k$  such that  $w_{L,I}^{l,j-l} \in \Gamma(X_{k',I}, \Omega_{X_{k'}}^l)$  for all  $0 \leq l \leq j$ . Taking an embedding  $\sigma' : k' \hookrightarrow \mathbb{C}$  such that  $\sigma'_k = \sigma$ , we then have

$$j_{\bullet} \circ \text{an}_{X_{\bullet}}^* w = j_{\bullet}^* ((m_l \cdot w_L^{l,j-l})_{0 \leq l \leq j}) = (w_{L,I,J}^{l,j-l})_{l,I,J} \in H^j \Gamma(X_{\bullet,\mathbb{C}}^{an} \cap \mathbb{D}_{\bullet}, \Omega_{X_{\mathbb{C}}}^{\bullet}).$$

where for each  $(I, J, l)$  with  $\text{card} I + \text{card} J + l = j$ ,

$$w_{L,I,J}^{l,j-l} := j_{I,J}^* w_{L,I}^{l,j-l} \in \Gamma(X_{I,\mathbb{C}}^{an} \cap \mathbb{D}_J, \Omega_{X_{\mathbb{C}}^{an}}^l).$$

We have by a standard computation, for each  $(I, J, l)$  with  $\text{card} I + \text{card} J + l = j$ ,

$$H_{\text{sing}}^*(X_{I,\mathbb{C}}^{an} \cap \mathbb{D}_J, \mathbb{Z}) = \langle \gamma_1, \dots, \gamma_{\text{card} I} \rangle,$$

where for  $1 \leq i \leq \text{card} I$ ,  $\gamma_i \in \text{Hom}(\Delta^*, X_{I,\mathbb{C}}^{an} \cap \mathbb{D}_J)$  are products of loops around the origin inside the pointed disc  $\mathbb{D}^1 \setminus \{0\}$ . On the other hand,

- $w_{L,I,J}^{l,j-l} = j_J^* (\sum_{\nu} df_{\nu_1}/f_{\nu_1} \wedge \cdots \wedge df_{\nu_l}/f_{\nu_l}) \in \Gamma(X_{I,\mathbb{C}}^{an} \cap \mathbb{D}_J, \Omega_{X_{\mathbb{C}}^{an}}^l)$  for  $1 \leq l \leq j$ ,
- $w_{L,I,J}^{0,j} = \lambda_I$  is a constant.

Hence, for  $\mu \in P([1, \dots, s])$  with  $\text{card} \mu = l$ , we get, for  $l = 0$   $H^l \text{ev}(X_{\mathbb{C}}^{an})_{I,J}(w_{L,I,J}^{0,j}) = 0$  and, for  $1 \leq l \leq j$ ,

$$H^l \text{ev}(X_{\mathbb{C}}^{an})_{I,J}(w_{L,I,J}^{l,j-l})(\gamma_{\mu}) = \sum_k \delta_{\nu,\mu} 2i\pi \in 2i\pi\mathbb{Z}.$$

where  $\gamma_{\mu} := \gamma_{\mu_1} \cdots \gamma_{\mu_l}$ . We conclude by [4] lemma 1.

(ii): It is a particular case of (i). See [4] proposition 1.  $\square$

Let  $k$  a field of characteristic zero. Let  $X \in \text{SmVar}(k)$ . Let  $X = \cup_{i=1}^s X_i$  an open affine cover with  $X_i := X \setminus D_i$  with  $D_i \subset X$  smooth divisors with normal crossing. Let  $\sigma : k \hookrightarrow \mathbb{C}$  an embedding. By proposition 4, we have a commutative diagram of graded algebras

$$\begin{array}{ccc} H_{DR}^*(X) & \xrightarrow{H^* \text{ev}(X)} & H_{\text{sing}}^*(X_{\mathbb{C}}^{an}, \mathbb{C}) \\ \uparrow \subset & & \uparrow H^* C^* \iota_{2i\pi\mathbb{Q}/\mathbb{C}}(X_{\mathbb{C}}^{an}) \\ H^* OL_X(\mathbb{H}_{et}^*(X, \Omega_{X_{et}, \log}^{\bullet})) \cap H_{DR}^*(X) & \xrightarrow{H^* \text{ev}(X)} & H_{\text{sing}}^*(X_{\mathbb{C}}^{an}, 2i\pi\mathbb{Q}) \end{array}$$

where

$$C^* \iota_{2i\pi\mathbb{Q}/\mathbb{C}}(X_{\mathbb{C}}^{an}) : C_{\text{sing}}^{\bullet}(X_{\mathbb{C}}^{an}, 2i\pi\mathbb{Q}) \hookrightarrow C_{\text{sing}}^{\bullet}(X_{\mathbb{C}}^{an}, \mathbb{C})$$

is the subcomplex consisting of  $\alpha \in C_{\text{sing}}^j(X_{\mathbb{C}}^{an}, \mathbb{C})$  such that  $\alpha(\gamma) \in 2i\pi\mathbb{Q}$  for all  $\gamma \in C_j^{\text{sing}}(X_{\mathbb{C}}^{an}, \mathbb{Q})$ . Recall that

$$H^*ev(X_{\mathbb{C}}) = H^*R\Gamma(X_{\mathbb{C}}^{an}, \alpha(X)) \circ \Gamma(X_{\mathbb{C}}^{an}, E_{zar}(\Omega(\text{an}_X))) : H_{DR}^*(X_{\mathbb{C}}) \xrightarrow{\sim} H_{\text{sing}}^*(X_{\mathbb{C}}^{an}, \mathbb{C})$$

is the canonical isomorphism induced by the analytical functor and  $\alpha(X) : \mathbb{C}_{X_{\mathbb{C}}^{an}} \hookrightarrow \Omega_{X_{\mathbb{C}}^{an}}^{\bullet}$ , which gives the periods elements  $H^*ev(X)(H_{DR}^*(X)) \subset H_{\text{sing}}^*(X_{\mathbb{C}}^{an}, \mathbb{C})$ . On the other side the induced map

$$H^*ev(X_{\mathbb{C}}) : H^*OL_X(\mathbb{H}_{\text{et}}^*(X_{\mathbb{C}}, \Omega_{X_{\mathbb{C}}^{\text{et}}, \log}^{\bullet})) \hookrightarrow H^* \iota_{2i\pi\mathbb{Q}/\mathbb{C}} H_{\text{sing}}^*(X_{\mathbb{C}}^{an}, 2i\pi\mathbb{Q})$$

is NOT surjective in general since the left hand side is invariant by the action of the group  $\text{Aut}(\mathbb{C})$  (the group of field automorphism of  $\mathbb{C}$ ) whereas the right hand side is not. The fact for a de Rham cohomology class of being logarithmic is algebraic and invariant under isomorphism of (abstract) schemes.

**Corollary 1.** *Let  $X \in \text{PSmVar}(\mathbb{C})$ . Then the Hodge conjecture holds for  $X$  if and only if the Hodge classes are given by logarithmic De Rham classes.*

*Proof.* Follows from theorem 1. □

### 3.3 Rigid GAGA for logarithmic de Rham classes

Let  $K$  a field of characteristic zero which is complete for a  $p$ -adic norm. Denote by  $\text{SmVar}(K)^{\text{pet}}$  the big pro-etale sheaf. We will consider

$$\Omega_{/K, \log, \mathcal{O}}^{\bullet, an} \in C(\text{SmVar}(K)^{\text{pet}}), X \mapsto \Omega_{\hat{X}^{(p)}, \log, \mathcal{O}}^{\bullet, an}(\hat{X}^{(p)}), (f : X' \rightarrow X) \mapsto \Omega(f) := f^*$$

and the embedding of  $C(\text{SmVar}(K)^{\text{pet}})$  (see definition 1(iii))

$$\begin{aligned} OL_{/K, an} &: \Omega_{/K, \log, \mathcal{O}}^{\bullet, an} \hookrightarrow \Omega_{/K}^{\bullet, an}, \text{ for } X \in \text{SmVar}(K), \\ OL_{/K, an}(X) &:= OL_{\hat{X}^{(p)}}(X) : \Omega_{\hat{X}^{(p)}, \log, \mathcal{O}}^{\bullet, an}(\hat{X}^{(p)}) \rightarrow \Omega_{\hat{X}^{(p)}}^{\bullet, an}(\hat{X}^{(p)}). \end{aligned}$$

We have the analytification map  $C(\text{SmVar}(K)^{\text{pet}})$  given by completion with respect to  $(p)$

$$c^* : \Omega_{/K}^{\bullet} \rightarrow \Omega_{/K}^{\bullet, an}, \text{ for } X \in \text{SmVar}(K), c^* : \Omega_X^{\bullet}(X) \rightarrow \Omega_{\hat{X}^{(p)}}^{\bullet, an}(\hat{X}^{(p)}),$$

which induces for each  $j \in \mathbb{Z}$  the morphism

$$c^* : H_{DR}^j(X) = \mathbb{H}_{\text{pet}}^j(X, \Omega_X^{\bullet}) \xrightarrow{\sim} \mathbb{H}_{\text{pet}}^j(X, \Omega_{\hat{X}^{(p)}}^{\bullet}) = \mathbb{H}_{\text{pet}}^j(\hat{X}^{(p)}, \Omega_{\hat{X}^{(p)}}^{\bullet})$$

which is an isomorphism by GAGA (c.f. EGA 3) and the properties of w-contractile rings (for the pro-etale topology  $c_* = Rc_*$ ).

The result of this section is the following :

**Proposition 5.** *Let  $X \in \text{PSmVar}(K)$ . Let  $K$  a field of characteristic zero which is complete for a  $p$ -adic norm. For each  $j, l \in \mathbb{Z}$ , the morphism induced by the analytification map*

$$c^* : H^j(m \circ (OL_X \otimes I))(H_{\text{pet}}^{j-l}(X, \Omega_{X^{\text{pet}}, \log, \mathcal{O}}^l \otimes \mathbb{Z}_p)) \rightarrow H^j OL_{\hat{X}^{(p)}}(H_{\text{pet}}^{j-l}(X, \Omega_{\hat{X}^{(p)}, \log, \mathcal{O}}^l))$$

*is an isomorphism.*

*Proof.* The presheaf

$$L^{l,j-l} := H^j OL_{/K,an}(H^{j-l} \Omega_{/K,log,\mathcal{O}}^{l,an}) \in \text{PSh}(\text{SmVar}(K))$$

is  $\mathbb{A}^1$  invariant and admits tranferts. Hence by a theorem of Voevodsky (c.f. [5] for example), it is  $\mathbb{A}^1$  local. This gives for  $Z \subset X$  a smooth subvariety of (pure) codimension  $d$ , by proposition 1(i) an isomorphism

$$L^{l,j-l}(P_{Z,X}) : L_Z^{l,j-l}(X) := H^j OL_{/K,an}(H^{j-l} \text{Hom}(\mathbb{Z}(X, X \setminus Z), E_{pet}^\bullet(\Omega_{/K,log,\mathcal{O}}^{l,an}))) \xrightarrow{\sim} L^{-d,j-l-d}(Z).$$

using the local to global spectral sequence for the pro-etale topology. Let

$$\alpha \in L^{l,j-l}(X) := H^j OL_{\hat{X}(p)}(H_{pet}^{j-l}(X, \Omega_{\hat{X}(p),log,\mathcal{O}}^l)).$$

There exists by [2] a  $w$ -contractile affine pro-etale over  $r = (r_i : X_i \rightarrow X)_{1 \leq i \leq r}$ . Denote  $j_i : r_i(X_i) \hookrightarrow X$  the Zariski open subset. Then by the properties of  $w$ -contractile schemes  $j_i^* \alpha = 0 \in L^{l,j-l}(r_i(X))$ . Considering a divisor such that  $X \setminus r_i(X_i) \subset D$ , we get  $\alpha = L^{l,j-l}(\gamma_D^\vee)(\alpha)$  with  $\alpha \in L_D^{l,j-l}(X)$ . Similarly to the proof of theorem 1, we get by induction (restricting to the smooth locus of the divisors) a closed subset  $Z' \subset X$  of codimension  $c = \min(l, j-l)$  such that  $\alpha = L^{l,j-l}(\gamma_{Z'}^\vee)(\alpha)$  with  $\alpha \in L_{Z'}^{l-c,j-l-c}(X)$ . Hence if  $j \neq 2l$ ,  $\alpha = 0$  and  $\alpha = [Z']$  if  $j = 2l$  since  $X$  is projective.  $\square$

## 4 Tate conjecture

Let  $k$  a field of finite type over  $\mathbb{Q}$ . Denote  $\bar{k}$  the algebraic closure of  $k$  and  $G = \text{Gal}(\bar{k}/k)$  the absolute Galois group of  $k$ . Let  $X \in \text{PSmVar}(k)$  be a smooth projective variety. Let  $p \in \mathbb{N}$  a prime number. Consider an embedding  $\sigma_p : k \hookrightarrow \mathbb{C}_p$ . Denote  $\hat{k}_{\sigma_p} \subset \mathbb{C}_p$  the  $p$ -adic completion of  $k$  with respect to  $\sigma_p$ . We consider the commutative diagram in  $C_{\mathbb{B}_{dr},fil,\hat{G}_{\sigma_p}}(X_{\mathbb{C}_p}^{an,pet})$

$$\begin{array}{ccc} (\mathbb{B}_{dr,X_{\mathbb{C}_p}}, F) & \xrightarrow{\alpha(X)} & (\Omega_{X_{\mathbb{C}_p}}^\bullet, F_b) \otimes_{O_{X_{\mathbb{C}_p}}} (O\mathbb{B}_{dr,X_{\mathbb{C}_p}}, F) \\ \uparrow \iota_{X_{\mathbb{C}_p}^{pet}} := (l \rightarrow l.1) & & \uparrow OL_X \otimes I := (w \otimes \lambda_n)_{n \in \mathbb{N}} \rightarrow (w \otimes \lambda_n)_{n \in \mathbb{N}} \\ \underline{\mathbb{Z}_p}_{X_{\mathbb{C}_p}} & \xrightarrow{\iota_{X_{\mathbb{C}_p}^{pet}} := (I, 0)} & (\Omega_{X_{\mathbb{C}_p},log,\mathcal{O}}^\bullet \otimes \mathbb{Z}_p, F_b) \end{array}$$

Let  $p \in \mathbb{N} \setminus \delta(k, X)$  a prime number. Consider an embedding  $\sigma_p : k \hookrightarrow \mathbb{C}_p$ . Then  $X_{\hat{k}_{\sigma_p}}$  has good reduction modulo  $p$  and let  $X_{\hat{k}_{\sigma_p}}^\mathcal{O} \in \text{Sch}/O_{\hat{k}_{\sigma_p}}$  be a smooth model, i.e.  $X_{\hat{k}_{\sigma_p}}^\mathcal{O} \otimes_{O_{\hat{k}_{\sigma_p}}} \hat{k}_{\sigma_p} = X_{\hat{k}_{\sigma_p}}$  and  $X_{\hat{k}_{\sigma_p}}^\mathcal{O}$  is smooth with smooth special fiber. The main result of [1] say in this case that the embedding in  $C((X_{\hat{k}_{\sigma_p}}^\mathcal{O})^{Falt})$

$$\alpha(X) : \mathbb{B}_{cris,X_{\hat{k}_{\sigma_p}}} \hookrightarrow a_{\bullet,\bullet} \Omega_{X_{\hat{k}_{\sigma_p}}^\mathcal{O},\bullet} \otimes_{O_{X^\mathcal{O}}} O\mathbb{B}_{cris,X_{\hat{k}_{\sigma_p}}^\bullet}$$

is a filtered quasi-isomorphism compatible with the action of  $\text{Gal}(\mathbb{C}_p/\hat{k}_{\sigma_p})$  and the action of the Frobenius  $\phi_p$ , and gives if  $X_{\hat{k}_{\sigma_p}}^\mathcal{O}$  has good reduction modulo  $p$ , for each  $j \in \mathbb{Z}$ , a filtered isomorphism of filtered abelian groups

$$\begin{aligned} H^j R\alpha(X) : H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{B}_{cris,\hat{k}_{\sigma_p}} &\xrightarrow{H^j T(a_X, \mathbb{B}_{cris})^{-1}} H_{et}^j((X)^{Falt})(\mathbb{B}_{cris,X_{\hat{k}_{\sigma_p}}}) \\ &\xrightarrow{H^j R\Gamma(X_{\hat{k}_{\sigma_p}}^\mathcal{O}, \alpha(X))} H_{DR}^j(X_{\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}} \mathbb{B}_{cris,\hat{k}_{\sigma_p}} \end{aligned}$$

compatible with the action of  $\text{Gal}(\mathbb{C}_p/\hat{k}_{\sigma_p})$  and of the Frobenius  $\phi_p$ .

**Definition 2.** Let  $k$  a field of finite type over  $\mathbb{Q}$ . Denote  $\bar{k}$  the algebraic closure of  $k$  and  $G = \text{Gal}(\bar{k}/k)$  the absolute Galois group of  $k$ . Let  $X \in \text{PSmVar}(k)$ . Let  $p \in \mathbb{N} \setminus \delta(k, X)$  a prime number. Consider an embedding  $\sigma_p : k \hookrightarrow \mathbb{C}_p$ . Denote  $\hat{k}_{\sigma_p} \subset \mathbb{C}_p$  the  $p$ -adic completion of  $k$  with respect to  $\sigma_p$ . For  $\alpha \in H_{\text{et}}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p)$ , we consider

$$w(\alpha) := H^j R\alpha(X)(\alpha \otimes 1) \in H_{DR}^j(X_{\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}} \mathbb{B}_{\text{cris}, \hat{k}_{\sigma_p}}.$$

the associated de Rham class by the  $p$ -adic periods. We recall

$$H^j R\alpha(X) : H_{\text{et}}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{B}_{\text{cris}, \hat{k}_{\sigma_p}} \xrightarrow{\sim} H_{DR}^j(X_{\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}} \mathbb{B}_{\text{cris}, \hat{k}_{\sigma_p}}$$

is the canonical filtered isomorphism compatible with the action of  $\hat{G}_{\sigma_p}$  and with the action of the Frobenius  $\phi_p$ .

We recall the following result from Illusie:

**Proposition 6.** Let  $k$  a field of finite type over  $\mathbb{Q}$ . Let  $X \in \text{SmVar}(k)$ . Let  $p \in \mathbb{N} \in \delta(k, X)$  a prime number. Consider an embedding  $\sigma_p : k \hookrightarrow \mathbb{C}_p$ . Denote  $\hat{k}_{\sigma_p} \subset \mathbb{C}_p$  the  $p$ -adic completion of  $k$  with respect to  $\sigma_p$ . Consider  $X_{\hat{k}_{\sigma_p}}^{\circ} \in \text{Sch}/O_{\hat{k}_{\sigma_p}}$  a smooth model of  $X_{\hat{k}_{\sigma_p}}$ , i.e.  $X_{\hat{k}_{\sigma_p}}^{\circ} \otimes_{O_{\hat{k}_{\sigma_p}}} \hat{k}_{\sigma_p} = X_{\hat{k}_{\sigma_p}}$  and  $X_{\hat{k}_{\sigma_p}}^{\circ}$  is smooth with smooth special fiber. Recall  $\hat{G} := \text{Gal}(\mathbb{C}_p/\hat{k}_{\sigma_p})$  denote the Galois group. Let  $l \in \mathbb{Z}$ . For each  $n \in \mathbb{N}$ , the sequence in  $C(X_{\hat{k}_{\sigma_p}}^{\text{et}})$

$$\begin{aligned} 0 \rightarrow a_{X_{\bullet}} * a_{X_{\bullet}}^* \Omega_{X_{\hat{k}_{\sigma_p}}^{\circ}}^{\bullet \geq l} / p^{n, \log} &\xrightarrow{((-) \otimes 1) \circ OL_{X^{\circ}/p^n}} a_{X_{\bullet}} * a_{X_{\bullet}}^* ((\Omega_{X_{\hat{k}_{\sigma_p}}^{\circ}}^{\bullet \geq l} \otimes_{O_{X^{\circ}}} O_{\mathbb{B}_{\text{cris}, X_{\hat{k}_{\sigma_p}}}}) / p^n)^{\hat{G}} \\ &\xrightarrow{\phi_p^{-I}} a_{X_{\bullet}} * a_{X_{\bullet}}^* ((\Omega_{X_{\hat{k}_{\sigma_p}}^{\circ}}^{\bullet \geq l} \otimes_{O_{X^{\circ}}} O_{\mathbb{B}_{\text{cris}, X_{\hat{k}_{\sigma_p}}}}) / p^n)^{\hat{G}} \rightarrow 0 \end{aligned}$$

is exact as a sequence of complexes of étale sheaves (i.e. we only have local surjectivity on the right), where  $a_{\bullet} : X_{\bullet} \rightarrow X$ ,  $a = (a_i : X_i \hookrightarrow X)_{1 \leq i \leq s}$  is an affine open cover of  $X$  such that we have a lift of the Frobenius  $\phi_p$ .

*Proof.* (i): It follows from [6] for  $n = 1$ . Indeed, since it is a local statement for étale cohomology, we may assume that  $X$  is affine and connected so that  $X' := X_{\hat{k}_{\sigma_p}}^{\circ} = \text{Spec } R$  with  $R = O_{\hat{k}_{\sigma_p}}[x_1, \dots, x_n]/I$ , and that there is a lift of the Frobenius  $\pi_p : R/pR \rightarrow R/pR$ . Denote  $\bar{R} := \varprojlim_{(R \rightarrow S)_{n.f.}} (R \rightarrow S)$  where n.f. means normal finite extensions. The sequence is then in this case

$$0 \rightarrow \Omega_{R/pR, \log}^{\bullet \geq l} \xrightarrow{((-) \times 1) \circ OL} (\Omega_{R/pR}^{\bullet \geq l} \otimes_{R/pR} \bar{R}/p\bar{R})^{\hat{G}} \xrightarrow{\phi_p^{-I}} (\Omega_{R/pR}^{\bullet \geq l} \otimes_{R/pR} \bar{R}/p\bar{R})^{\hat{G}} \rightarrow 0.$$

Since  $(\bar{R}/p\bar{R})^{\hat{G}} = R/pR$ , the sequence is then in this case

$$0 \rightarrow \Omega_{R/pR, \log}^{\bullet \geq l} \xrightarrow{((-) \times 1) \circ OL} \Omega_{R/pR}^{\bullet \geq l} \xrightarrow{\phi_p^{-I}} \Omega_{R/pR}^{\bullet \geq l} \rightarrow 0,$$

which is exact by [6]. It then follows for  $n \geq 2$  by devissage.  $\square$

We have the following key propositions (see [4]), for simplicity we state and prove them for smooth projective varieties, the case for smooth varieties is obtained in the same way using a smooth compactification with normal crossing divisors:

**Proposition 7.** Let  $k$  a field of finite type over  $\mathbb{Q}$ . Let  $X \in \text{PSmVar}(k)$ . Let  $p \in \mathbb{N} \setminus \delta(k, X)$  a prime number. Consider an embedding  $\sigma_p : k \hookrightarrow \mathbb{C}_p$ . Denote  $k \subset \hat{k}_{\sigma_p} \subset \mathbb{C}_p$  the  $p$ -adic completion of  $k$  with respect to  $\sigma_p$ .



(i) Let  $j, l \in \mathbb{Z}$ . We have, see definition 1(iii),

$$\begin{aligned} F^l H_{DR}^j(X_{\hat{k}_{\sigma_p}}) \cap H^j R\alpha(X)(H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p)) &= (c^*)^{-1}(H^j OL_{\hat{X}_{\hat{k}_{\sigma_p}}^{(p)}}(\mathbb{H}_{pet}^j(X, \Omega_{\hat{X}_{\hat{k}_{\sigma_p}}^{(p)}, \log, \mathcal{O}}^{\bullet \geq l}))) \\ &\subset H_{DR}^j(X_{\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}} \mathbb{B}_{cris, \hat{k}_{\sigma_p}}, \end{aligned}$$

where we recall  $c : \hat{X}_{\hat{k}_{\sigma_p}}^{(p)} \rightarrow X_{\hat{k}_{\sigma_p}}$  is the completion with respect to  $(p)$ , and

$$c^* : \mathbb{H}_{pet}^j(X_{\hat{k}_{\sigma_p}}, \Omega_{X_{\hat{k}_{\sigma_p}}}^{\bullet}) \xrightarrow{\sim} \mathbb{H}_{pet}^j(X_{\hat{k}_{\sigma_p}}, \Omega_{\hat{X}_{\hat{k}_{\sigma_p}}^{(p)}}^{\bullet})$$

is an isomorphism by GAGA and the properties of  $w$ -contractile open covers.

(ii) Let  $j, l \in \mathbb{Z}$ . We have, see definition 1(iii),

$$\begin{aligned} F^l H_{DR}^j(X_{\hat{k}_{\sigma_p}}) \cap H^j R\alpha(X)(H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p)) &= H^j(m \circ (OL_X \otimes I))(\mathbb{H}_{pet}^j(X, \Omega_{\hat{X}_{\hat{k}_{\sigma_p}}^{(p)}, \log, \mathcal{O}}^{\bullet \geq l} \otimes \mathbb{Z}_p)) \\ &\subset H_{DR}^j(X_{\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}} \mathbb{B}_{cris, \hat{k}_{\sigma_p}}, \end{aligned}$$

(ii)' For  $\alpha \in H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p)$  such that  $w(\alpha) \in F^l H_{DR}^j(X_{\hat{k}_{\sigma_p}})$  (see definition 2), there exist

$$(\lambda_i)_{1 \leq i \leq n} \in \mathbb{Z}_p \text{ and } (w_{Li})_{1 \leq i \leq n} \in \mathbb{H}_{pet}^j(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}, \log, \mathcal{O}}^{\bullet \geq l})$$

such that

$$w(\alpha) = \sum_{1 \leq i \leq n} \lambda_i \cdot w_{Li} \in \mathbb{H}_{pet}^j(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}}^{\bullet \geq l}) = F^l H_{DR}^j(X_{\mathbb{C}_p}), \quad w_{Li} := H^j OL_X(w_{Li}).$$

*Proof.* (i): Consider  $X_{\hat{k}_{\sigma_p}}^{\mathcal{O}} \in \text{Sch}/O_{\hat{k}_{\sigma_p}}$  a smooth model of  $X_{\hat{k}_{\sigma_p}}$ , i.e.  $X_{\hat{k}_{\sigma_p}}^{\mathcal{O}} \otimes_{O_{\hat{k}_{\sigma_p}}} \hat{k}_{\sigma_p} = X_{\hat{k}_{\sigma_p}}$  and  $X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}$  is smooth. Consider  $c : \hat{X}_{\hat{k}_{\sigma_p}}^{\mathcal{O}, (p)} \rightarrow X_{\hat{k}_{\sigma_p}}^{\mathcal{O}}$  the morphism in RTop which is the formal completion along the ideal  $(p)$ . By [1],  $\alpha \in H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p)$  is such that  $w(\alpha) \in F^l H_{DR}^j(X_{\hat{k}_{\sigma_p}})$  if and only if there exists an  $w$ -contractile pro-etale affine cover  $(r_i : X_i \rightarrow X)_{1 \leq i \leq r}$  such that

$$w(\alpha) \in H^j \Gamma(X_{\bullet, \hat{k}_{\sigma_p}}, \Omega_{\hat{X}_{\hat{k}_{\sigma_p}}^{\mathcal{O}}}^{\bullet \geq l} \otimes_{O_{X^{\mathcal{O}}}} O_{\mathbb{B}_{cris, X_{\hat{k}_{\sigma_p}}}})^{\hat{G}, \phi_p} = H^j \Gamma(X_{\bullet, \hat{k}_{\sigma_p}}, \Omega_{\hat{X}_{\hat{k}_{\sigma_p}}^{\mathcal{O}, (p)}}^{\bullet \geq l} \otimes_{O_{\hat{X}_{\hat{k}_{\sigma_p}}^{\mathcal{O}, (p)}}} O_{\mathbb{B}_{cris, X_{\hat{k}_{\sigma_p}}}})^{\hat{G}, \phi_p}$$

where the equality follows from GAGA (c.f. EGA 3). Recall that the action of  $\hat{G}$  commutes with the action of  $\phi_p$  since elements of  $\hat{G}$  are by definition (continuous) field automorphisms. Note that the differentials of the De Rham complex  $DR(X_{\hat{k}_{\sigma_p}})(O_{\mathbb{B}_{cris, X_{\hat{k}_{\sigma_p}}}})$  commutes with the Frobenius action. On the other hand, the sequence

$$\begin{aligned} 0 \rightarrow \Gamma(X_{\bullet, \hat{k}_{\sigma_p}}, \Omega_{\hat{X}_{\hat{k}_{\sigma_p}}^{\mathcal{O}, (p)}, \log, \mathcal{O}}^{\bullet \geq l}) &\xrightarrow{\Gamma(X_{\bullet, \hat{k}_{\sigma_p}}, OL_{\hat{X}_{\hat{k}_{\sigma_p}}^{(p)}})} \Gamma(X_{\bullet, \hat{k}_{\sigma_p}}, (\Omega_{\hat{X}_{\hat{k}_{\sigma_p}}^{\mathcal{O}, (p)}}^{\bullet \geq l} \otimes_{O_{X^{\mathcal{O}}}} O_{\mathbb{B}_{cris, X_{\hat{k}_{\sigma_p}}}})^{\hat{G}}) \\ &\xrightarrow{\Gamma(X_{\bullet, \hat{k}_{\sigma_p}}, \phi_p - I)} \Gamma(X_{\bullet, \hat{k}_{\sigma_p}}, \Omega_{\hat{X}_{\hat{k}_{\sigma_p}}^{\mathcal{O}, (p)}}^{\bullet \geq l} \otimes_{O_{X^{\mathcal{O}}}} O_{\mathbb{B}_{cris, X_{\hat{k}_{\sigma_p}}}})^{\hat{G}} \rightarrow 0 \end{aligned}$$

is exact by proposition 6 and by the properties of  $w$ -contractile pro-etale affine covers (see [2]). Hence  $\alpha \in H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p)$  is such that  $w(\alpha) \in F^l H_{DR}^j(X_{\hat{k}_{\sigma_p}})$  if and only if

$$\begin{aligned} w(\alpha) &\in H^j \Gamma(X_{\bullet, \hat{k}_{\sigma_p}}, \Omega_{\hat{X}_{\hat{k}_{\sigma_p}}^{\mathcal{O}}}^{\bullet \geq l} \otimes_{O_{X^{\mathcal{O}}}} O_{\mathbb{B}_{cris, X_{\hat{k}_{\sigma_p}}}})^{\hat{G}, \phi_p} \\ &= H^j \Gamma(X_{\bullet, \hat{k}_{\sigma_p}}, OL_{\hat{X}_{\hat{k}_{\sigma_p}}^{(p)}})(H^j \Gamma(X_{\bullet, \hat{k}_{\sigma_p}}, \Omega_{\hat{X}_{\hat{k}_{\sigma_p}}^{\mathcal{O}, (p)}, \log, \mathcal{O}}^{\bullet \geq l})) = OL_{\hat{X}_{\hat{k}_{\sigma_p}}^{(p)}}(\mathbb{H}_{pet}^j(X_{\hat{k}_{\sigma_p}}, \Omega_{\hat{X}_{\hat{k}_{\sigma_p}}^{\mathcal{O}, (p)}, \log, \mathcal{O}}^{\bullet \geq l})), \end{aligned}$$

for the first equality note that  $\Omega_{X_{\hat{k}_{\sigma_p}}^{\bullet \geq l}} \otimes_{O_{X \circ}} O_{\mathbb{B}_{cris, X_{\hat{k}_{\sigma_p}}}} \in C(X_{\hat{k}_{\sigma_p}})$  is acyclic for the Galois action.

(ii): Follows from (i) and proposition 5.

(ii)': By (ii), if  $(r_i : X_i \rightarrow X)_{1 \leq i \leq r}$  is a w-contractile affine pro-etale cover

$$w(\alpha) = \left( \sum_{1 \leq i \leq n_J} \lambda_{iJ} \cdot w_{LiJ} \right)_{J \subset [1, \dots, r], \text{card} J = j} \in H^j \Gamma(X_{\bullet, \hat{k}_{\sigma_p}}, m \circ (OL_X \otimes I)) (H^j \Gamma(X_{\bullet, \hat{k}_{\sigma_p}}, \Omega_{X_{\hat{k}_{\sigma_p}}}^{\bullet \geq l, \log, \mathcal{O}} \otimes \mathbb{Z}_p)),$$

with  $w_{LiJ} \in \Gamma(X_J, \Omega_{X_{\hat{k}_{\sigma_p}}}^{\bullet \geq l', \log, \mathcal{O}})$ ,  $l' \geq l$  and  $\lambda_{iJ} \in \mathbb{Z}_p$ . Since  $w(\alpha)$  is a Chech etale cycle (i.e. closed for the Chech differential) without torsion and since the topology consists of etale covers of  $X \in \text{PSmVar}(k)$  (which we way assume connected),  $n_J = n$  and  $\lambda_{iJ} = \lambda_i \in \mathbb{Z}_p$  for each  $J$ , which gives

$$w(\alpha) = \sum_{1 \leq i \leq n} \lambda_i \cdot w_{Li} \in \mathbb{H}_{pet}^j(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}}^{\bullet \geq l}) = F^l H_{DR}^j(X_{\mathbb{C}_p}), \quad w_{Li} := H^j OL_X(w_{Li}).$$

□

**Proposition 8.** *Let  $k$  a field of finite type over  $\mathbb{Q}$ . Denote  $\bar{k}$  the algebraic closure of  $k$ . Denote  $G := \text{Gal}(\bar{k}/k)$  its absolute Galois group. Let  $X \in \text{PSmVar}(k)$  a smooth projective variety. Let  $p \in \mathbb{N}$  a prime number. Consider an embedding  $\sigma_p : k \hookrightarrow \mathbb{C}_p$ . Denote  $k \subset \hat{k}_{\sigma_p} \subset \mathbb{C}_p$  being the  $p$ -adic completion with respect to the  $p$  adic norm induced by  $\sigma_p$ . Then  $\hat{G}_{\sigma_p} := \text{Gal}(\mathbb{C}_p/\hat{k}_{\sigma_p}) \subset G := \text{Gal}(\bar{k}/k)$ . Let  $\alpha \in H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p)$ . Consider then its associated De Rham class (see definition 2)*

$$w(\alpha) := H^j R\alpha(X)(\alpha \otimes 1) \in H_{DR}^j(X_{\mathbb{C}_p}) \otimes_{\mathbb{C}_p} \mathbb{B}_{dr, \mathbb{C}_p}.$$

Then  $\alpha \in H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p)(l)^{\hat{G}_{\sigma_p}}$  if and only if  $w(\alpha) \in F^l H_{DR}^j(X_{\hat{k}_{\sigma_p}}) = F^l H_{DR}^j(X_{\mathbb{C}_p}) \cap H_{DR}^j(X_{\hat{k}_{\sigma_p}})$ . That is we have

$$H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p)(l)^{\hat{G}_{\sigma_p}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \langle F^l H_{DR}^j(X_{\hat{k}_{\sigma_p}}) \cap H^j R\alpha(X)(H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p)) \rangle_{\mathbb{Q}_p} \subset H_{et}^j(X_{\mathbb{C}_p}, \mathbb{Z}_p, X^{et}) \otimes_{\mathbb{Z}_p} \mathbb{B}_{dr, \mathbb{C}_p},$$

where  $\langle - \rangle_{\mathbb{Q}_p}$  denote the  $\mathbb{Q}_p$  vector space generated by  $(-)$ .

*Proof.* Follows immediately from the fact that  $H^j R\alpha(X)$  is a filtered quasi-isomorphism compatible with the Galois action of  $\hat{G}_{\sigma_p}$  by [1]. □

These propositions, together with the results of section 3 implies the Tate conjecture :

**Theorem 2.** (i) *Let  $k$  a field of finite type over  $\mathbb{Q}$ . Let  $X \in \text{PSmVar}(k)$ . Then the Tate conjecture holds for  $X$ . That is for  $p$  a prime number and  $d \in \mathbb{Z}$ , the cycle class map*

$$\mathcal{Z}^d(X) \otimes \mathbb{Q}_p \rightarrow H_{et}^{2d}(X_{\bar{k}}, \mathbb{Q}_p)^G, \quad Z \mapsto [Z]$$

*is surjective, where  $G = \text{Gal}(\bar{k}/k)$ .*

(ii) *Let  $p$  a prime number. Let  $K$  a  $p$ -adic field, that is a finite extension of  $\mathbb{Q}_p$ . Let  $X \in \text{PSmVar}(K)$ . Then the Tate conjecture holds for  $X$ . That is for  $d \in \mathbb{Z}$ , the cycle class map*

$$\mathcal{Z}^d(X) \otimes \mathbb{Q}_p \rightarrow H_{et}^{2d}(X_{\mathbb{C}_p}, \mathbb{Q}_p)^G, \quad Z \mapsto [Z]$$

*is surjective, where  $G = \text{Gal}(\mathbb{C}_p, K)$ .*

*Proof.* (i): Let  $p \in \mathbb{N}$  a prime number. Consider an embedding  $\sigma_p : k \hookrightarrow \mathbb{C}_p$ . Then  $k \subset \bar{k} \subset \mathbb{C}_p$  and  $k \subset \hat{k}_{\sigma_p} \subset \mathbb{C}_p$ , where  $\hat{k}_{\sigma_p}$  is the  $p$ -adic field which is the completion of  $k$  with respect the  $p$  adic norm given by  $\sigma_p$ . Let  $\alpha \in H_{et}^{2d}(X_{\bar{k}}, \mathbb{Z}_p)(d)^G$ . Using definition 2, by proposition 7 and proposition 8 there exists  $(\lambda_i)_{1 \leq i \leq n} \in \mathbb{Z}_p$  and

$$(w_{Li})_{1 \leq i \leq n} \in \mathbb{H}_{pet}^{2d}(X_{\mathbb{C}_p}, \Omega_{X_{\mathbb{C}_p}}^{\bullet \geq d, \log, \mathcal{O}})$$

such that

$$w(\alpha) = \sum_{1 \leq i \leq n} \lambda_i \cdot w_{Li} \in \mathbb{H}_{pet}^{2d}(X_{C_p}, \Omega_{X_{C_p}}^{\bullet \geq d}) = F^d H_{DR}^{2d}(X_{C_p}), \quad w_{Li} := H^{2d} OL_X(w_{Li}).$$

By theorem 1(ii) and (iii), there exists  $Z_i \in \mathcal{Z}^d(X_{C_p})$  such that  $w_{Li} = [Z_i] \in H_{DR}^{2d}(X_{C_p})$ . Hence,

$$Z := \sum_{1 \leq i \leq n} \lambda_i [Z_i] \in \mathcal{Z}^d(X_{C_p}) \otimes \mathbb{Q}_p$$

satisfies  $w(\alpha) = [Z] \in H_{DR}^{2d}(X_{C_p})$ . By [1], we get  $\alpha = [Z] \in H_{et}^{2d}(X_{\bar{k}}, \mathbb{Z}_p)$ . Since  $[Z]$  is  $G$  invariant,

$$Z' := 1/\text{card}(gZ, g \in G) \sum_{g \in G} gZ \in \mathcal{Z}^d(X) \otimes \mathbb{Q}_p$$

satisfy  $[Z'] = [Z] = \alpha \in H_{et}^{2d}(X_{\bar{k}}, \mathbb{Z}_p)$ .

(ii): Similar to (i). □

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