De Rham logarithmic classes and Tate conjecture

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Abstract

We introduce the notion of De Rham logarithmic classes. We show that the De Rham class of an algebraic cycle of a smooth algebraic variety over a field of characteristic zero is logarithmic and conversely that a logarithmic class of bidegree (d,d) is the De Rham class of an algebraic cycle (of codimension d). Moreover for smooth projective varieties, we show that there are non trivial logarithmic classes of bidegree (p,q) for $p \neq q$. For smooth algebraic varieties over a p-adic field, we also give an analytic version of this result. We deduce, from the analytic version, the Tate conjecture for smooth projective varieties over fields of finite type over $\mathbb{Q}$ and over p-adic fields for $\mathbb{Q}_p$ coefficients, under good reduction hypothesis.

1 Introduction

1.1 De Rham logarithmic classes

In this work, we introduce in a first part for X a noetherian scheme, the notion of logarithmic De Rham cohomology classes which are for each $j \in \mathbb{Z}$, the elements of the subgroup $H^jOL_X(\mathbb{H}^j_{et}(X,\Omega^\bullet_{X,\log})) \subset \mathbb{H}^j_{et}(X,\Omega^\bullet_X) =: H^j_{DR}(X)$ of the De Rham cohomology abelian group, where $OL_X : \Omega^\bullet_{X,\log} \hookrightarrow \Omega^\bullet_X$ is the subcomplex of abelian sheaves on X consisting of logarithmic forms introduced in definition 2. To our knowledge, the notion of logarithmic forms was introduced in the seventies by S.Bloch for varieties over perfect fields of charactersitic p ([7], §3F) in order to compute the Frobenius fixed part of the De Rham-Witt complex. Note that this notion is different from the notion of logarithmic forms along a divisor with normal crossing as introduced by Deligne for exemple in Théorie de Hodge II. The presheaves $\Omega^\bullet_{X,\log}$, as for O^*_X are not O_X modules, nor locally constant but they have good purity properties. Our main result is that if X is a smooth algebraic variety over a field of characteristic zero, a de Rham cohomology class which is logarithmic of type (d,d) is the class of an algebraic cycle of codimension d. More precisely, let X be a smooth algebraic variety over a field k of characteristic zero. To each algebraic cycle $Z \in \mathcal{Z}^d(X)$, we associate its De Rham cohomology class $[Z] \in H^{2d}_{DR}(X)$, which is by definition, as for any Weil cohomology theory, the image of the fundamental class $[Z] \in H^{2d}_{DR,Z}(X)$ by the canonical morphism $H^{2d}_{DR,Z}(X) \to H^{2d}(X)$. In section 3, we show (c.f. theorem 2(i)) that

- for $Z \in \mathcal{Z}^d(X)$, $[Z] \in H^{2d}_{DR}(X)$ is logarithmic of bidegree (d,d), that is

$$[Z] = H^{2d}OL_X([Z]_L) \in H^{2d}_{DR}(X),\ [Z]_L \in H^d_{et}(X,\Omega^d_{X,\log})$$

  where $H^d_{et}(X,\Omega^d_{X,\log}) \subset \mathbb{H}^{2d}_{et}(X,\Omega^\bullet_{X,\log})$ is the canonical subspace, as all the differentials of $\Omega^\bullet_{X,\log}$ vanishes since by definition a logarithmic form is closed. This fact is a consequence of the fact that motivic isomorphisms applied to De Rham cohomology preserve logarithmic classes (c.f. proposition 1). The key point is that logarithmic De Rham forms on algebraic varieties over k are closed, (trivially) $\mathbb{A}^1_k$ invariant and compatible with the transfers maps induced by finite morphisms of algebraic varieties and in particular finite correspondences. Since $\Omega^l_{\log} \subset \Omega^l$ are (trivially) $\mathbb{A}^1_k$ invariant presheaves with transfers on the category of smooth algebraic varieties over k, by a theorem of Voevodsky the cohomology presheaves of $\mathbb{A}^1_k$ invariant presheaves with transfers are $\mathbb{A}^1_k$ invariant.

- Conversely, we show (c.f. theorem 2(ii)), that a logarithmic class $w \in H^{2d}_{DR}(X)$ of bidegree (d,d) is the De Rham class of an algebraic cycle (of codimension d). In the case X projective, we also get a vanishing result (c.f. theorem 2(iii)') : $H^{2d}OL_X(H^{d-k}_{et}(X, \Omega^{d+k}_{X,\log})) = 0$ for $k > 0$.
  The proof works as follows : A logarithmic class of bidegree (p,q) is locally acyclic for the Zariski topology of X since it is the etale cohomology of a single sheaf. This allows us to proceed by a finite induction using the crucial fact that the purity isomorphism for De Rham cohomology preserve logarithmic classes (c.f. proposition 6). The proof of proposition 6 follows from the fact that the purity isomorphism is motivic (c.f. [5], see proposition 1), that the Euler class of a vector bundle of rank d over an algebraic variety is logarithmic of bidegree (d,d) (c.f. proposition 5) and that the motivic isomorphisms applied to De Rham cohomology preserve logarithmic classes (c.f. proposition 4). At the final step, we use, for simplicity, the fact that for a scheme Y, $H^1_{et}(Y, \Omega^1_{Y,\log}) = H^1(Y, O^*_Y)$ is the Picard group of Y.

For X an algebraic variety over a p-adic field, we also introduce the notion of logarithmic analytic De Rham cohomology classes. In proposition 8 we give an analytic analogue of theorem 2, that is if X is a smooth projective variety over a p-adic field with good reduction,

- a logarithmic analytic De Rham cohomology class of type (d,d) is the class of a codimension d algebraic cycle,
- there are no non trivial logarithmic analytic De Rham cohomology classes of type (p,q) for $p \neq q$, i.e. $H^{2d}OL_{\hat{X}}(H^{d-k}_{pet}(X, \Omega^{d+k}_{\hat{X}^{(p)},\log,\mathcal{O}})) = 0$ for $k \neq 0$.

The proof of proposition 8 is motivic and similar to the proof of theorem 2 : a logarithmic analytic class of bidegree (p,q) is acyclic for the pro-etale topology on each open subset $U \subset X$ such that there exists an etale map $e : U^{\mathcal{O}} \to \mathbb{G}_m^{d_U} \subset \mathbb{A}^{d_U}_{O_K}$ finite over $e(U^{\mathcal{O}})$, where $U^{\mathcal{O}} \subset X^{\mathcal{O}}$ are integral models of $U \subset X$ and since this class is the pro etale cohomology of a single subsheaf of the De Rham complex (proposition 2). This allows us to proceed by a finite induction using the crucial fact that the purity isomorphism for De Rham cohomology preserve logarithmic analytic classes since the purity isomorphism is motivic (c.f. [5], see proposition 1). The GAGA comparison theorem for the De Rham cohomology of X induce, by proposition 8 and theorem 2, a GAGA comparison result for logarithmic De Rham classes.

1.2 Tate conjecture

Let X be a smooth projective variety over a field k of finite type over $\mathbb{Q}$. Let p be a prime number unramified over k such that X has good reduction at p, $\sigma_p : k \hookrightarrow \mathbb{C}_p$ a fixed embedding and $\bar{k}$ the algebraic closure of k inside $\mathbb{C}_p$. We prove the Tate conjecture for X with $\mathbb{Q}_p$ coefficient (c.f. theorem 3). The proof works as follows : let $\hat{k}_{\sigma_p} \subset \mathbb{C}_p$ be the p-adic completion of k with respect to σ_p. We choose a smooth proper scheme $X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}$ over $O_{\hat{k}_{\sigma_p}}$ which is an integral model of X, i.e. such that $X^{\mathcal{O}} \times_{O_{\hat{k}_{\sigma_p}}} \hat{k}_{\sigma_p} = X_{\hat{k}_{\sigma_p}}$. Let $\alpha \in H^{2d}_{et}(X_{\bar{k}}, \mathbb{Z}_p)(d)^G$ be a Tate class of X In proposition 9, we show, using a result of [7] together with results of [11] on the pro-etale topology, that α gives by the p-adic crystalline comparison isomorphism

$$H^{2d}R\alpha(X) : H^{2d}_{et}(X_{\mathbb{C}_p}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{B}_{cris,\hat{k}_{\sigma_p}} \xrightarrow{\sim} H^{2d}_{DR}(X) \otimes_k \mathbb{B}_{cris,\hat{k}_{\sigma_p}},$$

an analytic logarithmic de Rham class

$$w(\alpha) := H^{2d}R\alpha(X)(\alpha) \in H^{2d}OL_X(\mathbb{H}^{2d}_{pet}(X_{\hat{k}_{\sigma_p}}, \Omega^{\bullet \geq d}_{\hat{X}^{(p)}_{\hat{k}_{\sigma_p}},\log,\mathcal{O}})).$$

Proposition 8 then implies that $w(\alpha) = [Z] \in H^{2d}_{DR}(X_{\hat{k}_{\sigma_p}})$, is the de Rham cohomology class of an algebraic cycle $Z \in \mathcal{Z}^d(X_{\hat{k}_{\sigma_p}}) \otimes \mathbb{Q}_p$ of codimension d. By the p-adic crystalline comparison isomorphism, we get $\alpha = [Z] \in H^{2d}_{et}(X_{\mathbb{C}_p}, \mathbb{Q}_p)$. Since α is G invariant, we get

$$\alpha = [Z] = [Z'] \in H^{2d}_{et}(X_{\mathbb{C}_p}, \mathbb{Q}_p), Z' := (1/\#gZ) \sum_{g \in G \cdot Z} gZ \in \mathcal{Z}^d(X) \otimes \mathbb{Q}_p.$$

It follows from the proof that we have the p-adic Tate conjecture, that is for p-adic field with $\mathbb{Q}_p$ coefficients. Some authors ([10],[6]) believe that the conjecture is only true under additional assumptions (for example total degeneracy). The standard example provided is the product of two non-isogenous elliptic curves with $\mathrm{Hom}(C, C') = 0$, say C and C', with the fact that

$$\mathrm{Hom}(C, C') \otimes \mathbb{Q}_p \to \mathrm{Hom}_G(V_p(C), V_p(C'))$$

is not surjective since $\mathrm{Hom}(C, C') \otimes \mathbb{Q}_p = 0$. However, this does not contradict the general conjecture because for C, C' two smooth projective curve of genus greater or equal to one or two abelian varieties over a p-adic field K, e.g. elliptic curves over K, we have $\dim NS^1(C \times C') \geq 3$ where

$$NS^1(C \times C') \subset H^2_{et}((C \times C')_{\mathbb{C}_p}, \mathbb{Q}_p)$$

is the Neron Severi group (see remark 5).

On the other side, for a p-adic field $K \subset \mathbb{C}_p$ and X a smooth projective variety over K,

$$\dim H^j_{et}(X_{\mathbb{C}_p}, \mathbb{Q}_\ell)(k)^G > \dim H^j_{et}(X_{\mathbb{C}_p}, \mathbb{Q}_p)(k)^G$$

for $\ell \neq p$ in general and it is known that the ℓ-adic Tate conjecture is not true. Hence there is no relation with the classical Tate conjecture over finite fields.

Let X be a smooth projective variety over $\mathbb{C}$. Then X is defined over a subfield $k \subset \mathbb{C}$ of finite type over $\mathbb{Q}$, that is $X = X_k \otimes_k \mathbb{C}$. Take an isomorphism $\mathbb{C} \simeq \mathbb{C}_p$ with $p \in \mathbb{N}$ a prime number such that $X_{\mathbb{C}_p}$ has good reduction at p. As the Tate conjecture holds for X_k with $\mathbb{Q}_p$ coefficients, the standard conjectures holds for X and any absolute Hodge class of X is the class of an algebraic cycle (corollary 2).

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2 Preliminaries and Notations

2.1 Notations

- Denote by Top the category of topological spaces and RTop the category of ringed spaces.
- Denote by Cat the category of small categories and RCat the category of ringed topos.
- For $\mathcal{S} \in \mathrm{Cat}$ and $X \in \mathcal{S}$, we denote $\mathcal{S}/X \in \mathrm{Cat}$ the category whose objects are $Y/X := (Y, f)$ with $Y \in \mathcal{S}$ and $f : Y \to X$ is a morphism in $\mathcal{S}$, and whose morphisms $\mathrm{Hom}((Y', f'), (Y, f))$ consists of $g : Y' \to Y$ in $\mathcal{S}$ such that $f \circ g = f'$.
- For $(\mathcal{S}, O_S) \in \mathrm{RCat}$ a ringed topos, we denote by
  - $\mathrm{PSh}(\mathcal{S})$ the category of presheaves of abelian group on $\mathcal{S}$ and $\mathrm{PSh}_{O_S}(\mathcal{S})$ the category of presheaves of O_S modules on $\mathcal{S}$, whose objects are
    $$\mathrm{PSh}_{O_S}(\mathcal{S})^0 := \{(M, m), M \in \mathrm{PSh}(\mathcal{S}), m : M \otimes O_S \to M\},$$
    together with the forgetful functor $o : \mathrm{PSh}(\mathcal{S}) \to \mathrm{PSh}_{O_S}(\mathcal{S})$, for $F \in \mathrm{PSh}(\mathcal{S})$ and $X \in \mathcal{S}$, we denote $F(X) := \Gamma(X, F)$ the abelian group of section over X,
  - $C(\mathcal{S}) = C(\mathrm{PSh}(\mathcal{S}))$ and $C_{O_S}(\mathcal{S}) = C(\mathrm{PSh}_{O_S}(\mathcal{S}))$ the big abelian category of complexes of presheaves of O_S modules on $\mathcal{S}$,
  - $C_{O_S(2)fil}(\mathcal{S}) := C_{(2)fil}(\mathrm{PSh}_{O_S}(\mathcal{S})) \subset C(\mathrm{PSh}_{O_S}(\mathcal{S}), F, W)$, the big abelian category of (bi)filtered complexes of presheaves of O_S modules on $\mathcal{S}$ such that the filtration is biregular and $\mathrm{PSh}_{O_S(2)fil}(\mathcal{S}) := (\mathrm{PSh}_{O_S}(\mathcal{S}), F, W)$.
- Let $(\mathcal{S}, O_S) \in \mathrm{RCat}$ a ringed topos with topology τ. For $F \in C_{O_S}(\mathcal{S})$, we denote by $k : F \to E_\tau(F)$ the canonical flasque resolution in $C_{O_S}(\mathcal{S})$ (see [3]). In particular for $X \in \mathcal{S}$, $H^*(X, E_\tau(F)) \xrightarrow{\sim} \mathbb{H}^*_\tau(X, F)$.

- For $f : \mathcal{S}' \to \mathcal{S}$ a morphism with $\mathcal{S}, \mathcal{S}' \in \mathrm{RCat}$, endowed with topology τ and τ' respectively, we denote for $F \in C_{O_S}(\mathcal{S})$ and each $j \in \mathbb{Z}$,
  - $f^* := H^j\Gamma(\mathcal{S}, k \circ \mathrm{ad}(f^*, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \to \mathbb{H}^j(\mathcal{S}', f^*F)$,
  - $f^* := H^j\Gamma(\mathcal{S}, k \circ \mathrm{ad}(f^{*mod}, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \to \mathbb{H}^j(\mathcal{S}', f^{*mod}F)$,

  the canonical maps.
- For $\mathcal{X} \in \mathrm{Cat}$ a (pre)site and p a prime number, we consider the full subcategory
$$\mathrm{PSh}_{\mathbb{Z}_p}(\mathcal{X}) \subset \mathrm{PSh}(\mathbb{N} \times \mathcal{X}), \;\; F = (F_n)_{n\in\mathbb{N}},\; p^n F_n = 0,\; F_{n+1}/p^n \xrightarrow{\sim} F_n$$
  $C_{\mathbb{Z}_p}(\mathcal{X}) := C(\mathrm{PSh}_{\mathbb{Z}_p}(\mathcal{X})) \subset C(\mathbb{N} \times \mathcal{X})$ and
$$\mathbb{Z}_p := \mathbb{Z}_{p,\mathcal{X}} := ((\mathbb{Z}/p^*\mathbb{Z})_{\mathcal{X}}) \in \mathrm{PSh}_{\mathbb{Z}_p}(\mathcal{X})$$
  the diagram of constant presheaves on $\mathcal{X}$.
- For $X \in \mathrm{Top}$ and $Z \subset X$ a closed subset, denoting $j : X\backslash Z \hookrightarrow X$ the open complementary, we will consider
$$\Gamma^\vee_Z\mathbb{Z}_X := \mathrm{Cone}(\mathrm{ad}(j_!, j^*)(\mathbb{Z}_X) : j_!j^*\mathbb{Z}_X \hookrightarrow \mathbb{Z}_X) \in C(X)$$
  and denote for short $\gamma^\vee_Z := \gamma^\vee_X(\mathbb{Z}_X) : \mathbb{Z}_X \to \Gamma^\vee_Z\mathbb{Z}_X$ the canonical map in $C(X)$.
- Denote by $\mathrm{Sch} \subset \mathrm{RTop}$ the subcategory of schemes (the morphisms are the morphisms of locally ringed spaces). We denote by $\mathrm{PSch} \subset \mathrm{Sch}$ the full subcategory of proper schemes. For a field k, we consider $\mathrm{Sch}\,/k := \mathrm{Sch}\,/\mathrm{Spec}\,k$ the category of schemes over $\mathrm{Spec}\,k$, that is whose object are $X := (X, a_X)$ with $X \in \mathrm{Sch}$ and $a_X : X \to \mathrm{Spec}\,k$ a morphism and whose objects are morphism of schemes $f : X' \to X$ such that $f \circ a_{X'} = a_X$. We then denote by
  - $\mathrm{Var}(k) = \mathrm{Sch}^{ft}\,/k \subset \mathrm{Sch}\,/k$ the full subcategory consisting of algebraic varieties over k, i.e. schemes of finite type over k,
  - $\mathrm{PVar}(k) \subset \mathrm{QPVar}(k) \subset \mathrm{Var}(k)$ the full subcategories consisting of quasi-projective varieties and projective varieties respectively,
  - $\mathrm{PSmVar}(k) \subset \mathrm{SmVar}(k) \subset \mathrm{Var}(k)$, $\mathrm{PSmVar}(k) := \mathrm{PVar}(k) \cap \mathrm{SmVar}(k)$, the full subcategories consisting of smooth varieties and smooth projective varieties respectively.

  For a morphism of commutative rings $\sigma : A \hookrightarrow B$, we have the extention of scalar functor
$$\otimes_A B : \mathrm{Sch}\,/A \to \mathrm{Sch}\,/B,\; X \mapsto X_B := X_{B,\sigma} := X \otimes_A B,\; (f : X' \to X) \mapsto (f_B := f \otimes I : X'_B \to X_B).$$
  which is left ajoint to the restriction of scalar
$$Res_{A/B} : \mathrm{Sch}\,/B \to \mathrm{Sch}\,/A,\; X = (X, a_X) \mapsto X = (X, \sigma \circ a_X),\; (f : X' \to X) \mapsto (f : X' \to X)$$
  For a morphism of fields $\sigma : k \hookrightarrow K$, the extention of scalar functor restricts to a functor
$$\otimes_k K : \mathrm{Var}(k) \to \mathrm{Var}(K),\; X \mapsto X_K := X_{K,\sigma} := X \otimes_k K,\; (f : X' \to X) \mapsto (f_K := f \otimes I : X'_K \to X_K).$$
  and for $X \in \mathrm{Var}(k)$ we have $\pi_{k/K}(X) : X_K \to X$ the projection in $\mathrm{Sch}\,/k$.
- For $X \in \mathrm{Sch}$ a noetherian scheme and $p \in \mathbb{N}$, we denote by $\mathcal{Z}^p(X)$ the free abelian group generated by closed subset of codimension p.
- For $X \in \mathrm{Sch}$ and p a prime number, we denote by $c : \hat{X}^{(p)} \to X$ the morphism in RTop which is the completion along the ideal generated by p.

- For K a field which is complete with respect to a p-adic norm, we denote by $\mathrm{RigVar}(K) \subset \mathrm{RTop}$ the subcategory of rigid analytic space (i.e. locally given by affinoid which are Tate algebra spectrum). We denote by $\mathrm{An}_p : \mathrm{RigVar}(K) \to \mathrm{Var}(K)$ the analytic functor and for $X \in \mathrm{Var}(K)$, $\mathrm{an}_{X,p} := \mathrm{An}_{p,|X} : \hat{X}^{(p)} := \hat{X}^{\mathcal{O},(p)} \times_{O_K} K \to X$ the corresponding morphism in RTop.

- Denote $\mathrm{Sch}^2 \subset \mathrm{RTop}^2$ the subcategory whose objects are couples (X, Z) with $X = (X, O_X) \in \mathrm{Sch}$ and $Z \subset X$ a closed subset and whose set of morphisms $\mathrm{Hom}((X', Z'), (X, Z))$ consists of $f : X' \to X$ of locally ringed spaces such that $f^{-1}(Z) \subset Z'$.

- Let k be a field of characteristic zero. Denote $\mathrm{SmVar}^2(k) \subset \mathrm{Var}^2(k) \subset \mathrm{Sch}^2/k$ the full subcategories whose objects are (X, Z) with $X \in \mathrm{Var}(k)$, resp. $X \in \mathrm{SmVar}(k)$, and $Z \subset X$ is a closed subset, and whose morphisms $\mathrm{Hom}((X', Z') \to (X, Z))$ consists of $f : X' \to X$ of schemes over k such that $f^{-1}(Z) \subset Z'$.

- Denote by $\mathrm{AnSp}(\mathbb{C}) \subset \mathrm{RTop}$ the full subcategory of analytic spaces over $\mathbb{C}$, and by $\mathrm{AnSm}(\mathbb{C}) \subset \mathrm{AnSp}(\mathbb{C})$ the full subcategory of smooth analytic spaces (i.e. complex analytic manifold). Denote by $\mathrm{CW} \subset \mathrm{Top}$ the full subcategory of CW complexes. Denote by $\mathrm{Diff}(\mathbb{R}) \subset \mathrm{RTop}$ the full subcategory of differentiable (real) manifold.

2.2 The pro-etale site of schemes

For $X \in \mathrm{Sch}$, we denote $X^{et} \subset \mathrm{Sch}/X$ the etale site and $X^{pet} \subset \mathrm{Sch}/X$ the pro etale site (see [2]) which is the full subcategory of Sch/X whose object consists of weakly etale maps $U \to X$ (that is flat maps $U \to X$ such that $\Delta_U : U \to U \times_X U$ is also flat) and whose topology is generated by fpqc covers. We then have the canonical morphism of site

$$\nu_X : X^{pet} \to X^{et}, (U \to X) \mapsto (U \to X)$$

For $F \in C(X^{et})$,

$$\mathrm{ad}(\nu_X^*, R\nu_{X*})(F) : F \to R\nu_{X*}\nu_X^* F$$

is an isomorphism in $D(X^{et})$, in particular, for each $n \in \mathbb{Z}$

$$\nu_X^* : \mathbb{H}^n_{et}(X, F) \xrightarrow{\sim} \mathbb{H}^n_{pet}(X, \nu_X^* F)$$

are isomorphisms, For $X \in \mathrm{Sch}$, we denote

- $\underline{\mathbb{Z}_p}_X := \varprojlim_{n \in \mathbb{N}} \nu_X^*(\mathbb{Z}/p^n\mathbb{Z})_{X^{et}} \in \mathrm{PSh}(X^{pet})$ the constant presheaf on $\mathcal{X}$,

- $l_{p,\mathcal{X}} := (p(*)) : \underline{\mathbb{Z}_p}_X \to \nu_X^*(\mathbb{Z}/p\mathbb{Z})_{X^{et}}$ the projection map in $\mathrm{PSh}(\mathbb{N} \times \mathcal{X}^{pet})$.

An affine scheme $U \in \mathrm{Sch}$ is said to be w-contractible if any faithfully flat weakly etale map $V \to U$, $V \in \mathrm{Sch}$, admits a section. We will use the facts that (see [2]):

- Any affine scheme $X \in \mathrm{Sch}$ admits a faithfully flat pro-etale map $r : U \to X$ with U w-contractile.

- Any scheme $X \in \mathrm{Sch}$ admits a pro-etale affine cover $(r_i : X_i \to X)_{i \in I}$ with for each $i \in I$, X_i a w-contractile affine scheme and $r_i : X_i \to X$ a weakly etale map. For $X \in \mathrm{Var}(k)$ with k a field, we may assume I finite since the topological space X is then quasi-compact.

- If $U \in \mathrm{Sch}$ is a w-contractible affine scheme, then for any sheaf $F \in \mathrm{Shv}(U^{pet})$, $H^i_{pet}(U, F) = 0$ for $i \neq 0$ since $\Gamma(U, -)$ is an exact functor.

2.3 Integral models and p-adic completion of algebraic varieties over a p-adic field

For K a field which is complete with respect to a p-adic norm, we consider $O_K \subset K$ the subring of K consisting of integral elements, that is $x \in K$ such that $|x| \leq 1$.

- For $X \in \mathrm{PVar}(K)$, we will consider $X^{\mathcal{O}} \in \mathrm{PSch}/O_K$ a (non canonical) integral model of X, i.e. satisfying $X^{\mathcal{O}} \otimes_{O_K} K = X$
- For $X \in \mathrm{Var}(K)$, we will consider $X^{\mathcal{O}} \in \mathrm{Sch}/O_K$ a (non canonical) integral model of X, i.e. $X^{\mathcal{O}} = \bar{X}^{\mathcal{O}} \backslash Z^{\mathcal{O}}$ for $\bar{X} \in \mathrm{PVar}(K)$ a compactification of X, $Z := \bar{X} \backslash X$, where $\bar{X}^{\mathcal{O}} \in \mathrm{PSch}/O_K$ is an integral model of $\bar{X}$ and $Z^{\mathcal{O}} := V(I_Z^{\mathcal{O}}) \subset \bar{X}^{\mathcal{O}}$ is an integral model of Z.

We consider $\mathrm{Sch}^{int}/O_K := O(\mathrm{PSch}^2/O_K) \subset \mathrm{Sch}^{ft}/O_K$ the full subcategory consisting of integral models of algebraic varieties over K, where $O : \mathrm{PSch}^2/O_K \to \mathrm{Sch}^{ft}/O_K$, $O(X, Z) = X \backslash Z$ is the canonical functor, and

$$\mathrm{Sch}^{int,sm}/O_K := \mathrm{Sch}^{int}/O_K \cap \mathrm{Sch}^{sm}/O_K \subset \mathrm{Sch}/O_K$$

the full subcategory consisting of integral models of (smooth) algebraic varieties over K which are smooth over O_K. For $X \in \mathrm{Var}(K)$, we will consider $X^{\mathcal{O}} \in \mathrm{Sch}/O_K$ a (non canonical) integral model of X, we then conisder $X^{\mathcal{O},et} \subset (\mathrm{Sch}^{int}/O_K)/X^{\mathcal{O}}$ the full subcategory consisting of $e : U = U^{\mathcal{O}} \to X^{\mathcal{O}}$ such that e is an etale morphism of schemes, we have then the commutative diagram of sites

$$\begin{array}{ccc} X^{pet} & \xrightarrow{r} & X^{\mathcal{O},pet} \\ \downarrow{\scriptstyle \nu_X} & & \downarrow{\scriptstyle \nu_{X^{\mathcal{O}}}} \\ X^{et} & \xrightarrow{r} & X^{\mathcal{O},et} \end{array} \quad , \quad r(t : U = U^{\mathcal{O}} \to X^{\mathcal{O}}) := (t \otimes_{O_K} K : U \otimes_{O_K} K \to X^{\mathcal{O}} \otimes_{O_K} K = X)$$

Let K be a field which is complete with respect to a p-adic norm and $X \in \mathrm{PVar}(K)$ projective. For $X^{\mathcal{O}} \in \mathrm{PSch}/O_K$ an integral model of X, i.e. satisfying $X^{\mathcal{O}} \otimes_{O_K} K = X$, we consider $\hat{X}^{(p)} := \hat{X}^{\mathcal{O},(p)} \otimes_{O_K} K \in \mathrm{RigVar}(K)$ and the morphism in RTop

$$\mathrm{an}_{X,p} : \hat{X}^{(p)} := \hat{X}^{\mathcal{O},(p)} \otimes_{O_K} K \to X^{\mathcal{O}} \otimes_{O_K} K = X.$$

given by the analytical functor. We have also the Raynaud generic fiber morphism in RTop.

$$\eta_{X,p} : \hat{X}^{(p)} := \hat{X}^{\mathcal{O},(p)} \otimes_{O_K} K \to \hat{X}^{\mathcal{O},(p)}.$$

We have then the commutative diagram in RTop

$$\begin{array}{ccc} \hat{X}^{(p)} & \xrightarrow{\eta_{X,p}} & \hat{X}^{\mathcal{O},(p)} \\ \downarrow{\scriptstyle \mathrm{an}_{X,p}} & & \downarrow{\scriptstyle c} \\ X & \xrightarrow{r} & X^{\mathcal{O}} \end{array}$$

Recall (see section 2.1) that $c : \hat{X}^{\mathcal{O},(p)} \to X^{\mathcal{O}}$ is the morphism in RTop which is the completion along the ideal generated by p. Then, by GAGA (c.f. EGA 3), for $F \in \mathrm{Coh}_{O_X}(X)$ a coherent sheaf of O_X module, for all $k \in \mathbb{Z}$, the canonical map $c^* : H^k(X^{\mathcal{O}}, F) \xrightarrow{\sim} H^k(\hat{X}^{\mathcal{O},(p)}, c^{*mod} F)$ is an isomorphisms. In particular,

$$an^*_{X,p} = \eta_X^* c^* : \mathbb{H}^k(X, \Omega_X^{\bullet \geq l}) \xrightarrow{\sim} \mathbb{H}^k(\hat{X}^{(p)}, \Omega^{\bullet \geq l}_{\hat{X}^{(p)}})$$

for all $k, l \in \mathbb{Z}$, where $\Omega^{\bullet}_X := \Omega^{\bullet}_{X/K}$ and $\Omega^{\bullet}_{\hat{X}^{(p)}} := \Omega^{\bullet}_{\hat{X}^{(p)}/K}$ are the De Rham complexes.

2.4 De Rham cohomology

We recall some properties of the De Rham cohomology.

- We have

$$\Omega^\bullet \in C(\mathrm{Sch}), X \mapsto \Omega^\bullet(X) := \Gamma(X, \Omega^\bullet_X),$$
$$(f : X' \to X) \mapsto \Omega^\bullet(f) := f^* : \Gamma(X, \Omega^\bullet_X) \to \Gamma(X', \Omega^\bullet_{X'})$$

  Let $X \in \mathrm{Sch}$. Considering its De Rham complex $\Omega^\bullet_X := DR(X)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H^j_{DR}(X) := \mathbb{H}^j(X, \Omega^\bullet_X)$.

- We will consider

$$\Omega^\bullet_{/k} \in C(\mathrm{Var}(k)), X \mapsto \Omega^\bullet_{/k}(X) := \Gamma(X, \Omega^\bullet_X),$$
$$(f : X' \to X) \mapsto \Omega^\bullet_{/k}(f) := f^* : \Gamma(X, \Omega^\bullet_X) \to \Gamma(X', \Omega^\bullet_{X'})$$

  and its restriction to $\mathrm{SmVar}(k) \subset \mathrm{Var}(k)$. Let $X \in \mathrm{Var}(k)$. Considering its De Rham complex $\Omega^\bullet_X := \Omega^\bullet_{X/k} := DR(X/k)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H^j_{DR}(X) := \mathbb{H}^j(X, \Omega^\bullet_X)$. The differentials of $\Omega^\bullet_X := \Omega^\bullet_{X/k}$ are by definition k-linear, thus $H^j_{DR}(X) := \mathbb{H}^j(X, \Omega^\bullet_X)$ has a structure of a k vector space.

If $X \in \mathrm{SmVar}(k)$, then $H^j_{DR}(X) = \mathbb{H}^j_{et}(X, \Omega^\bullet_X)$ since $\Omega^\bullet_{/k} \in C(\mathrm{SmVar}(k))$ is $\mathbb{A}^1$ local and admits transfers where (see [3]). Note that for $X \in \mathrm{Var}(k)$ singular, we also consider its realized De Rham cohomology

$$\tilde{H}^j_{DR}(X) := \mathbb{H}^j(X_\bullet, \Omega^\bullet_{X_\bullet}) = \mathbb{H}^j(\tilde{X}, R\Gamma_X \Omega^\bullet_{\tilde{X}})$$

where $\epsilon : X_\bullet \to X$ in $\mathrm{Fun}(\Delta, \mathrm{SmVar}(k))$ is a simplicial desingularization of X and $X \hookrightarrow \tilde{X}$ is a closed embedding with $\tilde{X} \in \mathrm{SmVar}(k)$, and note that $H^j_{DR}(X)$ is NOT isomorphic to $\tilde{H}^j_{DR}(X)$ in general since $\Omega^\bullet_{/k} \in C(\mathrm{Var}(k))$ does NOT satisfied cdh descent.

Let $X \in \mathrm{Var}(k)$. Let $X = \cup_{i=1}^s X_i$ an open affine cover. For $I \subset [1, \ldots, s]$, we denote $X_I := \cap_{i \in I} X_i$. We get $X_\bullet \in \mathrm{Fun}(P([1, \ldots, s]), \mathrm{Var}(k))$. Since quasi-coherent sheaves on affine noetherian schemes are acyclic, we have for each $j \in \mathbb{Z}$, $H^j_{DR}(X) = \Gamma(X_\bullet, \Omega^\bullet_{X_\bullet})$.

2.5 Singular chains

We denote $\mathbb{I}^n := [0, 1]^n \in \mathrm{Diff}(\mathbb{R})$ (with boundary). For $X \in \mathrm{Top}$ and R a ring, we consider its singular cochain complex

$$C^*_{\mathrm{sing}}(X, R) := (\mathbb{Z}\,\mathrm{Hom}_{\mathrm{Top}}(\mathbb{I}^*, X)^\vee) \otimes R$$

and for $l \in \mathbb{Z}$ its singular cohomology $H^l_{\mathrm{sing}}(X, R) := H^n C^*_{\mathrm{sing}}(X, R)$. For $f : X' \to X$ a continous map with $X, X' \in \mathrm{Top}$, we have the canonical map of complexes

$$f^* : C^*_{\mathrm{sing}}(X, R) \to C^*_{\mathrm{sing}}(X, R), \sigma \mapsto f^*\sigma := (\gamma \mapsto \sigma(f \circ \gamma)).$$

In particular, we get by functoriality the complex

$$C^*_{X, R\,\mathrm{sing}} \in C_R(X),\ (U \subset X) \mapsto C^*_{\mathrm{sing}}(U, R)$$

We will consider the canonical embedding

$$C^*\iota_{2i\pi\mathbb{Z}/\mathbb{C}}(X) : C^*_{\mathrm{sing}}(X, 2i\pi\mathbb{Z}) \hookrightarrow C^*_{\mathrm{sing}}(X, \mathbb{C}),\ \alpha \mapsto \alpha \otimes 1$$

whose image consists of cochains $\alpha \in C^j_{\mathrm{sing}}(X, \mathbb{C})$ such that $\alpha(\gamma) \in 2i\pi\mathbb{Z}$ for all $\gamma \in \mathbb{Z}\,\mathrm{Hom}_{\mathrm{Top}}(\mathbb{I}^*, X)$. We get by functoriality the embedding in $C(X)$

$$C^*\iota_{2i\pi\mathbb{Z}/\mathbb{C}, X} : C^*_{X, 2i\pi\mathbb{Z}, \mathrm{sing}} \hookrightarrow C^*_{X, \mathbb{C}, \mathrm{sing}},$$
$$(U \subset X) \mapsto (C^*\iota_{2i\pi\mathbb{Z}/\mathbb{C}}(U) : C^*_{\mathrm{sing}}(U, 2i\pi\mathbb{Z}) \hookrightarrow C^*_{\mathrm{sing}}(U, \mathbb{C}))$$

We recall we have

- For $X \in \mathrm{Top}$ locally contractile, e.g. $X \in \mathrm{CW}$, and R a ring, the inclusion in $C_R(X)$ $c_X : R_X \to C^*_{X,R\,\mathrm{sing}}$ is by definition an equivalence top local and that we get by the small chain theorem, for all $l \in \mathbb{Z}$, an isomorphism $H^l c_X : H^l(X, R_X) \xrightarrow{\sim} H^l_{\mathrm{sing}}(X, R)$.

- For $X \in \mathrm{Diff}(\mathbb{R})$, the restriction map

$$r_X : \mathbb{Z}\,\mathrm{Hom}_{\mathrm{Diff}(\mathbb{R})}(\mathbb{I}^*, X)^\vee \to C^*_{\mathrm{sing}}(X, R),\ w \mapsto w : (\phi \mapsto w(\phi))$$

  is a quasi-isomorphism by Whitney approximation theorem.

2.6 Algebraic cycles, motives and the theorem of Voevodsky

For $X \in \mathrm{Sch}$ noetherian irreducible and $d \in \mathbb{N}$, we denote by $\mathcal{Z}^d(X)$ the group of algebraic cycles of codimension d, which is the free abelian group generated by irreducible closed subsets of codimension d.

For $X, X' \in \mathrm{Sch}$ noetherian, with X' irreducible, we denote $\mathcal{Z}^{fs/X'}(X' \times X) \subset \mathcal{Z}_{d_{X'}}(X' \times X)$ which consist of algebraic cycles $\alpha = \sum_i n_i \alpha_i \in \mathcal{Z}_{d_{X'}}(X' \times X)$ such that, denoting $\mathrm{supp}(\alpha) = \cup_i \alpha_i \subset X' \times X$ its support and $p' : X' \times X \to X'$ the projection, $p'_{|\,\mathrm{supp}(\alpha)} : \mathrm{supp}(\alpha) \to X'$ is finite surjective.

- Let k be a field. We denote by $\mathrm{Cor}\,\mathrm{SmVar}(k)$ the category such that the objects are $\{X \in \mathrm{SmVar}(k)\}$ and such that $\mathrm{Hom}_{\mathrm{Cor}\,\mathrm{SmVar}(k)}(X', X) := \mathcal{Z}^{fs/X'}(X' \times X)$. See [5] for the composition law. We denote by $\mathrm{Tr} : \mathrm{Cor}\,\mathrm{SmVar}(k) \to \mathrm{SmVar}(k)$ the morphism of site given by the embedding $\mathrm{Tr} : \mathrm{SmVar}(k) \hookrightarrow \mathrm{Cor}\,\mathrm{SmVar}(k)$. Let $F \in \mathrm{PSh}(\mathrm{SmVar}(k))$. We say that F admits transfers if $F = \mathrm{Tr}_* F$ with $F \in \mathrm{PSh}(\mathrm{Cor}\,\mathrm{SmVar}(k))$.

- Let A a regular commutative noetherian ring. We denote by $\mathrm{Cor}\,\mathrm{Sch}^{sm}/A$ the category such that the objects are $\{X \in \mathrm{Sch}^{sm}/A\}$ and such that $\mathrm{Hom}_{\mathrm{Cor}\,\mathrm{Sch}^{sm}/A}(X', X) := \mathcal{Z}^{fs/X'}(X' \times X)$. See [5] for the composition law. We denote by $\mathrm{Tr} : \mathrm{Cor}\,\mathrm{Sch}^{sm}/A \to \mathrm{Sch}^{sm}/A$ the morphism of site given by the embedding $\mathrm{Tr} : \mathrm{Sch}^{sm}/A \hookrightarrow \mathrm{Cor}\,\mathrm{Sch}^{sm}/A$. Let $F \in \mathrm{PSh}(\mathrm{Sch}^{sm}/A)$. We say that F admits transfers if $F = \mathrm{Tr}_* F$ with $F \in \mathrm{PSh}(\mathrm{Cor}\,\mathrm{Sch}^{sm}/A)$.

We recall the following standard notion (see e.g. [5] or [3]) :

Definition 1. *(i) Let $F \in \mathrm{PSh}(\mathrm{Var}(k))$ or $F \in \mathrm{PSh}(\mathrm{SmVar}(k))$. We say that F is $\mathbb{A}^1$ invariant if for all $X \in \mathrm{Var}(k)$ (resp. $X \in \mathrm{SmVar}(k)$), $p^* := F(p) : F(X) \to F(X \times \mathbb{A}^1)$ is an isomorphism where $p : X \times \mathbb{A}^1 \to X$ is the projection.*

(ii) Let $F \in \mathrm{PSh}(\mathrm{Var}(k))$ or $F \in \mathrm{PSh}(\mathrm{SmVar}(k))$. We say that F is $\mathbb{A}^1$ local if for all $j \in \mathbb{Z}$ and all $X \in \mathrm{Var}(k)$ (resp. $X \in \mathrm{SmVar}(k)$), $p^* := H^j E_{et}(F)(p) : H^j_{et}(X, F) \to H^j_{et}(X \times \mathbb{A}^1, F)$ is an isomorphism.

(ii)' Let $F \in C(\mathrm{Var}(k))$ or $F \in C(\mathrm{SmVar}(k))$. We say that F is $\mathbb{A}^1$ local if for all $j \in \mathbb{Z}$ and all $X \in \mathrm{Var}(k)$ (resp. $X \in \mathrm{SmVar}(k)$), $p^* := H^j E_{et}(F)(p) : \mathbb{H}^j_{et}(X, F) \to \mathbb{H}^j_{et}(X \times \mathbb{A}^1, F)$ is an isomorphism. Note that (ii) is a particular case of (ii)'.

For $X \in \mathrm{Var}(k)$ and $Z \subset X$ a closed subset, denoting $j : X \backslash Z \hookrightarrow X$ the open complementary, we will consider

$$\Gamma^\vee_Z \mathbb{Z}_X := \mathrm{Cone}(\mathrm{ad}(j_\sharp, j^*)(\mathbb{Z}_X) : \mathbb{Z}_X \hookrightarrow \mathbb{Z}_X) \in C(\mathrm{Var}(k)^{sm}/X)$$

and denote for short $\gamma^\vee_Z := \gamma^\vee_Z(\mathbb{Z}_X) : \mathbb{Z}_X \to \Gamma^\vee_Z \mathbb{Z}_X$ the canonical map in $C(\mathrm{Var}(k)^{sm}/X)$. Denote $a_X : X \to \mathrm{Spec}\,k$ the structural map. For $X \in \mathrm{Var}(k)$ and $Z \subset X$ a closed subset, we have the motive of X with support in Z defined as

$$M_Z(X) := a_{X!} \Gamma^\vee_Z a^!_X \mathbb{Z} \in \mathrm{DA}(k).$$

If $X \in \mathrm{SmVar}(k)$, we will also consider

$$a_{X\sharp} \Gamma^\vee_Z \mathbb{Z}_X := \mathrm{Cone}(a_{X\sharp} \circ \mathrm{ad}(j_\sharp, j^*)(\mathbb{Z}_X) : \mathbb{Z}(U) \hookrightarrow \mathbb{Z}(X)) =: \mathbb{Z}(X, X \backslash Z) \in C(\mathrm{SmVar}(k)).$$

Then for $X \in \mathrm{SmVar}(k)$ and $Z \subset X$ a closed subset

$$M_Z(X) := a_{X!}\Gamma_Z^\vee a_X^! \mathbb{Z} = a_{X\sharp}\Gamma_Z^\vee \mathbb{Z}_X =: \mathbb{Z}(X, X\backslash Z) \in \mathrm{DA}(k).$$

- Let $(X, Z) \in \mathrm{Sch}^2$ with $X \in \mathrm{Sch}$ a noetherian scheme and $Z \subset X$ a closed subset. We have the deformation $(D_Z X, \mathbb{A}^1_Z) \to \mathbb{A}^1$, $(D_Z X, \mathbb{A}^1_Z) \in \mathrm{Sch}^2$ of (X, Z) by the normal cone $C_{Z/X} \to Z$, i.e. such that
$$(D_Z X, \mathbb{A}^1_Z)_s = (X, Z),\ s \in \mathbb{A}^1\backslash 0,\ (D_Z X, \mathbb{A}^1_Z)_0 = (C_{Z/X}, Z).$$
We denote by $i_1 : (X, Z) \hookrightarrow (D_Z X, \mathbb{A}^1_Z)$ and $i_0 : (C_{Z/X}, Z) \hookrightarrow (D_Z X, \mathbb{A}^1_Z)$ the closed embeddings in Sch^2.

- Let k be a field of characteristic zero. Let $X \in \mathrm{SmVar}(k)$. For $Z \subset X$ a closed subset of pure codimension c, consider a desingularisation $\epsilon : \tilde{Z} \to Z$ of Z and denote $n : \tilde{Z} \xrightarrow{\epsilon} Z \subset X$. We have then the morphism in $\mathrm{DA}(k)$
$$G_{Z,X} : M(X) \xrightarrow{D(\mathbb{Z}(n))} M(\tilde{Z})(c)[2c] \xrightarrow{\mathbb{Z}(\epsilon)} M(Z)(c)[2c]$$
where $D : \mathrm{Hom}_{\mathrm{DA}(k)}(M_c(\tilde{Z}), M_c(X)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{DA}(k)}(M(X), M(\tilde{Z})(c)[2c])$ is the duality isomorphism from the six functors formalism (moving lemma of Suzlin and Voevodsky) and $\mathbb{Z}(n) := \mathrm{ad}(n_!, n^!)(a_X^! \mathbb{Z})$, noting that $n_! = n_*$ since n is proper and that $a_X^! = a_X^*[d_X]$ and $a_{\tilde{Z}}^! = a_{\tilde{Z}}^*[d_Z]$ since X, resp. $\tilde{Z}$, are smooth (considering the connected components, we may assume X and $\tilde{Z}$ of pure dimension).

We recall the following facts (see [5] and [3]):

Proposition 1. *Let k be a field a characteristic zero. Let $X \in \mathrm{SmVar}(k)$ and $i : Z \subset X$ a smooth subvariety of pure codimension d. Then $C_{Z/X} = N_{Z/X} \to Z$ is a vector bundle of rank d. The closed embeddings $i_1 : (X, Z) \hookrightarrow (D_Z X, \mathbb{A}^1_Z)$ and $i_0 : (C_{Z/X}, Z) \hookrightarrow (D_Z X, \mathbb{A}^1_Z)$ in $\mathrm{SmVar}^2(k)$ induces isomorphisms of motives $\mathbb{Z}(i_1) : M_Z(X) \xrightarrow{\sim} M_{\mathbb{A}^1_Z}(D_Z X)$ and $\mathbb{Z}(i_0) : M_Z(N_{Z/X}) \xrightarrow{\sim} M_{\mathbb{A}^1_Z}(D_Z X)$ in $\mathrm{DA}(k)$. We get the excision isomorphism in $\mathrm{DA}(k)$*

$$P_{Z,X} := \mathbb{Z}(i_0)^{-1} \circ \mathbb{Z}(i_1) : M_Z(X) \xrightarrow{\sim} M_Z(N_{Z/X}).$$

We have

$$Th(N_{Z/X}) \circ P_{Z,X} \circ \gamma_Z^\vee(\mathbb{Z}_X) = G_{Z,X} := D(\mathbb{Z}(i)) : M(X) \to M(Z)(d)[2d].$$

Proof. See [5]. □

We will use the following theorem of Voevodsky :

Theorem 1. *(Voevodsky)Let k be a perfect field (e.g. k a field of characteristic zero). Let $F \in \mathrm{PSh}(\mathrm{SmVar}(k), \mathbb{Q})$. If F is $\mathbb{A}^1$ invariant and admits transfers, then for all $j \in \mathbb{Z}$, $H^j E_{et}(F) \in \mathrm{PSh}(\mathrm{SmVar}(k))$ are $\mathbb{A}^1$ invariant. That is, if F is $\mathbb{A}^1$ invariant and admits transfers then F is $\mathbb{A}^1$ local.*

Proof. By [9], $H^j E_{Nis}(F) \in \mathrm{PSh}(\mathrm{SmVar}(k))$ are $\mathbb{A}^1$ invariant. On the other hand since F takes values in $\mathbb{Q}$-vector spaces, $H^j E_{Nis}(F) = H^j E_{et}(F)$. □

2.7 The logaritmic De Rham complexes

We introduce the logarithmic De Rham complexes

Definition 2. *(i) Let $X = (X, O_X) \in \mathrm{RCat}$ a ringed topos, we have in $C(X)$ the subcomplex of presheaves of abelian groups*

$$OL_X : \Omega^\bullet_{X,\log} \hookrightarrow \Omega^\bullet_X,\ \textit{s.t. for } X^o \in X \textit{ and } p \in \mathbb{N},\ p \geq 1,$$
$$\Omega^p_{X,\log}(X^o) :=< df_{\alpha_1}/f_{\alpha_1} \wedge \cdots \wedge df_{\alpha_p}/f_{\alpha_p}, f_{\alpha_k} \in O_X^*(X^o) >\subset \Omega^p_X(X^o),$$

where $\Omega^\bullet_X := DR(X)(O_X) \in C(X)$ *is the De Rham complex and* $O^*_X(X^o) \subset O_X(X^o)$ *is the multiplicative group consisting of invertible elements for the multiplication, here* $<,>$ *stand for the sub-abelian group generated by. By definition, for* $w \in \Omega^p_X(X^o)$, $w \in \Omega^p_{X,\log}(X^o)$ *if and only if there exists* $(n_i)_{1\leq i\leq s} \in \mathbb{Z}$ *and* $(f_{i,\alpha_k})_{1\leq i\leq s, 1\leq k\leq p} \in O^*_X(X^o)$ *such that*

$$w = \sum_{1\leq i\leq s} n_i df_{i,\alpha_1}/f_{i,\alpha_1} \wedge \cdots \wedge df_{i,\alpha_p}/f_{i,\alpha_p} \in \Omega^p_X(X^o).$$

For $p = 0$, *we set* $\Omega^0_{X,\log} := \mathbb{Z}$ *if* $\mathbb{Z} \subset O_X$ *and* $\Omega^0_{X,\log} := \mathbb{Z}/n$ *if* O_X *is a ring of characteristic* n. *Let* $f : X' = (X', O_{X'}) \to X = (X, O_X)$ *a morphism with* $X, X' \in \mathrm{RCat}$. *Consider the morphism* $\Omega(f) : \Omega^\bullet_X \to f_*\Omega^\bullet_{X'}$ *in* $C(X)$. *Then,* $\Omega(f)(\Omega^\bullet_{X,\log}) \subset f_*\Omega^\bullet_{X',\log}$.

(ii) For k *a field, we get from (i), for* $X \in \mathrm{Var}(k)$, *the embedding in* $C(X)$

$$OL_X : \Omega^\bullet_{X,\log} \hookrightarrow \Omega^\bullet_X := \Omega^\bullet_{X/k},$$

such that, for $X^o \subset X$ *an open subset and* $w \in \Omega^p_X(X^o)$, $w \in \Omega^p_{X,\log}(X^o)$ *if and only if there exists* $(n_i)_{1\leq i\leq s} \in \mathbb{Z}$ *and* $(f_{i,\alpha_k})_{1\leq i\leq s, 1\leq k\leq p} \in O^*_X(X^o)$ *such that*

$$w = \sum_{1\leq i\leq s} n_i df_{i,\alpha_1}/f_{i,\alpha_1} \wedge \cdots \wedge df_{i,\alpha_p}/f_{i,\alpha_p} \in \Omega^p_X(X^o),$$

and for $p = 0$, $\Omega^0_{X,\log} := \mathbb{Z}$ *if* k *is of characteristic zero,* $\Omega^0_{X,\log} := \mathbb{Z}/p$ *if* k *is of characteristic* p. *We get an embedding in* $C(\mathrm{Var}(k))$

$$OL : \Omega^\bullet_{/k,\log} \hookrightarrow \Omega^\bullet_{/k}, \text{ given by, for } X \in \mathrm{Var}(k),$$
$$OL(X) := OL_X : \Omega^\bullet_{/k,\log}(X) := \Gamma(X, \Omega^\bullet_{X,\log}) \hookrightarrow \Gamma(X, \Omega^\bullet_X) =: \Omega^\bullet_{/k}(X)$$

and its restriction to $\mathrm{SmVar}(k) \subset \mathrm{Var}(k)$.

(iii) Let K *be a field of characteristic zero which is complete for a* p*-adic norm. Recall that* $O_K \subset K$ *denotes its ring of integers. Let* $X \in \mathrm{Var}(K)$. *Let* $X^{\mathcal{O}} \in \mathrm{Sch}/O_K$ *an integral model of* X, *in particular* $X^{\mathcal{O}} \otimes_{O_K} K = X$. *Consider the full subcategory* $X^{\mathcal{O},et} \subset (\mathrm{Sch}^{int}/O_K)/X^{\mathcal{O}}$. *We have then the morphisms of sites* $r : X^{et} \to X^{\mathcal{O},et}$ *and* $r : X^{pet} \to X^{\mathcal{O},pet}$ *such that* $\nu_{X^{\mathcal{O}}} \circ r = r \circ \nu_X$. *We then consider the embedding of* $C(X^{et})$

$$OL_X := OL_X \circ \iota : \Omega^\bullet_{X^{et},\log,\mathcal{O}} := r^*\Omega^\bullet_{X^{\mathcal{O},et},\log} \hookrightarrow \Omega^\bullet_{X^{et},\log} \hookrightarrow \Omega^\bullet_{X^{et}}$$

consisting of integral logarithimc De Rham forms, with $\iota : r^*\Omega^\bullet_{X^{\mathcal{O},et},\log} \hookrightarrow \Omega^\bullet_{X^{et},\log}$. *We will also consider the embedding of* $C(X^{\mathcal{O},pet})$

$$OL_{\hat{X}^{\mathcal{O},(p)}} := (OL_{X^{\mathcal{O}}/p^n})_{n\in\mathbb{N}} : \Omega^\bullet_{\hat{X}^{(p)},\log,\mathcal{O}} := \varprojlim_{n\in\mathbb{N}} \nu^*_{X^{\mathcal{O}}} a_{et} \Omega^\bullet_{X^{\mathcal{O},et}/p^n,\log}$$
$$\hookrightarrow \Omega^\bullet_{\hat{X}^{\mathcal{O},(p)}} := \varprojlim_{n\in\mathbb{N}} \nu^*_{X^{\mathcal{O}}} a_{et} \Omega^\bullet_{X^{\mathcal{O},et}/p^n/(O_K/p^n)}$$

where for $p = 0$, *we set* $\Omega^0_{X^{\mathcal{O},et}/p^n,\log} := \mathbb{Z}/p^n$, *and we recall* $c : \hat{X}^{\mathcal{O},(p)} \to X^{\mathcal{O}}$ *the morphism in* RTop *is given by the completion along the ideal generated by* p, *and* $a_{et} : \mathrm{PSh}(\hat{X}^{\mathcal{O},(p)}) \to \mathrm{Shv}_{et}(\hat{X}^{\mathcal{O},(p)})$ *is the sheaftification functor. We then get the embeddings of* $C(X^{pet})$

$$m \circ (OL_X \otimes I) : \Omega^\bullet_{X^{pet},\log,\mathcal{O}} \otimes \mathbb{Z}_p := \Omega^\bullet_{X^{pet},\log,\mathcal{O}} \otimes \underline{\mathbb{Z}_p}_X \hookrightarrow \Omega^\bullet_{X^{pet}},\ (w \otimes \lambda_n)_{n\in\mathbb{N}} \mapsto (\lambda_n)_{n\in\mathbb{N}} \cdot w$$

and

$$OL_{\hat{X}^{(p)}} := r^*OL_{\hat{X}^{\mathcal{O},(p)}} : r^*\Omega^\bullet_{\hat{X}^{(p)},\log,\mathcal{O}} \hookrightarrow r^*\Omega^\bullet_{\hat{X}^{\mathcal{O},(p)}} \hookrightarrow \Omega^\bullet_{\hat{X}^{(p)}} := \Omega^\bullet_{\hat{X}^{(p)}/K},$$

where we recall $\mathrm{an}_{X,p} : \hat{X}^{(p)} := \hat{X}^{\mathcal{O},(p)} \otimes_{O_K} K \to X$ *the morphism in* RTop *induced by the analytification functor and* $r \circ \mathrm{an}_{X,p} = c \circ \eta_{X,p}$. *Note that the inclusion* $\Omega^l_{X^{\mathcal{O},et},\log}/p^n \subset \Omega^{\bullet}_{X^{\mathcal{O},et}/p^n,\log}$ *is strict in general. Note that*

$$\Omega^{\bullet}_{X^{pet},\log,\mathcal{O}} := \nu_X^* \Omega^{\bullet}_{X^{et},\log,\mathcal{O}} \in C(X^{pet}),$$

but

$$\underline{\mathbb{Z}_p}_X := \varprojlim_{n\in\mathbb{N}} \nu_X^*(\mathbb{Z}/p^n\mathbb{Z})_{X^{et}} \in C(X^{pet}),\ \Omega^{\bullet}_{\hat{X}^{(p)}} \in C(X^{pet}),\ \textit{and}$$
$$r^*\Omega^{\bullet}_{\hat{X}^{(p)},\log,\mathcal{O}} := r^* \varprojlim_{n\in\mathbb{N}} \nu_{X^{\mathcal{O}}}^* a_{et} \Omega_{X^{\mathcal{O},et}/p^n,\log} \in C(X^{pet})$$

are NOT the pullback of etale sheaves by ν_X.

(iii)' We consider the morphism of sites

$$r : \mathrm{SmVar}(K) \to \mathrm{Sch}^{int,sm}/O_K,\ X^{\mathcal{O}} \in \mathrm{Sch}^{int,sm}/O_K \mapsto X := X^{\mathcal{O}} \otimes_{O_K} K,$$

and for $X^{\mathcal{O}} \in \mathrm{Sch}^{int,sm}/O_K$ *and* $X := X^{\mathcal{O}} \otimes_{O_K} K$ *the commutative diagram of sites*

$$\begin{array}{ccc} \mathrm{SmVar}(K) & \xrightarrow{r} & (\mathrm{Sch}^{int,sm}/O_K) \\ \downarrow o_X & & \downarrow o_X \\ X^{et} & \xrightarrow{r} & X^{\mathcal{O},et} \end{array}$$

with $o_X(U^{\mathcal{O}} \to X^{\mathcal{O}}) = U^{\mathcal{O}}$ *and* $o_X(U \to X) = U$. *We will consider the embedding of* $C(\mathrm{Sch}^{int}/O_K)$

$$OL_{/O_K,an} : \Omega^{\bullet,an}_{/K,\log,\mathcal{O}} \hookrightarrow \Omega^{\bullet,an}_{/O_K},\ \textit{for}\ X^{\mathcal{O}} \in (\mathrm{Sch}^{int,sm}/O_K),$$
$$OL_{/O_K,an}(X^{\mathcal{O}}) := OL_{\hat{X}^{\mathcal{O},(p)}}(X^{\mathcal{O}}) : \Omega^{\bullet}_{\hat{X}^{(p)},\log,\mathcal{O}}(X^{\mathcal{O}}) \hookrightarrow \Omega^{\bullet}_{\hat{X}^{\mathcal{O},(p)}}(X^{\mathcal{O}}) := \Omega^{\bullet}_{\hat{X}^{\mathcal{O},(p)}}(\hat{X}^{\mathcal{O},(p)}).$$

and its restriction to $\mathrm{Sch}^{int,sm}/O_K \subset \mathrm{Sch}^{int}/O_K$. *We get the embedding of* $C(\mathrm{SmVar}(K)^{et})$

$$OL_{/K,an} := r^* OL_{/O_K,an} : r^*\Omega^{\bullet,an}_{/K,\log,\mathcal{O}} \hookrightarrow r^*\Omega^{\bullet,an}_{/O_K} \hookrightarrow \Omega^{\bullet,an}_{/K}.$$

Lemma 1. *(Faltings Small open subsets) Let* K *be a field of characteristic zero which is complete for a* p*-adic norm. Recall that* $O_K \subset K$ *denotes its ring of integers. Let* $X \in \mathrm{SmVar}(K)$ *with good reduction. Let* $X^{\mathcal{O}} \in \mathrm{Sch}^{int,sm}/O_K$ *a smooth integral model of* X, *in particular* $X^{\mathcal{O}} \otimes_{O_K} K = X$ *and* $X^{\mathcal{O}}$ *is smooth over* O_K. *Let* $x \in X^{\mathcal{O}}$. *There exists a (non empty) open subset* $U^{\mathcal{O}} \subset X^{\mathcal{O}}$ *which is an integral model (i.e.* $U^{\mathcal{O}} \in \mathrm{Sch}^{int,sm}/O_K$*) such that* $x \in U^{\mathcal{O}}$ *and such that there exists an etale map* $e : U^{\mathcal{O}} \to \mathbb{G}_m^{d_U} \subset \mathbb{A}^{d_U}_{O_K}$ *with* $e : U^{\mathcal{O}} \to e(U^{\mathcal{O}})$ *finite etale.*

Proof. Let $V^{\mathcal{O}} \subset X^{\mathcal{O}}$ an affine open subset which is an integral model. Then by an integral version (over O_K) of Noether's normalization lemma or by taking a linear projection, there exists a quasi-finite surjective map $e : V^{\mathcal{O}} \to \mathbb{A}^{d_U}_{O_K}$ such that x does not belong to the branch locus of a R of a compactification $e : \bar{V^{\mathcal{O}}} \to \mathbb{A}^{d_U}_{O_K}$ of e. Then the complementary open subset $U^{\mathcal{O}} := \bar{V^{\mathcal{O}}} \backslash R$ is finite etale over $e(U^{\mathcal{O}})$. □

Let K be a field of characteristic zero which is complete for a p-adic norm. Recall that $O_K \subset K$ denotes its ring of integers. Let $U^{\mathcal{O}} \in \mathrm{Sch}^{int,sm}/O_K$ be an affine integral model of dimension d such that the (co)tangent bundle $\Omega^1_{U^{\mathcal{O}}} \in \mathrm{Shv}_O(U^{\mathcal{O}})$ is trivial. In particular

$$\Omega^1(U^{\mathcal{O}}) = \oplus_{i=1}^d O(U^{\mathcal{O}}) dx_i,\ \Omega^1(U^{\mathcal{O}}/p^n) = \oplus_{i=1}^d O(U^{\mathcal{O}}/p^n) dx_i,$$

(i) Consider a cartesian square in $\mathrm{Sch}^{int,sm}/O_K$

$$\begin{array}{ccc} V_1^{\mathcal{O}} \times_{U^{\mathcal{O}}} V_2^{\mathcal{O}} & \xrightarrow{p_2} & V_2^{\mathcal{O}} \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle r_2} \\ V_1^{\mathcal{O}} & \xrightarrow{r_1} & U^{\mathcal{O}} \end{array}$$

where $r_1 : V_1^{\mathcal{O}} \to U^{\mathcal{O}}$ and $r_2 : V_2^{\mathcal{O}} \to U^{\mathcal{O}}$ are etale. Then the cotangent bundles $\Omega^1_{V_1^{\mathcal{O}}} \in \mathrm{Shv}_O(V_1^{\mathcal{O}})$, $\Omega^1_{V_2^{\mathcal{O}}} \in \mathrm{Shv}_O(V_2^{\mathcal{O}})$, $\Omega^1_{V_1^{\mathcal{O}} \times_{U^{\mathcal{O}}} V_2^{\mathcal{O}}} \in \mathrm{Shv}_O(V_1^{\mathcal{O}} \times_{U^{\mathcal{O}}} V_2^{\mathcal{O}})$ are trivials and we have in particular the splittings

$$\Omega^1(V_1^{\mathcal{O}}/p^n) = \oplus_{i=1}^d O(V_1^{\mathcal{O}}/p^n)dx_i,\ \Omega^1(V_2^{\mathcal{O}}/p^n) = \oplus_{i=1}^d O(V_2^{\mathcal{O}}/p^n)dx_i,$$
$$\Omega^1(V_1^{\mathcal{O}} \times_{U^{\mathcal{O}}} V_2^{\mathcal{O}}/p^n) = \oplus_{i=1}^d O(V_1^{\mathcal{O}} \times_{U^{\mathcal{O}}} V_2^{\mathcal{O}}/p^n)dx_i.$$

Choose by induction on $n \in \mathbb{N}$, for each $1 \leq i \leq d$, compatible splittings of $\mathbb{Z}/p^n$ modules

$$O(V_1^{\mathcal{O}}/p^n) = (\sum_{f \in O(V_1^{\mathcal{O}}/p^n)^*} \partial_{x_i} f/f) \oplus A_{1i},\ O(V_2^{\mathcal{O}}/p^n) = (\sum_{f \in O(V_2^{\mathcal{O}}/p^n)^*} \partial_{x_i} f/f) \oplus A_{2i}.$$

Denote by

$$a(p_1) : O(V_1^{\mathcal{O}}/p^n) \hookrightarrow O(V_1^{\mathcal{O}} \times_{U^{\mathcal{O}}} V_2^{\mathcal{O}}/p^n),\ h \mapsto a(p_1)(h) := h \otimes 1,$$
$$a(p_2) : O(V_2^{\mathcal{O}}/p^n) \hookrightarrow O(V_1^{\mathcal{O}} \times_{U^{\mathcal{O}}} V_2^{\mathcal{O}}/p^n),\ h \mapsto a(p_2)(h) := h \otimes 1,$$

the canonical maps. We claim that for each $1 \leq i \leq d$

$$a(p_1)(A_{1i}) \cap (\sum_{f \in O(V_1^{\mathcal{O}} \times_{U^{\mathcal{O}}} V_2^{\mathcal{O}}/p^n)^*} \partial_{x_i} f/f) = 0,\ a(p_2)(A_{1i}) \cap (\sum_{f \in O(V_1^{\mathcal{O}} \times_{U^{\mathcal{O}}} V_2^{\mathcal{O}}/p^n)^*} \partial_{x_i} f/f) = 0.$$

Indeed, let $f \in O(V_1^{\mathcal{O}}/p^n)$. Let $\tilde{f} \in O_K/p^n[[x_1, \cdots, x_d]]$ such that $f = \partial_{x_i}\tilde{f}$. Then $f = (\partial_{x_i}(exp(\tilde{f})))/exp(\tilde{f})$ with

$$exp(\tilde{f}) := \sum_{n \in \mathbb{N}} (1/n!)\tilde{f}^n \in O_K/p^n[[x_1, \cdots, x_d]].$$

But then $exp(\tilde{f}) \in O_K/p^n[[x_1, \cdots, x_d]]$ does not belong to any algebraic extension $O(V_1^{\mathcal{O}}/p^n) \subset O(V) \subset O_K/p^n[[x_1, \cdots, x_d]]$, since if $exp(\tilde{f}) \in O_K/p^n[[x_1, \cdots, x_d]]$ is algebraic over $K(V_1^{\mathcal{O}}/p)$, that is $exp(\tilde{f})^{s+1} = \sum_{k=0}^s a_k exp(\tilde{f})^k$ with $a_k \in K(V_1^{\mathcal{O}}/p)$ and $s \in \mathbb{N}$ minimal, we get

$$f(s+1)exp(\tilde{f})^{s+1} = \sum_{k=0}^{s} (\partial_{x_i} a_k + k f a_k) exp(\tilde{f})^k$$

which implies $(1 - k/(s+1))a_k = (\partial_{x_i} a_k)/f(s+1)$ by the unicity of the minimal polynomial and thus $a_k = 0 = exp(\tilde{f})$ for $0 \leq k \leq s$ (there is no algebraic solution to this differential equation for each $0 \leq k \leq s$), in particular $exp(\tilde{f}) \in O_K/p^n[[x_1, \cdots, x_d]]$ does not belong to $O(V_1^{\mathcal{O}} \times_{U^{\mathcal{O}}} V_2^{\mathcal{O}}/p^n)$. Hence, we can choose by induction on $n \in \mathbb{N}$, for each $1 \leq i \leq d$, compatible splittings of $\mathbb{Z}/p^n$ modules

$$O(V_1^{\mathcal{O}} \times_{U^{\mathcal{O}}} V_2^{\mathcal{O}}/p^n) = (\sum_{f \in O(V_1^{\mathcal{O}} \times_{U^{\mathcal{O}}} V_2^{\mathcal{O}}/p^n)^*} \partial_{x_i} f/f) \oplus A_{12i}$$

such that $a(p_1)(A_{1i}) \subset A_{12i}$ and $a(p_2)(A_{2i}) \subset A_{12i}$. Now, set

$$NL_n(V_1^{\mathcal{O}}) := \oplus_{i=1}^d (A_{1i}dx_i \oplus (\oplus_{j \neq i} O(V_1^{\mathcal{O}}/p^n)dx_j)) \subset \Omega^1(V_1^{\mathcal{O}}/p^n),$$
$$NL_n(V_2^{\mathcal{O}}) := \oplus_{i=1}^d (A_{1i}dx_i \oplus (\oplus_{j \neq i} O(V_1^{\mathcal{O}}/p^n)dx_j)) \subset \Omega^1(V_2^{\mathcal{O}}/p^n),$$
$$NL_n(V_1^{\mathcal{O}} \times_{U^{\mathcal{O}}} V_2^{\mathcal{O}}) := \oplus_{i=1}^d (A_{12i}dx_i \oplus (\oplus_{j \neq i} O(V_1^{\mathcal{O}} \times_{U^{\mathcal{O}}} V_2^{\mathcal{O}}/p^n)dx_j)) \subset \Omega^1(V_1^{\mathcal{O}} \times_{U^{\mathcal{O}}} V_2^{\mathcal{O}}/p^n).$$

We then have

$$\Omega^1(V_1^{\mathcal{O}}/p^n) = \Omega^1_{V^{\mathcal{O}}/p^n,\log}(V_1^{\mathcal{O}}) \oplus NL_n(V_1^{\mathcal{O}}),\ \Omega^1(V_2^{\mathcal{O}}/p^n) = \Omega^1_{V^{\mathcal{O}}/p^n,\log}(V_2^{\mathcal{O}}) \oplus NL_n(V_2^{\mathcal{O}}),$$
$$\Omega^1(V_1^{\mathcal{O}} \times_{U^{\mathcal{O}}} V_2^{\mathcal{O}}/p^n) = \Omega^1_{V^{\mathcal{O}}/p^n,\log}(V_1^{\mathcal{O}} \times_{U^{\mathcal{O}}} V_2^{\mathcal{O}}) \oplus NL_n(V_1^{\mathcal{O}} \times_{U^{\mathcal{O}}} V_2^{\mathcal{O}}),$$
$$a(p_1)(NL_n(V_1^{\mathcal{O}})) \subset NL_n(V_1^{\mathcal{O}} \times_{U^{\mathcal{O}}} V_2^{\mathcal{O}}),\ a(p_2)(NL_n(V_2^{\mathcal{O}})) \subset NL_n(V_1^{\mathcal{O}} \times_{U^{\mathcal{O}}} V_2^{\mathcal{O}}).$$

(ii) Let $r = (r_i : V_i^{\mathcal{O}} \to U^{\mathcal{O}})_{i\in I}$ be an etale cover in $\mathrm{Sch}^{int,sm}/O_K$. Denote

$$V_\bullet^{\mathcal{O}} \in \mathrm{Fun}(\Delta(I), \mathrm{Sch}^{int,sm}/O_K),\ V^{\mathcal{O}}_{i_1,\cdots,i_l} := V^{\mathcal{O}}_{i_1} \times_{U^{\mathcal{O}}} \cdots \times_{U^{\mathcal{O}}} V^{\mathcal{O}}_{i_l}$$

the associated Cech simplicial scheme. We get by (i) a subcomplex of $\mathbb{Z}/p^n$ modules

$$NL_n(V_\bullet^{\mathcal{O}}) \subset \Omega^1_{V^{\mathcal{O}}/p^n}(V_\bullet^{\mathcal{O}})$$

satisfaying

$$\Omega^1_{V^{\mathcal{O}}/p^n}(V_\bullet^{\mathcal{O}}) = \Omega^1_{V^{\mathcal{O}}/p^n,\log}(V_\bullet^{\mathcal{O}}) \oplus NL_n(V_\bullet^{\mathcal{O}}).$$

such that for each $p, n \in \mathbb{N}$ we get a (non canonical) compatible splitting of complexes of $\mathbb{Z}/p^n$ modules

$$\Omega^p_{U^{\mathcal{O}}/p^n}(V_\bullet^{\mathcal{O}}) = \Omega^p_{U^{\mathcal{O}}/p^n,\log}(V_\bullet^{\mathcal{O}}) \oplus \Omega^p_{U^{\mathcal{O}}/p^n,nl}(V_\bullet^{\mathcal{O}}), \tag{1}$$
$$\Omega^1_{U^{\mathcal{O}}/p^n,nl}(V_\bullet^{\mathcal{O}}) := NL_n(V_\bullet^{\mathcal{O}}),\ \Omega^p_{U^{\mathcal{O}}/p^n,nl}(V_\bullet^{\mathcal{O}}) := NL_n(V_\bullet^{\mathcal{O}}) \wedge \Omega^{p-1}_{U^{\mathcal{O}}/p^n}(V_\bullet^{\mathcal{O}}). \tag{2}$$

For each $p \in \mathbb{N}$ we get a (non canonical) splitting of complexes of $\mathbb{Z}_p$ modules

$$\Omega^p_{\hat{U}^{\mathcal{O},(p)}}(V_\bullet^{\mathcal{O}}) = \Omega^p_{\hat{U}^{(p)},\log,\mathcal{O}}(V_\bullet^{\mathcal{O}}) \oplus \Omega^p_{\hat{U}^{\mathcal{O},(p)},nl}(V_\bullet^{\mathcal{O}}),\ \Omega^p_{\hat{U}^{\mathcal{O},(p)},nl}(V_\bullet^{\mathcal{O}}) := \varprojlim_{n\in\mathbb{N}} \Omega^p_{U^{\mathcal{O}}/p^n,nl}(V_\bullet^{\mathcal{O}}). \tag{3}$$

We have the following result :

Proposition 2. *Let K be a field of characteristic zero which is complete for a p-adic norm. Recall that $O_K \subset K$ denotes its ring of integers.*

(i0) Let $U \in \mathrm{SmVar}(K)$ with good reduction such that there exists an etale map $e : U^{\mathcal{O}} \to \mathbb{G}_m^{d_U} \subset \mathbb{A}^{d_U}_{O_K}$ with $e : U^{\mathcal{O}} \to e(U^{\mathcal{O}})$ finite etale, where $U^{\mathcal{O}} \in \mathrm{Sch}^{int,sm}/O_K$ is an integral model of U, in particular $U^{\mathcal{O}} \otimes_{O_K} K = U$. Then, for each $p, q \in \mathbb{Z}$, $q \neq 0$, $p \neq 0$, $H^q_{et}(U, \Omega^p_{\hat{U}^{(p)},\log,\mathcal{O}}) = 0$.

(i) Let $U \in \mathrm{SmVar}(K)$ with good reduction such that there exists an etale map $e : U^{\mathcal{O}} \to \mathbb{G}_m^{d_U} \subset \mathbb{A}^{d_U}_{O_K}$ with $e : U^{\mathcal{O}} \to e(U^{\mathcal{O}})$ finite etale, where $U^{\mathcal{O}} \in \mathrm{Sch}^{int,sm}/O_K$ is an integral model of U, in particular $U^{\mathcal{O}} \otimes_{O_K} K = U$. Then, for each $p, q \in \mathbb{Z}$, $q \neq 0$, $p \neq 0$, $H^q_{pet}(U, \Omega^p_{\hat{U}^{(p)},\log,\mathcal{O}}) = 0$.

(ii) Let $X \in \mathrm{SmVar}(K)$ with good reduction. Let $X^{\mathcal{O}} \in \mathrm{Sch}^{int,sm}/O_K$ a smooth integral model of X, in particular $X^{\mathcal{O}} \otimes_{O_K} K = X$ and $X^{\mathcal{O}}$ is smooth over O_K. Then, for each $p, q \in \mathbb{Z}$,

$$H^{p+q}T(E, \varprojlim)(X) : H^q_{et}(X, \Omega^p_{\hat{X}^{\mathcal{O},(p)},\log}) \to \varprojlim_{n\in\mathbb{N}} H^q_{et}(X, \Omega^p_{X^{\mathcal{O}}/p^n,\log}) = H^q_{pet}(X, \Omega^p_{\hat{X}^{\mathcal{O},(p)},\log})$$

is an isomorphism.

(ii)' Let $X \in \mathrm{SmVar}(K)$ with good reduction. Let Z a closed subset. Let $X^{\mathcal{O}} \in \mathrm{Sch}^{int,sm}/O_K$ a smooth integral model of X, in particular $X^{\mathcal{O}} \otimes_{O_K} K = X$ and $X^{\mathcal{O}}$ is smooth over O_K. Then, for each $p, q \in \mathbb{Z}$,

$$H^{p+q}T(E, \varprojlim)(X) : H^q_{et,Z}(X, \Omega^p_{\hat{X}^{\mathcal{O},(p)},\log}) \to \varprojlim_{n\in\mathbb{N}} H^q_{et,Z}(X, \Omega^p_{X^{\mathcal{O}}/p^n,\log}) = H^q_{pet,Z}(X, \Omega^p_{\hat{X}^{\mathcal{O},(p)},\log})$$

is an isomorphism.

Proof. (i0):Similar to the proof of (i).
(i):By a standard result of Bhatt and Scholze (proposition 5.6.2 of [2]), we have for $X \in \mathrm{Var}(K)$

$$\varprojlim_{n\in\mathbb{N}} H^q_{et}(X, \Omega^p_{X^{\mathcal{O}}/p^n,\log}) = H^q_{pet}(X, \Omega^p_{\hat{X}^{\mathcal{O},(p)},\log})$$

since for each $m > n$, $\Omega^p_{X^{\mathcal{O}}/p^m,\log} \to \Omega^p_{X^{\mathcal{O}}/p^n,\log}$ is surjective locally for the etale topology. Hence its suffice to show that $H^q_{et}(U, \Omega^p_{U^{\mathcal{O}}/p^n,\log}) = 0$ for $q \neq 0$ for each $n \in \mathbb{N}$. On the other hand, for each quasi-compact scheme X, and $F \in \mathrm{PSh}(X^{et})$, we have, for $q \in \mathbb{Z}$,

$$H^q(X^{et}, F) = \varinjlim_{(r_i: X_i \to X)_{i\in I}} H^q F(X_\bullet),$$
$$(r_i : X_i \to X)_{i\in I} \text{ etale cover},\ X_\bullet \in \mathrm{Fun}(\Delta(I), \mathrm{Sch}),\ X_{i_1,\cdots,i_l} := X_{i_1} \times_X \cdots \times_X X_{i_l}$$

that is Chech etale cohomology coincide with etale cohomology. Since $U^{\mathcal{O}}/p^n$ is an affine scheme, we have for $(r_i : V_i^{\mathcal{O}} \to U^{\mathcal{O}})_{i\in I}$ an etale cover $H^q\Omega^p_{U^{\mathcal{O}}/p^n}(V_\bullet^{\mathcal{O}}) = 0$ since $\Omega^p_{U^{\mathcal{O}}/p^n} \in \mathrm{PSh}_O(U^{\mathcal{O},et})$ a (quasi)coherent $O(U^{\mathcal{O}})/p^n$-module. Let $(r_i : V_i^{\mathcal{O}} \to U^{\mathcal{O}})_{i\in I}$ be an etale cover. Denote

$$V_\bullet^{\mathcal{O}} \in \mathrm{Fun}(\Delta(I), \mathrm{Sch}^{int,sm}/O_K),\ V^{\mathcal{O}}_{i_1,\cdots,i_l} := V^{\mathcal{O}}_{i_1} \times_{U^{\mathcal{O}}} \cdots \times_{U^{\mathcal{O}}} V^{\mathcal{O}}_{i_l}$$

the associated Chech simplicial scheme. Since the cotangent bundle $\Omega^1_{U^{\mathcal{O}}} \in \mathrm{Shv}_O(U^{\mathcal{O}})$ of $U^{\mathcal{O}}$ is trivial as $U^{\mathcal{O}}$ admits an etale map to $\mathbb{A}^{d_U}_{O_K}$, we have a (non canonical) splitting (1)

$$\Omega^p_{U^{\mathcal{O}}/p^n}(V_\bullet^{\mathcal{O}}) = \Omega^p_{U^{\mathcal{O}}/p^n,\log}(V_\bullet^{\mathcal{O}}) \oplus \Omega^p_{U^{\mathcal{O}}/p^n,nl}(V_\bullet^{\mathcal{O}}).$$

It induces for each $q \in \mathbb{Z}$ a splitting

$$H^q\Omega^p_{U^{\mathcal{O}}/p^n}(V_\bullet^{\mathcal{O}}) = H^q\Omega^p_{U^{\mathcal{O}}/p^n,\log}(V_\bullet^{\mathcal{O}}) \oplus H^q\Omega^p_{U^{\mathcal{O}}/p^n,nl}(V_\bullet^{\mathcal{O}}).$$

Hence

$$H^q_{et}(U, \Omega^p_{U^{\mathcal{O}}/p^n,\log}) = H^q\Omega^p_{U^{\mathcal{O}}/p^n,\log}(V_\bullet^{\mathcal{O}}) = 0.$$

This proves (i).
(ii):Take (see lemma 1) an open affine cover $X = \cup_{1\leq i\leq r} X_i$ of X such that for each $1 \leq i \leq r$ such that there exists an etale map $e_i : X_i^{\mathcal{O}} \to T^{d_{X_i}} \subset \mathbb{A}^{d_{X_i}}_{O_K}$ with $e_i : X_i^{\mathcal{O}} \to e_i(X_i^{\mathcal{O}})$ finite etale. Denote, for each $1 \leq i \leq r$, $j_i : X_i \hookrightarrow X$ the open embedding. Consider then the commutative diagram of abelian groups

$$\begin{array}{ccc}
\mathbb{H}^q_{et}(X_\bullet, \Omega^p_{\hat{X}^{(p)},\log,\mathcal{O}}) & \xrightarrow{H^{p+q}T(E,\varprojlim)(X_\bullet)} & \mathbb{H}^q_{pet}(X_\bullet, \Omega^p_{\hat{X}^{(p)},\log,\mathcal{O}}) \\
\uparrow{\scriptstyle \oplus_{1\leq i\leq r} E_{et}\Omega(j_i)} & & \uparrow{\scriptstyle \oplus_{1\leq i\leq r} E_{pet}\Omega(j_i)} \\
H^q_{et}(X, \Omega^p_{\hat{X}^{(p)},\log,\mathcal{O}}) & \xrightarrow{H^{p+q}T(E,\varprojlim)(X)} & H^q_{pet}(X, \Omega^p_{\hat{X}^{(p)},\log,\mathcal{O}})
\end{array},$$

where the vertical maps are isomorphisms since the Zariski topology is finer then the etale topology. By (i) the upper map is an isomorphism (using the spectral sequence associated to the trivial filtration of complexes). Hence the lower map is also an isomorphism. This proves (ii).
(ii)':Follows from (ii) applied to X and $X\backslash Z$ by the distinguish triangle in $Ho(C(\mathrm{SmVar}(K)))$

$$\mathbb{Z}(X\backslash Z) \xrightarrow{\mathbb{Z}(j)} \mathbb{Z}(X) \to \mathbb{Z}(X, X\backslash Z) \to \mathbb{Z}(X\backslash Z)[1],$$

where $j : X\backslash Z \hookrightarrow X$ is the open embedding. □

Remark 1. *Let k be a field of characteristic zero of transcendence degree lower or equal to the cardinal of $\mathbb{R}$. Let $V \subset \mathbb{A}^d_k$ be a Zariski open subset.*

(i) Let $e : U \to V$ be a finite etale map with $U \in \mathrm{Var}(k)$. Let $Z \in \mathcal{Z}^p(U)$. Then $[Z] = 0 \in H^{2p}_{DR}(U)$. Indeed, note that U is smooth. Considering an embedding $k \hookrightarrow \mathbb{C}$, we may assume $k = \mathbb{C}$. By the Leray spectral sequence for $\mathbb{C}_{U^{an}}$, we have the following commutative diagram, where l is the degree of e,

$$\begin{array}{ccccc} H^j_{DR,Z}(U) & \xrightarrow{\Omega(\gamma^\vee_{Z/e^{-1}(e(Z))})} & H^j_{DR,e^{-1}(e(Z))}(U) & \xrightarrow{\Omega(\gamma^\vee_{e^{-1}(e(Z))})} & H^j_{DR}(U) \\ & \searrow & \downarrow \sim & & \downarrow \sim \\ & & \oplus^l H^j_{DR,e(Z)}(V) & \xrightarrow{\Omega(\gamma^\vee_{e(Z)})^l} & \oplus^l H^j_{DR}(V) \end{array} .$$

But $[e(Z)] = 0 \in H^{2p}_{DR}(V)$.

(ii) More generally, let $e : U \to V$ be a faithfully flat etale map, $U \in \mathrm{Var}(k)$. Let $Z \in \mathcal{Z}^p(U)$. Then $[Z] = 0 \in H^{2p}_{DR}(U)$. Indeed, note that U is smooth. Let $e : U \xrightarrow{j} W \xrightarrow{\bar{e}} V$ be a compactitication of e, with $W \in \mathrm{Var}(k)$ normal, $\bar{e} : W \to V$ a finite morphism and j an open embedding. Let $R \subset W$ be the ramification locus of $\bar{e}$ and $F := W \backslash U$. We have $R \subset F$ since e is etale. Denote $B := \bar{e}(R) \subset V$ which is a divisor. Then $\bar{e}^{-1}(B) = R \cup R' \subset W$ is a divisor. Denote $l : V \backslash B \hookrightarrow V$ and $l' : U \backslash R' \hookrightarrow V$ the open embeddings. Consider the exact sequence

$$H^j_{DR,R'}(U) \xrightarrow{H^j\Omega(\gamma^\vee_{R'})} H^j_{DR}(U) \xrightarrow{l'^*} H^j_{DR}(U \backslash R').$$

Since $e : U \backslash R' \to V \backslash B$ is finite etale, $l'^*[Z] = 0$ by (i). That is

$$[Z] = H^{2p}\Omega(\gamma^\vee_{R'})([Z]),\ [Z] \in H^{2p}_{DR,R'}(U).$$

We have $Z = j^*\bar{Z}$, where $\bar{Z} \in \mathcal{Z}^p(W)$ is the closure, and then

$$[Z] = j^*[\bar{Z}], [\bar{Z}] \in H^{2p}\Omega(\gamma^\vee_{R'})([\bar{Z}]),\ [\bar{Z}] \in H^{2p}_{DR,R'}(W).$$

Consider then the following commutative diagram

$$\begin{array}{ccccc} H^{2p-1}_{DR}(U \backslash R') & \xrightarrow{\partial} & H^{2p}_{DR,F\cup R'}(W) & \xrightarrow{\Omega(\gamma^\vee_{F\cup R'})} & H^{2p}_{DR}(W) \\ \downarrow e_* & & \downarrow \bar{e}_* & & \downarrow \bar{e}_* \\ H^{2p-1}_{DR}(V \backslash B) & \xrightarrow{\partial} & H^{2p}_{DR,B}(V) & \xrightarrow{\Omega(\gamma^\vee_{R'})} & H^{2p}_{DR}(V) \end{array}$$

Note that $\bar{e}_*$ is well defined since V is smooth. Since $e : U \backslash R' \to V \backslash B$ is finite etale, e_* is surjective, hence $[\bar{Z}] = 0$, thus $[Z] = 0$.

We will use the following result from Illusie:

Proposition 3. *Let K be a p adic field. Let $X \in \mathrm{SmVar}(K)$ with good reduction. Consider $X^{\mathcal{O}} \in \mathrm{Sch}^{int,sm}/O_K$ a smooth integral model of X, in particular $X^{\mathcal{O}} \otimes_{O_K} K = X$ and $X^{\mathcal{O}}$ is smooth with smooth special fiber. Assume there exist lifts $\phi_n : X^{\mathcal{O}}/p^n \to X^{\mathcal{O}}/p^n$ of the Frobenius $\phi : X^{\mathcal{O}}/p \to X^{\mathcal{O}}/p$, such that for $n' > n$ the following diagram commutes*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & O_{X^{\mathcal{O}}/p^{n'-n}} & \xrightarrow{p^n\cdot} & O_{X^{\mathcal{O}}/p^{n'}} & \xrightarrow{/p^{n'-n}} & O_{X^{\mathcal{O}}/p^n} & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \phi_{n'-n}} & & \uparrow{\scriptstyle \phi_{n'}} & & \uparrow{\scriptstyle \phi_n} & & \\ 0 & \longrightarrow & O_{X^{\mathcal{O}}/p^{n'-n}} & \xrightarrow{p^n\cdot} & O_{X^{\mathcal{O}}/p^{n'}} & \xrightarrow{/p^{n'-n}} & O_{X^{\mathcal{O}}/p^n} & \longrightarrow & 0 \end{array}$$

For each $n \in \mathbb{N}$, the sequence in $C(X^{\mathcal{O},et})$, where we recall $X^{\mathcal{O},et} \subset (\mathrm{Sch}^{int,sm}/O_K)/X^{\mathcal{O}}$,

$$0 \to a_{et}\Omega^{\bullet}_{X^{\mathcal{O}}/p^n,\log} \xrightarrow{OL_{X^{\mathcal{O}}/p^n}} \Omega^{\bullet}_{X^{\mathcal{O}}/p^n} \xrightarrow{\phi_n - I} \Omega^{\bullet}_{X^{\mathcal{O}}/p^n} \to 0$$

is exact as a sequence of complexes of etale sheaves (i.e. we only have local surjectivity on the right), $a_{et} : \mathrm{PSh}(X^{\mathcal{O},et}) \to \mathrm{Shv}_{et}(X^{\mathcal{O},et})$ being the sheaftification functor.

Proof. It follows from [7] for $n = 1$ since $X^{\mathcal{O},et} \xrightarrow{\sim} (X^{\mathcal{O}}/p)^{et}$ by definition of integral models (factor an etale map of $U^{\mathcal{O}} \to X^{\mathcal{O}}$ as the composition of a finite etale map and open embeddings). It then follows for $n \geq 2$ by induction on n by a trivial devissage. □

3 De Rham logarithmic classes

3.1 The De Rham classes of algebraic cycles vs De Rham logarithmic classes

Let $X \in \mathrm{Sch}$. Recall we have the canonical sub-complex $OL_X : \Omega^{\bullet}_{X,\log} \hookrightarrow \Omega^{\bullet}_X$ in $C(X^{et})$ (c.f. definition 2). All the differential of $\Omega^{\bullet}_{X,\log}$ vanishes since by definition the logarithmic forms are closed. For $j \in \mathbb{Z}$, the De Rham logarithmic classes consist of the image

$$H^j OL_X(\mathbb{H}^j_{et}(X, \Omega^{\bullet}_{X,\log})) \subset \mathbb{H}^j_{et}(X, \Omega^{\bullet}_X) = H^j_{DR}(X).$$

The differentials of the filtered complex $\Gamma(X, E_{et}(\Omega^{\bullet}_{X^{et},\log}, F_b)) \in C_{fil}(\mathbb{Z})$ vanishes at the E_1 level since the logarithmic forms are closed, hence we have a canonical splitting

$$\mathbb{H}^j_{et}(X, \Omega^{\bullet}_{X,\log}) = \oplus_{0 \leq l \leq j} H^{j-l}_{et}(X, \Omega^l_{X,\log}).$$

Let $X \in \mathrm{Sch}$ a noetherian scheme. We have by definition the exact sequence in $C(X^{et})$

$$0 \to F^* \to O^*_X \xrightarrow{dlog} \Omega_{X,\log} \to 0$$

where F is a prime field (i.e. $F = \mathbb{Q}$ or $F = \mathbb{Z}/p\mathbb{Z}$ for p a prime number). Hence $H^1_{et}(X, \Omega_{X,\log}) = H^1_{et}(X, O^*_X)$ and $H^q_{et}(X, \Omega_{X,\log}) = H^q_{et}(X, O^*_X) = 0$ for $q \leq 2$. Let $X \in \mathrm{Sch}$ a noetherian proper scheme. We have $H^0(X, \Omega^l_{X,\log}) = 0$.

Let k be a field of characteristic zero. We get (see definition 2) the embedding in $C(\mathrm{SmVar}(k))$

$$OL : \Omega^{\bullet}_{/k,\log} \hookrightarrow \Omega^{\bullet}_{/k}, \text{ given by, for } X \in \mathrm{SmVar}(k),$$
$$OL(X) := OL_X : \Omega^{\bullet}_{/k,\log}(X) := \Gamma(X, \Omega^{\bullet}_{X,\log}) \hookrightarrow \Gamma(X, \Omega^{\bullet}_X) =: \Omega^{\bullet}_{/k}(X).$$

We have also the sheaves

$$O_k, O^*_k \in \mathrm{PSh}(\mathrm{SmVar}(k)),\ X \in \mathrm{SmVar}(k) \mapsto O_k(X) := O(X),\ O^*_k(X) := O(X)^*,$$
$$(g : Y \to X) \mapsto a_g(X) : O_X(X) \to O_Y(Y),\ O_X(X)^* \to O_Y(Y)^*$$

Lemma 2. *(i0) The sheaves $O^*_k \in \mathrm{PSh}(\mathrm{SmVar}(k))$ and $\Omega^1_{/k,\log} \in \mathrm{PSh}(\mathrm{SmVar}(k))$ admit transfers compatible with transfers on $\Omega^1_{/k} \in \mathrm{PSh}(\mathrm{SmVar}(k))$.*

(i) For each $l \in \mathbb{Z}$, the sheaf $\Omega^l_{/k,\log} \in \mathrm{PSh}(\mathrm{SmVar}(k))$ admits transfers compatible with transfers on $\Omega^l_{/k} \in \mathrm{PSh}(\mathrm{SmVar}(k))$, that is $\Omega^{\bullet}_{/k} \in C(\mathrm{Cor}\,\mathrm{SmVar}(k))$ (see [8]) and the inclusion $OL : \Omega^l_{/k,\log}[-l] \hookrightarrow \Omega^{\bullet}_{/k}$ in $C(\mathrm{SmVar}(k))$ is compatible with transfers.

(ii) For each $l \in \mathbb{Z}$, the sheaf $\Omega^l_{/k,\log} \in \mathrm{PSh}(\mathrm{SmVar}(k))$ is $\mathbb{A}^1$ invariant.

Proof. (i0):The sheaf $O_k^* \in \mathrm{PSh}(\mathrm{SmVar}(k))$ admits transfers : for $W \subset X' \times X$ with $X, X' \in \mathrm{SmVar}(k)$ and W finite over X' and $f \in O(X)^*$, $W^*f := N_{W/X'}(p_X^* f)$ where $p_X : W \hookrightarrow X' \times X \to X$ is the projection and $N_{W/X'} : k(W)^* \to k(X')^*$ is the norm map. This gives transfers on $\Omega^1_{/k,\log} \in \mathrm{PSh}(\mathrm{SmVar}(k))$ compatible with transfers on $\Omega^1_{/k} \in \mathrm{PSh}(\mathrm{SmVar}(k))$: for $W \subset X' \times X$ with $X, X' \in \mathrm{SmVar}(k)$ and W finite over X' and $f \in O(X)^*$,

$$W^*df/f := dW^*f/W^*f = Tr_{W/X'}(p_X^*(df/f)),$$

where where $p_X : W \hookrightarrow X' \times X \to X$ is the projection and $Tr_{W/X'} : O_W \to O_X$ is the trace map. Note that $d(fg)/fg = df/f + dg/g$.
(i): By (i0), we get transfers on

$$\otimes^l_{\mathbb{Q}} \Omega^1_{/k,\log},\ \otimes^l_{O_k} \Omega^1_{/k} \in \mathrm{PSh}(\mathrm{SmVar}(k))$$

since $\otimes^l_{\mathbb{Q}} \Omega^1_{/k,\log} = H^0(\otimes^{L,l}_{\mathbb{Q}} \Omega^1_{/k,\log})$ and $\otimes^l_{O_k} \Omega^1_{/k} = H^0(\otimes^{L,l}_{O_k} \Omega^1_{/k})$. This induces transfers on

$$\wedge^l_{\mathbb{Q}} \Omega^1_{/k,\log} := \mathrm{coker}(\oplus_{I_2 \subset [1,\dots,l]} \otimes^{l-1}_{\mathbb{Q}} \Omega^1_{/k,\log} \xrightarrow{\oplus_{I_2 \subset [1,\dots,l]} \Delta_{I_2} := (w \otimes w' \mapsto w \otimes w \otimes w')} \otimes^l_{\mathbb{Q}} \Omega^1_{/k,\log}) \in \mathrm{PSh}(\mathrm{SmVar}(k)).$$

and

$$\wedge^l_{O_k} \Omega^1_{/k} := \mathrm{coker}(\oplus_{I_2 \subset [1,\dots,l]} \otimes^{l-1}_{O_k} \Omega^1_{/k} \xrightarrow{\oplus_{I_2 \subset [1,\dots,l]} \Delta_{I_2} := (w \otimes w' \mapsto w \otimes w \otimes w')}) \otimes^l_{O_k} \Omega^1_{/k} \in \mathrm{PSh}(\mathrm{SmVar}(k)).$$

(ii): Follows from the fact that for $X \in \mathrm{Var}(k)$, $O^*(X \times \mathbb{A}^1) = O^*(X)$ since for a commutative ring A, $(A[X])^* = A^*$. □

Let $X \in \mathrm{Var}(k)$. We have by definition the exact sequence in $C(X^{et})$

$$0 \to k^* \to O_X^* \xrightarrow{dlog} \Omega_{X,\log} \to 0$$

Hence $H^1_{et}(X, \Omega_{X,\log}) = H^1_{et}(X, O_X^*)$ and $H^q_{et}(X, \Omega_{X,\log}) = H^q_{et}(X, O_X^*) = 0$ for $q \leq 2$. For $X \in \mathrm{PVar}(k)$, we have $H^0(X, \Omega^l_{X,\log}) = 0$.

Proposition 4. *For each $j \in \mathbb{Z}$, the excision isomorphism induced by (i)*

$$H^j\Omega(P_{Z,X}) := H^j E_{et}(\Omega^\bullet_{/k})(\mathbb{Z}(i_0)) \circ H^j E_{et}(\Omega^\bullet_{/k})(\mathbb{Z}(i_1))^{-1} :$$
$$H^j_{DR,Z}(X) := H^j\Gamma_Z(X, E_{et}(\Omega^\bullet_X)) \xrightarrow{\sim} H^j_{DR,Z}(N_{Z/X}) := H^j\Gamma_Z(N_{Z/X}, E_{et}(\Omega^\bullet_{N_{Z/X}}))$$

preserve logarithmic De Rham classes, that is for each $l \in \mathbb{Z}$,

$$H^j\Omega(P_{Z,X})(H^j OL_X(H^{j-l}_{et,Z}(X, \Omega^l_{X,\log}))) = H^j OL_{N_{Z/X}}(H^{j-l}_{et,Z}(N_{Z/X}, \Omega^l_{N_{Z/X},\log}))$$

Proof. Since $\Omega^l_{/k,\log} \in \mathrm{PSh}(\mathrm{SmVar}(k))$ is $\mathbb{A}^1_k$ invariant and admits transfers by lemma 2, for each $j \in \mathbb{Z}$, the presheaves $H^{j-l}E_{et}(\Omega^l_{/k,\log}) \in \mathrm{PSh}(\mathrm{SmVar}(k))$ are $\mathbb{A}^1_k$ invariant by a theorem 1. It then follows from proposition 1 that

$$H^j\Omega_{\log}(P_{Z,X}) := H^j E_{et}(\Omega^\bullet_{/k,\log})(\mathbb{Z}(i_0)) \circ H^j E_{et}(\Omega^\bullet_{/k,\log})(\mathbb{Z}(i_1))^{-1} :$$
$$H^{j-l}_{et,Z}(X, \Omega^l_{X,\log}) \xrightarrow{\sim} H^{j-l}_{et,Z}(N_{Z/X}, \Omega^l_{N_{Z/X},\log}).$$

Hence,

$$\begin{aligned} H^j\Omega(P_{Z,X})(H^j OL_X(H^{j-l}_{et,Z}(X, \Omega^l_{X,\log}))) &= H^j OL_{N_{Z/X}}(\Omega_{\log}(P_{Z,X})(H^{j-l}_{et,Z}(X, \Omega^l_{X,\log}))) \\ &= H^j OL_{N_{Z/X}}(H^{j-l}_{et,Z}(N_{Z,X}, \Omega^l_{N_{Z/X},\log})). \end{aligned}$$

□

Remark 2. *Let k be a field a characteristic zero.*

(i) The statement of proposition 4 also holds for the subcomplex $\Omega^{\bullet,\partial=0}_{/k} \hookrightarrow \Omega^{\bullet}_{/k}$ of closed forms by the same argument since the inclusion $\Omega^{l,\partial=0}_{/k}[-l] \hookrightarrow \Omega^{\bullet}_{/k}$ in $C(\mathrm{SmVar}(k))$ is compatible with transfers.

(ii) The statement of proposition 4 does NOT hold for presheaves which do not admits transfers (they are $\mathbb{A}^1$ invariant but not $\mathbb{A}^1$ local in general). Note that the result of proposition 4 does NOT hold for the embedding in $C(\mathrm{SmVar}(k))$, associated to an embedding $\sigma : k \hookrightarrow \mathbb{C}$,

$$\mathcal{H}^{l,j-l}[-l] \hookrightarrow \mathcal{A}^{\bullet},\ X \in \mathrm{SmVar}(k),\ \Gamma(X, \mathcal{H}^{l,j-l}_{X^{an}_{\mathbb{C}}})[-l] \hookrightarrow \Gamma(X, \mathcal{A}^{\bullet}_{X^{an}_{\mathbb{C}}})$$

of the subsheaf of harmonic differential forms, the sheaves of differential forms $\mathcal{A} \in C(\mathrm{SmVar}(k))$ does NOT admits transfers (finite algebraic correspondences are not smooth and for $f : X' \to X$ a morphism with $X', X \in \mathrm{Diff}(\mathbb{R})$ vector fields on X only lift if f is a smooth morphism i.e. if the differential of df is surjective), transfers maps are only defined on cohomology. Recall that we do NOT have the Hodge decomposition for open complex varieties.

Let k be a field of characteristic zero. Let $X \in \mathrm{SmVar}(k)$ and $Z \subset X$ a smooth subvariety of pure codimension d. We have for each $j \in \mathbb{Z}$, the purity isomorphism given by $H^j\Omega(P_{Z,X})$ (see proposition 1) and the cup product with the Euler class of of the normal tangent bundle $N_{Z/X} \to Z$:

$$H^j DR(P_{Z,X}) : H^j_{DR,Z}(X) \xrightarrow{H^j\Omega(P_{Z,X})} H^j_{DR,Z}(N_{Z/X}) \xrightarrow{((-)\cdot e(N_{Z/X}))^{-1}} H^{j-2d}_{DR}(Z).$$

Now, we have the following :

Proposition 5. *Let k be a field of characteristic zero. Let $p : E \to X$ a vector bundle of rank $d \in \mathbb{N}$ with $X, E \in \mathrm{SmVar}(k)$ connected. Then*

(i) the Euler class $e(E) \in H^{2d}_{DR,X}(E)$ is logarithmic of type (d,d), that is $e(E) = H^{2d}OL_E(e(E)) \in H^{2d}_{DR,X}(E)$ with $e(E) \in H^d_{X,et}(E, \Omega^d_{E,\log})$,

(ii) the Euler class $e(E) \in H^{2d}_{DR,X}(E)$ induces for each $i, j \in \mathbb{Z}$ an isomorphism

$$((-)\cdot e(E)) : H^{i+j}OL_X(H^j_{et}(X, \Omega^i_{X,\log})) \xrightarrow{\sim} H^{2d+i+j}OL_E(H^{j+d}_{et}(E, \Omega^{i+d}_{E,\log})).$$

Proof. (i):Let $X = \cup_i X_i$ an open affine cover such that $E_i := E_{|X_i}$ is trivial : $(s_1, \cdots, s_d) : E_{|X_i} \xrightarrow{\sim} X_i \times \mathbb{A}^d$ with $s_j \in \Gamma(X_i, E)$. Then

$$e(E)_{|X_i} = \partial(ds_1/s_1 \wedge \cdots \wedge ds_d/s_d) \in H^{2d}OL_{E_i}(H^d_{X_i}(E_i, \Omega^d_{E,\log}))$$

where $\partial : H^{d-1}(E_i \backslash X_i, \Omega^d_{E,\log}) \to H^d_{X_i}(E_i, \Omega^d_{E,\log})$ is the boundary map. Hence

$$e(E) \in H^{2d}OL_E(H^0(E, H^d_X\Omega^d_{E,\log})) = H^d_{et,X}(E, \Omega^d_{E,\log}) \subset H^0(E, H^{2d}_X\Omega^{\bullet \geq d}_E) \subset H^{2d}_{DR,X}(E).$$

(ii):Follows from (i) and Kunneth formula for De Rham cohomology : let $X = \cup_i X_i$ an open affine cover such that $E_i := E_{|X_i}$ is trivial and consider the morphism of bi-complexes of abelian groups

$$((-)\cdot e(E)) : OL_X(\Gamma(X_\bullet, E_{et}(\Omega^i_{X,\log}))) \to OL_E(\Gamma(E_\bullet, E_{et}(\Omega^{i+d}_{E,\log})))[2d].$$

By (i) an Kunneth formula for De Rham cohomology, $((-)\cdot e(E))$ induces an isomorphism on the E_1 level of the spectral sequences for the filtrations associated to bi-complex structures on the total complexes. Hence, $((-)\cdot e(E))$ is a quasi-isomorphism. This proves (ii). □

We deduce from proposition 4 and proposition 5, the following key proposition

Proposition 6. *Let k be a field of characteristic zero. Let $X \in \mathrm{SmVar}(k)$ and $Z \subset X$ a smooth subvariety of codimension d. For each $j \in \mathbb{Z}$, the purity isomorphism*

$$H^jDR(P_{Z,X}) : H^j_{DR,Z}(X) \xrightarrow{H^j\Omega(P_{Z,X})} H^j_{DR,Z}(N_{Z/X}) \xrightarrow{((-)\cdot e(N_{Z/X}))^{-1}} H^{j-2d}_{DR}(Z)$$

preserve logarithmic De Rham classes, that is for each $l \in \mathbb{Z}$,

$$H^jDR(P_{Z,X})(H^jOL_X(H^{j-l}_{et,Z}(X, \Omega^l_{X,\log}))) = H^jOL_Z(H^{j-l}_{et}(Z, \Omega^l_{Z,\log})).$$

Proof. Follows from proposition 4 and proposition 5(ii). □

Let k be a field of characteristic zero. For $X \in \mathrm{SmVar}(k)$, we have the cycle class map for De Rham cohomology

$$Z \in \mathcal{Z}^d(X) \mapsto [Z] := H^{2d}\Omega(\gamma_Z^\vee)([Z]) \subset H^{2d}_{DR}(X),\ [Z] \in H^{2d}_{DR,|Z|}(X),$$
$$H^{2d}\Omega(\gamma_Z^\vee) : H^{2d}_{DR,|Z|}(X) \to H^{2d}_{DR}(X)$$

where

$$\Omega(\gamma_Z^\vee) := \mathrm{Hom}(\gamma_Z^\vee(\mathbb{Z}_X), E_{et}(\Omega^\bullet_X)) : \Gamma_Z(X, E_{et}(\Omega^\bullet_X)) \hookrightarrow \Gamma(X, E_{et}(\Omega^\bullet_X))$$

and, as for any Weil cohomology theory, we have the isomorphism given by purity : if $X^o \subset X$ is an open subset such that $Z^o := X^o \cap Z$ has smooth components we have the isomorphism

$$H^{2d}DR(P_{Z^o,X^o}) : H^{2d}_{DR,|Z|}(X) = H^{2d}_{DR,|Z^o|}(X^o) = H^{2d}_{DR,|Z^o|}(N_{Z^o/X^o}) \xrightarrow{\sim} H^0_{DR}(|Z^o|).$$

where the first equality follows from dimension reason : for $X \in \mathrm{SmVar}(k)$ and $Z' \subset X$ smooth, we have $H^i_{DR,Z'}(X) = 0$ for $i < 2\,\mathrm{codim}(Z', X)$ by the purity isomorphism $H^iDR(P_{Z',X})$.

The main result of this section is the following :

Theorem 2. *Let k be a field of characteristic zero. Let $X \in \mathrm{SmVar}(k)$. Let $d \in \mathbb{N}$.*

(i) The De Rham cohomology class of an algebraic cycle is logarithmic and is of type (d,d), that is, for $Z \in \mathcal{Z}^d(X)$

$$[Z] := H^{2d}\Omega(\gamma_Z^\vee)([Z]) \subset H^{2d}OL_X(H^d_{et}(X, \Omega^d_{X,\log})) \subset H^{2d}_{DR}(X).$$

(ii) Conversely, any $w \in H^{2d}OL_X(H^d_{et}(X, \Omega^d_{X,\log}))$ is the class of an algebraic cycle $Z \in \mathcal{Z}^d(X)$, i.e. $w = [Z]$. Note that it implies $w \in W_0H^{2d}_{DR}(X)$ as it is easily seen for $d = 1$ by the Chow moving lemma since $j^* : \mathrm{Pic}(\bar{X}) \to \mathrm{Pic}(X)$ is surjective if $j : X \hookrightarrow \bar{X}$ is a smooth compactification of X, $\bar{X} \in \mathrm{PSmVar}(k)$ and since $H^1_{et}(Y, \Omega^1_{Y,\log}) = \mathrm{Pic}(Y)$ for $Y \in \mathrm{Var}(k)$.

(iii) We have $H^jOL_X(H^{j-l}_{et}(X, \Omega^l_{X,\log})) = 0$ for $j, l \in \mathbb{Z}$ such that $2l < j$.

(iii)' If $X \in \mathrm{PSmVar}(k)$, we also have $H^jOL_X(H^{j-l}_{et}(X, \Omega^l_{X,\log})) = 0$ for $j, l \in \mathbb{Z}$ such that $2l > j$. That is, if $X \in \mathrm{PSmVar}(k)$, we only have logarithmic classes in bidegree (d,d) for $d \in \mathbb{N}$, in particular there is no non trivial logarithmic classes for odd degree De Rham cohomology $H^{2d+1}_{DR}(X)$.

Proof. (i):We have

$$[Z] := H^{2d}\Omega(\gamma_Z^\vee)([Z]) = H^{2d}OL_X \circ H^d\Omega^d_{\log}(\gamma_Z^\vee)([Z]) \subset H^{2d}_{DR}(X),\ [Z] \in H^d_{et,|Z|}(X, \Omega^d_{X,\log})$$

where

$$\Omega^d_{\log}(\gamma_Z^\vee) := \mathrm{Hom}(\gamma_Z^\vee(\mathbb{Z}_X), E_{et}(\Omega^d_{X,\log})) : \Gamma_Z(X, E_{et}(\Omega^d_{X,\log})) \hookrightarrow \Gamma(X, E_{et}(\Omega^d_{X,\log})),$$

since if $X^o \subset X$ is a Zariski open subset such that $Z^o := X^o \cap Z$ has smooth components and $N_{Z^o/X^o} \to Z^o$ is the normal tangent bundle, we have by proposition 6 the isomorphism

$$\begin{aligned} H^{2d}DR(P_{Z^o,X^o}) : H^{2d}OL_X(H^d_{et,|Z|}(X, \Omega^d_{X,\log})) &= H^{2d}OL_X(H^d_{et,|Z^o|}(X^o, \Omega^d_{X^o,\log})) \\ &= H^{2d}OL_X(H^d_{et,|Z^o|}(N_{Z^o/X^o}, \Omega^d_{N_{Z^o/X^o},\log})). \end{aligned}$$

(ii):By assumption we have

$$w = H^{2d}OL_X(w) \in H^{2d}_{DR}(X),\ w \in H^d_{et}(X, \Omega^d_{X,\log})$$

As $\Omega^d_{X,\log} \in \mathrm{PSh}(X^{et})$ consist of a single presheaf, that is a complex of presheaves concentrated one degree, there exist an etale open cover $r = r(w) = (r_i : X_i \to X)_{1\leq i\leq s}$, (depending on w or take X_i such that there exists an etale map $X_i \to \mathbb{A}^{d_{X_i}}$ which works for all w), such that $r_i^* w = 0 \in H^d_{et}(X_i, \Omega^d_{X,\log})$ for each i. Choose $i = 1$ and denote $j : U := r_1(X_1) \hookrightarrow X$ the corresponding (Zariski) open subset. As $\Omega^d_{X,\log}$ has no torsion and admits transfers, we have $j^*w = 0 \in H^d_{et}(U, \Omega^d_{X,\log})$. Hence, denoting $D := X\backslash U$, we have

$$w = H^{2d}OL_X(w), w \in H^d\Omega^d_{\log}(\gamma_D^\vee)(H^d_{et,D}(X, \Omega^d_{X,\log})).$$

We may assume, up to shrinking U, that $D \subset X$ is a divisor. Denote $D^o \subset D$ its smooth locus and $l : X^o \hookrightarrow X$ a Zariski open subset such that $X^o \cap D = D^o$. We then have by proposition 6

$$l^*w \in H^{2d}DR(P_{D^o,X^o})(H^{2d-2}OL_{D^o}(H^{d-1}_{et}(D^o, \Omega^{d-1}_{D^o,\log}))).$$

We repeat this procedure with each connected components of D^o instead of X. By a finite induction of d steps, we get

$$w = H^{2d}OL_X(w),\ w \in H^d\Omega^d_{\log}(\gamma_Z^\vee)(H^d_{et,Z}(X, \Omega^d_{X,\log})),$$

with $Z := D_d \subset \cdots \subset D \subset X$ a pure codimension d (Zariski) closed subset and $w = \sum_i n_i[Z_i] \in H^{2d}_{DR}(X)$, where $n_i \in \mathbb{Z}$ and $(Z_i)_{1\leq i\leq t} \subset Z$ are the irreducible components of Z, using in the final step the isomorphism

$$H^1_{et}(D^o_{d-1}, \Omega_{D^o_{d-1},\log}) = H^1_{et}(D^o_{d-1}, O^*_{D_{d-1}}) = Pic(D^o_{d-1}).$$

(iii):Let $j > 2l$. Let $w \in H^jOL_X(H^{j-l}_{et}(X, \Omega^l_{X,\log}))$. By the proof of (ii) there exists $Z \subset X$ a closed subset of pure codimension l such that

$$w = H^jOL_X(w),\ w \in H^{j-l}\Omega^l_{\log}(\gamma_Z^\vee)(H^{j-l}_{et,Z}(X, \Omega^l_{X,\log})),$$

By a finite induction of $d_X - l$ steps, restricting to the smooth locus of closed subsets of Z, $w = 0$ since by proposition 1, $H^jOL_X(H^{j-l}_{et,Z'}(X, \Omega^l_{X,\log})) = 0$ for all $Z' \in \mathrm{SmVar}(k)$ such that $\mathrm{codim}(Z', X) \geq l$.
(iii)':Let $j < 2l$. Let $w \in H^jOL_X(H^{j-l}_{et}(X, \Omega^l_{X,\log}))$. By the proof of (ii) there exists $Z \subset X$ a closed subset of pure codimension $j - l$ such that

$$w = (H^jOL_X \circ H^{j-l}\Omega^l_{\log}(\gamma_Z^\vee))(w),\ w \in H^{j-l}_{et,Z}(X, \Omega^l_{X,\log}),$$

For $Z' \subset X$ a closed subset of pure codimension c, consider a desingularisation $\epsilon : \tilde{Z}' \to Z'$ of Z' and denote $n : \tilde{Z}' \xrightarrow{\epsilon} Z' \subset X$. The morphism in $\mathrm{DA}(k)$

$$G_{Z',X} : M(X) \xrightarrow{D(\mathbb{Z}(n))} M(\tilde{Z}')(c)[2c] \xrightarrow{\mathbb{Z}(\epsilon)} M(Z')(c)[2c]$$

where $D : \mathrm{Hom}_{\mathrm{DA}(k)}(M_c(\tilde{Z}'), M_c(X)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{DA}(k)}(M(X), M(\tilde{Z}')(c)[2c])$ is the duality isomorphism from the six functor formalism (moving lemma of Suzlin and Voevodsky) and $\mathbb{Z}(n) := \mathrm{ad}(n_!, n^!)(a^!_X\mathbb{Z})$, is given by a morphism in $C(\mathrm{SmVar}(k))$

$$\hat{G}_{Z',X} : \mathbb{Z}_{tr}(X) \to E_{et}(C_*\mathbb{Z}_{tr}(Z'))(c)[2c].$$

Let $l : X^o \hookrightarrow X$ be an open embedding such that $Z^o := Z \cap X^o$ is the smooth locus of Z. We then have the following commutative diagram of abelian groups

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{j-l}_{et,Z}(X, \Omega^l_{X,\log}) & \xrightarrow{l^*} & H^{j-l}_{et,Z^o}(X^o, \Omega^l_{X,\log}) & \xrightarrow{\partial} & H^{j-l+1}_{et,Z\backslash Z^o}(X, \Omega^l_{X,\log}) & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle \Omega(\hat{G}_{Z,X})} & & \uparrow{\scriptstyle DR(P_{Z^o,X^o})} & & \uparrow{\scriptstyle \Omega(\hat{G}_{Z,X})} & & \\
0 & \longrightarrow & H^0_{et}(Z, \Omega^{2l-j}_{Z,\log}) & \xrightarrow{l*} & H^0_{et}(Z^o, \Omega^{2l-j}_{Z^o,\log}) & \xrightarrow{\partial} & H^1_{et,Z\backslash Z^o}(Z, \Omega^{2l-j}_{Z,\log}) & \longrightarrow & \cdots
\end{array}$$

whose rows are exact sequences. Consider

$$l^*w = DR(P_{Z^o,X^o})(w^o) \in H^{j-l}_{et,Z^o}(X^o, \Omega^l_{X,\log}),\ w^o \in H^0_{et}(Z^o, \Omega^{2l-j}_{Z^o,\log}).$$

Since $\partial l^* w = 0 \in H^{j-l+1}_{et,Z\backslash Z^o}(X, \Omega^l_{X,\log})$, we get using proposition 6 applied to $((Z\backslash Z^o)^o, X^{oo})$, where $(Z\backslash Z^o)^o \subset Z\backslash Z^o$ is the smooth locus and $X^{oo} \subset X$ is an open subset such that $X \cap (Z\backslash Z^o)^o = X^{oo} \cap (Z\backslash Z^o)$,

$$\partial w^o = 0 \in H^1_{et,Z\backslash Z^o}(Z, \Omega^{2l-j}_{Z,\log}),$$

since $H^1_{et,(Z\backslash Z^o)\backslash(Z\backslash Z^o)^o}(Z, \Omega^{2l-j}_{Z,\log}) = 0$ for dimension reasons, that is

$$w = \Omega(\hat{G}_{Z,X})(w), \text{ with } w \in H^0_{et}(Z, \Omega^{2l-j}_{Z,\log}).$$

Hence $w = 0$ since $H^0(Z', \Omega^k_{Z',\log}) = 0$ for all $k > 0$ and all $Z' \in \mathrm{PVar}(k)$. □

3.2 Complex integral periods

Let k be a field of characteristic zero.

Let $X \in \mathrm{SmVar}(k)$ a smooth variety. Let $X = \cup_{i=1}^s X_i$ an open affine cover. We have for $\sigma : k \hookrightarrow \mathbb{C}$ an embedding, the evaluation period embedding map which is the morphism of bi-complexes

$$ev(X)^\bullet_\bullet : \Gamma(X_\bullet, \Omega^\bullet_{X_\bullet}) \to \mathbb{Z}\,\mathrm{Hom}_{\mathrm{Diff}}(\mathbb{I}^\bullet, X^{an}_{\mathbb{C},\bullet})^\vee \otimes \mathbb{C},$$

$$w^l_I \in \Gamma(X_I, \Omega^l_{X_I}) \mapsto (ev(X)^l_I(w^l_I) : \phi^l_I \in \mathbb{Z}\,\mathrm{Hom}_{\mathrm{Diff}}(\mathbb{I}^l, X^{an}_{\mathbb{C},I})^\vee \otimes \mathbb{C} \mapsto ev^l_I(w^l_I)(\phi^l_I) := \int_{\mathbb{I}^l} \phi^{l*}_I w^l_I)$$

given by integration. By taking all the affine open cover $(j_i : X_i \hookrightarrow X)$ of X, we get for $\sigma : k \hookrightarrow \mathbb{C}$, the evaluation period embedding map

$$ev(X) := \varinjlim_{(j_i : X_i \hookrightarrow X)} ev(X)^\bullet_\bullet : \varinjlim_{(j_i : X_i \hookrightarrow X)} \Gamma(X_\bullet, \Omega^\bullet_{X_\bullet}) \to \varinjlim_{(j_i : X_i \hookrightarrow X)} \mathbb{Z}\,\mathrm{Hom}_{\mathrm{Diff}(\mathbb{R})}(\mathbb{I}^\bullet, X^{an}_{\mathbb{C},\bullet})^\vee \otimes \mathbb{C}$$

It induces in cohomology, for $j \in \mathbb{Z}$, the evaluation period map

$$H^j ev(X) = H^j ev(X)^\bullet_\bullet : H^j_{DR}(X) = H^j\Gamma(X_\bullet, \Omega^\bullet_{X_\bullet}) \to H^j_{\mathrm{sing}}(X^{an}_{\mathbb{C}}, \mathbb{C}) = H^j(\mathrm{Hom}_{\mathrm{Diff}(\mathbb{R})}(\mathbb{I}^\bullet, X^{an}_{\mathbb{C},\bullet})^\vee \otimes \mathbb{C}).$$

which does NOT depend on the choice of the affine open cover by acyclicity of quasi-coherent sheaves on affine noetherian schemes for the left hand side and from Mayer-Vietoris quasi-isomorphism for singular cohomology of topological spaces and Whitney approximation theorem for differential manifolds for the right hand side.

Proposition 7. *Let k be a field of characteristic zero. Let $X \in \mathrm{SmVar}(k)$. Let $\sigma : k \hookrightarrow \mathbb{C}$ an embedding.*

(i) Let $w \in H^j_{DR}(X) := \mathbb{H}^j(X, \Omega^\bullet_X) = \mathbb{H}^j_{pet}(X, \Omega^\bullet_X)$. If

$$w \in H^j OL_X(\mathbb{H}^j_{pet}(X_{\mathbb{C}}, \Omega^\bullet_{X^{et}_{\mathbb{C}},\log}))$$

then $H^j ev(X)(w) \in H^j_{\mathrm{sing}}(X^{an}_{\mathbb{C}}, 2i\pi\mathbb{Q})$.

(ii) Let $p \in \mathbb{N}$ a prime number and $\sigma_p : k \hookrightarrow \mathbb{C}_p$ an embedding. Let $j \in \mathbb{Z}$. Let $w \in H^j_{DR}(X) := \mathbb{H}^j(X, \Omega^\bullet_X) = \mathbb{H}^j_{pet}(X, \Omega^\bullet_X)$. If

$$w := \pi_{k/\mathbb{C}_p}(X)^* w \in H^j OL_X(\mathbb{H}^j_{et}(X_{\mathbb{C}_p}, \Omega^\bullet_{X^{et}_{\mathbb{C}_p},\log,\mathcal{O}}))$$

then $H^j ev(X)(w) \in H^j_{\mathrm{sing}}(X^{an}_{\mathbb{C}}, 2i\pi\mathbb{Q})$. Recall that $\mathbb{H}^j_{pet}(X_{\mathbb{C}}, \Omega^\bullet_{X^{pet}_{\mathbb{C}_p},\log,\mathcal{O}}) = \mathbb{H}^j_{et}(X_{\mathbb{C}}, \Omega^\bullet_{X^{et}_{\mathbb{C}_p},\log,\mathcal{O}})$.

Proof. (i): Let

$$w \in H^j_{DR}(X) := \mathbb{H}^j(X, \Omega^\bullet_X) = H^j\Gamma(X_\bullet, \Omega^\bullet_X).$$

where $(r_i : X_i \to X)_{1\leq i\leq s}$ is an affine etale cover. Let $X^{an}_{\mathbb{C}} = \cup_{i=1}^r \mathbb{D}_i$ an open cover with $\mathbb{D}_i \simeq D(0,1)^d$. Denote $j_{IJ} : X_I \cap \mathbb{D}_J \hookrightarrow X_I$ the open embeddings. Then by definition $H^j ev(X)(w) = H^j ev(X^{an}_{\mathbb{C}})(j^*_\bullet \circ \mathrm{an}^*_{X_\bullet} w)$ with

$$j^*_\bullet \circ \mathrm{an}^*_{X_\bullet} w \in H^j\Gamma(X^{an}_{\bullet,\mathbb{C}} \cap \mathbb{D}_\bullet, \Omega^\bullet_{X^{an}_{\mathbb{C}}}).$$

Now, if $w = H^j OL_X(\mathbb{H}^j_{et}(X_{\mathbb{C}}, \Omega^\bullet_{X^{et}_{\mathbb{C}},\log}))$, we have a canonical splitting

$$w = \sum_{l=0}^{j} w^{l,j-l}_L = \sum_{l=d}^{j} w^{l,j-l} \in H^j_{DR}(X_{\mathbb{C}}),\ w^{l,j-l}_L \in H^{j-l}(X_{\mathbb{C}}, \Omega^l_{X^{et}_{\mathbb{C}},\log}),\ w^{l,j-l} := H^j OL_{X^{et}_{\mathbb{C}}}(w^{l,j-l}_L).$$

Let $0 \leq l \leq j$. Using an affine w-contractile pro-etale cover of X, we see that there exists an affine etale cover $r = r(w^{l,j-l}) = (r_i : X_i \to X)_{1\leq i\leq n}$ of X (depending on $w^{l,j-l}$) such that

$$w^{l,j-l} = [(w^{l,j-l}_{L,I})_I] \in H^j OL_{X^{et}_{\mathbb{C}}}(H^{j-l}\Gamma(X_{\mathbb{C},\bullet}, \Omega^l_{X_{\mathbb{C}},\log})) \subset \mathbb{H}^j\Gamma(X_{\mathbb{C},\bullet}, \Omega^\bullet_{X_{\mathbb{C}}}).$$

Note that since X is an algebraic variety, this also follows from a comparison theorem between Chech cohomology of etale covers and etale cohomology. By [4] lemma 2, we may assume, up to take a desingularization $\pi : X' \to X$ of $(X, \cup_i(r_i(X\backslash X_i)))$ and replace w with $\pi^* w$, that $r_i(X_i) = r_i(X_i(w)) = X\backslash D_i$ with $D_i \subset X$ smooth divisors with normal crossing For $1 \leq l \leq j$, we get

$$w^{l,j-l}_{L,I} = \sum_\nu df_{\nu_1}/f_{\nu_1} \wedge \cdots \wedge df_{\nu_l}/f_{\nu_l} \in \Gamma(X_{\mathbb{C},I}, \Omega^l_{X_{\mathbb{C}}}).$$

For $l = 0$, we get

$$w^{0,j} = [(\lambda_I)] \in H^j\Gamma(X_{\mathbb{C},I}, O_{X_{\mathbb{C}_p,I}}),\ \lambda_I \in \Gamma(X_{\mathbb{C},I}, \mathbb{Z}_{X^{et}_{\mathbb{C},I}})$$

There exists $k' \subset \mathbb{C}$ containing k such that $w^{l,j-l}_{L,I} \in \Gamma(X_{k',I}, \Omega^l_{X_{k'}})$ for all $0 \leq l \leq j$. Taking an embedding $\sigma' : k' \hookrightarrow \mathbb{C}$ such that $\sigma'_{|k} = \sigma$, we then have

$$j^*_\bullet \circ \mathrm{an}^*_{X_\bullet} w = j^*_\bullet((m_l \cdot w^{l,j-l}_L)_{0\leq l\leq j}) = (w^{l,j-l}_{L,I,J})_{l,I,J} \in H^j\Gamma(X^{an}_{\bullet,\mathbb{C}} \cap \mathbb{D}_\bullet, \Omega^\bullet_{X^{an}_{\mathbb{C}}}).$$

where for each (I, J, l) with $cardI + cardJ + l = j$,

$$w^{l,j-l}_{L,I,J} := j^*_{IJ} w^{l,j-l}_{L,I} \in \Gamma(X^{an}_{I,\mathbb{C}} \cap \mathbb{D}_J, \Omega^l_{X^{an}_{\mathbb{C}}}).$$

We have by a standard computation, for each (I, J, l) with $cardI + cardJ + l = j$,

$$H^*_{\mathrm{sing}}(X^{an}_{I,\mathbb{C}} \cap \mathbb{D}_J, \mathbb{Z}) =< \gamma_1, \cdots, \gamma_{cardI} >,$$

where for $1 \leq i \leq cardI$, $\gamma_i \in \mathrm{Hom}(\Delta^*, X^{an}_{I,\mathbb{C}} \cap \mathbb{D}_J)$ are products of loops around the origin inside the pointed disc $\mathbb{D}^1\backslash 0$. On the other hand,

- $w^{l,j-l}_{L,I,J} = j^*_J(\sum_\nu df_{\nu_1}/f_{\nu_1} \wedge \cdots \wedge df_{\nu_l}/f_{\nu_l}) \in \Gamma(X^{an}_{I,\mathbb{C}} \cap \mathbb{D}_J, \Omega^l_{X^{an}_{\mathbb{C}}})$ for $1 \leq l \leq j$,
- $w^{0,j}_{L,I,J} = \lambda_I$ is a constant.

Hence, for $\mu \in P([1, \cdots, s])$ with $card\mu = l$, we get, for $l = 0$ $H^l ev(X^{an}_{\mathbb{C}})_{I,J}(w^{0,j}_{L,I,J}) = 0$ and, for $1 \leq l \leq j$,

$$H^l ev(X^{an}_{\mathbb{C}})_{I,J}(w^{l,j-l}_{L,I,J})(\gamma_\mu) = \sum_k \delta_{\nu,\mu} 2i\pi \in 2i\pi\mathbb{Z}.$$

where $\gamma_\mu := \gamma_{\mu_1} \cdots \gamma_{\mu_l}$. We conclude by [4] lemma 1.
(ii):It is a particular case of (i). See [4] proposition 1. □

Let k be a field of characteristic zero. Let $X \in \mathrm{SmVar}(k)$. Let $X = \cup_{i=1}^{s} X_i$ an open affine cover with $X_i := X \backslash D_i$ with $D_i \subset X$ smooth divisors with normal crossing. Let $\sigma : k \hookrightarrow \mathbb{C}$ an embedding. By proposition 7, we have a commutative diagram of graded algebras

$$\begin{array}{ccc} H^*_{DR}(X) & \xrightarrow{H^*ev(X)} & H^*_{\mathrm{sing}}(X^{an}_{\mathbb{C}}, \mathbb{C}) \\ \subset \uparrow & & \uparrow H^*C^*\iota_{2i\pi\mathbb{Q}/\mathbb{C}}(X^{an}_{\mathbb{C}}) \\ H^*OL_X(\mathbb{H}^*_{et}(X, \Omega^{\bullet}_{X^{et},\log})) \cap H^*_{DR}(X) & \xrightarrow{H^*ev(X)} & H^*_{\mathrm{sing}}(X^{an}_{\mathbb{C}}, 2i\pi\mathbb{Q}) \end{array}$$

where

$$C^*\iota_{2i\pi\mathbb{Q}/\mathbb{C}}(X^{an}_{\mathbb{C}}) : C^{\bullet}_{\mathrm{sing}}(X^{an}_{\mathbb{C}}, 2i\pi\mathbb{Q}) \hookrightarrow C^{\bullet}_{\mathrm{sing}}(X^{an}_{\mathbb{C}}, \mathbb{C})$$

is the subcomplex consiting of $\alpha \in C^j_{\mathrm{sing}}(X^{an}_{\mathbb{C}}, \mathbb{C})$ such that $\alpha(\gamma) \in 2i\pi\mathbb{Q}$ for all $\gamma \in C^{\mathrm{sing}}_j(X^{an}_{\mathbb{C}}, \mathbb{Q})$. Recall that

$$H^*ev(X_{\mathbb{C}}) = H^*R\Gamma(X^{an}_{\mathbb{C}}, \alpha(X))^{-1} \circ \Gamma(X^{an}_{\mathbb{C}}, E_{zar}(\Omega(\mathrm{an}_X))) : H^*_{DR}(X_{\mathbb{C}}) \xrightarrow{\sim} H^*_{\mathrm{sing}}(X^{an}_{\mathbb{C}}, \mathbb{C})$$

is the canonical isomorphism induced by the analytical functor and $\alpha(X) : \mathbb{C}_{X^{an}_{\mathbb{C}}} \hookrightarrow \Omega^{\bullet}_{X^{an}_{\mathbb{C}}}$, which gives the periods elements $H^*ev(X)(H^*_{DR}(X)) \subset H^*_{\mathrm{sing}}(X^{an}_{\mathbb{C}}, \mathbb{C})$. On the other side the induced map

$$H^*ev(X_{\mathbb{C}}) : H^*OL_X(\mathbb{H}^*_{et}(X_{\mathbb{C}}, \Omega^{\bullet}_{X^{et}_{\mathbb{C}},\log})) \hookrightarrow H^*\iota_{2i\pi\mathbb{Q}/\mathbb{C}}H^*_{\mathrm{sing}}(X^{an}_{\mathbb{C}}, 2i\pi\mathbb{Q})$$

is NOT surjective in general since the left hand side is invariant by the action of the group $Aut(\mathbb{C})$ (the group of field automorphism of $\mathbb{C}$) whereas the right hand side is not. The fact for a de Rham cohomology class of being logarithmic is algebraic and invariant under isomorphism of (abstract) schemes.

Corollary 1. *Let $X \in \mathrm{PSmVar}(\mathbb{C})$. Then the Hodge conjecture holds for X if and only if the Hodge classes are given by logarithmic De Rham classes.*

Proof. Follows from theorem 2. □

3.3 Rigid GAGA for logarithmic de Rham classes

Let K be a field of characteristic zero which is complete for a p-adic norm. Denote by $O_K \subset K$ its ring of integer. We consider $\mathrm{Sch}^{int}/O_K := O(\mathrm{PSch}^2/O_K) \subset \mathrm{Sch}^{ft}/O_K$ the full subcategory consisting of integral models of algebraic varieties over K, where $O : \mathrm{PSch}^2/O_K \to \mathrm{Sch}^{ft}/O_K$, $O(X, Z) = X \backslash Z$ is the canonical functor, and

$$\mathrm{Sch}^{int,sm}/O_K := \mathrm{Sch}^{int}/O_K \cap \mathrm{Sch}^{sm}/O_K \subset \mathrm{Sch}/O_K$$

the full subcategory consisting of integral models of (smooth) algebraic varieties over K which are smooth over O_K. We then have the morphism of etale sites

$$r : \mathrm{SmVar}(K) \to \mathrm{Sch}^{int,sm}/O_K,\ X^{\mathcal{O}} \in \mathrm{Sch}^{int,sm}/O_K \mapsto X^{\mathcal{O}} \otimes_{O_K} K$$

We will consider for each $n \in \mathbb{N}$

$$\Omega^{\bullet}_{/(O_K/p^n),\log} = a_{et}\Omega^{\bullet}_{/(O_K/p^n),\log} \in C(\mathrm{Sch}^{int,sm}/O_K),$$
$$X^{\mathcal{O}} \mapsto \Omega^{\bullet}_{X^{\mathcal{O}}/p^n,\log}(X^{\mathcal{O}}),\ \ (f : X' \to X) \mapsto \Omega(f) := f^*$$

which gives

$$\Omega^{\bullet,an}_{/K,\log,\mathcal{O}} := \varprojlim_{n\in\mathbb{N}} a_{et}\Omega^{\bullet}_{/(O_K/p^n),\log} \in C(\mathrm{Sch}^{int,sm}/O_K),$$
$$X^{\mathcal{O}} \mapsto \Omega^{\bullet}_{\hat{X}^{(p)},\log,\mathcal{O}}(\hat{X}^{\mathcal{O},(p)}),\ \ (f : X' \to X) \mapsto \Omega(f) := f^*$$

together with the embedding in $C(\mathrm{Sch}^{int,sm}/O_K)$ (see definition 2(iii))

$$OL := OL_{/O_K,an} : \Omega^{\bullet,an}_{/K,\log,\mathcal{O}} \hookrightarrow \Omega^{\bullet,an}_{/O_K}, \text{ for } X^{\mathcal{O}} \in (\mathrm{Sch}^{int,sm}/O_K),$$
$$OL_{/O_K,an}(X^{\mathcal{O}}) := OL_{\hat{X}^{\mathcal{O},(p)}}(X^{\mathcal{O}}) : \Omega^{\bullet}_{\hat{X}^{\mathcal{O},(p)},\log,\mathcal{O}}(\hat{X}^{\mathcal{O},(p)}) \to \Omega^{\bullet}_{\hat{X}^{\mathcal{O},(p)}}(\hat{X}^{\mathcal{O},(p)}).$$

which induces the embedding of $C(\mathrm{SmVar}(K))$ (see definition 2(iii))

$$OL := OL_{/K,an} : r^*\Omega^{\bullet,an}_{/K,\log,\mathcal{O}} \xrightarrow{r^*OL_{/O_K,an}} r^*\Omega^{\bullet,an}_{/O_K} \hookrightarrow \Omega^{\bullet,an}_{/K}, \text{ for } X \in \mathrm{SmVar}(K),$$
$$OL_{/K,an}(X) := (\Omega(e) \circ OL_{\hat{Y}^{(p)}}(Y)) :$$
$$\varinjlim_{Y^{\mathcal{O}} \in (\mathrm{Sch}^{int,sm}/O_K), e: X \to Y^{\mathcal{O}} \times_{O_K} K = Y} \Omega^{\bullet}_{\hat{Y}^{(p)},\log,\mathcal{O}}(\hat{Y}^{(p)}) \to \Omega^{\bullet}_{\hat{X}^{(p)}}(\hat{X}^{(p)}).$$

We have also, for each $n \in \mathbb{N}$, the sheaves

$$O_n, O_n^* \in \mathrm{PSh}((\mathrm{Sch}^{int,sm}/O_K)),\ X^{\mathcal{O}} \mapsto O_n(X^{\mathcal{O}}) := O(X^{\mathcal{O}}_{/p^n}),\ O_n^*(X^{\mathcal{O}}) := O(X^{\mathcal{O}}_{/p^n})^*,$$
$$(g : Y \to X) \mapsto a_g(X^{\mathcal{O}}_{/p^n}) : O_n(X^{\mathcal{O}}) \to O_n(Y^{\mathcal{O}}),\ O_n(X^{\mathcal{O}})^* \to O_n(Y^{\mathcal{O}})^*,$$

and $O_n := r^*O_n, O_n^* := r^*O_n^* \in \mathrm{PSh}(\mathrm{SmVar}(K))$. We will also consider the De Rham-Witt complex of Illusie ([7])

$$W\Omega^{\bullet}_{/O_K} \in C(\mathrm{Sch}^{int,sm}/O_K),$$
$$X^{\mathcal{O}} \in \mathrm{Sch}^{int,sm}/O_K \mapsto W\Omega^{\bullet}_{X^{\mathcal{O}}/p}(X^{\mathcal{O}}),\ (f : Y^{\mathcal{O}} \to X^{\mathcal{O}}) \mapsto W(\Omega)(f)$$

and the map in $C(\mathrm{Sch}^{int,sm}/O_K)$

$$OL^l_W := OL^l : \Omega^{l,an}_{/K,\log,\mathcal{O}}[-l] \xrightarrow{OL^l} \Omega^{\bullet,an}_{/O_K} \xrightarrow{IW} W\Omega^{\bullet}_{/O_K},$$

which induces in particular the map in $\mathrm{PSh}(\mathrm{SmVar}(K))$

$$H^lOL^l_W := H^l(k \circ r^*OL^l_W) : r^*\Omega^{l,an}_{/K,\log,\mathcal{O}}[-l] \to H^lE_{et}(r^*W\Omega^{\bullet,an}_{/O_K}).$$

We will use the following :

Lemma 3. *(i0) For each $l, n \in \mathbb{N}$, the sheaf $\Omega^l_{/(O_K/p^n)} \in \mathrm{PSh}(\mathrm{Sch}^{sm}/O_K)$ admits transfers*

(i1) For each $n \in \mathbb{N}$, the presheaves $O_n^* \in \mathrm{PSh}(\mathrm{Sch}^{sm}/O_K)$ and $\Omega^1_{/(O_K/p^n),\log} \in \mathrm{PSh}(\mathrm{Sch}^{sm}/O_K)$ admit transfers compatible with transfers on $\Omega^1_{/(O_K/p^n)} \in \mathrm{PSh}(\mathrm{Sch}^{sm}/O_K)$.

(i2) For each $l, n \in \mathbb{N}$, the sheaf $a_{et}\Omega^l_{/(O_K/p^n),\log} \in \mathrm{PSh}(\mathrm{Sch}^{sm}/O_K)$ admits transfers compatible with transfers on $\Omega^l_{/(O_K/p^n)} \in \mathrm{PSh}(\mathrm{Sch}^{sm}/O_K)$, that is $\Omega^{\bullet}_{/(O_K/p^n)} \in C(\mathrm{Cor}\,\mathrm{Sch}^{sm}/O_K)$ and the inclusion $OL : \Omega^l_{/(O_K/p^n),\log}[-l] \hookrightarrow \Omega^{\bullet}_{/(O_K/p^n)}$ in $C(\mathrm{Sch}^{sm}/O_K)$ is compatible with transfers.

(i3) The sheaf $r^*\Omega^{l,an}_{/K,\log,\mathcal{O}} \in \mathrm{PSh}(\mathrm{SmVar}(K))$ admits transfers.

(ii0) Let $o : \mathrm{Sch}^{int,ea}/O_K \hookrightarrow \mathrm{Sch}^{int,sm}/O_K$ be the full subcategory consisting of $X^{\mathcal{O}} \in \mathrm{Sch}^{int,sm}/O_K$ such that there exists a finite etale map $e : X^{\mathcal{O}} \to e(X^{\mathcal{O}}) \subset \mathbb{A}^{d_X}_{O_K}$, where $e(X^{\mathcal{O}}) \subset \mathbb{A}^{d_X}_{O_K}$ is an open subset. Denote $o : \mathrm{Sch}^{int,sm}/O_K \to \mathrm{Sch}^{int,ea}/O_K$, $o : \mathrm{SmVar}(K) \to \mathrm{SmVar}(K)^{ea}$, $\mathrm{SmVar}(K)^{ea} := r^{-1}(\mathrm{Sch}^{int,ea}/O_K)$, the corresponding morphisms of sites, with $o(U^{\mathcal{O}}) = U^{\mathcal{O}}$, $o(U) = U$, so that $r \circ o = o \circ r$. Then the map in $\mathrm{PSh}(\mathrm{SmVar}(K)^{ea})$

$$o_*H^lOL^l_W : o_*r^*\Omega^{l,an}_{/K,\log,\mathcal{O}} \to o_* \ker((\phi - I) : H^lE_{et}(r^*W\Omega^{\bullet}_{/O_K}) \to H^lE_{et}(r^*W\Omega^{\bullet}_{/O_K}))$$
$$\xrightarrow{=:} o_*r^*(X^{\mathcal{O}} \in \mathrm{Sch}^{int,sm}/O_K \mapsto \ker((\phi - I) : H^l_{DR}(X) \to H^l_{DR}(X))),$$

is an isomorphism.

(ii) For each $l \in \mathbb{N}$,

$$a_{et}r^*\Omega^{l,an}_{/K,\log,\mathcal{O}} := a_{et}r^* \varprojlim_{n\in\mathbb{N}} a_{et}\Omega^l_{/(O_K/p^n),\log} = r^* a_{et} \varprojlim_{n\in\mathbb{N}} a_{et}\Omega^l_{/(O_K/p^n),\log} \in \mathrm{PSh}(\mathrm{SmVar}(K))$$

is $\mathbb{A}^1$ invariant, where $a_{et} : \mathrm{PSh}(\mathrm{SmVar}(K)) \to \mathrm{Shv}_{et}(\mathrm{SmVar}(K))$ is the sheaftification functor.

Proof. (i0):The sheaves $\Omega^1_{/(O_K/p^n)} \in \mathrm{PSh}(\mathrm{Sch}^{sm}/O_K)$ admit transfers. Indeed, let $W \subset X' \times X$ finite over X', with $X, X' \in \mathrm{Sch}^{sm}/O_K$. Take $n : \tilde{W} \to W$ the normalisation of W. Since X' is of finite type over O_K it is an excellent scheme, thus n is a finite surjective morphism. Hence $m := p_{X'} \circ n : \tilde{W} \to X'$ is a finite surjective morphism. Since X' is smooth and $\tilde{W}$ is normal, m is a finite flat morphism. By base change $m_{/p^n} : \tilde{W}_{/p^n} \to X'_{/p^n}$ are finite flat, since m is finite flat. We thus have a canonical trace map $Tr(m_{/p^n}) : O_{\tilde{W}_{/p^n}} \to O_{X'_{/p^n}}$. We get transfers on $\Omega^l_{/(O_K/p^n)} \in \mathrm{PSh}(\mathrm{Sch}^{sm}/O_K)$ by induction on l (see the proof of (i2)).
(i1):By (i0), the sheaves $\Omega^1_{/(O_K/p^n)} \in \mathrm{PSh}(\mathrm{Sch}^{sm}/O_K)$ admit transfers. On the other hand the sheaf $O^*_1 \in \mathrm{PSh}(\mathrm{Sch}^{sm}/O_K)$ admits transfers : for $W \subset X' \times X$ with $X, X' \in \mathrm{Sch}^{sm}/O_K$ and W finite over X' and $f \in O(X^{\mathcal{O}}_{/p})^*$, $W^*f := N_{W/X'}(p^*_X f)$ where $p_X : W \hookrightarrow X' \times X \to X$ is the projection and $N_{W/X'} : O_K/p(W)^* \to O_K/p(X')^*$ is the norm map. This gives transfers on $\Omega^1_{/(O_K/p),\log} \in \mathrm{PSh}(\mathrm{Sch}^{sm}/O_K)$ compatible with transfers on $\Omega^1_{/(O_K/p)} \in \mathrm{PSh}(\mathrm{Sch}^{sm}/O_K)$: for $W \subset X' \times X$ with $X, X' \in \mathrm{Sch}^{sm}/O_K$ and W finite over X' and $f \in O(X^{\mathcal{O}}_{/p^n})^*$,

$$W^*df/f := dW^*f/W^*f = Tr_{W/X'}(p^*_X(df/f)),$$

where where $p_X : W \hookrightarrow X' \times X \to X$ is the projection and $Tr_{W/X'} : O_W \to O_X$ is the trace map. Note that $d(fg)/fg = df/f + dg/g$. Considering the commutative diagram in $\mathrm{PSh}(\mathrm{Sch}^{sm}/O_K)$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^1_{/(O_K/p^{n-1})} & \longrightarrow & \Omega^1_{/(O_K/p^n)} & \xrightarrow{\Omega(/p)} & \Omega^1_{/(O_K/p)} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle OL} & & \uparrow{\scriptstyle OL} & & \uparrow{\scriptstyle OL} & & \\
0 & \longrightarrow & a_{et}\Omega^1_{/(O_K/p^{n-1}),\log} & \longrightarrow & a_{et}\Omega^1_{/(O_K/p^n),\log} & \xrightarrow{\Omega(/p)} & a_{et}\Omega^1_{/(O_K/p),\log} & \longrightarrow & 0
\end{array}$$

we see by induction on $n \in \mathbb{N}$ that the transfers on $\Omega^1_{/(O_K/p^n)} \in \mathrm{PSh}(\mathrm{Sch}^{sm}/O_K)$ send logarithmic forms to logarithimc forms. Hence the transfers on $\Omega^1_{/(O_K/p^n)} \in \mathrm{PSh}(\mathrm{Sch}^{sm}/O_K)$ send logarithmic forms to logarithimc forms. This proves (i0).
(i2): By (i1), we get transfers on

$$\otimes^l_{\mathbb{Z}/p^n}\Omega^1_{/(O_K/p^n),\log}, \otimes^l_{O_n}\Omega^1_{/(O_K/p^n)} \in \mathrm{PSh}(\mathrm{Sch}^{sm}/O_K)$$

since $\otimes^l_{\mathbb{Z}/p^n}\Omega^1_{/K,\log} = H^0(\otimes^{L,l}_{\mathbb{Z}/p^n}\Omega^1_{/(O_K/p^n),\log})$ and $\otimes^l_{O_n}\Omega^{1,an}_{/(O_K/p^n)} = H^0(\otimes^{L,l}_{\hat{O}}\Omega^{1,an}_{/(O_K/p^n)})$. This induces transfers on

$$\wedge^l_{\mathbb{Z}/p^n}\Omega^1_{/(O_K/p^n),\log} := \mathrm{coker}(\oplus_{I_2\subset[1,\dots,l]} \otimes^{l-1}_{\mathbb{Z}/p^n} \Omega^1_{/(O_K/p^n),\log} \xrightarrow{\oplus_{I_2\subset[1,\dots,l]}\Delta_{I_2}:=(w\otimes w'\mapsto w\otimes w\otimes w')} \otimes^l_{\mathbb{Z}/p^n}\Omega^1_{/(O_K/p^n),\log}) \in \mathrm{PSh}(\mathrm{Sch}^{sm}/O_K).$$

and

$$\wedge^l_{O_n}\Omega^1_{/(O_K/p^n)} := \mathrm{coker}(\oplus_{I_2\subset[1,\dots,l]} \otimes^{l-1}_{O_n} \Omega^1_{/(O_K/p^n)} \xrightarrow{\oplus_{I_2\subset[1,\dots,l]}\Delta_{I_2}:=(w\otimes w'\mapsto w\otimes w\otimes w')}) \otimes^l_{O_n} \Omega^1_{/(O_K/p^n)} \in \mathrm{PSh}(\mathrm{Sch}^{sm}/O_K).$$

(i3):The presheaves for each $l \in \mathbb{Z}$

$$\Omega^{l,an}_{/K,\log,\mathcal{O}} := \varprojlim_{n\in\mathbb{N}} a_{et}\Omega^{l}_{/(O_K/p^n),\log} \in \mathrm{PSh}(\mathrm{Sch}^{int,sm}/O_K)$$

admits transfers by (i2). Hence, the presheaves for each $l \in \mathbb{Z}$

$$a_{et}r^*\Omega^{l,an}_{/K,\log,\mathcal{O}} := a_{et}r^* \varprojlim_{n\in\mathbb{N}} a_{et}\Omega^{l}_{/(O_K/p^n),\log} \in \mathrm{PSh}(\mathrm{SmVar}(K))$$

admit transfers since if $\Gamma \subset X' \times X''$ with $X', X'' \in \mathrm{SmVar}(K)$ is finite surjective over X', then, $\bar{\Gamma} \subset X'^{\mathcal{O}} \times X''^{\mathcal{O}}$ with $X'^{\mathcal{O}}, X''^{\mathcal{O}} \in \mathrm{Sch}^{int,sm}/O_K$ such that $X' = X'^{\mathcal{O}} \times_{O_K} K, X'' = X''^{\mathcal{O}} \times_{O_K} K$ is finite surjective over $X'^{\mathcal{O}}$ ($\bar{\Gamma}$ is proper and affine over $X'^{\mathcal{O}}$ and recall that a proper affine morphism is finite).
(ii0):Let $X^{\mathcal{O}} \in \mathrm{Sch}^{int,sm}/O_K$ such that there exists a finite etale map $e : X^{\mathcal{O}} \to e(X^{\mathcal{O}}) \subset \mathbb{A}^{d_X}_{O_K}$ to an open subset of the affine scheme. By proposition 3 and since the pro-etale site is a replete topos by [2], for each $l \in \mathbb{Z}$, the sequence of abelian groups

$$\mathbb{H}^l_{pet}(X, \Omega^{\bullet}_{\hat{X}^{(p)},\log,\mathcal{O}}) \xrightarrow{r^*OL_{\hat{X}^{\mathcal{O}}}} H^l_{DR}(\hat{X}^{(p)}) \xrightarrow{\phi - I} H^l_{DR}(\hat{X}^{(p)})$$

is exact. Hence, by proposition 2(i) and since a logarithmic form is exact if and only if it vanishes, we get an isomorphism

$$r^*OL^l_{X^{\mathcal{O}}} : \Gamma(X^{\mathcal{O}}, a_{et}\Omega^l_{\hat{X}^{(p)},\log,\mathcal{O}}) \xrightarrow{\sim} \ker((\phi - I) : H^l_{DR}(\hat{X}^{(p)}) \to H^l_{DR}(\hat{X}^{(p)})).$$

(ii): The question is local, hence we may consider $X^{\mathcal{O}} \in \mathrm{Sch}^{int,sm}/O_K$ such that there exists a finite etale map $e : X^{\mathcal{O}} \to e(X^{\mathcal{O}}) \subset \mathbb{A}^{d_X}_{O_K}$. Then by (ii0),

$$o_*a_{et} \varprojlim_{n\in\mathbb{N}} a_{et}\Omega^{l}_{/(O_K/p^n),\log}(X^{\mathcal{O}}) = a_{et}o_*\Omega^{l,an}_{/K,\log,\mathcal{O}}(X^{\mathcal{O}})$$

does not depend on the integral model $X^{\mathcal{O}} \in \mathrm{Sch}^{int,sm}/O_K$ of X, since in $\mathrm{PSh}(\mathrm{SmVar}(K)^{ea})$

$$o_*H^lOL^l_W : o_*r^*\Omega^{l,an}_{/K,\log,\mathcal{O}} \xrightarrow{\sim} o_*r^*(X^{\mathcal{O}} \in \mathrm{Sch}^{int,sm}/O_K \mapsto \ker((\phi - I) : H^l_{DR}(\hat{X}^{(p)}) \to H^l_{DR}(\hat{X}^{(p)}))).$$

Hence, considering an integral model of $X \times \mathbb{A}^1$ of the form $X^{\mathcal{O}} \times \mathbb{A}^1_{O_K}$, we get that $o_*r^*\Omega^{l,an}_{/K,\log,\mathcal{O}} \in \mathrm{PSh}(\mathrm{SmVar}(K)^{ea})$ is $\mathbb{A}^1$ invariant since for a commutative ring A, $(A[X])^* = A^*$.. Since moreover $o_*r^*\Omega^{l,an}_{/K,\log,\mathcal{O}}$ admits transfers, we get that $o_*a_{et}r^*\Omega^{l,an}_{/K,\log,\mathcal{O}} = a_{et}o_*r^*\Omega^{l,an}_{/K,\log,\mathcal{O}} \in \mathrm{PSh}(\mathrm{SmVar}(K)^{ea})$ is also $\mathbb{A}^1$ invariant by theorem 1. This proves (ii). □

The morphism of site

$$\mathrm{An}_p : \mathrm{RigVar}(K) \to \mathrm{Var}(K),\ X \mapsto \hat{X} := \hat{X}^{\mathcal{O},(p)} \otimes_{O_K} K,\ X^{\mathcal{O}} \in \mathrm{Sch}^{int}/O_K, \text{ s.t } X^{\mathcal{O}} \otimes_{O_K} K = X$$

induces the map in $C(\mathrm{Var}(K)^{pet})$

$$\mathrm{An}^*_p : \Omega^{\bullet}_{/K} \to \Omega^{\bullet,an}_{/K} := \mathrm{An}_{p*}\,\Omega^{\bullet,an}_{/K},\ \text{for } X \in \mathrm{Var}(K),\ An^*_p : \Omega^{\bullet}_X(X) \to \Omega^{\bullet}_{\hat{X}^{(p)}}(\hat{X}^{(p)}),$$

which induces for each $j \in \mathbb{Z}$ and $X \in \mathrm{Var}(K)$ the morphism

$$\mathrm{An}^*_p : H^j_{DR}(X) = \mathbb{H}^j_{et}(X, \Omega^{\bullet}_X) \to \mathbb{H}^j_{et}(X, \Omega^{\bullet}_{\hat{X}^{(p)}}) = \mathbb{H}^j_{et}(\hat{X}^{(p)}, \Omega^{\bullet}_{\hat{X}^{(p)}})$$

which is an isomorphism for $X \in \mathrm{SmVar}(K)$ or $X \in \mathrm{PVar}(K)$ by GAGA (c.f. EGA 3), and for each $j \in \mathbb{Z}$ and $X \in \mathrm{Var}(K)$ the morphism

$$\mathrm{An}^*_p : H^j_{DR}(X) = \mathbb{H}^j_{pet}(X, \Omega^{\bullet}_X) \to \mathbb{H}^j_{pet}(X, \Omega^{\bullet}_{\hat{X}^{(p)}}) = \mathbb{H}^j_{pet}(\hat{X}^{(p)}, \Omega^{\bullet}_{\hat{X}^{(p)}})$$

which is thus an isomorphism for $X \in \mathrm{SmVar}(K)$ since (see [11])

$$\mathbb{H}^j_{pet}(\hat{X}^{(p)}, \Omega^\bullet_{\hat{X}^{(p)}}) = \mathbb{H}^j_{et}(\hat{X}^{(p)}, \Omega^\bullet_{\hat{X}^{(p)}}), \text{ that is, } \mathbb{H}^j_{pet}(X, \Omega^\bullet_{\hat{X}^{(p)}}) = \mathbb{H}^j_{et}(X, \Omega^\bullet_{\hat{X}^{(p)}}).$$

by considering an open cover $X = \cup_i X_i$ such that there exist etale maps $X_i^{\mathcal{O}} \to \mathbb{G}_m^{d_{X_i}} \subset \mathbb{A}^{d_{X_i}}_{O_K}$ which are the composite of a finite etale map and an open embedding. The result of this section is the following ([4] proposition 2 (i)), we recall and give some precisions to the proof :

Proposition 8. *Let K be a field of characteristic zero which is complete for a p-adic norm. Let $X \in \mathrm{PSmVar}(K)$ with good reduction. Let $X^{\mathcal{O}} \in \mathrm{PSch}/O_K$, smooth over O_K such that $X^{\mathcal{O}} \otimes_{O_K} K = X$. Consider the morphism $\mathrm{an}_{X,p} : \hat{X}^{(p)} \to X$ in RTop given by completion of $X_{\mathcal{O}}$ with respect to (p).*

(i) The analytic De Rham cohomology class of an algebraic cycle is logarithmic and is of type (d,d), that is, for $Z \in \mathcal{Z}^d(X)$

$$[Z] := H^{2d}\Omega(\gamma_Z^\vee)([Z]) \subset H^{2d}OL_{\hat{X}}(H^d_{pet}(X, \Omega^d_{\hat{X}^{(p)},\log,\mathcal{O}})) \subset H^{2d}_{DR}(\hat{X}^{(p)}) = H^{2d}_{DR}(X).$$

(ii) Conversely, any $w \in H^{2d}OL_{\hat{X}}(H^d_{pet}(X, \Omega^d_{\hat{X}^{(p)},\log,\mathcal{O}}))$ is the class of an algebraic cycle $Z \in \mathcal{Z}^d(X) \otimes \mathbb{Q}_p$, i.e. $w = [Z]$.

(iii) We have $H^jOL_{\hat{X}}(H^{j-l}_{pet}(X, \Omega^l_{\hat{X}^{(p)},\log,\mathcal{O}})) = 0$ for $j, l \in \mathbb{Z}$ such that $2l \neq j$. That is, we only have analytic logarithmic classes in bidegree (d,d) for $d \in \mathbb{N}$, in particular there is no non trivial logarithmic classes for odd degree De Rham cohomology $H^{2d+1}_{DR}(\hat{X}^{(p)}) = H^{2d+1}_{DR}(X)$.

(iv) ([4] proposition 2 (i)). For each $j \in \mathbb{Z}$, the isomorphism

$$\mathrm{an}^*_{X,p} : H^j_{DR}(X) = \mathbb{H}^j_{pet}(X, \Omega^\bullet_X) \xrightarrow{\sim} \mathbb{H}^j_{pet}(X, \Omega^\bullet_{\hat{X}^{(p)}}) = H^j_{DR}(\hat{X}^{(p)})$$

and its inverse preserve logarithmic classes, that is for each $j, l \in \mathbb{Z}$,

$$\mathrm{an}^*_{X,p}(H^j(m \circ (OL_X \otimes I))(H^{j-l}_{pet}(X, \Omega^l_{X^{pet},\log,\mathcal{O}} \otimes \mathbb{Z}_p))) = H^jOL_{\hat{X}^{(p)}}(H^{j-l}_{pet}(X, \Omega^l_{\hat{X}^{(p)},\log,\mathcal{O}})).$$

Proof. Consider, for $j, l \in \mathbb{Z}$ and $X \in \mathrm{SmVar}(K)$ with good reduction,

$$L^{l,j-l}(X) := H^jOL_{\hat{X}^{(p)}}(H^{j-l}_{pet}(X, \Omega^l_{\hat{X}^{(p)},\log,\mathcal{O}})).$$

Consider also for $j, l \in \mathbb{Z}$, $X \in \mathrm{SmVar}(K)$ with good reduction and $Z \subset X$ a closed subset,

$$L^{l,j-l}_Z(X) := H^jOL_{\hat{X}^{(p)}}(H^{j-l}_{pet,Z}(X, \Omega^l_{\hat{X}^{(p)},\log,\mathcal{O}})).$$

By proposition 2(ii) and (ii)', we have

$$L^{l,j-l}(X) = H^jOL_{\hat{X}^{(p)}}(H^{j-l}_{et}(X, \Omega^l_{\hat{X}^{(p)},\log,\mathcal{O}})) = H^jOL_{/K,an}(H^{j-l}\,\mathrm{Hom}(\mathbb{Z}(X), E_{et}(r^*\Omega^{l,an}_{/K,\log,\mathcal{O}}))).$$

and

$$L^{l,j-l}_Z(X) = H^jOL_{\hat{X}^{(p)}}(H^{j-l}_{et,Z}(X, \Omega^l_{\hat{X}^{(p)},\log,\mathcal{O}})) = H^jOL_{/K,an}(H^{j-l}\,\mathrm{Hom}(\mathbb{Z}(X, X\backslash Z), E_{et}(r^*\Omega^{l,an}_{/K,\log,\mathcal{O}}))),$$

for the second equalities of $L^{l,j-l}(X)$ and $L^{l,j-l}_Z(X)$, note that by lemma 3 (ii0) and its proof

$$(a_{et}H^lOL^l_W)_{|X^{et}} : a_{et}(r^*\Omega^{l,an}_{/K,\log,\mathcal{O}})_{|X^{et}} \xrightarrow{\sim}$$
$$a_{et}((X' \to X) \in X^{et} \mapsto \ker((\phi - I) : H^l_{DR}(X'^{\mathcal{O}}) \to H^l_{DR}(X'^{\mathcal{O}}))) \xrightarrow{\sim} a_{et}(r^*\Omega_{\hat{X}^{(p)},\log,\mathcal{O}})$$

where $X^{et} \subset \mathrm{SmVar}(K)$ is the small etale site and $a_{et} : \mathrm{PSh}(X^{et}) \to \mathrm{Shv}_{et}(X^{et})$ is the sheaftification functor. By lemma 3 (i3), the presheaves for each $l \in \mathbb{Z}$

$$a_{et} r^* \Omega^{l,an}_{/K,\log,\mathcal{O}} := a_{et} r^* \varprojlim_{n \in \mathbb{N}} a_{et} \Omega^l_{/(O_K/p^n),\log} \in \mathrm{PSh}(\mathrm{SmVar}(K))$$

admit transfers. The presheaves for each $l \in \mathbb{Z}$

$$(a_{et} r^* \Omega^{l,an}_{/K,\log,\mathcal{O}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p := (a_{et} r^* \varprojlim_{n \in \mathbb{N}} a_{et} \Omega^l_{/(O_K/p^n),\log}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \in \mathrm{PSh}(\mathrm{SmVar}(K))$$

are also $\mathbb{A}^1$ invariant by lemma 3 (ii). Hence by theorem 1, they are $\mathbb{A}^1$ local since they are $\mathbb{A}^1$ invariant and admit transfers. This gives in particular, for $Z \subset X$ a smooth subvariety of (pure) codimension d with good reduction, by proposition 1 an isomorphism

$$E_{et} r^* \Omega^{l,an}_{/K,\log,\mathcal{O}}(P_{Z,X}) : L^{l,j-l}_Z(X) \xrightarrow{\sim} L^{l,j-l}_Z(N_{Z/X}) \xrightarrow{\sim} L^{l-d,j-l-d}(Z).$$

Let

$$\alpha = OL_{\hat{X}^{(p)}}(\alpha) \in L^{l,j-l}(X) := H^j OL_{\hat{X}^{(p)}}(H^{j-l}_{pet}(X, \Omega^l_{\hat{X}^{(p)},\log,\mathcal{O}})) = H^{j-l} \mathrm{Hom}(\mathbb{Z}(X), E_{et}(r^* \Omega^{l,an}_{/K,\log,\mathcal{O}})).$$

with $j - l \neq 0$. Let (see lemma 1) $U \subset X$ a (non empty) open subset such that there exists an etale map $e : U^{\mathcal{O}} \to \mathbb{G}_m^{d_U} \subset \mathbb{A}^{d_U}_{O_K}$ with $e : U^{\mathcal{O}} \to e(U^{\mathcal{O}})$ finite etale, where $U^{\mathcal{O}} := X^{\mathcal{O}} \backslash V(\mathcal{I}^{\mathcal{O}}_{X \backslash U})$. Denote $j : U \hookrightarrow X$ the open embedding. By proposition 2(i), we have

$$j^* \alpha = 0 \in H^{j-l}_{et}(U, \Omega^l_{\hat{X}^{(p)},\log,\mathcal{O}}) = 0.$$

Considering a divisor $X \backslash U \subset D \subset X$, we get

$$\alpha = H^{j-l} E_{et}(r^* \Omega^{l,an}_{/K,\log,\mathcal{O}})(\gamma_D^\vee)(\alpha),\ \alpha \in L^{l,j-l}_D(X).$$

- Proof of (i):Similar to the proof of theorem 2(i).
- Proof of (ii):By assumption we have $w \in L^{d,d}(X)$. Let $U \subset X$ an open subset such that there exists an etale map $e : U^{\mathcal{O}} \to \mathbb{G}_m^{d_U} \subset \mathbb{A}^{d_U}_{O_K}$ with $e : U^{\mathcal{O}} \to e(U^{\mathcal{O}})$ finite etale. Considering a divisor $X \backslash U \subset D \subset X$, we get

  $$w = H^d E_{et}(r^* \Omega^{d,an}_{/K,\log,\mathcal{O}})(\gamma_D^\vee)(w),\ \alpha \in L^{d,d}_D(X).$$

  Denote $D^o \subset D$ its smooth locus with good reduction and $l : X^o \hookrightarrow X$ a Zariski open subset such that $X^o \cap D = D^o$. We then have

  $$l^* w \in L^{d,d}_{D^o}(X^o) = L^{d-1,d-1}(D^o)$$

  We repeat this procedure with each connected components of D^o instead of X. By a finite induction of d steps, we get

  $$w = H^d E_{et}(r^* \Omega^{d,an}_{/K,\log,\mathcal{O}})(\gamma_Z^\vee)(w),\ w \in L^{d,d}_Z(X) = L^{0,0}(Z^o).$$

  with $Z := D_d \subset \cdots \subset D \subset X$ a pure codimension d (Zariski) closed subset, thus $\alpha = \sum_i n_i [Z_i] \in H^{2d}_{DR}(X)$, where $n_i \in \mathbb{Q}_p$ and $(Z_i)_{1 \leq i \leq t} \subset Z$ are the irreducible components of Z,
- Proof of (iii) part 1: Let $j > 2l$. Let $w \in L^{l,j-l}(X)$. By the proof of (ii) there exists $Z \subset X$ a closed subset of pure codimension l such that

  $$w = H^{j-l} E_{et}(r^* \Omega^{l,an}_{/K,\log,\mathcal{O}})(\gamma_Z^\vee)(w),\ w \in L^{l,j-l}_Z(X).$$

  By a finite induction of $d_X - l$ steps, restricting to the smooth locus with good reduction of closed subsets of Z, $w = 0$ since $L^{l,j-l}_{Z'}(X) = 0$ for all $Z' \in \mathrm{SmVar}(k)$ with good reduction such that $\mathrm{codim}(Z', X) \geq l$.

- Proof of (iii) part 2: Let $j < 2l$. Let $w \in L^{l,j-l}(X)$. By the proof of (ii) there exists $Z \subset X$ a closed subset of pure codimension $j-l$ such that

$$w = H^{j-l}E_{et}(r^*\Omega^{l,an}_{/K,\log,\mathcal{O}})(\gamma_Z^\vee)(w),\ w \in L^{l,j-l}_Z(X).$$

For $Z' \subset X$ a closed subset of pure codimension c, consider a desingularisation $\epsilon : \tilde{Z}' \to Z'$ of Z' and denote $n : \tilde{Z}' \xrightarrow{\epsilon} Z' \subset X$. The morphism in $\mathrm{DA}(k)$

$$G_{Z',X} : M(X) \xrightarrow{D(\mathbb{Z}(n))} M(\tilde{Z}')(c)[2c] \xrightarrow{\mathbb{Z}(\epsilon)} M(Z')(c)[2c]$$

where $D : \mathrm{Hom}_{\mathrm{DA}(k)}(M_c(\tilde{Z}'), M_c(X)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{DA}(k)}(M(X), M(\tilde{Z}')(c)[2c])$ is the duality isomorphism from the six functor formalism (moving lemma of Suzlin and Voevodsky) and $\mathbb{Z}(n) := \mathrm{ad}(n_!, n^!)(a^!_X\mathbb{Z})$, is given by a morphism in $C(\mathrm{SmVar}(k))$

$$\hat{G}_{Z',X} : \mathbb{Z}_{tr}(X) \to E_{et}(C_*\mathbb{Z}_{tr}(Z'))(c)[2c].$$

Let $l : X^o \hookrightarrow X$ be an open embedding such that $Z^o := Z \cap X^o$ is the smooth locus of Z with good reduction. We consider $Z^{\mathcal{O}} := V(I^{\mathcal{O}}_Z) \subset X^{\mathcal{O}}$ so that $Z^{\mathcal{O}} \in \mathrm{Sch}^{int}/O_K$ is an integral model of Z. We then have the following commutative diagram of abelian groups

$$\begin{array}{ccccccc}
0 \longrightarrow & H^{j-l}_{et,Z}(X, \Omega^l_{\hat{X}^{(p)},\log,\mathcal{O}}) & \xrightarrow{l^*} & H^{j-l}_{et,Z^o}(X^o, \Omega^l_{\hat{X}^{(p)},\log,\mathcal{O}}) & \xrightarrow{\partial} & H^{j-l+1}_{et,Z\setminus Z^o}(X, \Omega^l_{\hat{X}^{(p)},\log,\mathcal{O}}) & \longrightarrow \cdots \\
& \uparrow{\scriptstyle \Omega(\hat{G}_{Z,X})} & & \uparrow{\scriptstyle \Omega(P_{Z^o,X^o})} & & \uparrow{\scriptstyle \Omega(\hat{G}_{Z,X})} & \\
0 \longrightarrow & H^0_{et}(Z, \Omega^{2l-j}_{\hat{Z}^{(p)},\log,\mathcal{O}}) & \xrightarrow{l*} & H^0_{et}(Z^o, \Omega^{2l-j}_{\hat{Z}^{o(p)},\log,\mathcal{O}}) & \xrightarrow{\partial} & H^1_{et,Z\setminus Z^o}(Z, \Omega^{2l-j}_{\hat{Z}^{(p)},\log,\mathcal{O}}) & \longrightarrow \cdots
\end{array}$$

whose rows are exact sequences. Consider

$$l^*w = \Omega(P_{Z^o,X^o})(w^o) \in H^{j-l}_{et,Z^o}(X^o, \Omega^l_{\hat{X}^{(p)},\log,\mathcal{O}}),\ w^o \in H^0_{et}(Z^o, \Omega^{2l-j}_{\hat{Z}^{o(p)},\log,\mathcal{O}}).$$

Since $\partial l^*w = 0 \in H^{j-l+1}_{et,Z\setminus Z^o}(X, \Omega^l_{\hat{X}^{(p)},\log,\mathcal{O}})$, we get, considering the smooth locus with good reduction $(Z\setminus Z^o)^o \subset Z\setminus Z^o$ and $X^{oo} \subset X$ is an open subset such that $X \cap (Z\setminus Z^o)^o = X^{oo} \cap (Z\setminus Z^o)$,

$$\partial w^o = 0 \in H^1_{et,Z\setminus Z^o}(Z, \Omega^{2l-j}_{\hat{Z}^{(p)},\log,\mathcal{O}}),$$

since $H^1_{et,(Z\setminus Z^o)\setminus(Z\setminus Z^o)^o}(Z, \Omega^{2l-j}_{\hat{Z}^{(p)},\log,\mathcal{O}}) = 0$ for dimension reasons, that is

$$w = \Omega(\hat{G}_{Z,X})(w),\ \text{with}\, w \in H^0_{et}(Z, \Omega^{2l-j}_{\hat{Z}^{(p)},\log,\mathcal{O}}).$$

Hence $w = 0$ since $H^0(Z', \Omega^k_{Z'^{(p)},\log,\mathcal{O}}) = 0$ for all $k > 0$ and all $Z' \in \mathrm{PVar}(K)$.

(iv):Follows from (ii),(iii) on the one hand, and on the other hand theorem 2(ii)and (iii). □

Remark 3. *Let K be a field of characteristic zero which is complete for a p-adic norm. Let $X \in \mathrm{PSmVar}(K)$ with good reduction. Let $X^{\mathcal{O}} \in \mathrm{PSch}/O_K$, smooth over O_K such that $X^{\mathcal{O}} \otimes_{O_K} K = X$. The canonical map*

$$(H^jE_{et}(\Omega(/p^n))) : \mathbb{H}^j_{et}(X, \Omega^\bullet_{X^{\mathcal{O}},\log}) \to \varprojlim_{n\in\mathbb{N}} \mathbb{H}^j_{et}(X, \Omega^\bullet_{X^{\mathcal{O}}/p^n,\log}) = \mathbb{H}^j_{pet}(X, \Omega^\bullet_{\hat{X}^{\mathcal{O},(p)},\log})$$

is NOT surjective and NOT injective in general.

4 Tate conjecture

Let k be a field of finite type over $\mathbb{Q}$. Denote $\bar{k}$ the algebraic closure of k and $G = Gal(\bar{k}/k)$ the absolute Galois group of k.

Let $X \in \mathrm{SmVar}(k)$ be a smooth variety. Let $p \in \mathbb{N}$ a prime number. Consider an embedding $\sigma_p : k \hookrightarrow \mathbb{C}_p$. Denote $\hat{k}_{\sigma_p} \subset \mathbb{C}_p$ the p-adic completion of k with respect to σ_p. We have the commutative diagram in $C_{\mathbb{B}_{dr}fil,\hat{G}_{\sigma_p}}(X^{an,pet}_{\mathbb{C}_p})$

$$\begin{array}{ccc} (\mathbb{B}_{dr,X_{\mathbb{C}_p}},F) & \xrightarrow{\alpha(X)} & (\Omega^\bullet_{X_{\mathbb{C}_p}},F_b)\otimes_{O_{X_{\mathbb{C}_p}}}(O\mathbb{B}_{dr,X_{\mathbb{C}_p}},F) \\ {\scriptstyle \iota'_{X^{pet}_{\mathbb{C}_p}}:=l\mapsto l.1}\uparrow & & \uparrow{\scriptstyle OL_X\otimes I:=(w\otimes\lambda_n)_{n\in\mathbb{N}}\mapsto(w\otimes\lambda_n)_{n\in\mathbb{N}}} \\ \underline{\mathbb{Z}_p}_{X_{\mathbb{C}_p}} & \xrightarrow{\iota_{X^{pet}_{\mathbb{C}_p}}:=(I,0)} & (\Omega^\bullet_{X_{\mathbb{C}_p},\log,\mathcal{O}}\otimes\mathbb{Z}_p,F_b) \end{array} .$$

Consider an integral model $X^{\mathcal{O}}_{\hat{k}_{\sigma_p}} \in \mathrm{Sch}^{int}/O_{\hat{k}_{\sigma_p}}$ of $X_{\hat{k}_{\sigma_p}}$. If $X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}$ has good reduction modulo p, we have (see e.g. [1]) the embedding in $C((X^{\mathcal{O}}_{\hat{k}_{\sigma_p}})^{Falt})$

$$\alpha(X) : \mathbb{B}_{cris,X_{\hat{k}_{\sigma_p}}} \hookrightarrow a_{\bullet *}\Omega^\bullet_{X^{\mathcal{O},\bullet}_{\hat{k}_{\sigma_p}}} \otimes_{O_{X^{\mathcal{O}}}} O\mathbb{B}_{cris,X^\bullet_{\hat{k}_{\sigma_p}}}$$

which is a filtered quasi-isomorphism compatible with the action of $Gal(\mathbb{C}_p/\hat{k}_{\sigma_p})$ and the action of the Frobenius ϕ_p, where $(X^{\mathcal{O}}_{\hat{k}_{\sigma_p}})^{Falt}$ denote the Falting site.

For k a field of finite type over $\mathbb{Q}$ and $X \in \mathrm{SmVar}(k)$, we denote $\delta(k,X) \subset N$ the finite set consisting of prime numbers such that if $p \in \mathbb{N}\backslash\delta(k,X)$ is a prime number, k is unramified at p and there exists an integral model $X^{\mathcal{O}}_{\hat{k}_{\sigma_p}} \in \mathrm{Sch}^{int}/O_{\hat{k}_{\sigma_p}}$ of $X_{\hat{k}_{\sigma_p}}$ with good reduction modulo p for all embeddings $\sigma_p : k \hookrightarrow \mathbb{C}_p$, $\hat{k}_{\sigma_p} \subset \mathbb{C}_p$ being the p-adic completion of k with respect to σ_p.

Remark 4. *To see that $\delta(k,X)$ is a finite set, considering $k = k_0(x_1,\cdots,x_d)(x_{d+1})$ where $x_1,\cdots x_d$ are algebraicaly independent and k_0 is a number field one can take an integral model $Y \to O_{k_0}[x_1,\cdots,x_{d+1}]/f_{d+1}$ of X, a desingularization of $Y' \to Y$ of Y and then see that $\delta(k,X)$ is contained in the discriminant of the family $Y' \to Y \to O_{k_0}[x_1,\cdots,x_{d+1}]/f_{d+1}$.*

Let $X \in \mathrm{PSmVar}(k)$ be a smooth projective variety. Let $p \in \mathbb{N}\backslash\delta(k,X)$ a prime number. Consider an embedding $\sigma_p : k \hookrightarrow \mathbb{C}_p$. Then $X_{\hat{k}_{\sigma_p}}$ has good reduction modulo p and let $X^{\mathcal{O}}_{\hat{k}_{\sigma_p}} \in \mathrm{PSch}/O_{\hat{k}_{\sigma_p}}$ be a smooth model, i.e. $X^{\mathcal{O}}_{\hat{k}_{\sigma_p}} \otimes_{O_{\hat{k}_{\sigma_p}}} \hat{k}_{\sigma_p} = X_{\hat{k}_{\sigma_p}}$ and $X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}$ is smooth with smooth special fiber. The main result of [1] say in this case that the embedding in $C((X^{\mathcal{O}}_{\hat{k}_{\sigma_p}})^{Falt})$

$$\alpha(X) : \mathbb{B}_{cris,X_{\hat{k}_{\sigma_p}}} \hookrightarrow a_{\bullet *}\Omega^\bullet_{X^{\mathcal{O},\bullet}_{\hat{k}_{\sigma_p}}} \otimes_{O_{X^{\mathcal{O}}}} O\mathbb{B}_{cris,X^\bullet_{\hat{k}_{\sigma_p}}}$$

induces a filtered quasi-isomorphism compatible for each $j \in \mathbb{Z}$, a filtered isomorphism of filtered abelian groups

$$\begin{aligned} H^jR\alpha(X) : H^j_{et}(X_{\mathbb{C}_p},\mathbb{Z}_p)\otimes_{\mathbb{Z}_p}\mathbb{B}_{cris,\hat{k}_{\sigma_p}} &\xrightarrow{H^jT(a_X,\mathbb{B}_{cris})^{-1}} H^j_{et}((X)^{Falt})(\mathbb{B}_{cris,X_{\hat{k}_{\sigma_p}}}) \\ &\xrightarrow{H^jR\Gamma(X^{\mathcal{O}}_{\hat{k}_{\sigma_p}},\alpha(X))} H^j_{DR}(X_{\hat{k}_{\sigma_p}})\otimes_{\hat{k}_{\sigma_p}}\mathbb{B}_{cris,\hat{k}_{\sigma_p}} \end{aligned}$$

compatible with the action of $Gal(\mathbb{C}_p/\hat{k}_{\sigma_p})$ and of the Frobenius ϕ_p.

Definition 3. *Let k be a field of finite type over $\mathbb{Q}$. Denote $\bar{k}$ the algebraic closure of k and $G = Gal(\bar{k}/k)$ the absolute Galois group of k. Let $X \in \mathrm{PSmVar}(k)$. Let $p \in \mathbb{N}\backslash\delta(k, X)$ a prime number. Consider an embedding $\sigma_p : k \hookrightarrow \mathbb{C}_p$. Denote $\hat{k}_{\sigma_p} \subset \mathbb{C}_p$ the p-adic completion of k with respect to σ_p. For $\alpha \in H^j_{et}(X_{\mathbb{C}_p}, \mathbb{Z}_p)$, we consider*

$$w(\alpha) := H^j R\alpha(X)(\alpha \otimes 1) \in H^j_{DR}(X_{\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}} \mathbb{B}_{cris,\hat{k}_{\sigma_p}}.$$

the associated de Rham class by the p-adic periods. We recall

$$H^j R\alpha(X) : H^j_{et}(X_{\mathbb{C}_p}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{B}_{cris,\hat{k}_{\sigma_p}} \xrightarrow{\sim} H^j_{DR}(X_{\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}} \mathbb{B}_{cris,\hat{k}_{\sigma_p}}$$

is the canonical filtered isomorphism compatible with the action of $\hat{G}_{\sigma_p}$ and with the action of the Frobenius ϕ_p.

We have the following key proposition (the projective case of [4]), we state and prove it for smooth projective varieties, the case for smooth varieties is obtained in the same way using a smooth compactification with normal crossing divisors. The projective case suffices for our purpose:

Proposition 9. *(projective case of [4] proposition 4(i)). Let k be a field of finite type over $\mathbb{Q}$. Let $X \in \mathrm{PSmVar}(k)$. Let $p \in \mathbb{N}\backslash\delta(k, X)$ be a prime number. Consider an embedding $\sigma_p : k \hookrightarrow \mathbb{C}_p$. Denote by $k \subset \hat{k}_{\sigma_p} \subset \mathbb{C}_p$ the p-adic completion of k with respect to σ_p. Consider $X^{\mathcal{O}}_{\hat{k}_{\sigma_p}} \in \mathrm{PSch}/O_{\hat{k}_{\sigma_p}}$ a smooth integral model of $X_{\hat{k}_{\sigma_p}}$, i.e. $X^{\mathcal{O}}_{\hat{k}_{\sigma_p}} \otimes_{O_{\hat{k}_{\sigma_p}}} \hat{k}_{\sigma_p} = X_{\hat{k}_{\sigma_p}}$ and $X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}$ is smooth with smooth special fiber. Let $j \in \mathbb{Z}$. We have, see definition 2(iii),*

$$H^j_{DR}(X_{\hat{k}_{\sigma_p}}) \cap H^j R\alpha(X)(H^j_{et}(X_{\mathbb{C}_p}, \mathbb{Z}_p)) = (\mathrm{an}^*_{X,p})^{-1}(H^j OL_{\hat{X}^{(p)}_{\hat{k}_{\sigma_p}}}(\mathbb{H}^j_{pet}(X_{\hat{k}_{\sigma_p}}, \Omega^\bullet_{\hat{X}^{(p)}_{\hat{k}_{\sigma_p}},\log,\mathcal{O}})))$$
$$\subset H^j_{DR}(X_{\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}} \mathbb{B}_{cris,\hat{k}_{\sigma_p}},$$

where we recall $\mathrm{an}_{X,p} : \hat{X}^{(p)}_{\hat{k}_{\sigma_p}} \to X_{\hat{k}_{\sigma_p}}$ is the morphism in RTop *given by the completion of $X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}$ with respect to (p), and*

$$\mathrm{an}^*_{X,p} : \mathbb{H}^j_{pet}(X_{\hat{k}_{\sigma_p}}, \Omega^\bullet_{X_{\hat{k}_{\sigma_p}}}) \xrightarrow{\sim} \mathbb{H}^j_{pet}(X_{\hat{k}_{\sigma_p}}, \Omega^\bullet_{\hat{X}^{(p)}_{\hat{k}_{\sigma_p}}})$$

is an isomorphism by GAGA and considering an open cover $X = \cup_i X_i$ such that there exist etale maps $X^{\mathcal{O}}_i \to \mathbb{G}^{d_{X_i}}_m \subset \mathbb{A}^{d_{X_i}}_{O_{\hat{k}_{\sigma_p}}}$ which are the composite of a finite etale map and an open embedding of integral model.

Proof. Consider $c : \hat{X}^{\mathcal{O},(p)}_{\hat{k}_{\sigma_p}} \to X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}$ the morphism in RTop which is the formal completion along the ideal (p). Take a Zariski or etale cover $r = (r_i : X_i \to X^{\mathcal{O}})_{1 \leq i \leq r}$ such that for each i there exists an etale map $e_i : X_i \to \mathbb{G}^{d_{X_i}}_m \subset \mathbb{A}^{d_{X_i}}_{O_{\hat{k}_{\sigma_p}}}$ with $e_i : X_i \to e_i(X_i)$ finite etale. Then, by [1], we have for each i explicit lifts of Frobenius $\phi^i_n : X^{\mathcal{O}}_{i,\hat{k}_{\sigma_p}}/p^n \to X^{\mathcal{O}}_{i,\hat{k}_{\sigma_p}}/p^n$ of the Frobenius $\phi^i_1 : X^{\mathcal{O}}_{i,\hat{k}_{\sigma_p}}/p \to X^{\mathcal{O}}_{i,\hat{k}_{\sigma_p}}/p$, such that for $n' > n$ the following diagram commutes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & O_{X^{\mathcal{O}}_{i,\hat{k}_{\sigma_p}}/p^{n'-n}} & \xrightarrow{p^n\cdot} & O_{X^{\mathcal{O}}_{i,\hat{k}_{\sigma_p}}/p^{n'}} & \xrightarrow{/p^{n'-n}} & O_{X^{\mathcal{O}}_{i,\hat{k}_{\sigma_p}}/p^n} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \phi^i_{n'-n}} & & \uparrow{\scriptstyle \phi^i_{n'}} & & \uparrow{\scriptstyle \phi^i_n} & & \\
0 & \longrightarrow & O_{X^{\mathcal{O}}_{i,\hat{k}_{\sigma_p}}/p^{n'-n}} & \xrightarrow{p^n\cdot} & O_{X^{\mathcal{O}}_{i,\hat{k}_{\sigma_p}}/p^{n'}} & \xrightarrow{/p^{n'-n}} & O_{X^{\mathcal{O}}_{i,\hat{k}_{\sigma_p}}/p^n} & \longrightarrow & 0
\end{array}$$

and such that the action of ϕ_n^i on $\Omega^\bullet_{X^{\mathcal{O}}_{i,\hat{k}_{\sigma_p}}/p^n}$ is a morphism of complex, i.e. commutes with the differentials. On the other hand, by [7], we have action of the Frobenius on $H^j_{DR}(X_{\hat{k}_{\sigma_p}}) = H^j_{DR}(X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}) \otimes_{O_{\hat{k}_{\sigma_p}}} \hat{k}_{\sigma_p}$ by

$$\phi : H^j_{DR}(X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}) \xrightarrow{H^j IW} \mathbb{H}^j(X_{\hat{k}_{\sigma_p}}, W\Omega^\bullet_{X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}}) \xrightarrow{\phi^*_{W(X_{\hat{k}_{\sigma_p}})}} \mathbb{H}^j(X_{\hat{k}_{\sigma_p}}, W\Omega^\bullet_{X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}}) \xrightarrow{H^j IW^{-1}} H^j_{DR}(X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}),$$

where $IW = (IW_n)_{n\in\mathbb{N}}$ is given by the morphisms of complexes in $C(X^{\mathcal{O}}_{\hat{k}_{\sigma_p}})$

$$IW_n : \Omega^\bullet_{X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}/p^n} \xrightarrow{\Omega(JW_n)} \Omega^\bullet_{W_n(X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}/p)} \to W_n\Omega^\bullet_{X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}/p}$$

induced by the morphism of rings $JW_n : O_{X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}/p^n} \to W_n O(X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}/p)$. We then have the following commutative diagram, where $R := [1,\dots,r]$ and $X_R := X_1 \times_X \cdots \times_X X_r$,

$$\begin{array}{ccccc}
W\Omega^\bullet_{X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}} & \xrightarrow{r_i^*} & \oplus_{i=1}^r r_{i*} W\Omega^\bullet_{X^{\mathcal{O}}_{i,\hat{k}_{\sigma_p}}} & \xrightarrow{r_I^*} & r_{R*} W\Omega^\bullet_{X^{\mathcal{O}}_{R,\hat{k}_{\sigma_p}}} \\
{\scriptstyle I-\phi}\uparrow & & {\scriptstyle I-\phi^i}\uparrow & & {\scriptstyle I-\phi^R}\uparrow \\
W\Omega^\bullet_{X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}} & \xrightarrow{r_i^*} & \oplus_{i=1}^r r_{i*} W\Omega^\bullet_{X^{\mathcal{O}}_{i,\hat{k}_{\sigma_p}}} & \xrightarrow{r_I^*} & r_{R*} W\Omega^\bullet_{X^{\mathcal{O}}_{R,\hat{k}_{\sigma_p}}} \\
{\scriptstyle r^*OL_{\hat{X}^{\mathcal{O},(p)}_{\hat{k}_{\sigma_p}}} \circ IW_*}\uparrow & & {\scriptstyle r^*OL_{\hat{X}^{\mathcal{O},(p)}_{i,\hat{k}_{\sigma_p}}} \circ IW_*}\uparrow & & {\scriptstyle r^*OL_{\hat{X}^{\mathcal{O},(p)}_{R,\hat{k}_{\sigma_p}}} \circ IW_*}\uparrow \\
\Omega^\bullet_{\hat{X}^{(p)}_{\hat{k}_{\sigma_p}},\log,\mathcal{O}} & \xrightarrow{r_i^*} & \oplus_{i=1}^r r_{i*} \Omega^\bullet_{\hat{X}^{(p)}_{i,\hat{k}_{\sigma_p}},\log,\mathcal{O}} & \xrightarrow{r_I^*} & r_{R*} \Omega^\bullet_{\hat{X}^{(p)}_{R,\hat{k}_{\sigma_p}},\log,\mathcal{O}}
\end{array}$$

and $(I-\phi)\circ r^*OL_{\hat{X}^{\mathcal{O},(p)}_{\hat{k}_{\sigma_p}}} \circ IW_* = 0$. It induces the following commutative diagram, where $R := [1,\dots,r]$ and $X_R := X_1 \times_X \cdots \times_X X_r$,

$$\begin{array}{ccccccc}
\cdots \longrightarrow & H^{j-1}_{DR}(X^{\mathcal{O}}_{R,\hat{k}_{\sigma_p}}) & \xrightarrow{\partial} & H^j_{DR}(X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}) & \xrightarrow{r_i^*} & \oplus_{i=1}^r H^j_{DR}(X^{\mathcal{O}}_{i,\hat{k}_{\sigma_p}}) & \xrightarrow{r_I^*} \cdots \\
& {\scriptstyle I-\phi^R}\uparrow & & {\scriptstyle I-\phi}\uparrow & & {\scriptstyle I-\phi^i}\uparrow & \\
\cdots \longrightarrow & H^{j-1}_{DR}(X^{\mathcal{O}}_{R,\hat{k}_{\sigma_p}}) & \xrightarrow{\partial} & H^j_{DR}(X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}) & \xrightarrow{r_i^*} & \oplus_{i=1}^r H^j_{DR}(X^{\mathcal{O}}_{i,\hat{k}_{\sigma_p}}) & \xrightarrow{r_I^*} \cdots \\
& {\scriptstyle H^{j-1}r^*OL_{\hat{X}^{\mathcal{O},(p)}_{R,\hat{k}_{\sigma_p}}}}\uparrow & & {\scriptstyle H^j r^*OL_{\hat{X}^{\mathcal{O},(p)}_{\hat{k}_{\sigma_p}}}}\uparrow & & {\scriptstyle H^j r^*OL_{\hat{X}^{\mathcal{O},(p)}_{i,\hat{k}_{\sigma_p}}}}\uparrow & \\
\cdots \longrightarrow & \mathbb{H}^{j-1}_{pet}(X_{R,\hat{k}_{\sigma_p}}, \Omega^\bullet_{\hat{X}^{(p)}_{\hat{k}_{\sigma_p}},\log,\mathcal{O}}) & \xrightarrow{\partial} & \mathbb{H}^j_{pet}(X_{\hat{k}_{\sigma_p}}, \Omega^\bullet_{\hat{X}^{(p)}_{\hat{k}_{\sigma_p}},\log,\mathcal{O}}) & \xrightarrow{r_i^*} & \oplus_{i=1}^r \mathbb{H}^j_{pet}(X_{i,\hat{k}_{\sigma_p}}, \Omega^\bullet_{\hat{X}^{(p)}_{\hat{k}_{\sigma_p}},\log,\mathcal{O}}) & \xrightarrow{r_I^*} \cdots
\end{array} \tag{4}$$

whose rows are exact sequences and $(I-\phi)\circ H^j r^*OL_{\hat{X}^{\mathcal{O},(p)}_{\hat{k}_{\sigma_p}}} = 0$. By [1], $\alpha \in H^j_{et}(X_{\mathbb{C}_p}, \mathbb{Z}_p)$ is such that $w(\alpha) \in H^j_{DR}(X_{\hat{k}_{\sigma_p}})$ if and only if

$$w(\alpha) \in \ker(I-\phi : H^j_{DR}(X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}) \to H^j_{DR}(X^{\mathcal{O}}_{\hat{k}_{\sigma_p}})).$$

On the other hand, considering $X^{\mathcal{O},pet} \subset ((\mathrm{Sch}^{int,sm}/O_K)/X^{\mathcal{O}})^{pet}$, for each $I \subset [1,\dots,r]$, the sequence in $C(X^{\mathcal{O},pet}_I)$

$$0 \to \Omega^\bullet_{\hat{X}^{(p)}_{I,\hat{k}_{\sigma_p}},\log,\mathcal{O}} \xrightarrow{OL_{\hat{X}^{\mathcal{O},(p)}_{\hat{k}_{\sigma_p}}}} \Omega^\bullet_{\hat{X}^{\mathcal{O},(p)}_{I,\hat{k}_{\sigma_p}}} \xrightarrow{\phi^I - I := (\phi_n^I - I)_{n\in\mathbb{N}}} \Omega^\bullet_{\hat{X}^{\mathcal{O},(p)}_{I,\hat{k}_{\sigma_p}}} \to 0$$

is exact for the pro-etale topology by proposition 3 and since for each $l, n \in \mathbb{N}$ the map in $\mathrm{PSh}(X^{\mathcal{O},pet})$

$$\Omega(/p) : \Omega^l_{X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}/p^n} \to \Omega^l_{X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}/p^{n+1}}$$

are surjective for the etale topology and since the pro-etale site is a replete topos by [2]. Hence, for each $I \subset [1, \dots, r]$, by applying r^* where $r : X^{pet}_I \to X^{\mathcal{O},pet}_I$, the sequence in $C(X^{pet}_I)$

$$0 \to r^*\Omega^\bullet_{\hat{X}^{(p)}_{I,\hat{k}_{\sigma_p}},\log,\mathcal{O}} \xrightarrow{r^*OL_{\hat{X}^{\mathcal{O},(p)}_{\hat{k}_{\sigma_p}}}} r^*\Omega^\bullet_{\hat{X}^{\mathcal{O},(p)}_{I,\hat{k}_{\sigma_p}}} \xrightarrow{\phi^I - I := (\phi^I_n - I)_{n\in\mathbb{N}}} r^*\Omega^\bullet_{\hat{X}^{\mathcal{O},(p)}_{I,\hat{k}_{\sigma_p}}} \to 0$$

is exact for the pro-etale topology. Hence the columns

$$\mathbb{H}^q_{pet}(X_{I,\hat{k}_{\sigma_p}}, \Omega^\bullet_{\hat{X}^{(p)}_{\hat{k}_{\sigma_p}},\log,\mathcal{O}}) \xrightarrow{H^q r^*OL_{\hat{X}^{\mathcal{O},(p)}_{\hat{k}_{\sigma_p}}}} H^q_{DR}(X^{\mathcal{O}}_{I,\hat{k}_{\sigma_p}}) \xrightarrow{I-\phi} H^q_{DR}(X^{\mathcal{O}}_{I,\hat{k}_{\sigma_p}})$$

of the diagram (4) are exact. Using proposition 2(i) and the fact that a logarithmic form is exact if and only if it vanishes, we get

$$r^*OL_{\hat{X}^{\mathcal{O},(p)}_{\hat{k}_{\sigma_p}}} : \Gamma(X_{I,\hat{k}_{\sigma_p}}, \Omega^q_{\hat{X}^{(p)}_{\hat{k}_{\sigma_p}},\log,\mathcal{O}}) \xrightarrow{\sim} (\ker(I-\phi)(H^q_{DR}(X^{\mathcal{O}}_{I,\hat{k}_{\sigma_p}}) \to H^q_{DR}(X^{\mathcal{O}}_{I,\hat{k}_{\sigma_p}})))$$

from which we see that the remaining columns are also exact. This proves the proposition. □

This proposition together with proposition 8 implies the Tate conjecture for smooth projective varieties over fields of finite type over $\mathbb{Q}$:

Theorem 3. *(i) Let k be a field of finite type over $\mathbb{Q}$. Let $X \in \mathrm{PSmVar}(k)$. Let $p \in \mathbb{N}\backslash\delta(k, X)$ be a prime number. Then the Tate conjecture holds for X. That is for $d \in \mathbb{Z}$, the cycle class map*

$$\mathcal{Z}^d(X) \otimes \mathbb{Q}_p \to H^{2d}_{et}(X_{\bar{k}}, \mathbb{Q}_p)(d)^G,\ Z \mapsto [Z]$$

is surjective, where $G = Gal(\bar{k}/k)$.

(ii) Let $K \subset \mathbb{C}_p$ be a p-adic field which is unramified. Let $X \in \mathrm{PSmVar}(K)$ with good reduction. Then the Tate conjecture holds for X. That is for $d \in \mathbb{Z}$, the cycle class map

$$\mathcal{Z}^d(X) \otimes \mathbb{Q}_p \to H^{2d}_{et}(X_{\mathbb{C}_p}, \mathbb{Q}_p)(d)^G,\ Z \mapsto [Z]$$

is surjective, where $G = Gal(\mathbb{C}_p/K)$.

Proof. (i):Consider an embedding $\sigma_p : k \hookrightarrow \mathbb{C}_p$. Then $k \subset \bar{k} \subset \mathbb{C}_p$ and $k \subset \hat{k}_{\sigma_p} \subset \mathbb{C}_p$, where $\hat{k}_{\sigma_p}$ is the p-adic field which is the completion of k with respect the p adic norm given by σ_p. Consider $X^{\mathcal{O}}_{\hat{k}_{\sigma_p}} \in \mathrm{PSch}/O_{\hat{k}_{\sigma_p}}$ a smooth model of $X_{\hat{k}_{\sigma_p}}$, i.e. $X^{\mathcal{O}}_{\hat{k}_{\sigma_p}} \otimes_{O_{\hat{k}_{\sigma_p}}} \hat{k}_{\sigma_p} = X_{\hat{k}_{\sigma_p}}$ and $X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}$ is smooth with smooth special fiber. Let $\alpha \in H^{2d}_{et}(X_{\bar{k}}, \mathbb{Z}_p)(d)^G$. Using definition 3, by proposition 9

$$w(\alpha) \in H^{2d}OL_{\hat{X}^{(p)}}(\mathbb{H}^{2d}_{pet}(X_{\hat{k}_{\sigma_p}}, \Omega^\bullet_{\hat{X}^{(p)}_{\hat{k}_{\sigma_p}},\log,\mathcal{O}})) \subset H^{2d}_{DR}(\hat{X}^{(p)}_{\hat{k}_{\sigma_p}}) = H^{2d}_{DR}(X_{\hat{k}_{\sigma_p}})$$

where we recall $\mathrm{an}_{X,p} : \hat{X}^{(p)}_{\hat{k}_{\sigma_p}} \to X_{\hat{k}_{\sigma_p}}$ is the morphism in RTop given by the completion of $X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}$ with respect to (p). Considering the decomposition

$$\mathbb{H}^{2d}_{pet}(X_{\hat{k}_{\sigma_p}}, \Omega^\bullet_{\hat{X}^{(p)}_{\hat{k}_{\sigma_p}},\log,\mathcal{O}}) = \oplus^{2d}_{l=0} H^{2d-l}_{pet}(X_{\hat{k}_{\sigma_p}}, \Omega^l_{\hat{X}^{(p)}_{\hat{k}_{\sigma_p}},\log,\mathcal{O}}),$$

by proposition 8 (ii) and (iii) there exists $Z \in \mathcal{Z}^d(X_{\hat{k}_{\sigma_p}}) \otimes \mathbb{Q}_p$ such that $w(\alpha) = [Z] \in H^{2d}_{DR}(X_{\hat{k}_{\sigma_p}})$. By [1], considering the commutative diagram of abelian groups whose rows are exact, where $j : X\backslash|Z| \hookrightarrow X$ denote the open embedding,

$$\begin{array}{ccccc} H^{2d}_{et,Z}(X_{\bar{k}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{B}_{cris} & \xrightarrow{B_{cris,X}(\gamma_Z^\vee)} & H^{2d}_{et}(X_{\bar{k}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{B}_{cris} & \xrightarrow{j^*} & H^{2d}_{et}((X\backslash Z)_{\bar{k}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{B}_{cris} \\ \downarrow Ra_{X*}\Gamma_Z\alpha(X) & & \downarrow R\alpha(X) & & \downarrow R\alpha(X\backslash Z) \\ H^{2d}_{DR,Z}(X_{\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}} \mathbb{B}_{cris} & \xrightarrow{DR(X)(OB_{cris,X})(\gamma_Z^\vee)} & H^{2d}_{DR}(X_{\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}} \mathbb{B}_{cris} & \xrightarrow{j^*} & H^{2d}_{DR}((X\backslash Z)_{\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}} \mathbb{B}_{cris} \end{array},$$

we get $\alpha = [Z] \in H^{2d}_{et}(X_{\bar{k}}, \mathbb{Z}_p)$. Since $[Z]$ is G invariant,

$$Z' := 1/\#(gZ, g \in G) \sum_{g \in G/G_Z} gZ \in \mathcal{Z}^d(X) \otimes \mathbb{Q}_p$$

satisfy $[Z'] = [Z] = \alpha \in H^{2d}_{et}(X_{\bar{k}}, \mathbb{Z}_p)$.
(ii):Similar to (i).

□

Remark 5. *Let $K \subset \mathbb{Q}_p$ be a p-adic field. Denote by $G := Gal(\bar{\mathbb{Q}_p}/K)$ the absolute Galois group. Consider an isomorphism $\bar{\mathbb{Q}_p} \simeq \mathbb{C}$ so that $K \subset \mathbb{C}$.*

(i) Let $C, C' \in \mathrm{PSmVar}(K)$ be two smooth projective curve of genus $g(C) \geq 1$ greater or equal to one or two abelian varieties. Since $g(C) \geq 1$ in the case of curves, there exists a non trivial Hodge class $\alpha \neq 0 \in F^1H^2((C \times C')^{an}_{\mathbb{C}}, \mathbb{Q})$. By the Hodge conjecture for divisors there exists $Z \in \mathcal{Z}^1((C \times C')_{\mathbb{C}})$ such that

$$\alpha = [Z] \in H^2((C \times C')^{an}_{\mathbb{C}}, \mathbb{Q}).$$

Consider the splitting

$$Z = Z_1 - Z_2 \in \mathcal{Z}^1((C \times C')_{\mathbb{C}}),$$

where Z_1 and Z_2 are effective cycle. Then since $\alpha \neq 0$, either $[Z_1] \neq 0$ or $[Z_2] \neq 0$. Hence we may assume for instance that $[Z_1] \neq 0$. Consider that the effective cycle

$$Z' := \sum_{g \in G} gZ_1 \in \mathcal{Z}^1((C \times C')_{\bar{\mathbb{Q}_p}})$$

and $[Z'] = T(C \times C')([Z']) \in H^2_{et}((C \times C')_{\bar{\mathbb{Q}_p}}, \mathbb{Q}_p)$, where

$$T(C \times C') := (/p^n)_{n \in \mathbb{N}} : H^2((C \times C')^{an}_{\mathbb{C}}, \mathbb{Q}) \to H^2_{et}((C \times C')_{\bar{\mathbb{Q}_p}}, \mathbb{Q}_p)$$

is the canonical map. Then $[Z'] \neq 0 \in H^2_{et}((C \times C')_{\bar{\mathbb{Q}_p}}, \mathbb{Q}_p)(1)^G$. Hence the Neron Severi group

$$< e \times C, C \times e, [Z'] > \subset NS^1(C \times C) \subset H^2_{et}((C \times C')_{\bar{\mathbb{Q}_p}}, \mathbb{Q}_p)(1)^G$$

is of rank $\dim NS^1(C \times C') \geq 3$.

(ii) Let $X \in \mathrm{PSmVar}(K)$. Let $j \in \mathbb{Z}$. Denote

- *$G^{2j}_{mot}(X)$ is the motivic Galois group of $M^{2j}(X)$ and $G^{2j}_{mot}(X_{\mathbb{C}})$ is the motivic Galois group of $M^{2j}(X_{\mathbb{C}})$,*
- *$G_{MT}(X_{\mathbb{C}}) := \overline{h_X(\mathbb{S}_1)}^{\mathbb{Q}}$ the Munford Tate group of $X_{\mathbb{C}}$ where $h_X : \mathbb{S}_1 \to G_{mot}(X_{\mathbb{C}})$ is the morphism given by the Hodge structure of $H^{2j}(X^{an}_{\mathbb{C}}, \mathbb{Q})$.*

The Hodge conjecture for X implies that $G_{mot}^{2j}(X_{\mathbb{C}})/G_{MT}^{2j}(X_{\mathbb{C}})$ acts trivially on $H^{2j}(X_{\mathbb{C}}^{an},\mathbb{Q})^{G_{MT}^{j}(X_{\mathbb{C}})}$. Hence we expect that the action of $G_{mot}(X)$ on $H_{et}^{2j}(X_{\mathbb{C}_p},\mathbb{Q}_p)$ is given by a torus.

It is well known that theorem 3 implies the following :

Corollary 2. *Let $X \in \mathrm{PSmVar}(\mathbb{C})$.*

(i) The standard conjectures holds for X.

(ii) Let $k \subset \mathbb{C}$ be a subfield of finite type over $\mathbb{Q}$ over which X is defined, that is $X \simeq X_k \otimes_k \mathbb{C}$ in $\mathrm{PSmVar}(\mathbb{C})$ with $X_k \in \mathrm{PSmVar}(k)$. For $\theta \in Aut(\mathbb{C}/\mathbb{Q})$, we have the isomorphism $\theta : X \xrightarrow{\sim} X_\theta$ in Sch. *Then $\theta \in Aut(\mathbb{C}/\mathbb{Q})$ induces, for each $j \in \mathbb{Z}$, an isomorphism*

$$\theta : H^j_{\mathrm{sing}}(X^{an},\mathbb{C}) \xrightarrow{H^j ev(X)^{-1}} H^j_{DR}(X) \xrightarrow{\theta^*} H^j_{DR}(X_\theta) \xrightarrow{H^j ev(X_\theta)} H^j_{\mathrm{sing}}(X_\theta^{an},\mathbb{C}).$$

Let $d \in \mathbb{N}$. Let $\alpha \in F^dH^{2d}(X^{an},\mathbb{Q})$ where $F^dH^{2d}(X^{an},\mathbb{Q}) := F^dH^{2d}_{DR}(X) \cap H^{2d}_{\mathrm{sing}}(X^{an},\mathbb{Q}) \subset H^{2d}(X^{an},\mathbb{C})$. If $\theta(\alpha) \in H^{2d}_{\mathrm{sing}}(X_\theta^{an},\mathbb{Q})$ for all $\theta \in Aut(\mathbb{C}/k)$, then $\alpha = [Z]$ with $Z \in \mathcal{Z}^d(X)$.

Proof. Standard. We consider, for $j \in \mathbb{Z}$, $k' \subset \mathbb{C}$ a subfield and $Y \in \mathrm{Var}(k')$, the canonical morphism

$$T(Y) := T^j(Y) : H^i_{\mathrm{sing}}(Y_{\mathbb{C}}^{an},\mathbb{Q}) \xrightarrow{(/p^n)_{n\in\mathbb{N}}} H^j_{\mathrm{sing}}(Y_{\mathbb{C}}^{an},\mathbb{Q}_p) \xrightarrow{\sim} H^j_{et}(Y_{\mathbb{C}},\mathbb{Q}_p) \xrightarrow{\sim} H^j_{et}(Y_{\bar{k}'},\mathbb{Q}_p).$$

For $Y \in \mathrm{Var}(k')$ and $\theta \in Aut(\mathbb{C}/k')$, we have the commutative diagrams in $C(Y \times \mathbb{N})$

$$\begin{array}{ccccc}
\mathrm{An}_* E_{usu}(\mathbb{Z}) & \xrightarrow{(/p^n)_{n\in\mathbb{N}}} & \mathrm{An}_* E_{usu}(\mathbb{Z}_p) & \xleftarrow{\mathrm{ad}(\mathrm{An}^*,\mathrm{An}_*)(\mathbb{Z}_p)} & E_{et}\mathbb{Z}_{p,Y_{\mathbb{C}}} \\
\downarrow{\scriptstyle \mathrm{ad}(\theta^*,\theta_*)(-)} & & \downarrow{\scriptstyle \mathrm{ad}(\theta^*,\theta_*)(-)} & & \downarrow{\scriptstyle \mathrm{ad}(\theta^*,\theta_*)(-)} \\
\theta_*\theta^* \mathrm{An}_* E_{usu}(\mathbb{Z}) & \xrightarrow{(/p^n)_{n\in\mathbb{N}}} & \theta_*\theta^* \mathrm{An}_* E_{usu}(\mathbb{Z}_p) & \xleftarrow{\theta_*\theta^* \mathrm{ad}(\mathrm{An}^*,\mathrm{An}_*)(\mathbb{Z}_p)} & \theta_*\theta^* E_{et}\mathbb{Z}_{p,Y_{\mathbb{C}}} = E_{et}\mathbb{Z}_{p,Y_{\mathbb{C},\theta}} \\
 & \searrow^{\mathrm{ad}(\mathrm{An}_\theta^*,\mathrm{An}_{\theta*})(\mathbb{Z}_p)} & & & \\
\mathrm{An}_{\theta*} E_{usu}(\mathbb{Z}) & \xrightarrow{(/p^n)_{n\in\mathbb{N}}} & \mathrm{An}_{\theta*} E_{usu}(\mathbb{Z}_p) & &
\end{array}$$

and

$$\begin{array}{ccccc}
\mathrm{An}_* E_{usu}(\mathbb{Z}) & \xrightarrow{\iota_{2i\pi\mathbb{Z}/\mathbb{C}}(Y_{\mathbb{C}}^{an})} & \mathrm{An}_* E_{usu}(\Omega^\bullet_{Y_{\mathbb{C}}^{an}}) & \xleftarrow{\Omega(\mathrm{An})} & E_{et}\Omega^\bullet_{Y_{\mathbb{C}}} \\
\downarrow{\scriptstyle \mathrm{ad}(\theta^*,\theta_*)(-)} & & \downarrow{\scriptstyle \mathrm{ad}(\theta^*,\theta_*)(-)} & & \downarrow{\scriptstyle \mathrm{ad}(\theta^*,\theta_*)(-)} \\
\theta_*\theta^* \mathrm{An}_* E_{usu}(\mathbb{Z}) & \xrightarrow{\iota_{2i\pi\mathbb{Z}/\mathbb{C}}(Y_{\mathbb{C}}^{an})} & \theta_*\theta^* \mathrm{An}_* E_{usu}(\Omega^\bullet_{Y_{\mathbb{C}}^{an}}) & \xleftarrow{\theta_*\theta^*\Omega(\mathrm{An})} & \theta_*\theta^* E_{et}\Omega^\bullet_{Y_{\mathbb{C}}} = E_{et}\Omega^\bullet_{Y_{\mathbb{C},\theta}} \\
 & \searrow^{\Omega(\mathrm{An}_\theta)} & & & \\
\mathrm{An}_{\theta*} E_{usu}(\mathbb{Z}) & \xrightarrow{\iota_{2i\pi\mathbb{Z}/\mathbb{C}}(Y_{\mathbb{C},\theta}^{an})} & \mathrm{An}_{\theta*} E_{usu}(\Omega^\bullet_{Y_{\mathbb{C},\theta}^{an}}) & &
\end{array}$$

where $\mathrm{An} : Y_{\mathbb{C}}^{an} \to Y$ and $\mathrm{An}_\theta : Y_{\mathbb{C},\theta}^{an} \to Y$ are the analytical functors, usu denote the usual complex topology and E_{usu} is the canonical flasque resolution (see section 2.1). Hence for $Y \in \mathrm{SmVar}(k')$ and $\alpha \in H^j_{\mathrm{sing}}(Y_{\mathbb{C}}^{an},\mathbb{Q})$ such that $\theta(\alpha) \in H^j_{\mathrm{sing}}(Y_{\mathbb{C},\theta}^{an},\mathbb{Q})$ for some $\theta \in Aut(\mathbb{C}/k')$, where

$$\theta : H^j_{\mathrm{sing}}(Y_{\mathbb{C}}^{an},\mathbb{C}) \xrightarrow{H^j ev(Y)^{-1}} H^j_{DR}(Y_{\mathbb{C}}) \xrightarrow{\theta^*} H^j_{DR}(Y_{\mathbb{C},\theta}) \xrightarrow{H^j ev(Y_{\mathbb{C},\theta})} H^j_{\mathrm{sing}}(Y_{\mathbb{C},\theta}^{an},\mathbb{C}),$$

we have

$$\theta^* T(Y)(\alpha) = T(Y_\theta)(\theta(\alpha)) \in H^j_{et}(Y_{\bar{k}'},\mathbb{Q}_p).$$

(i): Let $k \subset \mathbb{C}$ be a subfield of finite type over $\mathbb{Q}$ such that X is defined. Denote $G = Gal(\bar{k}/k)$ the Galois group. Denote $d_X = \dim(X)$ and for short $X^{an} = X$. Let $L^j \in H^{2d_X-2j}_{\text{sing}}(X \times X, \mathbb{Q})$ the class inducing the cup product with of the intersection of j hyperplane sections on $H^{d_X-j}_{\text{sing}}(X, \mathbb{Q})$ and zero on $H^i_{\text{sing}}(X, \mathbb{Q})$, $i \neq d_X - j$. Let $\Lambda^j \in H^{2d_X+2j}_{\text{sing}}(X \times X, \mathbb{Q})$ inducing the inverse of L_j on $H^{d_X+j}_{\text{sing}}(X, \mathbb{Q})$ and zero on $H^i_{\text{sing}}(X, \mathbb{Q})$, $i \neq d_X + j$. Consider an isomorphism $\sigma_p : \mathbb{C} \xrightarrow{\sim} \mathbb{C}_p$ such that $\hat{k}_{\sigma_p} \subset \mathbb{C}_p$ is unramified and X has good reduction. We have $T(X \times X)(L^j) \in H^{2d_X-2j}_{et}((X \times X)_{\bar{k}}, \mathbb{Q}_p)^G$ since it is the class of an algebraic cycle. Hence,

$$T(X \times X)(\Lambda^j) \in H^{2d_X+2j}_{et}((X \times X)_{\bar{k}}, \mathbb{Q}_p)(d_X + j)^G$$

since the inverse of a Galois invariant morphism is Galois invariant. Hence, by theorem 3(i),

$$\Lambda^j = [Z], \text{ with } Z \in \mathcal{Z}^{d_X+j}(X \times X) \otimes \mathbb{Q}.$$

(ii): As $\alpha \in H^{2d}\Gamma(X_\bullet, \Omega_X)$, where $X = \cup_{i=1}^s X_i$ is an open affine cover, there exists a subfield $k \subset k' \subset \mathbb{C}$ of finite type over $\mathbb{Q}$ over which α is defined, that is $\alpha \in H^{2d}_{DR}(X_{k'}) \subset H^{2d}_{DR}(X)$, where $X_{k'} := X_k \otimes_k k' \in \text{PSmVar}(k')$ satisfy $X_{k'} \otimes_{k'} \mathbb{C} \simeq X$ in $\text{PSmVar}(\mathbb{C})$. Let $p \in \mathbb{N} \backslash \delta(k', X_{k'})$ be a prime number. Consider

$$T(X_{k'})(\alpha) \in H^{2d}(X_{\bar{k}'}, \mathbb{Q}_p).$$

If $\theta(\alpha) \in H^{2d}_{\text{sing}}(X^{an}_\theta, \mathbb{Q})$ for all $\theta \in Aut(\mathbb{C}/k')$, we get

$$T(X_{k'})(\alpha) = 1/\#(Aut(\mathbb{C}/k')\alpha) \sum_{\theta \in Aut(\mathbb{C}/k')} \theta^{-1,*} T(X_{k',\theta})(\theta(\alpha)) \in H^{2d}_{et}(X_{\bar{k}'}, \mathbb{Q}_p) = H^{2d}_{et}(X, \mathbb{Q}_p),$$

hence, for $g \in Gal(\bar{k}'/k')$, we get

$$\begin{aligned} g \cdot T(X_{k'})(\alpha) &= 1/\#(Aut(\mathbb{C}/k')\alpha) \sum_{\theta \in Aut(\mathbb{C}/k')} g \cdot \theta^{-1,*} T(X_{k',\theta})(\theta(\alpha)) \\ &= 1/\#(Aut(\mathbb{C}/k')\alpha) \sum_{\theta \in Aut(\mathbb{C}/k')} \theta^{-1,*} T(X_{k',\theta})(\theta(\alpha)) = T(X_{k'})(\alpha) \in H^{2d}_{et}(X_{\bar{k}'}, \mathbb{Q}_p), \end{aligned}$$

that is, $T(X_{k'})(\alpha) \in H^{2d}(X_{\bar{k}'}, \mathbb{Q}_p)(d)^G$, with $G := Gal(\bar{k}'/k)$. Hence, if $\theta(\alpha) \in H^{2d}_{\text{sing}}(X^{an}_\theta, \mathbb{Q})$ for all $\theta \in Aut(\mathbb{C}/k)$, then in particular $\theta(\alpha) \in H^{2d}_{\text{sing}}(X^{an}_\theta, \mathbb{Q})$ for all $\theta \in Aut(\mathbb{C}/k')$ and we get by theorem 3(i),

$$T(X)(\alpha) = T(X_{k'})(\alpha) = [Z], \text{ with } Z \in \mathcal{Z}^d(X) \otimes \mathbb{Q}.$$

□

We get the following. See [12] for conditions for families where the hypothesis of the theorem holds.

Theorem 4. *Let $X \in \text{PSmVar}(\mathbb{C})$, $X = V(I) \subset \mathbb{P}^N_{\mathbb{C}}$. Consider the canonical deformation $f : \mathcal{X} = V(\tilde{I}) \subset \mathbb{P}^N_{\mathbb{Q}} \times S \to S \subset \bar{S}$, with $S \subset \bar{S}$ the open subset over which f is smooth, where $\bar{S} := \bar{0_X}^{\mathbb{Q}} \subset \mathbb{A}^r_{\mathbb{Q}}$, $\mathcal{X} \in \text{SmVar}(\mathbb{Q})$, and $X = \mathcal{X}_s$ with $s := 0_X \in S_{\mathbb{C}}$. Denote $(E^{2d}_{DR}(\mathcal{X}/S), F) := H^{2d} \int_f (O_{\mathcal{X}}, F_b) \in \text{Vect}_{\mathcal{D}}(S)$ and*

$$HL^{d,2d}(\mathcal{X}/S) := F^d E^{2d}_{DR}(\mathcal{X}_{\mathbb{C}}/S_{\mathbb{C}}) \cap f_* ev(\mathcal{X})^{-1}(R^{2d} f_* \mathbb{Q}_{\mathcal{X}^{an}_{\mathbb{C}}}) \subset E^{2d}_{DR}(\mathcal{X}_{\mathbb{C}}/S_{\mathbb{C}})$$

the locus of Hodge classes. Let $\lambda \in F^d H^{2d}(X^{an}, \mathbb{Q})$. If the irreducible components $W \subset HL^{d,2d}(\mathcal{X}/S)$ of the locus of Hodge classes such that $\lambda \in W$ are defined over $\bar{\mathbb{Q}}$ and if their Galois conjugates $\sigma(W)$ with $\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ are also components of $HL^{d,2d}(\mathcal{X}/S)$, then $\lambda = [Z] \in H^{2d}(X^{an}, \mathbb{Q})$ with $Z \in \mathcal{Z}^d(X, \mathbb{Q})$.

Proof. We have $\lambda \in \cup_{i=1}^{s} W_i$ where $W_i \subset HL^{d,2d}(\mathcal{X}/S)$ are the irreducible components passing through λ. If the irreducible components $W \subset HL^{d,2d}(\mathcal{X}/S)$ of the locus of Hodge classes such that $\lambda \in W$ are defined over $\bar{\mathbb{Q}}$ and if their Galois conjugates $\sigma(W)$ with $\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ are also components of $HL^{d,2d}(\mathcal{X}/S)$, we get

$$\bar{\lambda}^{\mathbb{Q}} \subset \cup_{i=1}^{s} \cup_{\sigma} \sigma(W_i)$$

since $\cup_{i=1}^{s} \cup_{\sigma} (W_i)$ is then defined over $\mathbb{Q}$. Since $\bar{\lambda}^{\mathbb{Q}} = \overline{\{\theta^* \lambda, \theta \in Aut(\mathbb{C}, \mathbb{Q})\}}$ we get

$$\overline{\{\theta^* \lambda, \theta \in Aut(\mathbb{C}, \mathbb{Q})\}} \subset HL^{d,2d}(\mathcal{X}/S).$$

In particular, for all $\theta \in Aut(\mathbb{C}, \mathbb{Q})$, $\theta(\lambda) \in H^{2d}_{\text{sing}}(X_\theta^{an}, \mathbb{Q})$. Hence, by corollary 2(ii) $\lambda = [Z] \in H^{2d}(X^{an}, \mathbb{Q})$ with $Z \in \mathcal{Z}^d(X, \mathbb{Q})$. □

References

[1] F.Andreatta, A.Iovitta, *Semi-stable sheaves and comparaison isomorphisms in the semi-stable case*, Rencontre de seminaire mathematique de l'universite de Padova, Tome 128, 2012, p.131-286.

[2] B.Bhatt, P.Scholze, *The pro-etale topology for schemes*,Asterisque No 369, 2015, 99-201.

[3] J.Bouali, *The De Rham, complex and p adic Hodge realization functor for relative motives of algebraic varieties over a field of characteristic zero*, arxiv 2203.10875, preprint, 2022.

[4] J.Bouali *Complex vs etale Abel Jacobi map for higher Chow groups and algebraicity of the zero locus of etale normal functions*, arxiv 2211.15317, preprint, 2022.

[5] D.C.Cisinski, F.Deglise, *Triangulated categories of mixed motived*, arxiv preprint 0912.2110, 2009-arxiv.org.

[6] , O.Gregory, C.Liedtke, *p-adic Tate conjectures and abeloid varieties*, Doc. Math. 24, 1879-1934, 2019.

[7] L.Illusie, *Complexe de de Rham-Witt et cohomologie cristalline*, Annales scientifiques de l'ENS, tome 12, 1979, p.501-661.

[8] F.Lecomte, N.Wach, *Réalisation de Hodge des motifs de Voevodsky*, manuscripta mathematica.141,663-697, Springer-Verlag, Berlin, 2012.

[9] C.Mazza, V.Voevodsky, C.Weibel, *Lecture Notes on Motivic Cohomology*, Clay mathematics monographs, Vol 2, AMS, Providence RI, 2006.

[10] W.Raskind, *A generalized Hodge-Tate conjecture for algebraic varieties with totally degenerate reduction over p-adic fields*, Tandon, R. (eds) Algebra and Number Theory, 2005.

[11] P.Scholze, *p-adic Hodge theory for rigid analytic varieties*, Forum of Mathematics, Pi,1,el, 2013.

[12] C.Voisin, Hodge loci and absolute Hodge classes, Compositio Math. 143, 945-958, 2007.

LAGA UMR CNRS 7539
Université Paris 13, Sorbonne Paris Cité, 99 av Jean-Baptiste Clement,
93430 Villetaneuse, France,
bouali@math.univ-paris13.fr