De Rham logarithmic classes and Tate conjecture

Johann Bouali

De Rham logarithmic classes and Tate conjecture

Johann Bouali

March 17, 2023

Abstract

We introduce the definition of De Rham logarithmic classes. We show that the De Rham class of an algebraic cycle of an algebraic variety over a field of characteristic zero is logarithmic and conversely that a logarithmic class of bidegree (d,d) is the De Rham class of an algebraic cycle (of codimension d). We deduce from a previous work the Tate conjecture for smooth projective varieties over fields of finite type over $\mathbb{Q}$, over p-adic fields and over field of characteristic p, p being a prime number.

1 Introduction

In this work, we introduce for X a noetherian scheme, the definition of logarithmic De Rham cohomology classes which are for each $j \in \mathbb{Z}$, the subgroup $H^jOL_X(\mathbb{H}^j_{et}(X,\Omega^\bullet_{X,\log})) \subset \mathbb{H}^j_{et}(X,\Omega^\bullet_X) =: H^j_{DR}(X)$ of the De Rham cohomology abelian group. Let X a smooth connected variety over a field of characteristic zero. To an algebraic cycle $Z \in \mathcal{Z}^d(X)$ one can associate its De Rham cohomology class $[Z] \in H^{2d}_{DR}(X)$, which is by definition, as for any Weil cohomology theory, the image of the fundamental class $[Z] \in H^{2d}_{DR,Z}(X)$ by the canonical morphism $H^{2d}_{DR,Z}(X) \to H^{2d}(X)$. In section 3, we show (c.f. theorem 1(i)) that $[Z] \in H^{2d}_{DR}(X)$is logarithmic of bidegree (d,d), that is

$$[Z] = H^{2d}OL_X([Z]) \in H^{2d}_{DR}(X),\ [Z] \in H^{2d}OL_X(H^d_{et}(X,\Omega^d_{X,\log}))$$

where $H^d_{et}(X,\Omega^d_{X,\log}) \subset \mathbb{H}^{2d}_{et}(X,\Omega^\bullet_{X,\log})$ is the canonical subspace. The main ingredient is that the Euler class of a vector bundle of rank d over an algebraic variety is logarithmic of bidegree (d,d) (c.f. proposition 1). Conversely, we show (c.f. theorem 1(ii)), that a logarithmic class $w \in H^{2d}_{DR}(X)$ of bidegree (d,d) is the De Rham class of an algebraic cycle (of codimension d). The point is that a logarithmic class of bidegree (p,q) is locally acyclic for the Zariski topology of X since it is the etale cohomology of a single sheaf. This allows us to proceed by a finite induction and we use at the final step the fact that for a scheme X', $H^1_{et}(X',\Omega^1_{X',\log}) = H^1(X',O^*_{X'})$ is the Picard group of X'. In the case X projective, we also get a vanishing result (c.f. theorem 1(iii)) : $H^{2d}OL_X(H^{d-k}_{et}(X,\Omega^{d-k}_{X,\log})) = 0$ for $k > 0$.

In a previous work (see [4], prop 2,3), we show that Tate classes $\alpha \in H^{2d}(X_{\bar{k}},\mathbb{Z}_p)(d)^G$ of a smooth algebraic variety X over a subfield of $\sigma_p : k \hookrightarrow \mathbb{C}_p$ gives by the comparison isomorphism

$$H^{2d}R\alpha(X) : H^{2d}_{et}(X_{\mathbb{C}_p},\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{B}_{st,\mathbb{C}_p} \xrightarrow{\sim} H^{2d}_{DR}(X) \otimes_k \mathbb{B}_{st,\mathbb{C}_p},$$

a sum of $\mathbb{Z}_p$ multiple of logarithmic classes

$$w(\alpha) := H^{2d}R\alpha(X)(\alpha) := \sum_{i=1}^{s} \lambda_i w(\alpha)_{L,i} \in H^{2d}_{DR}(X_{\mathbb{C}_p}),\ \lambda_i \in \mathbb{Z}_p$$

with for each $1 \leq i \leq s$, $w(\alpha)_{L,i} \in H^{2d}OL_X\mathbb{H}^{2d}_{et}(X_{\mathbb{C}_p},\Omega^{\geq d}_{X_{\mathbb{C}_p},\log})$. We deduce, for X projective, by theorem 1(ii) and (iii), the Tate cojecture (c.f. theorem 2).

I am grateful for professor F.Mokrane for help and support during this work.

2 Preliminaries and Notations

- Denote by Top the category of topological spaces and RTop the category of ringed spaces.
- Denote by Cat the category of small categories and RCat the category of ringed topos.
- For $\mathcal{S} \in \mathrm{Cat}$ and $X \in \mathcal{S}$, we denote $\mathcal{S}/X \in \mathrm{Cat}$ the category whose objects are $Y/X := (Y, f)$ with $Y \in \mathcal{S}$ and $f : Y \to X$ is a morphism in $\mathcal{S}$, and whose morphisms $\mathrm{Hom}((Y', f'), (Y, f))$ consists of $g : Y' \to Y$ in $\mathcal{S}$ such that $f \circ g = f'$.
- For $\mathcal{S} \in \mathrm{Cat}$ denote $\mathrm{Gr}\,\mathcal{S} := \mathrm{Fun}(\mathbb{Z}, \mathcal{S})$ is the category of graded objects.
- Denote by Ab the category of abelian groups. For R a ring denote by $\mathrm{Mod}(R)$ the category of (left) R modules. We have then the forgetful functor $o_R : \mathrm{Mod}(R) \to \mathrm{Ab}$.
- Denote by AbCat the category of small abelian categories.
- For $(\mathcal{S}, O_S) \in \mathrm{RCat}$ a ringed topos, we denote by
  - $\mathrm{PSh}(\mathcal{S})$ the category of presheaves of O_S modules on $\mathcal{S}$ and $\mathrm{PSh}_{O_S}(\mathcal{S})$ the category of presheaves of O_S modules on $\mathcal{S}$, whose objects are $\mathrm{PSh}_{O_S}(\mathcal{S})^0 := \{(M, m), M \in \mathrm{PSh}(\mathcal{S}), m : M \otimes O_S \to M\}$, together with the forgetful functor $o : \mathrm{PSh}(\mathcal{S}) \to \mathrm{PSh}_{O_S}(\mathcal{S})$,
  - $C(\mathcal{S}) = C(\mathrm{PSh}(\mathcal{S}))$ and $C_{O_S}(\mathcal{S}) = C(\mathrm{PSh}_{O_S}(\mathcal{S}))$ the big abelian category of complexes of presheaves of O_S modules on $\mathcal{S}$,
  - $C_{O_S(2)fil}(\mathcal{S}) := C_{(2)fil}(\mathrm{PSh}_{O_S}(\mathcal{S})) \subset C(\mathrm{PSh}_{O_S}(\mathcal{S}), F, W)$, the big abelian category of (bi)filtered complexes of presheaves of O_S modules on $\mathcal{S}$ such that the filtration is biregular and $\mathrm{PSh}_{O_S(2)fil}(\mathcal{S}) = (\mathrm{PSh}_{O_S}(\mathcal{S}), F, W)$.
- Let $(\mathcal{S}, O_S) \in \mathrm{RCat}$ a ringed topos with topology τ. For $F \in C_{O_S}(\mathcal{S})$, we denote by $k : F \to E_\tau(F)$ the canonical flasque resolution in $C_{O_S}(\mathcal{S})$ (see [3]). In particular for $X \in \mathcal{S}$, $H^*(X, E_\tau(F)) \xrightarrow{\sim} \mathbb{H}^*_\tau(X, F)$.
- For $f : \mathcal{S}' \to \mathcal{S}$ a morphism with $\mathcal{S}, \mathcal{S}' \in \mathrm{RCat}$, endowed with topology τ and τ' respectively, we denote for $F \in C_{O_S}(\mathcal{S})$ and each $j \in \mathbb{Z}$,
  - $f^* := H^j\Gamma(\mathcal{S}, k \circ \mathrm{ad}(f^*, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \to \mathbb{H}^j(\mathcal{S}', f^*F)$,
  - $f^* := H^j\Gamma(\mathcal{S}, k \circ \mathrm{ad}(f^{*mod}, f_*)(F)) : \mathbb{H}^j(\mathcal{S}, F) \to \mathbb{H}^j(\mathcal{S}', f^{*mod}F)$,

  the canonical maps.
- For $\mathcal{X} \in \mathrm{Cat}$ a (pre)site and p a prime number, we consider the full subcategory

$$\mathrm{PSh}_{\mathbb{Z}_p}(\mathcal{X}) \subset \mathrm{PSh}(\mathbb{N} \times \mathcal{X}),\ \ F = (F_n)_{n\in\mathbb{N}},\ p^n F_n = 0,\ F_{n+1}/p^n \xrightarrow{\sim} F_n$$

  $C_{\mathbb{Z}_p}(\mathcal{X}) := C(\mathrm{PSh}_{\mathbb{Z}_p}(\mathcal{X})) \subset C(\mathbb{N} \times \mathcal{X})$ and

$$\mathbb{Z}_p := \mathbb{Z}_{p,\mathcal{X}} := ((\mathbb{Z}/p^*\mathbb{Z})_{\mathcal{X}}) \in \mathrm{PSh}_{\mathbb{Z}_p}(\mathcal{X})$$

  the diagram of constant presheaves on $\mathcal{X}$.
- Denote by $\mathrm{Sch} \subset \mathrm{RTop}$ the subcategory of schemes (the morphisms are the morphisms of locally ringed spaces). For $X \in \mathrm{Sch}$, we denote by
  - $\mathrm{Sch}^{ft}/X \subset \mathrm{Sch}/X$ the full subcategory consisting of objects $X'/X = (X', f) \in \mathrm{Sch}/X$ such that $f : X' \to X$ is an morphism of finite type
  - $X^{et} \subset \mathrm{Sch}^{ft}/X$ the full subcategory consisting of objects $U/X = (X, h) \in \mathrm{Sch}/X$ such that $h : U \to X$ is an etale morphism.

– $X^{sm} \subset \mathrm{Sch}^{ft}/X$ the full subcategory consisting of objects $U/X = (X, h) \in \mathrm{Sch}/X$ such that $h : U \to X$ is a smooth morphism.

For a field k, we consider $\mathrm{Sch}/k := \mathrm{Sch}/\operatorname{Spec} k$ the category of schemes over $\operatorname{Spec} k$. We then denote by

– $\mathrm{Var}(k) = \mathrm{Sch}^{ft}/k \subset \mathrm{Sch}/k$ the full subcategory consisting of algebraic varieties over k, i.e. schemes of finite type over k,
– $\mathrm{PVar}(k) \subset \mathrm{QPVar}(k) \subset \mathrm{Var}(k)$ the full subcategories consisting of quasi-projective varieties and projective varieties respectively,
– $\mathrm{PSmVar}(k) \subset \mathrm{SmVar}(k) \subset \mathrm{Var}(k)$ the full subcategories consisting of smooth varieties and smooth projective varieties respectively.

For a morphism of field $\sigma : k \hookrightarrow K$, we have the extention of scalar functor

$$\otimes_k K : \mathrm{Sch}/k \to \mathrm{Sch}/K,\ X \mapsto X_K := X_{K,\sigma} := X \otimes_k K,\ (f : X' \to X) \mapsto (f_K := f \otimes I : X'_K \to X_K).$$

which is left ajoint to the restriction of scalar

$$Res_{k/K} : \mathrm{Sch}/K \to \mathrm{Sch}/k,\ X = (X, a_X) \mapsto X = (X, \sigma \circ a_X),\ (f : X' \to X) \mapsto (f : X' \to X)$$

The adjonction maps are

– for $X \in \mathrm{Sch}/k$, the projection $\pi_{k/K}(X) : X_K \to X$ in Sch/k, for $X = \cup_i X_i$ an affine open cover with $X_i = \operatorname{Spec}(A_i)$ we have by definition $\pi_{k/K}(X_i) = n_{k/K}(A_i)$,
– for $X \in \mathrm{Sch}/K$, $I \times \Delta_K : X \hookrightarrow X_K = X \times_K K \otimes_k K$ in Sch/K, where $\Delta_K : K \otimes_k K \to K$ is the diagonal which is given by for $x, y \in K$, $\Delta_K(x, y) = x - y$.

The extention of scalar functor restrict to a functor

$$\otimes_k K : \mathrm{Var}(k) \to \mathrm{Var}(K),\ X \mapsto X_K := X_{K,\sigma} := X \otimes_k K,\ (f : X' \to X) \mapsto (f_K := f \otimes I : X'_K \to X_K).$$

and for $X \in \mathrm{Var}(k)$ we have $\pi_{k/K}(X) : X_K \to X$ the projection in Sch/k. An algebraic variety $X \in \mathrm{Var}(K)$ is said to be defined over k if there exists $X_0 \in \mathrm{Var}(k)$ such that $X \simeq X_0 \otimes_k K$ in $\mathrm{Var}(K)$. For $X = (X, a_X) \in \mathrm{Var}(k)$, we have $\mathrm{Sch}^{ft}/X = \mathrm{Var}(k)/X$ since for $f : X' \to X$ a morphism of schemes of finite type, $(X', a_X \circ f) \in \mathrm{Var}(k)$ is the unique structure of variety over k of $X' \in \mathrm{Sch}$ such that f becomes a morphism of algebraic varieties over k, in particular we have

– $X^{et} \subset \mathrm{Sch}^{ft}/X = \mathrm{Var}(k)/X$,
– $X^{sm} \subset \mathrm{Sch}^{ft}/X = \mathrm{Var}(k)/X$.

A morphism $f : X' \to X$ with $X, X' \in \mathrm{Var}(K)$ is said to be defined over k if $X \simeq X_0 \otimes_k K$ and $X' \simeq X'_0 \otimes_k K$ are defined over k and $\Gamma_f = \Gamma_{f_0} \otimes_k K \subset X' \times X$ is defined over k, so that $f_0 \otimes_k K = f$ with $f_0 : X'_0 \to X_0$.

- For $X \in \mathrm{Sch}$, we denote $X^{pet} \subset \mathrm{Sch}/X$ the pro etale site (see [2]) which is the full subcategory of Sch/X whose object consists of weakly etale maps $U \to X$ (that is flat maps $U \to X$ such that $\Delta_U : U \to U \times_X U$ is also flat) and whose topology is generated by fpqc covers. We then have the canonical morphism of site

$$\nu_X : X^{pet} \to X, (U \to X) \mapsto (U \to X)$$

For $F \in C(X^{et})$,

$$\mathrm{ad}(\nu_X^*, R\nu_{X*})(F) : F \to R\nu_{X*}\nu_X^* F$$

is an isomorphism in $D(X^{et})$, in particular, for each $n \in \mathbb{Z}$

$$\nu_X^* : \mathbb{H}^n_{et}(X, F) \xrightarrow{\sim} \mathbb{H}^n_{pet}(X, \nu_X^* F)$$

are isomorphisms, For $X \in \mathrm{Sch}$, we denote

- $\underline{\mathbb{Z}_p}_X := \varprojlim_{n\in\mathbb{N}} \nu_X^*(\mathbb{Z}/p^n\mathbb{Z})_{X^{et}} \in \mathrm{PSh}(X^{pet})$ the constant presheaf on $\mathcal{X}$,
- $l_{p,\mathcal{X}} := (p(*)) : \underline{\mathbb{Z}_p}_X \to \nu_X^*(\mathbb{Z}/p\mathbb{Z})_{X^{et}}$ the projection map in $\mathrm{PSh}(\mathbb{N}\times\mathcal{X}^{pet})$.

- An affine scheme $U \in \mathrm{Sch}$ is said to be w-contractible if any faithfully flat weakly etale map $V \to U$, $V \in \mathrm{Sch}$, admits a section. We will use the facts that (see [2]):
  - Any scheme $X \in \mathrm{Sch}$ admits a pro-etale affine cover $(r_i : X_i \to X)_{i\in I}$ with for each $i \in I$, X_i a w-contractile affine scheme and $r_i : X_i \to X$ a weakly etale map. For $X \in \mathrm{Var}(k)$ with k a field, we may assume I finite since the topological space X is then quasi-compact.
  - If $U \in \mathrm{Sch}$ is a w-contractible affine scheme, then for any sheaf $F \in \mathrm{Shv}(U^{pet})$, $H^i_{pet}(U, F) = 0$ for $i \neq 0$ since $\Gamma(U, -)$ is an exact functor.

- Let $X = (X, O_X) \in \mathrm{RCat}$ a ringed topos, we have in $C(X)$ the subcomplex of presheaves of abelian group

$$OL_X : \Omega^\bullet_{X,\log} \hookrightarrow \Omega^\bullet_X, \text{ s.t. for } X^o \in X \text{ and } p \in \mathbb{N},$$
$$\Omega^p_{X,\log}(X^0) :=< df_{\alpha_1}/f_{\alpha_1} \wedge \cdots \wedge df_{\alpha_p}/f_{\alpha_p}, f_\alpha \in \Gamma(X^o, O_X)^* >\subset \Omega^p_X(X^0),$$

  where $\Omega^\bullet_X := DR(X)(O_X) \in C(X)$ is the De Rham complex and $\Gamma(X^o, O_X)^* \subset \Gamma(X^o, O_X)$ is the subring consisting of inversible elements for the multiplication.

- Let $X \in \mathrm{Sch}$. Considering its De Rham complex $\Omega^\bullet_X := DR(X)(O_X)$, we have for $j \in \mathbb{Z}$ its De Rham cohomology $H^j_{DR}(X) := \mathbb{H}^j(X, \Omega^\bullet_X)$. If $X \in \mathrm{SmVar}(k)$, then $H^j_{DR}(X) = \mathbb{H}^j_{et}(X, \Omega^\bullet_X)$ since $\Omega^\bullet \in C(\mathrm{SmVar}(k))$ is $\mathbb{A}^1$ local and admits transfert (see [3]).

- Let $X \in \mathrm{Var}(k)$. Let $X = \cup_{i=1}^s X_i$ an open affine cover. For $I \subset [1, \ldots, s]$, we denote $X_I := \cap_{i\in I} X_i$. We get $X_\bullet \in \mathrm{Fun}(P([1, \ldots, s]), \mathrm{Var}(k))$. Since quasi-coherent sheaves on affine noetherian schemes are acyclic, we have for each $j \in \mathbb{Z}$, $H^j_{DR}(X) = \Gamma(X_\bullet, \Omega^\bullet_{X_\bullet})$.

- For $X \in \mathrm{Sch}$ noetherian irreducible and $d \in \mathbb{N}$, we denote by $\mathcal{Z}^d(X)$ the group of algebraic cycles of codimension d, which is the free abelian group generated by irreducible closed subsets of codimension d.

- Denote by $\mathrm{AnSp}(\mathbb{C}) \subset \mathrm{RTop}$ the full subcategory of analytic spaces over $\mathbb{C}$, and by $\mathrm{AnSm}(\mathbb{C}) \subset \mathrm{AnSp}(\mathbb{C})$ the full subcategory of smooth analytic spaces (i.e. complex analytic manifold). Denote by $\mathrm{CW} \subset \mathrm{Top}$ the full subcategory of CW complexes. Denote by $\mathrm{Diff}(\mathbb{R}) \subset \mathrm{RTop}$ the full subcategory of differentiable (real) manifold.

- For $X \in \mathrm{Top}$ and $Z \subset X$ a closed subset, denoting $j : X\backslash Z \hookrightarrow X$ the open complementary, we will consider
$$\Gamma^\vee_Z\mathbb{Z}_X := \mathrm{Cone}(\mathrm{ad}(j_!, j^*)(\mathbb{Z}_X) : j_!j^*\mathbb{Z}_X \hookrightarrow \mathbb{Z}_X) \in C(X)$$
  and denote for short $\gamma^\vee_Z := \gamma^\vee_X(\mathbb{Z}_X) : \mathbb{Z}_X \to \Gamma^\vee_Z\mathbb{Z}_X$ the canonical map in $C(X)$.

3 De Rham logarithmic classes

Let $X \in \mathrm{Sch}$. Recall we have the canonical subcomplex $OL_X : \Omega^\bullet_{X,\log} \hookrightarrow \Omega^\bullet_X$ in $C(X^{et})$. For $j \in \mathbb{Z}$, the De Rham logarithmic classes consist of the image

$$H^jOL_X(\mathbb{H}^j_{et}(X, \Omega^\bullet_{X,\log})) \subset \mathbb{H}^j_{et}(X, \Omega^\bullet_X) = H^j_{DR}(X).$$

The filtered complex $\Gamma(X, E_{et}(\Omega^\bullet_{X^{et},\log}, F_b)) \in C_{fil}(\mathbb{Z})$ has its differentials at the E_1 level trivial since all the logarithmic forms are closed, hence we have a canonical splitting

$$\mathbb{H}^j_{et}(X, \Omega^\bullet_{X,\log}) = \oplus_{0\leq l\leq j} H^{j-l}_{et}(X, \Omega^l_{X,\log}).$$

Let $X \in \mathrm{Sch}$ a noetherian scheme. We have by definition the exact sequence in $C(X^{et})$

$$0 \to \mathbb{Z}_X \to O_X^* \xrightarrow{dlog} \Omega_{X,\log} \to 0$$

Hence $H^1_{et}(X, \Omega_{X,\log}) = H^1_{et}(X, O_X^*)$ and $H^q_{et}(X, \Omega_{X,\log}) = H^q_{et}(X, O_X^*) = 0$ for $q \leq 2$. Let $X \in \mathrm{Sch}$ a noetherian proper scheme. We have $H^0(X, \Omega^l_{X,\log}) = 0$.

We will use the following lemma :

Lemma 1. *Let $X \in \mathrm{Sch}$. Let $X = \cup_{i\in I} X_i$ an open cover. Denote $j_i : X_i \hookrightarrow X$ the open embeddings. Let $w \in H^j_{DR}(X)$. Then $w \in H^j OL_X(H^{j-l}_{et}(X, \Omega^l_{X,\log}))$ if and only if $j_i^* w \in H^j OL_X(H^{j-l}_{et}(X_i, \Omega^l_{X,\log}))$ for all $i \in I$.*

Proof. The only if part is obvious. Considering the morphism of bicomplexes of abelian groups

$$\Gamma(X_\bullet, E^\bullet_{et}(\Omega^l_{X,\log})) \hookrightarrow \Gamma(X_\bullet, E^\bullet_{et}(\Omega^l_X))$$

the if part follows from Mayer-Vietoris (i.e. etale hence Zariski descent of flasque etale sheaves). □

Let $X \in \mathrm{Sch}$ a smooth connected noetherian scheme (X is quasi-compact since X is noetherian) and $Z \subset X$ a smooth subscheme of codimension d we have for each $j \in \mathbb{Z}$, the purity isomorphism given by excision (using the deformation $(\tilde{X}', \tilde{Z}') \to \mathbb{A}^1$, $(\tilde{X}', \tilde{Z}') \in \mathrm{Sch}^2$ of a pair $(X', Z') \in \mathrm{Sch}^2$ by the normal cone $C_{Z'/X'} \to Z'$, i.e. such that $(\tilde{X}', \tilde{Z}')_s = (X', Z')$ for $s' \in \mathbb{A}^1 \backslash 0$ and $(\tilde{X}', \tilde{Z}')_0 = (C_{Z'/X}, Z')$) and the cup product with the Euler class of of the normal tangent bundle $N_{Z/X} \to Z$:

$$((-) \cdot e(N_{Z/X}))^{-1} : H^j_{DR,Z}(X) = H^j_{DR,Z^o}(N_{Z/X}) \xrightarrow{\sim} H^{j-2d}_{DR}(Z).$$

Let k a field of characteristic zero. For $X \in \mathrm{SmVar}(k)$ connected, we have the cycle class map for De Rham cohomology

$$Z \in \mathcal{Z}^d(X) \mapsto [Z] := H^{2d}\Omega(\gamma_Z^\vee)([Z]) \subset H^{2d}_{DR}(X),\ [Z] \in H^{2d}_{DR,|Z|}(X),$$
$$H^{2d}\Omega(\gamma_Z^\vee) : H^{2d}_{DR,|Z|}(X) \to H^{2d}_{DR}(X)$$

where

$$\Omega(\gamma_Z^\vee) := \mathrm{Hom}(\gamma_Z^\vee(\mathbb{Z}_X), E_{et}(\Omega^\bullet_X)) : \Gamma_Z(X, E_{et}(\Omega^\bullet_X)) \hookrightarrow \Gamma(X, E_{et}(\Omega^\bullet_X))$$

and, as for any Weil cohomology theory, we have the purity isomorphism given by excision and the cup product with the Euler class of of the normal tangent bundle $N_{Z^o/X^o} \to Z^o$, $X^o \subset X$ being a open subset such that $Z^o := X^o \cap Z$ has smooth components :

$$((-) \cdot e(N_{Z^o/X^o}))^{-1} : H^{2d}_{DR,|Z|}(X) = H^{2d}_{DR,|Z^o|}(X^o) = H^{2d}_{DR,|Z^o|}(N_{Z^o/X^o}) \xrightarrow{\sim} H^0_{DR}(|Z^o|).$$

where the first equality follows from dimension reason : for $X \in \mathrm{SmVar}(k)$ and $Z' \subset X$ smooth, we have $H^i_{DR,Z'}(X) = 0$ for $i < 2\,\mathrm{codim}(Z', X)$ by the purity isomorphism.

We have the following key proposition:

Proposition 1. *Let k a field of characteristic zero. Let $p : E \to X$ a vector bundle of rank $d \in \mathbb{N}$ with $X, E \in \mathrm{SmVar}(k)$ connected. Then*

(i) the Euler class $e(E) \in H^{2d}_{DR,X}(E)$ is logarithmic of type (d, d), that is $e(E) = H^{2d}OL_E(e(E)) \in H^{2d}_{DR,X}(E)$ with $e(E) \in H^d_{X,et}(E, \Omega^d_{E,\log})$,

(ii) the Euler class $e(E) \in H^{2d}_{DR,X}(E)$ induces for each $i, j \in \mathbb{Z}$ an isomorphism

$$((-) \cdot e(E)) : H^{i+j}OL_X(H^j_{et}(X, \Omega^i_{X,\log})) \xrightarrow{\sim} H^{2d+i+j}OL_E(H^{j+d}_{et}(E, \Omega^{i+d}_{E,\log})).$$

Proof. (i):Let $X = \cup_i X_i$ an open affine cover such that $E_i := E_{|X_i}$ is trivial : $(s_1, \cdots, s_d) : E_{|X_i} \xrightarrow{\sim} X_i \times \mathbb{A}^d$ with $s_j \in \Gamma(X_i, E)$. Then

$$e(E)_{|X_i} = \partial(ds_1/s_1 \wedge \cdots \wedge ds_d/s_d) \in H^{2d}OL_{E_i}(H^d_{X_i}(E_i, \Omega^d_{E,\log}))$$

where $\partial : H^{d-1}(E_i \backslash X_i, \Omega^d_{E,\log}) \to H^d_{X_i}(E_i, \Omega^d_{E,\log})$ is the boundary map. Hence

$$e(E) \in H^{2d}OL_E(H^0(E, H^d_X\Omega^d_{E,\log})) = H^d_{et,X}(E, \Omega^d_{E,\log}) \subset H^0(E, H^{2d}_X\Omega^{\bullet \geq d}_E) \subset H^{2d}_{DR,X}(E).$$

(ii):Follows from (i) and Kunneth formula for De Rham cohomology together with lemma 1. □

The result of this section is the following :

Theorem 1. *Let k a field of characteristic zero. Let $X \in \mathrm{SmVar}(k)$ connected. Let $d \in \mathbb{N}$.*

(i) The De Rham cohomology class of an algebraic cycle is logarithmic and is of type (d,d), that is, for $Z \in \mathcal{Z}^d(X)$

$$[Z] := H^{2d}\Omega(\gamma_Z^\vee)([Z]) \subset H^{2d}OL_X(H^d_{et}(X, \Omega^d_{X,\log})) \subset H^{2d}_{DR}(X).$$

(ii) Conversely, any $w \in H^{2d}OL_X(H^d_{et}(X, \Omega^d_{X,\log}))$ is the class of an algebraic cycle.

(iii) If $X \in \mathrm{PSmVar}(k)$, $H^{2d}OL_X(H^{d-k}_{et}(X, \Omega^{d+k}_{X,\log})) = 0$ for $k > 0$.

Proof. (i):We have

$$[Z] := H^{2d}\Omega(\gamma_Z^\vee)([Z]) = H^{2d}OL_X \circ H^d\Omega^d_{\log}(\gamma_Z^\vee)([Z]) \subset H^{2d}_{DR}(X),\ [Z] \in H^d_{et,|Z|}(X, \Omega^d_{X,\log})$$

where

$$\Omega^d_{\log}(\gamma_Z^\vee) := \mathrm{Hom}(\gamma_Z^\vee(\mathbb{Z}_X), E_{et}(\Omega^d_{X,\log})) : \Gamma_Z(X, E_{et}(\Omega^d_{X,\log})) \hookrightarrow \Gamma(X, E_{et}(\Omega^d_{X,\log})),$$

since if $X^o \subset X$ is a Zariski open subset such that $Z^o := X^o \cap Z$ has smooth components and $N_{Z^o/X^o} \to Z^o$ is the normal tangent bundle, proposition 1 implies that the Euler class of N_{Z^o/X^o} induces the isomorphism

$$\begin{aligned}((-) \cdot e(N_{Z^o/X^o}))^{-1} : H^{2d}OL_X(H^d_{et,|Z|}(X, \Omega^d_{X,\log})) &= H^{2d}OL_X(H^d_{et,|Z^o|}(X^o, \Omega^d_{X^o,\log})) \\ &= H^{2d}OL_X(H^d_{et,|Z^o|}(N_{Z^o/X^o}, \Omega^d_{N_{Z^o/X^o},\log})).\end{aligned}$$

Note that the excision isomorphism being motivic preserve logarithmic classes.
(ii):By assumption we have

$$w = H^{2d}OL_X(w) \in H^{2d}_{DR}(X),\ w \in H^d_{et}(X, \Omega^d_{X,\log})$$

As $\Omega^d_{X,\log} \in \mathrm{PSh}(X^{et})$ consist of a single presheaf, that is a complex of presheaves concentrated one degree, there exist an etale open cover $r = r(w) = (r_i : X_i \to X)_{1 \leq i \leq s}$, depending on w, such that $r_i^* w = 0 \in H^d_{et}(X_i, \Omega^d_{X,\log})$ for each i. Choose $i = 1$ and denote $j : U := r_1(X_1) \hookrightarrow X$ the corresponding Zariski open subset. As $\Omega^d_{X,\log}$ has no torsion and admits transfert, we have $j^* w = 0 \in H^d_{et}(U, \Omega^d_{X,\log})$. Hence, denoting $D := X \backslash U$, we have

$$w = H^{2d}OL_X(w), w \in \Omega^d_{\log}(\gamma_D^\vee)(H^d_{et,D}(X, \Omega^d_{X,\log})).$$

We may assume, up to shinking U, that $D \subset X$ is a divisor. Denote $D^o \subset D$ its smooth locus and $l : X^o \hookrightarrow X$ a Zariski open subset such that $X^o \cap D = D^o$. We then have by proposition 1

$$l^* w \in ((-) \cdot e(N_{D^o/X^o}))(H^{2d-2}OL_D(H^{d-1}_{et}(D^o, \Omega^{d-1}_{D^o,\log}))).$$

We repeat this procedure with each connected components of D^o instead of X. By a finite induction of d steps, we get

$$w = H^{2d}OL_X(w),\ w \in \Omega^d_{\log}(\gamma_Z^\vee)(H^d_{et,Z}(X, \Omega^d_{X,\log})),$$

with $Z := D_d \subset \cdots \subset D \subset X$ a codimension d Zariski closed subset and $w = [Z] \in H^{2d}_{DR}(X)$, using in the final step the isomorphism

$$H^1_{et}(D^o_{d-1}, \Omega_{D^o_{d-1},\log}) = H^1_{et}(D^o_{d-1}, O^*_{D_{d-1}}) = Pic(D^o_{d-1}).$$

(iii):Follows from the proof of (ii) and the fact that for $X' \in \mathrm{PSmVar}(k)$, $H^0(X', \Omega^l_{X',\log}) = 0$ for all $l > 0$. □

Let $\sigma_p : k \hookrightarrow \mathbb{C}_p$ a subfield. Let $X \in \mathrm{Var}(k)$. We will write for short

$$OL_X := OL_{X^{et}_{\mathbb{C}_p},\mathcal{O}} : \Omega^\bullet_{X^{et}_{\mathbb{C}_p},\log,\mathcal{O}} := \Omega^\bullet_{X^{et}_{O_{\mathbb{C}_p}},\log} \hookrightarrow \Omega^\bullet_{X^{et}_{\mathbb{C}_p}}$$

the embedding in $C(X^{et}_{\mathbb{C}_p})$. We will also consider

$$m \circ (OL_X \otimes I) : \Omega^\bullet_{X^{pet}_{\mathbb{C}_p},\log,\mathcal{O}} \otimes \mathbb{Z}_p := \Omega^\bullet_{X^{pet}_{\mathbb{C}_p},\log,\mathcal{O}} \otimes \underline{\mathbb{Z}_p}_X \to \Omega^\bullet_{X^{pet}_{\mathbb{C}_p}},\ w \otimes \lambda \mapsto \lambda \cdot w$$

in $C(X^{pet}_{\mathbb{C}_p})$. Note that

$$\Omega^\bullet_{X^{pet}_{\mathbb{C}_p},\log,\mathcal{O}} := \nu_X^* \Omega^\bullet_{X^{et}_{\mathbb{C}_p},\log,\mathcal{O}} \in C(X^{pet}),$$

but

$$\underline{\mathbb{Z}_p}_X := \varprojlim_{n\in\mathbb{N}} \nu_X^*(\mathbb{Z}/p^n\mathbb{Z})_{X^{et}} \in C(X^{pet}),\ \text{and } \Omega^\bullet_{X^{pet}_{\mathbb{C}_p},\log,\mathcal{O}} \otimes \mathbb{Z}_p \in C(X^{pet})$$

are NOT the pullback of etale sheaves by ν_X.

Proposition 2. *Let k a field of characteristic zero. Let $X \in \mathrm{SmVar}(k)$. Let $\sigma : k \hookrightarrow \mathbb{C}$ an embedding.*

(i) Let $w \in H^j_{DR}(X) := \mathbb{H}^j(X, \Omega^\bullet_X) = \mathbb{H}^j_{pet}(X, \Omega^\bullet_X)$. If

$$w \in H^j OL_X(\mathbb{H}^j_{pet}(X_{\mathbb{C}}, \Omega^\bullet_{X^{et}_{\mathbb{C}},\log}))$$

then $H^j ev(X)(w) \in H^j_{\mathrm{sing}}(X^{an}_{\mathbb{C}}, 2i\pi\mathbb{Q})$.

(ii) Let $p \in \mathbb{N}$ a prime number and $\sigma_p : k \hookrightarrow \mathbb{C}_p$ an embedding. Let $j \in \mathbb{Z}$. Let $w \in H^j_{DR}(X) := \mathbb{H}^j(X, \Omega^\bullet_X) = \mathbb{H}^j_{pet}(X, \Omega^\bullet_X)$. If

$$w := \pi_{k/\mathbb{C}_p}(X)^* w \in H^j OL_X(\mathbb{H}^j_{et}(X_{\mathbb{C}_p}, \Omega^\bullet_{X^{et}_{\mathbb{C}_p},\log,\mathcal{O}}))$$

then $H^j ev(X)(w) \in H^j_{\mathrm{sing}}(X^{an}_{\mathbb{C}}, 2i\pi\mathbb{Q})$. Recall that $\mathbb{H}^j_{pet}(X_{\mathbb{C}}, \Omega^\bullet_{X^{pet}_{\mathbb{C}_p},\log,\mathcal{O}}) = \mathbb{H}^j_{et}(X_{\mathbb{C}}, \Omega^\bullet_{X^{et}_{\mathbb{C}_p},\log,\mathcal{O}})$.

Proof. (i): Let

$$w \in H^j_{DR}(X) := \mathbb{H}^j(X, \Omega^\bullet_X) = H^j\Gamma(X_\bullet, \Omega^\bullet_X).$$

where $(r_i : X_i \to X)_{1\le i\le s}$ is an affine etale cover. Let $X^{an}_{\mathbb{C}} = \cup_{i=1}^r \mathbb{D}_i$ an open cover with $\mathbb{D}_i \simeq D(0,1)^d$. Denote $j_{IJ} : X_I \cap \mathbb{D}_J \hookrightarrow X_I$ the open embeddings. Then by definition $H^j ev(X)(w) = H^j ev(X^{an}_{\mathbb{C}})(j^*_\bullet \circ \mathrm{an}^*_{X_\bullet} w)$ with

$$j^*_\bullet \circ \mathrm{an}^*_{X_\bullet} w \in H^j\Gamma(X^{an}_{\bullet,\mathbb{C}} \cap \mathbb{D}_\bullet, \Omega^\bullet_{X^{an}_{\mathbb{C}}}).$$

Now, if $w = H^j OL_X(\mathbb{H}^j_{et}(X_{\mathbb{C}}, \Omega^\bullet_{X^{et}_{\mathbb{C}},\log}))$, we have a canonical splitting

$$w = \sum_{l=0}^j w_L^{l,j-l} = \sum_{l=d}^j w^{l,j-l} \in H^j_{DR}(X_{\mathbb{C}}),\ w_L^{l,j-l} \in H^{j-l}(X_{\mathbb{C}}, \Omega^l_{X^{et}_{\mathbb{C}},\log}),\ w^{l,j-l} := H^j OL_{X^{et}_{\mathbb{C}}}(w_L^{l,j-l}).$$

Let $0 \leq l \leq j$. Using an affine w-contractile pro-etale cover of X, we see that there exists an affine etale cover $r = r(w^{l,j-l}) = (r_i : X_i \to X)_{1 \leq i \leq n}$ of X (depending on $w^{l,j-l}$) such that

$$w^{l,j-l} = [(w^{l,j-l}_{L,I})_I] \in H^j OL_{X^{et}_{\mathbb{C}}}(H^{j-l}\Gamma(X_{\mathbb{C},\bullet}, \Omega^l_{X_{\mathbb{C}},\log})) \subset \mathbb{H}^j\Gamma(X_{\mathbb{C},\bullet}, \Omega^\bullet_{X_{\mathbb{C}}}).$$

Note that since X is an algebraic variety, this also follows from a comparison theorem between Chech cohomology of etale covers and etale cohomology. By [4] lemma , we may assume, up to take a desingularization $\pi : X' \to X$ of $(X, \cup_i(r_i(X \backslash X_i)))$ and replace w with $\pi^* w$, that $r_i(X_i) = r_i(X_i(w)) = X \backslash D_i$ with $D_i \subset X$ smooth divisors with normal crossing For $1 \leq l \leq j$, we get

$$w^{l,j-l}_{L,I} = \sum_\nu df_{\nu_1}/f_{\nu_1} \wedge \cdots \wedge df_{\nu_l}/f_{\nu_l} \in \Gamma(X_{\mathbb{C},I}, \Omega^l_{X_{\mathbb{C}}}).$$

For $l = 0$, we get

$$w^{0,j} = [(\lambda_I)] \in H^j\Gamma(X_{\mathbb{C},I}, O_{X_{\mathbb{C}_p,I}}),\ \lambda_I \in \Gamma(X_{\mathbb{C},I}, \mathbb{Z}_{X^{et}_{\mathbb{C},I}})$$

There exists $k' \subset \mathbb{C}$ containing k such that $w^{l,j-l}_{L,I} \in \Gamma(X_{k',I}, \Omega^l_{X_{k'}})$ for all $0 \leq l \leq j$. Take an embedding $\sigma' : k' \hookrightarrow \mathbb{C}$ such that $\sigma'_{|k} = \sigma$. We then have

$$j^*_\bullet \circ \mathrm{an}^*_{X_\bullet} w = j^*_\bullet((m_l \cdot w^{l,j-l}_L)_{0 \leq l \leq j}) = (w^{l,j-l}_{L,I,J})_{l,I,J} \in H^j\Gamma(X^{an}_{\bullet,\mathbb{C}} \cap \mathbb{D}_\bullet, \Omega^\bullet_{X^{an}_{\mathbb{C}}}).$$

where for each (I, J, l) with $cardI + cardJ + l = j$,

$$w^{l,j-l}_{L,I,J} := j^*_{IJ} w^{l,j-l}_{L,I} \in \Gamma(X^{an}_{I,\mathbb{C}} \cap \mathbb{D}_J, \Omega^l_{X^{an}_{\mathbb{C}}}).$$

We have by a standard computation, for each (I, J, l) with $cardI + cardJ + l = j$,

$$H^*_{\mathrm{sing}}(X^{an}_{I,\mathbb{C}} \cap \mathbb{D}_J, \mathbb{Z}) =< \gamma_1, \cdots, \gamma_{cardI} > .$$

On the other hand,

- $w^{l,j-l}_{L,I,J} = j^*_J(\sum_\nu df_{\nu_1}/f_{\nu_1} \wedge \cdots \wedge df_{\nu_l}/f_{\nu_l}) \in \Gamma(X^{an}_{I,\mathbb{C}} \cap \mathbb{D}_J, \Omega^l_{X^{an}_{\mathbb{C}}})$ for $1 \leq l \leq j$,
- $w^{0,j}_{L,I,J} = \lambda_I$ is a constant.

Hence, for $\mu \in P([1, \cdots, s])$ with $card\mu = l$, we get, for $l = 0$ $H^l ev(X^{an}_{\mathbb{C}})_{I,J}(w^{0,j}_{L,I,J}) = 0$ and, for $1 \leq l \leq j$,

$$H^l ev(X^{an}_{\mathbb{C}})_{I,J}(w^{l,j-l}_{L,I,J})(\gamma_\mu) = \sum_k \delta_{\nu,\mu} \in 2i\pi\mathbb{Z}.$$

where $\gamma_\mu := \gamma_{\mu_1} \cdots \gamma_{\mu_l}$. We conclude by [4] lemma.
(ii):See [4] proposition 1. Note that it follows immediately from (i). □

Let k a field of characteristic zero. Let $X \in \mathrm{SmVar}(k)$. Let $X = \cup_{i=1}^s X_i$ an open affine cover with $X_i := X \backslash D_i$ with $D_i \subset X$ smooth divisors with normal crossing. Let $\sigma : k \hookrightarrow \mathbb{C}$ an embedding. Let $p \in \mathbb{N}$ a prime number and $\sigma_p : k \hookrightarrow \mathbb{C}_p$ an embedding. By proposition 2, we have a commutative diagram of graded algebras

$$\begin{array}{ccc} H^*_{DR}(X) & \xrightarrow{H^* ev(X)} & H^*_{\mathrm{sing}}(X^{an}_{\mathbb{C}}, \mathbb{C}) \\ \subset \uparrow & & \uparrow H^* C^* \iota_{2i\pi\mathbb{Q}/\mathbb{C}}(X^{an}_{\mathbb{C}}) \\ H^* OL_X(\mathbb{H}^*_{pet}(X_{\mathbb{C}_p}, \Omega^\bullet_{X^{pet}_{\mathbb{C}_p},\log,\mathcal{O}})) \cap H^*_{DR}(X) & \xrightarrow{H^* ev(X)} & H^*_{\mathrm{sing}}(X^{an}_{\mathbb{C}}, 2i\pi\mathbb{Q}) \end{array}$$

where

$$C^*\iota_{2i\pi\mathbb{Q}/\mathbb{C}}(X^{an}_{\mathbb{C}}) : C^{\bullet}_{\text{sing}}(X^{an}_{\mathbb{C}}, 2i\pi\mathbb{Q}) \hookrightarrow C^{\bullet}_{\text{sing}}(X^{an}_{\mathbb{C}}, \mathbb{C})$$

is the subcomplex consiting of $\alpha \in C^j_{\text{sing}}(X^{an}_{\mathbb{C}}, \mathbb{C})$ such that $\alpha(\gamma) \in 2i\pi\mathbb{Q}$ for all $\gamma \in C^{\text{sing}}_j(X^{an}_{\mathbb{C}}, \mathbb{Q})$. Recall that

$$H^*ev(X_{\mathbb{C}}) = H^*R\Gamma(X^{an}_{\mathbb{C}}, \alpha(X)) \circ \Gamma(X^{an}_{\mathbb{C}}, E_{zar}(\Omega(\text{an}_X))) : H^*_{DR}(X_{\mathbb{C}}) \xrightarrow{\sim} H^*_{\text{sing}}(X^{an}_{\mathbb{C}}, \mathbb{C})$$

is the canonical isomorphism induced by the analytical functor and $\alpha(X) : \mathbb{C}_{X^{an}_{\mathbb{C}}} \hookrightarrow \Omega^{\bullet}_{X^{an}_{\mathbb{C}}}$, which gives the periods elements $H^*ev(X)(H^*_{DR}(X)) \subset H^*_{\text{sing}}(X^{an}_{\mathbb{C}}, \mathbb{C})$. On the other side the induced map

$$H^*ev(X_{\mathbb{C}}) : H^*OL_X(\mathbb{H}^*_{et}(X_{\mathbb{C}}, \Omega^{\bullet}_{X^{et}_{\mathbb{C}},\log})) \hookrightarrow H^*\iota_{2i\pi\mathbb{Q}/\mathbb{C}}H^*_{\text{sing}}(X^{an}_{\mathbb{C}}, 2i\pi\mathbb{Q})$$

is NOT surjective in general since the left hand side is invariant by the action of the group $Aut(\mathbb{C})$ (the group of field automorphism of $\mathbb{C}$) whereas the right hand side is not. The fact for a de Rham cohomology class of being logarithmic is algebraic and invariant under isomorphism of (abstract) schemes.

Corollary 1. *Let $X \in \text{PSmVar}(\mathbb{C})$. Then the Hodge conjecture holds for X if and only if the Hodge classes are given by logarithmic De Rham classes.*

Proof. Follows from theorem 1. □

4 Tate conjecture

Let k a field of finite type over $\mathbb{Q}$. Denote $\bar{k}$ the algebraic closure of k and $G = Gal(\bar{k}/k)$ the absolute Galois group of k. Let $X \in \text{SmVar}(k)$ a smooth variety. Take a compactification $\bar{X} \in \text{PSmVar}(k)$ of X such that $D := \bar{X}\backslash X \subset X$ is a normal crossing divisor, and denote $j : X \hookrightarrow \bar{X}$ the open embedding. Let $p \in \mathbb{N}$ a prime number. Consider an embedding $\sigma_p : k \hookrightarrow \mathbb{C}_p$. Then $k \subset \bar{k} \subset \mathbb{C}_p$ and $k \subset \hat{k}_{\sigma_p} \subset \mathbb{C}_p$, where $\hat{k}_{\sigma_p}$ is the p-adic field which is the completion of k with respect the p adic norm given by σ_p. Denote $\hat{G}_{\sigma_p} = Gal(\mathbb{C}_p/\hat{k}_{\sigma_p}) = Gal(\bar{\mathbb{Q}}_p/\hat{k}_{\sigma_p})$ the Galois group of $\hat{k}_{\sigma_p}$. Recall (see section 2) that $\underline{\mathbb{Z}_p}_{X_{\mathbb{C}_p}} := \varprojlim_{n\in\mathbb{N}} \nu^*_X(\mathbb{Z}/p^n\mathbb{Z})_{X^{et}_{\mathbb{C}_p}} \in \text{Shv}(X^{pet}_{\mathbb{C}_p})$ and $\Omega^{\bullet}_{X^{pet}_{\mathbb{C}_p},\log,\mathcal{O}} := \Omega^{\bullet}_{X^{pet}_{O_{\mathbb{C}_p}},\log} \in C(X^{pet}_{\mathbb{C}_p})$. We have then the commutative diagram in $C_{\mathbb{B}_{dr}fil,\hat{G}_{\sigma_p}}(\bar{X}^{an,pet}_{\mathbb{C}_p})$

$$\begin{array}{ccc} j_*E_{pet}(\mathbb{B}_{dr,X_{\mathbb{C}_p}}, F) & \xrightarrow{j_*E_{pet}(\alpha(X))} & E_{pet}((\Omega^{\bullet}_{X_{\mathbb{C}_p}}(\log D_{\mathbb{C}_p}), F_b) \otimes_{O_{\bar{X}_{\mathbb{C}_p}}} (O\mathbb{B}_{dr,\bar{X}_{\mathbb{C}_p},\log}, F)) \\ {\scriptstyle E_{pet}(j_*\iota'_{X^{pet}_{\mathbb{C}_p}})_j := E_{pet}(l\mapsto l.1)_j}\uparrow & & \uparrow{\scriptstyle E_{pet}(m\circ(OL_X\otimes I)) := E_{pet}((w\otimes\lambda)\mapsto(w\otimes\lambda))} \\ j_*E_{pet}(\underline{\mathbb{Z}_p}_{X_{\mathbb{C}_p}}) & \xrightarrow{j_*E_{pet}(\iota_{X^{pet}_{\mathbb{C}_p}})} & j_*E_{pet}(\Omega^{\bullet}_{X_{\mathbb{C}_p},\log,\mathcal{O}} \otimes \underline{\mathbb{Z}_p}_{X_{\mathbb{C}_p}}, F_b) \end{array},$$

where for $j' : U' \hookrightarrow X'$ an open embedding with $X' \in \text{RTop}$ and τ a topology on RTop we denote for $m : j_*Q \to Q'$ with $Q \in \text{PSh}_O(U')$, $Q' \in \text{PSh}_O(X')$ the canonical map in $C_O(X')$

$$E^0_\tau(m)_j : j_*E^0_\tau(Q) \to E^0_\tau(j_*Q) \xrightarrow{E^0_\tau(m)} E^0_\tau(Q'),$$

giving by induction the canonical map $E_\tau(m)_j : j_*E_\tau(Q) \to E_\tau(Q')$ in $C_O(X')$. Let $p \in \mathbb{N} \in \delta(k, X)$ a prime number (any but finitely many) so that $\hat{k}_{\sigma_p} = Frac(W(O_{\hat{k}_{\sigma_p}}/pO_{\hat{k}_{\sigma_p}}))$ and so that the canonical model $(\bar{X}^{\mathcal{O}}, D^{\mathcal{O}}) \in \text{Sch}^2/O_{\hat{k}_{\sigma_p}}$ of $(\bar{X}_{\hat{k}_{\sigma_p}}, D_{\hat{k}_{\sigma_p}}) \in \text{PSmVar}^2(\hat{k}_{\sigma_p})$ has good or semi-stable reduction modulo p. We have then the morphisms of sites

$$v_{X,N} : (X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}, N_{U,\mathcal{O}})^{Falt} \to (X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}, N_{U,\mathcal{O}})^{ket},\ u_{X,N} : (X_{\mathbb{C}_p}, M_{Z_{\mathbb{C}_p}})^{ket} \to (X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}, N_{U,\mathcal{O}})^{Falt}$$

where $(X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}, N_{U,\mathcal{O}})^{Falt}$ denote the Falting site, and for $(Y,N) \in \log\mathrm{Sch}$, $(Y,N)^{ket} \subset \log\mathrm{Sch}/(Y,N)$ is the small Kummer etale site. If $(X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}, N_{U,\mathcal{O}})$ is log smooth, we consider an hypercover

$$a_\bullet : (X^{\mathcal{O},\bullet}_{\hat{k}_{\sigma_p}}, N_{U,\mathcal{O}}) \to (X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}, N_{U,\mathcal{O}})$$

in $\mathrm{Fun}(\Delta, \log\mathrm{Sch})$ by small log schemes in sense of [1]. The main result of [1] say that the embedding in $C((\bar{X}^{\mathcal{O}}_{\hat{k}_{\sigma_p}})^{Falt})$

$$\alpha(X) : \mathbb{B}_{st,\bar{X}_{\hat{k}_{\sigma_p}},\log} \hookrightarrow a_{\bullet *}\Omega^\bullet_{X^{\mathcal{O},\bullet}_{\hat{k}_{\sigma_p}}}(\log D_{\hat{k}_{\sigma_p}}) \otimes_{O_{X^{\mathcal{O}}}} O\mathbb{B}_{st,\bar{X}^\bullet_{\hat{k}_{\sigma_p}},\log}$$

is a filtered quasi-isomorphism compatible with the action of the Frobenius ϕ_p and the monodromy N. This gives if $(X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}, N_{U,\mathcal{O}})$ is log smooth, for each $j \in \mathbb{Z}$, a filtered isomorphism of filtered abelian groups

$$H^j R\alpha(X) : H^j_{et}(X_{\mathbb{C}_p}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{B}_{st,\hat{k}_{\sigma_p}} \xrightarrow{H^j T(a_X, \mathbb{B}_{st})^{-1}} H^j_{et}((X,N)^{Falt})(\mathbb{B}_{st,\bar{X}_{\hat{k}_{\sigma_p}},\log})$$
$$\xrightarrow{H^j R\Gamma((\bar{X}^{\mathcal{O}}_{\hat{k}_{\sigma_p}}, D_{\hat{k}_{\sigma_p}}), \alpha(X))} H^j_{DR}(X_{\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}} \mathbb{B}_{st,\hat{k}_{\sigma_p}}$$

compatible with the action of $Gal(\mathbb{C}_p/\hat{k}_{\sigma_p})$, of the Frobenius ϕ_p and the monodromy N.

Definition 1. *Let k a field of finite type over $\mathbb{Q}$. Denote $\bar{k}$ the algebraic closure of k and $G = Gal(\bar{k}/k)$ the absolute Galois group of k. Let $X \in \mathrm{SmVar}(k)$ a smooth variety. Take a compactification $\bar{X} \in \mathrm{PSmVar}(k)$ of X such that $D := \bar{X}\backslash X \subset X$ is a normal crossing divisor, and denote $j : X \hookrightarrow \bar{X}$ the open embedding. Let $p \in \mathbb{N}\backslash\delta(k,X)$ a prime number. Consider an embedding $\sigma_p : k \hookrightarrow \mathbb{C}_p$. Then $k \subset \bar{k} \subset \mathbb{C}_p$ and $k \subset \hat{k}_{\sigma_p} \subset \mathbb{C}_p$, where $\hat{k}_{\sigma_p}$ is the p-adic field which is the completion of k with respect the p adic norm given by σ_p. For $\alpha \in H^j_{et}(X_{\mathbb{C}_p}, \mathbb{Z}_p)$, we consider*

$$w(\alpha) := H^j R\alpha(X)(\alpha \otimes 1) \in H^j_{DR}(X_{\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}} \mathbb{B}_{st,\hat{k}_{\sigma_p}}.$$

the associated de Rham class by the p adic periods. We recall

$$H^j R\alpha(X) : H^j_{et}(X_{\mathbb{C}_p}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{B}_{st,\hat{k}_{\sigma_p}} \xrightarrow{\sim} H^j_{DR}(X_{\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}} \mathbb{B}_{st,\hat{k}_{\sigma_p}}$$

is the canonical filtered isomorphism compatible with the action of $\hat{G}_{\sigma_p}$, of the Frobenius ϕ_p and the monodromy N.

We have the following key propositions (see [4]):

Proposition 3. *Let k a field of finite type over $\mathbb{Q}$. Let $X \in \mathrm{SmVar}(k)$. Take a compactification $\bar{X} \in \mathrm{PSmVar}(k)$ of X such that $D := \bar{X}\backslash X \subset X$ is a normal crossing divisor. Let $p \in \mathbb{N} \in \delta(k,X)$ a prime number. Consider an embedding $\sigma_p : k \hookrightarrow \mathbb{C}_p$. Let $j,l \in \mathbb{Z}$. We have then*

$$F^l H^j_{DR}(X_{\mathbb{C}_p}) \cap H^j_{et}(X_{\mathbb{C}_p}, \mathbb{Z}_p) = H^j(m \circ (OL_X \otimes I))(\Omega^{\bullet \geq l}_{\bar{X}_{\mathbb{C}_p},\log,\mathcal{O}} \otimes \mathbb{Z}_p) \subset H^j_{DR}(X_{\hat{k}_{\sigma_p}}) \otimes_{\hat{k}_{\sigma_p}} \mathbb{B}_{st,\hat{k}_{\sigma_p}}$$

where we recall the map in $C_{fil}(X^{pet}_{\mathbb{C}_p})$

$$m \circ (OL_X \otimes I) : \Omega^\bullet_{X_{\mathbb{C}_p},\log,\mathcal{O}} \otimes \mathbb{Z}_p \to \Omega^\bullet_{X_{\mathbb{C}_p}}(\log D_{\mathbb{C}_p}),\ w \otimes \lambda \mapsto \lambda \cdot w$$

given in section 2. Hence for $\alpha \in H^j_{et}(X_{\mathbb{C}_p}, \mathbb{Z}_p)$ such that $w(\alpha) \in F^l H^j_{DR}(X_{\mathbb{C}_p})$ (see definition 1), there exists $(\lambda_i)_{1 \leq i \leq n} \in \mathbb{Z}_p$ and

$$(w_{Li})_{1 \leq i \leq n} \in \mathbb{H}^j_{pet}(X_{\mathbb{C}_p}, \Omega^{\bullet \geq l}_{\bar{X}_{\mathbb{C}_p},\log,\mathcal{O}})$$

such that

$$w(\alpha) = \sum_{1 \leq i \leq n} \lambda_i \cdot w_{Li} \in \mathbb{H}^j_{pet}(X_{\mathbb{C}_p}, \Omega^{\bullet \geq l}_{\bar{X}_{\mathbb{C}_p}}(\log D_{\mathbb{C}_p})) = F^l H^j_{DR}(X_{\mathbb{C}_p}),\ w_{Li} := H^j OL_X(w_{Li}).$$

Proof. See [4] proposition 2. We recall the proof. By [1], $\alpha \in H^j_{et}(X_{\mathbb{C}_p}, \mathbb{Z}_p)$ is such that $w(\alpha) \in F^l H^j_{DR}(X_{\mathbb{C}_p})$ if and only if there exists an w-contractile pro etale cover $(r_i : X_i \to X)_{1\leq i\leq r}$ such that

$$w(\alpha) \in H^j\Gamma(X_{\bullet,\mathbb{C}_p}, \Omega^{\bullet\geq l}_{X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}}(\log D^{\mathcal{O}}_{\hat{k}_{\sigma_p}}) \otimes_{O_{\bar{X}^{\mathcal{O}}}} O\mathbb{B}_{st,\bar{X}_{\hat{k}_{\sigma_p}},\log})^{\phi_p},$$

note that the differentials of the De Rham complex $DR(X_{\mathbb{C}_p})(O\mathbb{B}_{st,\bar{X}_{\hat{k}_{\sigma_p}},\log})$ commutes with the Frobenius action. On the other hand, the sequence

$$0 \to \Gamma(X_{\bullet,\mathbb{C}_p}, \Omega^{\bullet\geq l}_{X_{\mathbb{C}_p},\log,\mathcal{O}} \otimes \mathbb{Z}_p) \xrightarrow{\Gamma(X_{\bullet,\mathbb{C}_p}, m\circ(OL_X\otimes I))} \Gamma(X_{\bullet,\mathbb{C}_p}, \Omega^{\bullet\geq l}_{X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}}(\log D^{\mathcal{O}}_{\hat{k}_{\sigma_p}}) \otimes_{O_{X^{\mathcal{O}}}} O\mathbb{B}_{st,X_{\hat{k}_{\sigma_p}}})$$
$$\xrightarrow{\Gamma(X_{\bullet,\mathbb{C}_p}, (\phi_p - I)\otimes_{\hat{k}_{\sigma_p}} \mathbb{C}_p)} \Gamma(X_{\bullet,\mathbb{C}_p}, \Omega^{\bullet\geq l}_{X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}}(\log D^{\mathcal{O}}_{\hat{k}_{\sigma_p}}) \otimes_{O_{X^{\mathcal{O}}}} O\mathbb{B}_{st,X_{\hat{k}_{\sigma_p}}}) \to 0$$

is exact since it is exact modulo p^n for each $n \in \mathbb{N}$, $n \leq 1$ by [5] for $n = 1$ and devissage for $n \leq 2$, and by the properties of w-contractile pro-etale affine covers (see [2]). Hence $\alpha \in H^j_{et}(X_{\mathbb{C}_p}, \mathbb{Z}_p)$ is such that $w(\alpha) \in F^l H^j_{DR}(X_{\mathbb{C}_p})$ if and only if

$$w(\alpha) \in H^j\Gamma(X_{\bullet,\mathbb{C}_p}, \Omega^{\bullet\geq l}_{X^{\mathcal{O}}_{\hat{k}_{\sigma_p}}}(\log D^{\mathcal{O}}_{\hat{k}_{\sigma_p}}) \otimes_{O_{\bar{X}^{\mathcal{O}}}} O\mathbb{B}_{st,\bar{X}_{\hat{k}_{\sigma_p}},\log})^{\phi_p}$$
$$= H^j\Gamma(X_\bullet, m\circ(OL_X\otimes I))(H^j\Gamma(X_{\bullet,\mathbb{C}_p}, \Omega^{\bullet\geq l}_{X_{\mathbb{C}_p},\log,\mathcal{O}} \otimes \mathbb{Z}_p)) = \mathbb{H}^j_{pet}(X_{\mathbb{C}_p}, \Omega^{\bullet\geq l}_{X_{\mathbb{C}_p},\log,\mathcal{O}} \otimes \mathbb{Z}_p).$$

□

Proposition 4. *Let k a field of finite type over $\mathbb{Q}$. Denote $\bar{k}$ the algebraic closure of k. Denote $G := Gal(\bar{k}/k)$ its absolute Galois group. Let $X \in \mathrm{SmVar}(k)$ a smooth variety. Take a compactification $\bar{X} \in \mathrm{PSmVar}(k)$ of X such that $D := \bar{X}\backslash X \subset X$ is a normal crossing divisor, and denote $j : X \hookrightarrow \bar{X}$ the open embedding. Let $p \in \mathbb{N}$ a prime number. Consider an embedding $\sigma_p : k \hookrightarrow \mathbb{C}_p$. Denote $k \subset \hat{k}_{\sigma_p} \subset \mathbb{C}_p$ being the p-adic competion with respect to the p adic norm induced by σ_p. Then $\hat{G}_{\sigma_p} := Gal(\mathbb{C}_p/\hat{k}_{\sigma_p}) \subset G := Gal(\bar{k}/k)$. Let $\alpha \in H^j_{et}(X_{\mathbb{C}_p}, \mathbb{Z}_p)$. Consider then its associated De Rham class (see definition 1)*

$$w(\alpha) := H^j R\alpha(X)(\alpha \otimes 1) \in H^j_{DR}(X_{\mathbb{C}_p}) \otimes_{\mathbb{C}_p} \mathbb{B}_{dr,\mathbb{C}_p}.$$

Then $\alpha \in H^j_{et}(X_{\mathbb{C}_p}, \mathbb{Z}_p)(l)^{\hat{G}_{\sigma_p}}$ if and only if $w(\alpha) \in F^l H^j_{DR}(X_{\hat{k}_{\sigma_p}}) = F^l H^j_{DR}(X_{\mathbb{C}_p}) \cap H^j_{DR}(X_{\hat{k}_{\sigma_p}})$. That is we have

$$H^j_{et}(X_{\mathbb{C}_p}, \mathbb{Z}_p)(l)^{\hat{G}_{\sigma_p}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p =< F^l H^j_{DR}(X_{\hat{k}_{\sigma_p}}) \cap H^j_{et}(X_{\mathbb{C}_p}, \mathbb{Z}_p) >_{\mathbb{Q}_p} \subset H^j_{et}(X_{\mathbb{C}_p}, \mathbb{Z}_{p,X^{et}}) \otimes \mathbb{B}_{dr,\mathbb{C}_p},$$

where $< - >_{\mathbb{Q}_p}$ denote the $\mathbb{Q}_p$ vector space generated by $(-)$. Note that $H^j_{et}(X_{\mathbb{C}_p}, \mathbb{Z}_p)$ and $F^l H^j_{DR}(X_{\hat{k}_{\sigma_p}})$ are canonically embedded as subabelian groups of $H^j_{et}(X_{\mathbb{C}_p}, \mathbb{Z}_{p,X^{et}}) \otimes \mathbb{B}_{dr,\mathbb{C}_p}$ by $(-)\otimes 1$ and $\alpha(X)\circ((-)\otimes 1)$ respectively.

Proof. Follows immediately from the fact that $H^j R\alpha(X)$ is a filtered quasi-isomorphism compatible with the Galois action of $\hat{G}_{\sigma_p}$ by [1]. □

These propositions, together with the results of section 3 implies the Tate conjecture :

Theorem 2. *(i) Let k a field of finite type over $\mathbb{Q}$. Let $X \in \mathrm{PSmVar}(k)$. Then the Tate conjecture holds for X.*

(ii) Let p a prime number. Let K a p-adic field, that is a finite extension of $\mathbb{Q}_p$. Let $X \in \mathrm{PSmVar}(K)$. Then the Tate conjecture holds for X.

(iii) Let p a prime number. Let k_0 a field of characteristic p. Let $X_0 \in \mathrm{PSmVar}(k_0)$. Then the Tate conjecture holds for X_0.

Proof. (i):Let $p \in \mathbb{N}$ a prime number. Consider an embedding $\sigma_p : k \hookrightarrow \mathbb{C}_p$. Then $k \subset \bar{k} \subset \mathbb{C}_p$ and $k \subset \hat{k}_{\sigma_p} \subset \mathbb{C}_p$, where $\hat{k}_{\sigma_p}$ is the p-adic field which is the completion of k with respect the p adic norm given by σ_p. Let $\alpha \in H^{2d}_{et}(X_{\bar{k}}, \mathbb{Z}_p)(d)^G$. Using definition 1, by proposition 3 and proposition 4 there exists $(\lambda_i)_{1\leq i\leq n} \in \mathbb{Z}_p$ and

$$(w_{Li})_{1\leq i\leq n} \in \mathbb{H}^{2d}_{pet}(X_{\mathbb{C}_p}, \Omega^{\bullet\geq d}_{X_{\mathbb{C}_p},\log,\mathcal{O}})$$

such that

$$w(\alpha) = \sum_{1\leq i\leq n} \lambda_i \cdot w_{Li} \in \mathbb{H}^{2d}_{pet}(X_{\mathbb{C}_p}, \Omega^{\bullet\geq d}_{X_{\mathbb{C}_p}}) = F^d H^{2d}_{DR}(X_{\mathbb{C}_p}),\ w_{Li} := H^{2d}OL_X(w_{Li}).$$

By theorem 1(ii) and (iii), there exists $Z_i \in \mathcal{Z}^d(X_{\mathbb{C}_p})$ such that $w_{Li} = [Z_i] \in H^{2d}_{DR}(X_{\mathbb{C}_p})$. Hence,

$$Z := \sum_{1\leq i\leq n} \lambda_i [Z_i] \in \mathcal{Z}^d(X_{\mathbb{C}_p}) \otimes \mathbb{Q}_p$$

satisfies $w(\alpha) = [Z] \in H^{2d}_{DR}(X_{\mathbb{C}_p})$. By [1], we get $\alpha = [Z] \in H^{2d}_{et}(X_{\bar{k}}, \mathbb{Z}_p)$. Since $[Z]$ is G invariant,

$$Z' := 1/card(gZ, g \in G) \sum_{g\in G} gZ \in \mathcal{Z}^d(X) \otimes \mathbb{Q}_p$$

satisfy $[Z'] = [Z] = \alpha \in H^{2d}_{et}(X_{\bar{k}}, \mathbb{Z}_p)$.
(ii):Similar to (i).
(iii):Follows from (ii) by taking $X \in \mathrm{PSmVar}(K)$ with $K := Frac(W(k_0))$, such that $X \otimes_K k_0 = X_0$. □

References

[1] F.Andreatta, A.Iovitta, *Semi-stable sheaves and comparaison isomorphisms in the semi-stable case*, Rencontre de seminaire mathematique de l'universite de Padova, Tome 128, 2012, p.131-286.

[2] B.Bhatt, P.Scholze, *The pro-etale topology for schemes*,Asterisque No 369, 2015, 99-201.

[3] J.Bouali, *The De Rham, complex and p adic Hodge realization functor for relative motives of algebraic varieties over a field of characteristic zero*, arxiv 2203.10875, preprint, 2022.

[4] J.Bouali *Complex vs etale Abel Jacobi map for higher Chow groups and algebraicity of the zero locus of etale normal functions*, arxiv 2211.15317, preprint, 2022.

[5] L.Illusie, *Complexe de de Rham-Witt et cohomologie cristalline*, Annales scientifiques de l'ENS, tome 12, 1979, p.501-661.

LAGA UMR CNRS 7539
Université Paris 13, Sorbonne Paris Cité, 99 av Jean-Baptiste Clement,
93430 Villetaneuse, France,
bouali@math.univ-paris13.fr