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Robust energy a posteriori estimates for nonlinear elliptic problems

André Harnist^{†§1} Koondanibha Mitra^{‡2} Ari Rappaport^{†§3} Martin Vohralík^{†§4}

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Abstract

In this paper, we design a posteriori estimates for finite element approximations of nonlinear elliptic problems satisfying strong-monotonicity and Lipschitz-continuity properties. These estimates include, and build on, any iterative linearization method that satisfies a few clearly identified assumptions; this encompasses the Picard, Newton, and Zarantonello linearizations. The estimates give a guaranteed upper bound on an augmented energy difference (reliability with constant one), as well as a lower bound (efficiency up to a generic constant). We prove that for the Zarantonello linearization, this generic constant only depends on the space dimension, the mesh shape regularity, and possibly the approximation polynomial degree in four or more space dimensions, making the estimates robust with respect to the strength of the nonlinearity. For the other linearizations, there is only a computable dependence on the local variation of the linearization operators. Numerical experiments illustrate and validate the theoretical results, for both smooth and singular solutions.

Key words: nonlinear elliptic problem, finite elements, iterative linearization, energy difference, a posteriori error estimate, robustness, equilibrated flux reconstruction

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1 Introduction

Nonlinear elliptic problems are of paramount importance in a broad range of domains such as physics, mechanics, economics, biology, and medicine, see, e.g. [2, 22, 32, 39, 40]. Numerical discretization methods then serve to deliver approximate solutions, upon employing iterative linearizations to resolve the arising discrete nonlinear systems, see, e.g., [7, 16, 18, 34, 43] and the references therein.

Given a numerical approximation, there arises the important question of the error with respect to the exact solution. This is practically handled by the so-called a posteriori estimates. For nonlinear problems, these have been proposed, amongst others, in [3, 10, 11, 17, 21, 26, 28, 29, 33, 35, 37, 38, 39, 43]. One step further, the estimates can be used to adaptively steer the numerical approximation, and recently, convergence and optimality results have been obtained in [5, 9, 24, 25, 27, 30], see also the references therein.

A crucial property of an a posteriori estimate is its robustness, assessing whether its quality is independent of the parameters. In the present setting, we specifically use the term robustness if the chosen error measure and the associated estimate are uniformly equivalent, for any strength of the nonlinearity. This resumes to the same overestimation factor (effectivity index) for linear, mildly nonlinear, and highly nonlinear problems. Unfortunately, robustness is often not achieved; we illustrate this in Figure 1. There, we present the effectivity indices for three common error measures: the energy norm (L^2 norm of the difference of the gradients), the (square root of the) difference of the energies, and the dual norm of the residual (cf., e.g., [23] for their mutual comparisons). We employ guaranteed equilibrated flux estimates following [8, 12, 15], for the nonlinear problem of example of Section 5 below. We can observe that the estimate in the energy norm setting is not robust with respect to the strength of the nonlinearity (the effectivity indice explodes as the ratio a_c/a_m from (2.1) below grows). The dual norm of the residual leads to robustness, as proven in [10, 17, 21]. Though the dual norm of the residual is indeed localizable, cf. [6] and the references therein, it may be criticised as it does not take into account the nonlinearity (an incorporation has recently been addressed in [36]). The energy difference is therefore showing a numerical robustness, but the theoretical estimates are depending of the ratio a_c/a_m . Our aim is to obtain robust error estimates with respect to this ratio.

In this paper, we focus on nonlinear elliptic problems of the form: find $u : \Omega \rightarrow \mathbb{R}$ such that

$$-\nabla \cdot (a(\cdot, |\nabla u|) \nabla u) = f \quad \text{in } \Omega, \tag{1.1a}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.1b}$$

where $a : \Omega \times [0, \infty) \rightarrow (0, \infty)$ is a nonlinear function, satisfying assumptions of Lipschitz continuity and strong monotonicity (cf. (2.1) below). We employ a finite element approximation of (1.1) and an iterative linearization, yielding the approximation u_ℓ^k on each mesh \mathcal{T}_ℓ and linearization step k . Our aim is to provide an a posteriori estimate on the error between u and u_ℓ^k that is robust with respect to the strength of the nonlinearity, i.e., the quality of the estimate is independent of the nonlinear function a . The iterative linearization method needs to satisfy a few clearly identified assumptions. We will show that this is satisfied for usual linearizations such as Picard, Newton, or Zarantonello.

We consider the energy $\mathcal{E}_{N,\ell}^k$ of the nonlinear problem (cf. (3.9a)) and its a posteriori estimator $\eta_{N,\ell}^k$ (cf. (3.9b)). In order to achieve our goals, in addition to the energy functional given by the minimization

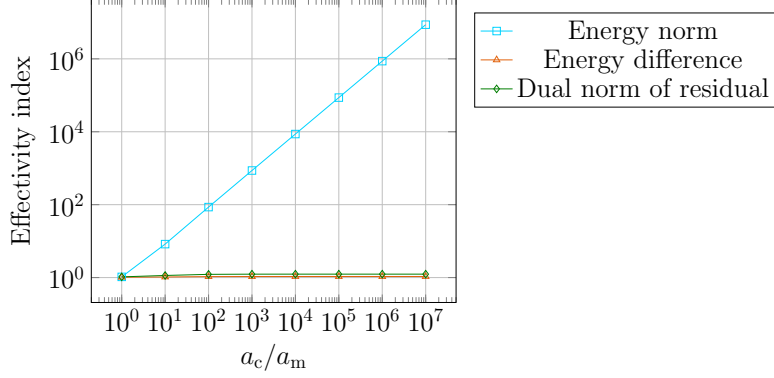


Figure 1: [Exponential nonlinearity (5.5), smooth solution (5.3), Newton solver, 25 DOFs] Comparison of the effectivity index (given as the ratio of the estimate over error) of different error measures and associated a posteriori estimates.

problem associated with (1.1). We consider a linearized energy difference $\mathcal{E}_{L,\ell}^k$ (cf. (3.11a)) and the associated estimator $\eta_{L,\ell}^k$ (cf. (3.11b)). We thus augment $\mathcal{E}_{N,\ell}^k$ by $\mathcal{E}_{L,\ell}^k$ to form \mathcal{E}_ℓ^k and similarly for the estimators. Our main result, Theorem 3.4, can be stated as follows. Assuming f piecewise polynomial for simplicity, if the iterative index k satisfies a computable stopping criterion, we have

$$\mathcal{E}_\ell^k \leq \eta_\ell^k \lesssim C_\ell^k \mathcal{E}_\ell^k, \quad (1.2)$$

where the hidden constant depends only on the space dimension, the mesh shape-regularity, and possibly on the polynomial degree of the finite element method when the spatial dimension is greater than or equal to 4. Here, C_ℓ^k is a computable constant depending on local variation of the linearization matrices. We show that $C_\ell^k = 1$ in the case of the Zarantonello iteration, making the estimate robust with respect to the strength of the nonlinearity in any case. For the other linearizations, the estimates are robust if the computed C_ℓ^k is small and insensitive of a (a posteriori verification of robustness for each given case).

An additional a posteriori estimate given in Theorem 4.2, which can be summarized as follows. Assuming f piecewise polynomial for simplicity, if the mesh index ℓ and the linearization index k satisfy an identified stopping criterion, we have

$$\mathcal{E}_{N,\ell}^k \leq \eta_{N,\ell}^k \lesssim C_\ell^k \tilde{C}_\ell^k \mathcal{E}_{N,\ell}^k, \quad (1.3)$$

with the same dependence as in (1.2) for the hidden constant, and where \tilde{C}_ℓ^k has a local dependence on the nonlinearity.

The rest of the paper is organized as follows. In Section 2, we detail the assumptions on a . We next write the continuous weak formulation with its equivalent energy minimization problem. Then, we announce the discrete weak formulation with its associated discrete energy minimization problem, and finally the iterative linearization. In Section 3, we define the augmented error and estimator, and state our main result, Theorem 3.4, referring to (1.2). In Section 4, we study the original energy difference, which leads us to an a posteriori error estimate of Theorem 4.2 referring to (1.3). In Section 5, we present a series of numerical experiments in order to illustrate the theoretical findings, for both smooth and singular solutions. In Section 6, we give a proof of Theorem 3.4, then, in Section 7, we give the proof of Theorem 4.2. Finally, we summarize, in Appendix A, some useful properties on the nonlinear function and the assumptions required, and, in Appendix B, we show some technical results to determine the eigenvalues of the Newton iteration.

2 Weak formulation, energy minimization, finite element discretization, and iterative linearization

Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be an open polytope with Lipschitz boundary $\partial\Omega$. We consider problem (1.1), where $f \in L^2(\Omega)$ represents a volumetric force term, while a is the diffusion coefficient which depends on the potential $u : \Omega \rightarrow \mathbb{R}$ only through the Euclidean norm of its gradient $|\nabla u|$.

We consider the following assumption for the nonlinear function a (see, e.g., [43] for more details).

Assumption 2.1 (Nonlinear function a). *We assume that the function $a : \Omega \times [0, \infty) \rightarrow (0, \infty)$ is measurable and that there exist constants $a_m \leq a_c \in (0, \infty)$ such that, a.e. in Ω and for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$,*

$$|a(\cdot, |\mathbf{x}|)\mathbf{x} - a(\cdot, |\mathbf{y}|)\mathbf{y}| \leq a_c |\mathbf{x} - \mathbf{y}| \quad (\text{Lipschitz continuity}), \quad (2.1a)$$

$$(a(\cdot, |\mathbf{x}|)\mathbf{x} - a(\cdot, |\mathbf{y}|)\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) \geq a_m |\mathbf{x} - \mathbf{y}|^2 \quad (\text{strong monotonicity}). \quad (2.1b)$$

2.1 Weak formulation and equivalent energy minimization

The weak formulation of problem (1.1) reads: find $u \in H_0^1(\Omega)$ such that

$$(a(\cdot, |\nabla u|)\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega), \quad (2.2a)$$

where (\cdot, \cdot) is the inner product of $L^2(\Omega)$. We also denote by $\|\cdot\|$ its corresponding norm.

Referring to [43], the weak formulation (2.2a) is equivalent to the following minimization problem:

$$u = \arg \min_{v \in H_0^1(\Omega)} \mathcal{J}(v), \quad (2.2b)$$

with the energy functional $\mathcal{J} : H_0^1(\Omega) \rightarrow \mathbb{R}$ defined as,

$$\mathcal{J}(v) := \int_{\Omega} \phi(\cdot, |\nabla v|) - (f, v), \quad v \in H_0^1(\Omega), \quad (2.3)$$

where the function $\phi : \Omega \times [0, \infty) \rightarrow [0, \infty)$ is defined such that, a.e. in Ω and for all $r \in [0, \infty)$,

$$\phi(\cdot, r) := \int_0^r a(\cdot, s) s \, ds. \quad (2.4)$$

It is shown in [43] that, under Assumption 2.1, there exists a unique solution to problem (2.2). We refer to Appendix A for more details about equivalent assumptions on the nonlinear functions a and ϕ .

2.2 Finite element discretization

Let $\ell \geq 0$ be a mesh level index. We consider a simplicial triangulation \mathcal{T}_{ℓ} of the domain Ω satisfying the following shape-regularity property. There exists a constant $\kappa_{\mathcal{T}} > 0$, such that for all $\ell \geq 0$ and all $K \in \mathcal{T}_{\ell}$, $h_K/\rho_K \leq \kappa_{\mathcal{T}}$, where h_K is the diameter of K and ρ_K is the diameter of the largest inscribed ball in K .

For a polynomial degree $p \geq 1$, denoting $\mathcal{P}_p(\mathcal{T}_{\ell})$ the space of piecewise polynomials on the mesh \mathcal{T}_{ℓ} of total degree at most p , we define the discrete finite element space $V_{\ell}^p := \mathcal{P}_p(\mathcal{T}_{\ell}) \cap H_0^1(\Omega)$. The finite element approximation of (2.2a) would be $u_{\ell} \in V_{\ell}^p$ such that

$$(a(\cdot, |\nabla u_{\ell}|)\nabla u_{\ell}, \nabla v_{\ell}) = (f, v_{\ell}) \quad \forall v_{\ell} \in V_{\ell}^p. \quad (2.5a)$$

Again, $u_{\ell} \in V_{\ell}^p$ satisfies the minimization problem

$$u_{\ell} = \arg \min_{v_{\ell} \in V_{\ell}^p} \mathcal{J}(v_{\ell}). \quad (2.5b)$$

2.3 Iterative linearization

In this work, we immediately consider an iterative linearization of (2.5a), which is anyhow necessary for a practical solution of (2.5a). Let $u_{\ell}^0 \in V_{\ell}^p$ be a given initial guess. For an iterative linearization index $k \geq 1$, consider $\mathbf{A}_{\ell}^{k-1} : \Omega \rightarrow \mathbb{R}^{d \times d}$ and $\mathbf{b}_{\ell}^{k-1} : \Omega \rightarrow \mathbb{R}^d$, arising from a suitable linearization; details and examples are given in the next section. We define the linearized finite element approximation: $u_{\ell}^k \in V_{\ell}^p$ to be such that

$$(\mathbf{A}_{\ell}^{k-1} \nabla u_{\ell}^k, \nabla v_{\ell}) = (f, v_{\ell}) + (\mathbf{b}_{\ell}^{k-1}, \nabla v_{\ell}) \quad \forall v_{\ell} \in V_{\ell}^p. \quad (2.6a)$$

As in (2.5b), this is equivalent to the discrete minimization problem

$$u_{\ell}^k = \arg \min_{v_{\ell} \in V_{\ell}^p} \mathcal{J}_{\ell}^{k-1}(v_{\ell}) \quad (2.6b)$$

with the linearized energy functional $\mathcal{J}_{\ell}^{k-1} : H_0^1(\Omega) \rightarrow \mathbb{R}$ defined for all $v \in H_0^1(\Omega)$ by

$$\mathcal{J}_{\ell}^{k-1}(v) := \frac{1}{2} \left\| (\mathbf{A}_{\ell}^{k-1})^{\frac{1}{2}} \nabla v \right\|^2 - (f, v) - (\mathbf{b}_{\ell}^{k-1}, \nabla v). \quad (2.7)$$

2.3.1 Assumptions on iterative linearization schemes

Let $k \geq 1$. We will suppose that the functions $\mathbf{A}_\ell^{k-1} : \Omega \rightarrow \mathbb{R}^{d \times d}$ and $\mathbf{b}_\ell^{k-1} : \Omega \rightarrow \mathbb{R}^d$ from (2.6) satisfy the following assumption. For the sake of conciseness, we also assume that they are well-defined everywhere in Ω . We will use the following notation of the derivatives in the second argument of the functions a , ϕ , and others (cf. (3.6)–(3.7)): for all $r \in [0, \infty)$,

$$a'(\cdot, r) := \frac{\partial}{\partial r} a(\mathbf{x}, r) \quad \text{and} \quad \phi'(\cdot, r) := \frac{\partial}{\partial r} \phi(\mathbf{x}, r). \quad (2.8)$$

Assumption 2.2 (Iterative linearization). *For all points $\mathbf{x} \in \Omega$, we assume that $\mathbf{A}_\ell^{k-1}(\mathbf{x}) \in \mathbb{R}^{d \times d}$ is a bounded symmetric positive definite matrix. Specifically, denoting by $A_{m,\ell}^{k-1}(\mathbf{x})$ and $A_{c,\ell}^{k-1}(\mathbf{x})$ respectively its smallest and largest eigenvalues, we suppose for all $\boldsymbol{\xi} \in \mathbb{R}^d$*

$$|\mathbf{A}_\ell^{k-1}(\mathbf{x})\boldsymbol{\xi}| \leq A_{c,\ell}^{k-1}(\mathbf{x})|\boldsymbol{\xi}| \quad (\text{boundedness}), \quad (2.9a)$$

$$(\mathbf{A}_\ell^{k-1}(\mathbf{x})\boldsymbol{\xi}) \cdot \boldsymbol{\xi} \geq A_{m,\ell}^{k-1}(\mathbf{x})|\boldsymbol{\xi}|^2 \quad (\text{positive definiteness}). \quad (2.9b)$$

Moreover, we suppose uniformity, i.e., that there exists $A_m, A_c \in (0, \infty)$ independent of k, ℓ , and \mathbf{x} such that

$$A_m \leq A_{m,\ell}^{k-1}(\mathbf{x}) \leq A_{c,\ell}^{k-1}(\mathbf{x}) \leq A_c. \quad (2.9c)$$

Finally, we explicitly define $\mathbf{b}_\ell^{k-1}(\mathbf{x}) \in \mathbb{R}^d$ for all $\mathbf{x} \in \Omega$ by

$$\mathbf{b}_\ell^{k-1}(\mathbf{x}) := \mathbf{A}_\ell^{k-1}(\mathbf{x})\nabla u_\ell^{k-1}(\mathbf{x}) - a(\mathbf{x}, |\nabla u_\ell^{k-1}(\mathbf{x})|)\nabla u_\ell^{k-1}(\mathbf{x}). \quad (2.10)$$

In the following, we use the boldface font to denote the spaces of multi-dimensional functions, e.g., $\mathbf{L}^2(\Omega)$.

Remark 2.3 (Assumption 2.2). *Equality (2.10) implies that (2.6a) can be equivalently written as a problem for the increment $u_\ell^k - u_\ell^{k-1}$ on the left-hand side and the residual of u_ℓ^{k-1} on the right-hand side:*

$$(\mathbf{A}_\ell^{k-1}\nabla(u_\ell^k - u_\ell^{k-1}), \nabla v_\ell) = (f, v_\ell) - (a(\cdot, |\nabla u_\ell^{k-1}|)\nabla u_\ell^{k-1}, \nabla v_\ell) \quad \forall v_\ell \in V_\ell^p, \quad (2.11)$$

which is the form used in, e.g., [36]. Therefore, equality (2.10) ensures that the discrete problem (2.6a) is consistent with the discrete problem (2.5a), i.e., $\mathbf{A}_\ell^{k-1}\nabla u_\ell^k - \mathbf{b}_\ell^{k-1} \rightarrow a(\cdot, |\nabla u_\ell|)\nabla u_\ell$ in $\mathbf{L}^2(\Omega)$ when $u_\ell^k \rightarrow u_\ell$ in $H_0^1(\Omega)$. Indeed, (2.9a) implies that $\mathbf{A}_\ell^{k-1}\nabla(u_\ell^k - u_\ell^{k-1}) \rightarrow \mathbf{0}$ in $\mathbf{L}^2(\Omega)$, and $a(\cdot, |\nabla u_\ell^{k-1}|)\nabla u_\ell^{k-1} \rightarrow a(\cdot, |\nabla u_\ell|)\nabla u_\ell$ in $\mathbf{L}^2(\Omega)$ thanks to (2.1a). Finally, we recall that the positive definiteness of \mathbf{A}_ℓ^{k-1} implies that $(\mathbf{A}_\ell^{k-1})^{-1}$ and $(\mathbf{A}_\ell^{k-1})^{\frac{1}{2}}$ exist, which are used below.

2.3.2 Examples of iterative linearization schemes

We now present some examples of linearization methods satisfying Assumption 2.2.

Example 2.4 (Picard). *The Picard iteration, see, e.g. [16], is defined as*

$$\mathbf{A}_\ell^{k-1} = a(\cdot, |\nabla u_\ell^{k-1}|)\mathbf{I}_d \quad \text{with} \quad \mathbf{b}_\ell^{k-1} = \mathbf{0} \quad \text{in } \Omega. \quad (2.12)$$

It satisfies Assumption 2.2 with $A_{m,\ell}^{k-1} = A_{c,\ell}^{k-1} = a(\cdot, |\nabla u_\ell^{k-1}|)$, which leads to $A_m = a_m$ and $A_c = a_c$ thanks to (A.3).

Example 2.5 (Zarantonello). *The Zarantonello iteration, introduced in [42], is defined as*

$$\mathbf{A}_\ell^{k-1} = \gamma \mathbf{I}_d \quad \text{with} \quad \mathbf{b}_\ell^{k-1} = (\gamma - a(\cdot, |\nabla u_\ell^{k-1}|)) \nabla u_\ell^{k-1} \quad \text{in } \Omega, \quad (2.13)$$

where $\gamma \in (0, \infty)$ is a constant parameter. To ensure contraction, one needs to assume that $\gamma \geq \frac{a_c^2}{a_m}$. The Zarantonello iteration converges linearly, but the convergence is slow as γ takes large values. It satisfies Assumption 2.2 with $A_{m,\ell}^{k-1} = A_{c,\ell}^{k-1} = \gamma$, which leads to $A_m = A_c = \gamma$.

Example 2.6 ((Damped) Newton). *The (damped) Newton iteration, see, e.g. [16], is defined as*

$$\mathbf{A}_\ell^{k-1} = a(\cdot, |\nabla u_\ell^{k-1}|) \mathbf{I}_d + \theta \frac{a'(\cdot, |\nabla u_\ell^{k-1}|)}{|\nabla u_\ell^{k-1}|} \nabla u_\ell^{k-1} \otimes \nabla u_\ell^{k-1} \quad (2.14)$$

$$\text{with } \mathbf{b}_\ell^{k-1} = \theta a'(\cdot, |\nabla u_\ell^{k-1}|) |\nabla u_\ell^{k-1}| \nabla u_\ell^{k-1} \text{ in } \Omega,$$

where $\theta \in [0, 1]$ is the damping parameter. Observe that $\theta = 1$ gives the Newton iteration whereas $\theta = 0$ corresponds to the Picard iteration. If $\theta = 1$, the Newton method converges quadratically. However, it might not always converge. The damped Newton iteration satisfies Assumption 2.2 with, if $d = 1$,

$$\begin{aligned} A_{\mathbf{m}, \ell}^{k-1} &= A_{\mathbf{c}, \ell}^{k-1} = a(\cdot, |\nabla u_\ell^{k-1}|) + \theta a'(\cdot, |\nabla u_\ell^{k-1}|) |\nabla u_\ell^{k-1}| \\ &\stackrel{(A.8)}{=} (1 - \theta) a(\cdot, |\nabla u_\ell^{k-1}|) + \theta \phi''(\cdot, |\nabla u_\ell^{k-1}|), \end{aligned} \quad (2.15)$$

and, if $d > 1$,

$$A_{\mathbf{m}, \ell}^{k-1} = (1 - \theta) a(\cdot, |\nabla u_\ell^{k-1}|) + \theta \min(a(\cdot, |\nabla u_\ell^{k-1}|), \phi''(\cdot, |\nabla u_\ell^{k-1}|)), \quad (2.16a)$$

$$A_{\mathbf{c}, \ell}^{k-1} = (1 - \theta) a(\cdot, |\nabla u_\ell^{k-1}|) + \theta \max(a(\cdot, |\nabla u_\ell^{k-1}|), \phi''(\cdot, |\nabla u_\ell^{k-1}|)). \quad (2.16b)$$

Indeed, denoting the spectrum of a matrix $\mathbf{A} \in \mathbb{R}^{d \times d}$ by $\text{Spec}(\mathbf{A})$, we infer (2.16) by writing,

$$\begin{aligned} \text{Spec}(\mathbf{A}_\ell^{k-1}) &\stackrel{(B.1)}{=} \{a(\cdot, |\nabla u_\ell^{k-1}|), a(\cdot, |\nabla u_\ell^{k-1}|) + \theta a'(\cdot, |\nabla u_\ell^{k-1}|) |\nabla u_\ell^{k-1}|\} \\ &\stackrel{(A.8)}{=} \{(1 - \theta) a(\cdot, |\nabla u_\ell^{k-1}|) + \theta a(\cdot, |\nabla u_\ell^{k-1}|), \\ &\quad (1 - \theta) a(\cdot, |\nabla u_\ell^{k-1}|) + \theta \phi''(\cdot, |\nabla u_\ell^{k-1}|)\}. \end{aligned}$$

Finally, we can set $A_{\mathbf{m}} = a_{\mathbf{m}}$ and $A_{\mathbf{c}} = a_{\mathbf{c}}$ thanks to (A.3) and (A.5).

3 A posteriori estimate of an augmented energy difference

In this section, we define the energy-differences-based error and a posteriori estimator of the finite element discretization u_ℓ^k given by (2.6) on mesh \mathcal{T}_ℓ , $\ell \geq 0$, and linearization step $k \geq 1$.

3.1 Flux equilibration

Let \mathcal{V}_ℓ be the set of all mesh vertices and, for each vertex $\mathbf{a} \in \mathcal{V}_\ell$, define the patch $\mathcal{T}_\ell^\mathbf{a}$ which corresponds to the collection of the simplicies of \mathcal{T}_ℓ sharing the vertex \mathbf{a} , as well as the patch subdomain $\omega_\ell^\mathbf{a}$ corresponding to the open subdomain of $\mathcal{T}_\ell^\mathbf{a}$. For all $\mathbf{a} \in \mathcal{V}_\ell$, we define the space $\mathbf{V}_\ell^\mathbf{a} := \mathbf{RTN}_p(\mathcal{T}_\ell^\mathbf{a}) \cap \mathbf{H}_0(\text{div}, \omega_\ell^\mathbf{a})$, where $\mathbf{RTN}_p(\mathcal{T}_\ell^\mathbf{a})$ denotes the broken space consisting of p -th order Raviart–Thomas–Nédélec space on each simplex $\mathbf{RTN}_p(K) := [\mathcal{P}(K)]^d + \mathbf{xP}(K)$. We also introduce, for all $\mathbf{a} \in \mathcal{V}_\ell$, the hat function $\psi_\ell^\mathbf{a}$ corresponding to the continuous, piecewise affine function equal to 1 on \mathbf{a} and 0 on $\mathcal{V}_\ell \setminus \{\mathbf{a}\}$. We recall that we have in particular, for all $\mathbf{x} \in \Omega$, the partition of unity

$$\sum_{\mathbf{a} \in \mathcal{V}_\ell} \psi_\ell^\mathbf{a}(\mathbf{x}) = 1. \quad (3.1)$$

We denote by $\Pi_\ell^\mathcal{P}$ the L^2 -orthogonal projection from $L^2(\Omega)$ to $\mathcal{P}_p(\mathcal{T}_\ell)$ and by $\Pi_\ell^{\mathbf{RTN}}$ the \mathbf{L}^2 -orthogonal projection from $\mathbf{L}^2(\Omega)$ to $\mathbf{RTN}_p(\mathcal{T}_\ell)$. Finally, we consider the equilibrated flux associated with u_ℓ^k by

$$\boldsymbol{\sigma}_\ell^k := \sum_{\mathbf{a} \in \mathcal{V}_\ell} \boldsymbol{\sigma}_\ell^{\mathbf{a}, k}, \quad (3.2a)$$

where, for all $\mathbf{a} \in \mathcal{V}_\ell$,

$$\begin{aligned} \boldsymbol{\sigma}_\ell^{\mathbf{a}, k} &:= \arg \min_{\mathbf{w}_\ell \in \mathbf{V}_\ell^\mathbf{a}} \|(\mathbf{A}_\ell^{k-1})^{-\frac{1}{2}} (\psi_\ell^\mathbf{a} \Pi_\ell^{\mathbf{RTN}}(\mathbf{A}_\ell^{k-1} \nabla u_\ell^k - \mathbf{b}_\ell^{k-1}) + \mathbf{w}_\ell)\|_{\omega_\ell^\mathbf{a}}. \\ &\quad \nabla \cdot \mathbf{w}_\ell = \Pi_\ell^\mathcal{P}(\psi_\ell^\mathbf{a} f - \nabla \psi_\ell^\mathbf{a} \cdot (\mathbf{A}_\ell^{k-1} \nabla u_\ell^k - \mathbf{b}_\ell^{k-1})) \end{aligned} \quad (3.2b)$$

Combining (3.2) and (3.1), we infer, as in, e.g., [20],

$$\begin{aligned} \nabla \cdot \boldsymbol{\sigma}_\ell^k &= \sum_{\mathbf{a} \in \mathcal{V}_\ell} \nabla \cdot \boldsymbol{\sigma}_\ell^{\mathbf{a}, k} = \sum_{\mathbf{a} \in \mathcal{V}_\ell} \Pi_\ell^\mathcal{P}(\psi_\ell^\mathbf{a} f) - \sum_{\mathbf{a} \in \mathcal{V}_\ell} \Pi_\ell^\mathcal{P}(\nabla \psi_\ell^\mathbf{a} \cdot (\mathbf{A}_\ell^{k-1} \nabla u_\ell^k - \mathbf{b}_\ell^{k-1})) \\ &= \Pi_\ell^\mathcal{P} \sum_{\mathbf{a} \in \mathcal{V}_\ell} (\psi_\ell^\mathbf{a} f) = \Pi_\ell^\mathcal{P} f, \end{aligned} \quad (3.3)$$

and, in particular, we have with the Green theorem that

$$-(\sigma_\ell^k, \nabla u_\ell^k) = (f, u_\ell^k). \quad (3.4)$$

3.2 Convex conjugate function and dual energies

Recalling the primal energy \mathcal{J} of (2.3), the corresponding dual energy $\mathcal{J}^* : \mathbf{H}(\text{div}, \Omega) \rightarrow \mathbb{R}$, cf. [39, 43], is defined by

$$\mathcal{J}^*(\mathbf{w}) := - \int_{\Omega} \phi^*(\cdot, |\mathbf{w}|), \quad \mathbf{w} \in \mathbf{H}(\text{div}, \Omega), \quad (3.5)$$

where $\phi^* : \Omega \times [0, \infty) \rightarrow [0, \infty)$ is the convex conjugate of ϕ (also known as the Legendre dual or the Fenchel conjugate), which is defined such that, a.e. in Ω , for all $s \in [0, \infty)$,

$$\phi^*(\cdot, s) := \sup_{r \in [0, \infty)} (sr - \phi(\cdot, r)), \quad (3.6a)$$

or equivalently, for all $s \in [0, \infty)$,

$$\phi^*(\cdot, s) = \int_0^s \phi'^{-1}(\cdot, r) \, dr = s\phi'^{-1}(\cdot, s) - \phi(\cdot, \phi'^{-1}(\cdot, s)). \quad (3.6b)$$

We refer to [32] for more details and recall that the construction of ϕ^* yields

$$\phi^{*'} = \phi'^{-1} \quad \text{and} \quad \phi^{*''} = \frac{1}{\phi'' \circ \phi'^{-1}}. \quad (3.7)$$

Consequently, under Assumption 2.1, ϕ^* is convex thanks to Remark A.3 below.

Finally, we define the discrete dual energy functional $\mathcal{J}_\ell^{*, k-1} : \mathbf{H}(\text{div}, \Omega) \rightarrow \mathbb{R}$ such that for all $\mathbf{w} \in \mathbf{H}(\text{div}, \Omega)$,

$$\mathcal{J}_\ell^{*, k-1}(\mathbf{w}) := -\frac{1}{2} \left\| (\mathbf{A}_\ell^{k-1})^{-\frac{1}{2}} (\mathbf{w} - \mathbf{b}_\ell^{k-1}) \right\|^2. \quad (3.8)$$

3.3 Energy difference and estimator of the nonlinear problem

We define the square root of the energy difference $\mathcal{E}_{\text{N}, \ell}^k$ corresponding to the nonlinear problem (2.2) by

$$\mathcal{E}_{\text{N}, \ell}^k := (2(\mathcal{J}(u_\ell^k) - \mathcal{J}(u)))^{\frac{1}{2}}. \quad (3.9a)$$

Note that $\mathcal{E}_{\text{N}, \ell}^k$ is well-defined from (2.2b) and the fact that $u_\ell^k \in H_0^1(\Omega)$. Actually, $\mathcal{E}_{\text{N}, \ell}^k \geq 0$ and 0 if and only if $u_\ell^k = u$. We then define the estimator $\eta_{\text{N}, \ell}^k$ corresponding to the nonlinear problem (2.2) as in, e.g., [3, 39, 43], by

$$\eta_{\text{N}, \ell}^k := (2(\mathcal{J}(u_\ell^k) - \mathcal{J}^*(\sigma_\ell^k)))^{\frac{1}{2}}. \quad (3.9b)$$

Note that $\eta_{\text{N}, \ell}^k$ is again well-defined.

3.4 Energy difference and estimator of the linearized problem

In order to derive robust estimates (recall Figure 1), we further consider the linearized problem (2.6). We introduce the abstract linearization on the continuous level: $u_{(\ell)}^k \in H_0^1(\Omega)$ to be such that

$$(\mathbf{A}_\ell^{k-1} \nabla u_{(\ell)}^k, \nabla v) = (f, v) + (\mathbf{b}_\ell^{k-1}, \nabla v) \quad \forall v \in H_0^1(\Omega), \quad (3.10a)$$

which is a linear problem. Note that (3.10a) is equivalent to

$$u_{(\ell)}^k := \arg \min_{v \in H_0^1(\Omega)} \mathcal{J}_\ell^{k-1}(v), \quad (3.10b)$$

based on the linearized energy (2.7). Analogously to the nonlinear energy difference $\mathcal{E}_{\text{N}, \ell}^k$ (3.9a), we then define the energy difference $\mathcal{E}_{\text{L}, \ell}^k$ of the linearized problem (3.10) by

$$\mathcal{E}_{\text{L}, \ell}^k := \left(2(\mathcal{J}_\ell^{k-1}(u_\ell^k) - \mathcal{J}_\ell^{k-1}(u_{(\ell)}^k)) \right)^{\frac{1}{2}} \stackrel{(4.4a)}{=} \|(\mathbf{A}_\ell^{k-1})^{\frac{1}{2}} \nabla(u_\ell^k - u_{(\ell)}^k)\|. \quad (3.11a)$$

Here, $\mathcal{E}_{L,\ell}^k \geq 0$ again and 0 if and only if $u_\ell^k = u_{(\ell)}^k$ is obvious. Finally, we define the estimator $\eta_{L,\ell}^k$ of the linearized problem (3.10) by

$$\begin{aligned} \eta_{L,\ell}^k &:= \left(2(\mathcal{J}_\ell^{k-1}(u_\ell^k) - \mathcal{J}_\ell^{*,k-1}(\sigma_\ell^k)) \right)^{\frac{1}{2}} \\ &\stackrel{(4.5a)}{=} \|(\mathbf{A}_\ell^{k-1})^{-\frac{1}{2}}(\mathbf{A}_\ell^{k-1}\nabla u_\ell^k - \mathbf{b}_\ell^{k-1} + \sigma_\ell^k)\| \\ &\stackrel{(2.10)}{=} \|(\mathbf{A}_\ell^{k-1})^{-\frac{1}{2}}(\mathbf{A}_\ell^{k-1}\nabla(u_\ell^k - u_\ell^{k-1}) + a(\cdot, |\nabla u_\ell^{k-1}|)\nabla u_\ell^{k-1} + \sigma_\ell^k)\|, \end{aligned} \quad (3.11b)$$

so that $\eta_{L,\ell}^k$ is again well-defined.

3.5 Augmented energy difference and estimator

We define the shifted estimator $\hat{\eta}_{L,\ell}^k$ of the linearized problem (3.10) by

$$\hat{\eta}_{L,\ell}^k := \|(\mathbf{A}_\ell^{k-1})^{-\frac{1}{2}}(a(\cdot, |\nabla u_\ell^k|)\nabla u_\ell^k + \sigma_\ell^k)\|, \quad (3.12)$$

and we consider the computable weight λ_ℓ^k defined from the estimators (3.9b) and (3.12) by

$$\lambda_\ell^k = \frac{\eta_{N,\ell}^k}{\hat{\eta}_{L,\ell}^k}. \quad (3.13)$$

Remark 3.1 (Weight λ_ℓ^k). *The role of the weight λ_ℓ^k is to ensure a balance between $\mathcal{E}_{N,\ell}^k$ and $\lambda_\ell^k \mathcal{E}_{L,\ell}^k$, see Remark 3.2. We employ $\hat{\eta}_{L,\ell}^k$ (and not $\eta_{L,\ell}^k$ from (3.11b)), since then λ_ℓ^k is uniformly bounded for every $k \geq 1$. Indeed, we have with equation (7.2) below that, using the notation from (4.7),*

$$\lambda_\ell^k = \frac{\eta_{N,\ell}^k}{\hat{\eta}_{L,\ell}^k} \leq \frac{\|(a_{m,\ell}^k)^{-\frac{1}{2}}(a_\ell^k \nabla u_\ell^k + \sigma_\ell^k)\|}{\|(\mathbf{A}_\ell^{k-1})^{-\frac{1}{2}}(a_\ell^k \nabla u_\ell^k + \sigma_\ell^k)\|} \stackrel{(2.9a)}{\leq} \operatorname{ess\,sup}_\Omega \left(\frac{A_{c,\ell}^{k-1}}{a_{m,\ell}^k} \right)^{\frac{1}{2}} \stackrel{(4.8)}{\leq} \sqrt{\frac{A_c}{a_m}}. \quad (3.14)$$

Now, we define the augmented energy difference \mathcal{E}_ℓ^k by

$$\mathcal{E}_\ell^k := \frac{1}{2} (\mathcal{E}_{N,\ell}^k + \lambda_\ell^k \mathcal{E}_{L,\ell}^k) \quad (3.15a)$$

and the corresponding augmented estimator η_ℓ^k by

$$\eta_\ell^k := \frac{1}{2} (\eta_{N,\ell}^k + \lambda_\ell^k \eta_{L,\ell}^k). \quad (3.15b)$$

Remark 3.2 (Roles of \mathcal{E}_ℓ^k and η_ℓ^k). *The objective of introducing \mathcal{E}_ℓ^k and η_ℓ^k is to augment the energy difference $\mathcal{E}_{N,\ell}^k$ and estimator $\eta_{N,\ell}^k$ of the nonlinear problem (2.2) (where $\mathcal{E}_{N,\ell}^k \leq \eta_{N,\ell}^k$ but not necessarily robustly, see Figure 1) in order to obtain an a posteriori estimate that will be robust with respect to the nonlinear function a . The added linearization components $\mathcal{E}_{L,\ell}^k$ and $\eta_{L,\ell}^k$ are multiplied by the weight λ_ℓ^k whose purpose is to make them comparable in size to $\mathcal{E}_{N,\ell}^k$ and $\eta_{N,\ell}^k$, respectively. Other options for λ_ℓ^k , see for example (4.8) below, are possible.*

Lemma 3.3 (Error consistency). *Let the error \mathcal{E}_ℓ^k be given by (3.15a). We have $\mathcal{E}_\ell^k \rightarrow 0$ if and only if $u_\ell^k \rightarrow u$ in $H_0^1(\Omega)$.*

Proof. Assuming $\mathcal{E}_\ell^k \rightarrow 0$ implies $\mathcal{E}_{N,\ell}^k \rightarrow 0$ since $\mathcal{E}_{N,\ell}^k, \mathcal{E}_{L,\ell}^k, \lambda_\ell^k \geq 0$. Furthermore, using the same arguments as in [24, Lemma 5.1], we have

$$a_m \|\nabla(u_\ell^k - u)\|^2 \leq 2(\mathcal{J}(u_\ell^k) - \mathcal{J}(u)) \stackrel{(3.9a)}{=} (\mathcal{E}_{N,\ell}^k)^2 \leq a_c \|\nabla(u_\ell^k - u)\|^2, \quad (3.16)$$

hence $u_\ell^k \rightarrow u$ in $H_0^1(\Omega)$.

Reciprocally, assume that $u_\ell^k \rightarrow u$ in $H_0^1(\Omega)$. We have again from (3.16) that $\mathcal{E}_{N,\ell}^k \rightarrow 0$. Furthermore, we have for all $k \geq 1$,

$$\begin{aligned}
A_m \|\nabla(u_{\langle \ell \rangle}^k - u_\ell^{k-1})\|^2 &\stackrel{(2.9)}{\leq} (\mathbf{A}_\ell^{k-1} \nabla(u_{\langle \ell \rangle}^k - u_\ell^{k-1}), \nabla(u_{\langle \ell \rangle}^k - u_\ell^{k-1})) \\
&\stackrel{(2.10)}{=} (\mathbf{A}_\ell^{k-1} \nabla u_{\langle \ell \rangle}^k - \mathbf{b}_\ell^{k-1}, \nabla(u_{\langle \ell \rangle}^k - u_\ell^{k-1})) \\
&\quad - (a(\cdot, |\nabla u_\ell^{k-1}|) \nabla u_\ell^{k-1}, \nabla(u_{\langle \ell \rangle}^k - u_\ell^{k-1})) \\
&\stackrel{(3.10a)}{=} (f, u_{\langle \ell \rangle}^k - u_\ell^{k-1}) - (a(\cdot, |\nabla u_\ell^{k-1}|) \nabla u_\ell^{k-1}, \nabla(u_{\langle \ell \rangle}^k - u_\ell^{k-1})) \\
&\stackrel{(2.2a)}{=} (a(\cdot, |\nabla u|) \nabla u - a(\cdot, |\nabla u_\ell^{k-1}|) \nabla u_\ell^{k-1}, \nabla(u_{\langle \ell \rangle}^k - u_\ell^{k-1})) \\
&\stackrel{(2.1a)}{\leq} a_c \|\nabla(u - u_\ell^{k-1})\| \|\nabla(u_{\langle \ell \rangle}^k - u_\ell^{k-1})\|.
\end{aligned} \tag{3.17}$$

Hence, dividing (3.17) by $\|\nabla(u_{\langle \ell \rangle}^k - u_\ell^{k-1})\|$ shows that $u_{\langle \ell \rangle}^k$ converges to the same limit of u_ℓ^k in $H_0^1(\Omega)$, i.e. u . Thus, the relation (4.4a) below together with (2.9) show that $\mathcal{E}_{L,\ell}^k \rightarrow 0$. Hence, since λ_ℓ^k is bounded (cf. (3.14)), we infer $\mathcal{E}_\ell^k \rightarrow 0$. \square

3.6 Data oscillation and quadrature estimators

Following, e.g. [20], we define the data oscillation estimator $\eta_{\text{osc},\ell}$ and the data oscillation-quadrature estimator $\eta_{\text{osc,quad},\ell}^k$ such that

$$(\eta_{\text{osc},\ell})^2 := \sum_{K \in \mathcal{T}_\ell} \left[\frac{h_K}{\pi} \|f - \Pi_\ell^\mathcal{P} f\|_K \right]^2, \quad (\eta_{\text{osc,quad},\ell}^k)^2 := \sum_{\mathbf{a} \in \mathcal{V}_\ell} (\eta_{\text{osc,quad},\ell}^{\mathbf{a},k})^2, \tag{3.18a}$$

where, for all $\mathbf{a} \in \mathcal{V}_\ell$,

$$\begin{aligned}
(\eta_{\text{osc,quad},\ell}^{\mathbf{a},k})^2 &:= \sum_{K \in \mathcal{T}_\ell^\mathbf{a}} \left[\frac{h_K}{\pi} \|\psi_\ell^\mathbf{a} f - \Pi_\ell^\mathcal{P}(\psi_\ell^\mathbf{a} f)\|_K \right. \\
&\quad + \|\nabla \psi_\ell^\mathbf{a} \cdot (\mathbf{A}_\ell^{k-1} \nabla u_\ell^k - \mathbf{b}_\ell^{k-1}) - \Pi_\ell^\mathcal{P}(\nabla \psi_\ell^\mathbf{a} \cdot (\mathbf{A}_\ell^{k-1} \nabla u_\ell^k - \mathbf{b}_\ell^{k-1}))\|_K \\
&\quad \left. + \|\psi_\ell^\mathbf{a} (\mathbf{A}_\ell^{k-1} \nabla u_\ell^k - \mathbf{b}_\ell^{k-1} - \Pi_\ell^{RTN}(\mathbf{A}_\ell^{k-1} \nabla u_\ell^k - \mathbf{b}_\ell^{k-1}))\|_K \right]^2,
\end{aligned} \tag{3.18b}$$

where the two parts correspond respectively to the oscillation in the data f and quadrature errors possibly arising from the nonlinear function a .

3.7 A posteriori estimate of the augmented energy difference

We now present our main result giving an a posteriori estimate based on the augmented energy difference and estimator defined in (3.15).

Theorem 3.4 (A posteriori estimate of the augmented energy difference). *Suppose Assumption 2.1 and let u be the weak solution of (2.2). Let u_ℓ^k be its finite element approximation given by (2.6) on mesh \mathcal{T}_ℓ , $\ell \geq 0$, and linearization step $k \geq 1$, for any iterative linearization satisfying Assumption 2.2. Let $u_{\langle \ell \rangle}^k$ be given by (3.10), the augmented error \mathcal{E}_ℓ^k by (3.15a), and the estimator η_ℓ^k by (3.15b). We have*

$$\mathcal{E}_\ell^k \leq \eta_\ell^k + \frac{1 + \lambda_\ell^k}{2} \eta_{\text{osc},\ell}. \tag{3.19}$$

Moreover, assume that $(u_\ell^k)_{k \geq 1}$ converges to u_ℓ in $H_0^1(\Omega)$. If k satisfies the criterion

$$\widehat{\eta}_{L,\ell}^k \leq 2\eta_{L,\ell}^k, \tag{3.20}$$

then

$$\eta_\ell^k \lesssim C_\ell^k (\mathcal{E}_\ell^k + \lambda_\ell^k \eta_{\text{osc,quad},\ell}^k), \tag{3.21}$$

where the hidden constant only depends on the space dimension d , the mesh shape-regularity $\kappa_{\mathcal{T}}$, and possibly, when $d \geq 4$, the polynomial degree p , with

$$C_\ell^k := \max_{\mathbf{a} \in \mathcal{V}_\ell} \left(\frac{\sup_{\omega_\ell^\mathbf{a}} A_{\mathbf{c},\ell}^{k-1}}{\inf_{\omega_\ell^\mathbf{a}} A_{\mathbf{m},\ell}^{k-1}} \right)^{\frac{1}{2}} \begin{cases} = 1 & \text{for the Zarantonello linearization,} \\ \leq \left(\frac{A_{\mathbf{c}}}{A_{\mathbf{m}}} \right)^{\frac{1}{2}} & \text{in general.} \end{cases} \quad (3.22)$$

Proof. See Section 6. □

Remark 3.5 (Robustness with the Zarantonello linearization). *In the Zarantonello linearization case, since $C_\ell^k = 1$, we obtain an estimation of the augmented energy difference \mathcal{E}_ℓ^k by the estimator η_ℓ^k (under small oscillations $\eta_{\text{osc},\ell}$ and $\eta_{\text{osc,quad},\ell}^k$) whose quality is independent of the nonlinear function a .*

Remark 3.6 (Robustness in the general case). *The constant C_ℓ^k can be easily calculated in practice, i.e. without knowing the continuous solution u of (2.2), and allows us to quantify the quality of the estimates in any specific situation. It in particular only relies on patchwise variations of the linearization matrix \mathbf{A}_ℓ^{k-1} (recall (2.9)). For the Picard and damped Newton iteration cases of Examples 2.4 and 2.6, in particular, the local ratio of the functions $A_{\mathbf{c},\ell}^{k-1}$ and $A_{\mathbf{m},\ell}^{k-1}$ is smaller than that of the global constants $A_{\mathbf{c}}$ and $A_{\mathbf{m}}$ given in (2.9c), typically bringing C_ℓ^k close to 1 as in the Zarantonello case. This is indeed observed in the numerical experiments of Section 5 below. Importantly, whenever C_ℓ^k is small, we can affirm robustness a posteriori.*

Remark 3.7 (Criterion (3.20)). *The computable criterion (3.20) allows us to determine when k is large enough to obtain (3.21). Indeed, using the notation from (4.7), we have*

$$\frac{\widehat{\eta}_{\mathbf{L},\ell}^k}{\eta_{\mathbf{L},\ell}^k} \stackrel{(3.12),(3.11b)}{=} \frac{\|(\mathbf{A}_\ell^{k-1})^{-\frac{1}{2}}(a_\ell^k \nabla u_\ell^k + \boldsymbol{\sigma}_\ell^k)\|}{\|(\mathbf{A}_\ell^{k-1})^{-\frac{1}{2}}(\mathbf{A}_\ell^{k-1} \nabla(u_\ell^k - u_\ell^{k-1}) + a_\ell^{k-1} \nabla u_\ell^{k-1} + \boldsymbol{\sigma}_\ell^k)\|}. \quad (3.23)$$

With the same reasoning as in Remark 2.3, $\widehat{\eta}_{\mathbf{L},\ell}^k$ and $\eta_{\mathbf{L},\ell}^k$ become equivalent for large k when $u_\ell^k \rightarrow u_\ell$ in $H_0^1(\Omega)$. Hence, if $\eta_{\mathbf{L},\ell}^k$ (and $\widehat{\eta}_{\mathbf{L},\ell}^k$) does not vanish, then (3.20) holds for k large enough. Otherwise, $\eta_{\mathbf{N},\ell}^k$ vanishes too (cf. (7.2) below), so that η_ℓ^k vanishes. In this particular case, if $\eta_{\text{osc},\ell} > 0$, then $\eta_{\text{osc,quad},\ell}^k$ does not vanish, and (3.20) holds for k large enough. Otherwise, if $\eta_{\text{osc},\ell} = 0$, then (3.19) implies $\mathcal{E}_\ell^k \rightarrow 0$ which is equivalent to $u_\ell^k \rightarrow u$ thanks to Lemma 3.3, and (3.21) is then not necessary.

4 A posteriori estimate of the energy difference

This section gives an a posteriori estimate for the energy difference $\mathcal{E}_{\mathbf{N},\ell}^k$ by the estimator $\eta_{\mathbf{N},\ell}^k$ of the nonlinear problem. To prove robustness, we use a different way to augment the linear energy parts. We will use the fact that for all $\mathbf{v} \in \mathbf{L}^2(\Omega)$ and $\mathbf{W} \in [\mathbf{L}^2(\Omega)]^{d \times d}$,

$$\|\mathbf{v}\|^2 \geq \frac{1}{d+1} \sum_{\mathbf{a} \in \mathcal{V}_\ell} \|\mathbf{v}\|_{\omega_\ell^\mathbf{a}}^2, \quad (4.1a)$$

$$\|\mathbf{v} + \mathbf{W} \boldsymbol{\sigma}_\ell^k\|^2 \leq (d+1) \sum_{\mathbf{a} \in \mathcal{V}_\ell} \|\psi_\ell^\mathbf{a} \mathbf{v} + \mathbf{W} \boldsymbol{\sigma}_\ell^{\mathbf{a},k}\|_{\omega_\ell^\mathbf{a}}^2. \quad (4.1b)$$

Indeed, since every simplex $K \in \mathcal{T}_\ell$ has $(d+1)$ vertices, collected in the set \mathcal{V}_K , we get

$$\sum_{\mathbf{a} \in \mathcal{V}_\ell} \|\mathbf{v}\|_{\omega_\ell^\mathbf{a}}^2 = \sum_{K \in \mathcal{T}_\ell} \sum_{\mathbf{a} \in \mathcal{V}_K} \|\mathbf{v}\|_K^2 \leq (d+1) \sum_{K \in \mathcal{T}_\ell} \|\mathbf{v}\|_K^2 = (d+1) \|\mathbf{v}\|^2. \quad (4.2)$$

On the other hand, with the same reasoning, we infer

$$\begin{aligned} \|\mathbf{v} + \mathbf{W} \boldsymbol{\sigma}_\ell^k\|^2 &= \sum_{K \in \mathcal{T}_\ell} \|\mathbf{v} + \mathbf{W} \boldsymbol{\sigma}_\ell^k\|_K^2 \stackrel{(3.1),(3.2a)}{=} \sum_{K \in \mathcal{T}_\ell} \left\| \sum_{\mathbf{a} \in \mathcal{V}_K} (\psi_\ell^\mathbf{a} \mathbf{v} + \mathbf{W} \boldsymbol{\sigma}_\ell^{\mathbf{a},k}) \right\|_K^2 \\ &\leq (d+1) \sum_{K \in \mathcal{T}_\ell} \sum_{\mathbf{a} \in \mathcal{V}_K} \|\psi_\ell^\mathbf{a} \mathbf{v} + \mathbf{W} \boldsymbol{\sigma}_\ell^{\mathbf{a},k}\|_K^2 = (d+1) \sum_{\mathbf{a} \in \mathcal{V}_\ell} \|\psi_\ell^\mathbf{a} \mathbf{v} + \mathbf{W} \boldsymbol{\sigma}_\ell^{\mathbf{a},k}\|_{\omega_\ell^\mathbf{a}}^2. \end{aligned} \quad (4.3)$$

Therefore, recalling the definition (2.7) of \mathcal{J}_ℓ^{k-1} , we can lower-bound $\mathcal{E}_{L,\ell}^k$ given by (3.11a) such that

$$\begin{aligned}
(\mathcal{E}_{L,\ell}^k)^2 &\stackrel{(2.7)}{=} \left\| (\mathbf{A}_\ell^{k-1})^{\frac{1}{2}} \nabla u_\ell^k \right\|^2 - \left\| (\mathbf{A}_\ell^{k-1})^{\frac{1}{2}} \nabla u_{\langle \ell \rangle}^k \right\|^2 \\
&\quad - 2(\mathbf{b}_\ell^{k-1}, \nabla(u_\ell^k - u_{\langle \ell \rangle}^k)) - 2(f, u_\ell^k - u_{\langle \ell \rangle}^k) \\
&\stackrel{(3.10a)}{=} \left\| (\mathbf{A}_\ell^{k-1})^{\frac{1}{2}} \nabla u_\ell^k \right\|^2 - \left\| (\mathbf{A}_\ell^{k-1})^{\frac{1}{2}} \nabla u_{\langle \ell \rangle}^k \right\|^2 \\
&\quad - 2(\mathbf{A}_\ell^{k-1} \nabla u_{\langle \ell \rangle}^k, \nabla(u_\ell^k - u_{\langle \ell \rangle}^k)) \\
&= \left\| (\mathbf{A}_\ell^{k-1})^{\frac{1}{2}} \nabla(u_\ell^k - u_{\langle \ell \rangle}^k) \right\|^2 \stackrel{(4.1a)}{\geq} \frac{1}{d+1} \sum_{\mathbf{a} \in \mathcal{V}_\ell} (\mathcal{E}_{L,\ell}^{\mathbf{a},k})^2.
\end{aligned} \tag{4.4a}$$

where for all $\mathbf{a} \in \mathcal{V}_\ell$,

$$\mathcal{E}_{L,\ell}^{\mathbf{a},k} := \|(\mathbf{A}_\ell^{k-1})^{\frac{1}{2}} \nabla(u_\ell^k - u_{\langle \ell \rangle}^k)\|_{\omega_\ell^\mathbf{a}}. \tag{4.4b}$$

Similarly, using the definition (3.8) of $\mathcal{J}_\ell^{*,k-1}$, we can upper-bound $\eta_{L,\ell}^k$ of (3.11b) such that

$$\begin{aligned}
(\eta_{L,\ell}^k)^2 &= \left\| (\mathbf{A}_\ell^{k-1})^{\frac{1}{2}} \nabla u_\ell^k \right\|^2 + \left\| (\mathbf{A}_\ell^{k-1})^{-\frac{1}{2}} (\boldsymbol{\sigma}_\ell^k - \mathbf{b}_\ell^{k-1}) \right\|^2 \\
&\quad - 2(\mathbf{b}_\ell^{k-1}, \nabla u_\ell^k) - 2(f, u_\ell^k) \\
&= \left\| (\mathbf{A}_\ell^{k-1})^{-\frac{1}{2}} (\mathbf{A}_\ell^{k-1} \nabla u_\ell^k - \mathbf{b}_\ell^{k-1} + \boldsymbol{\sigma}_\ell^k) \right\|^2 - 2(\boldsymbol{\sigma}_\ell^k, \nabla u_\ell^k) - 2(f, u_\ell^k) \\
&\stackrel{(3.4)}{=} \left\| (\mathbf{A}_\ell^{k-1})^{-\frac{1}{2}} (\mathbf{A}_\ell^{k-1} \nabla u_\ell^k - \mathbf{b}_\ell^{k-1} + \boldsymbol{\sigma}_\ell^k) \right\|^2 \stackrel{(4.1b)}{\leq} (d+1) \sum_{\mathbf{a} \in \mathcal{V}_\ell} (\eta_{L,\ell}^{\mathbf{a},k})^2.
\end{aligned} \tag{4.5a}$$

where, for all $\mathbf{a} \in \mathcal{V}_\ell$,

$$\eta_{L,\ell}^{\mathbf{a},k} := \|(\mathbf{A}_\ell^{k-1})^{-\frac{1}{2}} (\psi_\ell^\mathbf{a} (\mathbf{A}_\ell^{k-1} \nabla u_\ell^k - \mathbf{b}_\ell^{k-1}) + \boldsymbol{\sigma}_\ell^{\mathbf{a},k})\|_{\omega_\ell^\mathbf{a}}. \tag{4.5b}$$

Again, we can upper-bound $\hat{\eta}_{L,\ell}^k$ from (3.12) such that

$$(\hat{\eta}_{L,\ell}^k)^2 \leq (d+1) \sum_{\mathbf{a} \in \mathcal{V}_\ell} (\hat{\eta}_{L,\ell}^{\mathbf{a},k})^2 \tag{4.6a}$$

$$\text{with } \hat{\eta}_{L,\ell}^{\mathbf{a},k} := \|(\mathbf{A}_\ell^{k-1})^{-\frac{1}{2}} (\psi_\ell^\mathbf{a} a(\cdot, |\nabla u_\ell^k|) \nabla u_\ell^k + \boldsymbol{\sigma}_\ell^{\mathbf{a},k})\|_{\omega_\ell^\mathbf{a}} \quad \text{for all } \mathbf{a} \in \mathcal{V}_\ell. \tag{4.6b}$$

4.1 Locally augmented energy differences and estimator

With respect to Section 3.5, we now make a principal change, and this on the weight λ_ℓ^k defined in (3.13). Denoting, a.e. in Ω ,

$$a_\ell^k := a(\cdot, |\nabla u_\ell^k|) \quad \text{and} \quad a_{m,\ell}^k := \min \left(a_\ell^k, \operatorname{ess\,inf}_{r \in (\phi'^{-1}(\cdot, |\boldsymbol{\sigma}_\ell^k|), |\nabla u_\ell^k|)} \phi''(\cdot, r) \right), \tag{4.7}$$

we consider, for all $\mathbf{a} \in \mathcal{V}_\ell$, the patch-local weight $\lambda_\ell^{\mathbf{a},k}$ defined by

$$\lambda_\ell^{\mathbf{a},k} := \operatorname{ess\,sup}_{\omega_\ell^\mathbf{a}} \left(\frac{A_{c,\ell}^{k-1}}{a_{m,\ell}^k} \right)^{\frac{1}{2}} \stackrel{(2.9), (A.3), (A.5)}{\in} \left[\left(\frac{A_m}{a_c} \right)^{\frac{1}{2}}, \left(\frac{A_c}{a_m} \right)^{\frac{1}{2}} \right]. \tag{4.8}$$

We define the locally augmented energy difference $\tilde{\mathcal{E}}_\ell^k$ by

$$\tilde{\mathcal{E}}_\ell^k := \frac{1}{2} \left(\mathcal{E}_{N,\ell}^k + \left(\sum_{\mathbf{a} \in \mathcal{V}_\ell} (\lambda_\ell^{\mathbf{a},k} \mathcal{E}_{L,\ell}^{\mathbf{a},k})^2 \right)^{\frac{1}{2}} \right), \tag{4.9a}$$

and the corresponding locally augmented estimator $\tilde{\eta}_\ell^k$ by

$$\tilde{\eta}_\ell^k := \frac{1}{2} \left(\eta_{N,\ell}^k + \left(\sum_{\mathbf{a} \in \mathcal{V}_\ell} (\lambda_\ell^{\mathbf{a},k} \eta_{L,\ell}^{\mathbf{a},k})^2 \right)^{\frac{1}{2}} \right). \tag{4.9b}$$

Introducing the weighted patch-local efficiency oscillation-quadrature estimator

$$\tilde{\eta}_{\lambda, \text{osc}, \text{quad}, \ell}^k := \left(\sum_{\mathbf{a} \in \mathcal{V}_\ell} (\lambda_\ell^{\mathbf{a}, k} \eta_{\text{osc}, \text{quad}, \ell}^{\mathbf{a}, k})^2 \right)^{\frac{1}{2}}, \quad (4.10)$$

we can now state a patch-local version of the estimate (3.21).

Lemma 4.1 (Locally augmented a posteriori estimate). *Let $u_{(\ell)}^k$ be given by (3.10), the augmented error $\tilde{\mathcal{E}}_\ell^k$ by (4.9a), and the estimator $\tilde{\eta}_\ell^k$ by (4.9b). Assume that u_ℓ^k converges to u_ℓ in $H_0^1(\Omega)$ for $k \rightarrow \infty$. If $k \geq 1$ satisfies the criterion: for all $\mathbf{a} \in \mathcal{V}_\ell$,*

$$\hat{\eta}_{\mathbf{L}, \ell}^{\mathbf{a}, k} \leq 2\eta_{\mathbf{L}, \ell}^{\mathbf{a}, k}, \quad (4.11)$$

then

$$\tilde{\eta}_\ell^k \lesssim C_\ell^k (\tilde{\mathcal{E}}_\ell^k + \tilde{\eta}_{\lambda, \text{osc}, \text{quad}, \ell}^k). \quad (4.12)$$

Proof. See Section 7. □

4.2 A posteriori estimate of the energy difference

Denoting a.e. in Ω ,

$$a_u := a(\cdot, |\nabla u|), \quad a_{c, \ell}^k := \max \left(a_\ell^k, \operatorname{ess\,sup}_{r \in (|\nabla u|, |\nabla u_\ell^k|)} \phi''(\cdot, r) \right), \quad (4.13a)$$

$$\text{and } \phi_{\text{sup}, \ell}^{\text{inf}, k} := \frac{\operatorname{ess\,inf}_{r \in (|\nabla u|, |\nabla u_\ell^k|)} \phi''(\cdot, r)}{\operatorname{ess\,sup}_{r \in (|\nabla u|, |\nabla u_\ell^k|)} \phi''(\cdot, r)} \stackrel{(A.5)}{\in} \left[\frac{a_m}{a_c}, 1 \right], \quad (4.13b)$$

we consider for all $\mathbf{a} \in \mathcal{V}_\ell$, the weight $\mu_\ell^{\mathbf{a}, k}$ defined by

$$\mu_\ell^{\mathbf{a}, k} := \operatorname{ess\,inf}_{\omega_\ell^{\mathbf{a}}} \left(\frac{A_{m, \ell}^{k-1}}{a_{c, \ell}^k} \phi_{\text{sup}, \ell}^{\text{inf}, k} \right)^{\frac{1}{2}} \stackrel{(2.9), (A.3), (A.5)}{\in} \left[\left(\frac{A_m a_m}{a_c^2} \right)^{\frac{1}{2}}, \left(\frac{A_c}{a_m} \right)^{\frac{1}{2}} \right]. \quad (4.14)$$

We now present our second main result, giving an a posteriori estimate based on the energy difference and estimator defined in (3.9).

Theorem 4.2 (A posteriori estimate of the energy difference). *Suppose Assumption 2.1 and let $u \in H_0^1(\Omega)$ be the weak solution of (2.2). Let u_ℓ^k be its finite element approximation given by (2.6) on mesh \mathcal{T}_ℓ , $\ell \geq 0$, and linearization scheme step $k \geq 1$, for any iterative linearization satisfying Assumption 2.2. Then, for all $k \geq 1$,*

$$\mathcal{E}_{\mathbf{N}, \ell}^k \leq \eta_{\mathbf{N}, \ell}^k + \eta_{\text{osc}, \ell}^k. \quad (4.15)$$

Moreover, assume that u_ℓ^k converges to u_ℓ in $H_0^1(\Omega)$ as $k \rightarrow \infty$ and that u_ℓ converges to u in $H_0^1(\Omega)$ as $\ell \rightarrow \infty$. If ℓ and k satisfy the criteria (4.11) and, for all $\mathbf{a} \in \mathcal{V}_\ell$,

$$\| (A_\ell^{k-1})^{\frac{1}{2}} \nabla (u_\ell^k - u_{(\ell)}^k) \|_{\omega_\ell^{\mathbf{a}}} \leq 2 \| (A_\ell^{k-1})^{-\frac{1}{2}} (a_\ell^k \nabla u_\ell^k - a_u \nabla u) \|_{\omega_\ell^{\mathbf{a}}}, \quad (4.16)$$

then

$$\eta_{\mathbf{N}, \ell}^k \lesssim C_\ell^k \left(\tilde{C}_\ell^k \mathcal{E}_{\mathbf{N}, \ell}^k + \tilde{\eta}_{\lambda, \text{osc}, \text{quad}, \ell}^k \right), \quad (4.17)$$

where the hidden constants only depends on the space dimension d , the mesh shape-regularity $\kappa_{\mathcal{T}}$, and possibly, when $d \geq 4$, the polynomial degree p . Here, C_ℓ^k is given in (3.22) and

$$\tilde{C}_\ell^k := \max_{\mathbf{a} \in \mathcal{V}_\ell} \left(\frac{\lambda_\ell^{\mathbf{a}, k}}{\mu_\ell^{\mathbf{a}, k}} \right) = \max_{\mathbf{a} \in \mathcal{V}_\ell} \left(\frac{\operatorname{ess\,sup}_{\omega_\ell^{\mathbf{a}}} \left(\frac{A_{c, \ell}^{k-1}}{a_{c, \ell}^k} \right)}{\operatorname{ess\,inf}_{\omega_\ell^{\mathbf{a}}} \left(\frac{A_{m, \ell}^{k-1}}{a_{c, \ell}^k} \phi_{\text{sup}, \ell}^{\text{inf}, k} \right)} \right)^{\frac{1}{2}} \leq \frac{A_c^{\frac{1}{2}} a_c}{A_m^{\frac{1}{2}} a_m}. \quad (4.18)$$

Proof. See Section 7. □

Remark 4.3 (Robustness). *We can also make the same statement as in Remark 3.6, since the ratios $A_{c,\ell}^{k-1}/a_{m,\ell}^k$, $A_{m,\ell}^{k-1}/a_{c,\ell}^k$, and $\phi_{\text{sup},\ell}^{\text{inf},k}$ on each patch ω_ℓ^a are respectively sharper than the global constant ratios A_c/a_m , A_m/a_c , and a_m/a_c . They can compensate each other (according to the problem and the mesh), bringing \tilde{C}_ℓ^k close to 1. In fact, we can observe that $\phi_{\text{sup},\ell}^{\text{inf},k}$ approaches 1 when u_ℓ^k is converging to u in $H_0^1(\Omega)$, which means that, in the worst situation, $C_\ell^k \tilde{C}_\ell^k$ is close to $A_c a_c^{1/2}/(A_m a_m^{1/2})$. However, in contrast to C_ℓ^k , the constant \tilde{C}_ℓ^k cannot be fully computed since ∇u appears. We also remark that we need assumptions on ℓ and k in Theorem 4.2, whereas Theorem 3.4 is valid for every ℓ and k . Finally, \tilde{C}_ℓ^k is not purely given by the linearization method, cf. Remark 3.5.*

Remark 4.4 (Criterion (4.16)). *A global version of the criterion (4.16) holds for all $\ell \geq 0$ and only needs a large assumption on k , since*

$$\begin{aligned}
& \|(\mathbf{A}_\ell^{k-1})^{\frac{1}{2}} \nabla(u_\ell^k - u_{\langle \ell \rangle}^k)\|^2 \\
&= (\mathbf{A}_\ell^{k-1} \nabla(u_\ell^k - u_{\langle \ell \rangle}^k), \nabla(u_\ell^k - u_{\langle \ell \rangle}^k)) \\
&= (\mathbf{A}_\ell^{k-1} \nabla u_\ell^k - \mathbf{b}_\ell^{k-1}, \nabla(u_\ell^k - u_{\langle \ell \rangle}^k)) - (\mathbf{A}_\ell^{k-1} \nabla u_{\langle \ell \rangle}^k - \mathbf{b}_\ell^{k-1}, \nabla(u_\ell^k - u_{\langle \ell \rangle}^k)) \\
&\stackrel{(3.10a)}{=} (\mathbf{A}_\ell^{k-1} \nabla u_\ell^k - \mathbf{b}_\ell^{k-1}, \nabla(u_\ell^k - u_{\langle \ell \rangle}^k)) - (f, u_\ell^k - u_{\langle \ell \rangle}^k) \\
&\stackrel{(2.2a)}{=} (\mathbf{A}_\ell^{k-1} \nabla u_\ell^k - \mathbf{b}_\ell^{k-1}, \nabla(u_\ell^k - u_{\langle \ell \rangle}^k)) - (a_u \nabla u, \nabla(u_\ell^k - u_{\langle \ell \rangle}^k)) \\
&= ((\mathbf{A}_\ell^{k-1})^{-\frac{1}{2}} (\mathbf{A}_\ell^{k-1} \nabla u_\ell^k - \mathbf{b}_\ell^{k-1} - a_u \nabla u), (\mathbf{A}_\ell^{k-1})^{\frac{1}{2}} \nabla(u_\ell^k - u_{\langle \ell \rangle}^k)) \\
&\stackrel{(CS)}{\leq} \|(\mathbf{A}_\ell^{k-1})^{-\frac{1}{2}} (\mathbf{A}_\ell^{k-1} \nabla u_\ell^k - \mathbf{b}_\ell^{k-1} - a_u \nabla u)\| \|(\mathbf{A}_\ell^{k-1})^{\frac{1}{2}} \nabla(u_\ell^k - u_{\langle \ell \rangle}^k)\| \\
&\leq 2 \|(\mathbf{A}_\ell^{k-1})^{-\frac{1}{2}} (a_\ell^k \nabla u_\ell^k - a_u \nabla u)\| \|(\mathbf{A}_\ell^{k-1})^{\frac{1}{2}} \nabla(u_\ell^k - u_{\langle \ell \rangle}^k)\|,
\end{aligned} \tag{4.19}$$

where the last line comes from the fact that $\mathbf{A}_\ell^{k-1} \nabla u_\ell^k - \mathbf{b}_\ell^{k-1}$ is close to $a_\ell^k \nabla u_\ell^k$ in $\mathbf{L}^2(\Omega)$ when k is large enough (cf. Remark 3.7). However, the patch-local version (4.16) seems to also need ℓ large enough to ensure that u_ℓ^k is closer even locally to $u_{\langle \ell \rangle}^k$ than u_ℓ^k is to u in $H_0^1(\omega_\ell^a)$.

5 Numerical results

In this section, we present numerical experiments that serve to illustrate Theorems 3.4 and 4.2. Thus, we will primarily be interested in the effectivity indices

$$I_\ell^k := \frac{\eta_\ell^k}{\mathcal{E}_\ell^k}, \quad I_{N,\ell}^k := \frac{\eta_{N,\ell}^k}{\mathcal{E}_{N,\ell}^k}, \quad I_{L,\ell}^k := \frac{\eta_{L,\ell}^k}{\mathcal{E}_{L,\ell}^k}. \tag{5.1}$$

In particular, our numerical experiments study the robustness of our estimates with respect to the ratio a_c/a_m where we consider $a_c/a_m = 10^i$, $i \in \{0, \dots, 7\}$. We present results at convergence for the three linearization methods, i.e., Examples 2.4, 2.5, and 2.6 after the convergence criterion

$$\|\nabla(u_\ell^{k-1} - u_{\langle \ell \rangle}^k)\| < 10^{-6} \tag{5.2}$$

is satisfied.

In addition to V_ℓ^p as defined in Section 2.2, we also consider a richer discrete space $V_\ell^{\tilde{p}}$ obtained by refining the mesh \mathcal{T}_ℓ and using higher order polynomials, $p < \tilde{p}$ and $\ell < \tilde{\ell}$. This space serves as an approximation to $H_0^1(\Omega)$ so that we can approximately compute $u_{\langle \ell \rangle}^k$ defined in (3.10b) at each iteration k of the linearization method. This only serves here for the evaluation of the error $\mathcal{E}_{L,\ell}^k$ but is not needed to evaluate our estimators.

For all examples, we use the method of manufactured solutions, i.e. we choose a solution u and construct f through (1.1). The boundary conditions are then enforced by the true solution. All experiments were conducted using the `Gridap.jl` finite element software package [1, 41].

5.1 Smooth solution

We consider a unit square domain $\Omega = (0, 1)^2$. We set for all $(x, y) \in \Omega$,

$$u(x, y) := 10x(x-1)y(y-1). \quad (5.3)$$

For the space V_ℓ^p , we use a polynomial degree $p = 1$ and a uniform triangular mesh consisting of 8192 elements for a total of 3969 DOFs.

We now consider three different nonlinear functions $a : [0, \infty) \rightarrow (0, \infty)$ satisfying Assumption 2.1. We first consider the following example, in which the function a is decreasing.

Example 5.1 (Mean curvature nonlinearity). *The mean curvature nonlinearity (cf. [14]) is defined such that for all $r \in [0, \infty)$,*

$$a(r) := a_m + \frac{a_c - a_m}{\sqrt{1 + r^2}}, \quad (5.4)$$

where $a_m, a_c \in (0, \infty)$ with $a_m \leq a_c$. Observe that Assumption 2.1 holds for the mean curvature nonlinearity. Indeed, Proposition A.2 holds since we have for all $r \in [0, \infty)$,

$$\phi''(r) = a_m + \frac{a_c - a_m}{(1 + r^2)^{\frac{3}{2}}} \in [a_m, a_c].$$

The results for the effectivity indices (5.1) are presented in Figure 2, taking $a_m = 1$. We first remark that they vary only very mildly with respect to the ratio a_m/a_c and that all values are below 1.2. The robustness is anticipated for Zarantonello based on Remark 3.5. The fact that the ratios are so good for the Newton and Picard linearizations is the consequence of the fact that the constant C_ℓ^k of (3.22) is quite close to 1. We next remark that for all three solvers, the nonlinear effectivity index $I_{N,\ell}^k$ are close to 1 regardless of linearization method, which is to be expected since the results are computed at convergence criterion (5.2). Next, we remark that the results for the Picard iteration and Zarantonello are very similar for the linearization effectivity $I_{L,\ell}^k$ whereas the index for Newton behaves more like the nonlinear effectivity index. This could be explained by the fact that the constant C_ℓ^k in (3.22) is more complicated for the Newton iteration.

We now consider the following nonmonotone nonlinear function similar to the example given in [31, Section 5.3.2].

Example 5.2 (Exponential nonlinearity). *The exponential nonlinearity is defined such that for all $r \in [0, \infty)$,*

$$a(r) := a_m + (a_c - a_m) \frac{1 - e^{-\frac{3}{2}r^2}}{1 + 2e^{-\frac{3}{2}r^2}}, \quad (5.5)$$

where again $a_m, a_c \in (0, \infty)$ with $a_m \leq a_c$. Observe that Assumption 2.1 holds for the exponential nonlinearity. Indeed, Proposition A.2 holds since we have by construction that for all $r \in [0, \infty)$,

$$\phi''(r) = a_m + (a_c - a_m) \frac{1 + (3r^2 - 1)e^{-\frac{3}{2}r^2}}{1 + 2e^{-\frac{3}{2}r^2}} \in [a_m, a_c].$$

The results again with $a_m = 1$, are presented in Figure 3. The Picard linearization is not included because the solver does not converge for large values of the ratio a_c/a_m . We observe that the results for the Zarantonello iteration are similar to those of Figure 2. However, the effectivity indices of the Newton iteration follow an exponential law for large enough values of the ratio a_m/a_c . We can see that the reason is that the constant C_ℓ^k is becoming very large. This constant is thus here a good indicator that the robustness cannot be obtained when it is large. For this example, we also present, in Figure 4, the component errors $\lambda_\ell^k \mathcal{E}_{L,\ell}^k$ and $\mathcal{E}_{N,\ell}^k$, as well as the factor λ_ℓ^k . We observe that the quantities $\lambda_\ell^k \mathcal{E}_{L,\ell}^k$ and $\mathcal{E}_{N,\ell}^k$ stay very close, independently of the ratio a_c/a_m , which was our design. Remark, though, that $\lambda_\ell^k \simeq 1$ for Newton, whereas $\lambda_\ell^k \simeq (a_c/a_m)^{\frac{1}{2}}$ for Zarantonello.

5.2 Singular solution

We consider the L-shaped domain defined such that $\Omega = (-1, 1)^2 \setminus ([0, 1] \times (-1, 0])$ and the singular solution u defined in polar coordinates such that for all $(\rho, \theta) \in [0, \infty) \times [0, 2\pi]$,

$$u(\rho, \theta) = \rho^\alpha \sin(\alpha\theta) \quad (5.6)$$

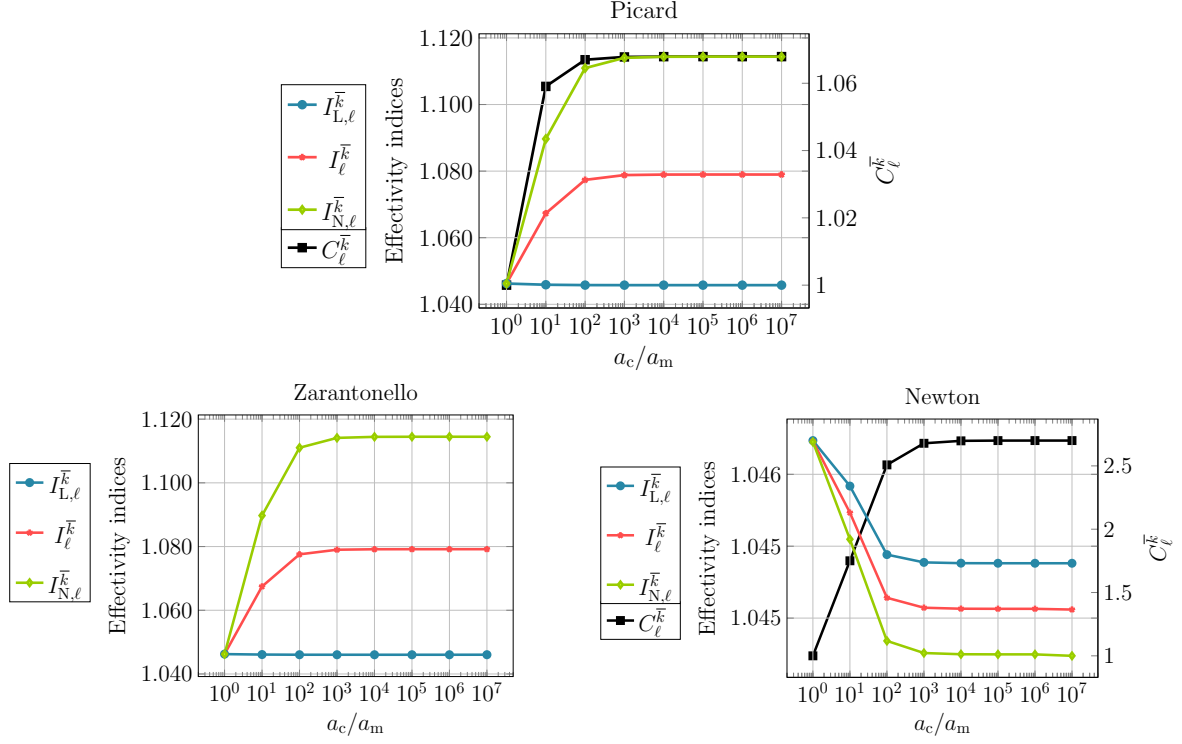


Figure 2: [Mean curvature nonlinearity (5.4), smooth solution (5.3), unit square domain, 3969 DOFs] Effectivity indices for the components and total quantities and the computable constant $C_\ell^{\bar{k}}$ from (3.22)

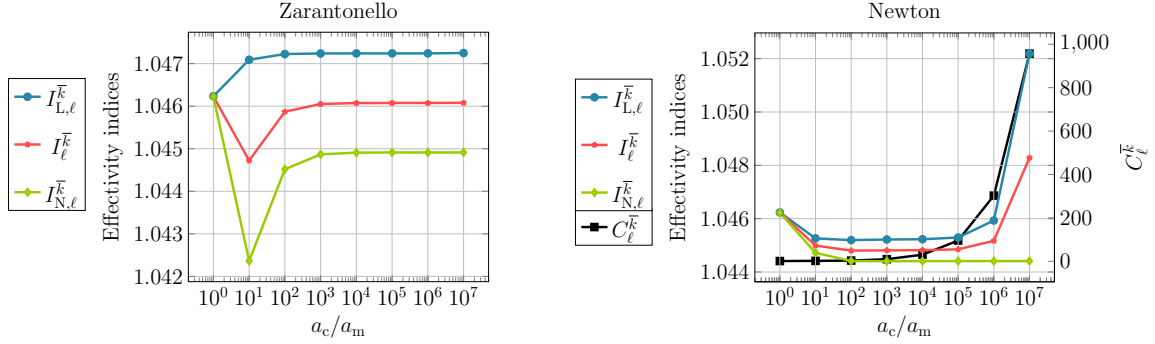


Figure 3: [Exponential nonlinearity (5.5), smooth solution (5.3), unit square domain, 3969 DOFs] Effectivity indices for the components and total quantities and the computable constant $C_\ell^{\bar{k}}$ from (3.22)

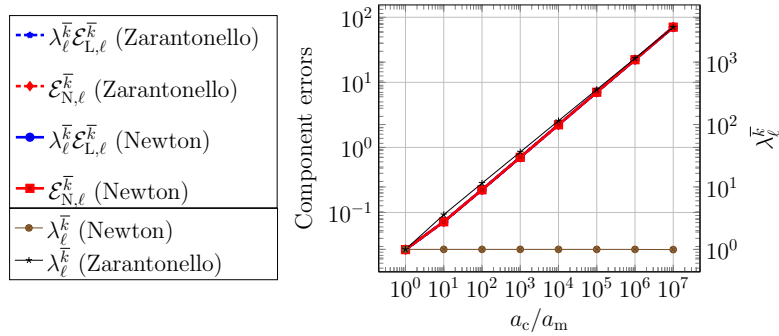


Figure 4: [Exponential nonlinearity (5.5), smooth solution (5.3), unit square domain, 3969 DOFs]

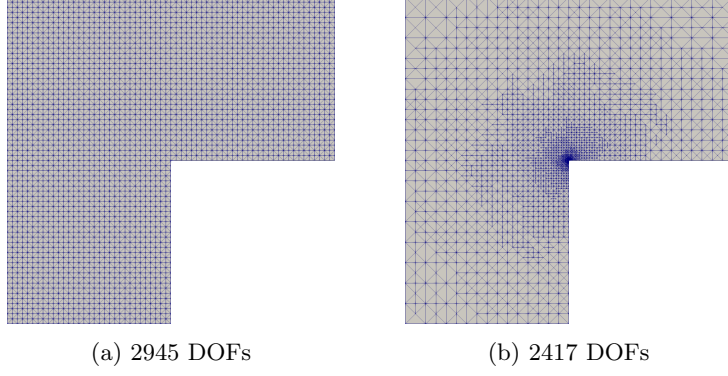


Figure 5: Uniformly (left) and adaptive (right) refined meshes for the L-shaped domain with the singular solution (5.6). The adaptive mesh corresponds to the 28th iteration of the algorithm.

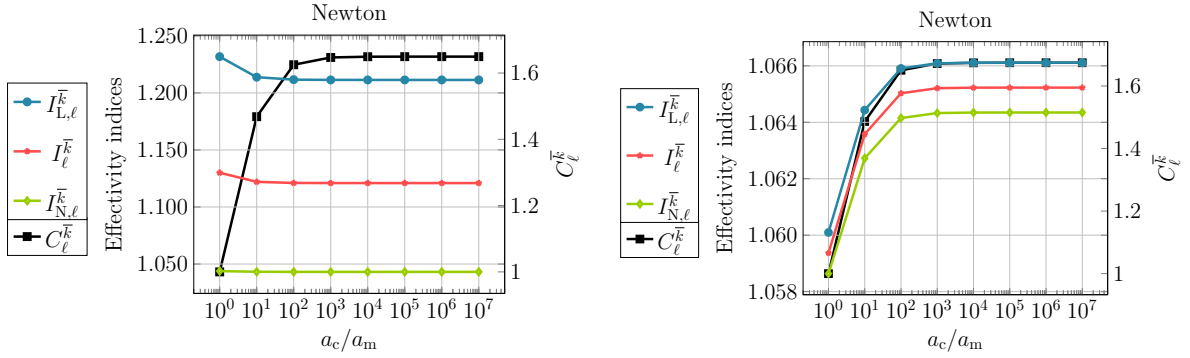


Figure 6: [Exponential nonlinearity (5.5), singular solution (5.6), L-shaped domain] Effectivity indices for the components and total quantities and the computable constant C_ℓ^k from (3.22), for the uniform mesh (left) and the adaptive mesh (right) shown in Figure 5.

with $\alpha := \frac{2}{3}$, so that $u \in H^{1+\frac{2}{3}-\varepsilon}(\Omega)$ for all $\varepsilon > 0$. We consider the exponential nonlinearity of Example 5.2 again with $a_m = 1$; this choice of solution ensures that the right-hand side f belongs to $L^2(\Omega)$ despite the singularity in the norm of the gradient for the L-shaped solution (5.6).

In particular, we consider two different meshes to analyze the results, see Figure 5. One mesh is obtained by taking an initial uniform triangulation of Ω , while the other one is adaptive following Algorithm 5.3.

The results are present in Figure 6. The Newton iteration presents, for both meshes, effectivity indices close to 1, which stabilize for large enough values of the ratio a_m/a_c . Moreover, the effectivity indices corresponding to the adaptive meshes are closer to 1 than those corresponding to the uniform meshes.

5.3 Convergence on adaptively refined meshes

Due to the singularity in the solution of the previous section, it is of interest to consider a local adaptive mesh refinement strategy. We need to compute the estimator $\eta_{N,\ell}^k$ locally on each element and so that the quantity $\eta_{N,\ell}^k$ is positive, as discussed in, e.g., [4, Proposition 4.9]. In order to do that, we rewrite the estimator $\eta_{N,\ell}^k$ such that

$$\eta_{N,\ell}^k \stackrel{(7.2)}{=} \left(2 \int_{\Omega} (\phi^*(\cdot, |\sigma_\ell^k|) + \phi(\cdot, |\nabla u_\ell^k|) + \sigma_\ell^k \cdot \nabla u_\ell^k) \right)^{\frac{1}{2}} = \left(\sum_{K \in \mathcal{T}_\ell} (\eta_{N,K}^k)^2 \right)^{\frac{1}{2}} \quad (5.7a)$$

$$\text{with, for } K \in \mathcal{T}_\ell, \quad (\eta_{N,K}^k)^2 := \int_K 2 (\phi^*(\cdot, |\sigma_\ell^k|) + \phi(\cdot, |\nabla u_\ell^k|) + \sigma_\ell^k \cdot \nabla u_\ell^k). \quad (5.7b)$$

Hence, recalling the generalized Young inequality, cf. [32]: for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$,

$$\phi(\cdot, |\mathbf{x}|) + \phi^*(\cdot, |\mathbf{y}|) + \mathbf{x} \cdot \mathbf{y} \geq 0, \quad (5.8)$$

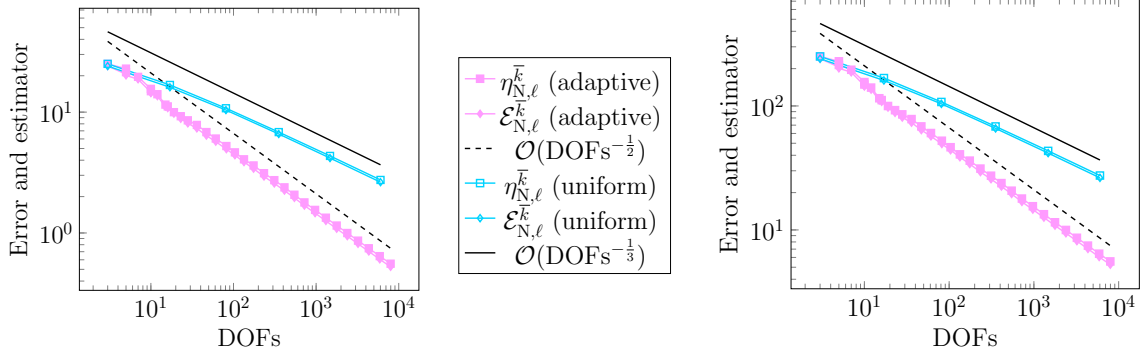


Figure 7: [Exponential nonlinearity (5.5), singular solution (5.6), L-shaped domain, uniform vs adaptive mesh refinement] Convergence rates of $\mathcal{E}_{N,\ell}^k$ and $\eta_{N,\ell}^k$ for uniform and adaptive mesh refinement for a_c/a_m equal to 10^3 (left) and 10^6 (right).

it follows that for all $K \in \mathcal{T}_\ell$, $\eta_{N,K}^k \geq 0$. We then use the standard newest vertex bisection algorithm, cf. [9], as follows.

Algorithm 5.3 (Adaptive refinement). *Let $\varepsilon_{\text{STOP}}$, $\theta \in (0, 1)$ be parameters, and let \mathcal{T}_0 be a conforming triangulation of Ω . Let $u_0^1 \in V_0^1$ be an initial guess. For $\ell \geq 0$:*

1. **Solve:** Solve the linearized problems (2.6) until the convergence criterion (5.2) is satisfied.
2. **Estimate:** Compute the elementwise estimators $(\eta_{N,K}^k)_{K \in \mathcal{T}_\ell}$ of (5.7b).
If $\eta_{N,\ell}^k < \varepsilon_{\text{STOP}}$, then stop.
3. **Mark:** Choose a set $\mathcal{M}_\ell \subset \mathcal{T}_\ell$ with minimal cardinality such that

$$\sum_{K \in \mathcal{M}_\ell} (\eta_{N,K}^k)^2 \geq \theta^2 \sum_{K \in \mathcal{T}_\ell} (\eta_{N,K}^k)^2. \quad (5.9)$$

4. **Refine:** Perform the newest vertex situation. Set $\ell := \ell + 1$ and restart with **Solve**.

The results of the refinement study are displayed in Figure 7, for the exponential nonlinearity (5.5) and the singular solution (5.6). We consider two values of the parameter a_c/a_m , namely 10^3 and 10^6 . We note that for both values of the ratio, the asymptotic rates for the estimator and error agree with the theoretical optimal rate of $(\text{DOFs})^{1/2}$ for the adaptive algorithm.

We made an analogous study on the augmented error \mathcal{E}_ℓ^k and estimator η_ℓ^k , with the same strategy of refinement, using the local version $\eta_K^k := \eta_{N,K}^k + \lambda_\ell^k \eta_{L,K}^k$, for all $K \in \mathcal{T}_\ell$, of the estimator η_ℓ^k . The results are displayed in Figure 8. We observe a similar behavior of the asymptotic rates than in Figure 7.

In conclusion, the adaptive method is more efficient than the uniform method since it requires a smaller number of DOFs for the same precision.

6 Proof of Theorem 3.4

For any $x, y \in [0, \infty)$, we use the notation $x \lesssim y$ when $x \leq Cy$ with $C \geq 0$ only depending on the space dimension d , the mesh shape-regularity $\kappa_\mathcal{T}$, and possibly when $d \geq 4$, the polynomial degree p .

Proof of Theorem 3.4. Let $k \geq 1$. Proceeding as in [19, 21], we have that $\mathcal{E}_{N,\ell}^k \leq \eta_{N,\ell}^k + \eta_{\text{osc},\ell}^k$ and $\mathcal{E}_{L,\ell}^k \leq \eta_{L,\ell}^k + \eta_{\text{osc},\ell}^k$, respectively, which yields

$$\begin{aligned} \mathcal{E}_\ell^k &\stackrel{(3.15a)}{=} \frac{1}{2}(\mathcal{E}_{N,\ell}^k + \lambda_\ell^k \mathcal{E}_{L,\ell}^k) \\ &\leq \frac{1}{2}(\eta_{N,\ell}^k + \eta_{\text{osc},\ell}^k + \lambda_\ell^k(\eta_{L,\ell}^k + \eta_{\text{osc},\ell}^k)) \stackrel{(3.15b)}{=} \eta_\ell^k + \frac{1 + \lambda_\ell^k}{2} \eta_{\text{osc},\ell}^k, \end{aligned} \quad (6.1)$$

hence (3.19). It remains to prove (3.21).

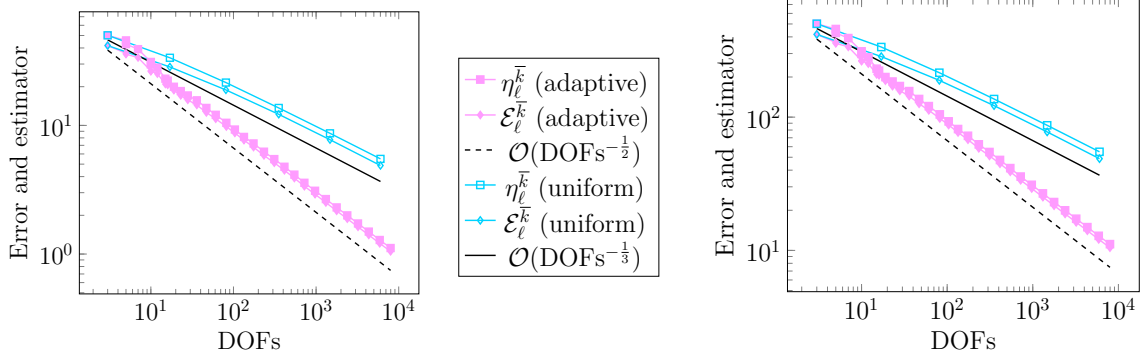


Figure 8: [Exponential nonlinearity (5.5), singular solution (5.6), L-shaped domain, uniform vs adaptive mesh refinement] Convergence rates of \mathcal{E}_ℓ^k and η_ℓ^k for uniform and adaptive mesh refinement for a_c/a_m equal to 10^3 (left) and 10^6 (right).

From arguments of [20] for the linearized problems (2.6a) and (3.10a), we have for all $\mathbf{a} \in \mathcal{V}_\ell$,

$$\eta_{L,\ell}^{\mathbf{a},k} \lesssim \left(\frac{\sup_{\omega_\ell^\mathbf{a}} A_{c,\ell}^{k-1}}{\inf_{\omega_\ell^\mathbf{a}} A_{m,\ell}^{k-1}} \right)^{\frac{1}{2}} \left(\mathcal{E}_{L,\ell}^{\mathbf{a},k} + \eta_{\text{osc,quad},\ell}^{\mathbf{a},k} \right). \quad (6.2)$$

Therefore, recalling the definition (3.22) of C_ℓ^k , we obtain

$$\begin{aligned} \eta_\ell^k &\stackrel{(3.13),(3.15b)}{=} \frac{1}{2} \left(\hat{\eta}_{L,\ell}^k + 1 \right) \lambda_\ell^k \eta_{L,\ell}^k \stackrel{(3.20)}{\lesssim} \lambda_\ell^k \eta_{L,\ell}^k \stackrel{(4.5)}{\lesssim} \lambda_\ell^k \left(\sum_{\mathbf{a} \in \mathcal{V}_\ell} (\eta_{L,\ell}^{\mathbf{a},k})^2 \right)^{\frac{1}{2}} \\ &\stackrel{(6.2)}{\lesssim} \lambda_\ell^k C_\ell^k \left(\sum_{\mathbf{a} \in \mathcal{V}_\ell} \left(\mathcal{E}_{L,\ell}^{\mathbf{a},k} + \eta_{\text{osc,quad},\ell}^{\mathbf{a},k} \right)^2 \right)^{\frac{1}{2}} \\ &\stackrel{(4.4),(3.18)}{\leq} (d+1)^{\frac{1}{2}} \lambda_\ell^k C_\ell^k \left(\mathcal{E}_{L,\ell}^k + \eta_{\text{osc,quad},\ell}^k \right) \stackrel{(3.15a)}{\lesssim} C_\ell^k \left(\mathcal{E}_\ell^k + \lambda_\ell^k \eta_{\text{osc,quad},\ell}^k \right), \end{aligned} \quad (6.3)$$

where we passed to the last line using the triangle inequality on $\ell_2(\mathbb{R}^{|\mathcal{V}_\ell|})$. Finally, we get $C_\ell^k = 1$ for the Zarantonello iteration since $A_{m,\ell}^{k-1} = A_{c,\ell}^{k-1} = \gamma$ in Ω in this case. \square

7 Proof of Theorem 4.2

Before proving Theorem 4.2, we give the proof of Lemma 4.1.

Proof of Lemma 4.1. Observing that by integration by parts (IBP), $\frac{\partial}{\partial r}((|\boldsymbol{\sigma}_\ell^k| - r)\phi^{*'}(\cdot, r)) = (|\boldsymbol{\sigma}_\ell^k| - r)\phi^{*''}(\cdot, r) - \phi^{*'}(\cdot, r)$, we obtain, a.e. in Ω ,

$$\begin{aligned} \phi^*(\cdot, |\boldsymbol{\sigma}_\ell^k|) - \phi^*(\cdot, a_\ell^k |\nabla u_\ell^k|) &= \int_{a_\ell^k |\nabla u_\ell^k|}^{|\boldsymbol{\sigma}_\ell^k|} \phi^{*'}(\cdot, r) \, dr \\ &\stackrel{(\text{IBP})}{=} \int_{a_\ell^k |\nabla u_\ell^k|}^{|\boldsymbol{\sigma}_\ell^k|} (|\boldsymbol{\sigma}_\ell^k| - r) \phi^{*''}(\cdot, r) \, dr - \left[(|\boldsymbol{\sigma}_\ell^k| - r) \phi^{*'}(\cdot, r) \right]_{a_\ell^k |\nabla u_\ell^k|}^{|\boldsymbol{\sigma}_\ell^k|} \\ &\stackrel{(3.7)}{=} \int_{a_\ell^k |\nabla u_\ell^k|}^{|\boldsymbol{\sigma}_\ell^k|} \frac{|\boldsymbol{\sigma}_\ell^k| - r}{\phi'' \circ \phi'^{-1}(\cdot, r)} \, dr + (|\boldsymbol{\sigma}_\ell^k| - a_\ell^k |\nabla u_\ell^k|) |\nabla u_\ell^k|, \end{aligned} \quad (7.1)$$

where, we used the fact that $\phi'^{-1}(\cdot, a_\ell^k |\nabla u_\ell^k|) = \phi'^{-1}(\cdot, \phi'(\cdot, |\nabla u_\ell^k|)) = |\nabla u_\ell^k|$ thanks to (A.2). Therefore,

recalling the definitions (2.3) of \mathcal{J} and (3.5) of \mathcal{J}^* , we have

$$\begin{aligned}
(\eta_{N,\ell}^k)^2 &\stackrel{(3.9b)}{=} 2 \int_{\Omega} (\phi^*(\cdot, |\sigma_{\ell}^k|) + \phi(\cdot, |\nabla u_{\ell}^k|)) - (f, u_{\ell}^k) \\
&\stackrel{(3.4)}{=} 2 \int_{\Omega} (\phi^*(\cdot, |\sigma_{\ell}^k|) + \phi(\cdot, |\nabla u_{\ell}^k|) + \sigma_{\ell}^k \cdot \nabla u_{\ell}^k) \\
&\stackrel{(3.6b)}{=} 2 \int_{\Omega} (\phi^*(\cdot, |\sigma_{\ell}^k|) - \phi^*(\cdot, a_{\ell}^k |\nabla u_{\ell}^k|) + a_{\ell}^k |\nabla u_{\ell}^k|^2 + \sigma_{\ell}^k \cdot \nabla u_{\ell}^k) \\
&\stackrel{(7.1)}{=} 2 \int_{\Omega} \left[\int_{a_{\ell}^k |\nabla u_{\ell}^k|}^{|\sigma_{\ell}^k|} \frac{|\sigma_{\ell}^k| - r}{\phi'' \circ \phi'^{-1}(\cdot, r)} dr + \frac{1}{a_{\ell}^k} (|\sigma_{\ell}^k| a_{\ell}^k |\nabla u_{\ell}^k| + \sigma_{\ell}^k \cdot (a_{\ell}^k \nabla u_{\ell}^k)) \right] \\
&\stackrel{(4.7)}{\leq} \int_{\Omega} \frac{1}{a_{m,\ell}^k} [(|\sigma_{\ell}^k| - a_{\ell}^k |\nabla u_{\ell}^k|)^2 + 2(|\sigma_{\ell}^k| a_{\ell}^k |\nabla u_{\ell}^k| + \sigma_{\ell}^k \cdot (a_{\ell}^k \nabla u_{\ell}^k))] \\
&\stackrel{(A.4a)}{=} \int_{\Omega} \frac{1}{a_{m,\ell}^k} |a_{\ell}^k \nabla u_{\ell}^k + \sigma_{\ell}^k|^2,
\end{aligned} \tag{7.2}$$

where, in using (3.6b), we have set $s = a_{\ell}^k |\nabla u_{\ell}^k|$, and where we used the fact that $|\sigma_{\ell}^k| a_{\ell}^k |\nabla u_{\ell}^k| + \sigma_{\ell}^k \cdot (a_{\ell}^k \nabla u_{\ell}^k) \geq 0$ thanks to the Cauchy-Schwarz inequality. Hence, we estimate $\eta_{N,\ell}^k$ by writing

$$\begin{aligned}
(\eta_{N,\ell}^k)^2 &\stackrel{(4.1b)}{\lesssim} \sum_{\mathbf{a} \in \mathcal{V}_{\ell}} \|(a_{m,\ell}^k)^{-\frac{1}{2}} (\psi_{\ell}^{\mathbf{a}} a_{\ell}^k \nabla u_{\ell}^k + \sigma_{\ell}^{\mathbf{a},k})\|_{\omega_{\ell}^{\mathbf{a}}}^2 \\
&\stackrel{(4.8)}{\leq} \sum_{\mathbf{a} \in \mathcal{V}_{\ell}} (\lambda_{\ell}^{\mathbf{a},k})^2 \|(A_{c,\ell}^{k-1})^{-\frac{1}{2}} (\psi_{\ell}^{\mathbf{a}} a_{\ell}^k \nabla u_{\ell}^k + \sigma_{\ell}^{\mathbf{a},k})\|_{\omega_{\ell}^{\mathbf{a}}}^2 \\
&\stackrel{(2.9a),(4.6b)}{\leq} \sum_{\mathbf{a} \in \mathcal{V}_{\ell}} (\lambda_{\ell}^{\mathbf{a},k} \hat{\eta}_{L,\ell}^{\mathbf{a},k})^2 \stackrel{(4.11)}{\leq} 4 \sum_{\mathbf{a} \in \mathcal{V}_{\ell}} (\lambda_{\ell}^{\mathbf{a},k} \eta_{L,\ell}^{\mathbf{a},k})^2.
\end{aligned} \tag{7.3}$$

Finally, as in the proof of Theorem 3.4, recalling the definition (4.9b) of $\tilde{\eta}_{\ell}^k$, we obtain

$$\begin{aligned}
\tilde{\eta}_{\ell}^k &\stackrel{(7.3)}{\lesssim} \left(\sum_{\mathbf{a} \in \mathcal{V}_{\ell}} (\lambda_{\ell}^{\mathbf{a},k} \eta_{L,\ell}^{\mathbf{a},k})^2 \right)^{\frac{1}{2}} \stackrel{(6.2)}{\lesssim} C_{\ell}^k \left(\sum_{\mathbf{a} \in \mathcal{V}_{\ell}} (\lambda_{\ell}^{\mathbf{a},k})^2 \left(\mathcal{E}_{L,\ell}^{\mathbf{a},k} + \eta_{\text{osc,quad},\ell}^{\mathbf{a},k} \right)^2 \right)^{\frac{1}{2}} \\
&\stackrel{(4.9a),(4.10)}{\leq} 4C_{\ell}^k \left(\tilde{\mathcal{E}}_{\ell}^k + \tilde{\eta}_{\lambda,\text{osc,quad},\ell}^k \right),
\end{aligned} \tag{7.4}$$

where we used the triangle inequality on $\ell_2(\mathbb{R}^{|\mathcal{V}_{\ell}|})$ to conclude. \square

Proof of Theorem 4.2. Inequality (4.15) is already contained in the proof of Theorem 3.4. It remains to prove (4.17). First, we have, a.e. in Ω ,

$$\frac{\int_{|\nabla u|}^{|\nabla u_{\ell}^k|} (|\nabla u_{\ell}^k| - r) \phi''(\cdot, r) dr}{\left(\int_{|\nabla u|}^{|\nabla u_{\ell}^k|} \phi''(\cdot, r) dr \right)^2} \geq \frac{\text{ess inf}_{[|\nabla u|, |\nabla u_{\ell}^k|]} \phi''}{2 \left(\text{ess sup}_{[|\nabla u|, |\nabla u_{\ell}^k|]} \phi'' \right)} \stackrel{(4.13)}{\geq} \frac{\phi_{\text{sup},\ell}^{\text{inf},k}}{2a_{c,\ell}^k}. \tag{7.5a}$$

Second, recalling that $a_u := a(\cdot, |\nabla u|)$ and $a_{\ell}^k := a(\cdot, |\nabla u_{\ell}^k|)$ and using the fact that $|a_{\ell}^k \nabla u_{\ell}^k| |a_u \nabla u| - (a_{\ell}^k \nabla u_{\ell}^k) \cdot (a_u \nabla u) \geq 0$ thanks to the Cauchy-Schwarz inequality together with the fact that $\phi_{\text{sup},\ell}^{\text{inf},k} \leq 1$ from (4.13b), we infer, a.e. in Ω ,

$$\begin{aligned}
&a_u (|\nabla u_{\ell}^k| |\nabla u| - \nabla u_{\ell}^k \cdot \nabla u) \\
&\stackrel{(4.13a)}{\geq} \frac{\phi_{\text{sup},\ell}^{\text{inf},k}}{a_{c,\ell}^k} [a_{\ell}^k \nabla u_{\ell}^k |a_u \nabla u| - (a_{\ell}^k \nabla u_{\ell}^k) \cdot (a_u \nabla u)] \\
&\stackrel{(A.4a)}{=} \frac{\phi_{\text{sup},\ell}^{\text{inf},k}}{2a_{c,\ell}^k} [a_{\ell}^k \nabla u_{\ell}^k - a_u \nabla u]^2 - (|a_{\ell}^k \nabla u_{\ell}^k| - |a_u \nabla u|)^2 \\
&\stackrel{(A.2)}{=} \frac{\phi_{\text{sup},\ell}^{\text{inf},k}}{2a_{c,\ell}^k} \left[a_{\ell}^k \nabla u_{\ell}^k - a_u \nabla u \right]^2 - \left(\int_{|\nabla u|}^{|\nabla u_{\ell}^k|} \phi''(\cdot, r) dr \right)^2.
\end{aligned} \tag{7.5b}$$

Hence, observing that $\frac{\partial}{\partial r}(|\nabla u_\ell^k| - r)\phi'(\cdot, r) = (|\nabla u_\ell^k| - r)\phi''(\cdot, r) - \phi'(\cdot, r)$, using a similar integration by parts (IBP) as in (7.2), we obtain

$$\begin{aligned}
(\mathcal{E}_{N,\ell}^k)^2 &\stackrel{(3.9a),(2.3)}{=} 2 \int_{\Omega} (\phi(\cdot, |\nabla u_\ell^k|) - \phi(\cdot, |\nabla u|)) - 2(f, u_\ell^k - u) \\
&\stackrel{(2.2a)}{=} 2 \int_{\Omega} \int_{|\nabla u|}^{|\nabla u_\ell^k|} \phi'(\cdot, r) dr - 2(a_u \nabla u, \nabla(u_\ell^k - u)) \\
&\stackrel{(IBP)}{=} 2 \int_{\Omega} \left(\int_{|\nabla u|}^{|\nabla u_\ell^k|} (|\nabla u_\ell^k| - r)\phi''(\cdot, r) dr - \left[(|\nabla u_\ell^k| - r)\phi'(\cdot, r) \right]_{|\nabla u|}^{|\nabla u_\ell^k|} \right) \\
&\quad - 2(a_u \nabla u, \nabla(u_\ell^k - u)) \\
&\stackrel{(A.2)}{=} 2 \int_{\Omega} \left(\int_{|\nabla u|}^{|\nabla u_\ell^k|} (|\nabla u_\ell^k| - r)\phi''(\cdot, r) dr + a_u(|\nabla u_\ell^k| |\nabla u| - \nabla u_\ell^k \cdot \nabla u) \right) \\
&\stackrel{(7.5)}{\geq} \int_{\Omega} \frac{\phi_{\sup,\ell}^{\inf,k}}{a_{c,\ell}^k} |a_\ell^k \nabla u_\ell^k - a_u \nabla u|^2 \\
&\stackrel{(4.1a),(4.14)}{\geq} \frac{1}{d+1} \sum_{\alpha \in \mathcal{V}_\ell} (\mu_\ell^{\alpha,k})^2 \|(\mathbf{A}_\ell^{k-1})^{-\frac{1}{2}} (a_\ell^k \nabla u_\ell^k - a_u \nabla u)\|_{\omega_\ell^\alpha}^2.
\end{aligned} \tag{7.6}$$

Therefore, we infer

$$\begin{aligned}
\sum_{\alpha \in \mathcal{V}_\ell} (\lambda_\ell^{\alpha,k} \mathcal{E}_{L,\ell}^{\alpha,k})^2 &\stackrel{(4.4b)}{=} \sum_{\alpha \in \mathcal{V}_\ell} (\lambda_\ell^{\alpha,k})^2 \|(\mathbf{A}_\ell^{k-1})^{\frac{1}{2}} \nabla(u_\ell^k - u_{\langle \ell \rangle}^k)\|_{\omega_\ell^\alpha}^2 \\
&\stackrel{(4.16)}{\leq} 2 \sum_{\alpha \in \mathcal{V}_\ell} (\lambda_\ell^{\alpha,k})^2 \|(\mathbf{A}_\ell^{k-1})^{-\frac{1}{2}} (a_\ell^k \nabla u_\ell^k - a_u \nabla u)\|_{\omega_\ell^\alpha}^2 \\
&\stackrel{(4.18)}{\leq} 2(\tilde{C}_\ell^k)^2 \sum_{\alpha \in \mathcal{V}_\ell} (\mu_\ell^{\alpha,k})^2 \|(\mathbf{A}_\ell^{k-1})^{-\frac{1}{2}} (a_\ell^k \nabla u_\ell^k - a_u \nabla u)\|_{\omega_\ell^\alpha}^2 \\
&\stackrel{(7.6)}{\leq} 2(d+1)(\tilde{C}_\ell^k \mathcal{E}_{N,\ell}^k)^2.
\end{aligned} \tag{7.7}$$

Hence, from Lemma 4.1, we obtain

$$\eta_{N,\ell}^k \stackrel{(7.3)}{\lesssim} \tilde{\eta}_\ell^k \stackrel{(4.12)}{\lesssim} C_\ell^k (\tilde{\mathcal{E}}_\ell^k + \tilde{\eta}_{\lambda,\text{osc,quad},\ell}^k) \stackrel{(7.7)}{\lesssim} C_\ell^k \left(\tilde{C}_\ell^k \mathcal{E}_{N,\ell}^k + \tilde{\eta}_{\lambda,\text{osc,quad},\ell}^k \right). \tag{7.8}$$

□

A Equivalent assumptions on the nonlinear function

In this section, we show some useful properties of the nonlinear function. In particular, we show that inequalities (2.1) admit equivalent versions with the function ϕ , preserving the same constants a_c and a_m .

Proposition A.1 (Equivalent assumption on ϕ'). *Inequalities (2.1) are equivalent to the following ones: a.e. in Ω , for all $r, s \in [0, \infty)$,*

$$|\phi'(\cdot, r) - \phi'(\cdot, s)| \leq a_c |r - s|, \tag{A.1a}$$

$$(\phi'(\cdot, r) - \phi'(\cdot, s))(r - s) \geq a_m (r - s)^2. \tag{A.1b}$$

Proof. Differentiating (2.4) gives a.e. in Ω , for all $r \in [0, \infty)$,

$$\phi'(\cdot, r) = a(\cdot, r)r. \tag{A.2}$$

Thus, assuming and using (2.1) (with $\mathbf{x} = (r, 0, \dots, 0)^t$ and $\mathbf{y} = (s, 0, \dots, 0)^t$) together with (A.2) yields (A.1).

Reciprocally, assuming (A.1), we have in particular (with $s = 0$) together with (A.2) that a.e. in Ω , for all $r \in [0, \infty)$,

$$a_m \leq a(\cdot, r) \leq a_c. \quad (\text{A.3})$$

We conclude by using the fact that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, and all $\alpha, \beta \in [0, \infty)$, we have

$$|\mathbf{x} - \mathbf{y}|^2 = (|\mathbf{x}| - |\mathbf{y}|)^2 + 2(|\mathbf{x}||\mathbf{y}| - \mathbf{x} \cdot \mathbf{y}), \quad (\text{A.4a})$$

$$(\alpha \mathbf{x} - \beta \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) = (\alpha |\mathbf{x}| - \beta |\mathbf{y}|)(|\mathbf{x}| - |\mathbf{y}|) + (\alpha + \beta)(|\mathbf{x}||\mathbf{y}| - \mathbf{x} \cdot \mathbf{y}), \quad (\text{A.4b})$$

to obtain a.e. in Ω , for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$,

$$\begin{aligned} & |a(\cdot, |\mathbf{x}|)\mathbf{x} - a(\cdot, |\mathbf{y}|)\mathbf{y}|^2 \\ & \stackrel{(\text{A.4a})}{=} (a(\cdot, |\mathbf{x}|)|\mathbf{x}| - a(\cdot, |\mathbf{y}|)|\mathbf{y}|)^2 + 2a(\cdot, |\mathbf{x}|)a(\cdot, |\mathbf{y}|)(|\mathbf{x}||\mathbf{y}| - \mathbf{x} \cdot \mathbf{y}) \\ & \stackrel{(\text{A.2})}{=} (\phi'(\cdot, |\mathbf{x}|) - \phi'(\cdot, |\mathbf{y}|))^2 + 2a(\cdot, |\mathbf{x}|)a(\cdot, |\mathbf{y}|)(|\mathbf{x}||\mathbf{y}| - \mathbf{x} \cdot \mathbf{y}) \\ & \stackrel{(\text{A.1}), (\text{A.3})}{\leq} a_c^2[(|\mathbf{x}| - |\mathbf{y}|)^2 + 2(|\mathbf{x}||\mathbf{y}| - \mathbf{x} \cdot \mathbf{y})] = a_c^2|\mathbf{x} - \mathbf{y}|^2 \end{aligned}$$

and

$$\begin{aligned} & (a(\cdot, |\mathbf{x}|)\mathbf{x} - a(\cdot, |\mathbf{y}|)\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) \\ & \stackrel{(\text{A.4b})}{=} (a(\cdot, |\mathbf{x}|)|\mathbf{x}| - a(\cdot, |\mathbf{y}|)|\mathbf{y}|)(|\mathbf{x}| - |\mathbf{y}|) + (a(\cdot, |\mathbf{x}|) + a(\cdot, |\mathbf{y}|))(|\mathbf{x}||\mathbf{y}| - \mathbf{x} \cdot \mathbf{y}) \\ & \stackrel{(\text{A.2})}{=} (\phi'(\cdot, |\mathbf{x}|) - \phi'(\cdot, |\mathbf{y}|))(|\mathbf{x}| - |\mathbf{y}|) + (a(\cdot, |\mathbf{x}|) + a(\cdot, |\mathbf{y}|))(|\mathbf{x}||\mathbf{y}| - \mathbf{x} \cdot \mathbf{y}) \\ & \stackrel{(\text{A.1}), (\text{A.3})}{\geq} a_m[(|\mathbf{x}| - |\mathbf{y}|)^2 + 2(|\mathbf{x}||\mathbf{y}| - \mathbf{x} \cdot \mathbf{y})] = a_m|\mathbf{x} - \mathbf{y}|^2, \end{aligned}$$

hence (2.1). \square

Proposition A.2 (Equivalent assumption on ϕ''). *Inequalities (A.1) are equivalent to the facts that ϕ' is weakly differentiable and a.e. in Ω , for a.e. $r \in [0, \infty)$,*

$$a_m \leq \phi''(\cdot, r) \leq a_c. \quad (\text{A.5})$$

Proof. Assuming (A.1), since ϕ' is Lipschitz continuous thanks to (A.1a), ϕ' is weakly differentiable. Furthermore, defining the difference quotient a.e. in Ω , for all $r, s \in [0, \infty)$, $r \neq s$, by

$$\tau(\cdot, r, s) := \frac{\phi'(\cdot, r) - \phi'(\cdot, s)}{r - s},$$

inequalities (A.1) are equivalent to the fact that a.e. in Ω , for all $r, s \in [0, \infty)$, $r \neq s$,

$$a_m \leq \tau(\cdot, r, s) \leq a_c. \quad (\text{A.6})$$

Hence, letting s tend to r in (A.6) gives (A.5).

On the other hand, assuming and integrating (A.5) gives a.e. in Ω , for a.e. $r, s \in [0, \infty)$ with $r > s$,

$$a_m(r - s) \leq \phi'(\cdot, r) - \phi'(\cdot, s) \leq a_c(r - s), \quad (\text{A.7})$$

and dividing (A.7) by $r - s$ gives (A.6) which becomes also true for $r < s$ by symmetry. \square

Remark A.3 (Convexity). *Under Assumption 2.1, inequality (A.1b) implies that ϕ' is nondecreasing, i.e., ϕ is convex. Moreover, from (A.2), we have a.e. in Ω , for a.e. $r \in [0, \infty)$,*

$$\phi''(\cdot, r) = a(\cdot, r) + a'(\cdot, r)r. \quad (\text{A.8})$$

B Spectral properties of the tensor product

Lemma B.1 (Spectral properties of tensor product). *The following properties hold:*

$$\text{Spec}(\alpha \mathbf{I}_d + \beta \mathbf{A}) = \{\alpha\} + \beta \text{Spec}(\mathbf{A}) \quad \forall \alpha, \beta \in \mathbb{R}, \forall \mathbf{A} \in \mathbb{R}^{d \times d}, \quad (\text{B.1a})$$

$$\text{Spec}(\boldsymbol{\xi} \otimes \boldsymbol{\xi}) = \{0, |\boldsymbol{\xi}|^2\} \quad \forall \boldsymbol{\xi} \in \mathbb{R}^d, d > 1. \quad (\text{B.1b})$$

Proof. We refer to [13] for more details about the tools used in the following. Denoting $P_{\mathbf{A}}$ the characteristic polynomial of \mathbf{A} , we obtain (B.1a) by writing for all $\lambda \in \mathbb{R}$,

$$P_{\alpha \mathbf{I}_d + \beta \mathbf{A}}(\alpha + \beta \lambda) = \det((\alpha + \beta \lambda) \mathbf{I}_d - (\alpha \mathbf{I}_d + \beta \mathbf{A})) = \beta^d \det(\lambda \mathbf{I}_d - \mathbf{A}) = \beta^d P_{\mathbf{A}}(\lambda).$$

Moving to (B.1b), since $(\boldsymbol{\xi} \otimes \boldsymbol{\xi})\boldsymbol{\tau} = (\boldsymbol{\xi} \cdot \boldsymbol{\tau})\boldsymbol{\xi}$ for all $\boldsymbol{\tau} \in \mathbb{R}^d$, $\dim(\text{Ker}(\boldsymbol{\xi} \otimes \boldsymbol{\xi})) = \dim(\text{Ker}(\langle \boldsymbol{\xi}, \cdot \rangle)) \geq d - 1$, i.e. $0 \in \text{Spec}(\boldsymbol{\xi} \otimes \boldsymbol{\xi})$ with a multiplicity of at least $d - 1$. Thus, the sum of the eigenvalues of $\boldsymbol{\xi} \otimes \boldsymbol{\xi}$, being $\text{tr}(\boldsymbol{\xi} \otimes \boldsymbol{\xi}) = |\boldsymbol{\xi}|^2$, is in $\text{Spec}(\boldsymbol{\xi} \otimes \boldsymbol{\xi})$. Hence, 0 and $|\boldsymbol{\xi}|^2$ are the only elements of $\text{Spec}(\boldsymbol{\xi} \otimes \boldsymbol{\xi})$, and we infer (B.1b). \square

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