Fields of Definition of Components of Hurwitz Spaces

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Abstract. For a fixed finite group G, we study the fields of definition of geometrically irreducible components of Hurwitz moduli schemes of marked branched G-covers of the projective line. The main focus is on determining whether components obtained by "gluing" two other components, both defined over a number field K, are also defined over K. The article presents a list of situations in which a positive answer is obtained. As an application, when G is a semi-direct product of symmetric groups or the Mathieu group M_{23}, components defined over $\mathbb{Q}$ of small dimension (6 and 4, respectively) are shown to exist.

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Contents

1. Introduction

Let G be a finite group and K be a field of characteristic zero. Hurwitz schemes are moduli spaces of branched G-covers of $\mathbb{P}^1$. Their K-points are particularly significant for number theory since they are tightly related to the inverse Galois problem for G over $K(t)$; see [RW06, Fri77, FV91].

When K is algebraically closed, Riemann's existence theorem implies that the theory reduces to topology. Specifically, $\mathbb{C}$-points of Hurwitz schemes correspond to isomorphism classes of topological G-covers of punctured Riemann spheres. A classical topological construction lets one "glue" two marked covers – one with r_1 branch points and one with r_2 branch points – into a single marked cover with $r_1 + r_2$ branch points. This gluing operation plays a central role in [EVW16].

When K is a discrete complete valued field, Harbater defined an analogous *patching* operation to construct covers defined over K with a specified monodromy group by patching together covers with smaller monodromy groups; see [Har03, HV96, Liu95]. This construction leads to a positive answer to the inverse Galois problem over $K(T)$.

For number theorists, the most interesting case is that of number fields. However, no gluing or patching operation is known in this case, although it would be a game-changing tool for inverse Galois theory. In this article, we focus not on G-covers themselves but on geometrically irreducible

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components of Hurwitz moduli schemes. Identifying components defined over $\mathbb{Q}$ is a crucial first step in finding rational points on these schemes. For this reason, the question of the fields of definition of these components is a well-studied topic; see [Cau12, DE06, EVW12, FV91].

Assume K is a number field. The gluing operation over $\overline{K}$ induces a monoid structure on the set $\mathrm{Comp}(G)$ of geometrically irreducible components of Hurwitz moduli schemes of marked branched G-covers of $\mathbb{P}^1$. In [Seg22], we studied this product operation and the corresponding monoid ring, introduced in [EVW16] under the name *ring of components*. To understand the arithmetic properties of the topological gluing operation, a prominent question is the following:

Question 1.1. Let $x, y \in \mathrm{Comp}(G)$ be components defined over K. Is the component xy, obtained by gluing x and y, defined over K?

Question 1.1 and related problems are the main focus of this article. Our main result is that the answer is positive in situations (i), (ii) and (iii) below:

Theorem 1.2. *Let $x, y \in \mathrm{Comp}(G)$ be components defined over K. Denote by H_1 and H_2 the monodromy groups of the covers contained in x and y respectively, and let $H = \langle H_1, H_2 \rangle$. Then:*

(i) If $H_1 H_2 = H$, then the glued component xy is defined over K.

(ii) If every conjugacy class of H that is a local monodromy class of the covers in the component xy occurs at at least M branch points (for some constant M which depends only on the group G), then xy is defined over K.

(iii) There are elements $\gamma, \gamma' \in H$ satisfying $\left\langle H_1^{\gamma}, H_2^{\gamma'} \right\rangle = H$ such that the component $x^{\gamma} y^{\gamma'}$, obtained by letting γ, γ' act on x, y and by gluing the resulting components, is defined over K.

In Section 2, we establish the notation and introduce the key objects. The three parts of Theorem 1.2 are then proved in three corresponding sections:

- Theorem 1.2 (i) (which is Theorem 3.3 (iii)) is proved using techniques introduced by [Cau12] and lemmas about the braid group action on tuples. In Section 3, we present this result, including a generalized version, and propose applications. Cases of interest include the situation where H_1 or H_2 is normal in H, notably if one is included in the other. A consequence of our ideas is presented in Subsection 3.4: we show that the Galois action on components is entirely determined by the Galois action on components with few branch points; precise statements are given in Propositions 3.8 and 3.9. An application of Theorem 1.2 (i) is given in Example 3.6: if G is a semi-direct product of symmetric groups, there is a component defined over $\mathbb{Q}$ of connected G-covers with six branch points. This improves on a similar example of Cau where twelve branch points were used.

- Theorem 1.2 (ii) (which is Theorem 4.7 (iii)) is proved using the *lifting invariant*, defined in [EVW12, Woo21] after ideas of Fried [Fri95], and a version of the Conway-Parker theorem found in these articles. In Section 4, we review this invariant and use it to determine the fields of definition of glued components.

- Theorem 1.2 (iii) (which is Theorem 5.4) is based on *patching results* over complete valued fields. We follow the algebraic patching approach from [HV96]. By patching covers over infinitely many complete valued fields, we obtain a result concerning the fields of definition of components in the non-complete case. Section 5 is concerned with the proof of this theorem. Application are given in Examples 5.5 and 5.6: when G is the Mathieu group M_{23} or the transitive group $\mathrm{PSL}_2(16) \rtimes \mathbb{Z}/2\mathbb{Z}$, there is a component defined over $\mathbb{Q}$ of connected G-covers with only four branch points.

We do not know if the answer to Question 1.1 is always positive. Finding counterexamples is difficult because there are few tools available to prove that components are not defined over $\mathbb{Q}$. For instance, the lifting invariant cannot be used to find a counterexample, as established by Theorem 4.10. Indeed, the lifting invariant of a product of components defined over K is invariant under the Galois action: from the point of view of this invariant, products of components defined over K are indistinguishable from components defined over K.

2. Preliminaries

In this section, we define key objects and review classical results concerning G-covers, their moduli spaces, their components and the Galois action. In Subsection 2.1, notational and terminological choices are presented. In Subsection 2.2, we introduce the main objects of this article, notably G-covers and Hurwitz spaces, both topological and algebraic. Then, a description of the Galois action of Γ_K on G-covers and their components is given in Subsection 2.3 and used to define fields of definitions of G-covers and of components of Hurwitz spaces. Finally, in Subsection 2.4, we give various versions of the *branch cycle lemma*, a fundamental result concerning the Galois action on covers and components.

2.1. Notation

We use the following notation and terminology throughout the article. In what follows, G is a finite group and K is a number field. Number fields are always equipped with an embedding into $\overline{\mathbb{Q}}$. We denote by Γ_K the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}} \mid K)$. The cyclotomic character is the group morphism $\chi : \Gamma_K \to \widehat{\mathbb{Z}}^\times$ determined by the Galois action on roots of unity: if $\zeta \in \overline{\mathbb{Q}}$ is an n-th root of unity and $\sigma \in \Gamma_K$, then $\sigma(\zeta) = \zeta^{\chi(\sigma) \mod n}$.

2.1.1. Conventions. The cardinality of a set X is denoted by $|X|$. We write $g^h = hgh^{-1}$ for conjugation in a group. We denote by $\mathrm{ord}(g)$ the order of an element g in a finite group H. If $g \in H$ and $\alpha \in \widehat{\mathbb{Z}}$ is a profinite integer, g^α is the well-defined element $g^{\alpha \mod \mathrm{ord}(g)} \in H$. Similarly, if $c \subseteq H$ is a conjugacy class, the order of its elements is denoted by $\mathrm{ord}(c)$, and c^α is the conjugacy class of the α-th powers of elements of c, where α is either an integer or a profinite integer.

Definition 2.1. A subset c of G is *K-rational* if for every $g \in c$ and $\sigma \in \Gamma_K$ we have $g^{\chi(\sigma)} \in c$.

If $K = \mathbb{Q}$, we have $\mathrm{Im}(\chi) = \widehat{\mathbb{Z}}^\times$. Therefore, $\mathbb{Q}$-rational subsets are subsets closed under n-th powers for all n coprime with $|G|$. In contrast, if K contains all $|G|$-th roots of unity, then the image of χ is trivial modulo $|G|$ and every subset of G is K-rational. Examples of sets which are always K-rational include G, $G \setminus \{1\}$, as well as any subset of G consisting of involutions.

2.1.2. Tuples. Tuples are denoted with underlined roman letters. Let $\underline{g} = (g_1, \ldots, g_n)$ be a tuple of elements of G. Then:

- Its *degree* or *size* $\deg(\underline{g})$ is the number n of elements in the tuple.
- Its *group* $\langle \underline{g} \rangle$ is the subgroup of G generated by $g_1, \ldots, g_n$. If $\underline{g}_1, \ldots, \underline{g}_s$ are tuples, we denote by $\langle \underline{g}_1, \ldots, \underline{g}_s \rangle$ the subgroup of G generated by the subgroups $\langle \underline{g}_1 \rangle, \ldots, \langle \underline{g}_s \rangle$.
- The *product* of $\underline{g}$ is $\pi\underline{g} = g_1 g_2 \cdots g_n \in G$. We say that $\underline{g}$ is a *product-one tuple* if $\pi\underline{g} = 1$.
- Let H be a subgroup of G containing $\langle \underline{g} \rangle$. A conjugacy class γ of H *appears in* $\underline{g}$ if there is some i for which $g_i \in \gamma$. The set of the conjugacy classes of H which appear in $\underline{g}$ is denoted by $D_H(\underline{g})$. We denote by $c_H(\underline{g})$ the conjugation-invariant subset of H obtained as the union of all classes in $D_H(\underline{g})$. If H is not specified in the notation, it is assumed that $H = \langle \underline{g} \rangle$.

- Let H be a subgroup of G which contains $\left\langle \underline{g} \right\rangle$ and c be a conjugation-invariant subset of H which contains $c_H(\underline{g})$. If γ is a conjugacy class of H contained in c, we denote by $\mu_{H,c}(\underline{g})(\gamma)$ the count of its appearances in $\underline{g}$, i.e.:

$$\mu_{H,c}(\underline{g})(\gamma) = \left| \left\{ i \in \{1, \dots, n\} \,\middle|\, g_i \in \gamma \right\} \right|.$$

  This defines an integer-valued map $\mu_{H,c}(\underline{g})$ on the set D of all conjugacy classes of H contained in c. We call this map the (H, c)-*multidiscriminant of* $\underline{g}$. If c is not specified in the notation $\mu_H(\underline{g})$, it is assumed that $c = c_H(\underline{g})$.

2.1.3. Schemes. Let L be a field. Here, L-*schemes* are separated schemes equipped with a morphism into $\mathrm{Spec}(L)$. Let $L' \mid L$ be a field extension and X be an L-scheme of finite type. We denote by $X_{L'}$ the L'-scheme $X \underset{\mathrm{Spec}(L)}{\times} \mathrm{Spec}(L')$ obtained by extending the scalars. The set $X(L')$ of L'-points of X is the set of morphisms of L-schemes from $\mathrm{Spec}(L')$ to X, or equivalently morphisms of L'-schemes from $\mathrm{Spec}(L')$ to $X_{L'}$. An L'-point $x \in X(L')$ is L-*rational* if there exists an L-point $x' \in X(L)$ such that the following diagram of L-schemes commutes:

$$\begin{array}{ccccc} \mathrm{Spec}(L') & \longrightarrow & \mathrm{Spec}(L) & \xrightarrow{x'} & X \\ & \searrow^{x} & & \nearrow & \end{array}$$

The point x' is called an L-*model* of the point x. Similarly, an L'-subscheme Y of $X_{L'}$ is *defined over* L if there exists an L-subscheme Y' of X such that the L'-subscheme $Y'_{L'}$ of $X_{L'}$ is equal to Y. We say that Y' is an L-*model* of Y.

Assume now $L' \mid L$ is Galois. An automorphism $\sigma \in \mathrm{Gal}(L' \mid L)$ induces an L-automorphism of $\mathrm{Spec}(L')$ which we denote by $\mathrm{Spec}(\sigma)$. The group $\mathrm{Gal}(L' \mid L)$ acts on an L'-point $x \in X(L')$ by the formula $\sigma.x = x \circ \mathrm{Spec}(\sigma)$ and on an L'-subscheme $Y \subseteq X_{L'}$ by pullback along $\mathrm{id}_X \underset{\mathrm{Spec}(L)}{\times} \mathrm{Spec}(\sigma)$:

$$\begin{array}{ccc} \mathrm{Spec}(L') & & \\ {\scriptstyle \mathrm{Spec}(\sigma)}\downarrow & \searrow^{\sigma.x} & \\ \mathrm{Spec}(L') & \xrightarrow[x]{} & X \end{array} \qquad \begin{array}{ccc} \sigma.Y & \longrightarrow & X_{L'} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{id}_X \underset{\mathrm{Spec}(L)}{\times} \mathrm{Spec}(\sigma)} \\ Y & \longrightarrow & X_{L'} \end{array}.$$

Proposition 2.2 (Galois descent)**.** *The following equivalences hold:*

- *An L'-point of X is L-rational if and only if it is invariant under the action of* $\mathrm{Gal}(L' \mid L)$.
- *An L'-subscheme Y of $X_{L'}$ is defined over L if and only if it is globally preserved by the action of* $\mathrm{Gal}(L' \mid L)$*, i.e. for every $\sigma \in \mathrm{Gal}(L' \mid L)$ there is an L'-automorphism σ' of Y such that the following diagram commutes:*

$$\begin{array}{ccc} Y & \xhookrightarrow{\subseteq} & X_{L'} \\ {\scriptstyle \sigma'}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_X \underset{\mathrm{Spec}(L)}{\times} \mathrm{Spec}(\sigma)} \\ Y & \xhookrightarrow{\subseteq} & X_{L'} \end{array}.$$

- *If L' is algebraically closed and Y is a reduced L'-subscheme of $X_{L'}$, then Y is defined over K if and only if the subset $Y(L')$ of $X(L')$ is globally preserved by the action of* $\mathrm{Gal}(L' \mid L)$.

2.2. Main objects

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In this section, we define the main objects which are used in the text. Configuration spaces (Paragraph 2.2.1), G-covers (Paragraphs 2.2.2 and 2.2.6) and Hurwitz spaces (Paragraphs 2.2.3 and 2.2.7) are introduced, both in the topological and in the algebraic settings, and the links between the two contexts are explicited. The combinatorial description of G-covers and their components is also stressed (Paragraphs 2.2.3 to 2.2.5).

2.2.1. Configurations and braid groups. A *configuration* $\underline{t} = \{t_1, \dots, t_n\}$ is an unordered list of n distinct complex numbers. Configurations form a topological space $\mathrm{Conf}_n(\mathbb{C})$, whose topology is inherited from the standard topology on $\mathbb{C}^n$ after removing tuples with equal elements and quotienting out by the action of $\mathfrak{S}_n$. The fundamental group of $\mathrm{Conf}_n(\mathbb{C})$ is the *Artin braid group* B_n. It admits a simple presentation: generators are given by the *elementary braids* $\sigma_1, \dots, \sigma_{n-1}$ satisfying the following generating set of relations:

- $\sigma_i\sigma_j = \sigma_j\sigma_i$ for all $i, j \in \{1, \dots, n-1\}$ satisfying $|i-j| > 1$;
- $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for all $i \in \{1, \dots, n-2\}$.

A configuration $\underline{t} \in \mathrm{Conf}_n(\mathbb{C})$ is *defined over* K if the elements $t_1, \dots, t_n$ are all algebraic and are permuted by the Galois action of $\mathrm{Gal}(\overline{\mathbb{Q}} \mid K)$. We denote by $\mathrm{Conf}_n(K)$ the set of configurations of $\mathrm{Conf}_n(\mathbb{C})$ defined over K.

The space of configurations has a scheme counterpart. Indeed, fixing a configuration $\underline{t} = \{t_1, \dots, t_n\}$ amounts to fixing the monic polynomial $(X - t_1) \cdots (X - t_n)$, of degree n with no double roots; we can parametrize these polynomials by their coefficients instead of their roots: the scheme Conf_n is the open subvariety of $\mathbb{A}^n$ obtained by removing the closed Zariski subset Δ defined by the polynomial equation "the discriminant of $X^n + a_1X^{n-1} + \ldots + a_{n-1}X + a_n$ cancels". The K-points of Conf_n are in natural bijection with the configurations of n points defined over K, and its $\mathbb{C}$-points are precisely the elements of $\mathrm{Conf}_n(\mathbb{C})$, which makes the notation unambiguous.

2.2.2. Topological G-covers and tuples. In this article, we consider branched G-covers of the projective line. Let $\underline{t} \in \mathrm{Conf}_n(\mathbb{C})$ be a configuration. Topological G-covers branched at $\underline{t}$ are covering maps $p : X \to \mathbb{P}^1(\mathbb{C}) \setminus \underline{t}$ equipped with a morphism $G \to \mathrm{Aut}(p)$ that induces a free transitive action on each fiber. Note that G-covers are necessarily Galois covers of degree $|G|$. We do not assume that these covers are connected, and we allow trivial ramification at the "branch points", even if the G-cover can be extended into a G-cover with less branch points.

A *marked G-cover* is a G-cover equipped with a marked point in the unramified fiber above the point at infinity ∞. The monodromy based at ∞ associates to every marked G-cover p branched at $\underline{t}$ a morphism:

$$\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{t_1, \dots, t_n\}, \infty) \to G$$

which uniquely characterizes the isomorphism class of the marked G-cover p.

Choose a topological bouquet $\gamma_1, \dots, \gamma_n$, as defined in [DE06, Paragraph 1.1]. This is a set of generators of $\pi_1(\mathbb{P}^1 \setminus \{t_1, \dots, t_n\}, \infty)$, where γ_i is the homotopy class of a loop which rotates once counterclockwise around t_i. The relations between these generators are generated by the single relation $\gamma_1 \cdots \gamma_n = 1$. The choice of a bouquet induces a bijection between isomorphism classes of marked G-covers branched at $\underline{t}$ and n-tuples $\underline{g} = (g_1, \dots, g_n)$ of elements of G which are *product-one*, i.e. such that $\pi\underline{g} = g_1 \cdots g_n = 1$. We say that $(g_1, \dots, g_n)$ is the *branch cycle description* of the marked G-cover. Via this description, we have a dictionary between geometric and group-theoretic/combinatorial objects:

- The monodromy group of the marked G-cover, i.e. the automorphism group of the connected component of its marked point, is $\left\langle \underline{g} \right\rangle = \langle g_1, \dots, g_n \rangle$. In particular, the G-cover is connected exactly when $g_1, \dots, g_n$ generate G.
- unmarked G-covers correspond to orbits of product-one tuples under the conjugation action of G:
$$(g_1, \dots, g_n)^\gamma = (g_1^\gamma, \dots, g_n^\gamma) \text{ for } \gamma \in G.$$
This conjugation action corresponds to the free transitive action of G on the fiber above the basepoint ∞. It amounts to a change of marked point.

Remark 2.3. We include non-connected G-covers, i.e. covers whose monodromy groups are proper subgroups of G, because we are interested in patching-like results. Typically, we want to construct components with monodromy group G by gluing components with smaller monodromy groups. If we do not take this phenomenon into account, the answer to Question 1.1 is "yes": the concatenation of two components defined over K of connected G-covers is always defined over K. This follows from Theorem 1.2 (i). In [Cau12], a different but equivalent choice is made: instead of considering components of marked non-connected G-covers, Cau considers components of unmarked connected H-covers where H is a subgroup of G. The links between these two approaches are discussed further in Paragraph 2.3.4.

2.2.3. Topological Hurwitz spaces and their components. Unless specified otherwise, Hurwitz spaces in this article are moduli spaces of *marked* G-covers, connected or not. We denote by $\mathrm{Hur}^*(G, n)$ the topological Hurwitz space of marked G-covers with n branch points. It is a covering space of $\mathrm{Conf}_n(\mathbb{C})$ whose fiber above some configuration $\underline{t} \in \mathrm{Conf}_n(\mathbb{C})$ consists of isomorphism classes of marked G-covers of $\mathbb{P}^1(\mathbb{C}) \setminus \underline{t}$.

Classically, there is an action of the braid group B_n on n-tuples of elements of G, induced by the following formula:
$$\sigma_i.(g_1, \dots, g_n) = (g_1, \dots, g_{i-1}, g_{i+1}^{g_i}, g_i, \dots, g_n).$$
We say that two n-tuples of elements of G are *braid equivalent* when they are in the same B_n-orbit for this action. The connected components of $\mathrm{Hur}^*(G, n)$ are in bijection with braid group orbits of product-one n-tuples $\underline{g} = (g_1, \dots, g_n)$ of elements of G. Whereas the branch cycle description $\underline{g}$ of a cover depends on the choice of a bouquet, the braid group orbit of $\underline{g}$ does not: this is due to the fact that B_n acts transitively on topological bouquets up to conjugation [DE06]. Thus, there is a *canonical* bijection between B_n-orbits of product-one n-tuples of elements of G and connected components of $\mathrm{Hur}^*(G, n)$.

2.2.4. The monoid of components. Via the description of connected components of $\mathrm{Hur}^*(G, n)$ as B_n-orbits of product-one n-tuples of elements of G, the concatenation of tuples induces a well-defined product operation on components of Hurwitz spaces:
$$(g_1, \dots, g_n)(g'_1, \dots, g'_{n'}) = (g_1, \dots, g_n, g'_1, \dots, g'_{n'}).$$
The graded *monoid of components* $\mathrm{Comp}(G)$ is:
$$\bigsqcup_{n \geq 0} \left(\left\{ \underline{g} \in G^n \,\middle|\, \pi \underline{g} = 1 \right\} / \mathrm{B}_n \right),$$
graded by the size n of a tuple, and equipped with the product operation induced by concatenation. Elements of degree n of $\mathrm{Comp}(G)$ are in bijection with connected components of $\mathrm{Hur}^*(G, n)$.[1] The

[1] For us, the *degree* of a component is its degree as an element of the graded monoid $\mathrm{Comp}(G)$, i.e. the number of branch points of the covers it contains, which is the size of its representing tuples. This is also equal to the *dimension* of the component since $\mathrm{Hur}^*(G, n)$ is a finite cover of the n-dimensional space $\mathrm{Conf}_n(\mathbb{C})$.

identity element of $\mathrm{Comp}(G)$ is the braid orbit of the empty tuple, i.e. the connected component of the trivial G-cover. The monoid $\mathrm{Comp}(G)$ is commutative and finitely generated, cf. [Seg22, Subsection 3.2]. We often abusively refer to an element of $\mathrm{Comp}(G)$ by one of its representing tuples $\underline{g} \in G^n$.

2.2.5. The braid group action. We recall results about the braid group action on tuples:

Proposition 2.4.

(i) The product, group, multidiscriminant of a tuple depend only on its braid group orbit.

(ii) The braid group orbit of a concatenation of tuples depends only on their braid group orbits.

(iii) If $\underline{g}$ and $\underline{g}'$ are product-one tuples, then the concatenated tuples $\underline{g}\underline{g}'$ and $\underline{g}'\underline{g}$ are braid equivalent.

(iv) If $\underline{g}_1, \underline{g}_2, \underline{g}_3$ are product-one tuples, then $\underline{g}_1\underline{g}_2\underline{g}_3$ is braid equivalent to $\underline{g}_1\underline{g}_2^{\gamma}\underline{g}_3$ for any γ which is either in $\left\langle \underline{g}_1, \underline{g}_3 \right\rangle$ or in $\left\langle \underline{g}_2 \right\rangle$.

Proofs can be found with this notation in [Seg22], or with different notation in [Cau12]. Together, points (i), (ii) and (iii) ensure that the monoid of components $\mathrm{Comp}(G)$ is a well-defined commutative monoid, and we can talk about the group and multidiscriminant of a component $x \in \mathrm{Comp}(G)$. We generalize the use of the symbols $\langle x \rangle, \mu_{H,c}(x), c_H(x)$ from Subsection 2.1 to elements $x \in \mathrm{Comp}(G)$.

For point (iv), see [Seg22, Corollary 3.5]. This result is central in the proofs of later results, notably Theorem 3.3. The case where $\underline{g}_1$ and $\underline{g}_3$ are both equal to the 0-tuple is used frequently: if $x \in \mathrm{Comp}(G)$ and $\gamma \in \langle x \rangle$, then $x = x^{\gamma}$.

2.2.6. Algebraic k-G-covers and Riemann's existence theorem. Let k be a field of characteristic relatively prime to $|G|$. An *algebraic cover* of $\mathbb{P}^1_k$ is a finite flat generically étale morphism from a smooth projective curve Y over k to $\mathbb{P}^1_k$. Moreover, we exclude from this definition algebraic covers ramified at the point at infinity. The $\overline{k}$-points of $\mathbb{P}^1$ over which an algebraic cover of $\mathbb{P}^1_k$ is ramified (i.e. the corresponding extension of local rings is ramified) form a finite configuration $\underline{t} \in \mathrm{Conf}_n(k)$, for some n.

A *k-G-cover* is an algebraic cover $p : Y \to \mathbb{P}^1_k$ equipped with a group morphism from G to the group $\mathrm{Aut}_{\mathbb{P}^1_k}(Y)$ of k-automorphisms of the cover, such that the induced action of G on every unramified geometric fiber is free and transitive.

If $p : Y \to \mathbb{P}^1_k$ is a k-G-cover with Y irreducible, then the induced extension of function fields $k(Y) \mid k(T)$ is Galois of group G. If Y is geometrically irreducible, this extension is *regular*, i.e. $k(Y) \cap \overline{k} = k$. Classically, this defines an equivalence of categories between Galois field extensions of $k(T)$ of group G and k-G-covers.

Riemann's existence theorem implies that the category of $\mathbb{C}$-G-covers (or $\overline{\mathbb{Q}}$-G-covers) branched at some configuration $\underline{t} \in \mathrm{Conf}_n(\mathbb{C})$ (resp. $\underline{t} \in \mathrm{Conf}_n(\overline{\mathbb{Q}})$) is equivalent to that of topological G-covers with the same branch points. We identify topological G-covers and k-G-covers over an algebraically closed field k of characteristic zero quite freely.

Assume now k is contained in $\overline{\mathbb{Q}}$. Since topological G-covers are well-understood, we look for *G-covers defined over k* instead of k-G-covers: a $\overline{\mathbb{Q}}$-G-cover $p : Y \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ is *defined over k* if it is isomorphic to the extension of scalars of a k-G-cover $p_k : Y' \to \mathbb{P}^1_k$. In that case, we say that p_k is a *k-model* of the G-cover p.

Remark 2.5. When k is not algebraically closed, we avoid the ambiguous expression "marked k-G-cover": it is not clear whether the cover is equipped with a geometric marked point above ∞ (these are the k-G-covers classified by the étale fundamental group of $\mathbb{P}^1_k \setminus \underline{t}$ based at ∞), or with a marked k-point above ∞ (these are the marked G-covers invariant under the action of $\mathrm{Gal}(\overline{\mathbb{Q}} \mid k)$).

We mostly use the latter notion, and we refer to these covers as *k-G-covers equipped with a marked k-point*, meaning that the marked point is in the fiber above the point at infinity.

When k is algebraically closed, we do use the terminology *marked k-G-cover* since a marked geometric point above ∞ is also a k-point.

We say that a marked $\overline{\mathbb{Q}}$-G-cover $(p, \star)$ is *defined over k* when it is invariant under the action of $\mathrm{Gal}(\overline{\mathbb{Q}} \mid k)$, i.e. when it is isomorphic, as a marked $\overline{\mathbb{Q}}$-G-cover, to the extension of scalars of k-G-cover equipped with a marked k-point. This will be defined again differently in Paragraph 2.3.2.

2.2.7. Hurwitz schemes. We denote by $\mathcal{H}^*(G, n)$ the Hurwitz scheme of marked G-covers of $\mathbb{P}^1$ with n branch points, unramified at ∞. Via the branch point morphism, the $\mathbb{Q}$-scheme $\mathcal{H}^*(G, n)$ is an étale cover of Conf_n. The K-points of $\mathcal{H}^*(G, n)$ correspond to algebraic K-G-covers branched at some configuration $\underline{t} \in \mathrm{Conf}_n(K)$, equipped with a marked K-point. The existence of this moduli space follows from the same arguments as [RW06, Theorem 4.11]. This is always a fine moduli space, unlike Hurwitz schemes of unmarked covers. The set of $\mathbb{C}$-points of $\mathcal{H}^*(G, n)$, equipped with the analytic topology, is homeomorphic to $\mathrm{Hur}^*(G, n)$. The geometrically irreducible components of $\mathcal{H}^*(G, n)$ are in bijection with the connected components of $\mathrm{Hur}^*(G, n)$, and consequently with B_n-orbits of product-one n-tuples of elements of G.

The more usual Hurwitz moduli scheme of branched unmarked G-covers of $\mathbb{P}^1$ is denoted by $\mathcal{H}(G, n)$. We use it exclusively in the discussion of Paragraph 2.3.4 which relates the fields of definition of components of $\mathcal{H}^*(G, n)$ to the more classical question of the fields of definition of components of geometrically connected unmarked covers. One thing to know is that contrary to $\mathcal{H}^*(G, n)$, the Hurwitz scheme $\mathcal{H}(G, n)$ is a *coarse* moduli scheme in general: its K-points do not correspond precisely to K-G-covers with n branch points. These issues are detailed in [DD97].

2.3. The Galois action

In this subsection, we describe the Galois action of $\Gamma_K = \mathrm{Gal}(\overline{\mathbb{Q}} \mid K)$ on the $\overline{\mathbb{Q}}$-points of the scheme $\mathcal{H}^*(G, n)$ and on its geometrically irreducible components.

We fix a configuration $\underline{t} \in \mathrm{Conf}_n(K)$, defined over K. We denote by $\pi_{1,\overline{\mathbb{Q}}}$ the étale fundamental group $\pi_1^{\text{ét}}(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \underline{t}, \infty)$ and by $\pi_{1,K}$ the étale fundamental group $\pi_1^{\text{ét}}(\mathbb{P}^1_K \setminus \underline{t}, \infty)$.

The group $\pi_{1,\overline{\mathbb{Q}}}$ is isomorphic to the profinite completion of $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \underline{t}, \infty)$. Since G is finite, there is a bijection between morphisms $\pi_{1,\overline{\mathbb{Q}}} \to G$ and morphisms $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \underline{t}, \infty) \to G$. Hence, isomorphism classes of marked G-covers branched at $\underline{t}$ may and will be seen as morphisms $\pi_{1,\overline{\mathbb{Q}}} \to G$.

2.3.1. The Galois action on covers. Let $\overline{\mathbb{Q}(T)}$ be an algebraic closure of $\mathbb{Q}(T)$ containing $\overline{\mathbb{Q}}$. Let $\Omega_{\underline{t}}$ be the maximal subfield of $\overline{\mathbb{Q}(T)}$ unramified outside of $\underline{t}$. We have the chain of field extensions:

$$\begin{array}{c} \Omega_{\underline{t}} \\ {\scriptstyle \pi_{1,\overline{\mathbb{Q}}}}\big| \quad \big) \, {\scriptstyle \pi_{1,K}} \\ \overline{\mathbb{Q}}(T) \\ {\scriptstyle \Gamma_K}\big| \\ K(T) \end{array}$$

which induces the following short exact sequence:

$$1 \longrightarrow \pi_{1,\overline{\mathbb{Q}}} \longrightarrow \pi_{1,K} \longrightarrow \Gamma_K \longrightarrow 1 \ . \tag{2.1}$$

The field $\Omega_{\underline{t}}$ embeds in $\overline{\mathbb{Q}(T)}$ which itself embeds in the field of Puiseux series over $\overline{\mathbb{Q}}$, denoted by $\overline{\mathbb{Q}}(((1/T)^{1/\infty}))$. The Galois group $\Gamma_K = \mathrm{Gal}(\overline{\mathbb{Q}} \mid K)$ acts on $\overline{\mathbb{Q}}(((1/T)^{1/\infty}))$ by acting on the

coefficients. Let $\sigma \in \Gamma_K$. Since the configuration $\underline{t}$ is defined over K, the image under σ of an extension unramified outside of $\underline{t}$ is still unramified outside of $\underline{t}$. This implies that the field $\Omega_{\underline{t}}$ is stable under the action of Γ_K. So there is an action of Γ_K on $\Omega_{\underline{t}}$, trivial on $K(T)$. This defines a morphism:

$$s : \Gamma_K \to \mathrm{Gal}\left(\Omega_{\underline{t}} \mid K(T)\right) \simeq \pi_{1,K}$$

associated with the choice of the basepoint (here, the point at infinity). The morphism s is a section of the short exact sequence of Equation (2.1):

$$1 \longrightarrow \pi_{1,\overline{\mathbb{Q}}} \longrightarrow \pi_{1,K} \underset{\leftarrow s}{\longrightarrow} \Gamma_K \longrightarrow 1.$$

Using the morphism s, we define an action of Γ_K on $\pi_{1,\overline{\mathbb{Q}}}$. For all $\sigma \in \Gamma_K$ and $\gamma \in \pi_{1,\overline{\mathbb{Q}}}$, we let:

$$\sigma.\gamma \overset{\text{def}}{=} \gamma^{s(\sigma)}$$

which belongs to $\pi_{1,\overline{\mathbb{Q}}}$ by normality. The action of an automorphism $\sigma \in \Gamma_K$ on a marked G-cover seen as a morphism $\varphi : \pi_{1,\overline{\mathbb{Q}}} \to G$ is defined by the following equality for all $\gamma \in \pi_{1,\overline{\mathbb{Q}}}$:

$$(\sigma.\varphi)(\gamma) = \varphi(\sigma.\gamma) = \varphi\left(\gamma^{s(\sigma)}\right).$$

This action does not change the monodromy group of a marked cover branched at $\underline{t}$. It satisfies $\sigma.(\varphi^g) = (\sigma.\varphi)^g$ for all $g \in G$, and so it induces a well-defined action of Γ_K on isomorphism classes of unmarked G-covers branched at $\underline{t}$.

2.3.2. Fields of definition of covers. Consider an isomorphism class of marked branched G-covers, seen as a morphism $\varphi : \pi_{1,\overline{\mathbb{Q}}} \to G$.

Definition 2.6. The marked G-cover associated to φ is *defined over* K if $\sigma.\varphi = \varphi$ for all $\sigma \in \Gamma_K$.

The equivalence with the definition given in Paragraph 2.2.6 (marked G-covers defined over K are obtained by extension of scalars of K-G-covers equipped with a marked K-point) follows from the properties of the étale fundamental group and from the following proposition[2]:

Proposition 2.7. *The marked cover associated to $\varphi : \pi_{1,\overline{\mathbb{Q}}} \to G$ is defined over K if and only if the morphism φ has an extension to $\pi_{1,K}$ which is trivial on $\mathrm{Im}(s)$, i.e. there exists a group morphism $\widetilde{\varphi} : \pi_{1,K} \to G$ such that the following diagram commutes:*

$$\begin{array}{ccc}
\pi_{1,\overline{\mathbb{Q}}} & & \\
\downarrow & \searrow^{\varphi} & \\
\pi_{1,K} & \xrightarrow[\widetilde{\varphi}]{} & G \\
\uparrow{\scriptstyle s} & & \uparrow \\
\Gamma_K & \longrightarrow & 1
\end{array}$$

The triviality of $\widetilde{\varphi}$ on $\mathrm{Im}(s)$ corresponds to the K-rationality of the marked point above ∞.

Proof. Give names to the morphisms in the exact sequence Equation (2.1):

$$\pi_{1,\overline{\mathbb{Q}}} \overset{\iota}{\hookrightarrow} \pi_{1,K} \overset{w}{\twoheadrightarrow} \Gamma_K$$

and remember that $s : \Gamma_K \to \pi_{1,K}$ is a section of w, i.e. $w \circ s = \mathrm{id}_{\pi_{1,K}}$.

[2] In some sense, this is also an instance of Proposition 2.2.

($\Leftarrow$) Assume there is a morphism $\widetilde{\varphi} : \pi_{1,K} \to G$ such that $\varphi = \widetilde{\varphi} \circ \iota$ and $\widetilde{\varphi} \circ s = 1$. For $x \in \pi_{1,\overline{\mathbb{Q}}}$ and $\sigma \in \Gamma_K$, compute:

$$\sigma.\varphi(x) = \varphi\left(x^{s(\sigma)}\right) = \widetilde{\varphi}\left(x^{s(\sigma)}\right) = \widetilde{\varphi}(x)^{\widetilde{\varphi}(s(\sigma))} = \widetilde{\varphi}(x) = \varphi(x).$$

($\Rightarrow$) Assume φ is defined over K. Let $x \in \pi_{1,K}$. We have:

$$w\left(xs(w(x))^{-1}\right) = w(x)w(s(w(x))^{-1} = w(x)w(x)^{-1} = 1,$$

which implies that $xs(w(x))^{-1} \in \pi_{1,\overline{\mathbb{Q}}}$ by exactness of Equation (2.1). Define the map:

$$\widetilde{\varphi} : \begin{cases} \pi_{1,K} & \to & G \\ x & \mapsto & \varphi(xs(w(x))^{-1}) \end{cases}.$$

If $x \in \pi_{1,\overline{\mathbb{Q}}}$, then $w(x) = 1$ and thus $\widetilde{\varphi}(x) = \varphi(x)$. If $\sigma \in \Gamma_K$, then $s(w(s(\sigma))) = s(\sigma)$ so $\widetilde{\varphi}(s(\sigma)) = \varphi(s(\sigma)s(\sigma)^{-1}) = \varphi(1) = 1$. So $\varphi = \widetilde{\varphi} \circ \iota$ and $\widetilde{\varphi} \circ s = 1$. It remains to check that $\widetilde{\varphi}$ is a morphism. Let $x, y \in \pi_{1,K}$ and compute:

$$\begin{aligned}
\widetilde{\varphi}(x)\widetilde{\varphi}(y) &= \varphi(xs(w(x))^{-1})\varphi(ys(w(y))^{-1}) && \text{by definition of } \widetilde{\varphi} \\
&= \varphi(xs(w(x))^{-1})(w(x).\varphi)(ys(w(y))^{-1}) && \text{because } \varphi \text{ is defined over } K \\
&= \varphi(xs(w(x))^{-1})\varphi\left(s(w(x))ys(w(y))^{-1}s(w(x))^{-1}\right) && \text{by definition of the } \Gamma_K\text{-action} \\
&= \varphi(xys(w(xy))^{-1}) = \widetilde{\varphi}(xy) && \text{by definition of } \widetilde{\varphi}.
\end{aligned}$$

This concludes the proof.

□

2.3.3. The Galois action on components and fields of definition. The Galois action on marked G-covers induces a well-defined Γ_K-action on the graded set $\mathrm{Comp}(G)$. Specifically, if $m \in \mathrm{Comp}(G)$, an automorphism $\sigma \in \Gamma_K$ maps marked G-covers in the component m to marked G-covers in the component $\sigma.m$. We do not claim that the Galois action is compatible with the monoid structure of $\mathrm{Comp}(G)$: this is precisely the difficulty of Question 1.1.

Definition 2.8. A component $m \in \mathrm{Comp}(G)$ is *defined over* K if for all $\sigma \in \Gamma_K$ we have $\sigma.m = m$.

2.3.4. Comparison between the marked and the unmarked cases. Let $m \in \mathrm{Comp}(G)$ be a component of $\mathcal{H}^*(G,n)_{\overline{\mathbb{Q}}}$, and $\widetilde{m}$ be the component of $\mathcal{H}(G,n)_{\overline{\mathbb{Q}}}$ obtained by forgetting the marked points. The component m is defined over K when $\sigma.m = m$ for all $\sigma \in \Gamma_K$. A weaker property, in general, is that $\widetilde{m}$ is defined over K, meaning that for all $\sigma \in \Gamma_K$ there is a $\gamma \in G$ such that $\sigma.m = m^\gamma$. There are two situations to consider:

- If $\langle m \rangle = G$, then, by Proposition 2.4 (iv), we have $m^\gamma = m$ for all $\gamma \in G$. In this instance, there is no difference between m and $\widetilde{m}$ being defined over K.
- If $\langle m \rangle$ is a proper subgroup H of G, we introduce the component m_H of $\mathcal{H}^*(H,n)_{\overline{\mathbb{Q}}}$ obtained by removing all connected components except for the component of the marked points from the covers in m, making them connected H-covers. We also introduce the component $\widetilde{m}_H$ of $\mathcal{H}(H,n)_{\overline{\mathbb{Q}}}$, obtained by forgetting the marked points of the connected H-covers in m_H.

  The fields of definition of m_H and $\widetilde{m}_H$ are the same, according to the previous point. Since the ambient group is not relevant in the definition of $\sigma.m$, it is also clear that m is defined over K if and only if m_H is defined over K.

The comparison between the situations is summarized in the figure below:

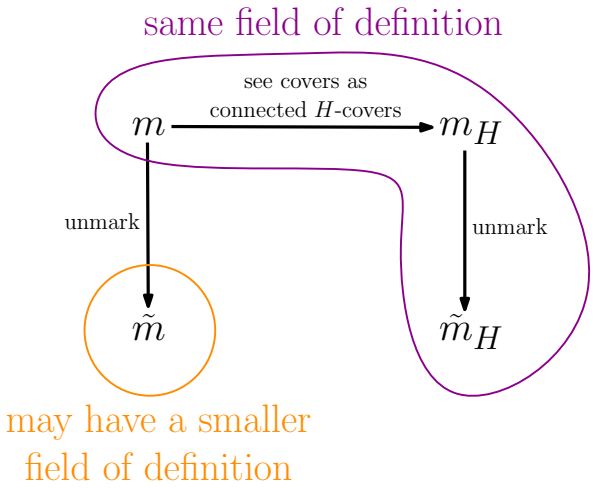


Figure 1: A summary of the situation

This discussion implies that considering the fields of definitions of components of marked G-covers with monodromy group H is equivalent to studying the fields of definitions of components of connected unmarked H-covers, which is the choice made in [Cau12]. However, we opt for the former approach because it allows for a unified treatment of these components and leads to a simpler algebraic structure.

There are still ways to relate the fields of definition of $\tilde{m}$ and m. For instance, the following lemma can be applied:

Lemma 2.9. *If $\tilde{m}$ is defined over K and H is either self-normalizing in G or has no outer automorphisms, then m is defined over K.*

Proof. Consider some $\sigma \in \Gamma_K$. The equality $\sigma.m = m^\gamma$ implies $H = H^\gamma$, i.e. conjugation by γ defines an automorphism of H.

- If H is self-normalizing, this implies $\gamma \in H$.
- If H has no outer automorphisms, conjugation by γ has to be an inner automorphism. So there is a $\gamma' \in H$ such that $h^\gamma = h^{\gamma'}$ for all $h \in H$, and in particular $m^\gamma = m^{\gamma'}$.

In both cases, Proposition 2.4 (iv) shows that $\sigma.m = m$. Therefore m is defined over K. □

2.4. The branch cycle lemma

The action of the Galois group Γ_K on multidiscriminants of G-covers or their components is precisely known. If $\sigma \in \Gamma_K$ is an automorphism, we denote by σ too the permutation of $\{1, \dots, n\}$ such that $\sigma.t_i = t_{\sigma(i)}$. Consider a marked G-cover branched at $\underline{t}$, seen as a morphism $\varphi : \pi_{1,\overline{\mathbb{Q}}} \to G$. Let $(\gamma_1, \dots, \gamma_n) \in \pi^n_{1,\overline{\mathbb{Q}}}$ be a bouquet associated to $\underline{t}$. The following result is classical, cf. [Fri77] (or [Cau12, Lemme 2.2] for a statement closer to ours):

Lemma 2.10. *For every $\sigma \in \Gamma_K$, the element $(\sigma.\varphi)(\gamma_i)$ is conjugate to $\left(\varphi(\gamma_{\sigma^{-1}(i)})\right)^{\chi(\sigma)^{-1}}$.*

We restate Lemma 2.10 in terms of multidiscriminants of components. Let $\underline{g} = (\varphi(\gamma_1), \dots, \varphi(\gamma_n))$ be the tuple associated to φ, and $\sigma.\underline{g} = (\varphi(\sigma.\gamma_1), \dots, \varphi(\sigma.\gamma_n))$ be the tuple associated to $\sigma.\varphi$ for all

$\sigma \in \Gamma_K$. Let H be a group containing $\left\langle \underline{g} \right\rangle$ and c be a K-rational (cf. Definition 2.1) conjugation-invariant subset of H containing $c_H(\underline{g})$. Denote by D the set of conjugacy classes of H contained in c and by p_σ the map $D \to D$ induced by the $\chi(\sigma)$-th power operation for $\sigma \in \Gamma_K$. Let $x \in \mathrm{Comp}(G)$ be the component represented by the tuple $\underline{g}$. Recall from Subsection 2.1 that the (H,c)-multidiscriminant $\mu_{H,c}(x)$ of x is the map that counts the occurrences in $\underline{g}$ of each conjugacy class in D.

Definition 2.11. We say that x *has a K-rational (H,c)-multidiscriminant* if for all $\sigma \in \Gamma_K$ we have:

$$\mu_{H,c}(x) = \mu_{H,c}(x) \circ p_\sigma,$$

i.e. every conjugacy class $\gamma \in D$ appears as many times in $\underline{g}$ as the classes $\gamma^{\chi(\sigma)}$ for $\sigma \in \Gamma_K$.

Note that a product of components with K-rational multidiscriminants has a K-rational multidiscriminant. Lemma 2.10 has the following corollary:

Corollary 2.12.

(i) For all $\sigma \in \Gamma_K$, the (H,c)-multidiscriminants of x and $\sigma.x$ are related by the equality:

$$\mu_{H,c}(\sigma.x) = \mu_{H,c}(x) \circ p_\sigma.$$

(ii) If x is defined over K, then x has a K-rational (H,c)-multidiscriminant.

(iii) If x has a K-rational (H,c)-multidiscriminant and H is abelian, then x is defined over K.

Point (ii) furnishes an easily checked necessary condition for a component to be defined over K. Point (iii) says that the implication of point (ii) is an equivalence in the abelian case: in this situation, fields of definition of components are well-understood.

Proof. (i) Let $\gamma \in D$ be a conjugacy class of H. Then $\mu_{H,c}(\sigma.x)(\gamma)$ is the number of occurrences of γ in $\sigma.\underline{g}$. Lemma 2.10 implies that this is also the number of occurrences of $\gamma^{\chi(\sigma)}$ in $\underline{g}$, which is precisely $\mu_{H,c}(x)(\gamma^{\chi(\sigma)})$.

(ii) Since x is defined over K, we have $\sigma.x = x$ for all automorphisms $\sigma \in \Gamma_K$. By (i), this implies $\mu_{H,c}(x) = \mu_{H,c}(x) \circ p_\sigma$, i.e. x has a K-rational (H,c)-multidiscriminant.

(iii) Since H is abelian, conjugacy classes of H and elements of H are "the same", and components are just unordered product-one tuples of elements of H (the braid group acts by permutation). Hence, two components are equal exactly when their (H,c)-multidiscriminants are equal.

Since x has a K-rational (H,c)-multidiscriminant, we know that for every $\sigma \in \Gamma_K$ we have $\mu_{H,c}(x) = \mu_{H,c}(x) \circ p_\sigma$, which is equal to $\mu_{H,c}(\sigma.x)$ by point (i). In other words, x and $\sigma.x$ are components with equal (H,c)-multidiscriminants, and therefore they are equal.

□

Example 2.13. Assume that the group G is abelian. Corollary 2.12 (iii) lets one determine the field of definition of components. A consequence is that the answer to Question 1.1 is always "yes" in this case. For example, a component represented by a tuple $\underline{g} \in G^n$ is defined over $\mathbb{Q}$ exactly when every element $g \in G$ appears as many times in $\underline{g}$ as the elements g^k for k relatively prime with $\mathrm{ord}(g)$. Let us give concrete examples:

- The component $(1,1,1) \in \mathrm{Comp}(\mathbb{Z}/3\mathbb{Z})$ is not defined over $\mathbb{Q}$, because 1 does not appear as many times as -1.
- The component $(1,-1) \in \mathrm{Comp}(\mathbb{Z}/n\mathbb{Z})$ is defined over $\mathbb{Q}$ for $n \in \{2,3,4,6\}$, and not defined over $\mathbb{Q}$ for $n = 5$ or $n \geq 7$.

Example 2.14. It follows from [Seg22, Theorem 10.6] that components of $\mathfrak{S}_d$-covers whose monodromy elements are transpositions are entirely determined by their monodromy group H and their H-multidiscriminant. Since transpositions are involutions, all conjugacy classes involved are $\mathbb{Q}$-rational, therefore Lemma 2.10 implies that the action of $\mathrm{Gal}(\overline{\mathbb{Q}} \mid \mathbb{Q})$ preserves multidiscriminants. Since components of $\mathfrak{S}_d$-covers whose monodromy elements are transpositions are determined by their multidiscriminants, which are $\mathbb{Q}$-rational, these components are all defined over $\mathbb{Q}$.

3. The group-theoretic approach

In this section, we propose new applications of ideas introduced in [Cau12], which we recall in Subsection 3.1. In Subsection 3.2, we prove the main result Theorem 3.3 (iii), which corresponds to Theorem 1.2 (i). This theorem is generalized in Subsection 3.5 and examples are given in Subsection 3.3. In Subsection 3.4, we use similar methods to reduce the Galois action to components of small degree, cf. Propositions 3.8 and 3.9. Our approach is based on braid manipulations and group-theoretic criteria.

3.1. Cau's theorem

Following [Cau12], if $x_1, \dots, x_n \in \mathrm{Comp}(G)$ and H is a subgroup of G which contains $\langle x_1, \dots, x_n \rangle$, we define the following subset of $\mathrm{Comp}(G)$:

$$\mathrm{ni}_H(x_1, \dots, x_n) = \left\{ \prod_{i=1}^{n} x_i^{\gamma_i} \;\middle|\; \gamma_i \in H \right\}.$$

We also introduce the following subset, which always contains $x_1 \cdots x_n$:

$$\mathrm{ni}_H^\natural(x_1, \dots, x_n) = \left\{ \prod_{i=1}^{n} x_i^{\gamma_i} \;\middle|\; \begin{array}{c} \gamma_i \in H \\ \langle x_1^{\gamma_1} \cdots x_n^{\gamma_n} \rangle = \langle x_1 \cdots x_n \rangle \end{array} \right\}.$$

When H is not specified, it is assumed that $H = \langle x_1 \cdots x_n \rangle$.

In Cau's terminology, a family of elements of $\mathrm{Comp}(G)$ corresponds to a *degenerescence structure* Δ, and elements of $\mathrm{ni}_H(\Delta)$ are called Δ-*components*. Cau gave a criterion to identify whether a given component is a Δ-component depending on the existence of a specific "Δ-admissible cover" on its boundary. This characterization is key for his proof of the following theorem, which is [Cau12, Théorème 3.2]:

Theorem 3.1. *Let $x_1, \dots, x_n$ be components, H a subgroup of G which contains $\langle x_1, \dots, x_n \rangle$, and $\sigma \in \Gamma_K$. Then the action of σ on components induces a bijection:*

$$\mathrm{ni}_H(x_1, \dots, x_n) \to \mathrm{ni}_H(\sigma.x_1, \dots, \sigma.x_n)$$

and the same statement holds if ni_H is replaced by $\mathrm{ni}_H^\natural$.

That Theorem 3.1 holds if ni_H is replaced with $\mathrm{ni}_H^\natural$ follows from the fact that the Galois action preserves the monodromy group. If X is a finite set of components and $\sigma \in \Gamma_K$, we write Theorem 3.1 under the form $\sigma.\mathrm{ni}(X) = \mathrm{ni}(\sigma.X)$, where $\sigma.X$ is a shorthand for $\{\sigma.x \mid x \in X\}$.

3.2. Permuting components

In [Cau12, Proposition 2.10] and [Cau16, Théorème 3.8], Cau applies Theorem 3.1 in situations where he shows $\mathrm{ni}(x_1, \dots, x_n) = \{x_1 \cdots x_n\}$. We introduce a different condition that will later be shown to imply $\mathrm{ni}^\natural(x_1, \dots, x_n) = \{x_1 \cdots x_n\}$:

Definition 3.2. Two components $x, y \in \mathrm{Comp}(G)$ of respective monodromy groups $H_1 = \langle x \rangle$ and $H_2 = \langle y \rangle$ are *permuting* if $H_1 H_2 = \langle H_1, H_2 \rangle$.

This terminology comes from the fact that subgroups H_1, H_2 of a group G are classically called *permuting* when $H_1 H_2$ is a subgroup of G. Two elements of $\mathrm{Comp}(G)$ are permuting exactly when their monodromy groups are permuting subgroups of G. This condition is neither stronger nor weaker than the completeness conditions considered by Cau. In Subsection 3.5, we give a condition that generalizes both Definition 3.2 and the hypothesis of [Cau12, Proposition 2.10].

Note that $x, y \in \mathrm{Comp}(G)$ are permuting whenever $\langle x \rangle$ or $\langle y \rangle$ is normal in $\langle x, y \rangle$, and in particular when one monodromy group contains the other. Cases of interest are $\langle x \rangle = \langle y \rangle$ as well as $\langle x \rangle = G$ or $\langle y \rangle = G$. Moreover, if x, y are permuting and $\sigma \in \Gamma_K$, then $\sigma.x$ and $\sigma.y$ are permuting.

We are now ready to prove Theorem 3.3, whose third point is Theorem 1.2 (i):

Theorem 3.3. *Let $x, y \in \mathrm{Comp}(G)$ be permuting components. Then:*

(i) The set $\mathrm{ni}^\natural(x, y)$ contains only the component xy.

(ii) For all $\sigma \in \Gamma_K$ we have $\sigma.(xy) = (\sigma.x)(\sigma.y)$.

(iii) If x and y are defined over K, then xy is defined over K.

Proof. (i) Let $\gamma, \gamma' \in H$ such that $\left\langle x^\gamma y^{\gamma'} \right\rangle = H$. We have to show $x^\gamma y^{\gamma'} = xy$. We use Proposition 2.4 (iv) to reduce to the case $\gamma = 1$: indeed, we have $x^\gamma y^{\gamma'} = x y^{\gamma^{-1}\gamma'}$ since $\gamma^{-1} \in \left\langle x^\gamma y^{\gamma'} \right\rangle = H$. Write $\gamma' = \gamma_1 \gamma_2$ with $\gamma_i \in H_i$. We have:

$$\begin{aligned} xy &= x y^{\gamma_2} && \text{by Proposition 2.4 (iv), because } \gamma_2 \in \langle y \rangle \\ &= x (y^{\gamma_2})^{\gamma_1} && \text{by Proposition 2.4 (iv), because } \gamma_1 \in \langle x \rangle \\ &= x y^{\gamma_1 \gamma_2} \\ &= x y^{\gamma'}. \end{aligned}$$

(ii) Let $\sigma \in \Gamma_K$. By Theorem 3.1, the component $\sigma.(xy)$ belongs to the set $\mathrm{ni}^\natural(\sigma.x, \sigma.y)$ and thus it is equal to $(\sigma.x)(\sigma.y)$, by point (i) applied to the permuting components $\sigma.x$ and $\sigma.y$.

(iii) Follows from point (ii).

□

A noteworthy corollary of Theorem 3.3 (iii) is the following:

Corollary 3.4. *If x is a component defined over K, then so is x^n for all $n \geq 0$.*

Remark 3.5. To deduce Theorem 3.3 (iii) from Theorem 3.3 (i), one can use Theorem 5.4 instead of Theorem 3.1.

3.3. Applications and examples

The rest of this section is concerned with applications of Theorem 3.3.

Example 3.6. Theorem 3.3 implies a slightly stronger version of [Cau12, Théorème 3.5]. Indeed, let $G = H_1 \ltimes H_2$ be a semi-direct product of groups. Assume that for $i = 1, 2$ there is a *rigid* (cf. [Cau12, Definition 2.4]) r_i-tuple $\underline{c}_i$ of $\mathbb{Q}$-rational conjugacy classes of H_i. Then for $i = 1, 2$ there is a unique component $m_i \in \mathrm{Comp}(G)$ such that $\langle m_i \rangle = H_i$ and the H_i-multidiscriminant of m_i counts the appearances of a class in $\underline{c}_i$. By the rigidity hypothesis and the branch cycle lemma (Corollary 2.12), these components are defined over $\mathbb{Q}$. Cau's results led him to observe that (H_1, H_1, H_2, H_2) is a complete family of subgroups of G and therefore $m_1^2 m_2^2$ is defined over $\mathbb{Q}$. We obtain a slightly better

result: since G is a semi-direct product of H_1 and H_2, the components m_1, m_2 are permuting and therefore the component $m_1 m_2$ is defined over K.

Assume G is a semi-direct product of symmetric groups: $G = \mathfrak{S}_n \ltimes \mathfrak{S}_m$. There are rigid $\mathbb{Q}$-rational triples of conjugacy classes of $\mathfrak{S}_n$ and $\mathfrak{S}_m$. The reasoning above shows that there is a component defined over $\mathbb{Q}$ of G-covers with six branch points, instead of the twelve needed by Cau, which already improved upon the thirty-two needed by Dèbes and Emsalem.

We now prove a result which can be perceived as the existence of a field norm for components:

Corollary 3.7. *Let $x \in \mathrm{Comp}(G)$ be a component, and $H = \langle x \rangle$. Let Γ_x be the subgroup of finite index of Γ_K consisting of elements $\sigma \in \Gamma_K$ such that $\sigma.x = x$. Then the following component, which has monodromy group H, is defined over K:*

$$\mathrm{N}_K(x) = \prod_{\sigma \in \Gamma_K/\Gamma_x} \sigma.x.$$

This result is a variant of [Cau12, Corollaire 1.1/Corollaire 3.4]: Cau shows that the concatenation of all components with a given degree is defined over $\mathbb{Q}$; here we are more precise by restricting our attention on a single Galois orbit, leading to a lower degree for the product component.

Proof. Let $\Gamma_K.x$ be the set $\{\sigma.x \,|\, \sigma \in \Gamma_K/\Gamma_x\}$. Since all components of the form $\sigma.x$ have group H, repeated applications of Theorem 3.3 (i) show that:

$$\mathrm{ni}_H^\natural(\Gamma_K.x) = \{\mathrm{N}_K(x)\}. \tag{3.1}$$

Consider an automorphism $\sigma \in \Gamma_K$. The action of σ permutes $\Gamma_K.x$. Finally:

$$\begin{aligned}
\{\mathrm{N}_K(x)\} &= \mathrm{ni}_H^\natural(\Gamma_K.x) && \text{by Equation (3.1)} \\
&= \mathrm{ni}_H^\natural(\sigma.(\Gamma_K.x)) && \text{because } \sigma \text{ permutes } \Gamma_K.x \\
&= \sigma.\mathrm{ni}_H^\natural(\Gamma_K.x) && \text{by Theorem 3.1} \\
&= \sigma.\{\mathrm{N}_K(x)\} && \text{by Equation (3.1)} \\
&= \{\sigma.\mathrm{N}_K(x)\}
\end{aligned}$$

and thus $N_K(x)$ is defined over K. □

3.4. Reduction of the Galois action to components of small degree

In this subsection we express the Galois action of Γ_K on components in terms on the action on components of small degree. Let $\psi(G)$ be the sum of the orders of the elements of G:

$$\psi(G) \stackrel{\text{def}}{=} \sum_{g \in G} \mathrm{ord}(g).$$

Consider a n-tuple $\underline{g} = (g_1, \ldots, g_n) \in G^n$, and let $H = \left\langle \underline{g} \right\rangle$. If $n > \psi(G)$, then there is an element $g \in G$ which appears at least $\mathrm{ord}(g) + 1$ times in the tuple $\underline{g}$. Usual braid manipulations allow one to move these occurrences of g to the beginning of the tuple. This shows that we have the following equality in $\mathrm{Comp}(G)$:

$$\underline{g} = (\underbrace{g, \ldots, g}_{\mathrm{ord}(g)})y$$

for some $y \in \mathrm{Comp}(G)$ of group H. Note that $(g, \ldots, g)$ and y are permuting components and that $\langle (g, \ldots, g) \rangle = \langle g \rangle$ is abelian. We have:

$$\begin{aligned}
\sigma.\underline{g} &= (\sigma.(g, \ldots, g))\,(\sigma.y) && \text{by Theorem 3.3 (ii)} \\
&= \left(g^{\chi(\sigma^{-1})}, \ldots, g^{\chi(\sigma^{-1})}\right)(\sigma.y) && \text{by the branch cycle lemma (Corollary 2.12 (i)).}
\end{aligned}$$

We can iterate this factorization process until the size of y is smaller than $\psi(G)$: this shows that the Galois action on components is entirely determined by the cyclotomic character and the Galois action on components of small degree. We turn this into a precise proposition:

Proposition 3.8. *Let $x \in \mathrm{Comp}(G)$ be a component and $H = \langle x \rangle$. There are elements $g_1, \dots, g_r \in H$ and a component $y \in \mathrm{Comp}(G)$ of group H with $\deg(y) \leq \psi(G)$ such that:*

$$x = \left(\prod_{i=1}^{r} (\underbrace{g_i, \dots, g_i}_{\mathrm{ord}(g_i)}) \right) y.$$

Moreover, once x is expressed under this form, the Galois action of an automorphism $\sigma \in \Gamma_K$ on x can be expressed in terms of the cyclotomic character χ, and of the Galois action on components of degree $\leq \psi(G)$:

$$\sigma.x = \left(\prod_{i=1}^{r} \left(\underbrace{g_i^{\chi(\sigma^{-1})}, \dots, g_i^{\chi(\sigma^{-1})}}_{\mathrm{ord}(g_i)} \right) \right) (\sigma.y).$$

Here is another example of this phenomenon. Let H be a subgroup of G and c a K-rational conjugation-invariant subset of H. Denote by $\mathcal{C}_{H,c}$ the set of components $x \in \mathrm{Comp}(G)$ such that $\langle x \rangle = H$ and $c_H(x) \subseteq c$. Then:

Proposition 3.9. *Assume every component $x \in \mathcal{C}_{H,c}$ of degree $\leq 2\,|c|\,\psi(G)$ with a K-rational (H,c)-multidiscriminant is defined over K. Then, every component $x \in \mathcal{C}_{H,c}$ with a K-rational (H,c)-multidiscriminant is defined over K.*

Proof. We prove the result by induction. Consider a component $x \in \mathcal{C}_{H,c}$ of degree $n > 2\,|c|\,\psi(G)$ with a K-rational (H,c)-multidiscriminant. Assume that every component in $\mathcal{C}_{H,c}$ of degree $< n$ with a K-rational (H,c)-multidiscriminant is defined over K. Choose a tuple $\underline{g} \in c^n$ representing x. Since $n > 2\,|c|\,\psi(G)$, there is some $g \in c$ which appears at least $2\mathrm{ord}(g)\,|c| + 1$ times in $\underline{g}$.

Let $g_1, \dots, g_r$ be the elements obtained as $g^{\chi(\sigma)}$ for some $\sigma \in \Gamma_K$. By Corollary 2.12 (iii), the following component, whose group is the abelian group $\langle g \rangle$, is defined over K:

$$y \overset{\mathrm{def}}{=} (\underbrace{g_1, \dots, g_1}_{\mathrm{ord}(g)}, \underbrace{g_2, \dots, g_2}_{\mathrm{ord}(g)}, \dots, \underbrace{g_r, \dots, g_r}_{\mathrm{ord}(g)}).$$

In particular, the component y has a K-rational (H,c)-multidiscriminant by Corollary 2.12 (ii).

We now show that there is a component z with $\langle z \rangle = H$ such that $x = yz$. For this, we apply the factorization result of [Seg22, Lemma 4.6]. Consider a conjugacy class γ of H which appears in y. Then:

- The conjugacy class γ is some $\chi(\sigma)$-th power of the conjugacy class of g, which appears at least $2\mathrm{ord}(g)\,|c|+1$ times in $\underline{g}$ because g itself does. Since x has a K-rational (H,c)-multidiscriminant, we have $\mu_{H,c}(x)(\gamma) \geq 2\mathrm{ord}(g)\,|c| + 1$.
- The conjugacy class γ appears at most $\mathrm{ord}(g)\,|c|$ times in y since $\deg(y) \leq \mathrm{ord}(g)\,|c|$.

Finally:

$$\begin{aligned} \mu_{H,c}(x)(\gamma) &\geq 2\mathrm{ord}(g)\,|c| + 1 \\ &\geq \mathrm{ord}(g)(|\gamma| + |c|) \\ &= \mathrm{ord}(\gamma)\,|\gamma| + \mathrm{ord}(g)\,|c| \\ &\geq \mathrm{ord}(\gamma)\,|\gamma| + \mu_{H,c}(y)(\gamma). \end{aligned}$$

By [Seg22, Lemma 4.6], there exists $z \in \mathrm{Comp}(G)$ such that $x = yz$ and $\langle z \rangle = H$, and in particular $z \in \mathcal{C}_{H,c}$. Since $x = yz$ and y both have K-rational (H,c)-multidiscriminants, the component z has a K-rational (H,c)-multidiscriminant too. By the induction hypothesis, z is defined over K. Moreover $\langle y \rangle \subseteq H$ so y and z are permuting, and thus $x = yz$ is defined over K by Theorem 3.3 (iii). We conclude by induction. □

Remark 3.10. When we have discussed the lifting invariant in Section 4, it will appear that the hypothesis "with a K-rational (H,c)-multidiscriminant" in Proposition 3.9 can be replaced by the more precise necessary condition "whose (H,c)-lifting invariant is Γ_K-invariant". The proof of Proposition 3.9 can be reproduced identically except for the two following details:

- That γ appears at least $2\mathrm{ord}(g)\,|c| + 1$ times in $\underline{g}$ follows from the fact that a component with a Γ_K-invariant (H,c)-lifting invariant also has a K-rational (H,c)-multidiscriminant. This follows directly from the definition of the Γ_K-action on lifting invariants in Subsection 4.3.
- To apply the induction hypothesis, we have to show that the component z obtained using the factorisation lemma has a Γ_K-invariant lifting invariant. At that point in the proof, we know that $x = yz$, and that x and y both have Γ_K-invariant lifting invariants. First notice that $x = yz$ implies:
$$\Pi_{H,c}(x) = \Pi_{H,c}(y)\Pi_{H,c}(z). \tag{3.2}$$
Now, consider an automorphism $\sigma \in \Gamma_K$. Theorem 4.10 together with the equality $x = yz$ imply:
$$\sigma.\Pi_{H,c}(x) = (\sigma.\Pi_{H,c}(y))(\sigma.\Pi_{H,c}(z))$$
i.e.:
$$\Pi_{H,c}(x) = \Pi_{H,c}(y)(\sigma.\Pi_{H,c}(z)). \tag{3.3}$$
Since the lifting invariant takes values in a group, Equation (3.2) and Equation (3.3) together imply $\Pi_{H,c}(z) = \sigma.\Pi_{H,c}(z)$. Hence the (H,c)-lifting invariant of z is Γ_K-invariant.

Remark 3.11. The constant $2\,|c|\,\psi(G)$ in Proposition 3.9 can easily be improved to:
$$\sum_{\gamma \in D} |\gamma| \left[\mathrm{ord}(\gamma) \Big(|\gamma| + \varphi\Big(\mathrm{ord}(\gamma)\Big) \Big) - 1 \right]$$
where D is the set of conjugacy classes of H contained in c and φ is Euler's totient function.

Example 3.12. In the situation of Example 2.14, where G is the symmetric group $\mathfrak{S}_d$ and c is the set of transpositions in G, checking that all components of degree $\leq \frac{1}{2}d^4$ are defined over $\mathbb{Q}$ would have been enough to prove that they are all defined over $\mathbb{Q}$.

3.5. Generalized permuting components

In this subsection, we prove Theorem 3.14, which generalizes both Theorem 3.3 and [Cau12, Proposition 2.10]. First, we introduce the following definition:

Definition 3.13. Let $x_1, \ldots, x_n \in \mathrm{Comp}(G)$ be components, let $H_i = \langle x_i \rangle$ and $H = \langle H_1, \ldots, H_n \rangle$. The family $(x_1, \ldots, x_n)$ is *permuting* when for all elements $\gamma_1, \ldots, \gamma_n \in H$ and for all $i \in \{2, \ldots, n\}$ we have:
$$\text{if } \left\langle H_1, H_2, \ldots, H_{i-1}, H_i, H_{i+1}^{\gamma_{i+1}}, \ldots, H_n^{\gamma_n} \right\rangle = H,$$
$$\text{then } \left\langle H_1, H_2, \ldots, H_{i-1}, H_{i+1}^{\gamma_{i+1}}, \ldots, H_n^{\gamma_n} \right\rangle H_i = H.$$

Theorem 3.14. *Let $(x_1, \ldots, x_n)$ be a permuting family of components. Then:*

(i) $\mathrm{ni}_H^\natural(x_1,\dots,x_n) = \{x_1\cdots x_n\}$.

(ii) For all automorphisms $\sigma \in \Gamma_K$, *we have* $\sigma.(x_1\cdots x_n) = (\sigma.x_1)\cdots(\sigma.x_n)$.

(iii) If $x_1,\dots,x_n$ *are defined over* K *then* $x_1\cdots x_n$ *is defined over* K.

The case $n = 2$ gives back Theorem 3.3. The hypothesis of Theorem 3.14 is slightly weaker than the one required to apply Theorem 3.3 multiple times recursively.

Proof. We focus on proving point (i), from which points (ii) and (iii) follow like in the proof of Theorem 3.3. Let $\gamma_1,\dots,\gamma_n \in G$ such that $\langle \prod x_i^{\gamma_i}\rangle = H$. First we can assume $\gamma_1 = 1$ as in the proof of Theorem 3.3 (i). We proceed by induction. Assume we have shown:

$$x_1\cdots x_n = x_1x_2\cdots x_{i-1}x_i x_{i+1}^{\gamma_{i+1}}\cdots x_n^{\gamma_n}.$$

In particular, we have $\left\langle H_1, H_2, \dots, H_{i-1}, H_i, H_{i+1}^{\gamma_{i+1}}, \dots, H_n^{\gamma_n}\right\rangle = H$. Since $(x_1,\dots,x_n)$ is permuting, we can write $\gamma_i = \gamma_i^{(1)}\gamma_i^{(2)}$ with $\gamma_i^{(1)} \in \left\langle H_1, H_2, \dots, H_{i-1}, H_{i+1}^{\gamma_{i+1}}, \dots, H_n^{\gamma_n}\right\rangle$ and $\gamma_i^{(2)} \in H_i$. Therefore:

$$\begin{aligned}
x_1\cdots x_n &= x_1\cdots x_{i-1}x_i^{\gamma_i^{(2)}}x_{i+1}^{\gamma_{i+1}}\cdots x_n^{\gamma_n} && \text{by Proposition 2.4 (iv), because } \gamma_i^{(2)} \in \langle x_i\rangle\\
&= x_1\cdots x_{i-1}\left(x_i^{\gamma_i^{(2)}}\right)^{\gamma_i^{(1)}}x_{i+1}^{\gamma_{i+1}}\cdots x_n^{\gamma_n} && \text{because } \gamma_i^{(1)} \in \left\langle x_1\cdots x_{i-1}, x_{i+1}^{\gamma_{i+1}}\cdots x_n^{\gamma_n}\right\rangle\\
&= x_1\cdots x_{i-1}x_i^{\gamma_i}x_{i+1}^{\gamma_{i+1}}\cdots x_n^{\gamma_n}
\end{aligned}$$

and we conclude by induction. □

We now give an application of Theorem 3.14:

Example 3.15. Let c be a K-rational conjugation-invariant set of G. Assume that c is *complete*, i.e. no proper subgroup of G intersects every conjugacy class contained in c (for example, Jordan's lemma implies that $c = G\setminus\{1\}$ is complete).

The following component (introduced in [EVW12, Paragraph 5.5]) is defined over K:

$$V = \prod_{g\in c}(\underbrace{g,\dots,g}_{\mathrm{ord}(g)}).$$

Indeed, consider an automorphism $\sigma \in \Gamma_K$. Since $\langle g\rangle$ is abelian, Corollary 2.12 (i) implies that $\sigma.(g,\dots,g) = (g^{\chi(\sigma^{-1})},\dots,g^{\chi(\sigma^{-1})})$. The profinite integer $\chi(\sigma^{-1})$ is invertible and so the action of σ permutes the factors of V. Now:

$$\sigma.V = \sigma.\prod_{g\in c}(g,\dots,g) \in \mathrm{ni}^\natural(\{(g,\dots,g)\,|\,g\in c\}).$$

We want to apply Theorem 3.14 (i) to show that $\mathrm{ni}^\natural(\{(g,\dots,g)\,|\,g\in c\})$ is a singleton, from which $\sigma.V = V$ follows. Consider an element $g \in c$ and elements $\gamma_{g'} \in G$ for all $g' \in c\setminus\{g\}$, such that G is generated by g together with the elements $(g')^{\gamma_{g'}}$ for $g' \in c\setminus\{g\}$. We want to show $\langle (g')^{\gamma_{g'}} \text{ for } g' \in c\setminus\{g\}\rangle\langle g\rangle = G$. There are two distinct cases:

- If $g \in Z(G)$, then this follows easily from, say, the fact that $\langle g\rangle$ is normal in G.
- If $g \notin Z(G)$, then there is a $g' \in c\setminus\{g\}$ such that g and g' are conjugate. Therefore $\langle g'^{\gamma_{g'}} \text{ for } g' \in c\setminus\{g\}\rangle$ is a subgroup of G that intersects every conjugacy class contained in c, and therefore it equals G by the completeness assumption.

4. The lifting invariant approach

In this section, we use the lifting invariant of [EVW12, Woo21] to study Question 1.1. We first recall known properties of this invariant (Subsection 4.1) and then give arithmetic applications (Subsection 4.2 and Subsection 4.3), including Theorem 4.7 (which is Theorem 1.2 (ii)).

4.1. Presentation of the lifting invariant

4.1.1. Definition and first properties. In this subsection, we present the lifting invariant. For exhaustivity and convenience of the reader, we include some proofs for known facts. In what follows, H is a subgroup of G and D is a set of conjugacy classes of H which together generate H. We denote by c the union of the conjugacy classes in D.

We define the group $U(H,c)$ in the following way: it is generated by elements $[g]$ for each $g \in c$, satisfying the relations $[g][h][g]^{-1} = [ghg^{-1}]$ for all $g, h \in c$.

Definition 4.1. Let $\underline{g} = (g_1, \dots, g_n) \in c^n$ be a tuple. Its *(H,c)-lifting invariant* is the following element of $U(H,c)$:

$$\Pi_{H,c}(\underline{g}) = [g_1] \cdots [g_n].$$

Proposition 4.2. *The (H,c)-lifting invariant $\Pi_{H,c}(\underline{g})$ depends only on the braid orbit of $\underline{g}$.*

Proof. The relation $[g][h][g]^{-1} = [ghg^{-1}]$ in $U(H,c)$ can be rewritten as $[g][h] = [h^g][g]$. So the (H,c)-lifting invariant is unchanged by elementary braids, which generate the braid group. □

Proposition 4.2 implies that we can talk about the (H,c)-lifting invariant $\Pi_{H,c}(x)$ of a component $x \in \mathrm{Comp}(G)$ as soon as $\langle x \rangle \subseteq H$ and $c_H(x) \subseteq c$. If (H,c) is not specified, the *lifting invariant* of a component is its (H,c)-lifting invariant with $H = \left\langle \underline{g} \right\rangle$ and $c = c_H(\underline{g})$.

We denote by π the morphism $U(H,c) \to H$ induced by the formula $[g] \mapsto g$. The use of the letter π is justified by the observation that for all $\underline{g} \in c^n$, we have $\pi(\Pi_{H,c}(\underline{g})) = \pi\underline{g}$. The kernel of the morphism π is denoted by $U_1(H,c)$.

Proposition 4.3. *The subgroup $U_1(H,c) = \ker(\pi)$ is central in $U(H,c)$.*

Proof. Let x be an element of $U_1(H,c)$ decomposed as $[g_1]^{\varepsilon_1} \cdots [g_n]^{\varepsilon_n}$ with $g_1^{\varepsilon_1} \cdots g_n^{\varepsilon_n} = 1$ and $\varepsilon_i \in \{-1, 1\}$. Now consider one of the generators $[h]$ of $U(H,c)$. We have:

$$\begin{aligned} x[h] &= [g_1]^{\varepsilon_1} \cdots [g_{n-1}]^{\varepsilon_{n-1}} [g_n]^{\varepsilon_n} [h] \\ &= [g_1]^{\varepsilon_1} \cdots [g_{n-1}]^{\varepsilon_{n-1}} [h^{g_n^{\varepsilon_n}}] [g_n]^{\varepsilon_n} \\ &= [h^{g_1^{\varepsilon_1} \cdots g_n^{\varepsilon_n}}] [g_1]^{\varepsilon_1} \cdots [g_{n-1}]^{\varepsilon_{n-1}} [g_n]^{\varepsilon_n} \\ &= [h]x. \end{aligned}$$

Hence x commutes with a generating set of elements of $U(H,c)$ and thus $x \in Z(U(H,c))$. This shows that $U_1(H,c)$ is a central subgroup of $U(H,c)$. □

It follows from Proposition 4.2 and Proposition 4.3 that $\Pi_{H,c}$ induces a morphism of monoids from the following submonoid of $\mathrm{Comp}(G)$:

$$\mathrm{Comp}(H,c) = \left\{ m \in \mathrm{Comp}(G) \;\middle|\; \begin{array}{c} \langle m \rangle \subseteq H \\ c_H(m) \subseteq c \end{array} \right\}$$

into the abelian group $U_1(H,c)$. In fact, the abelian group $U_1(H,c)$ is the Grothendieck group of $\mathrm{Comp}(H,c)$, i.e. $\Pi_{H,c}$ is universal among morphisms from $\mathrm{Comp}(H,c)$ into groups. This follows from [EVW12, Theorem 7.5.1]. In some sense, this means that the lifting invariant is the best possible group-valued multiplicative invariant for components.

The (H,c)-multidiscriminant defines a morphism from $\mathrm{Comp}(H,c)$ into the group $\mathbb{Z}^D$ of maps from D to $\mathbb{Z}$. By the universal property of the Grothendieck group, the (H,c)-multidiscriminant can be recovered from the (H,c)-lifting invariant. Indeed, the formula:

$$[g] \mapsto (0,0,\dots,0,1,0,\dots,0),$$

where the nonzero coefficient occurs at the coordinate corresponding to the conjugacy class of g, induces a morphism $U(H,c) \to Z^D$ whose restriction is the expected morphism $U_1(H,c) \to \mathbb{Z}^D$. So the (H,c)-lifting invariant refines the (H,c)-multidiscriminant.

The (G,G)-lifting invariant of a component $x \in \mathrm{Comp}(G)$ only depends on the component of unmarked covers obtained by forgetting the marked points of covers in x. This follows from the following proposition:

Proposition 4.4. *If $\gamma \in H$ and $x \in \mathrm{Comp}(H,c)$ then $\Pi_{H,c}(x) = \Pi_{H,c}(x^\gamma)$.*

Proof. Since c generates H, we can choose elements $\gamma_1,\dots,\gamma_n \in c$ such that $\gamma_1 \cdots \gamma_n = \gamma$. In $U(H,c)$, we have:

$$\begin{aligned}[\gamma_1]\cdots[\gamma_n]\Pi_{H,c}(x) &= [\gamma_1]\cdots[\gamma_{n-1}]\Pi_{H,c}(x^{\gamma_n})[\gamma_n] \\ &= \Pi_{H,c}(x^{\gamma_1\cdots\gamma_n})[\gamma_1]\cdots[\gamma_n] \\ &= \Pi_{H,c}(x^\gamma)[\gamma_1]\cdots[\gamma_n].\end{aligned}$$

By Proposition 4.3, the element $\Pi_{H,c}(x^\gamma) \in U_1(H,c)$ is central and thus we can cancel $[\gamma_1]\cdots[\gamma_n]$ in this equality. This concludes the proof. □

4.1.2. The structure of the group $U(H,c)$. We give a quick description of the group $U(H,c)$. The proofs for the facts stated here are contained in [Woo21, Paragraph 2.1]. The main result is that the group $U(H,c)$ is isomorphic to a fibered product:

$$U(H,c) \simeq S_c \underset{H^{\mathrm{ab}}}{\times} \mathbb{Z}^D$$

where S_c (a *reduced Schur cover* of G) is a finite group which fits in an exact sequence:

$$1 \to H_2(H,c) \to S_c \to H \to 1$$

for a specific quotient $H_2(H,c)$ of the second homology group $H_2(H,\mathbb{Z})$ of H. In particular, the central subgroup $U_1(H,c)$ is isomorphic to the following direct product:

$$H_2(H,c) \times \ker(\widetilde{\pi})$$

where the morphism $\widetilde{\pi} : \mathbb{Z}^D \to H^{\mathrm{ab}}$ is defined in the following way: when $\gamma \in D$ is a conjugacy class, denote by $\widetilde{\gamma}$ the (well-defined) image in H^{ab} of the elements of $\gamma \in D$; if $\psi \in \mathbb{Z}^D$, we then let:

$$\widetilde{\pi}(\psi) = \prod_{\gamma \in D} \widetilde{\gamma}^{\psi(\gamma)}.$$

4.1.3. The lifting invariant distinguishes "big" components. The following result, which is proved in [EVW12, Theorem 7.6.1] and [Woo21, Theorem 3.1], states that components are entirely determined by their (H,c)-lifting invariant as soon as each conjugacy class of H in c is represented enough times in their (H,c)-multidiscriminant. This is a stronger version of a result known as the Conway-Parker theorem.

To state the result, we introduce some notation. If $\psi \in \mathbb{Z}^D$, we denote by $|\psi|$ the value $\sum_{\gamma\in D}\psi(\gamma)$ and by $\min(\psi)$ the minimal value taken by $\psi(\gamma)$ for $\gamma \in D$. Then:

Theorem 4.5. *There is a constant $M_{H,c} \in \mathbb{N}$ such that for all $\psi \in \mathbb{Z}^D$ satisfying $\min(\psi) \geq M_{H,c}$, the morphism $\Pi_{H,c}$ induces a bijection:*

$$\left\{ \underline{g} \in G^{|\psi|} \;\middle|\; \begin{array}{c} \langle \underline{g} \rangle = H \\ c_H(\underline{g}) = c \\ \mu_{H,c}(\underline{g}) = \psi \end{array} \right\} \Big/ \mathrm{B}_{|\psi|} \xrightarrow{\sim} \left\{ x \in U(H,c) \;\middle|\; x \text{ has image } \psi \text{ in } \mathbb{Z}^D \right\}.$$

4.2. The lifting invariant and fields of definition of glued components

In this subsection, we use Theorem 4.5 to prove Theorem 1.2 (ii). The proof also makes use of Theorem 3.1.

First, remark that we may choose a (rough) constant M independent from (H, c) that satisfies the conclusion of Theorem 4.5:

$$M \overset{\text{def}}{=} \max_{(H,c)} M_{H,c},$$

where the maximum is taken over couples (H, c) where H is a subgroup of G and c is a conjugation-invariant subset of H that generates H. In what follows, the constant M is fixed in this way.

Definition 4.6. A tuple $\underline{g}$ of elements of G is *M-big* if every conjugacy class of $H = \langle \underline{g} \rangle$ that appears in $\underline{g}$ appears at least M times, i.e. $\min \mu_H(\underline{g}) \geq M$. A component $x \in \mathrm{Comp}(G)$ is *M-big* if its representing tuples are M-big.

Theorem 4.5 implies that M-big components are determined by their lifting invariant. Note that if $x \in \mathrm{Comp}(G)$ and $k \geq M$, the component x^k is always M-big. We now prove Theorem 4.7, which is Theorem 1.2 (ii):

Theorem 4.7. *Let $x_1, \ldots, x_n \in \mathrm{Comp}(G)$ be components such that $x_1 \cdots x_n$ is M-big. Then:*

(i) The set $\mathrm{ni}^\natural(x_1, \ldots, x_n)$ contains only the component $x_1 \cdots x_n$.

(ii) For all automorphisms $\sigma \in \Gamma_K$, we have $\sigma.(x_1 \cdots x_n) = (\sigma.x_1) \cdots (\sigma.x_n)$.

(iii) If $x_1, \ldots, x_n$ are defined over K, then $x_1 \cdots x_n$ is defined over K.

Proof. Let $H = \langle x_1 \cdots x_n \rangle$ and $c = c_H(x_1 \cdots x_n)$. It follows from Proposition 4.4 and from the multiplicativity of $\Pi_{H,c}$ that all elements of $\mathrm{ni}^\natural(x_1, \ldots, x_n)$ have the same (H, c)-lifting invariant. Moreover they are all M-big and have monodromy group H. By Theorem 4.5, they must be equal to each other. This proves point (i). Points (ii) and (iii) follow from point (i) and from Theorem 3.1, like in the proof of Theorem 3.3. □

Theorem 4.7 (iii) coupled with Corollary 3.4 imply that if $x_1, \ldots, x_n$ are defined over K, and if $k \geq M$, then $(x_1 \cdots x_n)^k$ is defined over K.

4.3. The Galois action on lifting invariants

Let H be a subgroup of G, c be a K-rational conjugation-invariant subset of H which generates H, and D be the set of conjugacy classes of H contained in c. In this subsection, we define a Galois action of Γ_K on the set $U(H, c)$. Proposition 4.8 implies that this action effectively describes the effect on lifting invariants of the Galois action on $\mathrm{Comp}(G)$. This generalizes the branch cycle lemma (Corollary 2.12 (i)). Moreover, in Theorem 4.10, we show that the Galois action on lifting invariants of elements of $\mathrm{Comp}(G)$ is compatible with multiplication.

Consider a Galois automorphism $\sigma \in \Gamma_K$. Since c is a K-rational subset, the $\chi(\sigma)$-th power operation defines a map $p_\sigma : D \to D$.

If $\gamma \in D$, choose an arbitrary element g_γ of γ and denote by $\widehat{g_\gamma}$ (resp. $\widehat{(g_\gamma)^{\chi(\sigma^{-1})}}$) the projection on S_c of the element $[g_\gamma] \in U(H,c)$ (resp. $\left[(g_\gamma)^{\chi(\sigma^{-1})}\right] \in U(H,c)$). Define the following element of S_c, which can be checked to be independent from the choice of g_γ:

$$w(\gamma,\sigma) \overset{\text{def}}{=} \widehat{g_\gamma}^{-\chi(\sigma^{-1})}\widehat{(g_\gamma)^{\chi(\sigma^{-1})}}.$$

Importantly, the element $w(\gamma,\sigma)$ is central in S_c (its image in H is $(g_\gamma)^{-\chi(\sigma^{-1})}(g_\gamma)^{\chi(\sigma^{-1})} = 1$, and thus Proposition 4.3 applies).

Consider an element $v \in U(H,c)$, decomposed as (h,ψ) via the isomorphism $U(H,c) \simeq S_c \underset{H^{\mathrm{ab}}}{\times} \mathbb{Z}^D$ (cf. Paragraph 4.1.2). We let:

$$\sigma.v = \left(h^{\chi(\sigma^{-1})} \prod_{\gamma\in D} w(\gamma,\sigma)^{\psi(c)} \quad , \quad \psi \circ p_\sigma \right).$$

This formula is shown to define an action of Γ_K on the set $U(H,c)$. This construction is taken from [Woo21, Paragraph 4.1], and the following proposition follows from [Woo21, Paragraph 6.1]:

Proposition 4.8. *Let $x \in \mathrm{Comp}(G)$ with $\langle x\rangle \subseteq H$ and $c_H(x) \subseteq c$. Then:*

$$\Pi_{H,c}(\sigma.x) = \sigma.\Pi_{H,c}(x).$$

By projection on $\mathbb{Z}^D$, Proposition 4.8 gives back the branch cycle lemma (Corollary 2.12 (i)). A consequence of Proposition 4.8 is the following necessary condition, which refines Corollary 2.12 (ii):

Corollary 4.9. *Let $x \in \mathrm{Comp}(G)$ with $\langle x\rangle \subseteq H$ and $c_H(x) \subseteq c$. If the component x is defined over K, then its (H,c)-lifting invariant is Γ_K-invariant.*

We now show that a product of Γ_K-invariant elements of $U_1(H,c)$ is Γ_K-invariant, and thus the lifting invariant cannot be used to detect negative answers to Question 1.1. This follows from the following fact:

Theorem 4.10. *The action of Γ_K on $U_1(H,c)$ is compatible with multiplication.*

Proof. Let $v, v' \in U_1(H,c)$, and decompose them as $v = (h,\psi)$, $v' = (h',\psi')$ with $h, h' \in H_2(H,c)$ and $\psi, \psi' \in \ker(\widetilde{\pi})$. We have $vv' = (hh', \psi + \psi')$. Let $\sigma \in \Gamma_K$. With notation as above, we have:

$$\begin{aligned}
\sigma.(vv') &= \left((hh')^{\chi(\sigma^{-1})} \prod_{\gamma\in D} w(\gamma,\sigma)^{(\psi+\psi')(c)} \quad , \quad (\psi+\psi')\circ p_\sigma \right) \\
&= \left(h^{\chi(\sigma^{-1})}(h')^{\chi(\sigma^{-1})} \prod_{\gamma\in D} w(\gamma,\sigma)^{\psi(c)} w(\gamma,\sigma)^{\psi'(c)} \quad , \quad \psi\circ p_\sigma + \psi'\circ p_\sigma \right) \\
&= \left(\left(h^{\chi(\sigma^{-1})} \prod_{\gamma\in D} w(\gamma,\sigma)^{\psi(c)} \right) \left((h')^{\chi(\sigma^{-1})} \prod_{\gamma\in D} w(\gamma,\sigma)^{\psi'(c)} \right) \quad , \quad \psi\circ p_\sigma + \psi'\circ p_\sigma \right) \\
&= (\sigma.v)(\sigma.v').
\end{aligned}$$

□

We have used repeatedly that $H_2(H,c)$ is abelian in the final computation: this proof does not apply to arbitrary elements of $U(H,c)$. However, the same proof shows that $\sigma.(vv') = (\sigma.v)(\sigma.v')$ holds as soon as v and v' commute in $U(H,c)$.

Theorem 4.10 implies positive answers to Question 1.1 in situations where the lifting invariant is shown to characterize components. For example, Theorem 4.7 (iii) can be deduced from Theorem 4.5 by using Theorem 4.10 instead of Theorem 3.1.

5. The patching approach

In this section, we use patching results over complete valued fields to study the fields of definition of components obtained by gluing two components $x, y \in \mathrm{Comp}(G)$ defined over the number field K. The main result is Theorem 5.4, which is Theorem 1.2 (iii). We now give a sketch of the argument, which also serves as an outline of the section.

In Subsection 5.1, we recall and use a version of Hilbert's irreducibility theorem (Theorem 5.1) to construct infinitely many field extensions $K_1, K_2, \dots$ of K, pairwise linearly disjoint, over which the components x and y both have points (Lemma 5.2). For each $n \in \mathbb{N}$, denote by f_n (resp. g_n) a $K_n((X))$-point of x (resp. y) obtained from a K_n-point of x (resp. y). Note that $K_n((X))$ is a complete valued field for the (X)-adic valuation.

In Subsection 5.2, we prove that for each $n \in \mathbb{N}$, the cover obtained by patching the $K_n((X))$-G-covers f_n and g_n is a $K_n((X))$-G-cover which lies in a component $m_n \in \mathrm{ni}^\natural(x, y)$ (Lemma 5.3). In particular, the field of definition of the component m_n is included in $\overline{\mathbb{Q}} \cap K_n((X)) = K_n$.

Finally, we observe that at least two components $m_n, m_{n'}$ have to be equal because $\mathrm{ni}^\natural(x, y)$ is finite. Such a component $m_n = m_{n'}$ has its field of definition included in $K_n \cap K_{n'} = K$. In other words, we have found a component defined over K in $\mathrm{ni}^\natural(x, y)$: this is precisely Theorem 5.4. The detailed proof is the focus of Subsection 5.3.

Note that the results of this section rely crucially on the fact that number fields are *Hilbertian*.

5.1. Constructing covers with linearly disjoint fields of definition

We will use the following form of Hilbert's irreducibility theorem, which is close to the statement in [BSFP14]:

Theorem 5.1 (Hilbert's irreducibility theorem)**.** *Let $L' \mid L$ be a finite extension of number fields and $p : X \to Y$ be a finite étale morphism from a variety X over L to an open subvariety Y of $\mathbb{A}^n_L$. Assume $X_{L'}$ is irreducible. Then there exists an L-point $t \in Y(L)$ such that the $\overline{\mathbb{Q}}$-points of X that are mapped to t by p lie in a single $\mathrm{Gal}(\overline{\mathbb{Q}} \mid L')$-orbit.*

When $L = L'$, this theorem is well-known. The fact that L' may be chosen larger than L follows from [FJ08, Corollary 12.2.3]. Using Theorem 5.1, we prove the following lemma:

Lemma 5.2. *Let $L' \mid L$ be a finite Galois extension of number fields and S be an irreducible component of the Hurwitz scheme $\mathcal{H}^*(G, n)_L$ which is geometrically irreducible. Then there exists a field extension $\widetilde{L} \mid L$ such that $\widetilde{L}$ and L' are linearly disjoint over L, and an $\widetilde{L}$-point $f \in S(\widetilde{L})$.*

Proof. Since S is geometrically irreducible, its extension $S_{L'}$ is irreducible. The branch point morphism $S \to (\mathrm{Conf}_n)_L$ is finite étale. By Hilbert's irreducibility theorem (Theorem 5.1), there is a configuration $\underline{t} \in \mathrm{Conf}_n(L)$ such that the fiber $F \subset S(\overline{\mathbb{Q}})$ above $\underline{t}$ consists of a single Galois orbit.

Let f be any of the points in F and $\widetilde{L}$ be the smallest extension of L over which the point f is rational. Note that $\widetilde{L}L'$ is the smallest extension of L' over which the point f is rational. The fiber F is the $\mathrm{Gal}(\overline{\mathbb{Q}} \mid L')$-orbit of f, hence the degree of the extension $\widetilde{L}L' \mid L'$ is the cardinality of F. The same argument shows that the degree of the extension $\widetilde{L} \mid L$ is also equal to the cardinality of F. The equality $[\widetilde{L}L' : L'] = [\widetilde{L} : L]$ implies that L' and $\widetilde{L}$ are linearly disjoint over L. □

5.2. Relating patching and gluing

Let $\mathcal{O}$ be a complete discrete valuation ring of characteristic zero and L its fraction field, which is a complete valued field. Since $\overline{\mathbb{Q}}$ is algebraically closed and $\overline{L}$ contains $\overline{\mathbb{Q}}$, extension of scalars induces a bijection between the irreducible components of $\mathcal{H}^*(G, n)_{\overline{\mathbb{Q}}}$ and those of $\mathcal{H}^*(G, n)_{\overline{L}}$. We denote by Φ_n this bijection. This lets us do the following slight terminological abuse: we say that a component $x \in \mathrm{Comp}(G)$ has an L-rational point if its extension $\Phi_{\deg(x)}(x)$ to $\overline{L}$ has an L-rational point.

Lemma 5.3. *Let $x_1, x_2 \in \mathrm{Comp}(G)$ be components which have L-rational points. Then there is a component $y \in \mathrm{ni}^\natural(x_1, x_2)$ which has an L-rational point.*

Proof.

- **Step 1: Setting things up**

  For $i = 1, 2$, let $r_i = \deg(x_i)$, $G_i = \langle x_i \rangle$ and fix an L-model $f_i \in \mathcal{H}^*(G, r_i)(L)$ of an L-rational point of $\Phi_{r_i}(x_i)$. The point f_i corresponds to an L-G-cover with a marked L-point above ∞. In the cover f_i, keep only the geometrically irreducible component of the marked point, which is defined over L since the marked point is L-rational. This turns f_i into a geometrically irreducible L-G_i-cover with a marked L-point. The cover f_i belongs to the component x_i' of $\mathcal{H}^*(G_i, r_i)_L$ obtained by keeping only the component of the marked points in the covers of x_i, like in Paragraph 2.3.4. Without loss of generality, we may assume $G = \langle G_1, G_2 \rangle$.

- **Step 2: Patching covers over L**

  We use the algebraic patching results of [HV96]. First define:

$$\begin{aligned} L\{z\} &= \left\{ \textstyle\sum_{i\geq 0} a_i z^i \in L[[z]] \,\middle|\, a_i \underset{i\to\infty}{\to} 0 \right\} & Q_1 &= \mathrm{Frac}(L\{z\}) \\ L\{z^{-1}\} &= \left\{ \textstyle\sum_{i\geq 0} a_i z^{-i} \in L[[z^{-1}]] \,\middle|\, a_i \underset{i\to\infty}{\to} 0 \right\} & Q_2 &= \mathrm{Frac}(L\{z^{-1}\}) \\ L\{z, z^{-1}\} &= \left\{ \textstyle\sum_{i\in\mathbb{Z}} a_i z^i \in L[[z, z^{-1}]] \,\middle|\, a_i \underset{i\to\pm\infty}{\to} 0 \right\} & \widehat{Q} &= \mathrm{Frac}(L\{z, z^{-1}\}). \end{aligned}$$

  Let also $Q_1' = Q_2$ and $Q_2' = Q_1$. From the point of view of rigid analytic geometry, Q_1 (resp. Q_2, and $\widehat{Q}$) is the algebra of analytic functions on the unit disk D_1 centered at 0 (resp. a disk D_2 centered at ∞, and the annulus $D_1 \cap D_2$):

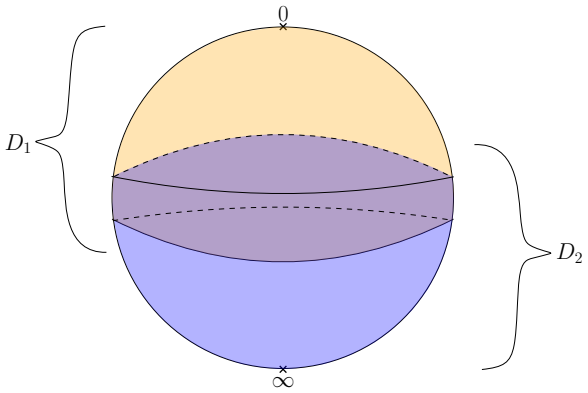


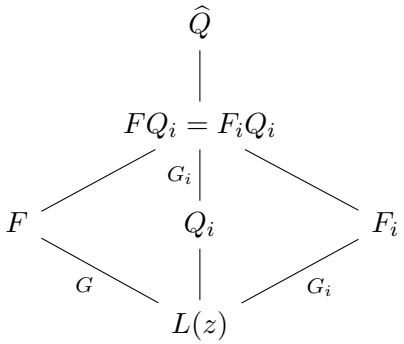
Figure 2: The rigid analytic projective line

  The marked points of the G-covers f_1, f_2 are L-points in an unramified fiber. Their existence ensures that the corresponding field extensions F_1, F_2 of $L(z)$ have an unramified prime of degree 1. By [HV96, Lemma 4.2], for $i = 1, 2$, we can then replace f_i by an isomorphic L-G_i-cover such that F_i is included in Q_i', and in particular the branch locus $\underline{t}_i \in \mathrm{Conf}_{r_i}(L)$ of f_i is included in a disk strictly smaller than D_i. [HV96, Proposition 4.3] implies that f_1 and f_2 can be patched into a geometrically irreducible L-G-cover f with an L-point.

- **Step 3: Restriction of the patched cover f to disks**

  Denote by F the field extension corresponding to f, i.e. the *compound* of F_1 and F_2 in the terminology of [HV96], which is included in $\widehat{Q}$. By [HV96, Lemma 3.6 (b)], we have the

equalities $FQ_i = F_iQ_i$ (for $i = 1, 2$) inside $\widehat{Q}$. Moreover, the morphism $\mathrm{Gal}(FQ_i \mid Q_i) \to \mathrm{Gal}(F \mid L(z))$ corresponds to the inclusion $G_i \hookrightarrow G$. We sum this up by the following diagram:



Geometrically, the equality $FQ_1 = F_1Q_1$ means that the cover f_1 is isomorphic to f as a rigid analytic cover when both are restricted to the unit disk D_1, and similarly for f_2 and f in restriction to D_2.

In consequence, the branch points of f are given by the configuration $\underline{t} = \underline{t}_1 \cup \underline{t}_2$. Let y be the component of $\mathcal{H}^*(G, r_1 + r_2)_{\overline{L}}$ containing f (seen as an $\overline{L}$-point). To show that the component y fits, it remains to check that $\Phi^{-1}_{r_1+r_2}(y) \in \mathrm{ni}(x_1, x_2)$.

- **Step 4: Admissibility of the special fiber $\overline{f}$ of the patched cover**

  Since $\underline{t}_1$ and $\underline{t}_2$ are included in disks strictly smaller than D_1, D_2, each of the configurations $\underline{t}_1, \underline{t}_2$ maps to a single element $\overline{a_1}, \overline{a_2}$ modulo the maximal ideal of $\mathcal{O}$, with $\overline{a_1} \neq \overline{a_2}$. The projective line $\mathbb{P}^1_L$ marked by $\underline{t} = \underline{t}_1 \cup \underline{t}_2$ has a semistable model $\widetilde{P}_{\underline{t}}$ over $\mathcal{O}$, whose special fiber $\overline{P}_{\underline{t}}$ is a "comb" with two teeth T_1, T_2, one for each coset $\overline{a_1}, \overline{a_2}$. For $i = 1, 2$, the points of the configuration $\underline{t}_i$ extend to sections which specialize to r_i distinct nonsingular points of the tooth T_i.

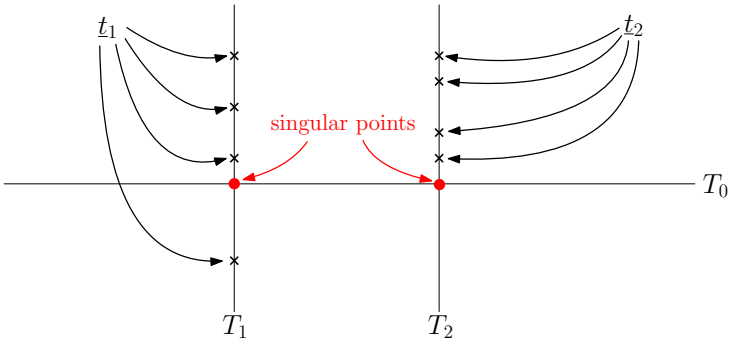


Figure 3: The comb with two teeth $\overline{P}_{\underline{t}}$

The cover f, branched at $\underline{t}$, extends to a cover $\widetilde{f}$ of the semistable model $\widetilde{P}_{\underline{t}}$, which is ramified along the sections of the points in $\underline{t}$. The special fiber $\overline{f}$ of $\widetilde{f}$ is a cover of the comb which lies on the "boundary" of the component y in the sense of the Wewers' compactification, see [DE06, Paragraph 1.2] or [Cau12, Paragraph 3.3.1].

To prove that the special fiber $\overline{f}$ of $\widetilde{f}$ is unramified at the singular points of the comb, we follow [DE06, Paragraph 2.3] closely. The restriction of f to D_1 extends to a cover (namely, f_1) of the rigid projective line which has no branch points outside D_1. By the arguments of [DE06, Proposition 2.3, (ii)⇒(i)⇒(iii)], the restricted cover $f_{|D_1}$ is trivial above the annulus ∂D_1. The same holds for $f_{|D_2}$. Hence, $\overline{f}$ is unramified at the singular points of the comb.

We conclude that $\overline{f}$ is a cover of the comb $\overline{P}_{\underline{t}}$ unramified at the singular points, whose restriction to the i-th tooth is isomorphic to the cover f_i – which belongs to the component x_i'.

- **Step 5: Conclusion**

  The conclusion of Step 4 implies that $\overline{f}$ is a Δ-admissible cover in the sense of [Cau12, Definition 3.7], where:

  $$\Delta = \Big(G, (G_1, G_2), (x_1', x_2')\Big)$$

  is the degenerescence structure associated to (x_1', x_2'). By [Cau12, Proposition 3.9], the component of f is a Δ-component, which in our terminology means that $\Phi_{r_1+r_2}^{-1}(y) \in \mathrm{ni}(x_1, x_2)$ as we noted in Subsection 3.1. This concludes the proof.

□

5.3. Proof of the theorem

Theorem 5.4. *Let $x, y \in \mathrm{Comp}(G)$ be components defined over K. Then $\mathrm{ni}^\natural(x, y)$ contains a component defined over K.*

Proof. Let $r_1 = \deg(x)$, $r_2 = \deg(y)$. Since the components x, y are defined over K, we fix $\begin{cases} \text{a } K\text{-model } X \subseteq \mathcal{H}^*(G, r_1)_K \text{ of } x \\ \text{a } K\text{-model } Y \subseteq \mathcal{H}^*(G, r_2)_K \text{ of } y \end{cases}$. Note that X and Y are geometrically irreducible. The proof consists of three steps:

1. First, we inductively construct two sequences of marked G-covers $(f_n)_{n\geq 1}$ and $(g_n)_{n\geq 1}$, as well as a sequence (K_n) of field extensions of K such that:

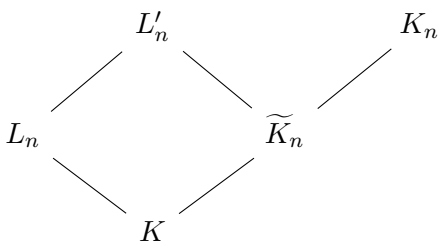
   - K_n is linearly disjoint with the Galois closure of $K_1 \cdots K_{n-1}$ over K.
   - f_n and g_n are K_n-points of X and Y respectively.

   For f_1 and g_1, choose arbitrary $\overline{\mathbb{Q}}$-points of X and Y respectively, and let K_1 be the smallest extension of K over which they are both rational.

   Assume we have constructed $K_1, \ldots, K_{n-1}$ and $f_1, g_1, \ldots, f_{n-1}, g_{n-1}$. Let L_n be the Galois closure of $K_1 \cdots K_{n-1}$ over K.

   Apply Lemma 5.2 with $L = K$, $L' = L_n$ and $S = X$. This yields a field extension $\widetilde{K}_n$ of K such that $\widetilde{K}_n$ and L_n are linearly disjoint over K, and a $\widetilde{K}_n$-point f_n of X. Let L_n' be the Galois closure of $L_n\widetilde{K}_n$. Apply once again Lemma 5.2 with $L = \widetilde{K}_n$, $L' = L_n'$ and $S = Y_{\widetilde{K}_n}$. This yields a field extension K_n of $\widetilde{K}_n$ such that K_n and L_n' are linearly disjoint over $\widetilde{K}_n$, and a K_n-point g_n of Y. Finally, replace the $\widetilde{K}_n$-point f_n by f_n seen as a K_n-point.

   The inclusions between the fields introduced above are summed up by the following diagram:

   By construction, we have $f_n \in X(K_n)$ and $g_n \in Y(K_n)$. Now:

   $$K_n \cap L_n = K_n \cap (L_n' \cap L_n) = (K_n \cap L_n') \cap L_n = \widetilde{K}_n \cap L_n = K.$$

Since $L_n \mid K$ is Galois, this is enough to conclude that K_n and L_n are linearly disjoint over K. We have verified that the constructed sequences (f_n), (g_n), (K_n) satisfy the desired properties.

2. Next, we show that for each n there is a component $z_n \in \mathrm{ni}^\natural(x,y)$ defined over K_n.

   Denote by $\widetilde{f}_n$ (resp. $\widetilde{g}_n$) the $K_n((X))$-point of X (resp. Y) obtained by seeing $f_n \in X(K_n)$ (resp. $g_n \in Y(K_n)$) as a $K_n((X))$-point. Since $F = K_n((X))$ is a complete valued field, Lemma 5.3 implies that there is a component $z_n \in \mathrm{ni}^\natural(x,y)$ which has a $K_n((X))$-rational point. In particular, the field of definition of z_n is included in $K_n((X)) \cap \overline{\mathbb{Q}} = K_n$.

   We have established that there is a component $z_n \in \mathrm{ni}^\natural(x,y)$ defined over K_n for all n.

3. Finally, since $\mathrm{ni}^\natural(x,y)$ is finite, there must be distinct integers n, n' such that $z_n = z_{n'}$. Fix such n, n'. Then, the field of definition of z_n is included in $K_n \cap K_{n'} = K$.

   This concludes the proof: there is a component $z_n \in \mathrm{ni}^\natural(x,y)$ defined over K.

□

Example 5.5. The Mathieu group M_{23} is the only sporadic simple group not known to be a Galois group over $\mathbb{Q}$. In [Cau16, Exemple 3.12], a component defined over $\mathbb{Q}$ of connected M_{23}-covers with 15 branch points is constructed. Theorem 5.4 improves upon this result. The group M_{23} is generated by two conjugate elements a, a^γ of order 3. Using GAP:

```
a := (1, 22, 14) (2, 13, 9) (3, 8, 6) (7, 16, 21) (10, 18, 19) (11, 23, 12);
b := (2, 4, 16) (3, 5, 7) (6, 11, 12) (8, 9, 14) (10, 21, 20) (15, 18, 17);
StructureDescription(Group(a, b)); # Output: "M23"
IsConjugate(Group(a, b), a, b); # Output: true
```

By the conclusions of Example 2.13, the component $x = (a, a^{-1})$ and its conjugate x^γ are defined over $\mathbb{Q}$. By Theorem 5.4, there are elements $\gamma_1, \gamma_2 \in M_{23}$ such that $x^{\gamma_1} x^{\gamma_2 \gamma}$ is a component defined over $\mathbb{Q}$ of connected M_{23}-covers with four branch points. The same is true of the component $x\widetilde{x^\gamma}$ where $\widetilde{\gamma} = \gamma_1^{-1}\gamma_2\gamma$. However, we know little about $\widetilde{\gamma} \in M_{23}$. There are many pairs of generators of M_{23} with orders in $\{2,3,4,6\}$, and consequently many other examples with four branch points.

Example 5.6. Similarly, the group $G = \mathrm{PSL}_2(16) \rtimes \mathbb{Z}/2\mathbb{Z}$ (labeled "17T7" on the Klüners-Malle database and on LMFDB), which is the transitive group of least degree not known to be a Galois group over $\mathbb{Q}$, is generated by two conjugate elements a, b of order 6:

```
a := (1, 11, 5, 13, 14, 17) (3, 15, 7, 12, 8, 6) (9, 10, 16);
b := (1, 2, 15, 12, 8, 5) (3, 14, 11,4, 9, 6) (7, 10, 17);
StructureDescription(Group(a, b)); # Output: "PSL(2,16) : C2"
```

Like in Example 5.5, we conclude by Theorem 5.4 that there is a component defined over $\mathbb{Q}$ of connected G-covers with 4 branch points.

The same holds for any group generated by two elements with orders in $\{2,3,4,6\}$. More generally, if one has a finite generating set of G and $m(i)$ is the number of generators of order i, then there is a component defined over $\mathbb{Q}$ of connected G-covers whose number of branch points is $2m(2) + \sum_{i\geq 3} \varphi(i)m(i)$.

References

[BSFP14] Lior Bary-Soroker, Arno Fehm, and Sebastian Petersen. "On Varieties of Hilbert Type". In: *Ann. Inst. Fourier, Grenoble* 64.5 (2014), pp. 1893–1901. URL: http://www.numdam.org/item/10.5802/aif.2899.pdf.

[Cau12] Orlando Cau. "Delta-composantes des espaces de modules de revêtements". fr. In: *Journal de Théorie des Nombres de Bordeaux* 24.3 (2012), pp. 557–582. DOI: 10.5802/jtnb.811. URL: http://www.numdam.org/articles/10.5802/jtnb.811/.

[Cau16] Orlando Cau. "Delta-composantes des modules de revêtements : corps de définition". In: *Bull. Soc. math. France* 144.2 (2016), pp. 145–162. URL: https://smf.emath.fr/system/files/2017-08/smf_bull_144_145-162.pdf.

[DD97] Pierre Dèbes and Jean-Claude Douai. "Algebraic covers : field of moduli versus field of definition". In: *Annales scientifiques de l'École Normale Supérieure* 30.3 (1997), pp. 303–338.

[DE06] Pierre Dèbes and Michel Emsalem. "Harbater-Mumford Components and Towers of Moduli Spaces". In: *Journal of the Institute of Mathematics of Jussieu* 5 (2006), pp. 351 – 371.

[EVW12] Jordan S. Ellenberg, Akshay Venkatesh, and Craig Westerland. "Homological stability for Hurwitz spaces and the Cohen-Lenstra conjecture over function fields, II." **retracted preprint**. 2012. URL: https://arxiv.org/pdf/1212.0923v1.pdf.

[EVW16] Jordan S. Ellenberg, Akshay Venkatesh, and Craig Westerland. "Homological stability for Hurwitz spaces and the Cohen-Lenstra conjecture over function fields". In: *Annals of Mathematics* 183.3 (2016), pp. 729–786. URL: https://www.jstor.org/stable/24735176.

[FJ08] Michael D. Fried and Moshe Jarden. *Field Arithmetic*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. Springer Berlin, Heidelberg, 2008. URL: https://link.springer.com/book/10.1007/978-3-540-77270-5.

[Fri77] Michael D. Fried. "Fields of definition of function fields and Hurwitz families—groups as Galois groups". In: *Communications in Algebra* 5.1 (1977), pp. 17–82. ISSN: 0092-7872. DOI: 10.1080/00927877708822158. URL: https://doi.org/10.1080/00927877708822158.

[Fri95] Michael D. Fried. "Introduction to modular towers: Generalizing dihedral group–modular curve connections". In: *Recent Developments in the Inverse Galois Problem (Seattle, WA, 1993)*. Ed. by RI Amer. Math. Soc. Providence. Vol. 186. Contemp. Math. 1995, 111–171.

[FV91] Michael D. Fried and Helmut Völklein. "The inverse Galois problem and rational points on moduli spaces". In: *Math. Ann.* 290.4 (1991), pp. 771–800. ISSN: 0025-5831. DOI: 10.1007/BF01459271. URL: https://doi.org/10.1007/BF01459271.

[Har03] David Harbater. "Patching and Galois theory". In: *MSRI Publication series* 41 (2003), pp. 313–424.

[HV96] Dan Haran and Helmut Völklein. "Galois groups over complete valued fields". In: *Israel Journal of Mathematics* 93 (1996), pp. 9–27. ISSN: 0021-2172. DOI: 10.1007/BF02761092.

[Liu95] Qing Liu. "Tout groupe fini est un groupe de Galois sur $\mathbb{Q}_p(T)$". In: *Recent developments in the Inverse Galois Problem*. Ed. by M. D. Fried. Vol. 186. Contemporary Mathematics. American Mathematical Society, 1995, pp. 261–265.

[RW06] Matthieu Romagny and Stefan Wewers. "Hurwitz Spaces". In: *Séminaire et Congrès* 13 (2006), pp. 313–341.

[Seg22] Béranger Seguin. *The Geometry of Rings of Components of Hurwitz Spaces*. 2022. eprint: arXiv:2210.12793.

[Woo21] Melanie Wood. "An algebraic lifting invariant of Ellenberg, Venkatesh, and Westerland". In: *Research in the Mathematical Sciences* 8 (June 2021). DOI: 10.1007/s40687-021-00259-2. URL: https://math.berkeley.edu/~mmwood/Publications/lifting.pdf.