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# Algorithmic aspects of broadcast independence

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## Abstract

An independent broadcast on a connected graph  $G$  is a function  $f : V(G) \rightarrow \mathbb{N}_0$  such that, for every vertex  $x$  of  $G$ , the value  $f(x)$  is at most the eccentricity of  $x$  in  $G$ , and  $f(x) > 0$  implies that  $f(y) = 0$  for every vertex  $y$  of  $G$  within distance at most  $f(x)$  from  $x$ . The broadcast independence number  $\alpha_b(G)$  of  $G$  is the largest weight  $\sum_{x \in V(G)} f(x)$  of an independent broadcast  $f$  on  $G$ .

We describe an efficient algorithm that determines the broadcast independence number of a given tree. Furthermore, we show NP-hardness of the broadcast independence number for planar graphs of maximum degree four, and hardness of approximation for general graphs. Our results solve problems posed by Dunbar, Erwin, Haynes, Hedetniemi, and Hedetniemi (2006), Hedetniemi (2006), and Ahmane, Bouchemakh, Sopena (2018).

**Keywords:** broadcast independence

# 1 Introduction

In his PhD thesis [7] Erwin introduced the notions of broadcast domination and broadcast independence in graphs. While broadcast domination was studied in detail, only little research has been done on broadcast independence [1,4,6,9], and several fundamental problems related to this notion remained open. After efficient algorithms for optimal broadcast domination were developed for restricted graph classes [3,5], Heggernes and Lokshtanov [10] showed the beautiful and surprising result that broadcast domination can be solved optimally in polynomial time for every graph. In contrast to that, Dunbar et al. [6] and Hedetniemi [9] explicitly ask about the complexity of broadcast independence and about efficient algorithms for trees. As pointed out recently by Ahmane et al. [1], the complexity of broadcast independence was unknown even for trees.

In the present paper we describe an efficient algorithm for optimal broadcast independence in trees. Furthermore, we show hardness of approximation for general graphs and NP-completeness for planar graphs of maximum degree four. Before stating our results precisely, we collect the necessary definitions. We consider finite, simple, and undirected graphs, and use standard terminology and notation. Let  $\mathbb{N}_0$  be the set of nonnegative integers, and let  $\mathbb{Z}$  be the set of integers. For a connected graph  $G$ , a function  $f : V(G) \rightarrow \mathbb{N}_0$  is an *independent broadcast on  $G$*  if

- (B1)  $f(x) \leq \text{ecc}_G(x)$  for every vertex  $x$  of  $G$ , where  $\text{ecc}_G(x)$  is the eccentricity of  $x$  in  $G$ , and
- (B2)  $\text{dist}_G(x, y) > \max\{f(x), f(y)\}$  for every two distinct vertices  $x$  and  $y$  of  $G$  with  $f(x), f(y) > 0$ , where  $\text{dist}_G(x, y)$  is the distance of  $x$  and  $y$  in  $G$ .

The *weight* of  $f$  is  $\sum_{x \in V(G)} f(x)$ . The *broadcast independence number*  $\alpha_b(G)$  of  $G$  is the maximum weight of an independent broadcast on  $G$ , and an independent broadcast on  $G$  of weight  $\alpha_b(G)$  is *optimal*. Let  $\alpha(G)$  be the usual independence number of  $G$  defined as the maximum cardinality of an independent set in  $G$ , which is a set of pairwise nonadjacent vertices of  $G$ . For an integer  $k$ , let  $[k]$  be the set of all positive integers at most  $k$ , and let  $[k]_0 = \{0\} \cup [k]$ .

Note that adding a universal vertex to a non-empty graph does not change its independence number but reduces its diameter to two, and that  $\alpha_b(G) = \alpha(G)$  for a graph  $G$  with diameter two and  $\alpha(G) \geq 3$ . These observations imply that Zuckerman's [11] hardness of approximation result for maximum clique immediately yields the following.

**Proposition 1.1.** *For every positive real number  $\epsilon$ , it is NP-hard to approximate the broadcast independence number of a given connected graph of order  $n$  to within  $n^{1-\epsilon}$ .*

As already stated, we show that computing the broadcast independence number remains hard even when restricted to instances with bounded maximum degree. In fact, we believe that it is hard even when restricted to cubic graphs.

**Theorem 1.1.** *For a given connected planar graph  $G$  of maximum degree 4 and a given positive integer  $k$ , it is NP-complete to decide whether  $\alpha_b(G) \geq k$ .*

Clearly,  $\alpha_b(G) \geq \alpha(G)$  for every connected graph  $G$ . In [2] we show  $\alpha_b(G) \leq 4\alpha(G)$  for every connected graph  $G$ , which yields efficient constant factor approximation algorithms for the broadcast independence number on every class of connected graphs on which the independence number can efficiently be approximated within a constant factor; in particular, on graphs of bounded maximum degree.

In the next section, we prove Theorem 1.1, and, in Section 3, we present the polynomial time algorithm to compute the broadcast independence number of a given tree, more precisely, we prove the following.

**Theorem 1.2.** *The broadcast independence number  $\alpha_b(T)$  of a given tree  $T$  of order  $n$  can be determined in  $O(n^9)$  time.*

## 2 NP-completeness of broadcast independence on planar graphs with maximum degree 5

In this section, we prove Theorem 1.1.

Since an independent broadcast can be encoded using polynomially many bits, and (B1) and (B2) can be checked in polynomial time, the considered decision problem is in NP. We show its NP-completeness by reducing to it the NP-complete problem [8] INDEPENDENT SET restricted to connected planar cubic graphs. Therefore, let  $(H, k)$  be an instance of INDEPENDENT SET, where  $H$  is a connected planar cubic graph. Recall that INDEPENDENT SET is the problem to decide whether  $\alpha(H) \geq k$ . In order to complete the proof, we describe a polynomial time construction of a planar graph  $G$  of maximum degree 4 such that

$$\alpha_b(G) = \alpha(H) + \frac{45}{2}n(H),$$

where  $n(H)$  is the order of  $H$ . First, let the graph  $H'$  arise by subdividing each edge of  $H$  exactly twice. It is well known and easy to see that  $\alpha(H') = \alpha(H) + m(H) = \alpha(H) + \frac{3}{2}n(H)$ , where  $m(H)$  is the number of edges of  $H$ . Now, the graph  $G$  arises from  $H'$  by

- adding, for every vertex  $x$  in  $V(H)$ , one copy  $K(x)$  of the star  $K_{1,4}$  of order 4 and connecting its center with  $x$ , and
- adding, for every vertex  $x$  in  $V(H') \setminus V(H)$ , two disjoint copies  $K_1(x)$  and  $K_2(x)$  of the star  $K_{1,4}$  of order 4 and connecting their two centers with  $x$ .

See Figure 1 for an illustration.

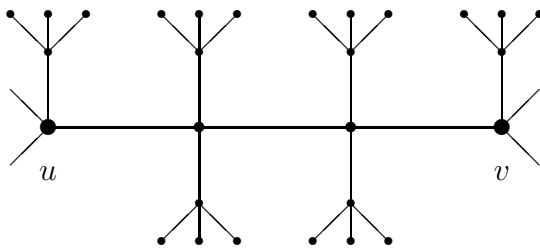


Figure 1: An edge  $uv$  of  $H$  after two subdivisions and the attachment of the disjoint stars. The vertices of  $H$  are shown largest and the vertices of the attached stars are shown smallest.

Note that  $G$  is connected and planar, has order  $32n(H)$  and maximum degree 4, and contains  $21n(H)$  endvertices. Since some maximum independent set in  $G$  contains all  $21n(H)$  endvertices of  $G$ , and removing the closed neighborhoods of all these endvertices yields  $H'$ , we have

$$\alpha(G) = \alpha(H') + 21n(H) = \alpha(H) + \frac{3}{2}n(H) + 21n(H) = \alpha(H) + \frac{45}{2}n(H),$$

that is, it remains to show that  $\alpha_b(G) = \alpha(G)$ .

Let  $f : V(G) \rightarrow \mathbb{N}_0$  be an optimal independent broadcast on  $G$ . For every vertex  $x$  of  $H'$ , let  $L(x)$  be the set of endvertices of  $G$  that are at distance 2 from  $x$ , and let  $L = \bigcup_{x \in V(H')} L(x)$ .

Note that  $|L(x)| = 3$  if  $x \in V(H)$ , and that  $|L(x)| = 6$  if  $x \in V(H') \setminus V(H)$ .

If there is some vertex  $x$  in  $V(G) \setminus L$  with  $f(x) = k$  for some  $k \geq 2$ , and  $y$  is a vertex in  $L$  that is closest to  $x$ , then changing the value of  $f(y)$  to  $k$ , and the value of  $f(x)$  to 0 yields an independent broadcast on  $G$  of the same weight as  $f$ . Applying this operation iteratively, we may assume that

$$f(x) \geq 2 \text{ only if } x \in L.$$

If there is some vertex  $y$  in  $V(H')$  such that  $f(x) \in \{2, 3\}$  for some vertex  $x$  in  $L(y)$ , then changing the value of  $f$  for the at least three vertices in  $L(y)$  to 1 yields an independent broadcast on  $G$  whose weight is at least the weight of  $f$ . Applying this operation iteratively, we may assume that

$$f(x) \notin \{2, 3\} \text{ for every } x \in V(G).$$

If there is some vertex  $y$  in  $V(H') \setminus V(H)$  such that  $f(x) \in \{4, 5, 6\}$  for some vertex  $x$  in  $L(y)$ , then changing the value of  $f$  for the six vertices in  $L(y)$  to 1 yields an independent broadcast on  $G$  whose weight is at least the weight of  $f$ . Applying this operation iteratively, we may assume that

$$f(x) \notin \{4, 5, 6\} \text{ for every } y \in V(H') \setminus V(H) \text{ and every } x \in L(y). \quad (1)$$

Let  $X = \{x \in V(G) : f(x) > 0\}$ . To every vertex  $x$  in  $X$ , we assign an independent set  $I(x)$  as follows:

- If  $f(x) = 1$ , then let  $I(x) = \{x\}$ .
- If  $f(x) = 4$ , then, by (1), there is some vertex  $y$  of  $H$  such that  $x \in L(y)$ . Let  $I(x) = \{y\} \cup L(y)$ .
- If  $f(x) = 5$ , then, by (1), there is some vertex  $y$  of  $H$  such that  $x \in L(y)$ . Let  $y'$  be some neighbor of  $y$  in  $H'$ , and let  $I(x) = L(y) \cup L(y')$ .
- If  $f(x) \geq 6$ , then there is some vertex  $y$  of  $H'$  such that  $x \in L(y)$ . By (B1), there is a shortest path  $P : x_0 \dots x_{\lfloor \frac{f(x)}{2} \rfloor}$  in  $G$  such that  $x_0 = x$  and  $x_2 = y$ . Let

$$I(x) = \bigcup_{i=2}^{\lfloor \frac{f(x)}{2} \rfloor} L(x_i).$$

By construction,  $I(x)$  is an independent set for every  $x$  in  $X$ . Furthermore, if  $f(x) \leq 5$  for some  $x$  in  $X$ , then  $|I(x)| \geq f(x)$  is easily verified. Now, if  $f(x) \geq 6$  for some  $x$  in  $X$ , and  $P$  is as above, then  $\lfloor \frac{f(x)}{2} \rfloor \geq 3$ , at least one of the two sets  $L(x_2)$  and  $L(x_3)$  contains six vertices, and, hence,  $|I(x)| \geq 3 \left( \lfloor \frac{f(x)}{2} \rfloor - 1 \right) + 3 \geq f(x)$ . Altogether,  $I(x)$  is an independent set of order at least  $f(x)$  for every  $x$  in  $X$ .

Suppose, for a contradiction, that there are two distinct vertices  $x$  and  $x'$  in  $X$  such that  $I(x)$  and  $I(x')$  intersect. If  $f(x) = 1$ , then, necessarily,  $f(x') = 4$ , and  $\text{dist}_G(x, x') \leq 2$ , contradicting (B2). If  $f(x), f(x') \in \{4, 5\}$ , then, by (1),  $\text{dist}_G(x, x') = 2$ , and if  $f(x) \in \{4, 5\}$  and  $f(x') \geq 6$ , then  $\text{dist}_G(x, x') \leq \lfloor \frac{f(x')}{2} \rfloor + 2 < f(x')$ , contradicting (B2). Finally, if  $f(x), f(x') \geq 6$ , then  $\text{dist}_G(x, x') \leq \lfloor \frac{f(x)}{2} \rfloor + \lfloor \frac{f(x')}{2} \rfloor \leq \max\{f(x), f(x')\}$ , again contradicting (B2). Hence, the sets  $I(x)$  for  $x$  in  $X$  are all disjoint.

Suppose, for a contradiction, that there are two distinct vertices  $x$  and  $x'$  in  $X$  such that  $G$  contains an edge between  $I(x)$  and  $I(x')$ . Since no two endvertices in  $G$  are adjacent, and  $I(x) \subseteq L$  for  $x$  in  $X$  with  $f(x) \geq 5$ , this implies  $f(x), f(x') \in \{1, 4\}$ , which easily implies a contradiction to (B2). Altogether, it follows that  $I = \bigcup_{x \in X} I(x)$  is an independent set of order at least  $\alpha_b(G)$ , which implies  $\alpha(G) \geq \alpha_b(G)$ . Since  $\alpha_b(G') \geq \alpha(G')$  holds for every graph  $G'$ , this complete the proof.

### 3 A polynomial time algorithm for trees

Throughout this section, let  $T$  be a fixed tree of order  $n$ .

Before we explain the details of our approach, which is based on dynamic programming, we collect some key observations. We will consider certain subtrees  $T(u, i)$  of  $T$  that contain a vertex  $u$  such that all edges of  $T$  between  $V(T(u, i))$  and  $V(T) \setminus V(T(u, i))$  are incident with  $u$ ,

that is,  $V(T(u, i))$  contains  $u$  and some connected components of  $T - u$ . For every independent broadcast  $f$  on  $T$ , the restriction of  $f$  to  $V(T(u, i))$  clearly satisfies

$$(C1) \quad f(x) \leq \text{ecc}_T(x) \text{ for every vertex } x \text{ of } T(u, i).$$

$$(C2) \quad \text{dist}_T(x, y) > \max\{f(x), f(y)\} \text{ for every two distinct vertices } x \text{ and } y \text{ of } T(u, i) \\ \text{with } f(x), f(y) > 0.$$

Furthermore, if  $y$  is a vertex in  $V(T) \setminus V(T(u, i))$  with  $f(y) > 0$ , then  $y$  imposes upper bounds on the possible values of  $f$  inside  $V(T(u, i))$ . More precisely,  $f(x)$  must be 0 for all vertices  $x$  of  $T(u, i)$  with  $\text{dist}_T(x, y) \leq f(y)$ , and  $f(x)$  can be at most  $\text{dist}_T(x, y) - 1$  for all vertices  $x$  of  $T(u, i)$  with  $\text{dist}_T(x, y) > f(y)$ . So, in short we have  $f(x) = 0$  if  $\text{dist}_T(x, y) \leq f(y)$  and  $f(x) \leq \text{dist}_T(x, y) - 1$  otherwise. Expressed as a function of  $\text{dist}_T(u, x)$  instead of  $\text{dist}_T(x, y)$ , and using the equality  $\text{dist}_T(u, x) = \text{dist}_T(x, y) - \text{dist}_T(u, y)$ , we obtain the condition

$$f(x) \leq g_{(p,q)}(\text{dist}_T(u, x)) \text{ for every vertex } x \text{ of } T(u, i),$$

where the function  $g_{(p,q)}(d) : \mathbb{Z} \rightarrow \mathbb{N}_0$  is such that

$$g_{(p,q)}(d) = \begin{cases} 0 & , \text{ if } d \leq p, \text{ and} \\ d - p + q - 1 & , \text{ if } d \geq p + 1, \end{cases} \quad (2)$$

$$p = f(y) - \text{dist}_T(u, y), \text{ and}$$

$$q = f(y).$$

Note that  $q$  is positive,  $p$  may be negative, and that  $|p|$  and  $q$  are both at most the diameter of  $T$ , which is at most  $n$ .

One key observation is the following simple lemma.

**Lemma 3.1.** *If  $(p_1, q_1), \dots, (p_k, q_k)$  are pairs of integers such that  $-n \leq p_i \leq n$  and  $1 \leq q_i \leq n$  for every  $i$  in  $[k]$ , then there is a pair  $(p_{\text{in}}, q_{\text{in}})$  of integers such that  $-n \leq p_{\text{in}} \leq n$ ,  $1 \leq q_{\text{in}} \leq n$ , and*

$$g_{(p_{\text{in}}, q_{\text{in}})}(d) = \min \left\{ g_{(p_1, q_1)}(d), \dots, g_{(p_k, q_k)}(d) \right\} \text{ for every nonnegative integer } d.$$

*Proof.* The statement follows for  $p_{\text{in}} = \max\{p_1, \dots, p_k\}$  and

$$q_{\text{in}} = \min \left\{ g_{(p_1, q_1)}(p_{\text{in}} + 1), \dots, g_{(p_k, q_k)}(p_{\text{in}} + 1) \right\}.$$

Clearly, we have  $-n \leq p_{\text{in}} \leq n$  and  $\min\{g_{(p_1, q_1)}(d), \dots, g_{(p_k, q_k)}(d)\} = 0 = g_{(p_{\text{in}}, q_{\text{in}})}(d)$  for  $d \leq p_{\text{in}}$ . Since  $g_{(p_i, q_i)}(p_{\text{in}} + 1) \geq g_{(p_i, q_i)}(p_i + 1) = q_i \geq 1$  for every  $i$  in  $[k]$ , we have  $q_{\text{in}} \geq 1$ . Furthermore, if  $i$  in  $[k]$  is such that  $p_{\text{in}} = p_i$ , then

$$q_{\text{in}} \leq g_{(p_i, q_i)}(p_{\text{in}} + 1) = g_{(p_i, q_i)}(p_i + 1) = (p_i + 1) - p_i + q_i - 1 = q_i,$$

which implies  $q_{\text{in}} \leq q_i \leq n$ . Finally, notice that for every  $p, q, p'$ , and  $d$  such that  $d \geq p + 1$  and  $p' \geq p$ , we have  $g_{(p,q)}(d) = d - (p' + 1) + g_{(p,q)}(p' + 1)$ . So, for every  $d \geq p_{\text{in}} + 1$ , we have

$$\begin{aligned} \min\{g_{(p_1,q_1)}(d), \dots, g_{(p_k,q_k)}(d)\} &= d - (p_{\text{in}} + 1) + \min\{g_{(p_1,q_1)}(p_{\text{in}} + 1), \dots, g_{(p_k,q_k)}(p_{\text{in}} + 1)\} \\ &= d - (p_{\text{in}} + 1) + q_{\text{in}} \\ &= g_{(p_{\text{in}},q_{\text{in}})}(d). \end{aligned}$$

□

Lemma 3.1 implies that the upper bounds on the possible values of  $f$  inside  $V(T(u, i))$  that are imposed by positive values of  $f$  in  $V(T) \setminus V(T(u, i))$  can be encoded by just two integers  $p_{\text{in}}$  and  $q_{\text{in}}$  with  $-n \leq p_{\text{in}} \leq n$  and  $1 \leq q_{\text{in}} \leq n$ . Symmetrically, the upper bounds on the possible values of  $f$  inside  $V(T) \setminus V(T(u, i))$  that are imposed by positive values of  $f$  in  $V(T(u, i))$ , again expressed as a function of the distance from  $u$  in  $T$ , can be encoded by just two integers  $p_{\text{out}}$  and  $q_{\text{out}}$  with  $-n \leq p_{\text{out}} \leq n$  and  $1 \leq q_{\text{out}} \leq n$ .

For all  $O(n^4)$  possible choices for  $((p_{\text{in}}, q_{\text{in}}), (p_{\text{out}}, q_{\text{out}}))$ , the algorithm determines the maximum contribution  $\sum_{x \in V(T(u, i))} f(x)$  to the weight of an independent broadcast  $f$  on  $T$  such that the following conditions hold:

$$(C3) \quad f(x) \leq g_{(p_{\text{in}}, q_{\text{in}})}(\text{dist}_T(u, x)) \text{ for every vertex } x \text{ of } T(u, i).$$

$$(C4) \quad \text{If } f(x) > 0 \text{ for some vertex } x \text{ of } T(u, i), \text{ then}$$

$$g_{(p_{\text{out}}, q_{\text{out}})}(d) \leq g_{(f(x), f(x))}(d + \text{dist}_T(u, x)) \text{ for every positive integer } d.$$

Intuitively speaking, (C3) means that every value of  $f$  assigned to some vertex of  $T(u, i)$  respects the upper bound encoded by  $g_{(p_{\text{in}}, q_{\text{in}})}$ , and (C4) means that every positive value of  $f$  assigned to some vertex of  $T(u, i)$  imposes an upper bound on the values of  $f$  outside  $V(T(u, i))$  that is at least the upper bound encoded by  $g_{(p_{\text{out}}, q_{\text{out}})}$ . The next lemma shows how to check condition (C4) in constant time for an individual vertex  $x$ .

**Lemma 3.2.** *If  $p, q, f$ , and  $\text{dist}$  are integers such that  $-n \leq p \leq n$  and  $q, f, \text{dist} \in [n]$ , then*

$$g_{(p,q)}(d) \leq g_{(f,f)}(d + \text{dist}) \text{ for every positive integer } d \tag{3}$$

*if and only if  $\max\{f - \max\{p, 0\}, q - p\} \leq \text{dist}$ .*

*Proof.* We consider two cases.

First, let  $p \leq 0$ . In this case,  $g_{(p,q)}(d)$  is positive for every positive  $d$ , and (3) holds if and only if

$$(i) \quad g_{(f,f)}(1 + \text{dist}) \text{ is positive, and}$$



(ii)  $g_{(f,f)}(1 + \text{dist})$  is at least  $g_{(p,q)}(1)$ .

(i) is equivalent to  $1 + \text{dist} \geq f + 1$ , and, by (i), (ii) is equivalent to  $(1 + \text{dist}) - f + f - 1 \geq 1 - p + q - 1$ , that is, (i) and (ii) together are equivalent to  $\max\{f, q - p\} \leq \text{dist}$ .

Next, let  $p \geq 1$ . In this case,  $g_{(p,q)}(d)$  is positive if and only if  $d \geq p + 1$ , and (3) holds if and only if

(i)  $g_{(f,f)}(p + 1 + \text{dist})$  is positive, and

(ii)  $g_{(f,f)}(p + 1 + \text{dist})$  is at least  $g_{(p,q)}(p + 1)$ .

(i) is equivalent to  $p + 1 + \text{dist} \geq f + 1$ , and, by (i), (ii) is equivalent to  $p + 1 + \text{dist} - f + f - 1 \geq p + 1 - p + q - 1 = q$ , that is, (i) and (ii) together are equivalent to  $\max\{f - p, q - p\} \leq \text{dist}$ .

The statement of the lemma summarizes both cases.  $\square$

Now, we explain which subtrees  $T(u, i)$  we consider. We select a vertex  $r$  of  $T$ , and consider  $T$  as rooted in  $r$ . For every vertex  $u$  of  $T$  that is not a leaf, we fix an arbitrary linear order on the set of children of  $u$ . If  $v_1, \dots, v_k$  are the children of  $u$  in their linear order, then for every  $i \in [k]_0$ , let  $T(u, i)$  be the subtree of  $T$  that contains  $u, v_1, \dots, v_i$  as well as all descendants of the vertices  $v_1, \dots, v_i$  in  $T$ . Note that  $T(u, 0)$  contains only  $u$ , and that  $u$  is the only vertex of  $T(u, i)$  that may have neighbors in  $T$  outside of  $V(T(u, i))$ . Altogether, there are at most  $(d_T(r) + 1) + \sum_{x \in V(T) \setminus \{r\}} d_T(x) = O(n)$  many choices for  $(u, i)$ . For brevity, let  $V(u, i)$  denote  $V(T(u, i))$ .

Let  $u$  be a vertex with  $k$  children, and let  $i \in [k]_0$ . Let  $p_{\text{in}}, p_{\text{out}}, q_{\text{in}}$ , and  $q_{\text{out}}$  be integers with  $-n \leq p_{\text{in}}, p_{\text{out}} \leq n$  and  $1 \leq q_{\text{in}}, q_{\text{out}} \leq n$ . A function

$$f : V(u, i) \rightarrow \mathbb{N}_0 \text{ is } ((p_{\text{in}}, q_{\text{in}}), (p_{\text{out}}, q_{\text{out}}))\text{-compatible}$$

if conditions (C1), (C2), (C3), and (C4) hold. Let

$$\beta((u, i), (p_{\text{in}}, q_{\text{in}}), (p_{\text{out}}, q_{\text{out}}))$$

be the maximum weight  $\sum_{x \in V(u, i)} f(x)$  of a function  $f : V(u, i) \rightarrow \mathbb{N}_0$  that is  $((p_{\text{in}}, q_{\text{in}}), (p_{\text{out}}, q_{\text{out}}))$ -compatible.

Our next lemma shows that the broadcast independence number can be extracted from these values.

**Lemma 3.3.**  $\alpha_b(T) = \beta((r, d_T(r)), (-1, n), (n, 1))$ .

*Proof.* By definition,  $T(r, d_T(r))$  equals  $T$ . Since  $g_{(-1,n)}(d) \geq n$  for every  $d$  in  $\mathbb{N}_0$ , condition (C3) is void in view of condition (C1). Similarly, condition (C4) is void by Lemma 3.2 and condition (C1). This clearly implies the statement.  $\square$

The following lemmas explain how to determine  $\beta((u, i), (p_{\text{in}}, q_{\text{in}}), (p_{\text{out}}, q_{\text{out}}))$  recursively for all  $O(n^5)$  possible choices for  $((u, i), (p_{\text{in}}, q_{\text{in}}), (p_{\text{out}}, q_{\text{out}}))$ . The next lemma specifies in particular the values for leaves.

**Lemma 3.4.**

$$\beta((u, 0), (p_{\text{in}}, q_{\text{in}}), (p_{\text{out}}, q_{\text{out}})) = \begin{cases} 0 & , \text{ if } q_{\text{out}} > p_{\text{out}}, \text{ and} \\ \min \left\{ \text{ecc}_T(x), g_{(p_{\text{in}}, q_{\text{in}})}(0), p_{\text{out}} \right\} & , \text{ if } q_{\text{out}} \leq p_{\text{out}}. \end{cases}$$

*Proof.* This follows immediately from (C1), (C3), and (C4) using Lemma 3.2,  $V(u, 0) = \{u\}$ , and  $\text{dist}_T(u, u) = 0$ .  $\square$

The next lemma is the technical key lemma. Recall that  $v_1, \dots, v_k$  denote the children of  $u$  in the chosen linear order. See Figure 2.

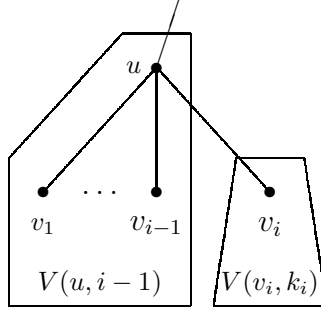


Figure 2: The situation considered in Lemma 3.5.

**Lemma 3.5.** *Let  $i \in [k]$ , and let  $v_i$  have  $k_i$  children. Let  $f : V(u, i) \rightarrow \mathbb{N}_0$ .*

*If  $f$  is  $((p_{\text{in}}, q_{\text{in}}), (p_{\text{out}}, q_{\text{out}}))$ -compatible, then there are integers*

$$p_{\text{in}}^{(0)}, p_{\text{out}}^{(0)}, p_{\text{in}}^{(1)}, p_{\text{out}}^{(1)}, q_{\text{in}}^{(0)}, q_{\text{out}}^{(0)}, q_{\text{in}}^{(1)}, q_{\text{out}}^{(1)},$$

*with  $-n \leq p_{\text{in}}^{(0)}, p_{\text{out}}^{(0)}, p_{\text{in}}^{(1)}, p_{\text{out}}^{(1)} \leq n$  and  $1 \leq q_{\text{in}}^{(0)}, q_{\text{out}}^{(0)}, q_{\text{in}}^{(1)}, q_{\text{out}}^{(1)} \leq n$  such that:*

- (i) *The restriction of  $f$  to  $V(u, i-1)$  is  $\left( (p_{\text{in}}^{(0)}, q_{\text{in}}^{(0)}), (p_{\text{out}}^{(0)}, q_{\text{out}}^{(0)}) \right)$ -compatible.*
- (ii) *The restriction of  $f$  to  $V(v_i, k_i)$  is  $\left( (p_{\text{in}}^{(1)}, q_{\text{in}}^{(1)}), (p_{\text{out}}^{(1)}, q_{\text{out}}^{(1)}) \right)$ -compatible.*
- (iii)  *$g_{(p_{\text{in}}^{(0)}, q_{\text{in}}^{(0)})}(d) = \min \left\{ g_{(p_{\text{in}}, q_{\text{in}})}(d), g_{(p_{\text{out}}^{(1)}, q_{\text{out}}^{(1)})}(d+1) \right\}$  for every nonnegative integer  $d$ .*
- (iv)  *$g_{(p_{\text{in}}^{(1)}, q_{\text{in}}^{(1)})}(d) = \min \left\{ g_{(p_{\text{in}}, q_{\text{in}})}(d+1), g_{(p_{\text{out}}^{(0)}, q_{\text{out}}^{(0)})}(d+1) \right\}$  for every nonnegative integer  $d$ .*
- (v)  *$g_{(p_{\text{out}}, q_{\text{out}})}(d) \leq \min \left\{ g_{(p_{\text{out}}^{(0)}, q_{\text{out}}^{(0)})}(d), g_{(p_{\text{out}}^{(1)}, q_{\text{out}}^{(1)})}(d+1) \right\}$  for every positive integer  $d$ .*

Conversely, if the integers

$$p_{\text{in}}^{(0)}, p_{\text{out}}^{(0)}, p_{\text{in}}^{(1)}, p_{\text{out}}^{(1)}, q_{\text{in}}^{(0)}, q_{\text{out}}^{(0)}, q_{\text{in}}^{(1)}, q_{\text{out}}^{(1)},$$

with  $-n \leq p_{\text{in}}^{(0)}, p_{\text{out}}^{(0)}, p_{\text{in}}^{(1)}, p_{\text{out}}^{(1)} \leq n$  and  $1 \leq q_{\text{in}}^{(0)}, q_{\text{out}}^{(0)}, q_{\text{in}}^{(1)}, q_{\text{out}}^{(1)} \leq n$  are such that conditions (i) to (v) hold, then  $f$  is  $((p_{\text{in}}, q_{\text{in}}), (p_{\text{out}}, q_{\text{out}}))$ -compatible.

*Proof.* First, let  $f : V(u, i) \rightarrow \mathbb{N}_0$  be  $((p_{\text{in}}, q_{\text{in}}), (p_{\text{out}}, q_{\text{out}}))$ -compatible.

By Lemma 3.1, there is a pair  $(p_{\text{out}}^{(0)}, q_{\text{out}}^{(0)})$  such that the function  $g_{(p_{\text{out}}^{(0)}, q_{\text{out}}^{(0)})}$  encodes the upper bounds on the possible values of  $f$  outside of  $V(u, i-1)$  that are imposed by the positive values of  $f$  in  $V(u, i-1)$  as a function of the distance to  $u$ . Similarly, there is a pair  $(p_{\text{out}}^{(1)}, q_{\text{out}}^{(1)})$  such that the function  $g_{(p_{\text{out}}^{(1)}, q_{\text{out}}^{(1)})}$  encodes the upper bounds on the possible values of  $f$  outside of  $V(v_i, k_i)$  that are imposed by the positive values of  $f$  in  $V(v_i, k_i)$  as a function of the distance to  $v_i$ . Since the distance of a vertex outside of  $V(u, i)$  to  $v_i$  is one more than its distance to  $u$ , condition (C4) implies (v).

By Lemma 3.1, the minimum in (iii) equals  $g_{(p_{\text{in}}^{(0)}, q_{\text{in}}^{(0)})}$  for a suitable choice of  $(p_{\text{in}}^{(0)}, q_{\text{in}}^{(0)})$ , where we use  $g_{(p,q)}(d+1) = g_{(p-1,q)}(d)$  for all integers  $p, d$  with  $q \geq 1$ . Similarly, the minimum in (iv) equals  $g_{(p_{\text{in}}^{(1)}, q_{\text{in}}^{(1)})}$  for a suitable choice of  $(p_{\text{in}}^{(1)}, q_{\text{in}}^{(1)})$ , that is, (iii) and (iv) hold.

Since  $f$  is  $((p_{\text{in}}, q_{\text{in}}), (p_{\text{out}}, q_{\text{out}}))$ -compatible, the value of  $f$  assigned to a vertex  $x$  in  $V(u, i-1)$  is at most  $g_{(p_{\text{in}}, q_{\text{in}})}(\text{dist}_T(u, x))$ . Furthermore, by (C2) for  $f$ , the value of  $f$  assigned to any vertex  $x$  in  $V(u, i-1)$  is at most  $g_{(p_{\text{out}}^{(1)}, q_{\text{out}}^{(1)})}(\text{dist}_T(v_i, x)) = g_{(p_{\text{out}}^{(1)}, q_{\text{out}}^{(1)})}(\text{dist}_T(u, x) + 1)$ . Altogether, (i) holds for  $g_{(p_{\text{in}}^{(0)}, q_{\text{in}}^{(0)})}$  as in (iii). By a completely symmetric argument, it follows that (ii) holds for  $g_{(p_{\text{in}}^{(1)}, q_{\text{in}}^{(1)})}$  as in (iv). This completes the proof of the first part of the statement.

Next, let

$$p_{\text{in}}^{(0)}, p_{\text{out}}^{(0)}, p_{\text{in}}^{(1)}, p_{\text{out}}^{(1)}, q_{\text{in}}^{(0)}, q_{\text{out}}^{(0)}, q_{\text{in}}^{(1)}, q_{\text{out}}^{(1)},$$

with  $-n \leq p_{\text{in}}^{(0)}, p_{\text{out}}^{(0)}, p_{\text{in}}^{(1)}, p_{\text{out}}^{(1)} \leq n$  and  $1 \leq q_{\text{in}}^{(0)}, q_{\text{out}}^{(0)}, q_{\text{in}}^{(1)}, q_{\text{out}}^{(1)} \leq n$  be such that conditions (i) to (v) hold.

By (i) and (ii),

$f$  satisfies (C1).

By (i) and (ii), the restriction of  $f$  to  $V(u, i-1)$  satisfies (C2), and the restriction of  $f$  to  $V(v_i, k_i)$  satisfies (C2). Let  $x \in V(u, i-1)$  and  $y \in V(v_i, k_i)$  be such that  $f(x), f(y) > 0$ . We obtain

$$\begin{aligned} f(x) &\stackrel{(i),(C3)}{\leq} g_{(p_{\text{in}}^{(0)}, q_{\text{in}}^{(0)})}(\text{dist}_T(u, x)) \\ &\stackrel{(iii)}{\leq} g_{(p_{\text{out}}^{(1)}, q_{\text{out}}^{(1)})}(\text{dist}_T(u, x) + 1) \\ &\stackrel{(ii),(C4)}{\leq} g_{(f(y), f(y))}(\text{dist}_T(u, x) + 1 + \text{dist}_T(v_i, y)) \\ &= g_{(f(y), f(y))}(\text{dist}_T(x, y)), \end{aligned}$$

which implies  $\text{dist}_T(x, y) > f(y)$ , because  $f(x)$  is positive. Symmetrically, we obtain

$$\begin{aligned}
f(y) &\stackrel{(ii),(C3)}{\leq} g_{(p_{\text{in}}^{(1)}, q_{\text{in}}^{(1)})}(\text{dist}_T(v_i, y)) \\
&\stackrel{(iv)}{\leq} g_{(p_{\text{out}}^{(0)}, q_{\text{out}}^{(0)})}(\text{dist}_T(v_i, y) + 1) \\
&\stackrel{(i),(C4)}{\leq} g_{(f(x), f(x))}(\text{dist}_T(v_i, y) + 1 + \text{dist}_T(u, x)) \\
&= g_{(f(x), f(x))}(\text{dist}_T(x, y)),
\end{aligned}$$

which implies  $\text{dist}_T(x, y) > f(x)$ , because  $f(y)$  is positive. Altogether, it follows that

$f$  satisfies (C2).

Since, by (iii),  $g_{(p_{\text{in}}^{(0)}, q_{\text{in}}^{(0)})}(d) \leq g_{(p_{\text{in}}, q_{\text{in}})}(d)$ , and, by (iv),  $g_{(p_{\text{in}}^{(1)}, q_{\text{in}}^{(1)})}(d) \leq g_{(p_{\text{in}}, q_{\text{in}})}(d + 1)$  for every nonnegative integer  $d$ , we have

$f$  satisfies (C3).

If  $x \in V(u, i - 1)$  is such that  $f(x) > 0$ , then,

$$\begin{aligned}
g_{(p_{\text{out}}, q_{\text{out}})}(d) &\stackrel{(v)}{\leq} g_{(p_{\text{out}}^{(0)}, q_{\text{out}}^{(0)})}(d) \\
&\stackrel{(i),(C4)}{\leq} g_{(f(x), f(x))}(d + \text{dist}_T(u, x)) \text{ for every positive integer } d.
\end{aligned}$$

Similarly, if  $x \in V(v_i, k_i)$  is such that  $f(x) > 0$ , then,

$$\begin{aligned}
g_{(p_{\text{out}}, q_{\text{out}})}(d) &\stackrel{(v)}{\leq} g_{(p_{\text{out}}^{(1)}, q_{\text{out}}^{(1)})}(d + 1) \\
&\stackrel{(ii),(C4)}{\leq} g_{(f(x), f(x))}(d + 1 + \text{dist}_T(v_i, x)) \\
&= g_{(f(x), f(x))}(d + \text{dist}_T(u, x)) \text{ for every positive integer } d.
\end{aligned}$$

Altogether,

$f$  satisfies (C4),

which completes the proof. □

The next lemma is a consequence of Lemma 3.5.

**Lemma 3.6.** *Let  $i \in [k]$ , and let  $v_i$  have  $k_i$  children.*

*$\beta((u, i), (p_{\text{in}}, q_{\text{in}}), (p_{\text{out}}, q_{\text{out}}))$  equals the maximum of*

$$\beta\left((u, i - 1), \left(p_{\text{in}}^{(0)}, q_{\text{in}}^{(0)}\right), \left(p_{\text{out}}^{(0)}, q_{\text{out}}^{(0)}\right)\right) + \beta\left((v_i, k_i), \left(p_{\text{in}}^{(1)}, q_{\text{in}}^{(1)}\right), \left(p_{\text{out}}^{(1)}, q_{\text{out}}^{(1)}\right)\right),$$

*where the maximum extends over all choices of  $p_{\text{in}}^{(0)}, p_{\text{out}}^{(0)}, p_{\text{in}}^{(1)}, p_{\text{out}}^{(1)}, q_{\text{in}}^{(0)}, q_{\text{out}}^{(0)}, q_{\text{in}}^{(1)}$ , and  $q_{\text{out}}^{(1)}$ , with*

$-n \leq p_{\text{in}}^{(0)}, p_{\text{out}}^{(0)}, p_{\text{in}}^{(1)}, p_{\text{out}}^{(1)} \leq n$  and  $1 \leq q_{\text{in}}^{(0)}, q_{\text{out}}^{(0)}, q_{\text{in}}^{(1)}, q_{\text{out}}^{(1)} \leq n$  that satisfy the conditions (iii), (iv), and (v) from Lemma 3.5.

*Proof.* This follows from Lemma 3.5 and the fact that  $V(u, i)$  is the disjoint union of  $V(u, i - 1)$  and  $V(v_i, k_i)$ . Notice that when considering the values  $\beta\left((u, i - 1), \left(p_{\text{in}}^{(0)}, q_{\text{in}}^{(0)}\right), \left(p_{\text{out}}^{(0)}, q_{\text{out}}^{(0)}\right)\right)$  and  $\beta\left((v_i, k_i), \left(p_{\text{in}}^{(1)}, q_{\text{in}}^{(1)}\right), \left(p_{\text{out}}^{(1)}, q_{\text{out}}^{(1)}\right)\right)$ , we know, by definition, that the corresponding restricted independent broadcasts on  $T(u, i - 1)$  and  $T(v_i, k_i)$  are  $\left(\left(p_{\text{in}}^{(0)}, q_{\text{in}}^{(0)}\right), \left(p_{\text{out}}^{(0)}, q_{\text{out}}^{(0)}\right)\right)$ -compatible and  $\left(\left(p_{\text{in}}^{(1)}, q_{\text{in}}^{(1)}\right), \left(p_{\text{out}}^{(1)}, q_{\text{out}}^{(1)}\right)\right)$ -compatible, respectively. As the choice of the parameters satisfies the conditions (iii), (iv), and (v) from Lemma 3.5, we have seen in the proof of that lemma that the common extension of the two restricted independent broadcasts is a restricted independent broadcast on  $T(u, i)$ .  $\square$

We are now ready for the following.

*Proof of Theorem 1.2.* By Lemmas 3.4 and 3.6, each of the  $O(n^5)$  values

$$\beta((u, i), (p_{\text{in}}, q_{\text{in}}), (p_{\text{out}}, q_{\text{out}}))$$

can be determined in time  $O(n^4)$ , by

- processing the vertices  $u$  of  $T$  in an order of nonincreasing depth within  $T$ ,
- considering the  $O(n)$  possible values for each of the four integers  $p_{\text{out}}^{(0)}, p_{\text{in}}^{(1)}, q_{\text{out}}^{(0)}$ , and  $q_{\text{out}}^{(1)}$ ,
- checking condition (v) from Lemma 3.5 in constant time,
- determining the four integers  $p_{\text{in}}^{(0)}, p_{\text{in}}^{(1)}, q_{\text{in}}^{(0)}$ , and  $q_{\text{in}}^{(1)}$  as in (iii) and (iv) from Lemma 3.5 in constant time, and
- adding  $\beta\left((u, i - 1), \left(p_{\text{in}}^{(0)}, q_{\text{in}}^{(0)}\right), \left(p_{\text{out}}^{(0)}, q_{\text{out}}^{(0)}\right)\right)$  and  $\beta\left((v_i, k_i), \left(p_{\text{in}}^{(1)}, q_{\text{in}}^{(1)}\right), \left(p_{\text{out}}^{(1)}, q_{\text{out}}^{(1)}\right)\right)$ .

Now, Lemma 3.3 implies the statement.  $\square$

It may be possible — yet tedious — to extend Theorem 1.2 to graphs of bounded treewidth.

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