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Alexander Zimmermann

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CLIFFORD'S THEOREM FOR ORBIT CATEGORIES

ALEXANDER ZIMMERMANN; ORCID: 0000-0001-8168-4624

Abstract. Clifford theory relates the representation theory of finite groups to those of a fixed normal subgroup by means of induction and restriction, which is an adjoint pair of functors. We generalize this result to the situation of a Krull-Schmidt category on which a finite group acts as automorphisms. This then provides the orbit category introduced by Cibils and Marcos, and studied intensively by Keller in the context of cluster algebras, and by Asashiba in the context of Galois covering functors. We formulate and prove Clifford's theorem for Krull-Schmidt orbit categories with respect to a finite group Γ of automorphisms, clarifying this way how the image of an indecomposable object in the original category decomposes in the orbit category. The pair of adjoint functors appears as the Kleisli category of the naturally appearing monad given by Γ.

1. Introduction

Clifford theory for finite groups links the representation theory of a normal subgroup N of G to the representation theory of G. It is known that large parts of this classical theory does not depend on the coefficient domain R. The crucial part is the notion of the inertia group $I_G(M)$ of an indecomposable RN-module M, which is defined as the subgroup of G formed by those elements $g \in G$, such that the twisted RN-module ${}^gM$ is isomorphic to M as an RN-module. The most elementary part of the theory shows that then for any indecomposable direct factor M_0 of the induced $RI_G(M)$-module $M \uparrow_N^{I_G(M)}$ we get that $M_0 \uparrow_{I_G(M)}^G$ is indecomposable. A Krull-Schmidt situation really is natural, and indeed necessary, for this statement.

Orbit categories arise in representation theory at three places at least. Cluster categories are constructed using orbit categories of triangulated categories, as it was made precise by Keller [26]. Similarly, Peng and Xiao [30, 31] used quotient categories for their construction of quantum groups as Hall algebras of derived categories of hereditary algebras. Further, Riedtmann [32], Cibils, Solotar and Redondo [14, 15, 16] study Gabriel's Galois covering technique in an abstract fashion. More systematically Cibils and Marcos [13], and later Asashiba [1] use orbit categories to explain and actually define clearly these categories. Further, we mention that the setting appears in the context of braided tensor categories and fusion categories (cf [20] and [21]).

The present paper uses arguments from Clifford theory of finite group representation to answer the question what is the decomposition of the image of indecomposable objects in the orbit category. First, orbit categories tend to be non idempotent complete. Hence one needs to consider the Karoubi envelope of the orbit categories. Then, we use techniques from category theory, namely the Kleisli adjunction of a monad, to give an analogue of Clifford's theory for orbit categories.

In a sense the present result can also be seen as a continuation of our previous results [37, 38] on a categorical framework for Green correspondence for adjoint functors. Clifford theory and Green correspondence are two of the main building blocks of modular representation theory of finite groups. Giving a framework in a category framework, opening hence the possible applications is a desirable task. Auslander-Kleiner [3] showed that classical Green correspondence can be formulated and proved as a property of a pair of adjoint functors between additive categories. This was then put in the framework of triangulated categories in [37, 38]. Alternative approaches were given by Benson, Carlson, Grime, Peng, Wheeler, Wang and Zhang in a series of joint papers [8, 24, 6, 9, 36]. Another different approach is given by Balmer and del-Ambrogio [5] in the context of tensor triangulated categories and Mackey 2-functors.

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We start with a Krull-Schmidt category $\mathcal{H}$ on which a finite group Γ acts. The orbit category $\mathcal{G} := \mathcal{H}[\Gamma]$ is then actually the Kleisli construction of the corresponding monad and provides a pair of adjoint functors (S, T)

$$\begin{tikzcd} \mathcal{H} \arrow[r, bend left, "S"] & \mathcal{G} \arrow[l, bend left, "T"] \end{tikzcd}$$

such that $TS = \bigoplus_{i\in I} E_i$ is the direct sum of automorphisms E_i of $\mathcal{H}$, which forms the group Γ. We replace the inertia group in the group algebra situation by the orbit category $\mathcal{H}[\Gamma_M]$ with respect to the subgroup Γ_M of those elements of Γ which fix the isomorphism class of a given object M. We first show that Γ lifts to a group of automorphisms of $\mathcal{G}$ and of $\mathcal{H}[\Gamma_M]$. We obtain that the lift $\widehat{\Gamma}$ of Γ as group of automorphisms of $\mathcal{G}$ fixes isomorphism classes of objects.

These orbit categories $\mathcal{H}[\Gamma_M]$ and $\mathcal{G}$ do not have split idempotents in general, but replacing them with their Karoubi envelope, we can show a precise analogue of Clifford's theorem. This is our main result Theorem 5.11. We note that our main Theorem 5.11 starts with an R-linear Krull-Schmidt category and an action of a finite group Γ on $\mathcal{H}$. Then Theorem 5.11 determines how indecomposable objects in $\mathcal{H}$ behave in the orbit category $\mathcal{H}[\Gamma]$ with respect to indecomposability.

The paper is organised as follows. In Section 2 we recall the necessary notations and background on monads, the Kleisli category, the Eilenberg-Moore adjunctions and their properties as far as we need them. The short Section 3 recalls the group algebra situation. In Section 4 we define categories which model a normal subgroup in our setting, and we prove our first main result, Theorem 4.8, which is the Clifford theorem for monads in the case when analogue of the inertia subgroup does not exceed the normal subgroup. This situation does not need all the hypotheses, and is therefore formulated in a more general setting. Section 5 then defines an analogue of the inertia group in our general situation, studies its properties and shows the main result Theorem 5.11. We use in particular properties on orbit categories, Karoubi envelopes and some general statements on adjoint functors. All these tools are recalled in this section. Finally in Section 6 we present examples from Galois modules and from fusion categories.

2. Monads revisited

Recall that if (S, T) is an adjoint pair, then the endofunctor TS of $\mathcal{H}$ together with the unit and the counit of the adjunction give a monad. There is extensive literature on monads.

Definition 2.1. [28] A monad (A, μ, η) on a category $\mathcal{C}$ is an endofunctor A of $\mathcal{C}$ with a natural transformation

$$\mu : A^2 \longrightarrow A$$

such that

$$\mu \circ (A\mu) = \mu \circ (\mu A) : A^3 \longrightarrow A$$

and

$$\eta : \mathrm{id}_{\mathcal{C}} \longrightarrow A$$

such that

$$\mu \circ (A\eta) = \mu \circ (\eta A).$$

If $\mathbf{A} := (A, \mu, \eta)$ is a monad, an $\mathbf{A}$-module in $\mathcal{C}$ is a pair (X, ρ) where X is an object in $\mathcal{C}$ and $\rho : A(X) \longrightarrow X$ is a morphism in $\mathcal{C}$ such that

$$\rho \circ (A\rho) = \rho \circ \mu_X : A^2(X) \longrightarrow X$$

and

$$\rho \circ \eta_X = \mathrm{id}_X : X \longrightarrow X.$$

If $\mathbf{A} := (A, \mu, \eta)$ is a monad and (X, ρ) and (X', ρ') are $\mathbf{A}$-modules, then a morphism of $\mathbf{A}$-modules

$$f : (X, \rho) \longrightarrow (X', \rho')$$

is an $f \in \mathcal{C}(X, X')$ such that

$$\rho' \circ A(f) = f \circ \rho : A(X) \longrightarrow X'.$$

$\mathbf{A} - \mathrm{Mod}_{\mathcal{C}}$ is the category (!) of $\mathbf{A}$-modules on $\mathcal{C}$.

Every adjunction induces a monad. In general many different adjunctions induce the same monad and all these form a category. There are two particular and extremal such adjunctions realising a given monad. The Eilenberg-Moore adjunction is a terminal object in this category of adjunctions realising a give monad and the Kleisli category is an initial object in this category of adjunctions realising a give monad.

2.1. **The Eilenberg-Moore adjunction.** The Eilenberg-Moore [19] adjunction is an adjoint pair $(F_{\mathbf{A}}, U_{\mathbf{A}})$ where $F_{\mathbf{A}} : \mathcal{C} \longrightarrow \mathbf{A} - \mathrm{Mod}_{\mathcal{C}}$ is defined as

$$F_{\mathbf{A}}(Y) := (A(Y), \mu_Y) \text{ and } F_{\mathbf{A}}(f) = A(f)$$

and

$$U_{\mathbf{A}}(X, \rho) = X \text{ and } U_{\mathbf{A}}(f) = f.$$

The objects in the image of $F_{\mathbf{A}}$ are called free $\mathbf{A}$-modules, and the full subcategory of $\mathbf{A} - \mathrm{Mod}_{\mathcal{C}}$ generated by free modules is denoted $\mathbf{A} - \mathrm{Free}_{\mathcal{C}}$. Restricting to the image of $F_{\mathbf{A}}$ one obtains the so-called Kleisli adjunction $(\widehat{F}_{\mathbf{A}}, \widehat{U}_{\mathbf{A}})$ where $\widehat{F}_{\mathbf{A}} : \mathcal{C} \longrightarrow \mathbf{A} - \mathrm{Free}_{\mathcal{C}}$.

If (S, T) is an adjoint pair with counit $\epsilon : ST \longrightarrow \mathrm{id}_{\mathcal{D}}$, and where $S : \mathcal{C} \longrightarrow \mathcal{D}$, then TS is a monad with $\mu = T\epsilon S$ and $\eta : \mathrm{id}_{\mathcal{C}} \longrightarrow TS$ is the unit.

Further, by [28, Chapter VI] if (S, T) is an adjoint pair with $S : \mathcal{C} \longrightarrow \mathcal{D}$ we get functors

$$K : \mathbf{A} - \mathrm{Free}_{\mathcal{C}} \longrightarrow \mathcal{D} \text{ and } E : \mathcal{D} \longrightarrow \mathbf{A} - \mathrm{Mod}_{\mathcal{C}}$$

such that

$$\widehat{U}_{\mathbf{A}} = T \circ K,\ S = K \circ \widehat{F}_{\mathbf{A}},\ F_{\mathbf{A}} = E \circ S,\ T = U_{\mathbf{A}} \circ E.$$

and in particular the diagram (†)

$$\begin{tikzcd}
 & \mathcal{C} \arrow[dl, "\widehat{F}_{\mathbf{A}}"'] \arrow[d, "S"] \arrow[dr, "F_{\mathbf{A}}"] & \\
\mathbf{A} - \mathrm{Free}_{\mathcal{C}} \arrow[r, "K"'] & \mathcal{D} \arrow[r, "E"'] & \mathbf{A} - \mathrm{Mod}_{\mathcal{C}}
\end{tikzcd}$$

is commutative, and such that $E \circ K$ is a fully faithful embedding. Here, K is defined by

$$\begin{array}{ccc}
\mathbf{A} - \mathrm{Free}_{\mathcal{C}} & \longrightarrow & \mathcal{D} \\
F_{\mathbf{A}}(Y) & \mapsto & S(Y)
\end{array}$$

and by

$$\begin{aligned}
\mathbf{A} - \mathrm{Free}_{\mathcal{C}}(F_{\mathbf{A}}(Y), F_{\mathbf{A}}(Y')) &\simeq \mathcal{C}(Y, AY') \\
&\simeq \mathcal{D}(SY, SY') \\
&\simeq \mathcal{D}(KF_{\mathbf{A}}Y, KF_{\mathbf{A}}Y')
\end{aligned}$$

on morphisms. Hence K is always fully faithful. Further, $E(Z) = (T(Z), T(\epsilon_Z))$ on objects and $E(f) = T(f)$ on morphisms. An adjunction (S, T) is called *monadic* if E is an equivalence. A monad (A, μ, η) on a category $\mathcal{C}$ is called *separable* if there is a natural transformation $\sigma : A \longrightarrow A^2$ such that $\mu \circ \sigma = \mathrm{id}_A$ and

$$(A\mu) \circ (\sigma A) = \sigma \circ \mu = (\mu A) \circ (A\sigma) : A^2 \longrightarrow A^2.$$

Remark 2.2. In the following Proposition 2.3 Balmer uses the concept of an equivalence up to direct summands. A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is called to be an equivalence up to direct summands if the induced functor $\widehat{F} : Kar(\mathcal{C}) \longrightarrow Kar(\mathcal{D})$ is an equivalence. For more details on the Karoubi idempotent completion Kar and the functor $\widehat{F}$ see Remark 5.7 below.

Proposition 2.3. [4, Lemma 2.10] *Let (S, T) be a pair of adjoint functors, $S : \mathcal{C} \longrightarrow \mathcal{D}$ such that the counit $\epsilon : ST \longrightarrow \mathrm{id}_{\mathcal{D}}$ has a section $\xi : \mathrm{id}_{\mathcal{D}} \longrightarrow ST$, i.e. $\xi \circ \epsilon = \mathrm{id}$. Then*

1. *The monad $A = TS$ is separable*
2. *The functors K and E are equivalences up to direct summands*
3. *If $\mathcal{C}$ and $\mathcal{D}$ are idempotent complete, then K and E are equivalences.*

2.2. **The Kleisli category.** Kleisli gave another direct construction realising the monad by an adjunction, the Kleisli category, which is actually isomorphic to $\mathbf{A}-\mathrm{Free}_{\mathcal{C}}$.

Given a monad (A,μ,η) on $\mathcal{H}$, then define the Kleisli category $\mathfrak{K}_A$ by the following construction: The objects of $\mathfrak{K}_A$ and of $\mathcal{H}$ coincide. Let X,Y be two objects. Then

$$\mathfrak{K}_A(X,Y) := \left\{ f \in \mathcal{H}(AX,AY) \;\middle|\; \begin{tikzcd} A^2X \arrow[r, "Af"] \arrow[d, "\mu_{AX}"'] & A^2Y \arrow[d, "\mu_{AY}"] \\ AX \arrow[r, "f"'] & AY \end{tikzcd} \text{ is commutative} \right\}$$

Composition is given by composition of maps in $\mathcal{H}$. This is well-defined since A is a functor. Then there are functors

$$T_{\mathfrak{K}_A} : \mathfrak{K}_A \longrightarrow \mathcal{H}$$

and

$$S_{\mathfrak{K}_A} : \mathcal{H} \longrightarrow \mathfrak{K}_A$$

given by the following:

$T_{\mathfrak{K}_A}(X) = AX$ for any object X and $T_{\mathfrak{K}_A}(f) = f$ for morphisms.

$S_{\mathfrak{K}_A}(X) = X$ for any object X and $S_{\mathfrak{K}_A}(f) = Af$ for morphisms.

Again, $(S_{\mathfrak{K}_A}, T_{\mathfrak{K}_A})$ is an adjoint pair inducing the monad A (cf e.g. [29, Chapter 2].)

The following lemma seems to be well-known to the specialist (cf e.g. the introduction into Teleiko [35]) but for the convenience of the reader we include the result and the short proof.

Lemma 2.4. *Let $\mathcal{H}$ be a category and let (A,μ,η) be a monad on $\mathcal{H}$. Then $\mathfrak{K}_A(X,Y) = \mathcal{H}(X,AY)$.*

Proof. We have the unit $\eta : \mathrm{id}_{\mathcal{H}} \longrightarrow A$ and get a map

$$\begin{array}{ccc} \mathfrak{K}_A(X,Y) & \xrightarrow{\phi} & \mathcal{H}(X,AY) \\ f & \mapsto & \eta_X \circ f \end{array}$$

This way the diagram

$$\begin{tikzcd} AX \arrow[r, "f"] & AY \\ X \arrow[u, "\eta_X"] \arrow[ur, "\phi(f)"'] & \end{tikzcd}$$

is commutative. We have a map in the opposite direction.

$$\begin{array}{ccc} \mathcal{H}(X,AY) & \xrightarrow{\psi} & \mathcal{H}(AX,AY) \\ g & \mapsto & \mu_Y \circ A(g) \end{array}$$

$$\begin{tikzcd} & A^2Y \arrow[d, "\mu_Y"] \\ AX \arrow[ur, "Ag"] \arrow[r, "\psi(g)"'] & AY \end{tikzcd}$$

We compose $\phi \circ \psi$ as in the diagram

$$\begin{tikzcd} & A^2Y \arrow[d, "\mu_Y"] \\ AX \arrow[ur, "Ag"] \arrow[r, "\psi(g)"'] & AY \\ X \arrow[u, "\eta_X"] \arrow[ur, "\phi(\psi(g))"'] & \end{tikzcd}$$

But since $\mu \circ A\eta = \mu \circ \eta A = \mathrm{id}$,

$$\mu_Y \circ A(g) \circ \eta_X = \mu_Y \circ \eta_{A(Y)} \circ g = g.$$

Hence $\phi \circ \psi = \mathrm{id}$.

In order to show that $\psi \circ \phi$ is the identity we consider $f \in \mathfrak{K}_A(X,Y)$.

$$\begin{tikzcd} AX \arrow[r, "f"] & AY \\ X \arrow[u, "\eta_X"] \arrow[ur, "\phi(f)"'] & \end{tikzcd}$$

gives the commutative diagram

$$\begin{tikzcd} A^2X \arrow[r, "Af"] & A^2Y \arrow[d, "\mu_Y"] \\ AX \arrow[u, "A\eta_X"] \arrow[ur, "A\phi(f)"] \arrow[r, "\psi(\phi(f))"'] & AY \end{tikzcd}$$

But since $f \in \mathfrak{K}_A(X,Y)$, the diagram

$$\begin{tikzcd} A^2X \arrow[r, "Af"] \arrow[d, "\mu_X"'] & A^2Y \arrow[d, "\mu_Y"] \\ AX \arrow[r, "f"'] & AY \end{tikzcd}$$

is commutative. Hence

$$\psi\phi(f) = \mu_Y \circ Af \circ A\eta_X = f \circ \mu_X \circ A\eta_X = f$$

since $\mu \circ A\eta = \mathrm{id}$. ■

Note that in order to compute the composition of two maps, the original definition of morphisms in the Kleisli construction is more appropriate.

2.3. **Universal property of the two Eilenberg-Moore and the Kleisli category.** Let (A,μ,η) be a monad on $\mathcal{H}$. Let moreover

$$\begin{tikzcd} \mathcal{H} \arrow[r, bend left, "S"] & \mathcal{G} \arrow[l, bend left, "T"] \end{tikzcd}$$

be functors such that (S,T) is an adjoint pair inducing the monad (A,μ,η) with $A = TS$. Then there are unique functors $K : \mathfrak{K}_A \longrightarrow \mathcal{G}$ and $L : \mathcal{G} \longrightarrow A - Mod_{\mathcal{H}}$ making the diagram

$$\begin{tikzcd} & A - Mod_{\mathcal{H}} \\ \mathcal{H} \arrow[ur, bend left, "S_{EM}"] \arrow[ur, bend right, "T_{EM}"'] \arrow[r, bend left, "S"] \arrow[r, bend right, "T"'] \arrow[dr, bend left, "S_{\mathfrak{K}_A}"] \arrow[dr, bend right, "T_{\mathfrak{K}_A}"'] & \mathcal{G} \arrow[u, "L"'] \\ & \mathfrak{K}_A \arrow[u, "K"'] \end{tikzcd}$$

commutative, in the sense that

$$S = KS_{\mathfrak{K}_A}; T_{\mathfrak{K}_A} = TK$$
$$S_{EM} = LS; T = T_{EM}L.$$

Indeed, on objects we have: $K(X) := S(X)$ for all objects X of $\mathfrak{K}_A$, which are the same objects as those in $\mathcal{H}$. As for morphisms we obtain

$$\begin{array}{ccccccc} \mathfrak{K}_{\mathcal{H}}(X,Y) & \hookrightarrow & \mathcal{H}(AX,AY) & \xrightarrow{\simeq} & \mathcal{G}(STSX,SY) & \longrightarrow & \mathcal{G}(SX,SY) \\ f & \longmapsto & & & & & L(f) \end{array}$$

where the first isomorphism is the adjointness property, and the latter map is given by the unit $\eta : \mathrm{id}_{\mathcal{G}} \longrightarrow TS$, which induces a natural transformation $S\eta : S \longrightarrow STS$, which is actually split, by the fundamental property of adjoint functors. For any morphism $f \in \mathfrak{K}_{\mathcal{H}}(X,Y)$ put $K(f)$ the image under this map.

As for L we put $L(X) := (T(X), \epsilon_Y)$ on objects, where $\epsilon : ST \longrightarrow \mathrm{id}_{\mathcal{H}}$ is the unit of the adjointness, and $L(f) := T(f)$ on morphisms.

Note that the above property, together with the diagram (†) in Section 2.1, show that the Kleisli category and $\mathbf{A} - \mathrm{Free}_{\mathcal{C}}$ are equivalent. For more details see e.g. [29, Section 2.3 Satz 1].

2.4. **A further property of adjoint functors.** We shall need a further property of adjoint functors. These statements are well-known and can be extracted implicitly from the proof of [22, Chapter 1, Proposition 1.3]. For the convenience of the reader we provide a short proof.

Lemma 2.5. *Let $\mathcal{H}$ and $\mathcal{G}$ be categories and let*

$$\mathcal{H} \underset{T}{\overset{S}{\rightleftarrows}} \mathcal{G}$$

be functors such that (S,T) is an adjoint pair. Then

(1) *S is faithful if and only if the unit $\mathrm{id} \longrightarrow TS$ is pointwise a monomorphism.*
(2) *T is faithful if and only if the counit $ST \longrightarrow \mathrm{id}$ is pointwise an epimorphism.*
(3) *S is full if and only if the unit $\mathrm{id} \longrightarrow TS$ is pointwise a split epimorphism.*
(4) *T is full if and only if the counit $ST \longrightarrow \mathrm{id}$ is pointwise a split monomorphism.*
(5) *S is fully faithful if and only if the unit $\mathrm{id} \longrightarrow TS$ is an isomorphism.*
(6) *T is fully faithful if and only if the counit $ST \longrightarrow \mathrm{id}$ is an isomorphism.*

Proof. Items $(2n)$ and $(2n-1)$ are dual to each other for all $n \in \{1,2,3\}$. We need to show one of the corresponding statements.

Item 2): T is faithful if and only if $\mathcal{G}(X,Y) \xrightarrow{T} \mathcal{H}(TX,TY)$ is injective. Now $\mathcal{H}(TX,TY) \simeq \mathcal{G}(STX,Y)$ and any map in the latter factors through the counit $\epsilon_X : STX \longrightarrow X$ (cf [28, IV.1 Theorem 1]). Now, ϵ_X is an epimorphism if and only if $\mathcal{G}(\epsilon_X, Y)$ is injective and this shows the statement.

Item 4): T is full if and only if $\mathcal{G}(X,Y) \xrightarrow{T} \mathcal{H}(TX,TY)$ is surjective. Again $\mathcal{H}(TX,TY) \simeq \mathcal{G}(STX,Y)$ and any map in the latter factors through the counit $\epsilon_X : STX \longrightarrow X$ (cf [28, IV.1 Theorem 1]). Hence T is full if and only if $\mathcal{G}(X,Y) \xrightarrow{\mathcal{G}(\epsilon_X,Y)} \mathcal{G}(STX,Y)$ is surjective for all X,Y. In particular, for $Y = STX$ we obtain that this is equivalent to ϵ_X being split monomorphism.

Item 6): An epimorphism which is in addition a split monomorphism is an isomorphism. ■

3. Recall Clifford's theorem from group algebras

We briefly recall the situation for group rings. Let G be a group and let k be a commutative ring. For a subgroup H of G and a kH-module M we denote by $M\uparrow_H^G$ the kG-module $kG \otimes_{kH} M$, where kG acts by multiplication on the left factor. Further, for a kG-module X we denote by $X\downarrow_H^G$ the kH-module obtained by restricting the action of G to an action of the subgroup H. Both $\uparrow_H^G$ and $\downarrow_H^G$ are functors between the respective module categories, and actually the are biadjoint one to the other in case G is a finite group.

If $N \trianglelefteq G$, then Mackey's formula shows

$$M\uparrow_N^G\downarrow_H^G = \bigoplus_{gN \in G/N} {}^gM$$

for each M, and actually we have an isomorphism of functors

$$- \uparrow_N^G \downarrow_N^G = \bigoplus_{gN \in G/N} {}^g - .$$

Remark 3.1. We should note that $M \uparrow_H^G = kG \otimes_{kH} M$ and $N \downarrow_G^H = Hom_{kH}(kG, N)$ both only depend on the group algebras, rather than the groups. Therefore, the decomposition is not canonical, and since there may be different groups with the same group algebra, the decomposition depends on the specific group one takes. We mention [23] for a striking example in characteristic 3 and [25] for an example over the integers.

If N is a normal subgroup of G and M an indecomposable kN-module. Then the inertia group $I_G(M)$ of M is the set of $g \in G$ such that ${}^gM \simeq M$ as kN-module

$$I_G(M) := \{g \in G \mid {}^gM \simeq M \text{ as } kN\text{-modules}\}.$$

Theorem 3.2. *(Clifford* [17]*) Let k be a field and let G be a group with normal subgroup $N \trianglelefteq G$ of finite index. Let M be an indecomposable kN-module and let M_0 be an indecomposable direct factor of $M \uparrow_N^{I_G(M)}$. Then $M_0 \uparrow_{I_G(M)}^G$ is an indecomposable kG-module.*

In the rest of the paper we show that this theorem is actually a result on orbit categories.

4. Categories modelling normal subgroups

Motivated by Mackey's theorem for group rings in the situation of normal subgroups we give

Definition 4.1. Let $\mathcal{H}$ and $\mathcal{G}$ be categories and let

$$\begin{tikzcd} \mathcal{H} \arrow[r, bend left, "S"] & \mathcal{G} \arrow[l, bend left, "T"] \end{tikzcd}$$

be functors such that (S, T) is an adjoint pair. Then S gives a *situation of normal subgroup categories* if

$$A = TS = \bigoplus_{i \in I} E_i$$

for self-equivalences E_i; $i \in I$; of $\mathcal{H}$.

Note that in the main result of Auslander and Kleiner [3] as well as in its triangulated category version [37] we started from the situation that (S, T) is an adjoint pair and $TS = \mathrm{id}_{\mathcal{H}} \oplus U$ for some endofunctor U of $\mathcal{H}$, and such that the unit $\mathrm{id}_{\mathcal{H}} \longrightarrow TS$ is the left inverse to the projection $TS \longrightarrow \mathrm{id}_{\mathcal{H}}$. Hence, if $\mathcal{H}$ is Krull-Schmidt, we may assume that for some index $i_0 \in I$ we have $E_{i_0} = \mathrm{id}_{\mathcal{H}}$. Note further that by Remark 3.1 it is not reasonable to assume the decomposition to be canonical in any sense. We only ask for the existence.

A particularly simple situation occurs in the classical case of Clifford's theorem if $N = I_G(M)$. In our categorical situation this is a favorable situation as well.

Proposition 4.2. *Let $S : \mathcal{H} \longrightarrow \mathcal{G}$ be an additive functor between abelian Krull-Schmidt categories admitting a right adjoint T. Assume further that S is a normal subgroup category situation. We have, by definition $TS = \bigoplus_{i \in I} E_i$ for self-equivalences E_i of $\mathcal{H}$, and for some i_0, $E_{i_0} = \mathrm{id}$. If $E_iM \not\simeq M$ for any $i \in I \setminus \{i_0\}$, then M* simple *implies that $End_{\mathcal{G}}(SM)$ is a skew field and in particular SM* is indecomposable *.*

Proof. We compute

$$\begin{aligned} End_{\mathcal{G}}(SM) &\simeq \mathcal{H}(M, TSM) \\ &\simeq \mathcal{H}(M, E_{i_0}M) \oplus \bigoplus_{i \in I \setminus \{i_0\}} \mathcal{H}(M, E_iM) \end{aligned}$$

Since E_i are all self-equivalences, M is simple if and only if E_iM is simple. Hence, $\mathcal{H}(M, E_iM) = 0$ if $i \in I \setminus \{i_0\}$. Further, by functoriality of the adjointness, the resulting isomorphism $End_{\mathcal{G}}(SM) \simeq End_{\mathcal{H}}(M)$ is an isomorphism of rings. Therefore $End_{\mathcal{G}}(SM)$ is a skew field and therefore SM is indecomposable. ■

Suppose that $E_iA \xrightarrow{\lambda_i} AE_i$ are natural equivalences. Then for all $f \in \mathcal{H}(X,Y)$ we have a commutative diagram

$$(\dagger)\qquad \begin{tikzcd} E_iAX \arrow[r, "E_iAf"] \arrow[d, "(\lambda_i)_X"'] & E_iAY \arrow[d, "(\lambda_i)_Y"] \\ AE_iX \arrow[r, "AE_if"] & AE_iY \end{tikzcd}$$

where the vertical maps are isomorphisms. As a consequence, if E_i is an automorphism, then

$$AE_i^{-1} \xrightarrow{E_i^{-1}(\lambda_i)E_i^{-1}} E_i^{-1}A$$

is a natural equivalence in the sense that

$$(\dagger\dagger)\qquad \begin{tikzcd} AE_i^{-1}X \arrow[r, "AE_i^{-1}f"] \arrow[d, "E_i^{-1}(\lambda_i)_{E_i^{-1}X}"'] & AE_i^{-1}Y \arrow[d, "E_i^{-1}(\lambda_i)_{E_i^{-1}Y}"] \\ E_i^{-1}X \arrow[r, "E_i^{-1}Af"] & E_i^{-1}AY \end{tikzcd}$$

is commutative.

Proposition 4.3. *Let $S : \mathcal{H} \longrightarrow \mathcal{G}$ be an additive functor between Krull-Schmidt categories admitting a right adjoint T. Assume further that S is a normal subgroup category situation. Then $TS = \bigoplus_{i\in I} E_i$ for self-equivalences E_i of $\mathcal{H}$. Further, assume that $E_{i_0} =: E_0 = \mathrm{id}_{\mathcal{H}}$. Denote by $A = TS$ the endofunctor of the monad $\mathbf{A}$ induced by this adjoint pair. Let $J \subseteq I$ and suppose that for each $j \in J$ there are natural equivalences $E_jA \xrightarrow{\lambda_j} AE_j$. Then for each $j \in J$ there is a self-equivalence of the Kleisli category $\widehat{E}_j$ of $\mathfrak{K}_A$ such that $E_j \circ T = T \circ \widehat{E}_j$ and $\widehat{E}_j \circ S = S \circ E_j$. If E_j is an automorphism, then $\widehat{E}_j$ is an automorphism. Analogous statements hold for the Eilenberg-Moore category $A-\mathrm{Mod}_{\mathcal{H}}$.*

Proof. Let $j \in J$. We shall prove that each E_j lifts to a self-equivalence $\widehat{E}_j$ of $\mathfrak{K}_A$.

We put $\widehat{E}_j(X) = E_j(X)$ for each object X of $\mathfrak{K}_A$. As for morphisms we may use Lemma 2.4. There we obtained $\mathcal{H}(X,AY) = \mathfrak{K}_{\mathbf{A}}(X,Y)$ and for $f \in \mathcal{H}(X,AY)$ we have $E_jf \in \mathcal{H}(E_jX, E_jAY)$, and therefore $(\lambda_j)_Y \circ E_jf \in \mathcal{H}(E_jX, AE_jY)$. Hence we may put $\widehat{E}_j(f) := (\lambda_j)_Y \circ E_jf$. If E_j is an automorphism, then the diagrams $(\dagger\dagger)$ and $(\dagger)$ show that $\widehat{E}_j$ is an automorphism as well. Indeed,

$$\widehat{E_j^{-1}}(f) := E_j^{-1}(\lambda_j)_{E_j^{-1}Y}{}^{-1} \circ E_j^{-1}(f)$$

gives

$$\begin{aligned} \widehat{E_j^{-1}}(\widehat{E}_j(f)) &= \widehat{E_j^{-1}}((\lambda_j)_Y \circ E_jf) \\ &= E_j^{-1}(\lambda_j)_{E_j^{-1}Y}{}^{-1} \circ E_j^{-1}(((\lambda_j)_Y \circ E_jf) \\ &= E_j^{-1}(\lambda_j)_{E_j^{-1}Y}{}^{-1} \circ E_j^{-1}(\lambda_j)_{E_j^{-1}Y} \circ f \\ &= f \end{aligned}$$

On the level of $\mathbf{A}-\mathrm{Mod}_{\mathcal{H}}$ the construction is as follows. We put $\rho_{E_jX} := E_j(\rho_X) \circ (\lambda_j)_X^{-1}$ for every X and we put $\widehat{E}_j(X,\rho_X) := (E_jX, \rho_{E_jX})$. If ρ_X is a splitting of the unit η_X, then ρ_{E_jX} is a splitting of η_{E_jX}. Hence, this is indeed an object of $\mathbf{A}-\mathrm{Mod}_{\mathcal{H}}$. Moreover, for any $f \in \mathcal{H}(X,Y)$, we have that $E_j(f) \in \mathcal{H}(E_jX, E_jY)$, and if f gives rise to a morphism in $\mathbf{A}-\mathrm{Mod}_{\mathcal{H}}$, then

$$\begin{tikzcd} AX \arrow[r, "Af"] \arrow[d, "\rho_X"'] & AY \arrow[d, "\rho_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is commutative, and therefore

$$\begin{tikzcd} E_jAX \arrow[r, "E_jAf"] \arrow[d, "E_j\rho_X"'] & E_jAY \arrow[d, "E_j\rho_Y"] \\ E_jX \arrow[r, "E_jf"] & E_jY \end{tikzcd}$$

is commutative as well. Now, recall the diagram (†) and get a commutative diagram

$$\begin{tikzcd} AE_jX \arrow[r, "AE_jf"] \arrow[d, "(\lambda_j)_X^{-1}"'] & AE_jY \arrow[d, "(\lambda_j)_Y^{-1}"] \\ E_jAX \arrow[r, "E_jAf"] \arrow[d, "E_j\rho_X"'] & E_jAY \arrow[d, "E_j\rho_Y"] \\ E_jX \arrow[r, "E_jf"] & E_jY. \end{tikzcd}$$

Since we were putting $\rho_{E_jX} = E_j(\rho_X) \circ (\lambda_j)_X^{-1}$ for every X we get that then also E_jf gives rise to a morphism in $A - \mathrm{Mod}_{\mathcal{H}}$.

The fact that $\widehat{E}_j$ is an equivalence is clear, since the construction holds as well for E_j^{-1}, producing a functor $\widehat{E_j^{-1}}$, which equals $\widehat{E}_j^{-1}$ as is verified by composing with $\widehat{E}_j$.

Let us verify that $S_KE_j = \widehat{E}_jS_K$. Indeed, $S_K(X) = X$ for any object X and $S_K(f) = Af$ for any morphism f. Then $S_KE_j(X) = \widehat{E}_jS_K(X)$ for any object. Further,

$$\widehat{E}_jS_Kf = \widehat{E}_jAf = \lambda_j^{-1} \circ E_jAf = \lambda_j^{-1}\lambda_jAE_jf = AE_jf.$$

Similarly, the facts that $T_K\widehat{E}_j = E_jT_K$, and that $T_{EM}\widehat{E}_j = E_jT_{EM}$ and $S_{EM}E_j = \widehat{E}_jS_{EM}$ follow by direct inspection. ■

Remark 4.4. If in the notation and under the hypotheses of Proposition 4.3 we have in addition $E_i \circ E_j = E_k$ for some $i, j, k \in I$, and if in addition

$$\lambda_k = \lambda_i \circ E_i(\lambda_j)$$

then we also get $\widehat{E}_i \circ \widehat{E}_j = \widehat{E}_k$. This follows by the explicit construction of $\widehat{E}_i$ in each of the two cases. The equation on the isomorphisms λ_* comes from the necessity that the diagram

$$\begin{tikzcd} E_iE_jA \arrow[r, "E_i(\lambda_j)"] \arrow[d, equal] & E_iAE_j \arrow[r, "\lambda_j"] & AE_iE_j \arrow[d, equal] \\ E_kA \arrow[rr, "\lambda_k"] & & AE_k \end{tikzcd}$$

commutes. Note that λ_* is a relation of vanishing of a 1-cocycle type. (Note that we did not assume yet that the $\{E_i \mid i \in I\}$ forms a group.) In general, for freely chosen λ_i for $i \in I$, the 1-cocycle above is not vanishing. However, by the usual argument it is sufficient that the 1-cocycle above is a 1-coboundary, that is there is α such that $\lambda_i = \alpha^{-1} \circ E_i(\alpha)$ for all $i \in I$. Then, by an elementary computation the vanishing of the 1-cocycle follows.

Further, if $\{E_i \mid i \in I\}$ is a group, then $E_iA \longrightarrow AE_i$ is actually the conjugation action on the summands of A. Since the conjugation action is associative, i.e. $(E_iE_j)A(E_iE_j)^{-1} = E_i(E_jAE_j^{-1})E_i^{-1}$, if $\{E_i \mid i \in I\}$ is a group, then $E_i \circ E_j = E_k$ implies $\widehat{E}_i \circ \widehat{E}_j = \widehat{E}_k$.

Proposition 4.5. *Let $S : \mathcal{H} \longrightarrow \mathcal{G}$ be an additive functor between Krull-Schmidt categories admitting a right adjoint T. Assume further that S is a normal subgroup category situation, i.e. $TS = \bigoplus_{i\in I} E_i$ for self-equivalences E_i of $\mathcal{H}$. Further, assume that $E_{i_0} =: E_0 = \mathrm{id}_{\mathcal{H}}$. Denote by $A = TS$ the endofunctor of the monad $\mathbf{A}$ induced by this adjoint pair. Then the counit $\epsilon : S_KT_K \longrightarrow \mathrm{id}_{\mathfrak{K}_A}$ of the Kleisli category associated to $\mathbf{A}$ splits pointwise, and we get $\widehat{E}_i(M) \simeq M$ for all indecomposable objects M of $\mathfrak{K}_A$ and the lift $\widehat{E}_i$ of E_i to $\mathfrak{K}_A$ following Proposition 4.3.*

Proof. In order to prove this statement we first analyse the monad (A,μ,ϵ) in the normal subgroup situation, in particular $\mu : A^2 \longrightarrow A$. Recall the construction of μ. Let $\epsilon : ST \longrightarrow \mathrm{id}_{\mathcal{G}}$ be the counit of the adjunction. Then

$$\mu = T\epsilon S : TSTS \longrightarrow T \circ \mathrm{id}_{\mathcal{G}} \circ S.$$

We study this map in particular for the Kleisli category. Then, for the category $\mathfrak{K}_A$ and the adjoint pair (S_K, T_K) giving the monad $A = TS$, we have

$$\begin{array}{rcl} S_K : \mathcal{H} & \longrightarrow & \mathfrak{K}_A \\ X & \mapsto & X \\ f & \mapsto & Af \end{array} \qquad \begin{array}{rcl} T_K : \mathfrak{K}_A & \longrightarrow & \mathcal{H} \\ X & \mapsto & AX \\ f & \mapsto & f \end{array}$$

where f denotes a morphism and X an object of $\mathcal{H}$. Hence, $S_K T_K(X) = AX$ and $S_K T_K(f) = Af$ for all objects X and all morphisms f. The choice $\epsilon : S_K T_K \longrightarrow \mathrm{id}_{\mathfrak{K}_A}$ given by η the projection of $A = 1 \oplus U$ onto 1 satisfies the necessary properties. Moreover, by definition the counit defined in this way splits pointwise.

The fact that $M \simeq \widehat{E}_i M$ in $\mathfrak{K}_{\mathbf{A}}$ follows basically from [13, Lemma 2.5]. Here is the adaption of the proof. By Lemma 2.4 we have

$$\mathfrak{K}_{\mathbf{A}}(X,Y) = \mathcal{H}(X, AY) = \mathcal{H}(X, \bigoplus_{i\in I} E_i Y).$$

Hence, for $X = \widehat{E}_i M$ and $Y = M$ we get that $\mathfrak{K}_{\mathbf{A}}(X,Y)$ contains the identity on $\widehat{E}_i M$, which is clearly invertible. Therefore $M \simeq \widehat{E}_i M$ in $\mathfrak{K}_{\mathbf{A}}$. ■

Corollary 4.6. *If (S,T) is another adjoint pair inducing this monad $\mathbf{A}$ (where $S : \mathcal{H} \longrightarrow \mathcal{G}$), then let $K : \mathfrak{K}_{\mathbf{A}} \longrightarrow \mathcal{G}$ be the functor from Section 2.3. If now there are self-equivalences E_i^0 of $\mathcal{G}$ such that $K\widehat{E}_i = E_i^0 K$ for all K, then also $E_i^0 V \simeq V$ for all V in the image of S.*

Proof. Recall that by Section 2.3 the Kleisli category is initial amongst all adjoint pairs inducing a fixed monad $\mathbf{A}$ in the sense that the functor K exists and satisfies $KS_K = S$. Then by Proposition 4.5 we have that $\widehat{E}_i V \simeq V$ for all V in $\mathfrak{K}_{\mathbf{A}}$. Recall that all objects in $\mathfrak{K}_{\mathbf{A}}$ are in the image of S_K. Hence $V = S_K W$ and

$$E_i^0 SW = E_i^0 KS_K W = K\widehat{E}_i S_K W = K\widehat{E}_i V \simeq KV = KS_K W = SW.$$

This shows the Corollary. ■.

Remark 4.7. Note that if the counit $\epsilon : S_K T_K \longrightarrow \mathrm{id}_{\mathfrak{K}_{\mathbf{A}}}$ (or actually even more generally the counit $\epsilon : ST \longrightarrow \mathrm{id}_{\mathcal{G}}$) splits functorially, then by Proposition 2.3 we have an equivalence $\mathcal{G} \simeq \mathbf{A} - \mathrm{Mod}_{\mathcal{H}}$ up to direct factors, an equivalence in case the categories are idempotent complete, and therefore we may assume that this equivalence is actually an equality. By the universal property of the Kleisli category, we also get that

$$\mathcal{G} \simeq \mathbf{A} - \mathrm{Mod}_{\mathcal{H}} \simeq \mathfrak{K}_{\mathbf{A}}$$

if the categories are idempotent complete.

Theorem 4.8. *Let $\mathcal{H}$ and $\mathcal{G}$ be additive Krull-Schmidt categories. Let*

$$\begin{tikzcd} \mathcal{H} \arrow[r, bend left, "S"] & \mathcal{G} \arrow[l, bend left, "T"] \end{tikzcd}$$

be functors giving a normal subgroup situation $TS = \bigoplus_{i\in I} E_i$, such that (S,T) is an adjoint pair, and such that $E_{i_0} = \mathrm{id}_{\mathcal{H}}$. Suppose that for any $i \in I$ there is a self-equivalence $\widehat{E}_i$ of $\mathcal{G}$ with $SE_i = \widehat{E}_i S$ and $T\widehat{E}_i = E_i T$. Then for any indecomposable object W of $\mathcal{H}$ with $E_j W \simeq W$ if and only if $j = i_0$, we get $S(W)$ is indecomposable as well.

Proof. If $S(W) = V_1 \oplus V_2$, then $TSW = TV_1 \oplus TV_2$ and since $\mathrm{id}_{\mathcal{H}}$ is a direct factor of TS, W is a direct factor of either TV_1 or TV_2, w.l.o.g. of TV_1, and we may assume that V_1 is indecomposable. Since for any indecomposable object W of $\mathcal{H}$ with $E_j W \simeq W$ if and only if $j = i_0$, we have $E_i W \not\simeq W$ for all $i \in I \smallsetminus \{i_0\}$. Since E_j is a self-equivalence, $E_j W$ is indecomposable again for all $j \in I$. But then $E_j W$ is a direct factor of $E_j TV_1 = T\widehat{E}_j V_1$. However, $\widehat{E}_j V_1 \simeq V_1$ by Proposition 4.5 and Corollary 4.6

and hence E_jW is a direct factor of TV_1 for all $j \in I$. Therefore $AW = TSW$ is a direct factor of TV_1. But $TSW = TV_1 \oplus TV_2$. Hence $TV_2 = 0$ by the Krull-Schmidt property of $\mathcal{H}$. But this implies

$$0 = \mathcal{H}(W, TV_2) \simeq \mathcal{G}(SW, V_2) = \mathcal{G}(V_1 \oplus V_2, V_2)$$

which has a direct factor $\mathcal{G}(V_2, V_2)$, containing the identity on V_2. Therefore $V_2 = 0$ and $S(W)$ is indecomposable. ■

Remark 4.9. Note that for $\mathcal{G}$ being the Kleisli category or the Eilenberg-Moore category of the adjunction, then Proposition 4.3 shows the existence of $\widehat{E}_i$ for all i.

5. The inertia category

We can now imitate the construction of an inertia group.

5.1. **On orbit categories and Kleisli categories.** Let (A, μ, η) be a monad on $\mathcal{H}$, and suppose $A = \bigoplus_{i \in I} E_i$ for some automorphisms E_i of $\mathcal{H}$. We suppose that $\{E_i \mid i \in I\}$ with the multiplication μ forms a group Γ of automorphisms of $\mathcal{H}$. If $\mu : A \circ A \longrightarrow A$ is the structure map, then writing $A = \bigoplus_{i \in I} E_i$ we get by restriction natural transformations $\mu^{i,j} : E_i \circ E_j \longrightarrow E_k$, and $\mu^{i,j} = 0$ for almost all k, and for a unique $k = k(i,j)$ this is an equivalence.

Let now (S, T) be a pair of adjoint functors

$$\begin{tikzcd} \mathcal{H} \arrow[r, bend left, "S"] & \mathcal{G} \arrow[l, bend left, "T"] \end{tikzcd}$$

inducing the monad A giving a normal subgroup situation $A = \bigoplus_{i \in I} E_i$ such that the structure map on A induces a group law on the E_i, as indicated above. Then the hypothesis $E_iA \simeq AE_i$ for all i from Proposition 4.3 holds true.

We recall the construction of the orbit category as is displayed in [13]. We use the notation introduced there, though we mention that in Keller's [26] the notion $\mathcal{C}/\Gamma$ is used for the category which is denoted by $\mathcal{C}[\Gamma]$ in [13].

Let $\mathcal{C}$ be a category admitting arbitrary direct sums and let Γ be a group of self-equivalences of $\mathcal{C}$. Then we may form the orbit category $\mathcal{C}[\Gamma]$. The objects of $\mathcal{C}$ and of $\mathcal{C}[\Gamma]$ coincide. Further,

$$\mathcal{C}[\Gamma](X, Y) = \bigoplus_{g \in \Gamma} \mathcal{C}(X, gY).$$

Note that we have

$$\mathcal{C}[\Gamma](X, Y) = \left(\prod_{(g_1, g_2) \in G \times G} \mathcal{C}(g_1X, g_2Y) \right)^G$$

where G acts diagonally. This observation can be found in e.g. Asashiba [1]. Composition of morphisms is then clear with the above formula.

The orbit category of a group is actually a special case of a Kleisli category.

Corollary 5.1. *Let $\mathbf{A} = (A, \mu, \eta)$ be a monad, suppose $A = \bigoplus_{i \in I} E_i$ for self-equivalences E_j of $\mathcal{H}$, and suppose that $\{E_i \mid i \in I\}$ with the structure map μ forms a group Γ of self-equivalences of $\mathcal{H}$. Then, the orbit category $\mathcal{H}[\Gamma]$ is naturally isomorphic to $\mathfrak{K}_{\mathbf{A}}$, the Kleisli category and $\mathbf{A}$ is the monad corresponding to an adjoint pair $(N[\Gamma], R[\Gamma])$*

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "N[\Gamma]"] & \mathcal{C}[\Gamma] \arrow[l, bend left, "R[\Gamma]"] \end{tikzcd}$$

Proof. This is a direct consequence of Lemma 2.4 where we showed that

$$\mathcal{H}[\Gamma] \simeq \mathfrak{K}_{\mathbf{A}}.$$

Moreover, $N[\Gamma]$ identifies with $S_{\mathfrak{K}_{\mathbf{A}}}$ and $R[\Gamma]$ identifies with $T_{\mathfrak{K}_{\mathbf{A}}}$. ■

We recall a concept from Asashiba [1].

Definition 5.2. [1] Let $\mathcal{C}$ be a category with an action of a group Γ in the sense that there is a set of self-equivalences E_g of $\mathcal{C}$ which form a group Γ. We denote by E_1 the identity self-equivalence.

Then for a functor $F:\mathcal{C}\longrightarrow\mathcal{D}$ a collection of natural transformations $\phi_g:F\longrightarrow FE_g$ for $E_g\in\Gamma$ is called Γ-adjuster if

$$\begin{tikzcd} F \arrow[r, "\phi_g"] \arrow[dr, "\phi_{hg}"'] & FE_g \arrow[d, "\phi_h E_g"] \\ & FE_{hg} \arrow[r, equal] & FE_hE_g \end{tikzcd}$$

is commutative. A functor F for which there is a Γ-adjuster is called a Γ-invariant functor.

Asashiba remarks that then $\phi_1=\mathrm{id}_F$ automatically.

Lemma 5.3. [1, Proposition 2.6] *Let $\mathcal{C}$ be a category with an action of a group Γ in the sense that there is a set of self-equivalences E_g of $\mathcal{C}$ which form a group Γ. Then*

(1) *The natural functor $N[\Gamma]:\mathcal{C}\longrightarrow\mathcal{C}[\Gamma]$ is Γ-invariant with adjuster $(\nu_g)_{g\in\Gamma}$, where for each $g\in\Gamma$ and object X of $\mathcal{C}$ we define*

$$\nu_{g,X}:=(\delta_{h_1,h_2g}\mathrm{id}_{h_1X})_{(h_1,h_2)\in\Gamma\times\Gamma}\in Hom_{\mathcal{C}[\Gamma]}(N[\Gamma]X,N[\Gamma]gX).$$

(2) *For each Γ-invariant functor $\mathcal{C}\overset{(F,\phi)}{\longrightarrow}\mathcal{D}$ there is a unique functor $\mathcal{C}[\Gamma]\overset{H}{\longrightarrow}\mathcal{D}$ such that $(F,\phi)=(HN[\Gamma],H\nu)$, i.e. in particular the diagram*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "N[\Gamma]"'] & \mathcal{D} \\ \mathcal{C}[\Gamma] \arrow[ur, "H"'] & \end{tikzcd}$$

is commutative.

(3) *For each Γ-invariant functor $\mathcal{C}\overset{(F,\phi)}{\longrightarrow}\mathcal{D}$ there is an up to isomorphism unique functor $\mathcal{C}[\Gamma]\overset{H}{\longrightarrow}\mathcal{D}$ such that $(F,\phi)\simeq(HN[\Gamma],H\nu)$,*

We have to deal with submonads and orbit categories with respect to a subgroup.

Proposition 5.4. *Let $\mathcal{H}$ be an additive category, let Γ be a group of automorphisms and let Γ_0 be a subgroup of Γ. Consider the associated adjoint pairs*

$$\begin{tikzcd} \mathcal{H} \arrow[r, bend left, "S_\Gamma"] & \mathcal{H}[\Gamma] \arrow[l, bend left, "T_\Gamma"] \end{tikzcd}$$

and

$$\begin{tikzcd} \mathcal{H} \arrow[r, bend left, "S_{\Gamma_0}"] & \mathcal{H}[\Gamma_0] \arrow[l, bend left, "T_{\Gamma_0}"] \end{tikzcd}$$

given by the Kleisli categories associated to the monads $A=\bigoplus_{\gamma\in\Gamma}\gamma$ respectively $A_0=\bigoplus_{\gamma\in\Gamma_0}\gamma$. Then there is an adjoint pair $(S_{\Gamma^0},T_{\Gamma^0})$

$$\begin{tikzcd} \mathcal{H}[\Gamma_0] \arrow[r, bend left, "S_{\Gamma^0}"] & \mathcal{H}[\Gamma] \arrow[l, bend left, "T_{\Gamma^0}"] \end{tikzcd}$$

such that

$$S_\Gamma=S_{\Gamma^0}\circ S_{\Gamma_0}\text{ and }T_\Gamma=T_{\Gamma_0}\circ T_{\Gamma^0}.$$

Proof. By Lemma 5.3.1 the functor $S_\Gamma:\mathcal{H}\longrightarrow\mathcal{H}[\Gamma]$ is Γ-invariant in the sense of Definition 5.2. Hence, by restriction to the subgroup Γ_0, S_Γ is Γ_0-invariant as well. By Lemma 5.3.2 factorizes through $\mathcal{H}[\Gamma_0]$. This shows the existence of S_{Γ^0}.

We give the functor explicitly.

$$\mathcal{H}[\Gamma_0]\overset{S_{\Gamma^0}}{\longrightarrow}\mathcal{H}[\Gamma]$$

satisfying $S_\Gamma = S_{\Gamma^0} \circ S_{\Gamma_0}$. We put $S_{\Gamma^0}(X) = X$ on objects. Then, since $S_{\Gamma_0}(X) = X$ on objects and $S_\Gamma(X) = X$ on objects we have $S_\Gamma = S_{\Gamma^0} \circ S_{\Gamma_0}$ on objects. On morphisms we have $S_\Gamma(f) = \bigoplus_{\gamma\in\Gamma} \gamma f$ and $S_{\Gamma_0}(f) = \bigoplus_{\gamma\in\Gamma_0} \gamma f$ for any $f \in Hom_{\mathcal{H}}(X,Y)$. Then

$$\mathcal{H}[\Gamma_0](X,Y) = \mathcal{H}(X, \bigoplus_{\gamma\in\Gamma_0} \gamma Y)$$

and for $f \in \mathcal{H}(X, \bigoplus_{\gamma\in\Gamma_0} \gamma Y)$ we put

$$S_{\Gamma^0}(f) := \bigoplus_{\Gamma_0\sigma\in\Gamma/\Gamma_0} \sigma f.$$

Then by definition $S_\Gamma = S_{\Gamma^0} \circ S_{\Gamma_0}$ on morphisms.

Similarly, $T_{\Gamma_0}(X) = \bigoplus_{\Gamma_0\sigma\in\Gamma/\Gamma_0} \sigma X$ and $T_{\Gamma_0}(f) = f$ for each morphism f. We need to explain briefly the meaning here. Note that

$$\mathcal{H}[\Gamma_0](X,Y) = \mathcal{H}(X, \bigoplus_{\gamma\in\Gamma_0} \gamma Y) = \mathcal{H}[\Gamma_0](X,Y) = \mathcal{H}(\bigoplus_{\gamma\in\Gamma_0} \gamma X, \bigoplus_{\gamma\in\Gamma_0} \gamma Y)^{\Gamma_0}$$

and let

$$f \in \mathcal{H}[\Gamma](X,Y) = \mathcal{H}(X, \bigoplus_{\gamma\in\Gamma} \gamma Y) = \mathcal{H}(\bigoplus_{\gamma\in\Gamma} \gamma X, \bigoplus_{\gamma\in\Gamma} \gamma Y)^{\Gamma}.$$

Then

$$\begin{aligned} T_{\Gamma_0}(f) \in \mathcal{H}[\Gamma_0](T_{\Gamma_0}X, T_{\Gamma_0}Y) &= \mathcal{H}[\Gamma_0](\bigoplus_{\Gamma_0\sigma\in\Gamma/\Gamma_0} \sigma X, \bigoplus_{\Gamma_0\sigma\in\Gamma/\Gamma_0} \sigma Y) \\ &= \mathcal{H}(\bigoplus_{\gamma\in\Gamma_0} \gamma \bigoplus_{\Gamma_0\sigma\in\Gamma/\Gamma_0} \sigma X, \bigoplus_{\gamma\in\Gamma_0} \gamma \bigoplus_{\Gamma_0\sigma\in\Gamma/\Gamma_0} \sigma Y)^{\Gamma_0} \\ &= \mathcal{H}(\bigoplus_{\gamma\in\Gamma} \gamma X, \bigoplus_{\gamma\in\Gamma} \gamma Y)^{\Gamma_0} \end{aligned}$$

and $T_{\Gamma_0}(f)$ just maps f in the space of Γ-fixed points to the space of Γ_0-fixed points. This is clearly a functor. Furthermore, since (S,T) is an adjoint pair, also $(S_{\Gamma^0}, T_{\Gamma^0})$ is an adjoint pair.

This shows the statement. ■

Remark 5.5. Note that the unit of the adjunction $(S_{\Gamma^0}, T_{\Gamma_0})$ leads to some Mackey formula involving double classes.

Remark 5.6. In [12], in case Γ is cyclic, Chávez gave a criterion when the orbit category is idempotent complete (i.e. all idempotents split). In general the orbit category is not idempotent complete. Indeed, let $A = Mat_2(K)$ for some field K. Conjugation by $M := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is an automorphism of order 2 of A, yielding an action of the cyclic group of order 2 on $A - mod$. Then, for a simple A-module S we get

$$Hom_{A-mod[C_2]}(S,S) = Hom_A(S, S \oplus {}^M S) = K \times K$$

has non trivial idempotents, whereas S is indecomposable. Moreover, in Krull-Schmidt categories endomorphism algebras of indecomposable objects are local. The above endomorphism algebra is not local.

Similarly, let A be the Kronecker algebra and σ be the automorphism swapping the two arrows of the Kronecker quiver. This induces an action of the cyclic group of order 2 on the module category. The simples are fixed under the automorphism.

Remark 5.7. Recall the Karoubi envelope of a category. Let $\mathcal{C}$ be a category. Then $Kar(\mathcal{C})$ has a objects the pairs (C,e) for objects C of $\mathcal{C}$ and $e^2 = e \in End_{\mathcal{C}}(C)$. Further,

$$Kar(\mathcal{C})((C,e),(D,f)) = \{\alpha \in \mathcal{C}(C,D) \mid \alpha \circ e = f \circ \alpha\}$$

Note that there is a fully faithful embedding

$$\mathcal{C} \hookrightarrow Kar(\mathcal{C})$$

given by mapping an object C to $(C, 1_C)$ and a morphism to this same morphism. It is well-known, and not difficult to show, that if $\mathcal{C}$ is idempotent complete, then the natural embedding $\mathcal{C} \hookrightarrow Kar(\mathcal{C})$ is an equivalence.

If $S:\mathcal{H}\longrightarrow\mathcal{G}$ is a functor. Then we define a functor

$$\widehat{S}:Kar(\mathcal{H})\longrightarrow Kar(\mathcal{G})$$

by $\widehat{S}(X,e):=(S(X),S(e))$ and $\widehat{S}(\beta):=S(\beta)$ for any morphism β. Since

$$S(\beta)S(e)=S(\beta e)=S(f\beta)=S(f)S(\beta)$$

with the obvious notations, this is indeed well-defined.

Remark 5.8. We should mention that an additive category, whose morphism spaces are R-modules of finite length over some commutative ring R, is Krull-Schmidt if and only if it has split idempotents (i.e. is idempotent complete). This was proved most recently by Amit Shah [33], and may have been known earlier to experts.

Lemma 5.9. *Let now (S,T) be a pair of adjoint functors*

$$\begin{tikzcd}\mathcal{H} \arrow[r, bend left, "S"] & \mathcal{G} \arrow[l, bend left, "T"]\end{tikzcd}$$

with unit $\eta:\mathrm{id}_{\mathcal{H}}\longrightarrow TS$ and counit $\epsilon:ST\longrightarrow\mathrm{id}_{\mathcal{G}}$. Then

$$\begin{tikzcd}Kar(\mathcal{H}) \arrow[r, bend left, "\widehat{S}"] & Kar(\mathcal{G}) \arrow[l, bend left, "\widehat{T}"]\end{tikzcd}$$

is again an adjoint pair with counit $\widehat{\epsilon}_{(X,e)}:=\epsilon_X$ and unit $\widehat{\eta}_{(X,e)}:=\eta_X$.

Proof. Indeed, we first observe by direct inspection that $\widehat{\epsilon}$ and $\widehat{\eta}$ are natural transformations. Since (S,T) is an adjoint pair, the compositions

$$S\xrightarrow{S\eta}STS\xrightarrow{\epsilon S}S$$

and

$$T\xrightarrow{\eta T}TST\xrightarrow{T\epsilon}T$$

are the identity. But then,

$$\widehat{S}\xrightarrow{\widehat{S}\widehat{\eta}}\widehat{S}\widehat{T}\widehat{S}\xrightarrow{\widehat{\epsilon}\widehat{S}}\widehat{S}$$

and

$$\widehat{T}\xrightarrow{\widehat{\eta}\widehat{T}}\widehat{T}\widehat{S}\widehat{T}\xrightarrow{\widehat{T}\widehat{\epsilon}}\widehat{T}$$

are the identity as well since the natural transformations on the Karoubi envelope are defined just as on the original categories. Hence, by [28, Chapter IV, Section 1, Theorem 2] $(\widehat{S},\widehat{T})$ is an adjoint pair with unit $\widehat{\eta}$ and counit $\widehat{\epsilon}$. ■

5.2. **Preparing adjoint functors for the Inertia category.**

Lemma 5.10. *Let (A,μ,η) be a monad giving a normal subgroup situation, i.e. $A=\bigoplus_{i\in I}E_i$ for automorphisms E_i, suppose that $\{E_i\mid i\in I\}$ with the structure map μ forms a group Γ of automorphisms. Let Γ_0 be a subgroup of Γ of index $|\Gamma:\Gamma_0|$. Suppose that*

$$\begin{tikzcd}\mathcal{H} \arrow[r, bend left, "S"] & \mathcal{H}[\Gamma] \arrow[l, bend left, "T"]\end{tikzcd}$$

is an adjoint pair of additive functors realising A, and suppose that the counit of the adjunction (S,T) is locally split. Then by Proposition 5.4 the adjoint pair (S,T) induces naturally an adjoint pair $(S[\Gamma^0],T[\Gamma^0])$

$$\begin{tikzcd}\mathcal{H}[\Gamma_0] \arrow[r, bend left, "S[\Gamma^0]"] & \mathcal{H}[\Gamma] \arrow[l, bend left, "T[\Gamma^0]"]\end{tikzcd}$$

satisfying $S = S[\Gamma^0] \circ S[\Gamma_0]$ *and* $T = T[\Gamma_0] \circ T[\Gamma^0]$. *Moreover, each* E_i *induces an automorphism* E_i^0 *of* $\mathcal{H}[\Gamma_0]$ *with* $E_i^0 N[\Gamma_0] = N[\Gamma_0]E_i$ *and* $E_i R[\Gamma_0] = R[\Gamma_0]E_i^0$. *Furthermore, on objects we get*

$$T[\Gamma^0] \circ S[\Gamma^0] = \bigoplus_{E_i\Gamma_0 \in \Gamma/\Gamma_0} E_i^0.$$

Proof. By construction

$$\mathcal{H}[\Gamma_0](X,Y) = \mathcal{H}(X, \bigoplus_{g\in\Gamma_0} gY)$$

and since the counit is locally split, by Proposition 4.3 each E_i induces an automorphism E_i^0 of $\mathcal{H}[\Gamma_0]$ with $E_i^0 N[\Gamma_0] = N[\Gamma_0]E_i$ and $E_i R[\Gamma_0] = R[\Gamma_0]E_i^0$. Similarly, each E_i induces an automorphism $\widehat{E}_i$ of $\mathcal{H}[\Gamma]$ with $\widehat{E}_i S = SE_i$ and $E_i T = T\widehat{E}_i$.

$$\begin{aligned}
\mathcal{H}[\Gamma_0](-, T[\Gamma^0]S[\Gamma^0]Y) &= \mathcal{H}[\Gamma](S[\Gamma^0]-, S[\Gamma^0]Y) \\
&= \mathcal{H}(-, \bigoplus_{E_i\in\Gamma} E_i Y) \\
&= \mathcal{H}(-, \bigoplus_{E_i\Gamma_0\in\Gamma/\Gamma_0} E_i \bigoplus_{E_j\in\Gamma_0} E_j Y) \\
&= \mathcal{H}(\bigoplus_{E_i\Gamma_0\in\Gamma/\Gamma_0} E_i^{-1}-, \bigoplus_{E_j\in\Gamma_0} E_j Y) \\
&= \mathcal{H}[\Gamma_0](\bigoplus_{E_i\Gamma_0\in\Gamma/\Gamma_0} (E_i^0)^{-1}-, Y) \\
&= \mathcal{H}[\Gamma_0](-, \bigoplus_{E_i\Gamma_0\in\Gamma/\Gamma_0} E_i^0 Y)
\end{aligned}$$

By Yoneda's lemma we have

$$\big(T[\Gamma^0]S[\Gamma^0]\big) \simeq \bigoplus_{E_i\Gamma_0\in\Gamma/\Gamma_0} E_i^0$$

on objects. ■

5.3. **Clifford's theorem for orbit categories, the main result.** We are now ready to prove our main result.

Theorem 5.11. *Let* R *be a commutative Noetherian ring. Suppose that*

$$\begin{tikzcd} \mathcal{H} \arrow[r, bend left, "S"] & \mathcal{G} \arrow[l, bend left, "T"] \end{tikzcd}$$

is an adjoint pair of R*-linear functors between* R*-linear idempotent complete categorie. Assume that that Hom-spaces are of finite* R*-length in* $\mathcal{H}$ *and in* $\mathcal{G}$*, and assume that* (S,T) *give a normal subgroup situation. Let* $TS = \bigoplus_{i\in I} E_i$ *for automorphisms* E_i*, suppose that* $\Gamma := \{E_i \mid i \in I\}$ *forms a group of automorphisms and for an indecomposable object* M *let* $I_S(M) := \{i \in I \mid E_i M \simeq M\}$*, and let* $\Gamma_M := \{E_i \mid i \in I_S(M)\}$*. Suppose that* $\mathcal{G}$ *is the Karoubi envelope of the Kleisli construction of the monad (or, what is the same, the orbit category with respect to* Γ*), and suppose that* Γ *is finite. Then we have a commutative diagram of adjoint pairs*

$$\begin{tikzcd}
\mathcal{H} \arrow[rr, bend left, "S"] \arrow[dr, bend right, "S[\Gamma_M]"] & & \mathcal{G} \arrow[ll, bend left, "T"] \arrow[dl, bend left, "S[\Gamma_M^0]"'] \\
& Kar(\mathcal{H}[\Gamma_M]) \arrow[ul, bend left, "T[\Gamma_M]"] \arrow[ur, bend right, "T[\Gamma_M^0]"'] &
\end{tikzcd}$$

and for any indecomposable direct factor M_0 *of* $S[\Gamma_M](M)$ *we have that* $S[\Gamma_M^0](M_0)$ *is indecomposable.*

Proof. We first suppose that $\mathcal{G}$ is actually the Kleisli construction. Since Γ is a group, $E_iA \simeq AE_i$ for all i and the isomorphisms satisfy the 1-cocycle condition. If $\mathcal{G}$ is the Kleisli construction of the monad, then, by Proposition 4.3 and Remark 4.4, we get that Γ_M acts on $\mathcal{G}$ as group of automorphisms $\widehat{\Gamma}_M$ with automorphisms $\widehat{E}_i^M$ of $\mathcal{G}$.

Moreover, by same argument, there are automorphisms E_i^M of $\mathcal{H}[\Gamma_M]$ forming a group Γ_M with

$$E_i^M S[\Gamma_M] = S[\Gamma_M]E_i \text{ and } T[\Gamma_M]E_i^M = E_iT[\Gamma_M].$$

By Lemma 5.1 we get that $(S[\Gamma_M], T[\Gamma_M])$ is an adjoint pair

$$\begin{tikzcd} \mathcal{H} \arrow[r, bend left, "S[\Gamma_M]"] & \mathcal{H}[\Gamma_M] \arrow[l, bend left, "T[\widehat{\Gamma}_M]"] \end{tikzcd}$$

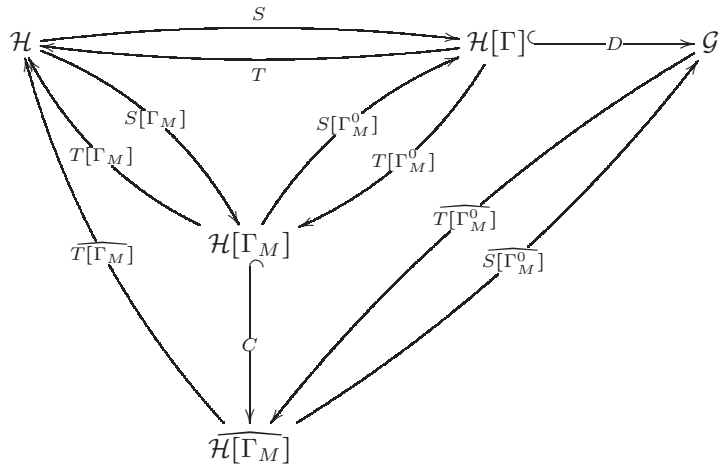
Since Γ is finite, Γ_M is finite as well. Further, since all homomorphisms are R-modules of finite length, the same holds true for the orbit category with respect to Γ_M and to Γ. Since R is Noetherian, this is still true for the Karoubi envelopes. Hence, by [33] the Karoubi envelopes of the orbit categories with respect to Γ_M and to Γ are Krull-Schmidt categories. By Lemma 5.10 we have a commutative diagram of adjoints

$$\begin{tikzcd} \mathcal{H} \arrow[rr, bend left, "S"] \arrow[rr, bend right, "T"'] \arrow[dr, bend right, "S[\Gamma_M]"] & & \mathcal{G} \arrow[ll, bend left, "T"] \arrow[dl, bend left, "S[\Gamma_M^0]"'] \\ & \mathcal{H}[\Gamma_M] \arrow[ul, bend left, "T[\Gamma_M]"] \arrow[ur, bend right, "T[\Gamma_M^0]"'] & \end{tikzcd}$$

Now, in case $\mathcal{G}$ is the Karoubi envelope of the Kleisli construction, since $\mathcal{H}$ and $\mathcal{G}$ are idempotent complete, we can replace $\mathcal{H}[\Gamma_M]$ by its Karoubi envelope $\widehat{\mathcal{H}[\Gamma_M]}$. Then, using Lemma 5.9, in the diagram

$$\begin{tikzcd} \mathcal{H} \arrow[rr, bend left, "S"] \arrow[rr, bend right, "T"'] \arrow[dr, bend right, "S[\Gamma_M]"] & & \mathcal{H}[\Gamma] \arrow[ll, bend left, "T"] \arrow[dl, bend left, "S[\Gamma_M^0]"'] \arrow[r, hook, "D"] & \mathcal{G} \\ & \mathcal{H}[\Gamma_M] \arrow[ul, bend left, "T[\Gamma_M]"] \arrow[ur, bend right, "T[\Gamma_M^0]"'] \arrow[d, hook, "C"] & & \\ & \widehat{\mathcal{H}[\Gamma_M]} & & \end{tikzcd}$$

the functor $T[\Gamma_M]$ and $S[\Gamma^0_M]$ lift to $\widehat{\mathcal{H}[\Gamma_M]}$ such that the diagram



is commutative in the natural sense. Moreover, $(\widehat{S[\Gamma_M]}, \widehat{T[\Gamma_M]})$ and $(\widehat{S[\Gamma^0_M]}, \widehat{T[\Gamma^0_M]})$ are adjoint pairs again. Furthermore, E_i^M extend to automorphisms of $\widehat{\mathcal{H}[\Gamma_M]}$, and each $\widehat{E}_i^M$ extend to an automorphism of $\mathcal{G}$, and in order to avoid an additional notational burden, we denote the extension of E_i^M to the Karoubi envelope by E_i^M again, and likewise for $\widehat{E}_i^M$.

Let

$$CS[\Gamma_M](M) = \bigoplus_{j\in J} M_j$$

for indecomposable objects M_j of $\widehat{\mathcal{H}[\Gamma_M]}$. Then, using Lemma 5.10,

$$\widehat{T[\Gamma^0_M]}\widehat{S[\Gamma^0_M]}(M_j) = \left(\bigoplus_{E_i\in\Gamma/\Gamma_M} E_i^M(M_j)\right).$$

Let M_j be an indecomposable direct factor of $CS[\Gamma_M](M)$.

By definition for each representative E_i of a class different from Γ_M in Γ/Γ_M we have $E_iM \not\simeq M$. By the definition of Γ_M,

$$\widehat{T[\Gamma_M]}CS[\Gamma_M]M \simeq T[\Gamma_M]S[\Gamma_M]M \simeq \bigoplus_{E_i\in\Gamma_M} E_iM \simeq \bigoplus_{E_i\in\Gamma_M} M.$$

The object M being indecomposable, M_j being a direct factor of $CS[\Gamma_M]M$, hence $\widehat{T[\Gamma_M]}M_j$ is a direct factor of $\bigoplus_{E_i\in\Gamma_M} M$. Therefore, by the Krull-Schmidt theorem on $\mathcal{H}$, $\widehat{T[\Gamma_M]}M_j$ is a direct sum of, n_j say, copies of M. In particular $E_i^M M_j \not\simeq M_j$ whenever $E_i^M \notin \Gamma_M$, and moreover, since

$$M^{|\Gamma_M|} = T[\Gamma_M]S[\Gamma_M]M = \widehat{T[\Gamma_M]}CS[\Gamma_M]M = \widehat{T[\Gamma_M]}(\bigoplus_{j\in J} M_j) = \bigoplus_{j\in J}\widehat{T[\Gamma_M]}M_j = \bigoplus_{j\in J} M^{n_j}$$

we get $\sum_{j\in J} n_j = |\Gamma_M|$.

Fix now $j \in J$ and consider the indecomposable direct factor M_j of $CS[\Gamma_M](M)$. Since M_j is a direct factor of $(\widehat{T[\Gamma^0_M]}\widehat{S[\Gamma^0_M]})(M_j)$, there is an indecomposable direct factor X_j of $\widehat{S[\Gamma^0_M]}(M_j) = X_j \oplus Y_j$ in $\mathcal{G}$ such that M_j is a direct factor of $\widehat{T[\Gamma^0_M]}(X_j)$ and M^{n_j} is a direct factor of $\widehat{T[\Gamma_M]}\widehat{T[\Gamma^0_M]}(X_j)$.

Recall that there are automorphisms $\widehat{E}_i^M$ of $\mathcal{G}$ such that

$$E_i^M\widehat{T[\Gamma^0_M]} = \widehat{T[\Gamma^0_M]}\widehat{E}_i^M$$

and by Proposition 4.5 we have $\widehat{E}_i^M(V) \simeq V$ for all $E_i \in \Gamma$ and objects V. Since M_j is a direct factor of $\widehat{T[\Gamma^0_M]}X_j$, we get that $E_i^M(M_j)$ is a direct factor of

$$E_i^M\widehat{T[\Gamma^0_M]}X_j = \widehat{T[\Gamma^0_M]}\widehat{E}_i^M X_j \simeq \widehat{T[\Gamma^0_M]}X_j$$

for all $E_i \in \Gamma$. Since for each non trivial class $E_i^M\Gamma_M$ of Γ/Γ_M we have $E_i^M(M_j) \not\simeq M_j$, as seen above, $\bigoplus_{E_i\Gamma_M\in\Gamma/\Gamma_M} E_i^M(M_j)$ is a direct factor of $\widehat{T[\Gamma^0_M]}(X_j)$, using the Krull-Schmidt property on

$\widehat{\mathcal{H}[\Gamma_M]}$. Furthermore

$$\widehat{T[\Gamma_M]}\left(\bigoplus_{E_i\Gamma_M\in\Gamma/\Gamma_M} E_i^M(M_j)\right) = \bigoplus_{E_i\Gamma_M\in\Gamma/\Gamma_M} E_i\widehat{T[\Gamma_M]}(M_j) = \bigoplus_{E_i\Gamma_M\in\Gamma/\Gamma_M} E_iM^{n_j}$$

is a direct factor of $\widehat{T[\Gamma_M]}\widehat{T[\Gamma_M^0]}(X_j)$.

However,

$$\begin{aligned}\widehat{T[\Gamma_M]}\widehat{T[\Gamma_M^0]}\widehat{S[\Gamma_M^0]}M_j &= \widehat{T[\Gamma_M]}\left(\bigoplus_{E_i^M\Gamma_M\in\Gamma/\Gamma_M} E_i^M\right)M_j\\ &= \left(\bigoplus_{E_i^M\Gamma_M\in\Gamma/\Gamma_M} E_i\right)\widehat{T[\Gamma_M]}M_j\\ &= \left(\bigoplus_{E_i^M\Gamma_M\in\Gamma/\Gamma_M} E_i\right)M^{n_j}\\ &= \bigoplus_{E_i^M\Gamma_M\in\Gamma/\Gamma_M} E_iM^{n_j}\end{aligned}$$

However, we have

$$\widehat{T[\Gamma_M]}\widehat{T[\Gamma_M^0]}\widehat{S[\Gamma_M^0]}M_j \simeq \widehat{T[\Gamma_M]}\bigoplus_{E_i\Gamma_M\in\Gamma/\Gamma_M} E_i^M(M_j).$$

The right hand side is a direct factor of $\widehat{T[\Gamma_M]}\widehat{T[\Gamma_M^0]}(X_j)$. We get that $\widehat{T[\Gamma_M]}\widehat{T[\Gamma_M^0]}\widehat{S[\Gamma_M^0]}M_j$ is a direct factor of $\widehat{T[\Gamma_M]}\widehat{T[\Gamma_M^0]}(X_j)$. But $\widehat{S[\Gamma_M^0]}M_j = X_j\oplus Y_j$ and therefore

$$\widehat{T[\Gamma_M]}\widehat{T[\Gamma_M^0]}\widehat{S[\Gamma_M^0]}M_j = \widehat{T[\Gamma_M]}\widehat{T[\Gamma_M^0]}X_j\oplus\widehat{T[\Gamma_M]}\widehat{T[\Gamma_M^0]}Y_j.$$

By the Krull-Schmidt property of $\mathcal{H}$, we conclude $\widehat{T[\Gamma_M]}\widehat{T[\Gamma_M^0]}(Y_j) = 0$. But $T = T[\Gamma_M]T[\Gamma_M^0] = \widehat{T[\Gamma_M]}\widehat{T[\Gamma_M^0]}D$ by Lemma 5.10. Now, Y_j is a direct factor of some $D(\widetilde{Y}_j)$. More precisely, $Y_j = (\widetilde{Y}_j, e)$ for some non trivial idempotent endomorphism e of $\widetilde{Y}_j$. Since by Proposition 4.5 the counit $ST \longrightarrow \mathrm{id}_{\mathcal{G}}$ is pointwise split, $\widetilde{Y}_j$ is a direct factor of $ST\widetilde{Y}_j$. Since $\widehat{T[\Gamma_M]}\widehat{T[\Gamma_M^0]}(Y_j) = 0$ we get $Te = \widehat{T[\Gamma_M]}\widehat{T[\Gamma_M^0]}D(e) = 0$ which is a contradiction. This proves the theorem. ■

Remark 5.12. Suppose that G is a group with finite index normal subgroup N. Then let $\mathcal{H} = kH-mod$ for a commutative ring k, and $\mathcal{G} = kG-mod$, the functors S and T being induction and restriction. The counit of the adjunction $ST \longrightarrow \mathrm{id}_{\mathcal{G}}$ is then the trace map, as was developed by Linckelmann [27]. Proposition 2.3 requires this map to be split, which is equivalent to $|\Gamma| = |G:H|$ being invertible in k. In this case all adjunctions giving this monad are then actually isomorphic. Theorem 5.11 hence generalises Clifford's theorem 3.2 for group representations for normal subgroups with index invertible in the ground ring.

Remark 5.13. Recently Asashiba developed his paper [1] further. He replaced the action of the group Γ by the Grothendieck construction [2]. We are not yet able to generalise Theorem 5.11 in this direction but we intend to do so in future work.

6. Examples and applications

6.1. **Galois modules.** In Remark 5.12 we observed that the classical Clifford theorem over fields of characteristic 0 is a special case of Theorem 5.11.

We consider a slight generalisation using Galois modules.

Let K be a field and suppose that $K \le L \le M$ is a sequence of finite field extensions.

- Suppose that M is Galois over K, denote $Gal(M:K) =: \Gamma$ and
- suppose that L is normal, i.e. $\Delta := Gal(M:L) \trianglelefteq Gal(M:K) = \Gamma$. Then, restriction gives an epimorphism
$$Gal(M:K) \longrightarrow Gal(L:K)$$
with kernel $\Delta = Gal(M:L)$.

- Suppose that G is a finite group and let $\varphi : G \longrightarrow \Gamma$ be a group homomorphism. Then, via φ we get that M is a KG-module, and via
$$\Gamma \longrightarrow \Gamma/\Delta \simeq Gal(L:K)$$
we obtain that also L is a KG-module.
- Let $H \trianglelefteq G$ be a normal subgroup, and suppose further that $[\varphi(H), \Delta] = 1$.

Consider the twisted group rings $M \rtimes G$, respectively $L \rtimes G$. Recall that $M \rtimes G$ is $M \otimes_K KG$ as abelian group, and the multiplicative law is given by
$$(m_1 \otimes g_1) \cdot (m_2 \otimes g_2) = (m_1 \cdot \varphi(g_1)(m_2)) \otimes g_1 g_2$$
for any $m_1, m_2 \in M$ and $g_1, g_2 \in G$.

Then $L \rtimes H$ is a subring of $M \rtimes G$. This situation then allows to apply our Theorem 5.11.

Lemma 6.1. *With the notations above, $M \rtimes G$ is a free $L \rtimes H$-module with basis parameterised by $\Delta \times G/H$.*

Proof. Since M is Galois over L, by the normal basis theorem we get
$$M = \bigoplus_{\delta \in \Delta} {}^{\delta}L.$$
Hence, since $KG = \bigoplus_{Hg \in G/H} KHg$,
$$\begin{aligned} M \rtimes G &\simeq \left(\bigoplus_{\delta \in \Delta} {}^{\delta}L \right) \rtimes G \\ &\simeq \bigoplus_{\delta \in \Delta} \left({}^{\delta}L \rtimes G \right) \text{ since } L \text{ is normal and hence } G \text{ acts on } L \\ &\simeq \bigoplus_{\delta \in \Delta} \bigoplus_{Hg \in G/H} \left({}^{\delta}L \rtimes H \right) g \\ &\simeq \bigoplus_{(\delta, Hg) \in \Delta \times G/H} \left({}^{\delta}L \rtimes H \right) g \end{aligned}$$
Since $\delta \in \Delta$, the automorphism δ fixes L pointwise, and hence
$$M \rtimes G \simeq (L \rtimes H)^{|\Delta \times G/H|}$$
as $L \rtimes H$-module. ■

G acts on Γ via φ, and then by conjugation. We may hence form the semidirect product $\Gamma \rtimes G$. As L is normal over K, we get $\Delta \trianglelefteq \Gamma$, and therefore by restriction we get a semidirect product $\Delta \rtimes G$. Further, $[\varphi(H), \Delta] = 1$ implies that this induces a semidirect product structure $\Delta \rtimes G/H$.

Lemma 6.2. *The monad given by the adjoint pair $(M \rtimes G \otimes_{L \rtimes H} -, res^{M \rtimes G}_{L \rtimes H})$ is a normal subgroup situation with group $\Delta \rtimes G/H$.*

Proof. Denote $(\Psi, \Phi) := (M \rtimes G \otimes_{L \rtimes H} -, res^{M \rtimes G}_{L \rtimes H})$ for short. From Lemma 6.1 we see that
$$\Phi \circ \Psi = \bigoplus_{(\delta, g) \in \Delta \rtimes G/H} ({}^{\delta}L \rtimes H) g.$$
Hence we may put
$$E_{(\delta, g)} = ({}^{\delta}L \rtimes H) g \otimes_{L \rtimes H} -.$$
Composition gives
$$E_{(\delta_1, g_1)} \circ E_{(\delta_2, g_2)} = E_{(\delta_3, g_3)}$$
for $(\delta_3, g_3) = (\delta_1 \cdot {}^{\varphi(g_1)}\delta_2 \, , \, g_1 g_2)$. This is precisely the structure of the group $\Delta \rtimes G/H$. ■

Remark 6.3. Note that we did not fully use that K, L and M are actually fields. As long as we have a normal basis theorem, such as for tame abelian extensions of algebraic integers in number fields (using Taylor's theorem [34]), the statement stays true.

We now consider the functor $\Psi := (M \rtimes G) \otimes_{(L \rtimes H)} -$

$$L \rtimes H - mod \longrightarrow M \rtimes G - mod$$

together with its right adjoint, the forgetful functor Φ. By Lemma 6.2 the adjoint pair (Ψ, Φ) determines the monad

$$\Phi \circ \Psi = \bigoplus_{(\gamma, gH) \in \Delta \rtimes G/H} E_{(\delta, g)}$$

where

$$E_{(\delta,g)} = ({}^{\delta}L \rtimes H)g \otimes_{L \rtimes H} -.$$

Hence, the hypotheses of Theorem 5.11 are satisfied.

A particularly interesting case occurs when K is a field of characteristic 0. Indeed, again the counit

$$\Psi \circ \Phi \longrightarrow \mathrm{id}$$

is the trace function. The hypothesis that K is a field of characteristic 0 then shows that the trace function, hence the counit of the adjunction, is split. Therefore, by Proposition 2.3 all adjunctions are isomorphic. Hence, the theorem gives a criterion how $M \rtimes G \otimes_{L \rtimes H} V$ decomposes in terms of the inertia group $I_{\Delta \rtimes G/H}(V)$ for a simple $L \rtimes H$-module V. This inertia group also depends on the Galois extension. However, $\Delta = Gal(M : L)$, and hence ${}^{\delta}L \simeq L$ as L-module for any $\delta \in \Delta$. Therefore $I_{\Delta \rtimes G/H}(V) = \Delta \rtimes U$ for some subgroup U of G containing H.

6.2. **De-equivariantization.** In the theory of tensor, braided and fusion categories the concept studied in our paper is known under the name of de-equivariantization. Recall the setting from [21, Section 8.22] there. For more ample details we refer to [21].

A fusion category is a finite semisimple tensor category such that the endomorphism algebra of the unit element is the base field K. Let $\mathcal{C}$ be a fusion category with an action of a group G. Then $KG - mod$ is a subcategory of the full subcategory of G-equivariant objects in $\mathcal{C}$, which is again a fusion category denoted $\mathcal{C}^G$. This embedding factors through the canonical embedding $Z(\mathcal{C}^G) \hookrightarrow \mathcal{C}^G$.

Let $\mathcal{D}$ be a fusion category and let G be a finite group acting on $\mathcal{D}$. Suppose that there is a braided tensor functor $KG - mod \longrightarrow Z(\mathcal{D})$ such that the composition with $Z(\mathcal{D}) \hookrightarrow \mathcal{D}$ is fully faithful. Then the image of the regular KG-module in $\mathcal{D}^G$ forms a monad A. Consider $A - Mod_{\mathcal{D}}$ (cf Section 2.1). Then $\mathcal{D}_G := A - Mod_{\mathcal{D}}$ is again a fusion category with an action of G such that $(\mathcal{D}_G)^G \simeq \mathcal{D}$. Observe that the monad A is precisely what we call a normal subgroup situation. We hence may apply the results of Theorem 5.11 to this setting.

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Université de Picardie,
Département de Mathématiques et LAMFA (UMR 7352 du CNRS),
33 rue St Leu,
F-80039 Amiens Cedex 1,
France
Email address: alexander.zimmermann@u-picardie.fr