Multi-centered dilatations, congruent isomorphisms and Rost double deformation space

Arnaud Mayeux

Multi-centered dilatations, congruent isomorphisms and Rost double deformation space

Arnaud Mayeux

Abstract. We introduce multi-centered dilatations of rings, schemes and algebraic spaces, a basic algebraic concept. Dilatations of schemes endowed with a structure (e.g. monoid, group or Lie algebra) are in favorable cases schemes endowed with the same structure. As applications, we use our new formalism to contribute to the understanding of mono-centered dilatations, to formulate and deduce some multi-centered congruent isomorphisms and to interpret Rost double deformation space as both "double-centered" and mono-centered dilatations.

Contents

Acknowledgement. The idea behind this project appeared while I was preparing a talk at the workshop *Arthur packets and related problems, University of Zagreb, June 2022, Croatia*, I thank Marcela Hanzer for the invitation. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 101002592). I am grateful to Anne-Marie Aubert, Adrien Dubouloz, Timo Richarz, Matthieu Romagny, João Pedro dos Santos and Simon Riche for useful discussions and suggestions. Some historical parts of the introduction were written in parallel with [DMdS23].

2020 Mathematics Subject Classification 13A99, 14A20, 14L15
Keywords: algebraic dilatations, dilatations of rings, dilatations of schemes, dilatations of algebraic spaces

1. Introduction

1.1 Motivation and goals

Dilatation of rings is a basic construction of commutative algebra, like localization or tensor product. It can be globalized so that it also makes sense on schemes or algebraic spaces. In fact dilatations generalize localizations.

Let A be a ring and let S be a multiplicative subset of A. Recall that the localization $S^{-1}A$ is an A-algebra such that for any A-algebra $A \to B$ such that the image of s is invertible for any $s \in S$, then $A \to B$ factors through $A \to S^{-1}A$. Intuitively, $S^{-1}A$ is the A-algebra obtained from A adding all fractions $\frac{a}{s}$ with $a \in A$ and $s \in S$. Formally, $S^{-1}A$ is made of classes of fractions $\frac{a}{s}$ where $a \in A$ and $s \in S$ (two representative $\frac{a}{s}$ and $\frac{b}{t}$ are identified if $atr = bsr$ for some $r \in S$), addition and multiplication are given by usual formulas. Now let us give for any element $s \in S$ an ideal M_s of A containing s. The dilatation of A relatively to the data $S, \{M_s\}_{s\in S}$ (introduced in this paper) is an A-algebra A' obtained intuitively by adding to A only the fractions $\frac{m}{s}$ with $s \in S$ and $m \in M_s$. The dilatation A' satisfies that for any $s \in S$, we have $sA' = M_sA'$ (intuitively any $m \in M_s$ belongs to sA', i.e. becomes a multiple of s, so that we have an element $\frac{m}{s}$ such that $m = s\frac{m}{s}$). As a consequence of the construction, the elements $s \in S$ become a non-zero-divisor in A' so that $\frac{m}{s}$ is well-defined (i.e. unique). It turns out that it is convenient, with dilatations of schemes in mind, to make a bit more flexible the above framework, namely to remove the conditions that S is multiplicative and that $s \in M_s$, so we use the following definition.

Definition. Let A be a ring. Let I be an index set. A multi-center in A indexed by I is a set of pairs $\{[M_i, a_i]\}_{i\in I}$ where for each i, M_i is an ideal of A and a_i is an element of A.

To each multi-center $\{[M_i, a_i]\}_{i\in I}$, one has the dilatation $A[\{\frac{M_i}{a_i}\}_{i\in I}]$, it is an A-algebra. We will define and study in details dilatations of rings in Section 2, in particular we will state formally the universal property they enjoy. We will see that $A[\{\frac{M_i}{a_i}\}_{i\in I}]$ is generated, as A-algebra, by $\{\frac{M_i}{a_i}\}_{i\in I}$. We will also see that if $M_i = A$ for all i, then $A[\{\frac{M_i}{a_i}\}_{i\in I}] = S^{-1}A$ where S is the multiplicative subset generated by $\{a_i\}_{i\in I}$. Reciprocally, we will see that any sub-A-algebra of a localization $S^{-1}A$ for a certain S is isomorphic to a dilatation of A.

Dilatations of schemes and algebraic spaces are obtained from dilatations of rings via glueing. We introduce the following definition (we restrict to schemes in this introduction).

Definition. Let X be a scheme. Let I be an index set. A multi-center in A indexed by I is a set of pairs $\{[Y_i, D_i]\}_{i\in I}$ such that Y_i and D_i are closed subschemes for each i and such that locally, all D_i are principal for $i \in I$.

Associated to each multi-center, one has the dilatation $\mathrm{Bl}_{\{\frac{D_i}{Y_i}\}_{i\in I}}X$, it is a scheme endowed with a canonical affine morphism $f : \mathrm{Bl}_{\{\frac{D_i}{Y_i}\}_{i\in I}}X \to X$. It satisfies, in a universal way, that $f^{-1}(D_i)$ is a cartier divisor (i.e. is locally defined by a non-zero-divisor) and that $f^{-1}(D_i) = f^{-1}(Y_i \cap D_i)$ for all $i \in I$. If $\#I = 1$, we use the terminology mono-centered dilatation. We will introduce formally and study several facets of this construction and show that it enjoys many wonderful properties.

As we explained, dilatations are a basic construction which can be easily encountered in specific situations. As a consequence, the theory of dilatations has deep and distinguished roots. Right from the start, we warn the reader that we could not present a comprehensive historical account. As soon as Cremona and Bertini started using quadratic transformations (or blowups)

in the framework of algebraic geometry over fields, "substitutions" of the form $x' = x$ and $y' = y/x$ started being made by algebraic geometers, see for example equation (8) in [No1884, Section 11] and Noether's acknowledgement, at the start of [No1884, Section 12], that these manipulations come from Cremona's point of view. Examples of dilatations appear frequently in some works of Zariski and Abhyankar, cf. [AZ55, Definition, p. 86] and [Za43, p499 proof of Th.4, case (b)]. Other forerunner examples of dilatations play a central role in several independent and unrelated works later, cf. [Da67], [Ner64, Section 25] and [Ar69, Section 4]. As far as we know, the terminology dilatations emerged in [BLR90, §3.2], where a section is devoted to study dilatations of schemes over discrete valuation rings systematically. In the context of schemes over a discrete valuation ring, we draw the reader's attention to [Ana73], [WW80] and [PY06]. The paper [KZ99] studies dilatations (under the name affine modifications) systematically in the framework of algebraic geometry over fields. Over two-dimensional base schemes dilatations also appear in [PZ13, p. 175]. In recent times, the authors of [Du05] and [MRR20] have set out to accommodate all these mono-centered constructions in a larger and unified frame, namely for arbitrary schemes. The paper [MRR20] introduces dilatations of arbitrary schemes in the mono-centered case and provides a systematic treatment of mono-centered dilatations of general schemes. An equivalent definition of mono-centered dilatations of general schemes, under the name affine modifications, was introduced earlier in [Du05, Définition 2.9] under few assumptions. Set aside localizations, mono-centered dilatations have been the main focus of mathematicians in the past. However, in the context of group schemes over discrete valuation rings, examples of multi-centered dilatations of rings and schemes that are not localizations or mono-centered dilatations appeared and were used in [SGA3, Exp. VIB Ex. 13.3], [PY06] and [DHdS18]. The present paper unifies all these constructions, it introduces dilatations of arbitrary rings, schemes and algebraic spaces for arbitrary multi-centers. Allowing multi-centers also leads naturally to the formulation of combinatorial isomorphisms on dilatations and gives birth to refined universal properties.

Beyond rings and algebraic spaces, the concept of dilatations makes sense for other structures and geometric settings. Let us indicate some constructions already available. Some dilatation constructions in the framework of complex analytic spaces were introduced in [Ka94]. For many other structures than rings, dilatations also make sense (e.g. categories, non-commutative rings, semirings), this is the topic of [Ma23c].

Recall that dilatations have distinguished roots, as a consequence, several other terminologies are used to call certains dilatations in literature. For examples the constructions named *affine blowups*, *affine modifications*, *automatic blowups*, *formal blowups*, *Kaliman-Zaidenberg modifications*, *localizations* and *Néron blowups* are examples of (eventually multi-centered) dilatations.

1.2 Results

Let A be a ring and let $\{[M_i, a_i]\}_{i\in I}$ be a multi-center. Put $L_i = M_i + (a_i)$ for $i \in I$. Let $Ring_A^{a\text{-reg}}$ be the full subcategory of A-algebras $f : A \to B$ such that $f(a_i)$ is a non-zero-divisor for all $i \in I$. Dilatations of rings enjoy several properties, the main ones are summarized in the following statement.

Proposition 1.1. *The following assertions hold.*

(i) *The functor from $Ring_A^{a\text{-reg}}$ to Set given by*

$$(f\colon A \to B) \longmapsto \begin{cases} \{*\}, \text{ if } f(a_i)B = f(L_i)B \text{ for } i \in \{1, \dots, k\}; \\ \varnothing, \text{ else.} \end{cases} \quad (2.26)$$

is representable by an A-algebra $A[\frac{M}{a}]$, also denoted $A[\{\frac{M_i}{a_i}\}_{i\in I}]$, called the dilatation of A with center $\{[M_i, a_i]\}_{i\in I}$.

(ii) *The image of a_i in $A[\frac{M}{a}]$ is a non-zero-divisor for any $i \in I$, moreover*

$$a_i A[\frac{M}{a}] = L_i A[\frac{M}{a}]. \quad (2.10, 2.25)$$

(iii) *Assume that $I = \{1, \dots, k\}$ is finite. Then we have a canonical identification of A-algebras*

$$A[\{\tfrac{M_i}{a_i}\}_{i\in I}] = A[\tfrac{\sum_{i\in I}(M_i \cdot \prod_{j\in I\setminus\{i\}} a_j)}{a_1 \cdots a_k}]. \quad (2.30)$$

(iv) *If $M_i = A$ for all i, then $A[\frac{M}{a}]$ identifies with the localization $S^{-1}A$ where S is the multiplicative subset of A generated by $\{a_i\}_{i\in I}$.* (2.11)

(v) *Any sub-A-algebra of a localization $S^{-1}A$ can be obtained as a multi-centered dilatation.* (2.15)

(vi) *If A is a domain and $a_i \neq 0$ for all i, then $A[\frac{M}{a}]$ is a domain.* (2.8)

(vii) *If A is reduced, then $A[\frac{M}{a}]$ is reduced.* (2.9)

(viii) *In regular cases, one has an explicit description of $A[\frac{M}{a}]$ as a quotient of a polynomial algebra.* (5.5)

Now let X be a scheme and let $\{[Y_i, D_i]\}_{i\in I}$ be a multi-center. Put $Z_i = Y_i \cap D_i$ for all $i \in I$. Let $Sch_X^{D\text{-reg}}$ be the full subcategory of X-schemes $f : T \to X$ such that $f^{-1}(D_i)$ is a Cartier divisor for all $i \in I$. Our main results on dilatations of schemes are summarized in the following theorem. Note that we work with general algebraic spaces later in the paper.

THEOREM 1.2. *The following assertions hold.*

(i) *The functor from $Sch_X^{D\text{-reg}}$ to Set given by*

$$(f\colon T \to X) \longmapsto \begin{cases} \{*\}, \text{ if } f|_{T\times_X D_i} \text{ factors through } Y_i \cap D_i \subset X \text{ for } i \in \{1, \dots, k\}; \\ \varnothing, \text{ else.} \end{cases}$$

is representable by an X-affine scheme $\mathrm{Bl}_Y^D X$, also denoted $\mathrm{Bl}_{\{Y_i\}_{i\in I}}^{\{D_i\}_{i\in I}} X$ and $\mathrm{Bl}_{\{Y_i\}_{i\in I}}^{\{D_i\}_{i\in I}} X$, and called the dilatation of X with multi-center $\{[Y_i, D_i]\}_{i\in I}$. (3.17)

(ii) *As closed subschemes of $\mathrm{Bl}_Y^D X$, one has, for all $i \in I$,*

$$\mathrm{Bl}_Y^D X \times_X Z_i = \mathrm{Bl}_Y^D X \times_X D_i,$$

which is an effective Cartier divisor on $\mathrm{Bl}_Y^D X$. (3.16)

(iii) *Let J be a subset of I and put $K = I \setminus J$. Then*

$$\mathrm{Bl}_{\{Y_i\}_{i\in I}}^{\{D_i\}_{i\in I}} X = \mathrm{Bl}_{\left\{Y_k \times_X \mathrm{Bl}_{\{Y_i\}_{i\in J}}^{\{D_i\}_{i\in J}} X\right\}_{k\in K}}^{\left\{D_k \times_X \mathrm{Bl}_{\{Y_i\}_{i\in J}}^{\{D_i\}_{i\in J}} X\right\}_{k\in K}} \mathrm{Bl}_{\{Y_i\}_{i\in J}}^{\{D_i\}_{i\in J}} X.$$

In particular, there is a unique X-morphism

$$\mathrm{Bl}_{\{Y_i\}_{i\in I}}^{\{D_i\}_{i\in I}} X \to \mathrm{Bl}_{\{Y_i\}_{i\in J}}^{\{D_i\}_{i\in J}} X. \quad (3.19)$$

(iv) *Assume that $\#I = k$ is finite. We fix an arbitrary bijection $I = \{1, \dots, k\}$. We have a canonical isomorphism of X-schemes*

$$\mathrm{Bl}_{\{Y_i\}_{i\in I}}^{\{D_i\}_{i\in I}} X \cong \mathrm{Bl}_{(\mathrm{Bl}\cdots)\times_X Y_k}^{(\mathrm{Bl}\cdots)\times_X D_k} \left(\cdots \mathrm{Bl}_{(\mathrm{Bl}\cdots)\times_X Y_3}^{(\mathrm{Bl}\cdots)\times_X D_3} \left(\mathrm{Bl}_{(\mathrm{Bl}_{Y_1}^{D_1} X)\times_X Y_2}^{(\mathrm{Bl}_{Y_1}^{D_1} X)\times_X D_2} \left(\mathrm{Bl}_{Y_1}^{D_1} X \right) \right) \right). \quad (3.30)$$

(v) *(Monopoly isomorphism) Assume that $\#I = k$ is finite. We fix an arbitrary bijection $I = \{1, \dots, k\}$. We have a canonical isomorphism of X-schemes*

$$\mathrm{Bl}^{\{D_i\}_{i\in I}}_{\{Y_i\}_{i\in I}} X \cong \mathrm{Bl}^{D_1+\ldots+D_k}_{\bigcap_{i\in I}(Y_i+D_1+\ldots+D_{i-1}+D_{i+1}+\ldots+D_k)} X. \quad (3.31)$$

(vi) *In many cases, multi-centered dilatations can be iterated in a compatible way with the addition law on closed subschemes. (3.1, 4.6)*

(vii) *Multi-centered dilatations whose multi-center $\{[Y_i, D_i]\}_{i\in I}$ satisfies that $\{D_i\}_{i\in I}$ are multiples of a single locally principal closed subscheme satisfy additional properties (preservation of flatness, smoothness). (cf. e.g. 5.5, 6.1 for precise statements)*

We also define dilatations in a relative setting as follows. Let S be a scheme and let X be a scheme over S. Let $C = \{C_i\}_{i\in I}$ be closed subschemes of S such that, locally, each C_i is principal. Put $D = \{C_i \times_S X\}_{i\in I}$. Let $Y = \{Y_i\}_{i\in I}$ be closed S-subspaces of X. We put $\mathrm{Bl}^C_Y X := \mathrm{Bl}^D_Y X$.

Fact 1.3. *$\mathrm{Bl}^C_Y X$ represents the functor from $Sch_S^{C\text{-reg}}$ to Set given by*

$$(f : T \to S) \mapsto \{x \in \mathrm{Hom}_S(T, X) |\ x_{|C_i} : T \times_S C_i \to X \times_S C_i \text{ factors through } Y_i \times_S C_i\}.$$

Remark 1.4. Fact 1.3 implies that for any $T \in Sch_S^{C\text{-reg}}$ (e.g. $T = S$ if each C_i is a Cartier divisor in S) we have a canonical inclusion on T-points $\mathrm{Bl}^C_Y X(T) \subset X(T)$. But in general $\mathrm{Bl}^C_Y X \to X$ is not a monomorphism in the full category of S-schemes.

We now describe our results regarding the behaviour of dilatations of schemes endowed with a structure. So let S be a base scheme and assume that $X = G$ is a monoid (resp. group, resp. Lie algebra) scheme (or any structure defined using products). Let $H_i \subset D_i = G|_{C_i}$ be a closed submonoid (resp. subgroup, resp. Lie subalgebra) schemes over C_i for all $i \in I$ and let $H = \{H_i\}_{i\in I}$. Let $\mathcal{G} := \mathrm{Bl}^D_H G \to G$ be the associated dilatation. The structure morphism $\mathcal{G} \to S$ defines an object in $Sch_S^{C\text{-reg}}$.

Proposition 1.5. *Let $\mathcal{G} \to S$ be the above dilatations.*

(i) *Recall that the scheme $\mathcal{G} \to S$ represents the contravariant functor $Sch_S^{C\text{-reg}} \to Set$ given for $T \to S$ by the set of all S-morphisms $T \to G$ such that the induced morphism $T|_{C_i} \to G|_{C_i}$ factors through $H_i \subset G|_{C_i}$ for all $i \in \{1, \dots, k\}$. (7.1)*

(ii) *Let $T \to S$ be an object in $Sch_S^{C\text{-reg}}$, then as subsets of $G(T)$*

$$\mathcal{G}(T) = \bigcap_{i\in I} \left(\mathrm{Bl}^{D_i}_{H_i} G\right)(T). \quad (7.1)$$

(iii) *The map $\mathcal{G} \to G$ is affine. Its restriction over C_i factors as $\mathcal{G}_i \to H_i \subset D_i$ for all $i \in I$. (7.1)*

(iv) *Assume the dilatation $\mathcal{G} \to S$ is flat, then $\mathcal{G} \to S$ is equipped with the structure of a monoid (resp. group, resp. Lie algebra) scheme over S such that $\mathcal{G} \to G$ is a morphism of S-monoid (resp. S-group, resp. S-Lie algebra) schemes. (7.2)*

(v) *Under flatness assumptions, dilatations commute with the formation of Lie algebra schemes in a natural sense*

$$\mathbb{L}ie(\mathrm{Bl}^{G|_C}_H G) \cong \mathrm{Bl}^{\{\mathbb{L}ie(G)\times_S C_i\}}_{\{\mathbb{L}ie(H_i)\}_{i\in I}} \mathbb{L}ie(G). \quad (7.4)$$

As an example of dilatations, let us explain a connexion between dilatations and Yu's famous construction of supercuspidal representations [Yu01] (cf. also [Yu15, §10]).

EXAMPLE 1.6. Assume in this example that $\mathcal{O}$ is the ring of integers of a non-Archimedean local field F and that π is the maximal ideal of $\mathcal{O}$. Let G be a split connected reductive group scheme over $\mathcal{O}$, i.e. a Demazure group scheme over $\mathcal{O}$. Let $\overrightarrow{G} = (G^0 \subset G^1 \subset \ldots \subset G^d = G)$ be a sequence of split Levi subgroups of G over $\mathcal{O}$. Put $\mathrm{G}_i = G_i \times_{\mathcal{O}} F$ for all $i \in \{0, \ldots, k\}$. Put $\overrightarrow{\mathrm{G}} = (\mathrm{G}^0 \subset \mathrm{G}^1 \subset \ldots \subset \mathrm{G}^d =: \mathrm{G})$, a sequence of split Levi subgroups of G over F. For $i \in \{0, \ldots, d\}$, let x_i be the special point in the Bruhat-Tits building of G_i such that G_i corresponds to x_i via Bruhat-Tits theory. Then x_i comes from x_0 via functoriality of buildings as in [Yu01, §1]. Let $0 \leqslant r_0 \leqslant r_1 \leqslant \ldots \leqslant r_d$ be integers. Recall that e_G denotes the trivial closed subgroup scheme of G. There is a canonical isomorphism of groups

$$\overrightarrow{\mathrm{G}}(F)_{x,\overrightarrow{r}} = \mathrm{Bl}^{r_0,r_1,r_2,\ldots,r_{i+1},\ldots,r_d}_{e_G,G^0,G^1,\ldots,G^i,\ldots,G^{d-1}} G(\mathcal{O}),$$

where $\overrightarrow{\mathrm{G}}(F)_{x,\overrightarrow{r}}$ is defined in [Yu01, §1 p584] and $\mathrm{Bl}\{^{t_i}_{H_i}\}_{i\in I} G$ denotes $\mathrm{Bl}\{^{\mathcal{O}/\pi_i^t}_{H_i}\}_{i\in I} G$ for integers $t_i \geqslant 0$.

In the paper, we prove the following result. It generalizes the fact that congruence subgroups are normal subgroups. It was also motivated by the fact that the proof of [Yu01, Lemma 1.4], related to Example 1.6, is not correct.

PROPOSITION 1.7. *Assume that $\#I = 1$, $C = C_1$ is a Cartier divisor in S and $G \to S$ is a flat group scheme. Let $\eta : K \to G$ be a morphism of group schemes over S such that $K \to S$ is flat. Assume that $H \subset G$ is a closed subgroup scheme over S such that $H \to S$ is flat and $\mathrm{Bl}^C_H G \to S$ is flat (and in particular a group scheme). Assume that K_C commutes with H_C in the sense that the morphism $K_C \times_C H_C \to G_C$, $(k, h) \mapsto \eta(k)h\eta(k)^{-1}$ equals the composition morphism $K_C \times_C H_C \to H_C \subset G_C$, $(k, h) \mapsto h$. Then K normalizes $\mathrm{Bl}^C_H G$, more precisely the solid composition map*

$$\begin{array}{ccccc} K \times_S \mathrm{Bl}^C_H G & \xrightarrow{Id\times \mathrm{Bl}} & K \times_S G & \xrightarrow{k,g\mapsto \eta(k)g\eta(k)^{-1}} & G \\ & \searrow & & \nearrow_{\mathrm{Bl}} & \\ & & \mathrm{Bl}^C_H G & & \end{array}$$

factors uniquely through $\mathrm{Bl}^C_H G$. (7.5)

Recall that mono-centered dilatations ($\#I = 1$) are already studied in many references, cf. e.g. [WW80], [BLR90], [KZ99], [Du05] and [MRR20]. We refer the reader to [DMdS23] for a survey paper on the topic of both mono and multi-centered dilatations.

We now discuss some applications of the above theoretical results. Our first application is a multi-centered congruent isomorphism. To explain it, let $(\mathcal{O}, \pi)$ be a henselian pair where $\pi \subset \mathcal{O}$ is an invertible ideal.

THEOREM 1.8 (Multi-centered congruent isomorphism). *Let G be a separated and smooth group scheme over S. Let $H_0, H_1, \ldots, H_k$ be closed subgroup schemes of G such that $H_0 = e_G$ is the trivial subgroup and such that $H_i \to S$ is smooth for all $i \in \{0, \ldots, k\}$. Let $s_0, s_1, \ldots, s_k$ and $r_0, r_1, \ldots, r_k$ be in $\mathbb{N}$ such that*

(i) *$s_i \geqslant s_0$ and $r_i \geqslant r_0$ for all $i \in \{0, \ldots, k\}$*

(ii) *$r_i \geqslant s_i$ and $r_i - s_i \leqslant s_0$ for all $i \in \{0, \ldots, k\}$.*

Assume that G is affine or $\mathcal{O}$ is local. Assume that a regularity condition (RC) is satisfied (cf. Definition 8.2). Then we have a canonical isomorphism of groups

$$\mathrm{Bl}^{s_0,\ s_1,\ \cdots,s_k}_{H_0,H_1,\ldots,H_k} G(\mathcal{O})/\mathrm{Bl}^{r_0,\ r_1,\ \cdots,r_k}_{H_0,H_1,\ldots,H_k} G(\mathcal{O}) \cong \mathrm{Lie}(\mathrm{Bl}^{s_0,\ s_1,\ \cdots,s_k}_{H_0,H_1,\ldots,H_k} G)(\mathcal{O})/\mathrm{Lie}(\mathrm{Bl}^{r_0,\ r_1,\ \cdots,r_k}_{H_0,H_1,\ldots,H_k} G)(\mathcal{O})$$

where $\mathrm{Bl}^{t_0,\ \cdots,t_k}_{H_0,\ldots,H_k} G$ *denotes* $\mathrm{Bl}^{\mathcal{O}/\pi^{t_0},\ldots,\mathcal{O}/\pi^{t_k}}_{H_0,\ \cdots\ ,H_k} G$ *for any* $t_0,\ldots,t_k \in \mathbb{N}$. *(8.1)*

Remark 1.9. Note that [Yu01, Lemma 1.3] provides a comparable "multi-centered" isomorphism, but in the framework of reductive groups over non-Archimedean local field ([Yu01, Lemma 1.3] does not involve dilatations). Note also that Theorem 1.8 extends [MRR20, Theorem 4.3]. Recall that [MRR20, Theorem 4.3] is related to the Moy-Prasad isomorphism in the setting of reductive groups over non-Archimedean local fields. The Moy-Prasad isomorphism for reductive groups is of fundamental importance in representation theory of reductive groups over non-Archimedean local fields. The proof of the Moy-Prasad isomorphism in the recent reference [KP22] use [MRR20, Theorem 4.3], cf. [KP22, Theorem 13.5.1 and its proof, Proposition A.5.19 (3) and its proof].

Our other application is an interpretation of Rost double deformation space in the language of dilatations. Rost double deformation space is a fundamental tool in intersection theory and motivic homotopy theory. Let $Z \to Y \to X$ be closed immersions of schemes (in [Ro96], all schemes are assumed to be defined over fields but we work with arbitrary schemes here). Let $\overline{D}(X,Y,Z)$ be the double deformation space as defined in [Ro96, §10].

Let $\mathbb{A}^2$ be $\mathrm{Spec}(\mathbb{Z}[s,t])$. Let D_s, D_{st} and D_{s^2t} be the locally principal closed subschemes of $\mathbb{A}^2$ defined by the ideals $(s),(st)$ and (s^2t). We now omit the subscript $_{\mathrm{Spec}(\mathbb{Z})}$ in fiber products.

Proposition 1.10. (i) *We have a canonical identification:*

$$\overline{D}(X,Y,Z) \cong \mathrm{Bl}^{(X\times D_{st}),\ (X\times D_s)}_{(Y\times\mathbb{A}^2),\ \ (Z\times\mathbb{A}^2)}(X\times\mathbb{A}^2).$$

In other words, Rost double deformation space is naturally interpreted as a "double-centered" dilatation. (9.1)

(ii) *Using the monopoly isomorphism (cf. Theorem 1.2), we obtain that Rost double deformation space is canonically isomorphic to a mono-centered dilatation as follows:*

$$\overline{D}(X,Y,Z) \cong \mathrm{Bl}^{X\times D_{s^2t}}_{\big((Y\times\mathbb{A}^2)+(X\times D_s)\big)\cap\big((Z\times\mathbb{A}^2)+(X\times D_{st})\big)}(X\times\mathbb{A}^2). \quad (9.2)$$

1.3 Structure of the paper

Section 2 introduces dilatations of rings. Section 3 introduces multi-centered dilatations. Section 4 deals with iterated dilatations. Section 5 focuses on the case where the multi-center $\{[Z_i,D_i]\}_{i\in I}$ satisfies that $\{D_i\}$ are given by multiples of a single D. Section 6 proves some flatness and smoothness results. Section 7 considers dilatations of monoid, group and Lie algebra schemes. Section 8 studies congruent isomorphisms. Section 9 interprets Rost double deformation space as dilatations.

2. Dilatations of rings

We introduce dilatations of commutative rings. Recall that dilatations of categories also make sense (cf. [Ma23c]), however dilatations of commutative rings behave specifically and it is better to treat them separately. From now on, rings are assumed to be unital and commutative.

2.1 Definition

Let A be a unital commutative ring. If M is an ideal of A and $a \in A$ is an element, we say that the pair $[M, a]$ is a center in A. Let I be an index set and let $\{[M_i, a_i]\}_{i\in I}$ be a set of centers indexed by I. For $i \in I$, we put $L_i = M_i + (a_i)$, an ideal of A. Let $\mathbb{N}_I$ be the monoid $\bigoplus_{i\in I} \mathbb{N}$. If $\nu = (\nu_1, \ldots, \nu_i, \ldots) \in \mathbb{N}_I$ we put $L^\nu = L_1^{\nu_1} \cdots L_i^{\nu_i} \cdots$ (product of ideals of A) and $a^\nu = a_1^{\nu_1} \cdots a_i^{\nu_i} \cdots$ (product of elements of A). We also put $a^{\mathbb{N}_I} = \{a^\nu | \nu \in \mathbb{N}_I\} \subset A$.

Definition and Proposition 2.1. *The dilatation of A with multi-center $\{[M_i, a_i]\}_{i\in I}$ is the unital commutative ring $A[\{\frac{M_i}{a_i}\}_{i\in I}]$ defined as follows:*

- *The underlying set of $A[\{\frac{M_i}{a_i}\}_{i\in I}]$ is the set of equivalence classes of symbols $\frac{m}{a^\nu}$ where $\nu \in \mathbb{N}_I$ and $m \in L^\nu$ under the equivalence relation*

$$\frac{m}{a^\nu} \equiv \frac{p}{a^\lambda} \Leftrightarrow \exists \beta \in \mathbb{N}_I \text{ such that } ma^{\beta+\lambda} = pa^{\beta+\nu} \text{ in } A.$$

From now on, we abuse notation and denote a class by any of its representative $\frac{m}{a^\nu}$ if no confusion is likely.

- *The addition law is given by $\frac{m}{a^\nu} + \frac{p}{a^\beta} = \frac{ma^\beta + pa^\nu}{a^{\beta+\nu}}$.*
- *The multiplication law is given by $\frac{m}{a^\nu} \times \frac{p}{a^\beta} = \frac{mp}{a^{\nu+\beta}}$.*
- *The additive neutral element is $\frac{0}{1}$ and the multiplicative neutral element is $\frac{1}{1}$.*

We have a canonical morphism of rings $A \to A[\{\frac{M_i}{a_i}\}_{i\in I}]$ given by $a \mapsto \frac{a}{1}$. We sometimes use the notation $A[\frac{M}{a}]$ to denote $A[\{\frac{M_i}{a_i}\}_{i\in I}]$.

Proof. Let us first prove that the relation is an equivalence relation. Assume

$$\frac{m}{a^\nu} \equiv \frac{p}{a^\lambda}$$
$$\frac{l}{a^\theta} \equiv \frac{p}{a^\lambda}$$

where $\nu, \lambda, \theta \in \mathbb{N}_I$, $(m, p, l) \in (L^\nu, L^\lambda, L^\theta)$. We want to prove that

$$\frac{m}{a^\nu} \equiv \frac{l}{a^\theta}.$$

By definition, there exist $\beta, \alpha \in \mathbb{N}_I$ such that

$$ma^{\beta+\lambda} = pa^{\beta+\nu}$$
$$la^{\alpha+\lambda} = pa^{\alpha+\theta}.$$

Put $\delta = \beta + \lambda + \alpha$. We get

$$ma^{\delta+\theta} = ma^{\beta+\lambda+\alpha+\theta} = pa^{\beta+\nu+\alpha+\theta} = la^{\beta+\nu+\alpha+\lambda} = la^{\delta+\nu},$$

so $\frac{m}{a^\nu} \equiv \frac{l}{a^\theta}$. The addition and multiplication laws are associative and commutative. The distributivity axiom is satisfied and the additive neutral element is absorbent for the multiplication. So $A[\frac{M}{a}]$ is a unital commutative ring. The formula $a \mapsto \frac{a}{1}$ provides a canonical morphism of rings $A \to A[\frac{M}{a}]$. □

The element $\frac{a}{1}$ of $A[\{\frac{M_i}{a_i}\}_{i\in I}]$ will sometimes be denoted by a if no confusion is likely.

Remark 2.2. Let $\{N_i\}_{i\in I}$ be ideals in A such that $N_i + (a_i) = L_i$ for all $i \in I$. Then we have identifications of A-algebras $A[\{\frac{M_i}{a_i}\}_{i\in I}] = A[\{\frac{N_i}{a_i}\}_{i\in I}] = A[\{\frac{L_i}{a_i}\}_{i\in I}]$.

REMARK 2.3. Note that Def. Prop. 2.1 and its proof shows that if A is assumed to be just a unital commutative semiring, then $A[\{\frac{M_i}{a_i}\}_{i\in I}]$ is a unital commutative semiring and $A \to A[\{\frac{M_i}{a_i}\}_{i\in I}]$ is a morphism of semirings. Note that most results of §2.1-2.2 extend to semirings.

REMARK 2.4. Let $\{E_i\}_{i\in I}$ be subsets of A, let P_i be the ideal generated by E_i for $i \in I$. Then one can define $A[\{\frac{E_i}{a_i}\}_{i\in I}]$ as being $A[\{\frac{P_i}{a_i}\}_{i\in I}]$.

DEFINITION 2.5. Let $f : A \to B$ be a morphism of rings, we say that f is a *dilatation map* or an *affine modification* if there exists a multi-center $\{[M_i, a_i]\}_{i\in I}$ in A such that $B \cong A[\{\frac{M_i}{a_i}\}_{i\in I}]$ as A-algebras (cf. also Fact 2.15 for an other caracterization).

2.2 Properties of dilatations

We proceed with the notation from §2.1.

REMARK 2.6. As A-algebra, $A[\{\frac{M_i}{a_i}\}_{i\in I}]$ is generated by $\{\frac{L_i}{a_i}\}_{i\in I}$. Since $L_i = M_i + (a_i)$, this implies that $A[\{\frac{M_i}{a_i}\}_{i\in I}]$ is generated by $\{\frac{M_i}{a_i}\}_{i\in I}$.

FACT 2.7. *The following assertions are equivalent.*

(i) *There exists $\nu \in \mathbb{N}_I$ such that $a^\nu = 0$ in A.*

(ii) *The ring $A[\{\frac{M_i}{a_i}\}_{i\in I}]$ is equal to the zero ring.*

Proof. Assume (i) holds. Let $\frac{m}{a^\beta} \in A[\frac{M}{a}]$ with $\beta \in \mathbb{N}_I$ and $m \in L^\beta$. Then $a^\nu m = 0$ in A and so $\frac{m}{a^\beta} = \frac{0}{1}$ in $A[\{\frac{M_i}{a_i}\}_{i\in I}]$. So (ii) holds. Reciprocally, assume (ii) holds. Then $\frac{1}{1} = \frac{0}{1}$ and so there exists $\nu \in \mathbb{N}_I$ such that $a^\nu = 0$ in A. So (i) holds. □

FACT 2.8. *Assume that A is a domain and $a_i \neq 0$ for all i, then $A[\{\frac{M_i}{a_i}\}_{i\in I}]$ is a domain.*

Proof. Assume that $\frac{m}{a^\nu}\frac{l}{a^\beta} = 0$ in $A[\{\frac{M_i}{a_i}\}_{i\in I}]$. Then there exists $\theta \in \mathbb{N}_I$ such that $a^\theta ml = 0$ in A. Since $a^\theta \neq 0$ and A is a domain, we get that $m = 0$ or $l = 0$. This finishes the proof. □

FACT 2.9. *Assume that A is reduced, then $A[\{\frac{M_i}{a_i}\}_{i\in I}]$ is reduced.*

Proof. Assume that, in $A[\{\frac{M_i}{a_i}\}_{i\in I}]$, $(\frac{m}{a^\nu})^N = 0$ for some $N \in \mathbb{N}$ then there exists $\beta \in \mathbb{N}_I$ such that $a^\beta m^N = 0$. We can assume that $\beta = N\theta$ with $\theta \in \mathbb{N}_I$. Then $(a^\theta m)^N = 0$ in A and so $a^\theta m = 0$ since A is reduced. So $\frac{m}{a^\nu} = 0$ in $A[\{\frac{M_i}{a_i}\}_{i\in I}]$. □

FACT 2.10. *Let ν be in $\mathbb{N}_I$. The image of a^ν in $A[\{\frac{M_i}{a_i}\}_{i\in I}]$ is a non-zero-divisor.*

Proof. Let $b \in A[\{\frac{M_i}{a_i}\}_{i\in I}]$ such that $a^\nu b = 0$ in $A[\{\frac{M_i}{a_i}\}_{i\in I}]$. Write $b = \frac{m}{a^\alpha}$, then we get $\frac{a^\nu m}{a^\alpha} = 0$ in $A[\{\frac{M_i}{a_i}\}_{i\in I}]$. This implies that there is $\beta \in \mathbb{N}_I$ such that $a^\beta a^\nu m = 0$. So $b = 0$ in $A[\{\frac{M_i}{a_i}\}_{i\in I}]$. So a^ν is a non-zero-divisor. □

FACT 2.11. *Assume $M_i = A$ for all $i \in I$. Then $A[\{\frac{M_i}{a_i}\}_{i\in I}] = (a^{\mathbb{N}_I})^{-1}A$ where $(a^{\mathbb{N}_I})^{-1}A$ is the localization of A relatively to the multiplicative monoid $a^{\mathbb{N}_I}$.*

Proof. For any $\nu \in \mathbb{N}_I$, we have $L^\nu = A$. Now the map $\frac{x}{a^\nu} \mapsto \frac{x}{a^\nu}$ provides an isomorphism of A-algebras $A[\{\frac{M_i}{a_i}\}_{i\in I}] = (a^{\mathbb{N}_I})^{-1}A$. □

REMARK 2.12. Dilatations of rings generalize entirely localizations of rings. Indeed, let A be a ring and let S be a multiplicative subset of A (i.e. a submonoid of $A, \times$). Let I be a set such that $S = \{s_i\}_{i\in I}$. Then $s^{\mathbb{N}_I} = S$ and Fact 2.11 says that $S^{-1}A = A[\{\frac{A}{s}\}_{s\in S}]$.

FACT 2.13. *Let $f : A \to B$ be a morphism of rings. Let $\{[N_i, b_i]\}_{i\in I}$ be centers of B such that $f(M_i) \subset N_i$ and $f(a_i) = b_i$ for all $i \in I$. Then we have a canonical morphism of A-algebras*

$$\phi : A[\{\tfrac{M_i}{a_i}\}_{i\in I}] \to B[\{\tfrac{N_i}{b_i}\}_{i\in I}] \ .$$

Proof. Put $\phi(\frac{m}{a^\nu}) = \frac{f(m)}{b^\nu}$. □

FACT 2.14. *Let $\{P_i\}_{i\in I}$ be ideals of A such that $P_i \subset M_i$ for all $i \in I$. Then we have a canonical injective morphism of A-algebras*

$$A[\{\tfrac{P_i}{a_i}\}_{i\in I}] \to A[\{\tfrac{M_i}{a_i}\}_{i\in I}].$$

Proof. Clear. □

FACT 2.15. *Let $f : A \to B$ be a morphism of rings. The following assertions are equivalent.*

(i) *The morphism f is a dilatation map (cf. Definition 2.5).*

(ii) *There exists a multiplicative subset S of A and a sub-A-algebra C of $S^{-1}A$ such that $B \cong C$ as A-algebras.*

Proof. The assertion (i) implies (ii) by Facts 2.11 and 2.14. Reciprocally, let C be a sub-A-algebra of $S^{-1}A$. Let I be the set defined as

$$I := \{i = (m_i, a_i) \in A \times S |\ \frac{m_i}{a_i} \text{ belongs to } C\} \subset A \times S$$

Then $C \cong A[\{\frac{(m_i)}{a_i}\}_{i\in I}]$, indeed we have a canonical morphism of A-algebras $\phi : C \to A[\{\frac{(m_i)}{a_i}\}_{i\in I}]$ sending $\frac{m_i}{a_i}$ to $\frac{m_i}{a_i}$. The morphism ϕ is injective, e.g. because $A[\{\frac{(m_i)}{a_i}\}_{i\in I}] \subset S^{-1}A$ since $S = \{a_i\}_{i\in I}$ because (a_i, a_i) belongs to I for any $a_i \in S$. The morphism ϕ is surjective by Remark 2.6. □

REMARK 2.16. Note that the concept of dilatations extend to categories, cf. [Ma23c]. However the analog of Fact 2.15 fails for categories, cf. [Ma23c].

FACT 2.17. *Let c be a non-zero-divisor element in A. Then $\frac{c}{1}$ is a non-zero-divisor in $A[\{\frac{M_i}{a_i}\}_{i\in I}]$.*

Proof. Let $\frac{m}{a^\nu} \in A[\{\frac{M_i}{a_i}\}_{i\in I}]$ such that $\frac{m}{a^\nu}\frac{c}{1} = 0$. Then there exists $\beta \in \mathbb{N}_I$ such that $a^\beta mc = 0$ in A. Because of c is a non-zero-divisor, this implies $a^\beta m = 0$ in A and so $\frac{m}{a^\nu} = 0$ in $A[\{\frac{M_i}{a_i}\}_{i\in I}]$. □

PROPOSITION 2.18. *Let $K \subset I$ put $J = I \setminus K$. Then we have a canonical morphism of A-algebras*

$$\varphi : A[\{\tfrac{M_i}{a_i}\}_{i\in K}] \to A[\{\tfrac{M_i}{a_i}\}_{i\in I}].$$

Moreover

(i) *if $M_i \subset (a_i)$ for all $i \in J$, then φ is surjective, and*

(ii) *if a_i is a non-zero-divisor in A for all $i \in J$, then φ is injective.*

Proof. We have a canonical injective morphism of monoids $\mathbb{N}_K \to \mathbb{N}_I$. Let $\frac{m}{a^\nu}$ with $\nu \in \mathbb{N}_K$ and $m \in L^\nu$, then we put $\varphi(\frac{m}{a^\nu}) = \frac{m}{a^\nu} \in A[\{\frac{M_i}{a_i}\}_{i\in I}]$. We now prove the listed properties.

(i) It is enough to show that $\frac{M_i}{a_i}$ is in the image of φ for all $i \in I$. This is obvious for all $i \in K$. So let $i \in J$ and let $\frac{m_i}{a_i} \in \frac{M_i}{a_i}$. Since $M_i \subset (a_i)$ we write $m_i = a_i x$ with $x \in A$. Then $\frac{m_i}{a_i} = \frac{x}{1}$ belongs to the image of φ. So φ is surjective.

(ii) Let $\frac{m}{a^\nu} \in A[\{\frac{M_i}{a_i}\}_{i\in K}]$ with $\nu \in \mathbb{N}_K$. Assume that $\frac{m}{a^\nu} = \frac{0}{1}$ in the image $A[\{\frac{M_i}{a_i}\}_{i\in I}]$. Then there exists $\beta \in \mathbb{N}_I$ such that $ma^\beta = 0$. Write $\beta = \nu' + \theta$ with $\nu' \in \mathbb{N}_K$ and $\theta \in \mathbb{N}_J$. Then we have $ma^{\nu'}a^\theta = 0$ in the ring A and a^θ is a non-zero-divisor, so $ma^{\nu'} = 0$ in the ring A and so $\frac{m}{a^\nu} = \frac{0}{1}$ in the source. So φ is injective.

□

COROLLARY 2.19. *Let $K \subset I$. Assume that, for all $j \in J = I \setminus K$, the element a_j belongs to the face A^* of invertible elements of the monoid $(A, \times)$, i.e. a_j is invertible for $\times$. Then*

$$A[\{\tfrac{M_i}{a_i}\}_{i\in I}] = A[\{\tfrac{M_i}{a_i}\}_{i\in K}].$$

Proof. This follows from Proposition 2.18. □

COROLLARY 2.20. *Let $J \subset I$ be such that for all $j \in J$, there is $i \in I \setminus J$ satisfying that $a_i = a_j$ and $M_j \subset M_i$, then we have a canonical identification of A-algebras $A[\{\frac{M_i}{a_i}\}_{i\in I}] = A[\{\frac{M_i}{a_i}\}_{i\in I\setminus J}]$.*

Proof. This follows from Proposition 2.18 and elementary arguments. □

COROLLARY 2.21. *Let $\{d_i\}_{i\in I}$ be positive integers. Let $I = \coprod_{j\in J} I_j$ be a partition of I. Assume that, for all $j \in J$, $M_i = M_{i'} =: M_j$ and $a_i = a_{i'} =: a_j$ for all $i, i' \in I_j$. Assume moreover that, for all $j \in J$, $\max_{i\in I_j} d_i =: d_j$ exists. Then we have a canonical identification*

$$A[\{\tfrac{M_i}{a_i{}^{d_i}}\}_{i\in I}] = A[\{\tfrac{M_j}{a_j{}^{d_j}}\}_{j\in J}].$$

Proof. This follows from Proposition 2.18 and elementary arguments. □

PROPOSITION 2.22. *Let $K \subset I$. Then we have a canonical isomorphism of $A[\{\frac{M_i}{a_i}\}_{i\in K}]$-algebras*

$$A[\{\tfrac{M_i}{a_i}\}_{i\in I}] = A[\{\tfrac{M_i}{a_i}\}_{i\in K}][\{\tfrac{A[\{\frac{M_i}{a_i}\}_{i\in I}]^{\frac{M_j}{1}}}{\frac{a_j}{1}}\}_{j\in I\setminus K}],$$

where $A[\{\frac{M_i}{a_i}\}_{i\in I}]^{\frac{M_j}{1}}$ is the ideal of $A[\{\frac{M_i}{a_i}\}_{i\in I}]$ generated by $\frac{M_j}{1} \subset A[\{\frac{M_i}{a_i}\}_{i\in I}]$.

Proof. We have a morphism of rings $A[\{\frac{M_i}{a_i}\}_{i\in K}] \to A[\{\frac{M_i}{a_i}\}_{i\in I}]$ given by Fact 2.18. The right-hand side of the equation in the statement of 2.22 is generated as $A[\{\frac{M_i}{a_i}\}_{i\in K}]$-algebra by $\{\frac{A[\{\frac{M_i}{a_i}\}_{i\in I}]M_j}{a_j}\}_{j\in I\setminus K}$. We now define an $A[\{\frac{M_i}{a_i}\}_{i\in K}]$-morphism from the right-hand side to the left-hand side sending $\frac{\frac{m_\nu}{a^\nu}m_j}{a_j^k}$ (with $\nu \in I$, $j \in I \setminus K$ and $k \in \mathbb{N}$) to $\frac{m_\nu m_j}{a^\nu a_j^k}$. This is well-defined and it is easy to check injectivity and surjectivity. □

COROLLARY 2.23. *Let S and S' be the multiplicative monoids in A and $A[\{\frac{M_i}{a_i}\}_{i\in I}]$ given by $\{a^\nu | \nu \in \mathbb{N}_I\}$. Then $S'^{-1}A[\{\frac{M_i}{a_i}\}_{i\in I}] = S^{-1}A$.*

Proof. Using, Fact 2.11, Proposition 2.22 and Corollary 2.20, we get

$$S'^{-1}A' = A[\{\tfrac{M_i}{a_i}\}_{i\in I}][\{\tfrac{A[\{\frac{M_i}{a_i}\}_{i\in I}]}{a_i}\}_{i\in I}] = A[\{\tfrac{M_i}{a_i}\}_{i\in I}, \{\tfrac{A}{a_i}\}_{i\in I}] = A[\{\tfrac{A}{a_i}\}_{i\in I}] = S^{-1}A\ .$$

□

PROPOSITION 2.24. *Assume that $a_i = a_j =: b$ for all $i, j \in I$, then*

$$A[\{\tfrac{M_i}{a_i}\}_{i\in I}] = A[\tfrac{\sum_{i\in I} M_i}{b}].$$

Proof. This follows from the identity $\Big(\sum_{\nu\in\mathbb{N}_I, \sum_{i\in I}\nu_i = n} M^\nu\Big) = (\sum_{i\in I} M_i)^n$. □

Fact 2.25. *Let $\nu \in \mathbb{N}_I$. We have $L^\nu A[\{\tfrac{M_i}{a_i}\}_{i\in I}] = a^\nu A[\{\tfrac{M_i}{a_i}\}_{i\in I}]$.*

Proof. Obviously $a^\nu A[\{\tfrac{M_i}{a_i}\}_{i\in I}] \subset L^\nu A[\{\tfrac{M_i}{a_i}\}_{i\in I}]$. Let $y \in L^\nu$ and $\tfrac{x}{a^\alpha} \in A[\{\tfrac{M_i}{a_i}\}_{i\in I}]$, the formula

$$y\frac{x}{a^\alpha} = a^\nu \frac{yx}{a^{\alpha+\nu}}$$

now shows that $L^\nu A[\{\tfrac{M_i}{a_i}\}_{i\in I}] = a^\nu A[\{\tfrac{M_i}{a_i}\}_{i\in I}]$. □

Proposition 2.26. *(Universal property) If $\chi : A \to B$ is a morphism of rings such that $\chi(a_i)$ is a non-zero-divisor and generates $\chi(L_i)B$ for all $i \in I$, then there exists a unique morphism χ' of A-algebras $A[\{\tfrac{M_i}{a_i}\}_{i\in I}] \to B$. The morphism χ' sends $\tfrac{l}{a^\nu}$ $(\nu \in \mathbb{N}_I, l \in L^\nu)$ to the unique element $b \in B$ such that $\chi(a^\nu)b = \chi(l)$.*

Proof. The element b in the statement is unique because $\chi(a^\nu)$ is a non-zero-divisor for all $\nu \in \mathbb{N}_I$. Clearly, the map χ' defined in the statement is a morphism of A-algebras. Now let ϕ be an other morphism of A-algebras $A[\{\tfrac{L_i}{a_i}\}_{i\in I}] \to B$. We have

$$\chi(a^\nu)\phi(\frac{l}{a^\nu}) = \phi(\frac{l}{a^\nu})\phi(a^\nu) = \phi(l) = \chi(l).$$

This implies $\chi'(\tfrac{l}{a^\nu}) = \phi(\tfrac{l}{a^\nu})$. □

Remark 2.27. The universal property of dilatations generalizes the universal property of localizations. Indeed, let S be a multiplicative subset of A and let $f : A \to B$ be an A-algebra such that $f(s)$ is invertible for any $s \in S$. Recall that by 2.11, we have $S^{-1}A = A[\{\tfrac{A}{s}\}_{s\in S}]$. Then obviously $f(s)$ is a non-zero-divisor and $f(s)$ generate $B = f(A)B$, so by the universal property of dilatations there exists a unique morphism f' of A-algebras $S^{-1}A \to B$. So dilatation is a construction that generalizes localization without the need to know localization. An other less uniform way to introduce dilatations of rings (or rings) is to first treat the case of localizations and then to define a dilatation as in the second assertion of Fact 2.15. At the end both points of view are equivalent by Fact 2.15.

Definition 2.28. The blowup algebra, or the Rees algebra, associated to A and $\{L_i\}_{i\in I}$ is the $\mathbb{N}_I$-graded A-algebra

$$\mathrm{Bl}_{\{L_i\}_{i\in I}}A = \bigoplus_{\nu\in\mathbb{N}_I} L^\nu$$

where the summand L^ν is placed in degree $\nu \in \mathbb{N}_I$.

Let $e_1 = (1, 0, \ldots), e_2 = (0, 1, 0, \ldots), \ldots, e_i = (0, \ldots, 0, 1, 0, \ldots), \ldots$ be the canonical basis of the free $\mathbb{N}$-semimodule $\mathbb{N}_I$. Recall that $a_i \in L_i$ for $i \in I$. Denote $a_{i,i}$ the element a_i seen as an element of degree e_i in the Rees algebra $\mathrm{Bl}_{\{L_i\}_{i\in I}}A$. Let S be the multiplicative subset of $\mathrm{Bl}_{\{L_i\}_{i\in I}}A$ generated by $\{a_{i,i}\}_{i\in I}$. Let $\big(\mathrm{Bl}_{\{L_i\}_{i\in I}}A\big)[S^{-1}]$ be the localization of the multi-Rees algebra relatively to S. This A-algebra inherits a $\mathbb{Z}_I$-grading given, for any $l \in L^\nu$, by

$$\deg(\frac{l}{{a_{1,1}}^{\alpha_1}\ldots{a_{i,i}}^{\alpha_i}}) = \sum_{i\in I}(\nu_i - \alpha_i)e_i \in \mathbb{Z}_I := \bigoplus_{i\in I}\mathbb{Z}.$$

FACT 2.29. *We have a canonical identification of A-algebras*

$$A[\{\tfrac{M_i}{a_i}\}_{i\in I}] = \Big[\big(\mathrm{Bl}_{\{L_i\}_{i\in I}}A\big)[S^{-1}]\Big]_{\deg=(0,\dots,0,\dots)}$$

where the right-hand part is obtained as degree zero elements in $\big(\mathrm{Bl}_{\{L_i\}_{i\in I}}A\big)[S^{-1}]$.

Proof. This is tautological. □

PROPOSITION 2.30. *Assume that $I=\{1,\dots,k\}$ is finite. Then we have a canonical isomorphism of A-algebras*

$$A[\{\tfrac{M_i}{a_i}\}_{i\in I}] \cong A[\tfrac{\sum_{i\in I}(M_i\cdot\prod_{j\in I\setminus\{i\}}a_j)}{a_1\cdots a_k}].$$

Proof. Let us provide a map

$$\phi : A[\tfrac{\sum_{i\in I}(M_i\cdot\prod_{j\in I\setminus\{i\}}a_j)}{a_1\cdots a_k}] \to A[\{\tfrac{M_i}{a_i}\}_{i\in I}].$$

The ring $A[\frac{\sum_{i\in I}(M_i\cdot\prod_{j\in I\setminus\{i\}}a_j)}{a_1\cdots a_k}]$ is generated as A-algebra by $\frac{\sum_{i\in I}(M_i\cdot\prod_{j\in I\setminus\{i\}}a_j)}{a_1\cdots a_k}$, we now define a map ϕ via (for $m_i\in M_i$, $i\in I$):

$$\phi(\frac{\sum_{i\in I}(m_i\cdot\prod_{j\in I\setminus\{i\}}a_j)}{a_1\cdots a_k}) = \frac{\sum_{i\in I}(m_i\cdot\prod_{j\in I\setminus\{i\}}a_j)}{a_1\cdots a_k}.$$

This is well-defined and ϕ is a morphism of A-algebras. It is easy to prove that ϕ is injective and surjective. □

REMARK 2.31. Assume that $A=\mathbb{Z}[X,Y]$. The formal symbol $\frac{X}{3}$ does not make sense in $A[\frac{(2X)+(3Y)}{6}]$. The formal symbol $\frac{2X}{6}$ makes sense and defines an element in $A[\frac{(2X)+(3Y)}{6}]$. The formal symbols $\frac{2X}{6}$ and $\frac{X}{3}$ make sense in $A[\frac{(X)}{3},\frac{(Y)}{2}]$ and define the same element. The canonical isomorphism of Proposition 2.30 sends $\frac{2X}{6}\in A[\frac{(2X)+(3Y)}{6}]$ to $\frac{2X}{6}=\frac{X}{3}\in A[\frac{(X)}{3},\frac{(Y)}{2}]$.

LEMMA 2.32. *Write $I=\mathrm{colim}_{J\subset I}J$ as a filtered colimit of sets. We have a canonical identification of A-algebras*

$$A[\{\tfrac{M_i}{a_i}\}_{i\in I}] = \mathrm{colim}_{J\subset I}A[\{\tfrac{M_j}{a_i}\}_{i\in J}]$$

where the transition maps are given by Fact 2.18.

Proof. For each $J\subset I$, Fact 2.18 gives a canonical morphism $A[\{\frac{M_i}{a_i}\}_{i\in J}]\to A[\{\frac{M_i}{a_i}\}_{i\in I}]$ of A-algebras. These morphisms are compatible with transition maps. So we have a canonical A-morphism

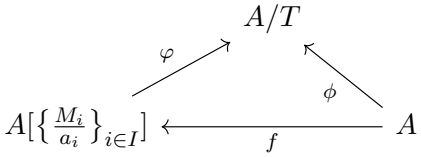
$$\phi : \mathrm{colim}_{J\subset I}A[\{\tfrac{M_i}{a_i}\}_{i\in J}] \to A[\{\tfrac{M_i}{a_i}\}_{i\in I}].$$

The map ϕ is surjective because for any $\nu\in\mathbb{N}_I$, there exists a subset $J\subset I$ such that $\nu\in\mathbb{N}_J$ (recall that $\mathbb{N}_J\subset\mathbb{N}_I$). It is easy to check injectivity. □

2.3 More properties of dilatations of rings

We proceed with the notation from §2.1.

PROPOSITION 2.33. *Let T be an ideal of A. Assume that we have a commutative diagram of A-algebras*



where ϕ is the quotient map. Assume that $\phi(a_i)$ is a non-zero-divisor for all $i \in I$. Then

$$\ker(\varphi) = \sum_{\nu \in \mathbb{N}_I} \frac{L^\nu \cap T}{a^\nu} \subset A[\{\tfrac{M_i}{a_i}\}_{i\in I}].$$

Proof. Let $\nu \in \mathbb{N}_I$ and $\frac{m}{a^\nu} \in A[\{\frac{M_i}{a_i}\}_{i\in I}]$, i.e. $m \in L^\nu$. We have

$$\phi(a^\nu)\varphi(\frac{m}{a^\nu}) = \varphi(\frac{a^\nu}{1})\varphi(\frac{m}{a^\nu}) = \varphi(\frac{m}{1}) = \varphi(f(m)) = \phi(m).$$

Now assume $\varphi(\frac{m}{a^\nu}) = 0$, then $\phi(m) = 0$ and so $m \in L^\nu \cap T$. This shows that $\ker(\varphi) \subset \sum_{\nu\in\mathbb{N}_I} \frac{L^\nu\cap T}{a^\nu}$. Reciprocally assume $m \in L^\nu \cap T$. Then $\phi(m) = 0$. This implies $\varphi(\frac{m}{a^\nu}) = 0$ because $\phi(a^\nu)$ is a non-zero-divisor by assumption. □

COROLLARY 2.34. *We proceed with the notation from Proposition 2.33 and assume that $I = \{i\}$, $a_i = b_i^k$ for some $k \in \mathbb{N}, b_i \in A$ and that $T = M_i$. Then $\ker(\varphi)$ is the ideal of $A[\frac{M_i}{a_i}]$ generated by $\frac{M_i}{a_i}$, moreover $A[\frac{M_i}{b_i^k}][\frac{\ker(\varphi)}{b_i^d}] = A[\frac{M_i}{b_i^{k+d}}]$ for any $d \in \mathbb{N}$.*

Proof. Clearly, $\frac{M_i}{a_i} \subset \ker(\varphi)$, so it is enough to prove that $\ker(\varphi)$ is included in the ideal generated by $\frac{M_i}{a_i}$ (that we denote in this proof by $\langle\frac{M_i}{a_i}\rangle$). So let $n \in \mathbb{N}$, we have to prove that $\frac{L_i^n\cap M_i}{a_i^n}$ is included in $\langle\frac{M_i}{a_i}\rangle$. An element $x \in L_i^n$ can be written as a sum $x = \sum_{k=0}^n m_k a_i^{n-k}$ with $m_k \in M_i^k$ (note that, if x belongs to $L_i^n \cap M_i$, then $m_0 a_i^n$ also belongs to M_i). Now we assume that x belongs to $L_i^n \cap M_i$, it is clear that for $k > 0$ the element $\frac{m_k a_i^{n-k}}{a_i^n} = \frac{m_k}{a_i^k}$ belongs to $\langle\frac{M_i}{a_i}\rangle$. Now for $k = 0$, using that a_i is a non-zero-divisor in A/M_i and that $m_0 a_i^n$ belongs to M_i, we get that m_0 belongs to M_i and it is now clear that $\frac{m_0 a_i^n}{a_i^n}$ belongs to $\langle\frac{M_i}{a_i}\rangle$. So x belongs to $\langle\frac{M_i}{a_i}\rangle$.

Now we deduce the equality $A[\frac{M_i}{b_i^k}][\frac{\ker(\varphi)}{b_i^d}] = A[\frac{M_i}{b_i^{k+d}}]$ and finish the proof:

$$\begin{aligned} A[\frac{M_i}{b_i^k}][\frac{\ker(\varphi)}{b_i^d}] &= A[\frac{M_i}{b_i^k}][\frac{\frac{M_i}{b_i^k}A[\frac{M_i}{b_i^k}]}{b_i^d}] \\ &= A[\frac{M_i}{b_i^k}][\frac{M_i A[\frac{M_i}{b_i^k}]}{b_i^{k+d}}] \\ \text{by Proposition 2.22} &= A[\frac{M_i}{b_i^k}, \frac{M_i}{b_i^{k+d}}] \\ \text{by Corollary 2.21} &= A[\frac{M_i}{b_i^{k+d}}]. \end{aligned}$$

□

PROPOSITION 2.35. *Let $f : A \to B$ be an A-ring. Put $N_i = f(M_i)B$ and $b_i = f(a_i)$ for $i \in I$. Then $B[\{\frac{N_i}{b_i}\}_{i\in I}]$ is the quotient of $B \otimes_A A[\{\frac{M_i}{a_i}\}_{i\in I}]$ by the ideal T_b of elements annihilated by some element in $b^{\mathbb{N}_I} := \{b^\nu | \nu \in \mathbb{N}_I\}$.*

Proof. We can assume that $a_i \in M_i$ and $b_i \in N_i$. Let B' be the quotient of $B \otimes_A A[\{\frac{M_i}{a_i}\}_{i\in I}]$ by T_b. The ring map

$$B \otimes_A A[\{\tfrac{M_i}{a_i}\}_{i\in I}] \to B[\{\tfrac{N_i}{b_i}\}_{i\in I}]$$

is surjective and annihilates $a^{\mathbb{N}_I}$-torsion as elements in $b^{\mathbb{N}_I}$ are non-zero-divisors in $B[\{\frac{N_i}{b_i}\}_{i\in I}]$. Hence we obtain a surjective map $B' \to B[\{\frac{N_i}{b_i}\}_{i\in I}]$. To see that the kernel is trivial, we construct an inverse map. Namely, let $z = \frac{y}{b^\nu}$ be an element of $B[\{\frac{N_i}{b_i}\}_{i\in I}]$, i.e $y \in N^\nu$ for some $\nu \in \mathbb{N}_I$. Write $y = \sum x_i s_i$ with $x_i \in M^\nu$ and $s_i \in B$. We map z to the class of $\sum s_i \otimes \frac{x_i}{a^\nu}$ in B'. This is well defined because we claim that an element of the kernel of the map $B \otimes_A M^\nu \to N^\nu$ is annihilated by a^ν hence maps to zero in B'. We now prove the claim of the previous assertion. Let $\sum_j (s_j \otimes m_j)$ be in the kernel of the map as before ($s_j \in B, m_j \in M^\nu$ for all j), so that $\sum_j s_j f(m_j) = 0$; we have $a^\nu \sum_j (s_j \otimes m_j) = \sum_j (s_j \otimes m_j a^\nu) = \sum_j (s_j f(m_j) \otimes a^\nu) = (\sum_j s_j f(m_j)) \otimes a^\nu = 0$. This finishes the proof. □

Corollary 2.36. *We proceed with the notation from Proposition 2.35 and assume $f : A \to B$ is flat. Then $T_b = 0$, in other words we have a canonical isomorphism*

$$B[\{\tfrac{N_i}{b_i}\}_{i\in I}] = B \otimes_A A[\{\tfrac{M_i}{a_i}\}_{i\in I}].$$

Proof. Since f is flat, the map $\phi : A[\{\frac{M_i}{a_i}\}_{i\in I}] \to B \otimes_A A[\{\frac{M_i}{a_i}\}_{i\in I}]$ is flat. Since ϕ is flat, the image of any non-zero-divisor element in $A[\{\frac{M_i}{a_i}\}_{i\in I}]$ under the map ϕ is a non-zero-divisor. So $T_b = 0$ and Proposition 2.35 finishes the proof. □

Definition 2.37. Assume that a_i is a non-zero divisor for all $i \in I$. Let J_i be the ideal of A generated by a_i, it is invertible. We consider the A-algebra

$$C_L^J A \overset{\text{def}}{=} \bigoplus_{\nu\in\mathbb{N}_I} L^\nu \otimes J^{-\nu},$$

and we call it the associated conic algebra.

Proposition 2.38. *We proceed with the notation from Definition 2.37. Let ζ be the ideal of $C_L^J A$ generated by elements $\rho^\alpha - 1$ for $\alpha \in \mathbb{N}_I$ where $\rho_i \in C_L^J A$ is the image of $1 \in A$ under $A \cong J_i \otimes J_i^{-1} \subset L_i \otimes J_i^{-1} \subset C_L^J A$ for any $i \in I$. We have a canonical isomorphism of A-algebras*

$$(C_L^J A)/\zeta \longrightarrow A[\{\tfrac{L_i}{a_i}\}_{i\in I}].$$

Proof. Let $t_i = a_i^\vee$ be the generator for J_i^{-1}, dual to a_i for all $i \in \mathbb{N}_I$. We have a natural morphism of rings given explicitly by

$$\psi : C_L^J A \longrightarrow A[\{\frac{L_i}{a_i}\}_{i\in I}], \sum\nolimits_{\nu\in\mathbb{N}_I} l_\nu \otimes t^\nu \mapsto \sum\nolimits_{\nu\in\mathbb{N}_I} \frac{l_\nu}{a^\nu}.$$

The morphism ψ is surjective and $\zeta \subset \ker\psi$. It is enough to prove that $\ker\psi \subset \zeta$. Let $X = \sum_{\nu\in c} l_\nu \otimes t^\nu \in C_L^J A$ where $l_\nu \in L^\nu$, and c is a finite subset of $\mathbb{N}_I$. Let $\beta \in \mathbb{N}_I$ defined by $\beta_i = \max_{\nu\in c}\nu_i$ for all $i \in I$. Then we have

$$\psi(X) = \sum_{\nu\in c} \frac{l_\nu}{a^\nu} = \frac{\sum_{\nu\in c} a^{\beta-\nu} l^\nu}{a^\beta}.$$

Assume $X \in \ker\psi$, then $\sum_{\nu\in c} a^{\beta-\nu} l^\nu = 0$ because a_i are non-zero-divisors $i \in I$. So we are allowed to write

$$X = \big(\sum_{\nu\in c} l_\nu \otimes t^\nu\big) - \big(\sum_{\nu\in c} a^{\beta-\nu} l^\nu\big) \otimes t^\beta = \sum_{\nu\in c} \Big[\big(l_\nu \otimes t^\nu\big)\big(1 - (a^{\beta-\nu} \otimes t^{\beta-\nu})\big)\Big].$$

This finishes the proof since $a^{\beta-\nu}\otimes t^{\beta-\nu}=\rho_1^{\beta_1-\nu_1}\cdots\rho_i^{\beta_i-\nu_i}\cdots$. □

REMARK 2.39. We note that the ideal ζ appearing in Proposition 2.38 is in fact generated by $\{\rho_i-1\}_{i\in I}$. To see this, use for example that in any ring and for any elements ρ,σ in the ring we have $\rho^n-1=(\rho-1)(\rho^{n-1}+\ldots+\rho+1)$ and $\rho\sigma-1=(\rho-1)(\sigma+1)+(\sigma-1)-(\rho-1)$.

FACT 2.40. *Let R be a ring and assume that $f:R\to A$ is a morphism of rings. Let $\{r_i\}_{i\in I}$ be elements in R and assume that $a_i=f(r_i)$ for all $i\in I$. Let $R_r=R[\{\frac{R}{r_i}\}_{i\in I}]$ be the localization of R at $\{r_i\}_{i\in I}$. For any $i\in I$, let $M_i\otimes r_i^{-1}\subset A\otimes_R R_r$. Then $A[\{\frac{M_i}{a_i}\}_{i\in I}]$ identifies with the A-subalgebra of $A\otimes_R R_r$ generated by $\{M_i\otimes r_i^{-1}\}_{i\in I}$ and A.*

Proof. By [StP, Tag 00DK] and Fact 2.11, we get $A\otimes_R R_r=(A\otimes_R R)_r=A[\{\frac{A}{a_i}\}_{i\in I}]$. Moreover $M_i\otimes r_i^{-1}\subset A\otimes_R R_r$ corresponds to $\frac{M_i}{a_i}\subset A[\{\frac{A}{a_i}\}_{i\in I}]$. Now 2.6 and 2.14 finish the proof. □

REMARK 2.41. We discussed before that dilatations of rings provide a formalism unifying localizations of rings and affine blowups of rings (recall that affine blowups are studied in [StP, Tag 052P]). It is easy to check on examples that dilatations strictly generalize localizations and affine blowups. In fact such examples already appeared in literature in the framework of flat group schemes of finite type over discrete valuation rings, cf. e.g. [SGA3, Exp. VIB Ex. 13.3], [PY06, §7, after Proposition 7.3], [DHdS18, Definitions 5.1 and 5.5].

2.4 Dilatations of modules

Recall that dilatations of rings generalize localizations of rings. Recall also that localizations of A-modules make sense. In this remark we explain that dilatations of A-modules also make sense. Let M be an A-module. The dilatation of M with multi-center $\{[M_i,a_i]\}_{i\in I}$ is the $A[\{\frac{M_i}{a_i}\}_{i\in I}]$-module $\mathrm{M}[\{\frac{M_i}{a_i}\}_{i\in I}]$ defined as follows:

- The underlying set of $\mathrm{M}[\{\frac{M_i}{a_i}\}_{i\in I}]$ is the set of equivalence classes of symbols $\frac{lm}{a^\nu}$ where $\nu\in\mathbb{N}_I$, $m\in\mathrm{M}$ and $l\in L^\nu$ under the equivalence relation

$$\frac{lm}{a^\nu}\equiv\frac{hp}{a^\lambda}\Leftrightarrow\exists\beta\in\mathbb{N}_I\text{ such that }lma^{\beta+\lambda}=hpa^{\beta+\nu}\text{ in }A.$$

From now on, we abuse notation and denote a class by any of its representative $\frac{lm}{a^\nu}$ if no confusion is likely.

- The addition law is given by $\frac{lm}{a^\nu}+\frac{hp}{a^\beta}=\frac{lma^\beta+hpa^\nu}{a^{\beta+\nu}}$.
- The action law is given by $\frac{l}{a^\nu}\frac{hp}{a^\beta}=\frac{lhp}{a^{\nu+\beta}}$.
- The additive neutral element is $\frac{0}{1}$.

We have a canonical morphism of A-modules from M to $\mathrm{M}[\{\frac{M_i}{a_i}\}_{i\in I}]$ given by $m\mapsto\frac{m}{1}$. We now put $\mathrm{M}'=\mathrm{M}[\{\frac{M_i}{a_i}\}_{i\in I}]$.

PROPOSITION 2.42. *Let $\nu\in\mathbb{N}_I$.*

(i) *Let $m\in\mathrm{M}'$. If $a^\nu m=0$, then $m=0$.*

(ii) *We have $a^\nu\mathrm{M}'=L^\nu\mathrm{M}'$.*

Proof. (i) Write $m=\frac{hp}{a^\beta}$. There exists γ such that $a^\gamma a^\nu hp=0$ in M, so $m=0$ in M'.

(ii) It is enough to prove that $a^\nu\mathrm{M}'\supset L^\nu\mathrm{M}'$. This follows from the identity, for $l\in L^\nu$:

$$l\frac{hp}{a^\beta}=a^\nu\frac{lhp}{a^{\beta+\nu}}.$$

□

PROPOSITION 2.43. *(Universal Property) Let A-$Mod_{\mathrm{M}}^{a\text{-}reg}$ be the category whose objects are morphisms of A-modules $F : \mathrm{M} \to \mathrm{M}'$ with source M such that a_i is a non-zero-divisor of M' for all $i \in I$, then*

$$\mathrm{Hom}_{A\text{-}Mod_{\mathrm{M}}^{a\text{-}reg}}(\mathrm{M}[\{\frac{M_i}{a_i}\}_{i\in I}], \mathrm{M}') = \begin{cases} \{*\}, \text{ if } L_i\mathrm{M}' = a_i\mathrm{M}' \text{ for } i \in I; \\ \varnothing, \text{ else.} \end{cases}$$

Dilatations of modules enjoy similar properties than dilatations of rings, we do not list all of them here.

REMARK 2.44. In general $\mathrm{M}[\{\frac{M_i}{a_i}\}_{i\in I}]$ is not equal to $\mathrm{M} \otimes_A A[\{\frac{M_i}{a_i}\}_{i\in I}]$. Indeed, let $A = \mathbb{Z}[X]$, $I = \{0\}$, $M_0 = (X)$, $a_0 = 2$ and $\mathrm{M} = A[\frac{M_0}{2}]$. Note that $A[\frac{M_0}{2}] = \mathbb{Z}[\frac{X}{2}] \subset \mathbb{Q}[X]$. Then $\mathrm{M}[\frac{M_0}{2}] \cong \mathrm{M}$, however $\mathrm{M} \otimes_A A[\frac{M_0}{2}] \ncong \mathrm{M}$. Indeed the element $0 \neq T := (\frac{X}{2} \otimes 1) + (-1 \otimes \frac{X}{2}) \in A[\frac{M_0}{2}] \otimes_A A[\frac{M_0}{2}]$ satisfies $2T = 0$ whereas M is 2-torsion free. This contrasts with the case of localizations where we always have $\mathrm{M} \otimes_A S^{-1}A \cong S^{-1}\mathrm{M}$, cf. e.g. [StP, Tag 00DK].

REMARK 2.45. Let $0 \to \mathrm{M}_1 \to \mathrm{M}_2 \to \mathrm{M}_3 \to 0$ be an exact sequence of A-modules. Then in general $\mathrm{M}_1[\{\frac{M_i}{a_i}\}_{i\in I}] \to \mathrm{M}_2[\{\frac{M_i}{a_i}\}_{i\in I}] \to \mathrm{M}_3[\{\frac{M_i}{a_i}\}_{i\in I}]$ is not exact. For example, Take $A = \mathbb{Z}[X]$, $I = \{0\}$, $M_0 = (X)$, $a_0 = 2$ and consider the exact sequence $0 \to \mathbb{Z}[X] \xrightarrow{m\mapsto 2m} \mathbb{Z}[X] \to \mathbb{Z}/2\mathbb{Z}[X] \to 0$. Then $\mathbb{Z}/2\mathbb{Z}[X][\frac{X}{2}] = 0$, however $\mathbb{Z}[\frac{X}{2}] \xrightarrow{m\mapsto 2m} \mathbb{Z}[\frac{X}{2}]$ is not surjective. This contrasts with the case of localizations where we always have preservation of exact sequences, cf. e.g. [StP, Tag 00CS].

3. Multi-centered dilatations in the absolute setting

In this section, we define multi-centered dilatations and prove some properties.

3.1 Definitions

Let S be a scheme. An S-space is an S-algebraic space. Let us fix an S-space X. For the convenience of the reader, we recall some basic notations and well-known facts.

NOTATION 3.1. *Let $Clo(X)$ be the set of closed S-subspaces of X. Recall that $Clo(X)$ corresponds to quasi-coherent ideals of $\mathcal{O}_X$ via [StP, Tag 03MB]. Let $IQCoh(\mathcal{O}_X)$ denote the set of quasi-coherent ideals of $\mathcal{O}_X$. It is clear that $(IQCoh(\mathcal{O}_X), +, \times, 0, \mathcal{O}_X)$ is a semiring. So we obtain a semiring structure on $Clo(X)$, usually denoted by $(Clo(X), \cap, +, X, \emptyset)$. For clarity, we now recall directly operations on $Clo(X)$. Given two closed subspaces Y_1, Y_2 given by ideals $\mathcal{J}_1, \mathcal{J}_2$, their sum $Y_1 + Y_2$ is defined as the closed subspace given by the ideal $\mathcal{J}_1\mathcal{J}_2$. Moreover, if $n \in \mathbb{N}$, we denote by nY_1 the n-th multiple of Y_1. The set of locally principal closed subspaces of X (cf. [StP, Tag 083B]), denoted $Pri(X)$, forms a submonoid of $(Clo(X), +)$. Effective Cartier divisors of X, denoted $Car(X)$, form a submonoid of $(Pri(X), +)$. Note that $Car(X)$ is a face of $Pri(X)$. We have an other monoid structure on $Clo(X)$ given by intersection, this law is denoted $\cap$. The operation $\cap$ corresponds to the sum of quasi-coherent sheaves of ideals . The set $Clo(X)$ endowed with $\cap, +$ is a semiring whose neutral element for $+$ is $\emptyset$ and whose neutral element for $\cap$ is X. Let $C \in Car(X)$, a non-zero-divisor (for $+$) in the semiring $Clo(X)$. Let $Y, Y' \in Clo(X)$. If $C + Y$ is a closed subspace of $C + Y'$, then Y is a closed subspace of Y'.*

Moreover if $C + Y = C + Y'$, then $Y = Y'$. Let $f : X' \to X$ be a morphism of S-spaces, then f induces a morphism of semirings $Clo(f) : Clo(X) \to Clo(X'), Y \mapsto Y \times_X X'$, moreover $Clo(f)$ restricted to $(Pri(X), +)$ factors through $(Pri(X'), +)$, this morphism of monoids is denoted $Pri(f)$. Let $Y_1, Y_2 \in Clo(X)$, we write $Y_1 \subset Y_2$ if Y_1 is a closed subspace of Y_2. We obtain a poset $(Clo(X), \subset)$. Let $Y_1, Y_2, Y_3 \in Clo(X)$, if $Y_1 \subset Y_2$ and $Y_1 \subset Y_3$ then $Y_1 \subset Y_2 \cap Y_3$. Let $Y_1, Y_2 \in Clo(X)$, then $(Y_1 \cap Y_2) \subset Y_1$ and $Y_1 \subset (Y_1 + Y_2)$. Finally, if $Y = \{Y_e\}_{e \in E}$ is a subset of $Clo(X)$ and if $\nu \in \mathbb{N}^E$, we put $Y^\nu = \{\nu_e Y_e\}_{e \in E}$ and if moreover $\nu \in \mathbb{N}_E$, we put $\nu Y = \sum_{e \in E} \nu_e Y_e$.

REMARK 3.2. Be careful that the operation $+$ on $Clo(X)$ is not the operation $\cup$ of [StP, Tag 0C4H]. Recall that $+$ corresponds to multiplication of ideals whereas $\cup$ corresponds to intersection of ideals.

REMARK 3.3. We proceed with the notation from Notation 3.1. In general the image of the map $Pri(f)|_{Car(X)}$ is not included in $Car(X')$.

DEFINITION 3.4. Let $D = \{D_i\}_{i \in I}$ be a subset of $Clo(X)$.

(i) Let $Spaces_X^{D\text{-reg}}$ be the category of S-algebraic spaces $f : T \to X$ over X such that for any $i \in I$, $T \times_X D_i$ is a Cartier divisor in T.

(ii) If $X = S$ is a scheme, let $Sch_X^{D\text{-reg}}$ be the category of X-schemes $f : T \to X$ such that for any $i \in I$, $T \times_X D_i$ is a Cartier divisor in T.

If $T' \to T$ is flat and $T \to X$ is an object in $Spaces_X^{D\text{-reg}}$ or $Sch_X^{D\text{-reg}}$, so is the composition $T' \to T \to X$ by [StP, Tag 083Z] and [StP, Tag 02OO]. In particular, the categories $Spaces_X^{D\text{-reg}}$ and $Sch_X^{D\text{-reg}}$ can be equipped with the fpqc/fppf/étale/Zariski Grothendieck topology so that the notion of sheaves is well-defined.

FACT 3.5. *Let $D = \{D_i\}_{i \in I}$ be a subset of $Clo(X)$.*

(i) *Let $f : T \to X$ be an object in $Spaces_X^{D\text{-reg}}$. Then for any $\nu \in \mathbb{N}_I$, the space $T \times_X \nu D$ is a Cartier divisor in T, namely $\nu(T \times_X D)$.*

(ii) *Assume that $\#I$ is finite, then $Spaces_X^{D\text{-reg}}$ equals $Spaces_X^{\sum_{i \in I} D_i}$.*

Proof. (i) This follows from the fact that $(Clo(f), +)$ is a morphism of monoids and the fact that $Car(X)$ is a submonoid of $Clo(X)$ (e.g. cf. the discussion in Notation 3.1).

(ii) This follows from the fact that $Car(X)$ is a face of the monoid $Pri(X)$ (e.g. cf. the discussion in Notation 3.1).

□

DEFINITION 3.6. A multi-center in X is a set $\{[Y_i, D_i]\}_{i \in I}$ such that

(i) Y_i and D_i belong to $Clo(X)$,

(ii) there exists an affine étale covering $\{U_\gamma \to X\}_{\gamma \in \Gamma}$ of X such that $D_i|_{U_\gamma}$ is principal for all $i \in I$ and $\gamma \in \Gamma$ (in particular D_i belongs to $Pri(X)$ for all i).

In other words a multi-center $\{[Y_i, D_i]\}_{i \in I}$ is a set of pairs of closed S-spaces such that locally each D_i is principal.

REMARK 3.7. Let $\{Y_i, D_i\}_{i \in I}$ such that $Y_i \in Clo(X)$ and $D_i \in Pri(X)$ for any $i \in I$. Assume that I is finite, then $\{[Y_i, D_i]\}_{i \in I}$ is a multi-center in X, i.e. the second condition in Definition 3.6 is satisfied.

We now fix a multi-center $\{[Y_i, D_i]\}_{i\in I}$ in X. Denote by $\mathcal{M}_i, \mathcal{J}_i$ the quasi-coherent sheaves of ideals in $\mathcal{O}_X$ so that $Y_i = V(\mathcal{M}_i), V(\mathcal{J}_i) = D_i$. We put $Z_i = Y_i \cap D_i$ and $\mathcal{L}_i = \mathcal{M}_i + \mathcal{J}_i$ so that $Z_i = V(\mathcal{L}_i)$ for any $i \in I$. We put $Y = \{Y_i\}_{i\in I}$, $D = \{D_i\}_{i\in I}$ and $Z = \{Z_i\}_{i\in I}$. We now introduce dilatations $\mathcal{O}_X$-algebras by glueing (cf. [StP, Tag 04TP]).

Definition and Proposition 3.8. *The dilatation of $\mathcal{O}_X$ with multi-center $\{[\mathcal{M}_i, \mathcal{J}_i]\}_{i\in I}$ is the quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{O}_X\Big[\Big\{\frac{\mathcal{M}_i}{\mathcal{J}_i}\Big\}_{i\in I}\Big]$ obtained by glueing as follows. The quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{O}_X\Big[\Big\{\frac{\mathcal{M}_i}{\mathcal{J}_i}\Big\}_{i\in I}\Big]$ is characterized by the fact that its restriction, on any étale S-morphism $\varphi : U \to X$ such that U is an affine scheme and each D_i is principal on U and generated by a_{iU}, is given by*

$$\Big(\mathcal{O}_X\Big[\Big\{\frac{\mathcal{M}_i}{\mathcal{J}_i}\Big\}_{i\in I}\Big]\Big)\Big|_U = \Gamma(U, \mathcal{O}_X)\Big[\Big\{\frac{\widetilde{\Gamma(U, \mathcal{M}_i)}}{a_{iU}}\Big\}_{i\in I}\Big]$$

where $\widetilde{\ }$ is given by [StP, Tag 01I7] and [StP, Tag 03DT] (we work with small étale sites).

Proof. By Definition 3.6, the affine schemes $U \to X$ satisfying the conditions in the statement form an étale covering of X. Now Proposition 3.8 follows from [StP, Tag 03M0] and Corollary 2.36. □

Let $\mathrm{Bl}_{\{\mathcal{L}_i\}_{i\in I}}\mathcal{O}_X = \bigoplus_{\nu\in\mathbb{N}_I}\mathcal{L}^\nu$ denote the multi-Rees algebra, it is a quasi-coherent $\mathbb{N}_I$-graded $\mathcal{O}_X$-algebra. By localization, we get a quasi-coherent $\mathcal{O}_X$-algebra $\big(\mathrm{Bl}_{\{\mathcal{L}_i\}_{i\in I}}\mathcal{O}_X\big)[\{\mathcal{J}_i^{-1}\}_{i\in I}]$ (locally, we invert a generator of $\mathcal{J}_i$, for each $i \in I$). This $\mathcal{O}_X$-algebra inherits a grading giving local generators of $\mathcal{J}_i$ degree e_i.

Fact 3.9. *We have a canonical identification of quasi-coherent $\mathcal{O}_X$-algebras*

$$\mathcal{O}_X\Big[\Big\{\frac{\mathcal{M}_i}{\mathcal{J}_i}\Big\}_{i\in I}\Big] = \Big[\big(\mathrm{Bl}_{\{\mathcal{L}_i\}_{i\in I}}\mathcal{O}_X\big)[\{\mathcal{J}_i^{-1}\}_{i\in I}]\Big]_{\deg=(0,\ldots,0,\ldots)},$$

where the right-hand side is obtained as the subsheaf of degree zero elements in $\big(\mathrm{Bl}_{\{\mathcal{L}_i\}_{i\in I}}\mathcal{O}_X\big)[\{\mathcal{J}_i^{-1}\}_{i\in I}]$. In particular $\mathcal{O}_X\Big[\Big\{\frac{\mathcal{M}_i}{\mathcal{J}_i}\Big\}_{i\in I}\Big] = \mathcal{O}_X\Big[\Big\{\frac{\mathcal{L}_i}{\mathcal{J}_i}\Big\}_{i\in I}\Big]$.

Proof. This follows from Fact 2.29. □

Definition 3.10. The dilatation of X with multi-center $\{[Y_i, D_i]\}_{i\in I}$ is the X-affine algebraic space over S

$$\mathrm{Bl}^D_Y X \stackrel{\text{def}}{=} \mathrm{Spec}_X\big(\mathcal{O}_X\Big[\Big\{\frac{\mathcal{M}_i}{\mathcal{J}_i}\Big\}_{i\in I}\Big]\big).$$

Remark 3.11. Fact 3.9 implies that $\mathrm{Bl}^D_Y X = \mathrm{Bl}^D_Z X$.

Fact 3.12. *Assume that $X = S$ is a scheme. Then $\mathrm{Bl}^D_Y X$ is a scheme.*

Proof. We have an affine morphism $\mathrm{Bl}^D_Y X \to X$, now the fact follows from [StP, Tag 03WG]. □

Notation 3.13. *We will also use the notation $\mathrm{Bl}^{\{D_i\}_{i\in I}}_{\{Y_i\}_{i\in I}} X$ and $\mathrm{Bl}^{\{D_i\}_{i\in I}}_{\{Y_i\}_{i\in I}} X$ to denote $\mathrm{Bl}^D_Y X$. If $I = \{i\}$ is a singleton we also use the notation $\mathrm{Bl}^{D_i}_{Y_i} X$. If $K \subset I$, we sometimes use the notation $\mathrm{Bl}^{\{D_i\}_{i\in K},\{D_i\}_{i\in I\setminus K}}_{\{Y_i\}_{i\in K},\{Y_i\}_{i\in I\setminus K}} X$. If $I = \{1, \ldots, k\}$, we use the notation $\mathrm{Bl}^{D_1,\ldots,D_k}_{Y_1,\ldots,Y_k} X$. Etc.*

Definition 3.14. Let X be a scheme or an algebraic space over a scheme S. We say that a morphism $f : X' \to X$ is a dilatation morphism if f is equal to $\mathrm{Bl}^{\{D_i\}_{i\in I}}_{\{Y_i\}_{i\in I}} X \to X$ for some multi-center $\{[Y_i, D_i]\}_{i\in I}$. The terminologies affine blowups and affine modifications are also used.

FACT 3.15. *If $Y_i = \emptyset$ is the empty closed subscheme defined by the ideal $\mathcal{O}_X$ for all $i \in I$, then we say that $\mathrm{Bl}^{\{D_i\}}_{\{\emptyset\}_{i\in I}} X \to X$ is a localization. Moreover if $\#I$ is finite, the dilatation morphism $\mathrm{Bl}^{\{D_i\}}_{\{\emptyset\}_{i\in I}} X \to X$ is an open immersion.*

Proof. This is local on X [StP, Tag 03M4] and follows from Fact 2.11. □

3.2 Exceptional divisors

We proceed with the notation from § 3.1.

PROPOSITION 3.16. *As closed subspaces of $\mathrm{Bl}^D_Y X$, one has, for all $\nu \in \mathbb{N}_I$,*

$$\mathrm{Bl}^D_Y X \times_X \nu Z = \mathrm{Bl}^D_Y X \times_X \nu D,$$

which is an effective Cartier divisor on $\mathrm{Bl}^D_Y X$.

Proof. Our claim is étale local on X. We reduce to the affine case and apply 2.10 and 2.25. □

3.3 Universal property

We proceed with the notation from § 3.1. As $\mathrm{Bl}^D_Y X \to X$ defines an object in $Spaces^{D\text{-reg}}_X$ by Proposition 3.16, the contravariant functor

$$Spaces^{D\text{-reg}}_X \to Set, \quad (T \to X) \mapsto \mathrm{Hom}_{X\text{-Spaces}}\big(T, \mathrm{Bl}^D_Y X\big) \tag{3.1}$$

together with $\mathrm{id}_{\mathrm{Bl}^D_Y X}$ determines $\mathrm{Bl}^D_Y X \to X$ uniquely up to unique isomorphism. The next proposition gives the universal property of dilatations.

PROPOSITION 3.17. *The dilatation $\mathrm{Bl}^D_Y X \to X$ represents the contravariant functor $Spaces^{D\text{-reg}}_X \to Set$ given by*

$$(f : T \to X) \longmapsto \begin{cases} \{*\}, \text{ if } f|_{T\times_X D_i} \text{ factors through } Y_i \subset X \text{ for } i \in I; \\ \varnothing, \text{ else.} \end{cases} \tag{3.2}$$

Proof. Note that the condition $f|_{T\times_X D_i}$ factors through $Y_i \subset X$ is equivalent to the condition $f|_{T\times_X D_i}$ factors through $Z_i \subset X$, because $Z_i = Y_i \cap D_i$. Let F be the functor defined by (3.2). If $T \to \mathrm{Bl}^D_Y X$ is a map of X-spaces, then the structure map $T \to X$ restricted to $T \times_X D_i$ factors through $Z_i \subset X$ by Proposition 3.16. This defines a map

$$\mathrm{Hom}_{X\text{-Spaces}}\big(\text{-}, \mathrm{Bl}^D_Y X\big) \longrightarrow F \tag{3.3}$$

of contravariant functors $Spaces^{D\text{-reg}}_X \to Set$. We want to show that (3.3) is bijective when evaluated at an object $T \to X$ in $Spaces^{D\text{-reg}}_X$. As (3.3) is a morphism of étale sheaves, we reduce to the case where both X and T are affine and J_i is principal for all $i \in I$. Now Proposition 2.26 finishes the proof. □

PROPOSITION 3.18. *Put $f : \mathrm{Bl}^D_Y X \to X$. Then the morphism of monoids $Clo(f)|_{Car(X)}$ factors through $Car(\mathrm{Bl}^D_Y X)$. In other words, any effective Cartier divisor $C \subset X$ is defined for f, i.e. the fiber product $C \times_X \mathrm{Bl}^D_Y X \subset \mathrm{Bl}^D_Y X$ is an effective cartier divisor (cf. [StP, Tag 01WV]).*

Proof. We reduce to the case where $X = \mathrm{Spec}(A)$ is affine and apply Fact 3.18. □

PROPOSITION 3.19. *Let J be a subset of I. There exists a unique X-morphism*

$$\varphi : \mathrm{Bl}^{\{D_j\}_{j\in I}}_{\{Y_j\}_{j\in I}} X \to \mathrm{Bl}^{\{D_j\}_{j\in J}}_{\{Y_j\}_{j\in J}} X.$$

Proof. This follows from Propositions 3.17 and 3.16. □

Remark 3.20. Proposition 3.19 is the spaces version of Fact 2.18.

Remark 3.21. Proposition 3.28 will refine Proposition 3.19 and show that φ is in fact a dilatation map (cf. Definition 3.14).

Proposition 3.22. *Let $K \subset I$ and assume $Z_i = D_i$ is a Cartier divisor in X, for all $i \in K$. Then*

$$\mathrm{Bl}^{\{D_j\}_{j\in I}}_{\{Z_j\}_{j\in I}} X = \mathrm{Bl}^{\{D_j\}_{j\in I\setminus K}}_{\{Z_j\}_{j\in I\setminus K}} X.$$

Proof. Both sides belong to $\mathit{Spaces}_X^{D\text{-reg}}$ by Propositions 3.18 and 3.16, so it is enough to show that they agree when evaluated at any $f : T \to X \in \mathit{Spaces}_X^{D\text{-reg}}$. We have

$$\begin{aligned}
\mathrm{Bl}^{\{D_j\}_{j\in I}}_{\{Z_j\}_{j\in I}} X(T) &= \begin{cases} \{*\}, \text{ if } f|_{T\times_X D_i} \text{ factors through } Z_i \subset X \text{ for } i \in I; \\ \varnothing, \text{ else.} \end{cases} \\
&= \begin{cases} \{*\}, \text{ if } f|_{T\times_X D_i} \text{ factors through } Z_i \subset X \text{ for } i \in I\setminus K; \\ \varnothing, \text{ else.} \end{cases} \\
&= \mathrm{Bl}^{\{D_j\}_{j\in I\setminus K}}_{\{Z_j\}_{j\in I\setminus K}} X(T).
\end{aligned}$$

□

Remark 3.23. Proposition 3.22 is the spaces version of Proposition 2.18.

Proposition 3.24. *Assume $D_i = D_j =: D$ for all $i, j \in I$. Then*

$$\mathrm{Bl}^{\{D\}}_{\{Y_i\}_{i\in I}} X = \mathrm{Bl}^{D}_{\cap_{i\in I} Y_i} X.$$

Proof. Both sides belong to $\mathit{Spaces}_X^{D\text{-reg}}$ by Proposition 3.16, so it is enough to show that they agree when evaluated at any $f : T \to X \in \mathit{Spaces}_X^{D\text{-reg}}$. We have

$$\begin{aligned}
\mathrm{Bl}^{\{D\}}_{\{Y_i\}_{i\in I}} X(T) &= \begin{cases} \{*\}, \text{ if } f|_{T\times_X D} \text{ factors through } Z_i \subset X \text{ for } i \in I; \\ \varnothing, \text{ else.} \end{cases} \\
&= \begin{cases} \{*\}, \text{ if } f|_{T\times_X D} \text{ factors through } \cap_{i\in I} Z_i \subset X \text{ ;} \\ \varnothing, \text{ else.} \end{cases} \\
&= \mathrm{Bl}^{D}_{\cap_{i\in I} Y_i} X(T).
\end{aligned}$$

□

Remark 3.25. Proposition 3.24 is the spaces version of Proposition 2.24.

Fact 3.26. *Let $I = \coprod_{j\in J} I_j$ be a partition of I. Assume that, for all $j \in J$, $Y_i = Y_{i'} =: Y_j$ and $D_i = D'_i$ for all $i, i' \in I_j$. Let ν be in $\mathbb{N}^I$ and assume that for all $j \in J$, the number $\max_{i\in I_j} \nu_i$ exists. Let $\nu \in \mathbb{N}^J$ defined by $\nu_j = \max_{i\in I_j} \nu_i$. Then*

$$\mathrm{Bl}^{\{\nu_j D_j\}}_{\{Y_j\}} {}_{j\in J} X = \mathrm{Bl}^{\{\nu_i D_i\}}_{\{Y_i\}} {}_{i\in I} X.$$

Proof. We have a canonical morphism $\mathrm{Bl}^{\{\nu_j D_j\}}_{\{Y_j\}} {}_{j\in J} X \to \mathrm{Bl}^{\{\nu_i D_i\}}_{\{Y_i\}} {}_{i\in I} X$ by Proposition 3.19, now we reduce to the affine case and apply Corollary 2.21. □

REMARK 3.27. Proposition 3.26 is the spaces version of Corollary 2.21.

PROPOSITION 3.28. *Let J be a subset of I and put $K = I \setminus J$. Then*

$$\mathrm{Bl}_{\{Y_i\}_{i\in I}}^{\{D_i\}_{i\in I}} X = \mathrm{Bl}_{\left\{Y_k \times_X \mathrm{Bl}_{\{Y_i\}_{i\in J}}^{\{D_i\}_{i\in J}} X\right\}_{k\in K}}^{\left\{D_k \times_X \mathrm{Bl}_{\{Y_i\}_{i\in J}}^{\{D_i\}_{i\in J}} X\right\}_{k\in K}} \mathrm{Bl}_{\{Y_i\}_{i\in J}}^{\{D_i\}_{i\in J}} X.$$

This in particular gives the unique X-morphism

$$\mathrm{Bl}_{\{Y_i\}_{i\in I}}^{\{D_i\}_{i\in I}} X \to \mathrm{Bl}_{\{Y_i\}_{i\in J}}^{\{D_i\}_{i\in J}} X$$

of Proposition 3.19.

Proof. The right hand side is well-defined (e.g. cf. [StP, Tag 053P]). Using Proposition 3.16 and Proposition 3.18, one obtains that the right hand side is in $Spaces_X^{D\text{-reg}}$. So it is enough to see that both sides coincide when evaluated at any $f : T \to X \in Spaces_X^{D\text{-reg}}$. This follows from Proposition 3.17. □

3.4 Multi-centered dilatations and mono-centered dilatations

We proceed with the notation from § 3.1.

PROPOSITION 3.29. *Write $I = \mathrm{colim}_{J\subset I} J$ as a filtered colimit of sets where transition maps are given by inclusions of subsets. We have a canonical identification*

$$\mathrm{Bl}_{\{Y_i\}_{i\in I}}^{\{D_i\}_{i\in I}} X = \lim_{J\subset I} \mathrm{Bl}_{\{Y_i\}_{i\in J}}^{\{D_i\}_{i\in J}} X$$

where transition maps are described in Propositions 3.19 and 3.28. On the right-hand side the direct limit is on the category of S-spaces over X.

Proof. By [StP, Tag 07SF] the limit exists. For each $J \subset I$, Propositions 3.19 and 3.28 give us a X-morphism $\mathrm{Bl}_{\{Y_i\}_{i\in I}}^{\{D_i\}_{i\in I}} X \to \mathrm{Bl}_{\{Y_i\}_{i\in J}}^{\{D_i\}_{i\in J}} X$, so we get an X-morphism $\phi : \mathrm{Bl}_{\{Y_i\}_{i\in I}}^{\{D_i\}_{i\in I}} X \to \lim_{J\subset I} \mathrm{Bl}_{\{Y_i\}_{i\in J}}^{\{D_i\}_{i\in J}} X$. To prove that ϕ is an isomorphism, we reduce to the affine case where the result follows from Proposition 2.32. □

PROPOSITION 3.30. *Assume that $\#I = k$ is finite. We fix an arbitrary bijection $I = \{1, \dots, k\}$. We have a canonical isomorphism of X-spaces*

$$\mathrm{Bl}_{\{Y_i\}_{i\in I}}^{\{D_i\}_{i\in I}} X \cong \mathrm{Bl}_{(\mathrm{Bl}\cdots)\times_X Y_k}^{(\mathrm{Bl}\cdots)\times_X D_k} \left(\cdots \mathrm{Bl}_{(\mathrm{Bl}\cdots)\times_X Y_3}^{(\mathrm{Bl}\cdots)\times_X D_3} \left(\mathrm{Bl}_{(\mathrm{Bl}_{Y_1}^{D_1} X)\times_X Y_2}^{(\mathrm{Bl}_{Y_1}^{D_1} X)\times_X D_2} \left(\mathrm{Bl}_{Y_1}^{D_1} X \right) \right) \right).$$

Proof. By induction on k using Proposition 3.28. □

PROPOSITION 3.31 (Monopoly isomorphism). *Assume that $\#I = k$ is finite. We fix an arbitrary bijection $I = \{1, \dots, k\}$. We have a canonical isomorphism of X-spaces*

$$\mathrm{Bl}_{\{Y_i\}_{i\in I}}^{\{D_i\}_{i\in I}} X \cong \mathrm{Bl}_{\bigcap_{i\in I}(Y_i + D_1 + \ldots + D_{i-1} + D_{i+1} + \ldots + D_k)}^{D_1+\ldots+D_k} X.$$

Proof. Since $Car(X)$ is a face of the monoid $Pri(X)$, the right-hand side belongs to $Spaces_X^{D\text{-reg}}$. Let $f : \mathrm{Bl}_{\bigcap_{i\in I} Z_i + D_1 + \ldots + D_{i-1} + D_{i+1} + \ldots + D_k}^{D_1+\ldots+D_k} X \to X$ be the dilatation morphism. Let us prove that $f^{-1}(D_i) \subset f^{-1}(Y_i)$ for all $i \in \{1, \dots, k\}$. By Proposition 3.17,

$$f^{-1}(D_1 + \ldots + D_k) \subset f^{-1}(\bigcap_{i\in I} Y_i + D_1 + \ldots + D_{i-1} + D_{i+1} + \ldots + D_k)$$

is a Cartier divisor in $f^{-1}(X)$. Moreover by the discussion in Notation 3.1

$$\begin{aligned} &f^{-1}(\bigcap_{i\in I} Y_i + D_1 + \ldots + D_{i-1} + D_{i+1} + \ldots + D_k) \\ =&\bigcap_{i\in I} f^{-1}(Y_i) + f^{-1}(D_1) + \ldots + f^{-1}(D_{i-1}) + f^{-1}(D_{i+1}) + \ldots + f^{-1}(D_k), \text{ and} \end{aligned}$$

$$f^{-1}(D_1 + \ldots + D_k) = f^{-1}(D_1) + \ldots + f^{-1}(D_k).$$

So for any $i \in \{1, \ldots, k\}$, we have

$$f^{-1}(D_1) + \ldots + f^{-1}(D_k) \subset f^{-1}(Y_i) + f^{-1}(D_1) + \ldots + f^{-1}(D_{i-1}) + f^{-1}(D_{i+1}) + \ldots + f^{-1}(D_k).$$

Since $f^{-1}(D_l)$ is a Cartier divisor for any $l \in \{1, \ldots, k\}$, this implies $f^{-1}(D_i) \subset f^{-1}(Y_i)$. So we obtain an X-morphism $\phi : \mathrm{Bl}^{D_1+\ldots+D_k}_{\bigcap_{i\in I}(Y_i+D_1+\ldots+D_{i-1}+D_{i+1}+\ldots+D_k)} X \to \mathrm{Bl}^{\{D_i\}_{i\in I}}_{\{Y_i\}_{i\in I}} X$. To check that ϕ is an isomorphism, it is enough to prove that there is an X-morphism

$$\varphi : \mathrm{Bl}^{\{D_i\}_{i\in I}}_{\{Y_i\}_{i\in I}} X \to \mathrm{Bl}^{D_1+\ldots+D_k}_{\bigcap_{i\in I}(Y_i+D_1+\ldots+D_{i-1}+D_{i+1}+\ldots+D_k)} X.$$

To build φ, we consider the map $f' : \mathrm{Bl}^{\{D_i\}_{i\in I}}_{\{Y_i\}_{i\in I}} X \to X$ and check that $f'^{-1}(D_1 + \ldots + D_k) \subset f'^{-1}(\bigcap_{i\in I}(Y_i + D_1 + \ldots + D_{i-1} + D_{i+1} + \ldots + D_k))$. This is easy because $f'^{-1}(D_i) \subset f'^{-1}(Y_i)$ for all i. An other method to prove Proposition 3.31 is to build ϕ or φ and then reduce to the affine case an apply Proposition 2.30.

□

3.5 Functoriality

We proceed with the notation from § 3.1. Let X' and $\{[Y'_i, D'_i]\}_{i\in I}$ be another datum as in § 3.1. As usual, put $Z'_i = Y'_i \cap D'_i$. A morphism $f : X' \to X$ such that, for all $i \in I$, its restriction to D'_i (resp. Z'_i) factors through D_i (resp. Z_i), and such that $f^{-1}(D_i) = D'_i$, induces a unique morphism $\mathrm{Bl}^{D'}_{Y'} X' \to \mathrm{Bl}^{D}_{Y} X$ such that the following diagram of S-spaces

$$\begin{array}{ccc} \mathrm{Bl}^{D'}_{Y'} X' & \longrightarrow & \mathrm{Bl}^{D}_{Y} X \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X. \end{array}$$

commutes. This follows directly from Proposition 3.17.

Remark 3.32. This is the spaces version of Fact 2.13.

3.6 Base change

We proceed with the notation from § 3.1. Let $X' \to X$ be a map of S-spaces, and denote by $Y'_i, Z'_i, D'_i \subset X'$ the preimage of $Y_i, Z_i, D_i \subset X$. Then $D'_i \subset X'$ is locally principal for any i so that the dilatation $\mathrm{Bl}^{D'}_{Y'} X' \to X'$ is well-defined. By § 3.5 there is a canonical morphism of X'-spaces

$$\mathrm{Bl}^{D'}_{Y'} X' \longrightarrow \mathrm{Bl}^{D}_{Y} X \times_X X'. \tag{3.4}$$

Lemma 3.33. *If $\mathrm{Bl}^{D}_{Y} X \times_X X' \to X'$ is an object of $\mathit{Spaces}^{D\text{-reg}}_{X'}$, then (3.4) is an isomorphism.*

Proof. Our claim is étale local on X and X'. We reduce to the case where both $X = \mathrm{Spec}(B)$, $X' = \mathrm{Spec}(B')$ are affine, and $J_i = (b_i)$ is principal for all i. We denote $Z'_i = \mathrm{Spec}(B'/L'_i)$ and

$D'_i = \mathrm{Spec}(B'/J'_i)$. Then $J'_i = (b'_i)$ is principal as well where b'_i is the image of b_i under $B \to B'$. We need to show that the map of B'-algebras $B' \otimes_B B\left[\frac{L}{b}\right] \longrightarrow B'\left[\frac{L'}{b'}\right]$ is an isomorphism. However, this map is surjective with kernel the $b'^{\mathbb{N}_I}$-torsion elements by Lemma 2.35. As $b'_1, \ldots, b'_i, \ldots$ are non-zero-divisors in $B' \otimes_B B\left[\frac{L}{b}\right]$ by assumption, the lemma follows. □

Corollary 3.34. *If the morphism $X' \to X$ is flat and satisfies a property $\mathcal{P}$ which is stable under base change, then $\mathrm{Bl}^{D'}_{Y'}X' \to \mathrm{Bl}^D_Y X$ is flat and satisfies $\mathcal{P}$.*

Proof. Since flatness is stable under base change the projection $p\colon \mathrm{Bl}^D_Y X \times_X X' \to \mathrm{Bl}^D_Y X$ is flat and has property $\mathcal{P}$. By Lemma 3.33, it is enough to check that the closed subspace $\mathrm{Bl}^D_Y X \times_X D'_i$ defines an effective Cartier divisor on $\mathrm{Bl}^D_Y X \times_X X'$ for all i. But this closed subscheme is the preimage of the effective Cartier divisor $\mathrm{Bl}^D_Y X \times_X D_i$ under the flat map p, and hence is an effective Cartier divisor as well by [StP, Tag 083Z]. □

3.7 Relation to multi-centered affine projecting cone

We proceed with the notation from § 3.1 and assume that $\{D_i\}_{i\in I}$ belong to $Car(X)$. In this case, we can also realize $\mathrm{Bl}^D_Y X$ as a closed subspace of the multi-centered affine projecting cone associated to X, Z and D.

Definition 3.35. The *affine projecting cone $\mathcal{O}_X$-algebra* with multi-center $\{[Z_i = V(\mathcal{L}_i), D_i = V(\mathcal{J}_i)]\}_{i\in I}$ is

$$\mathrm{C}^{\mathcal{J}}_{\mathcal{L}}\mathcal{O}_X \overset{\text{def}}{=} \bigoplus_{\nu\in\mathbb{N}_I} \mathcal{L}^{\nu} \otimes \mathcal{J}^{-\nu}.$$

The *affine projecting cone* of X with multi-center $\{[Z_i, D_i]\}_{i\in I}$ is

$$\mathrm{C}^D_Z X \overset{\text{def}}{=} \mathrm{Spec}\big(\mathrm{C}^{\mathcal{J}}_{\mathcal{L}}\mathcal{O}_X\big).$$

Proposition 3.36. *The dilatation $\mathrm{Bl}^D_Z X$ is the closed subspace of the affine projecting cone $\mathrm{C}^D_Z X$ defined by the equations $\{\varrho_i - 1\}_{i\in I}$, where for all $i \in I$, $\varrho_i \in \mathrm{C}^{\mathcal{J}}_{\mathcal{L}}\mathcal{O}_X$ is the image of $1 \in \mathcal{O}_X$ under the map*

$$\mathcal{O}_X \cong \mathcal{J}_i \otimes \mathcal{J}_i^{-1} \subset \mathcal{L}_i \otimes \mathcal{J}_i^{-1} \subset \mathrm{C}^{\mathcal{J}}_{\mathcal{L}}\mathcal{O}_X.$$

Proof. We may work locally and the proposition follows from Proposition 2.38 and Remark 2.39. □

Remark 3.37. Let X be a scheme and let D be a Cartier. Let $Z \subset D$ be a closed subscheme of finite type. Then the mono-centered dilatation $\mathrm{Bl}^D_Z X$ was already defined in [Du05, Definition 2.9] using the conic point of view.

3.8 Relation to Proj of multi-graded algebras and multi-centered blowups

We proceed with the notation from § 3.1 and assume that $X = \mathrm{Spec}(A)$ is an affine scheme but we expect the content of this section can be adapted to algebraic spaces. We assume moreover that $I = \{1, \ldots, k\}$ is finite. We have $\mathbb{N}_I = \mathbb{N}^k$. We refer to [BS07, §2] for the construction of the Proj associated to Ω-graded R-algebras where Ω is a finitely generated abelian group and R is a ring. This construction should globalize to arbitrary Ω-graded quasi-coherent algebras. It should also work for algebraic spaces, e.g. cf. [StP, Tag 085P] for a hint. We plan to study this later. Recall that $Z_1, \ldots, Z_k$ are closed subspaces of X with ideals $L_1, \ldots, L_k$. Recall that $Z_i \subset D_i$ for all i. We assume that each D_i is principal and given by elements $a_i \in A$. We introduce the following definition. Recall that $\mathrm{Bl}_{L_1,\ldots,L_k}A$ is canonically $\mathbb{N}^k$-graded, and in particular $\mathbb{Z}^r$-graded.

Definition 3.38. The blowup of X with multi-center $Z = (Z_1, \dots, Z_k)$ is the scheme

$$\mathrm{Bl}_Z X \overset{\text{def}}{=} \mathrm{Proj}\big(\mathrm{Bl}_{L_1,\dots,L_k} A\big).$$

Proposition 3.39. *The map $\mathrm{Bl}_{Z_1,\dots,Z_k} X \to X$ is a final object of $Sch_X^{Z_1,\dots,Z_k\text{-reg}}$.*

Proof. We adapt [GW20, Chap 13, pages 414-415]. By definition $\mathrm{Bl}_Z X = \bigcup_{\{f \text{ relevant}\}} D_+(f)$ (cf. [BS07, Def. 2.2]). We have $D_+(f) = \mathrm{Spec} A[\frac{L_1}{f}, \dots, \frac{L_k}{f}]$. Now 2.10, 2.25 and 2.26 finish the proof. □

Proposition 3.40. *Let $1 \leqslant r \leqslant k$ and let $Z_i' = \mathrm{Bl}_{Z_{r+1},\dots,Z_k} X \times_X Z_i$ for $1 \leqslant i \leqslant r$. Then we have a canonical isomorphism of X-schemes*

$$\mathrm{Bl}_{Z_1',\dots,Z_r'} \mathrm{Bl}_{Z_{r+1},\dots,Z_k} X = \mathrm{Bl}_{Z_1,\dots,Z_k} X.$$

Proof. The right-hand side belongs to $Sch_X^{Z_1,\dots,Z_k\text{-reg}}$. We now observe that the left-hand side also belongs to $Sch_X^{Z_1,\dots,Z_k\text{-reg}}$. Indeed, let g be the composition

$$\mathrm{Bl}_{Z_1',\dots,Z_r'} \mathrm{Bl}_{Z_{r+1},\dots,Z_k} X \to \mathrm{Bl}_{Z_{r+1},\dots,Z_k} X \to X.$$

Obviously $g^{-1}(Z_i)$ is a Cartier divisor in $\mathrm{Bl}_{Z_1',\dots,Z_r'} \mathrm{Bl}_{Z_{r+1},\dots,Z_k} X$ for $1 \leqslant i \leqslant r$. We claim that $g^{-1}(Z_i)$ is also a Cartier divisor in $\mathrm{Bl}_{Z_1',\dots,Z_r'} \mathrm{Bl}_{Z_{r+1},\dots,Z_k} X$ for $r+1 \leqslant i \leqslant k$. To see this, we observe that the preimage of any Cartier divisor of $\mathrm{Bl}_{Z_{r+1},\dots,Z_k} X$ in $\mathrm{Bl}_{Z_1',\dots,Z_r'} \mathrm{Bl}_{Z_{r+1},\dots,Z_k} X$ is a Cartier divisor, this follows from the local definition of Proj and Proposition 3.18. So both sides belong to $Sch_X^{Z_1,\dots,Z_k\text{-reg}}$. It is easy to construct an X-morphism $\mathrm{Bl}_{Z_1',\dots,Z_r'} \mathrm{Bl}_{Z_{r+1},\dots,Z_k} X \leftarrow \mathrm{Bl}_{Z_1,\dots,Z_k} X$. This shows that the left-hand side is also (cf. Proposition 3.39) a final object of $Sch_X^{Z_1,\dots,Z_k\text{-reg}}$. This finishes the proof. □

Proposition 3.41. *We have a canonical isomorphism of X-schemes*

$$\mathrm{Bl}_{Z_1+\dots+Z_k} X = \mathrm{Bl}_{Z_1,\dots,Z_k} X.$$

Proof. This follows from [StP, Tag 085Y]. □

Fact 3.42. *The dilatation $\mathrm{Bl}_Z^D X$ is the open subscheme of the blowup $\mathrm{Bl}_Z X$ defined by $D_+(a_1 \dots a_k)$ (cf. [BS07, page 6] for the notation $D_+(-)$).*

Proof. The identity $A[\frac{L_1}{a_1}, \dots, \frac{L_k}{a_k}] = (\mathrm{Bl}_{L_1,\dots,L_k} A)_{(a_1 \dots a_k)}$, given by Fact 2.29, is a proof. □

3.9 Dilatations of quasi-coherent $\mathcal{O}_X$-modules

We proceed with the notation from §3.1. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Working locally as in Definition 3.10 and using dilatations of modules (cf. §2.4), we obtain a canonical quasi-coherent sheaf on $\mathrm{Bl}_Y^D X$, denoted $\mathrm{Bl}_Y^D \mathcal{F}$ or $\mathcal{F}\Big[\Big\{\frac{\mathcal{M}_i}{\mathcal{J}_i}\Big\}_{i\in I}\Big]$. Note that in general $\mathrm{Bl}_Y^D \mathcal{F} \ncong \mathrm{Bl}^* \mathcal{F}$ where $\mathrm{Bl} : \mathrm{Bl}_Y^D X \to X$ is the dilatation map. This construction enjoys the following universal property. Let $QCoh(X)_{\mathcal{F}}^{D\text{-reg}}$ be the category whose objects are morphisms of quasi-coherent $\mathcal{O}_X$-modules $F : \mathcal{F} \to \mathcal{F}'$ with source $\mathcal{F}$ such that, locally, each $\mathcal{J}_i$ is defined by a non-zero-divisor of the module $\mathcal{F}'$, then

$$\mathrm{Hom}_{QCoh(X)_{\mathcal{F}}^{D\text{-reg}}}\Big(\mathcal{F}\Big[\Big\{\frac{\mathcal{M}_i}{\mathcal{J}_i}\Big\}_{i\in I}\Big], \mathcal{F}'\Big) = \begin{cases} \{*\}, \text{ if } \mathcal{L}_i \mathcal{F}' = \mathcal{J}_i \mathcal{F}' \text{ for } i \in I; \\ \varnothing, \text{ else.} \end{cases}$$

4. Iterated multi-centered dilatations

We proceed with the notation from § 3.1. Let $\nu, \theta \in \mathbb{N}^I$ such that $\theta \leqslant \nu$, i.e. $\theta_i \leqslant \nu_i$ for all $i \in I$.

PROPOSITION 4.1. *There is a unique X-morphism*

$$\varphi_{\nu,\theta} : \mathrm{Bl}_Y^{D^\nu} X \to \mathrm{Bl}_Y^{D^\theta} X.$$

Proof. Let $\varphi_\nu : \mathrm{Bl}_Y^{D^\nu} X \to X$, $\varphi_\theta : \mathrm{Bl}_Y^{D^\theta} X \to X$ be the dilatation maps. Let $i \in I$, if $\theta_i = 0$, then $\theta_i D_i = X$ and so $\varphi_\nu^{-1}(\theta_i D_i) = \mathrm{Bl}_Y^{D^\nu} X$ is a Cartier divisor in $\mathrm{Bl}_Y^{D^\nu} X$. If $\theta_i > 0$, then $\nu_i > 0$ and $\varphi_\nu^{-1}(\nu_i D_i) = \nu_i \varphi_\nu^{-1}(D_i)$ is Cartier and so $\varphi_\nu^{-1}(D_i)$ is Cartier because $Car(\mathrm{Bl}_Y^{D^\nu} X)$ is a face of $Pri(\mathrm{Bl}_Y^{D^\nu} X)$ (cf. the discussion in Notation 3.1). Consequently $\varphi_\nu^{-1}(\theta_i D_i) = \theta_i \varphi_\nu^{-1}(D_i)$ is a Cartier divisor. So we proved that $\mathrm{Bl}_Y^{D^\nu} X$ belongs to $Spaces_X^{D^\theta\text{-reg}}$. We now use Proposition 3.17. Let $i \in I$, we have $\theta_i D_i \subset \nu_i D_i$. So we have $\varphi_\nu^{-1}(\theta_i D_i) \subset \varphi_\nu^{-1}(\nu_i D_i) \subset \varphi_\nu^{-1}(Y \cap \nu_i D_i)$. so

$$\varphi_\nu^{-1}(\theta_i D_i) \subset \big(\varphi_\nu^{-1}(Y \cap \nu_i D_i)\big) \cap \big(\varphi_\nu^{-1}(\theta_i D_i)\big) = \varphi_\nu^{-1}(Y \cap \nu_i D_i \cap \theta_i D_i) = \varphi_\nu^{-1}(\theta_i D_i \cap Y).$$

Now we apply Proposition 3.17 and finish the proof. □

Assume now moreover that $\nu, \theta \in \mathbb{N}_I \subset \mathbb{N}^I$. We will prove that, under some assumptions, $\varphi_{\nu,\theta}$ is a dilatation morphism with explicit descriptions. We need the following observation.

PROPOSITION 4.2. *Assume that we have a commutative diagram of S-spaces*

$$\begin{array}{ccc} B & \xrightarrow{f'} & C \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle F} & \\ & X & \end{array} .$$

Assume that F is affine and f is a closed immersion. Then f' is a closed immersion.

Proof. A closed immersion is affine, so by [StP, Tag 08GB], f' is affine. Using [StP, Tag 03M4], we reduce to the case where B, C and X are affine (taking an étale covering of X by affine schemes). Now the assertion is clear because closed immersions of affine schemes correspond to surjective morphisms at the level of rings. □

COROLLARY 4.3. *Assume that we have a commutative diagram of S-spaces*

$$\begin{array}{ccc} B & \xrightarrow{f'} & \mathrm{Bl}_Y^D X \\ & {\scriptstyle f}\searrow \quad \swarrow & \\ & X & \end{array}$$

where the right-hand side morphism is the dilatation map. Assume that f is a closed immersion. Then f' is a closed immersion.

Proof. Clear by Proposition 4.2. □

We now assume that $Z_i \subset Y_i$ is a Cartier divisor inclusion for all $i \in I$. Let $\mathcal{D}_i$ be the canonical diagram of closed immersions

$$\begin{array}{ccc} Y_i & \longrightarrow & \mathrm{Bl}_{Y_i}^{\nu_i D_i} X \\ \uparrow & \square & \uparrow \\ Z_i & \longrightarrow & D_i \end{array}$$

obtained by Propositions 3.17 and 4.3.

LEMMA 4.4. *Assume $I = \{i\}$ and let $n_i \in \mathbb{N}$. We have an identification*

$$\mathrm{Bl}_{Y_i}^{(\nu_i+n_i)D_i} X = \mathrm{Bl}_{Y_i}^{n_i D_i} \mathrm{Bl}_{Y_i}^{\nu_i D_i} X.$$

Proof. Using Propositions 3.17 and 4.1 and the discussion in Notation 3.1, it is easy to prove that there is a unique $\mathrm{Bl}_{Y_i}^{\nu_i D_i} X$-morphism $\mathrm{Bl}_{Y_i}^{(\nu_i+n_i)D_i} X \to \mathrm{Bl}_{Y_i}^{n_i D_i} \mathrm{Bl}_{Y_i}^{\nu_i D_i} X$. To prove that it is an isomorphism, we reduce to the affine case and apply Corollary 2.34 □

Let f_i be the canonical morphism (e.g. cf. 3.19, 3.28 or 4.1)

$$\mathrm{Bl}_Y^{D^\nu} X \to \mathrm{Bl}_{Y_i}^{\nu_i D_i} X.$$

We denote by $Y_i \times_{\mathrm{Bl}_{Y_i}^{\nu_i D_i} X} \mathrm{Bl}_Y^{D^\nu} X$ the fiber product obtained via the arrows given by f_i and $\mathcal{D}_i$. We use similarly the notation $D_i \times_{\mathrm{Bl}_{Y_i}^{\nu_i D_i} X} \mathrm{Bl}_Y^{D^\nu} X$.

LEMMA 4.5. *Let $i \in I$ and let $n_i \in \mathbb{N}$. Let $\gamma_i \in \mathbb{N}_I$ be $(0, \ldots, 0, n_i, 0, \ldots)$ where n_i is in place i. We have an identification*

$$\mathrm{Bl}_Y^{D^{\nu+\gamma_i}} X = \mathrm{Bl}^{n_i D_i \times_{\mathrm{Bl}_{Y_i}^{\nu_i D_i} X} \mathrm{Bl}_Y^{D^\nu} X}_{Y_i \times_{\mathrm{Bl}_{Y_i}^{\nu_i D_i} X} \mathrm{Bl}_Y^{D^\nu} X} \mathrm{Bl}_Y^{D^\nu} X.$$

In particular we have a canonical dilatation morphism

$$\varphi_{\nu+\gamma_i,\nu} : \mathrm{Bl}_Y^{D^{\nu+\gamma_i}} X \to \mathrm{Bl}_Y^{D^\nu} X.$$

Proof. We have

$$\begin{aligned}
& \mathrm{Bl}^{n_i D_i \times_{\mathrm{Bl}_{Y_i}^{\nu_i D_i} X} \mathrm{Bl}_Y^{D^\nu} X}_{Y \times_{\mathrm{Bl}_{Y_i}^{\nu_i D_i} X} \mathrm{Bl}_Y^{D^\nu} X} \mathrm{Bl}_Y^{D^\nu} X \\
\text{by Proposition 3.28} = \ & \mathrm{Bl}^{n_i D_i \times_{\mathrm{Bl}_{Y_i}^{\nu_i D_i} X} \mathrm{Bl}_Y^{D^\nu} X}_{Y \times_{\mathrm{Bl}_{Y_i}^{\nu_i D_i} X} \mathrm{Bl}_Y^{D^\nu} X} \mathrm{Bl}^{\{\nu_j D_j \times_X \mathrm{Bl}_{Y_i}^{\nu_i D_i} X\}_{j \in I \setminus \{i\}}}_{\{Y_j \times_X \mathrm{Bl}_{Y_i}^{\nu_i D_i} X\}_{j \in I \setminus \{i\}}} \mathrm{Bl}_{Y_i}^{\nu_i D_i} X \\
\text{by Proposition 3.28} = \ & \mathrm{Bl}^{n_i D_i, \{\nu_j D_j \times_X \mathrm{Bl}_{Y_i}^{\nu_i D_i} X\}_{j \in I \setminus \{i\}}}_{Y_i, \{Y_j \times_X \mathrm{Bl}_{Y_i}^{\nu_i D_i} X\}_{j \in I \setminus \{i\}}} \mathrm{Bl}_{Y_i}^{\nu_i D_i} X \\
\text{by Proposition 3.28 and Lemma 4.4} = \ & \mathrm{Bl}^{\{\nu_j D_j \times_X \mathrm{Bl}_{Y_i}^{(\nu_i+n_i) D_i} X\}_{j \in I \setminus \{i\}}}_{\{Y_j \times_X \mathrm{Bl}_{Y_i}^{(\nu_i+n_i) D_i} X\}_{j \in I \setminus \{i\}}} \mathrm{Bl}_{Y_i}^{(\nu_i+n_i) D_i} X \\
\text{by Proposition 3.28} = \ & \mathrm{Bl}_Y^{D^{\nu+\gamma_i}} X.
\end{aligned}$$

□

PROPOSITION 4.6. *Recall that $\theta \leqslant \nu$. Put $\gamma = \nu - \theta$. Put $K = \{i \in I | \gamma_i > 0\}$. We have an identification*

$$\mathrm{Bl}_Y^{D^\nu} X = \mathrm{Bl}^{\{\gamma_i D_i \times_{\mathrm{Bl}_{Y_i}^{\theta_i D_i} X} \mathrm{Bl}_Y^{D^\theta} X\}_{i \in K}}_{\{Y_i \times_{\mathrm{Bl}_{Y_i}^{\theta_i D_i} X} \mathrm{Bl}_Y^{D^\theta} X\}_{i \in K}} \mathrm{Bl}_Y^{D^\theta} X.$$

In particular the unique X-morphism

$$\varphi_{\nu,\theta} : \mathrm{Bl}_Y^{D^\nu} X \to \mathrm{Bl}_Y^{D^\theta} X$$

of Proposition 4.1 is a dilatation map.

Proof. We prove the first assertion by induction on $\#\{i \in I | \nu_i > 0\}$. If $k = 1$ the assertion follows from Lemma 4.4. The passage from $k-1$ to k follows from Lemma 4.5 and Proposition 3.28. □

It is now natural to introduce the following terminology.

Definition 4.7. For any $\nu \in \mathbb{N}_I$, let us consider

$$\mathrm{Bl}_Y^{D^\nu} X = \mathrm{Bl}^{\{\nu_i D_i\}}_{\{Y_i\}_{i\in I}} X$$

and call it the ν-th iterated dilatation of X with multi-center $\{[Y_i, D_i]\}_{i\in I}$.

5. Multi-centered dilatations along multiples of a single divisor

Let X be an S-space. We fix a locally principal closed subscheme $D \subset X$. Let $Y_0, Y_1, \ldots, Y_i, \ldots, Y_k$ be closed S-subspaces of X such that $D \cap Y_i \subset Y_i$ is a Cartier divisor for all i. We assume moreover that $Y_0 \subset Y_i$ for $i \in \{1, \ldots, k\}$. Let $s_0, s_1, \ldots, s_k \in \mathbb{N}$ be integers. We claim that we have a canonical closed immersion $Y_0 \to \mathrm{Bl}^{s_0 D, \ldots, s_k D}_{Y_0, \ \ldots \ , Y_k} X$. This follows from Propositions 3.17 and 4.3 observing that the map $Y_0 \to X$ restricted to $s_i D$ factors through $Y_i \cap s_i D$ for any i. We now use the notation $\mathrm{Bl}^{s_0,\ldots,s_k}_{Y_0,\ldots,Y_k} X$ to denote $\mathrm{Bl}^{s_0 D, \ldots, s_k D}_{Y_0, \ \ldots \ , Y_k} X$. The following fact is a direct generalization of the first assertion of Corollary 2.34 to the present situation.

Fact 5.1. *Assume that $X = \mathrm{Spec}(A)$ is affine. Assume that $D = \mathrm{Spec}(A/(a))$ and $Y_i = \mathrm{Spec}(A/M_i)$ for $i \in I$. Then the ideal Q of $A' := A[\frac{M_0}{a^{s_0}}, \frac{M_1}{a^{s_1}} \cdots \frac{M_k}{a^{s_k}}]$ corresponding to the canonical closed immersion $Y_0 \to \mathrm{Bl}^{s_0,s_1,\ldots,s_k}_{Y_0,Y_1,\ldots,Y_k} X$ is the ideal $\langle \frac{M_0}{a^{s_0}}, \frac{M_1}{a^{s_1}}, \ldots, \frac{M_k}{a^{s_k}} \rangle$ of A' generated by $\frac{M_0}{a^{s_0}}, \frac{M_1}{a^{s_1}}, \ldots, \frac{M_k}{a^{s_k}}$.*

Proof. There is no difficulty to adapt the proof of 2.34. We provide details for the convenience of the reader. Using 2.33, it is clear that $\langle \frac{M_0}{a^{s_0}}, \frac{M_1}{a^{s_1}}, \ldots, \frac{M_k}{a^{s_k}} \rangle \subset Q$, so it is enough to prove that $Q \subset \langle \frac{M_0}{a^{s_0}}, \frac{M_1}{a^{s_1}}, \ldots, \frac{M_k}{a^{s_k}} \rangle$. So let $\nu \in \mathbb{N}^k$, we have to prove that $\frac{L^\nu \cap M_0}{(a^s)^\nu}$ is included in $\langle \frac{M_0}{a^{s_0}}, \frac{M_1}{a^{s_1}}, \ldots, \frac{M_k}{a^{s_k}} \rangle$. An element $x \in L^\nu$ can be written as a sum $x = \sum_{\nu=\beta+\alpha} m_\beta (a^s)^\alpha$ with $m_\beta \in M^\beta$ (note that, if x belongs to $L^\nu \cap M_0$, then $m_{(0,\ldots,0)}(a^s)^\nu$ also belongs to M_0). Now we assume that x belongs to $L^\nu \cap M_0$, it is clear that for $\beta \neq (0, \ldots, 0)$ the element $\frac{m_\beta (a^s)^\alpha}{(a^s)^\nu} = \frac{m_\beta}{(a^s)^\beta}$ belongs to $\langle \frac{M_0}{a^{s_0}}, \frac{M_1}{a^{s_1}}, \ldots, \frac{M_k}{a^{s_k}} \rangle$. Now for $\beta = (0, \ldots, 0)$, using that a_i is a non-zero-divisor in A/M_i for all i and that $m_{(0,\ldots,0)}(a^s)^\nu$ belongs to M_0, we get that $m_{(0,\ldots,0)}$ belongs to M_0 and it is now clear that $\frac{m_{(0,\ldots,0)}(a^s)^\nu}{(a^s)^\nu}$ belongs to $\langle \frac{M_0}{a^{s_0}}, \frac{M_1}{a^{s_1}}, \ldots, \frac{M_k}{a^{s_k}} \rangle$. So x belongs to $\langle \frac{M_0}{a^{s_0}}, \frac{M_1}{a^{s_1}}, \ldots, \frac{M_k}{a^{s_k}} \rangle$. □

Proposition 5.2. *Let $0 \leqslant t \leqslant s_0$ be an integer. We have a canonical identification*

$$\mathrm{Bl}^t_{Y_0} \mathrm{Bl}^{s_0,s_1,\ldots,s_k}_{Y_0,Y_1,\ldots,Y_k} X = \mathrm{Bl}^{s_0+t,s_1+t,\ldots,s_k+t}_{Y_0, \ Y_1, \ \ldots \ , Y_k} X.$$

Proof. Using Propositions 3.17 and 4.1 and the discussion in Notation 3.1, we get a morphism from left to right. We then reduce to the case where $X = \mathrm{Spec}(A)$ is affine and use the notation of Fact 5.1. It is enough to show that we have a canonical identification of rings

$$\big(A[\tfrac{M_0}{a^{s_0}}, \tfrac{M_1}{a^{s_1}} \cdots \tfrac{M_k}{a^{s_k}}]\big)[\tfrac{Q}{a^t}] = A[\tfrac{M_0}{a^{s_0+t}}, \tfrac{M_1}{a^{s_1+t}} \cdots \tfrac{M_k}{a^{s_k+t}}].$$

For this, there is no difficulty to adapt the proof of the second assertion of Corollary 2.34. Indeed, it is enough to apply Proposition 2.22 and Corollary 2.21. □

FACT 5.3. *Let A be a ring. Let $P = A[T, X_1, \ldots, X_n]$ be the polynomial algebra in $n+1$ variables. Let $d_1, \ldots, d_n \in \mathbb{N}$. Then there is a canonical identification of A-algebras*

$$P[x_1, \ldots, x_n]/(X_1 - T^{d_1}x_1, \ldots, X_n - T^{d_n}x_n) = P[\tfrac{(X_1)}{T^{d_1}}, \ldots, \tfrac{(X_n)}{T^{d_n}}].$$

Proof. The map given by $x_i \mapsto \frac{X_i}{T^{d_i}}$ is well-defined and surjective. The source and target of our map are T-torsion free and the map is an isomorphism after inverting T by Corollary 2.23. □

PROPOSITION 5.4. *Let A be a ring. Let $a \in A$. Let $g_1, \ldots, g_n$ be elements in A where $n \in \mathbb{N}$. Let $d_1, \ldots, d_n \in \mathbb{N}$. There is a surjection*

$$A[x_1, \ldots, x_n]/(g_1 - a^{d_1}x_1, \ldots, g_n - a^{d_n}x_n) \to A[\tfrac{(g_1)}{a^{d_1}}, \ldots, \tfrac{(g_n)}{a^{d_n}}]$$

whose kernel is the a-power torsion in the source.

Proof. Consider the map $P = A[T, X_1, \ldots, X_n] \to A$ sending T to a and X_i to g_i for $1 \leqslant i \leqslant n$. By Fact 5.3, we have $P[x_1, \ldots, x_n]/(X_1 - T^{d_1}x_1, \ldots, X_n - T^{d_n}x_n) = P[\frac{(X_1)}{T^{d_1}}, \ldots, \frac{(X_n)}{T^{d_n}}]$. Now we use Proposition 2.35 to finish the proof. □

PROPOSITION 5.5. *Let A be a ring. Let $a, g_1, \ldots, g_n$ be a H_1-regular sequence in A (cf. [StP, Tag 062E] for H_1-regularity). Let $d_1, \ldots, d_n$ be positive integers. Then the surjection of Proposition 5.4 is an isomorphism. In particular, the dilatation algebra identifies with a quotient of a polynomial algebra as follows*

$$A[\tfrac{(g_1)}{a^{d_1}}, \ldots, \tfrac{(g_n)}{a^{d_n}}] = A[x_1, \ldots, x_n]/(g_1 - a^{d_1}x_1, \ldots, g_n - a^{d_n}x_n).$$

Proof. We can assume that $d_i > 0$ for all i by Corollary 2.19. By Proposition 5.4, it is enough to show that the right-hand side is a-torsion free. We adapt the proof of [StP, Tag 0BIQ]. We claim that the sequence $(a, g_1 - a^{d_1}x_1, \ldots, g_n - a^{d_n}x_n)$ is H_1-regular in $A[x_1, \ldots, x_n]$. Namely, the map

$$(a, g_1 - a^{d_1}x_1, \ldots, g_n - a^{d_n}x_n) : A[x_1, \ldots, x_n]^{\oplus(1+n)} \to A[x_1, \ldots, x_n]$$

used to define the Koszul complex on $a, g_1 - a^{d_1}x_1, \ldots, g_n - a^{d_n}x_n$ is isomorphic to the map

$$(a, g_1, \ldots, g_n) : A[x_1, \ldots, x_n]^{\oplus(1+n)} \to A[x_1, \ldots, x_n]$$

used to define the Koszul complex on $a, g_1, \ldots, g_n$ via the isomorphism Θ

$$A[x_1, \ldots, x_n]^{\oplus(1+n)} \to A[x_1, \ldots, x_n]^{\oplus(1+n)}$$

sending $(P_0, P_1, \ldots, P_n)$ to

$$(P_0 - \sum_{i=1}^{n} a^{d_i - 1}x_iP_i, P_1, P_2, \ldots, P_n);$$

this follows from the identity

$$aP_0 + \sum_{i=1}^{n}(g_i - a^{d_i}x_i)P_i = a\big(P_0 - \sum_{i=1}^{n} a^{d_i-1}x_iP_i\big) + \sum_{i=1}^{n} g_iP_i.$$

By [StP, Tag 0624] these Koszul complexes are isomorphic. By [StP, Tag 0629] the Koszul complex K on $(a, g_1 - a^{d_1}x_1, \ldots, g_n - a^{d_n}x_n)$ is the cone on $a : L \to L$ where L is the Koszul complex on $(g_1 - a^{d_1}x_1, \ldots, g_n - a^{d_n}x_n)$, since $H_1(K) = 0$, we conclude that $a : H_0(L) \to H_0(L)$ is injective, so the right-hand side is a-torsion free. □

6. Some flatness and smoothness results

Let S be a scheme and let $C \subset S$ be a Cartier divisor in S. Let X be a scheme over S. Let D be the closed subscheme of X given by $X \times_S C$. Let $X_j \subset D$ be closed subchemes for $1 \leqslant j \leqslant d$. We are now making the following assumption. We assume that locally over S, X the following conditions are satisfied

(i) $S = \mathrm{Spec}(R)$, $C = \mathrm{Spec}(R/a)$ and $X = \mathrm{Spec}(A)$,

(ii) there exists a sequence $g_1, \ldots, g_n \in A$ such that $a, g_1, \ldots, g_n$ is a H_1-regular sequence in A,

(iii) there exists a sequence $1 \leqslant i_1 < i_2 < \ldots < i_j < \ldots < i_d = n$ such that

$$\begin{aligned}
X_1 &= \mathrm{Spec}\big(A/(g_1, \ldots, g_{i_1})\big)\\
X_2 &= \mathrm{Spec}\big(A/(\overset{\pm}{g_1}, \ldots, \overset{\pm}{g_{i_1}}, g_{i_1+1}, \ldots, g_{i_2})\big)\\
&\vdots \qquad \vdots\\
X_j &= \mathrm{Spec}\big(A/(\overset{\pm}{g_1}, \ldots, \overset{\pm}{g_{i_{j-1}}}, g_{i_{j-1}+1}, \ldots, g_{i_j})\big)\\
&\vdots \qquad \vdots\\
X_d &= \mathrm{Spec}\big(A/(\overset{\pm}{g_1}, \ldots, \overset{\pm}{g_{i_{d-1}}}, g_{i_{d-1}+1}, \ldots, g_n)\big)
\end{aligned}$$

where $\pm$ over a symbol means that this symbol possibly appears but not necessarily. We put $U_j = \mathrm{Spec}(A/(g_1, \ldots, g_{i_j}))$ for $1 \leqslant i \leqslant j$.

Let $m_1 \geqslant \ldots \geqslant m_j \geqslant \ldots \geqslant m_d \geqslant 0 \in \mathbb{N}$ be integers.

Proposition 6.1. (i) *If X/S is flat and if moreover one of the following holds:*

(a) *$X_j \to S$ is flat and S, X are locally noetherian for $1 \leqslant j \leqslant d$,*

(b) *$X_j \to S$ is flat and $X_j \to S$ is locally of finite presentation,*

(c) *the local rings of S are valuation rings,*

then $\mathrm{Bl}^{m_1,\ldots,m_d}_{X_1,\ldots,X_d} X \to S$ is flat.

(ii) *If $X \to S$ is smooth and (with the local notation of the assumption) $U_j \times_S \mathrm{Spec}(R/(a^{m_j})) \to \mathrm{Spec}(R/(a^{m_j}))$ is smooth for all $1 \leqslant j \leqslant d$, then $\mathrm{Bl}^{m_1,\ldots,m_d}_{X_1,\ldots,X_d} X \to S$ is smooth.*

Proof. This is local on S, X. We use notations used to state the assumption before the statement. Corollary 2.21 implies that we can and do assume that $X_j = U_j$ for all $1 \leqslant j \leqslant d$.

(i) We prove the assertions by induction on d. If $d = 1$ this follows from [MRR20, Proposition 2.16]. By Proposition 3.28 we have $\mathrm{Bl}^{m_1,\ldots,m_d}_{X_1,\ldots,X_d} X = \mathrm{Bl}^{m_d}_{X'_d} \mathrm{Bl}^{m_1,\ldots,m_{d-1}}_{X_1,\ldots,X_{d-1}} X$ where $X'_d = \mathrm{Bl}^{m_1,\ldots,m_{d-1}}_{X_1,\ldots,X_{d-1}} X \times_X X_d$. Proposition 5.5 implies that $\mathrm{Bl}^{m_1,\ldots,m_{d-1}}_{X_1,\ldots,X_{d-1}} X$ identifies with the spectrum of $P/\mathfrak{I}$ where $P = A[x_1, \ldots, x_{i_{d-1}}]$ and

$$\mathfrak{I} = (g_1 - a^{m_1} x_1, \ldots, g_{i_1} - a^{m_1} x_{i_1}, g_{i_1+1} - a^{m_2} x_{i_1+1}, \ldots, g_{i_2} - a^{m_2} x_{i_2}, \ldots, g_{i_{d-1}} - a^{m_{d-1}} x_{i_{d-1}}).$$

We claim that the sequence given by $g_{i_{d-1}+1}, \ldots, g_{i_d}$ is H_1-regular in $P/(\mathfrak{I} + (a^{m_d}))$. Let us prove the claim. Since $m_d \leqslant m_j$ for all $1 \leqslant j \leqslant d-1$, the ideal $\mathfrak{I} + (a^{m_d})$ of A is equal to $(a^{m_d}, g_1, \ldots, g_{i_{d-1}})$. So $P/(\mathfrak{I} + (a^{m_d}))$ identifies with $\big(A/(a^{m_d}, g_1, \ldots, g_{i_{d-1}})\big)[x_1, \ldots, x_{i_{d-1}}]$. Now since $(a, g_1, \ldots, g_n)$ is H_1-regular in A, we know that $(a^{m_d}, g_1, \ldots, g_n)$ is H_1-regular in A by [StP, Tag 062G]. So $g_{i_{d-1}+1}, \ldots, g_{i_d}$ is a H_1-regular sequence in $A/(a^{m_d}, g_1, \ldots, g_{i_{d-1}})$ by [StP, Tag 068L]. This implies that $g_{i_{d-1}+1}, \ldots, g_{i_d}$ is a H_1-regular sequence in $\big(A/(a^{m_d}, g_1, \ldots, g_{i_{d-1}})\big)[x_1, \ldots, x_{i_{d-1}}]$. This finishes to prove our claim. We now apply [MRR20, Proposition 2.16] to deduce the assertions for d.

(ii) We prove the assertion by induction on d. If $d = 1$, this is [MRR20, Proposition 2.16]. We now assume that the assertion is true for $d-1$. Using [MRR20, Proposition 2.16] and Proposition 3.28, it is enough to show that $Z'_d := (X_d \times_X \mathrm{Bl}^{m_1,\dots,m_d}_{X_1,\dots,X_d} X) \times_S (\mathrm{Spec}(R/(a^{m_d}))$ is smooth over $\mathrm{Spec}(R/(a^{m_d})$. We have

$$Z'_d = X_d \times_S \mathrm{Spec}(R/(a^{m_d})) \times_{\mathrm{Spec}(R/(a^{m_d}))} \mathrm{Bl}^{m_1,\dots,m_d}_{X_1,\dots,X_d} X \times_S \mathrm{Spec}(R/(a^{m_d}).$$

We computed $\mathrm{Bl}^{m_1,\dots,m_d}_{X_1,\dots,X_d} X \times_S \mathrm{Spec}(R/(a^{m_d})$ in (i) and proved that it is the spectrum of $A/(a^{m_d}, g_1, \dots, g_{i_{d-1}})[x_1, \dots, x_{i_{d-1}}]$. Now $X_d \times_S \mathrm{Spec}(R/(a^{m_d}))$ is the spectrum of $A/(a^{d_m}, g_1, \dots, g_n)$. Consequently Z'_d is the spectrum of $(A/(a^{d_m}, g_1, \dots, g_n))[x_1, \dots, x_{i_{d-1}}]$. So by assumption $A/(a^{d_m}, g_1, \dots, g_n)$ is smooth over $R/(a^{d_m})$. Moreover any polynomial algebra is smooth over its base ring. A composition of smooth morphisms is smooth. So the composition

$$R/(a^{d_m}) \to A/(a^{d_m}, g_1, \dots, g_n) \to (A/(a^{d_m}, g_1, \dots, g_n))[x_1, \dots, x_{i_{d-1}}]$$

is smooth. This finishes the proof.

□

7. Dilatations of monoid, group and Lie algebra schemes

We study dilatations of schemes endowed with a structure (cf. [SGA3, Exp. I §2.2]) in this section. We focus on monoid, group and Lie algebra structures. Let S be a scheme, and let $G \to S$ be a monoid (resp. group, resp. Lie algebra) scheme. Let $C = \{C_i\}_{i\in I} \subset S$ be locally principal closed subschemes. Put $D_i = G|_{C_i} = G \times_S C_i$ and $D = \{D_i\}_{i\in I}$. Let $H_i \subset D_i$ be a closed submonoid (resp. subgroup, resp. Lie subalgebra) scheme over C_i for all $i \in I$ and let $H = \{H_i\}_{i\in I}$. Let $\mathcal{G} := \mathrm{Bl}^D_H G \to G$ be the associated dilatation. By Lemma 3.16 the structure morphism $\mathcal{G} \to S$ defines an object in $Sch_S^{C\text{-reg}}$.

Fact 7.1. *Let $\mathcal{G} \to S$ be the above dilatations.*

(1) *The scheme $\mathcal{G} \to S$ represents the contravariant functor $Sch_S^{C\text{-reg}} \to Set$ given for $T \to S$ by the set of all S-morphisms $T \to G$ such that the induced morphism $T|_{C_i} \to G|_{C_i}$ factors through $H_i \subset G|_{C_i}$ for all $i \in \{1, \dots, k\}$.*

(2) *Let $T \to S$ be an object in $Sch_S^{C\text{-reg}}$, then as subsets of $G(T)$*

$$\mathcal{G}(T) = \bigcap_{i\in I} (\mathrm{Bl}^{D_i}_{H_i} G)(T).$$

(3) *The map $\mathcal{G} \to G$ is affine. Its restriction over S_i factors as $\mathcal{G}_i \to H_i \subset D_i$ for all $i \in I$.*

Proof. Part (1) is a reformulation of Proposition 3.17. Assertion (2) is immediate using (1). Finally (3) is immediate from Proposition 3.16. □

Proposition 7.2. *If the dilatation $\mathcal{G} \to S$ is flat, then it is equipped with the structure of a monoid (resp. group, resp. Lie algebra) scheme over S such that $\mathcal{G} \to G$ is a morphism of S-monoid (resp. S-group, resp. S-Lie algebra) schemes.*

Proof. By virtue of Fact 7.1 the (forgetful) map $\mathcal{G} \to G$ defines a submonoid (resp. subgroup, resp. Lie subalgebra) functor when restricted to the category $Sch_S^{C\text{-reg}}$. As $\mathcal{G} \to S$ is an object in $Sch_S^{C\text{-reg}}$, it is a monoid (resp. group, resp. Lie algebra) object in this category. Now if $X \to S$ and $Y \to S$ are two flat morphisms in $Sch_S^{C\text{-reg}}$, then the product of X and Y in the category

$Sch_S^{C\text{-reg}}$ exists and is equal to the product of X and Y in the full category of S-schemes. So $\mathcal{G} \to S$ is a monoid (resp. group, resp. Lie algebra) object in the full category of S-schemes. □

Dilatations of group schemes are often called Néron blowups. We note that dilatations preserve similarly structures defined using products and commutative diagrams (cf. [SGA3, Exp. I §2.2]).

REMARK 7.3. In general the product of two objects in $Sch_S^{C\text{-reg}}$ is not equal to the product of the same objects seen in the full category of S-schemes (in fact the product of these objects in the full category of S-schemes need not be in $Sch_S^{C\text{-reg}}$). Proposition 3.39 allows to compute products in $Sch_S^{C\text{-reg}}$.

PROPOSITION 7.4. *Assume that $C_i \subset S$ is a cartier divisor for all i. Assume that $G \to S$ and $H_i \to C_i$ are flat group schemes. Assume that $\mathrm{Bl}_H^D G \to S$ is flat (and so a group scheme by 7.2). Let $\mathbb{L}ie(G)/S$ (resp. $\mathbb{L}ie(H_i)/C_i$, resp. $\mathbb{L}ie(\mathrm{Bl}_H^D G)/S$) be the Lie algebra scheme of G/S (resp. H_i/C_i, resp. $\mathrm{Bl}_H^D G/S$) (cf. [SGA3, Exp. II Scholie 4.11.3]). Assume that $\mathrm{Bl}\{^{\mathbb{L}ie(G)\times_S C_i}_{\mathbb{L}ie(H_i)}\}_{i\in I}\mathbb{L}ie(G) \to S$ is flat (and so a Lie algebra scheme by 7.2). Then we have a canonical isomorphism of S-Lie algebra schemes:*

$$\mathbb{L}ie(\mathrm{Bl}_H^D G) \cong \mathrm{Bl}\{^{\mathbb{L}ie(G)\times_S C_i}_{\mathbb{L}ie(H_i)}\}_{i\in I}\mathbb{L}ie(G).$$

Proof. We have a morphism of S-group schemes $\mathrm{Bl}_H^D G \to G$, it induces a morphism of S-Lie algebra schemes $\mathbb{L}ie(\mathrm{Bl}_H^D G) \to \mathbb{L}ie(G)$. Using the universal property of dilatations of Lie algebras, we obtain a canonical morphism of S-Lie algebra schemes $\mathbb{L}ie(\mathrm{Bl}_H^D G) \to \mathrm{Bl}\{^{\mathbb{L}ie(G)\times_S C_i}_{\mathbb{L}ie(H_i)}\}_{i\in I}\mathbb{L}ie(G)$. We now show that it is an isomorphism. Using flatness and the assumptions on divisors C_i, we see that both sides belongs to $Sch_S^{C\text{-reg}}$. It is enough (to finish the proof) to evaluate both sides on a test scheme $T \to S$ in $Sch_S^{C\text{-reg}}$ and obtain an identification of sets. Recall that for any scheme U, $I_U = \mathrm{Spec}(\mathbb{Z}[X]/(X^2)) \times_{\mathrm{Spec}(\mathbb{Z})} U$ denotes the scheme of dual numbers over U. The scheme I_U is obviously flat over U. In particular $I_T \to S$ belongs to $Sch_S^{C\text{-reg}}$. We have a canonical morphism $T \to I_T$ induced by $\mathbb{Z}[X]/X^2 \to \mathbb{Z}, X \mapsto 0$. Using [SGA3, Exp. II Sch. 4.11.3, Cor. 3.9.0.2.] we get

$$\begin{aligned}
(\mathbb{L}ie(\mathrm{Bl}_H^D G))(T) &= \ker\big((\mathrm{Bl}_H^D G)(I_T) \to (\mathrm{Bl}_H^D G)(T)\big)\\
&= \big\{I_T \to \mathrm{Bl}_H^D G \in \mathrm{Hom}_S(I_T, \mathrm{Bl}_H^D G) |\ T \to I_T \to \mathrm{Bl}_H^D G \text{ is the unit } \big\}\\
&= \Big\{I_T \to G \in \mathrm{Hom}_S(I_T, G) |\ \begin{smallmatrix} T\to I_T\to G \text{ is the unit} \\ I_T|_{C_i}\to G|_{C_i} \text{ factors through } H_i \text{ for all } i\end{smallmatrix}\Big\}\\
&= \Big\{T \to \mathbb{L}ie(G) \in \mathrm{Hom}_S(T, \mathbb{L}ie(G)) |\ \ T|_{C_i}\to \mathbb{L}ie(G)|_{C_i} \text{ factors through } \mathbb{L}ie(H_i) \text{ for all } i\Big\}\\
&= \big(\mathrm{Bl}\{^{\mathbb{L}ie(G)|_{C_i}}_{\mathbb{L}ie(H_i)}\}_{i\in I}\mathbb{L}ie(G)\big)(T).
\end{aligned}$$

This finishes the proof. □

The following result generalizes the fact that congruence groups are normal subgroups, it is related to Example 1.6 (note that the proof of [Yu01, Lemma 1.4] is not correct).

PROPOSITION 7.5. *Assume that $I = \{1\}$, $C = C_1$ is a Cartier divisor in S and $G \to S$ is a flat group scheme. Let $\eta : K \to G$ be a morphism of group schemes over S such that $K \to S$ is flat. Assume that $H \subset G$ is a closed subgroup scheme over S such that $H \to S$ is flat and $\mathrm{Bl}_H^C G \to S$ is flat (and in particular a group scheme). Assume that K_C commutes with H_C in the sense that the morphism $K_C \times_C H_C \to G_C$, $(k, h) \mapsto \eta(k)h\eta(k)^{-1}$ equals the composition*

morphism $K_C \times_C H_C \to H_C \subset G_C$, $(k,h) \mapsto h$. Then K normalizes $\mathrm{Bl}^C_H G$, more precisely the solid composition map

$$\begin{array}{ccccc} K \times_S \mathrm{Bl}^C_H G & \xrightarrow{Id \times \mathrm{Bl}} & K \times_S G & \xrightarrow{k,g \mapsto \eta(k) g \eta(k)^{-1}} & G \\ & \searrow & & \nearrow_{\mathrm{Bl}} & \\ & & \mathrm{Bl}^C_H G & & \end{array}$$

factors uniquely through $\mathrm{Bl}^C_H G$.

Proof. Let ϕ be the solid composition map, we claim that it belongs to $Sch_G^{G_C\text{-reg}}$. Let us prove this claim. The map $\theta : K \times_S G \xrightarrow{k,g \mapsto \eta(k) g \eta(k)^{-1}} G$ is flat. Indeed it is the composition of an isomorphism, namely $K \times_S G \xrightarrow{(k,g) \mapsto (k, \eta(k) g \eta(k)^{-1})} K \times_S G$, with a flat morphism, namely the projection on the second factor $K \times_S G \to G$. So $\theta^{-1}(G_C)$ is a Cartier divisor in $K \times_S G$ (note that G_C is a Cartier divisor in G because $G \to S$ is flat). Now Lemma 3.33 and 3.18 shows that $(Id \times \mathrm{Bl})^{-1}(\theta^{-1}(G_C))$ is a Cartier divisor. This finishes to prove the claim. Now by Proposition 3.17, ϕ factors uniquely through $\mathrm{Bl}^C_H G$ if and only if $\phi|_{G_C}$ factors through H_C. The following diagram, obtained using Proposition 3.16, finishes the proof

$$\begin{array}{ccccc} (K \times_S \mathrm{Bl}^C_H G) \times_G G_C & \longrightarrow & (K \times_S G) \times_G G_C & \longrightarrow & G_C \\ \| & & \| & & \| \\ K_C \times_C (\mathrm{Bl}^C_H G \times_G G_C) & \longrightarrow & K_C \times_C G_C & \longrightarrow & G_C \\ & \searrow & \uparrow & & \uparrow \\ & & K_C \times_C H_C & \dashrightarrow & H_C. \end{array}$$

□

8. Congruent isomorphisms

Let $(\mathcal{O}, \pi)$ be an henselian pair where $\pi \subset \mathcal{O}$ is an invertible ideal. Let $S = \mathrm{Spec}(\mathcal{O})$ and let $C = \mathrm{Spec}(\mathcal{O}/\pi)$. If $G \to S$ is a group scheme $\mathrm{Lie}(G) \to S$ denote the underlying group schemes of $\mathbb{L}ie(G) \to S$ (cf. §7).

Theorem 8.1. *Let G be a separated and smooth goup scheme over S. Let $H_0, H_1, \dots, H_k$ be closed subgroup schemes of G such that $H_0 = e_G$ is the trivial subgroup. Let $s_0, s_1, \dots, s_k$ and $r_0, r_1, \dots, r_k$ be in $\mathbb{N}$ such that*

(i) *$s_i \geqslant s_0$ and $r_i \geqslant r_0$ for all $i \in \{0, \dots, k\}$*

(ii) *$r_i \geqslant s_i$ and $r_i - s_i \leqslant s_0$ for all $i \in \{0, \dots, k\}$.*

Assume that G is affine or $\mathcal{O}$ is local. Assume that the regularity condition (RC) introduced below is satisfied (cf. Definition 8.2). Then we have a canonical isomorphism of groups

$$\mathrm{Bl}^{s_0,\ s_1,\ \dots,s_k}_{H_0,H_1,\dots,H_k} G / \mathrm{Bl}^{r_0,\ r_1,\ \dots,r_k}_{H_0,H_1,\dots,H_k} G \cong \mathrm{Lie}(\mathrm{Bl}^{s_0,\ s_1,\ \dots,s_k}_{H_0,H_1,\dots,H_k} G) / \mathrm{Lie}(\mathrm{Bl}^{r_0,\ r_1,\ \dots,r_k}_{H_0,H_1,\dots,H_k} G).$$

Proof. For $i \in \{0, \dots, k\}$, put $t_i = r_i - s_i$. As a first step in our proof, we assume that $t_i = t_j =: t$ for all $i, j \in \{0, \dots, k\}$. Proposition 5.2 shows that

$$\mathrm{Bl}^{r_0,\ r_1,\ \dots,r_k}_{H_0,H_1,\dots,H_k} G = \mathrm{Bl}^{2t}_{H_0} \mathrm{Bl}^{s_0-t,\dots,s_k-t}_{H_0,H_1,\dots,H_k} G \text{ and } \mathrm{Bl}^{s_0,\ s_1,\ \dots,s_k}_{H_0,H_1,\dots,H_k} G = \mathrm{Bl}^{t}_{H_0} \mathrm{Bl}^{s_0-t,\dots,s_k-t}_{H_0,H_1,\dots,H_k} G.$$

Put $G' = \mathrm{Bl}^{s_0-t,\dots,s_k-t}_{H_0,H_1,\dots,H_k} G$. The scheme G' is smooth over S by (RC). By [MRR20, Theorem 4.3], we have a canonical isomorphism

$$\mathrm{Bl}^{t}_{H_0} G'/\mathrm{Bl}^{2t}_{H_0} G' \cong \mathrm{Lie}(\mathrm{Bl}^{t}_{H_0} G')/\mathrm{Lie}(\mathrm{Bl}^{2t}_{H_0} G').$$

This finishes the proof of the case where $t_i = t_j$ for all $i, j \in \{1, \dots, k\}$. Now we prove the general case. Put $t_m = \max_{i\in\{0,\dots,k\}} t_i$. The isomorphism [MRR20, Theorem 4.3] is functorial in G. Applying this functoriality to the morphism $\mathrm{Bl}^{\{s_i-t_m\}_{0\leqslant i\leqslant k}}_{\{H_i\}_{0\leqslant i\leqslant k}} G \to \mathrm{Bl}^{\{r_i-2t_m\}_{0\leqslant i\leqslant k}}_{\{H_i\}_{0\leqslant i\leqslant k}} G$ and with the integers given by the inequality $0 \leqslant \frac{2t_m}{2} \leqslant t_m \leqslant 2t_m$, we get a canonical commutative diagram

$$\begin{array}{ccc}
 & \mathrm{Bl}^{\{s_i\}_{0\leqslant i\leqslant k}}_{\{H_i\}_{0\leqslant i\leqslant k}} G(\mathcal{O})/\mathrm{Bl}^{\{r_i\}_{0\leqslant i\leqslant k}}_{\{H_i\}_{0\leqslant i\leqslant k}} G(\mathcal{O}) & \\
\nearrow & & \searrow \\
\mathrm{Bl}^{\{s_i\}_{0\leqslant i\leqslant k}}_{\{H_i\}_{0\leqslant i\leqslant k}} G(\mathcal{O})/\mathrm{Bl}^{\{s_i+t_m\}_{0\leqslant i\leqslant k}}_{\{H_i\}_{0\leqslant i\leqslant k}} G(\mathcal{O}) & \longrightarrow & \mathrm{Bl}^{\{r_i-t_m\}_{0\leqslant i\leqslant k}}_{\{H_i\}_{0\leqslant i\leqslant k}} G(\mathcal{O})/\mathrm{Bl}^{\{r_i\}_{0\leqslant i\leqslant k}}_{\{H_i\}_{0\leqslant i\leqslant k}} G(\mathcal{O}) \\
\| & & \| \\
\mathrm{Lie}(\mathrm{Bl}^{\{s_i\}_{0\leqslant i\leqslant k}}_{\{H_i\}_{0\leqslant i\leqslant k}} G)(\mathcal{O})/\mathrm{Lie}(\mathrm{Bl}^{\{s_i+t_m\}_{0\leqslant i\leqslant k}}_{\{H_i\}_{0\leqslant i\leqslant k}} G)(\mathcal{O}) & \hookrightarrow & \mathrm{Lie}(\mathrm{Bl}^{\{r_i-t_m\}_{0\leqslant i\leqslant k}}_{\{H_i\}_{0\leqslant i\leqslant k}} G)(\mathcal{O})/\mathrm{Lie}(\mathrm{Bl}^{\{r_i\}_{0\leqslant i\leqslant k}}_{\{H_i\}_{0\leqslant i\leqslant k}} G)(\mathcal{O}) \\
 & \searrow \quad \nearrow & \\
 & \mathrm{Lie}(\mathrm{Bl}^{\{s_i\}_{0\leqslant i\leqslant k}}_{\{H_i\}_{0\leqslant i\leqslant k}} G)(\mathcal{O})/\mathrm{Lie}(\mathrm{Bl}^{\{r_i\}_{0\leqslant i\leqslant k}}_{\{H_i\}_{0\leqslant i\leqslant k}} G)(\mathcal{O}). &
\end{array}$$

The injectivity of the two hookarrows follows from Remark 1.4. This identifies

$$\mathrm{Bl}^{\{s_i\}_{0\leqslant i\leqslant k}}_{\{H_i\}_{0\leqslant i\leqslant k}} G(\mathcal{O})/\mathrm{Bl}^{\{r_i\}_{0\leqslant i\leqslant k}}_{\{H_i\}_{0\leqslant i\leqslant k}} G(\mathcal{O})$$

and

$$\mathrm{Lie}(\mathrm{Bl}^{\{s_i\}_{0\leqslant i\leqslant k}}_{\{H_i\}_{0\leqslant i\leqslant k}} G)(\mathcal{O})/\mathrm{Lie}(\mathrm{Bl}^{\{r_i\}_{0\leqslant i\leqslant k}}_{\{H_i\}_{0\leqslant i\leqslant k}} G)(\mathcal{O})$$

inside the right part of the diagram. □

Definition 8.2. Let $G, \{H_i, s_i, r_i\}_{0\leqslant i\leqslant k}$ be as in Theorem 8.1. Put $t_m = \max_{i\in\{0,\dots,k\}}(r_i - s_i)$. We introduce the following regularity condition

$$\text{(RC) } \mathrm{Bl}^{\{s_i-t_m\}_{0\leqslant i\leqslant k}}_{\{H_i\}_{0\leqslant i\leqslant k}} G \text{ and } \mathrm{Bl}^{\{r_i-2t_m\}_{0\leqslant i\leqslant k}}_{\{H_i\}_{0\leqslant i\leqslant k}} G \text{ are smooth over } S.$$

We recall that Proposition 6.1 offers a way to check (RC) in many cases. We finish with the following result.

Corollary 8.3. *Let G be a separated and smooth goup scheme over S. Let $H_0 \subset H_1 \subset \ldots \subset H_k$ be closed subgroup schemes of G such that H_i is smooth over S for $0 \leqslant i \leqslant d$ and $H_0 = e_G$. Let $s_0, s_1, \dots, s_k$ and $r_0, r_1, \dots, r_k$ be in $\mathbb{N}$ such that*

(i) *$s_i \geqslant s_0$ and $r_i \geqslant r_0$ for all $i \in \{0, \dots, k\}$*

(ii) *$r_i \geqslant s_i$ and $r_i - s_i \leqslant s_0$ for all $i \in \{0, \dots, k\}$.*

Assume that G is affine or $\mathcal{O}$ is local. Then we have a canonical isomorphism of groups

$$\mathrm{Bl}^{s_0,\ s_1,\ \dots,s_k}_{H_0,H_1,\dots,H_k} G/\mathrm{Bl}^{r_0,\ r_1,\ \dots,r_k}_{H_0,H_1,\dots,H_k} G \cong \mathrm{Lie}(\mathrm{Bl}^{s_0,\ s_1,\ \dots,s_k}_{H_0,H_1,\dots,H_k} G)/\mathrm{Lie}(\mathrm{Bl}^{r_0,\ r_1,\ \dots,r_k}_{H_0,H_1,\dots,H_k} G).$$

Proof. By Theorem 8.1, it is enough to check the condition (RC), it follows from [SGA3, Exp. III, Proposition 4.15] and Proposition 6.1. □

Remark 8.4. Note that the result of the proof of [SGA3, Exp. III, Proposition 4.15] is stronger than its statement. Indeed the statement uses Koszul-regularity and the proof shows regularity (in the terminology of [StP, Tag 063J]).

9. Interpretation of Rost double deformation space as dilatations

We interpret Rost double deformation space [Ro96] in the language of dilatations. This section emerged after a question of A. Dubouloz. Let $Z \to Y \to X$ be closed immersions (in [Ro96], all schemes are assumed to be defined over fields but we work with arbitrary schemes here). Let $\overline{D}(X,Y,Z)$ be the double deformation space as defined in [Ro96, §10]. Let $\mathbb{A}^2$ be $\mathrm{Spec}(\mathbb{Z}[s,t])$. Let D_s, D_{st} and D_{s^2t} be the locally principal closed subschemes of $\mathbb{A}^2$ defined by the ideals $(s),(st)$ and (s^2t). We now omit the subscript $_{\mathrm{Spec}(\mathbb{Z})}$ in fiber products.

Proposition 9.1. *We have a canonical identification*

$$\overline{D}(X,Y,Z) \cong \mathrm{Bl}^{(X\times D_{st}),\ (X\times D_s)}_{(Y\times\mathbb{A}^2),\ (Z\times\mathbb{A}^2)}(X\times\mathbb{A}^2).$$

In other words, Rost double deformation space is canonically interpreted as a "double-centered" dilatation.

Proof. The definition of $\overline{D}(X,Y,Z)$ is given in [Ro96, 10.5] locally for affine schemes. So we reduce to the case where $X=\mathrm{Spec}(A)$, $Y=\mathrm{Spec}(A/I)$, $Z=\mathrm{Spec}(A/J)$ are affine. Then $\overline{D}(X,Y,Z)$ is defined as the spectrum of the ring $O_{\overline{D}}=\sum_{n,m} I^nJ^{m-n}t^{-n}s^{-m}\subset A[t,s,t^{-1},s^{-1}]\cong A\otimes_{\mathbb{Z}}\mathbb{Z}[t,s,t^{-1},s^{-1}]$ where $I^k=J^k=A$ for $k<0$ as in [Ro96, §10.2].

We claim that $O_{\overline{D}}$ is equal to the sub-$A[t,s]$-algebra of $A[t,s,t^{-1},s^{-1}]$ generated by $I(ts)^{-1}$ and Js^{-1}. Indeed, let $a,b\in\mathbb{N}$ and put $m=a+b$ and $n=a$. Then $(I(ts)^{-1})^a(Js^{-1})^b = I^nJ^{m-n}t^{-n}s^{-m}$. So $(A[t,s])[I(ts)^{-1},Js^{-1}]$ is included in $O_{\overline{D}}$. Reciprocally, let $n,m\in\mathbb{Z}$. Assume firstly that $n<0$ and put $l=-n>0$, then

$$I^nJ^{m-n}t^{-n}s^{-m}=J^{m+l}t^ls^{-m}\subset J^ms^{-m}t^l\subset A[t,s][Js^{-1}]\subset A[t,s][I(ts)^{-1},Js^{-1}].$$

Assume secondly that $n\geqslant 0$ and $m\geqslant n$ and put $a=n$ and $b=m-n\geqslant 0$, then

$$I^nJ^{m-n}t^{-n}s^{-m}=(I(ts)^{-1})^a(Js^{-1})^b\subset A[t,s][I(ts)^{-1},Js^{-1}].$$

Assume thirdly that $n\geqslant 0$ and $m-n<0$ and put $c=n-m>0$, then

$$I^nJ^{m-n}t^{-n}s^{-m}=(I(ts)^{-1})^ns^c\subset A[t,s][I(ts)^{-1}]\subset A[t,s][I(ts)^{-1},Js^{-1}].$$

So in all cases, $I^nJ^{m-n}t^{-n}s^{-m}\subset A[t,s][I(ts)^{-1},Js^{-1}$. This finishes to prove our claim. Now Fact 2.40 finishes the proof of Proposition 9.1. □

Corollary 9.2. *Rost double deformation space is canonically isomorphic to a mono-centered dilatation as follows:*

$$\overline{D}(X,Y,Z)\cong \mathrm{Bl}^{X\times D_{s^2t}}_{\big((Y\times\mathbb{A}^2)+(X\times D_s)\big)\cap\big((Z\times\mathbb{A}^2)+(X\times D_{st})\big)}(X\times\mathbb{A}^2).$$

Proof. This follows from Proposition 9.1, the monopoly isomorphism 3.31 and the identity

$$(X\times D_{st})+(X\times D_s)=X\times(D_{st}+D_s)=X\times D_{s^2t}.$$

□

References

Ana73 S. Anantharaman: *Schémas en groupes: espaces homogènes et espaces algébriques sur une base de dimension* 1, Sur les groupes algébriques, Soc. Math. France, Paris (1973), pp. 5-79. Bull. Soc. Math. France, Mém. **33**. 3

Ar69 M. Artin, *Algebraic approximation of structures over complete local rings.* Inst. Hautes Études Sci. Publ. Math. **36** (1969) 23 – 58. 3

Ar99 M. Artin: *Non commutative rings.* class notes Math 251 Berkeley 1999 https://math.mit.edu/~zyun/Artin_notes.pdf

AZ55 S. S. Abhyankar and O. Zariski, *Splitting of valuations in extensions of local domains.* Proc. Natl. Acad. Sci. USA, 41 (1955), 84–90. 3

BLR90 S. Bosch, W. Lütkebohmert and M. Raynaud: *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **21**, Berlin, New York: Springer-Verlag. 3, 6

BS07 H. Brenner, S. Schröer: *Ample families, multihomogeneous spectra, and algebraization of formal schemes*, Pacific Journal of Mathematics (2003), we use a new and corrected version (2007) available at https://arxiv.org/pdf/math/0012082.pdf. 24, 25

Da67 E. Davis: *Ideals of the principal class, R-sequences and a certain monoidal transformation.* Pacific J. Math. 20 (1967), 197–205. 3

DHdS18 N. D. Duong, P. H. Hai, J. P. dos Santos: *On the structure of affine flat group schemes over discrete valuation rings, I.* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 18 (2018), no. 3, 977–1032. 3, 16

DMdS23 A. Dubouloz, A. Mayeux, J. P. dos Santos: A survey on algebraic dilatations, 2023 1, 6

Du05 A. Dubouloz: *Quelques remarques sur la notion de modification affine.* https://arxiv.org/abs/math/0503142, 2005. 3, 6, 24

GW20 U. Görtz, T. Wedhorn: *Algebraic Geometry I*, Second edition, Springer, (2020). 25

JJ15 J. Jaiung: *Algebraic geometry over semi-structures and hyper-structures of characteristic one*, Ph.D. thesis at The Johns Hopkins University, (2015).

Ka94 S. Kaliman: *Exotic analytic structures and Eisenman intrinsic measures.* Israel J. Math. 88 (1994), no. 1-3, 411–423. 3

KP22 T. Kaletha, G. Prasad: *Bruhat-Tits Theory: A New Approach*, New Mathematical Monographs, Series Number 44, 2022 7

KZ99 S. Kaliman, M. Zaidenberg: *Affine modifications and affine hypersurfaces with a very transitive automorphism group.* Transform. Groups 4 (1999), no. 1, 53–95. 3, 6

Ma23c A. Mayeux: *Dilatations of categories*, https://arxiv.org/abs/2305.11303, 2023 3, 7, 10

MRR20 A. Mayeux, T. Richarz, M. Romagny: *Néron blowups and low-degree cohomological applications*, https://arxiv.org/abs/2001.03597, to appear in Algebraic Geometry. 3, 6, 7, 30, 31, 34

Ner64 A. Néron: *Modèles minimaux des variétés abéliennes sur les corps locaux et globaux*, Inst. Hautes Études Sci. Publ. Math. No. 21 (1964), 128 pp. 3

No1884 Max Noether, *Rationale Ausführung der Operationen in der Theorie der algebraischen Functionen.* Math. Ann. 23, 1884, 311–358. 3

PY06 G. Prasad, J.-K. Yu: *On quasi-reductive group schemes*, J. Algebraic Geom. **15** (2006), 507–549. With an appendix by Brian Conrad. 3, 16

PZ13 G. Pappas, X. Zhu: *Local models of Shimura varieties and a conjecture of Kottwitz*, Invent. Math. **194** (2013), 147–254. 3

Ro96 M. Rost: *Chow groups with coefficients.* Doc. Math.1(1996), No. 16, 319–393. 7, 35

SGA3 M. Demazure, A. Grothendieck: *Séminaire de Géométrie Algébrique du Bois Marie - 1962-64 - Schémas en groupes - (SGA 3) Tome I. Propriétés générales des schémas en groupes.* Revised and annotated edition of the 1970 French original, (2011). Edited by Philippe Gille and Patrick Polo. 3, 16, 31, 32, 35

StP The Stacks project authors: *The Stacks Project*, https://stacks.math.columbia.edu, (2022). 16, 17, 18, 19, 20, 22, 24, 25, 26, 29, 30, 35

WW80 W. C. Waterhouse, B. Weisfeiler: *One-dimensional affine group schemes*, J. Algebra **66** (1980), 550–568. 3, 6

Yu01 J.-K. Yu: *Construction of tame supercuspidal representations*, J. Amer. Math. Soc. 14 (2001). 5, 6, 7, 32

Yu15 J.-K. Yu: *Smooth models associated to concave functions in Bruhat-Tits Theory*, Autour des schémas en groupes. Vol. III, 227–258, Panor. Synthèses, 47, Soc. Math. France, Paris, 2015. 5

Za43 O. Zariski: *Foundations of a general theory of birational correspondences,* Trans. Amer. Math. Soc. 53 (1943). 3

Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem, Israel
E-mail address: arnaud.mayeux@mail.huji.ac.il