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# Multi-centered dilatations and congruent isomorphisms

Arnaud Mayeux

March 15, 2023

ABSTRACT. We introduce a multi-centered formalism in the context of dilatations of schemes and algebraic spaces. In particular, this provides new results on the behavior of [MRR20] dilatations. Furthermore, we use our new formalism to deduce some refined congruent isomorphisms.

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## 1. Introduction

### 1.1 Paradigm

1.1.1 *Intrinsic questions and results on dilatations.* Given a scheme  $X$  and closed subschemes  $Z \subset D \subset X$ , such that  $D$  is locally principal, [MRR20] introduces a scheme  $\mathrm{Bl}_Z^D X$

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and an affine morphism  $f : \mathrm{Bl}_Z^D X \rightarrow X$ . The scheme  $\mathrm{Bl}_Z^D X$  is called the dilatation of  $X$  with center  $Z, D$ . Dilatations provide a fundamental tool in mathematics (cf. [MRR20, Introduction]) and so it is natural to study the theory of dilatations intrinsically. The construction of  $\mathrm{Bl}_Z^D X$  in [MRR20] is both very explicit and conceptual and is characterized by the fact that it satisfies, in a universal way, that  $f^{-1}(D) = f^{-1}(Z)$  and that  $f^{-1}(D)$  is a Cartier divisor. Now given other pairs  $\{Z_i \subset D_i\}_{i \in I}$  of closed subschemes of  $X$  such that locally each  $D_i$  is principal, we can ask the following question:

*Is there a scheme  $X_?$  and a morphism  $f : X_? \rightarrow X$  characterized by the fact that it satisfies, in a universal way, that  $f^{-1}(D_i) = f^{-1}(Z_i)$  and that  $f^{-1}(D_i)$  is a Cartier divisor for all  $i \in I$  ?*

This article gives a positive answer to this question: there is a unique  $X_?$  (up to canonical  $X$ -isomorphism). It is natural to denote  $X_?$  by  $\mathrm{Bl}_{\{Z_i\}_{i \in I}}^{\{D_i\}_{i \in I}} X$  and we do it from now on. If  $\#I$  is finite, there are several non-trivial ways to describe  $\mathrm{Bl}_{\{Z_i\}_{i \in I}}^{\{D_i\}_{i \in I}} X$  using [MRR20] as follows.

*Assume  $\#I$  is finite and fix an arbitrary bijection  $I = \{1, \dots, k\}$ . The scheme  $\mathrm{Bl}_{\{Z_i\}_{i \in I}}^{\{D_i\}_{i \in I}} X$  exists and is unique up to canonical  $X$ -isomorphism. Moreover we have canonical  $X$ -isomorphisms*

$$\begin{aligned} \mathrm{Bl}_{\{Z_i\}_{i \in I}}^{\{D_i\}_{i \in I}} X &\stackrel{\dagger}{\cong} \mathrm{Bl}_{(\mathrm{Bl}^{\dots}) \times_X Z_k}^{(\mathrm{Bl}^{\dots}) \times_X D_k} \left( \dots \mathrm{Bl}_{(\mathrm{Bl}^{\dots}) \times_X Z_3}^{(\mathrm{Bl}^{\dots}) \times_X D_3} \left( \mathrm{Bl}_{(\mathrm{Bl}_{Z_1}^{D_1} X) \times_X Z_2}^{(\mathrm{Bl}_{Z_1}^{D_1} X) \times_X D_2} \left( \mathrm{Bl}_{Z_1}^{D_1} X \right) \right) \right) \\ &\stackrel{\dagger\dagger}{\cong} \mathrm{Bl}_{\bigcap_{i \in I} (Z_i + D_1 + \dots + D_{i-1} + D_{i+1} + \dots + D_k)}^{D_1 + \dots + D_k} X \end{aligned}$$

where  $+$  and  $\cap$  are the usual operations underlying the semiring structure on the set of closed subschemes of  $X$  (3.1, 3.16, 3.27, 3.28).

As we claimed before, in this article, we prove the existence of  $\mathrm{Bl}_{\{Z_i\}_{i \in I}}^{\{D_i\}_{i \in I}} X$  for arbitrary  $I$  (3.16). Moreover, we work in the natural framework of algebraic spaces (3.16). We call our formalism multi-centered dilatations. We prove several results leading to information on the arithmetic of multi-centered dilatations. We prove that dilatations are highly functorial and compatible with many operations (e.g. cf. 3.23, 3.25, 3.31, 4.6, 4.8, 5.2). We prove some results on smoothness and flatness of multi-centered dilatations (6.1). Thanks to the isomorphism  $\dagger\dagger$ , in the case of schemes and if  $\#I$  is finite, our results provide new informations on the arithmetic of [MRR20] dilatations. We would like to notice here that in the rest of the article, we remove the condition that  $Z_i$  is included in  $D_i$ , this is a convenient detail that does not affect the substantial arithmetic of dilatations (cf. Fact 3.11).

**1.1.2 Dilatations of rings and semirings.** To define and study multi-centered dilatations of schemes and algebraic spaces, we first introduce a new formalism in the framework of commutative algebra, called dilatations of rings and semirings. We believe this is the most natural and intelligible way to introduce multi-centered dilatations. Let  $A$  be a semiring, let  $\{M_i\}_{i \in I}$  be ideals in  $A$  and let  $\{a_i\}_{i \in I}$  be elements in  $A$ . In this situation we define an  $A$ -semiring  $A[\{\frac{M_i}{a_i}\}_{i \in I}]$  (2.1). Intuitively, this  $A$ -semiring is the most natural  $A$ -semiring that contains the symbols  $\frac{m_i}{a_i}$  with  $i \in I, m_i \in M_i$ . Dilatations of semirings (resp. rings) fully generalize localizations of semirings (resp. rings), indeed let  $S$  be a submonoid of  $(A, \times)$  (sometimes called a multiplicative subset), then (2.9)

$$S^{-1}A = A[\{\frac{A}{s}\}_{s \in S}].$$

If  $I = \{1, \dots, k\}$  is finite, we prove that we have (2.25)

$$(\ddagger) \quad A[\{\frac{M_i}{a_i}\}_{i \in I}] \cong A[\frac{\sum_{i \in I} (M_i \cdot \prod_{j \in I \setminus \{i\}} a_j)}{a_1 \cdots a_k}].$$

Note that  $\ddagger$  is the algebraic version of  $\dagger\dagger$  in some sense. If  $A$  is a ring,  $I = \{1\}$  and  $a_1 \in M_1$  then  $A[\frac{M_1}{a_1}]$  is defined in [StP, Tag 052P] and used in [MRR20] (this  $A$ -ring is called an affine blowup algebra in [StP]). We prove several results on dilatations of rings and semirings (cf. Section 2 and 5.1, 5.3, 5.4, 5.5). This finishes to introduce the first goal of this paper: to provide intrinsic results on the theory of dilatations.

## 1.2 Statement of some formal results and congruent isomorphisms

We now continue this introduction explaining a second goal of this article: to prove some congruent isomorphisms. To introduce congruent isomorphisms, we start with an arithmetical observation. Then, we summarize some preliminaries necessary to state our results regarding congruent isomorphisms.

**1.2.1 Arithmetical observation.** Let  $G$  be a smooth group scheme over  $\mathbb{Z}_p$  for a prime number  $p$ . Put  $\mathfrak{p} = p\mathbb{Z}_p$ . For any  $n \in \mathbb{N}$ , let  $G_n$  be the  $n$ -th congruence group of  $G$  (namely the dilatation  $\text{Bl}_{e_{G_{\mathbb{Z}_p/p^n}}}^{G_{\mathbb{Z}_p/p^n}} G$  in the sense of [MRR20]). Recall that it is a group scheme over  $\mathbb{Z}_p$  that satisfies

$$G_n(\mathbb{Z}_p) = \ker (G(\mathbb{Z}_p) \rightarrow G(\mathbb{Z}_p/\mathfrak{p}^n)).$$

For any pair of integers  $(s, r)$  such that  $0 \leq \frac{r}{2} \leq s \leq r$ , [MRR20, Theorem 4.3], based on the theory of dilatations, provides a canonical isomorphism

$$(\star) \quad G_s(\mathbb{Z}_p)/G_r(\mathbb{Z}_p) \cong \mathfrak{g}_s(\mathbb{Z}_p)/\mathfrak{g}_r(\mathbb{Z}_p)$$

where  $\mathfrak{g}_n$  is the Lie algebra of  $G_n$ . If  $G = GL_2/\mathbb{Z}_p$ ,  $G_n(\mathbb{Z}_p) = \begin{pmatrix} 1 + \mathfrak{p}^n & \mathfrak{p}^n \\ \mathfrak{p}^n & 1 + \mathfrak{p}^n \end{pmatrix} \subset GL_2(\mathbb{Z}_p)$  and  $\mathfrak{g}_n(\mathbb{Z}_p) = \begin{pmatrix} \mathfrak{p}^n & \mathfrak{p}^n \\ \mathfrak{p}^n & \mathfrak{p}^n \end{pmatrix} \subset M_2(\mathbb{Z}_p)$  for any  $n > 0$ . The isomorphism  $(\star)$  gives us, for pairs  $(r, s)$  such that  $0 < \frac{r}{2} \leq s \leq r$ , isomorphisms

$$(*) \quad \begin{pmatrix} 1 + \mathfrak{p}^s & \mathfrak{p}^s \\ \mathfrak{p}^s & 1 + \mathfrak{p}^s \end{pmatrix} / \begin{pmatrix} 1 + \mathfrak{p}^r & \mathfrak{p}^r \\ \mathfrak{p}^r & 1 + \mathfrak{p}^r \end{pmatrix} \cong \begin{pmatrix} \mathfrak{p}^s & \mathfrak{p}^s \\ \mathfrak{p}^s & \mathfrak{p}^s \end{pmatrix} / \begin{pmatrix} \mathfrak{p}^r & \mathfrak{p}^r \\ \mathfrak{p}^r & \mathfrak{p}^r \end{pmatrix}.$$

These maps are given by  $[1 + M] \mapsto [M]$ . Using the formula  $[1 + M] \mapsto [M]$ , it is elementary to check that we have other isomorphisms of abstract groups

$$(**) \quad \begin{pmatrix} 1 + \mathfrak{p}^3 & \mathfrak{p}^3 \\ \mathfrak{p}^3 & 1 + \mathfrak{p}^3 \end{pmatrix} / \begin{pmatrix} 1 + \mathfrak{p}^5 & \mathfrak{p}^6 \\ \mathfrak{p}^6 & 1 + \mathfrak{p}^5 \end{pmatrix} \cong \begin{pmatrix} \mathfrak{p}^3 & \mathfrak{p}^3 \\ \mathfrak{p}^3 & \mathfrak{p}^3 \end{pmatrix} / \begin{pmatrix} \mathfrak{p}^5 & \mathfrak{p}^6 \\ \mathfrak{p}^6 & \mathfrak{p}^5 \end{pmatrix},$$

$$(***) \quad \begin{pmatrix} 1 + \mathfrak{p}^3 & \mathfrak{p}^9 \\ \mathfrak{p}^3 & 1 + \mathfrak{p}^3 \end{pmatrix} / \begin{pmatrix} 1 + \mathfrak{p}^6 & \mathfrak{p}^9 \\ \mathfrak{p}^6 & 1 + \mathfrak{p}^6 \end{pmatrix} \cong \begin{pmatrix} \mathfrak{p}^3 & \mathfrak{p}^9 \\ \mathfrak{p}^3 & \mathfrak{p}^3 \end{pmatrix} / \begin{pmatrix} \mathfrak{p}^6 & \mathfrak{p}^9 \\ \mathfrak{p}^6 & \mathfrak{p}^6 \end{pmatrix}.$$

The new tool introduced in this article –multi-centered dilatations– allows to extend the generality offered by [MRR20, Theorem 4.3] so that the new result will also cover  $(**)$  and  $(***)$  in an intelligible way, cf. Theorem 1.5 and Example 1.7 below. We emphasize here that our proof of Theorem 1.5 relies on [MRR20, Theorem 4.3] and the arithmetic of multi-centered dilatations.

**1.2.2 Definition of multi-centered dilatations in the absolute setting.** Let  $S$  be a scheme and let  $X$  be an algebraic space over  $S$ . Let  $\{[Y_i, D_i]\}_{i \in I}$  be pairs of closed  $S$ -subspace of  $X$  such that

locally on  $X$ , each  $D_i, i \in I$  is principal. We define an  $S$ -algebraic space  $\mathrm{Bl}_Y^D X$  together with a morphism  $\mathrm{Bl}_Y^D X \rightarrow X$ . This construction enjoys the following properties.

**THEOREM 1.1.** *Let  $\mathrm{Spaces}_X^{D\text{-reg}}$  be the category of  $X$ -algebraic spaces  $f : T \rightarrow X$  such that  $T \times_X D_i$  is an effective Cartier divisor on  $T$  for any  $i \in I$ .*

(i)  $\mathrm{Bl}_Y^D X$  represents the functor from  $\mathrm{Spaces}_X^{D\text{-reg}}$  to Sets given by

$$(f : T \rightarrow X) \longmapsto \begin{cases} \{*\}, & \text{if } f|_{T \times_X D_i} \text{ factors through } Y_i \cap D_i \subset X \text{ for } i \in \{1, \dots, k\}; \\ \emptyset, & \text{else.} \end{cases} \quad (3.16)$$

(ii) The structural morphism  $\mathrm{Bl}_Y^D X \rightarrow X$  is affine (3.8).

(iii) If  $X$  is a scheme, then  $\mathrm{Bl}_Y^D X$  is a scheme (3.12).

(iv) For  $\alpha \in \mathbb{N}^I$ , we put  $D^\alpha = \{\alpha_i D_i\}_{i \in I}$ . Let  $\nu, \beta \in \mathbb{N}^I$  such that  $\nu \leq \beta$  (i.e.  $\nu_i \leq \beta_i$  for all  $i$ ), then there is a unique morphism of  $X$ -algebraic spaces (4.1)

$$\varphi_{\beta, \nu} : \mathrm{Bl}_Y^{D^\beta} X \rightarrow \mathrm{Bl}_Y^{D^\nu} X.$$

Assume moreover that  $\alpha, \nu \in \mathbb{N}_I := \bigoplus_{i \in I} \mathbb{N} \subset \mathbb{N}^I$  and that  $Y_i \cap D_i \subset Y_i$  is a Cartier divisor for all  $i$ , then  $\varphi_{\beta, \nu}$  is a dilatation map (3.14, 4.5).

**1.2.3 Multi-centered dilatations in the relative setting.** We also define dilatations in a relative setting as follows. Let  $S$  be a scheme and let  $X$  be an algebraic space over  $S$ . Let  $C = \{C_i\}_{i \in I}$  be closed subschemes of  $S$  such that, locally, each  $C_i$  is principal. Put  $D = \{C_i \times_S X\}_{i \in I}$ . Let  $Y = \{Y_i\}_{i \in I}$  be closed  $S$ -subspaces of  $X$ . We put  $\mathrm{Bl}_Y^C X := \mathrm{Bl}_Y^D X$ .

**FACT 1.2.**  $\mathrm{Bl}_Y^C X$  represents the functor from  $\mathrm{Sch}_S^{C\text{-reg}}$  to Sets given by

$$(f : T \rightarrow S) \mapsto \{x \in \mathrm{Hom}_S(T, X) \mid x|_{C_i} : T \times_S C_i \rightarrow X \times_S C_i \text{ factors through } Y_i \times_S C_i\}.$$

**REMARK 1.3.** Fact 1.2 implies that for any  $T \in \mathrm{Sch}_S^{C\text{-reg}}$  (e.g.  $T = S$  if each  $C_i$  is a Cartier divisor in  $S$ ) we have a canonical inclusion on  $T$ -points  $\mathrm{Bl}_Y^C X(T) \subset X(T)$ . But in general  $\mathrm{Bl}_Y^C X \rightarrow X$  is not a monomorphism in the full category of  $S$ -schemes.

**1.2.4 The case of group schemes: multi-centered Néron blowups.** Let  $S$  be a scheme and let  $G$  be a group algebraic space over  $S$ . Let  $C = \{C_i\}_{i \in I}$  be as in §1.2.3. Let  $H = \{H_i\}_{i \in I}$  be closed subgroup spaces over  $S$  in  $G$ . The dilatation  $\mathrm{Bl}_H^C G$  is called the Néron blowup of  $G$  with multi-center  $H, C$ . Now assume that  $S = \mathrm{Spec}(\mathcal{O})$  and  $C = \mathrm{Spec}(\mathcal{O}/\pi)$  where  $(\mathcal{O}, \pi)$  is an henselian pair and  $\pi \subset \mathcal{O}$  is an invertible ideal. Assume  $I = \{1, \dots, k\}$ . In the following we use the notation  $\mathrm{Bl}_{H_1, \dots, H_k}^{s_1, \dots, s_k} G$  to denote a Néron blowup  $\mathrm{Bl}_{H_1, \dots, H_k}^{s_1 C_1, \dots, s_k C_k} G$  where  $s_1, \dots, s_k \in \mathbb{N}^k$ . We think that multi-centered dilatations and Néron blowups are very natural. As an evidence let us indicate a connexion with Yu's famous construction of supercuspidal representations [Yu01].

**EXAMPLE 1.4.** Assume in this example that  $\mathcal{O}$  is the ring of integers of a non-Archimedean local field  $F$  and that  $\pi$  is the maximal ideal of  $\mathcal{O}$ . Let  $G$  be a split connected reductive group scheme over  $\mathcal{O}$ , i.e. a Demazure group scheme over  $\mathcal{O}$ . Let  $\vec{G} = (G^0 \subset G^1 \subset \dots \subset G^d = G)$  be a sequence of split Levi subgroups of  $G$  over  $\mathcal{O}$ . Put  $G_i = G_i \times_{\mathcal{O}} F$  for all  $i \in \{0, \dots, k\}$ . Put  $\vec{G} = (G^0 \subset G^1 \subset \dots \subset G^d =: G)$ , a sequence of split Levi subgroups of  $G$  over  $F$ . For  $i \in \{0, \dots, d\}$ , let  $x_i$  be the special point in the Bruhat-Tits building of  $G_i$  such that  $G_i$  corresponds to  $x_i$  via Bruhat-Tits theory. Then  $x_i$  comes from  $x_0$  via functoriality of buildings

as in [Yu01, §1]. Let  $0 \leq r_0 \leq r_1 \leq \dots \leq r_d$  be integers. Recall that  $e_G$  denotes the trivial closed subgroup scheme of  $G$ . There is a canonical isomorphism of groups

$$\vec{G}(F)_{x, \vec{r}} = \mathrm{Bl}_{e_G, G^0, G^1, \dots, G^i, \dots, G^{d-1}}^{r_0, r_1, r_2, \dots, r_{i+1}, \dots, r_d} G(\mathcal{O}),$$

where  $\vec{G}(F)_{x, \vec{r}}$  is defined in [Yu01, §1 p584]. Details and other examples will appear elsewhere.

**THEOREM 1.5.** (*Congruent isomorphism*) *Let  $G$  be a separated and smooth group scheme over  $S$ . Let  $H_0, H_1, \dots, H_k$  be closed subgroup schemes of  $G$  such that  $H_0 = e_G$  is the trivial subgroup and such that  $H_i \rightarrow S$  is smooth for all  $i \in \{0, \dots, k\}$ . Let  $s_0, s_1, \dots, s_k$  and  $r_0, r_1, \dots, r_k$  be in  $\mathbb{N}$  such that*

- (i)  $s_i \geq s_0$  and  $r_i \geq r_0$  for all  $i \in \{0, \dots, k\}$
- (ii)  $r_i \geq s_i$  and  $r_i - s_i \leq s_0$  for all  $i \in \{0, \dots, k\}$ .

*Assume that  $G$  is affine or  $\mathcal{O}$  is local. Assume that a regularity condition (RC) is satisfied (cf. Definition 8.2). Then we have a canonical isomorphism of groups (8.1)*

$$\mathrm{Bl}_{H_0, H_1, \dots, H_k}^{s_0, s_1, \dots, s_k} G(\mathcal{O}) / \mathrm{Bl}_{H_0, H_1, \dots, H_k}^{r_0, r_1, \dots, r_k} G(\mathcal{O}) \cong \mathrm{Lie}(\mathrm{Bl}_{H_0, H_1, \dots, H_k}^{s_0, s_1, \dots, s_k} G)(\mathcal{O}) / \mathrm{Lie}(\mathrm{Bl}_{H_0, H_1, \dots, H_k}^{r_0, r_1, \dots, r_k} G)(\mathcal{O}).$$

**REMARK 1.6.** [MRR20, Theorem 4.3] is exactly Theorem 1.5 with  $k = 0$  in the notation of 1.5.

**EXAMPLE 1.7.** Let  $G$  be  $GL_2/\mathbb{Z}_p$ . Let  $e_G \subset G$  be the trivial subgroup. Let  $T \simeq D(\mathbb{Z}^2)_{\mathbb{Z}_p}$  be the diagonal split torus in  $G$ . Let  $B$  be the lower triangular Borel in  $G$  over  $\mathbb{Z}_p$ .

- (i) The isomorphism (\*\*\*) above is given by Theorem 1.5 with  $S = \mathrm{Spec}(\mathbb{Z}_p)$ ,  $C = \mathrm{Spec}(\mathbb{Z}_p/\mathfrak{p})$ ,  $H_0 = e_G$ ,  $H_1 = T$ ,  $s_0 = 3$ ,  $s_1 = 3$ ,  $r_0 = 5$  and  $r_1 = 6$ .
- (ii) The isomorphism (\*\*\*) above is given by Theorem 1.5 with  $S = \mathrm{Spec}(\mathbb{Z}_p)$ ,  $C = \mathrm{Spec}(\mathbb{Z}_p/\mathfrak{p})$ ,  $H_0 = e_G$ ,  $H_1 = B$ ,  $s_0 = 3$ ,  $s_1 = 9$ ,  $r_0 = 6$  and  $r_1 = 9$ .

### 1.3 Structure of the paper

Section 2 introduces dilatations of rings and semirings. Section 3 introduces multi-centered dilatations. Section 4 deals with iterated dilatations. Section 5 focuses on the case where the multi-center  $\{[Z_i, D_i]\}_{i \in I}$  satisfies that  $\{D_i\}$  are given by multiples of a single  $D$ . Section 6 proves some flatness and smoothness results. Section 7 considers Néron blowups. Section 8 studies congruent isomorphisms.

## 2. Dilatations of semirings

We introduce dilatations of semirings and rings. If  $A$  is a semiring, an  $A$ -semiring is a morphism of semirings  $A \rightarrow B$ . If  $A$  is a ring, an  $A$ -ring is a morphism of rings  $A \rightarrow B$ . Recall that  $A$ -rings are sometimes called  $A$ -algebras. In other parts of this works, relying on [StP], we use dilatations of rings only. Note that semirings are used in other references, e.g. [JJ15, §2.2].

### 2.1 Definition

Let  $A$  be a unital commutative semiring. If  $M$  is an ideal of  $A$  and  $a \in A$  is an element, we say that the pair  $[M, a]$  is a center in  $A$ . Let  $I$  be an index set and let  $\{[M_i, a_i]\}_{i \in I}$  be a set of centers indexed by  $I$ . For  $i \in I$ , we put  $L_i = M_i + (a_i)$ , an ideal of  $A$ . Let  $\mathbb{N}_I$  be the monoid defined as  $\bigoplus_{i \in I} \mathbb{N}$ . If  $\nu = (\nu_1, \dots, \nu_i, \dots) \in \mathbb{N}_I$  we put  $L^\nu = L_1^{\nu_1} \cdots L_i^{\nu_i} \cdots$  (product of ideals of  $A$ ) and  $a^\nu = a_1^{\nu_1} \cdots a_i^{\nu_i} \cdots$  (product of elements of  $A$ ). We also put  $a^{\mathbb{N}_I} = \{a^\nu | \nu \in \mathbb{N}_I\}$ .

DEFINITION AND PROPOSITION 2.1. *The dilatation of  $A$  with multi-center  $\{[M_i, a_i]\}_{i \in I}$  is the unital commutative semiring  $A[\{\frac{M_i}{a_i}\}_{i \in I}]$  defined as follows:*

- *The underlying set of  $A[\{\frac{M_i}{a_i}\}_{i \in I}]$  is the set of equivalence classes  $\frac{m}{a^\nu}$  where  $\nu \in \mathbb{N}_I$  and  $m \in L^\nu$  under the relation*

$$\frac{m}{a^\nu} \equiv \frac{p}{a^\lambda} \Leftrightarrow \exists \beta \in \mathbb{N}_I \text{ such that } ma^{\beta+\lambda} = pa^{\beta+\nu} \text{ in } A.$$

- *The addition law is given by  $\frac{m}{a^\nu} + \frac{p}{a^\beta} = \frac{ma^{\beta+\nu} + pa^\nu}{a^{\beta+\nu}}$ .*
- *The multiplication law is given by  $\frac{m}{a^\nu} \times \frac{p}{a^\beta} = \frac{mp}{a^{\nu+\beta}}$ .*
- *The additive neutral element is  $\frac{0}{1}$  and the multiplicative neutral element is  $\frac{1}{1}$ .*

We have a canonical morphism of semirings  $A \rightarrow A[\{\frac{M_i}{a_i}\}_{i \in I}]$  given by  $a \mapsto \frac{a}{1}$ . We sometimes use the notation  $A[\frac{M}{a}]$  to denote  $A[\{\frac{M_i}{a_i}\}_{i \in I}]$ .

*Proof.* Let us first prove that the relation is an equivalence relation. Assume

$$\begin{aligned} \frac{m}{a^\nu} &\equiv \frac{p}{a^\lambda} \\ \frac{l}{a^\theta} &\equiv \frac{p}{a^\lambda} \end{aligned}$$

where  $\nu, \lambda, \theta \in \mathbb{N}_I$ ,  $(m, p, l) \in (M^\nu, M^\lambda, M^\theta)$ . We want to prove that

$$\frac{m}{a^\nu} \equiv \frac{l}{a^\theta}.$$

By definition, there exist  $\beta, \alpha \in \mathbb{N}_I$  such that

$$\begin{aligned} ma^{\beta+\lambda} &= pa^{\beta+\nu} \\ la^{\alpha+\lambda} &= pa^{\alpha+\theta}. \end{aligned}$$

Put  $\delta = \beta + \lambda + \alpha$ . We get

$$ma^{\delta+\theta} = ma^{\beta+\lambda+\alpha+\theta} = pa^{\beta+\nu+\alpha+\theta} = la^{\beta+\nu+\alpha+\lambda} = la^{\delta+\nu},$$

so  $\frac{m}{a^\nu} \equiv \frac{l}{a^\theta}$ . The addition and multiplication laws are associative and commutative. The distributivity axiom is satisfied and the additive neutral element is absorbent for the multiplication. So  $A[\frac{M}{a}]$  is a unital commutative semiring. The formula  $a \mapsto \frac{a}{1}$  provides a canonical morphism of semirings  $A \rightarrow A[\frac{M}{a}]$ .  $\square$

We sometimes omit  $\bar{1}$  in the notations when there is no ambiguity. In other words, we sometimes denote by  $a$  the image  $\frac{a}{1}$  of an element  $a \in A$  in  $A[\{\frac{M_i}{a_i}\}_{i \in I}]$ .

FACT 2.2. *With the notation of Definition 2.1, if  $A$  is a ring then  $A[\{\frac{M_i}{a_i}\}_{i \in I}]$  is an  $A$ -ring.*

*Proof.* If  $A$  is a ring, then the monoid  $(A, +)$  is a group and for any  $\frac{m}{a^\nu}$  with  $\nu \in \mathbb{N}_I, m \in M^\nu$ , we get  $\frac{m}{a^\nu} + \frac{-m}{a^\nu} = 0$ .  $\square$

REMARK 2.3. Let  $\{N_i\}_{i \in I}$  be ideals in  $A$  such that  $N_i + (a_i) = L_i$  for all  $i \in I$ . Then we have identifications of  $A$ -semirings  $A[\{\frac{M_i}{a_i}\}_{i \in I}] = A[\{\frac{N_i}{a_i}\}_{i \in I}] = A[\{\frac{L_i}{a_i}\}_{i \in I}]$ .

REMARK 2.4. Let  $\{E_i\}_{i \in I}$  be subsets of  $A$ , let  $P_i$  be the ideal generated by  $E_i$  for  $i \in I$ . Then one can define  $A[\{\frac{E_i}{a_i}\}_{i \in I}]$  as being  $A[\{\frac{P_i}{a_i}\}_{i \in I}]$ . This is the most natural  $A$ -semiring containing all symbols  $\frac{e_i}{a_i}$  for  $i \in I, e_i \in E_i$ .

## 2.2 Properties of dilatations

We proceed with the notation from §2.1.

REMARK 2.5. As  $A$ -semiring,  $A[\{\frac{M_i}{a_i}\}_{i \in I}]$  is generated by  $\{\frac{L_i}{a_i}\}_{i \in I}$ . Since  $L_i = M_i + (a_i)$ , this implies that  $A[\{\frac{M_i}{a_i}\}_{i \in I}]$  is generated by  $\{\frac{M_i}{a_i}\}_{i \in I}$ .

FACT 2.6. *The following assertions are equivalent.*

- (i) *There exists  $\nu \in \mathbb{N}_I$  such that  $a^\nu = 0$  in  $A$ .*
- (ii) *The ring  $A[\{\frac{M_i}{a_i}\}_{i \in I}]$  is equal to the zero ring.*

*Proof.* Assume (i) holds. Let  $\frac{m}{a^\beta} \in A[\frac{M}{a}]$  with  $\beta \in \mathbb{N}_I$  and  $m \in L^\beta$ . Then  $a^\nu m = 0$  in  $A$  and so  $\frac{m}{a^\beta} = \frac{0}{1}$  in  $A[\{\frac{M_i}{a_i}\}_{i \in I}]$ . So (ii) holds. Reciprocally, assume (ii) holds. Then  $\frac{1}{1} = \frac{0}{1}$  and so there exists  $\nu \in \mathbb{N}_I$  such that  $a^\nu = 0$  in  $A$ . So (i) holds.  $\square$

FACT 2.7. *Let  $\nu$  be in  $\mathbb{N}_I$ . The image of  $a^\nu$  in  $A[\{\frac{M_i}{a_i}\}_{i \in I}]$  is a non-zero-divisor.*

*Proof.* Let  $b \in A[\{\frac{M_i}{a_i}\}_{i \in I}]$  such that  $a^\nu b = 0$  in  $A[\{\frac{M_i}{a_i}\}_{i \in I}]$ . Write  $b = \frac{i}{a^\alpha}$ , then we get  $\frac{a^\nu i}{a^\alpha} = 0$  in  $A[\{\frac{M_i}{a_i}\}_{i \in I}]$ . This implies that there is  $\beta \in \mathbb{N}_I$  such that  $a^\beta a^\nu i = 0$ . So  $b = 0$  in  $A[\{\frac{M_i}{a_i}\}_{i \in I}]$ . So  $a^\nu$  is a non-zero-divisor.  $\square$

FACT 2.8. *Assume  $M_i = A$  for all  $i \in I$ . Then  $A[\{\frac{M_i}{a_i}\}_{i \in I}] = (a^{\mathbb{N}_I})^{-1}A$  where  $(a^{\mathbb{N}_I})^{-1}A$  is the localization of  $A$  relatively to the multiplicative monoid  $a^{\mathbb{N}_I}$ .*

*Proof.* For any  $\nu \in \mathbb{N}_I$ , we have  $M^\nu = A$ . Now the map  $\frac{x}{a^\nu} \mapsto \frac{x}{a^\nu}$  provides an isomorphism of  $A$ -semirings  $A[\{\frac{M_i}{a_i}\}_{i \in I}] = (a^{\mathbb{N}_I})^{-1}A$ .  $\square$

REMARK 2.9. Dilatations of semirings generalize entirely localizations of semirings. Indeed, let  $A$  be a semiring and let  $S$  be a multiplicative subset of  $A$  (i.e. a submonoid of  $A, \times$ ). Let  $I$  be a set such that  $S = \{s_i\}_{i \in I}$ . Then  $s^{\mathbb{N}_I} = S$  and Fact 2.8 says that  $S^{-1}A = A[\{\frac{A}{s}\}_{s \in S}]$ .

FACT 2.10. *Let  $K \subset I$ . Assume that, for all  $k \in K$ , the element  $a_k$  belongs to the face  $A^*$  of invertible elements of the monoid  $(A, \times)$ , i.e.  $a_k$  is invertible for  $\times$ . Then*

$$A[\{\frac{M_i}{a_i}\}_{i \in I}] = A[\{\frac{M_i}{a_i}\}_{i \in I \setminus K}].$$

*Proof.* For  $k \in K$ , let  $c_k \in A$  such that  $a_k c_k = 1$ . As  $A$ -semiring,  $A[\{\frac{M_i}{a_i}\}_{i \in I}]$  is generated by  $\{\frac{M_i}{a_i}\}_{i \in I}$ . So it is enough to show that  $\frac{M_k}{a_k}$  belongs to the image of  $A \rightarrow A[\{\frac{M_i}{a_i}\}_{i \in I}]$  for any  $k \in K$ . This follows from the identity  $\frac{m}{a_k} = \frac{c_k m}{1}$  in  $A[\{\frac{M_i}{a_i}\}_{i \in I}]$ .  $\square$

FACT 2.11. *Let  $f : A \rightarrow B$  be a morphism of semirings. Let  $\{[N_i, b_i]\}_{i \in I}$  be centers of  $B$  such that  $f(M_i) \subset N_i$  and  $f(a_i) = b_i$  for all  $i \in I$ . Then we have a canonical morphism of  $A$ -semirings*

$$\phi : A[\{\frac{M_i}{a_i}\}_{i \in I}] \rightarrow B[\{\frac{N_i}{b_i}\}_{i \in I}].$$

*Proof.* Put  $\phi(\frac{m}{a^\nu}) = \frac{f(m)}{b^\nu}$ .  $\square$

FACT 2.12. *Let  $\{P_i\}_{i \in I}$  be ideals of  $A$  such that  $P_i \subset M_i$  for all  $i \in I$ . Then we have a canonical injective morphism of  $A$ -semirings*

$$A[\{\frac{P_i}{a_i}\}_{i \in I}] \rightarrow A[\{\frac{M_i}{a_i}\}_{i \in I}].$$



*Proof.* Clear.  $\square$

FACT 2.13. Let  $c$  be a non-zero-divisor element in  $A$ . Then  $\frac{c}{1}$  is a non-zero-divisor in  $A[\{\frac{M_i}{a_i}\}_{i \in I}]$ .

*Proof.* Let  $\frac{m}{a^\nu} \in A[\{\frac{M_i}{a_i}\}_{i \in I}]$  such that  $\frac{m}{a^\nu} \frac{c}{1} = 0$ . Then there exists  $\beta \in \mathbb{N}_I$  such that  $a^\beta m c = 0$  in  $A$ . Because of  $c$  is a non-zero-divisor, this implies  $a^\beta m = 0$  in  $A$  and so  $\frac{m}{a^\nu} = 0$  in  $A[\{\frac{M_i}{a_i}\}_{i \in I}]$ .  $\square$

PROPOSITION 2.14. Let  $K \subset I$  put  $J = I \setminus K$ . Then we have a canonical morphism of  $A$ -semirings

$$\varphi : A[\{\frac{M_i}{a_i}\}_{i \in K}] \rightarrow A[\{\frac{M_i}{a_i}\}_{i \in I}].$$

Moreover

- (i) if  $M_i \subset (a_i)$  for all  $i \in J$ , then  $\varphi$  is surjective, and
- (ii) if  $a_i$  is a non-zero-divisor in  $A$  for all  $i \in J$ , then  $\varphi$  is injective.

*Proof.* We have a canonical injective morphism of monoids  $\mathbb{N}_K \rightarrow \mathbb{N}_I$ . Let  $\frac{m}{a^\nu}$  with  $\nu \in \mathbb{N}_K$  and  $m \in L^\nu$ , then we put  $\varphi(\frac{m}{a^\nu}) = \frac{m}{a^\nu} \in A[\{\frac{M_i}{a_i}\}_{i \in I}]$ . We now prove the listed properties.

- (i) It is enough to show that  $\frac{M_i}{a_i}$  is in the image of  $\varphi$  for all  $i \in I$ . This is obvious for all  $i \in K$ . So let  $i \in J$  and let  $\frac{m_i}{a_i} \in \frac{M_i}{a_i}$ . Since  $M_i \subset (a_i)$  we write  $m_i = a_i x$  with  $x \in A$ . Then  $\frac{m_i}{a_i} = \frac{x}{1}$  belongs to the image of  $\varphi$ . So  $\varphi$  is surjective.
- (ii) Let  $\frac{m}{a^\nu} \in A[\{\frac{M_i}{a_i}\}_{i \in K}]$  with  $\nu \in \mathbb{N}_K$ . Assume that  $\frac{m}{a^\nu} = \frac{0}{1}$  in the image  $A[\{\frac{M_i}{a_i}\}_{i \in I}]$ . Then there exists  $\beta \in \mathbb{N}_I$  such that  $m a^\beta = 0$ . Write  $\beta = \nu' + \theta$  with  $\nu' \in \mathbb{N}_K$  and  $\theta \in \mathbb{N}_J$ . Then we have  $m a^{\nu'} a^\theta = 0$  in the semiring  $A$  and  $a^\theta$  is a non-zero-divisor, so  $m a^{\nu'} = 0$  in the semiring  $A$  and so  $\frac{m}{a^\nu} = \frac{0}{1}$  in the source. So  $\varphi$  is injective.  $\square$

FACT 2.15. Let  $J \subset I$  be such that for all  $j \in J$ , there is  $i \in I \setminus J$  satisfying that  $a_i = a_j$  and  $M_j \subset M_i$ , then we have a canonical identification of  $A$ -semirings  $A[\{\frac{M_i}{a_i}\}_{i \in I}] = A[\{\frac{M_i}{a_i}\}_{i \in I \setminus J}]$ .

*Proof.* By Proposition 2.14, we have a canonical morphism  $\phi : A[\{\frac{M_i}{a_i}\}_{i \in I \setminus J}] \rightarrow A[\{\frac{M_i}{a_i}\}_{i \in I}]$ . By assumption, we have  $a^{\mathbb{N}_J} \subset a^{\mathbb{N}_{I \setminus J}}$ , this implies  $a^{\mathbb{N}_I} = a^{\mathbb{N}_{I \setminus J}}$  and the injectivity of  $\phi$ . The surjectivity is clear.  $\square$

PROPOSITION 2.16. Let  $K \subset I$ . Then we have a canonical isomorphism of  $A[\{\frac{M_i}{a_i}\}_{i \in K}]$ -semirings

$$A[\{\frac{M_i}{a_i}\}_{i \in I}] = A[\{\frac{M_i}{a_i}\}_{i \in K}] \left[ \left\{ \frac{A[\{\frac{M_i}{a_i}\}_{i \in I}] \frac{M_j}{1}}{a_j} \right\}_{j \in I \setminus K} \right],$$

where  $A[\{\frac{M_i}{a_i}\}_{i \in I}] \frac{M_j}{1}$  is the ideal of  $A[\{\frac{M_i}{a_i}\}_{i \in I}]$  generated by  $\frac{M_j}{1} \in A[\{\frac{M_i}{a_i}\}_{i \in I}]$ .

*Proof.* We have a morphism of semirings  $A[\{\frac{M_i}{a_i}\}_{i \in K}] \rightarrow A[\{\frac{M_i}{a_i}\}_{i \in I}]$  given by Fact 2.14. The right-hand side of the equation in the statement of 2.16 is generated as  $A[\{\frac{M_i}{a_i}\}_{i \in K}]$ -semiring by  $\left\{ \frac{A[\{\frac{M_i}{a_i}\}_{i \in I}] M_j}{a_j} \right\}_{j \in I \setminus K}$ . We now define an  $A$ -morphism from the right-hand side to the left-hand side sending  $\frac{a^\nu m_j}{a_j^k}$  (with  $\nu \in I$ ,  $j \in I \setminus K$  and  $k \in \mathbb{N}$ ) to  $\frac{m_\nu m_j}{a^\nu a_j^k}$ . This is well-defined and it is easy to check injectivity and surjectivity.  $\square$

COROLLARY 2.17. Let  $S$  and  $S'$  be the multiplicative monoids in  $A$  and  $A[\{\frac{M_i}{a_i}\}_{i \in I}]$  given by  $\{a^\nu | \nu \in \mathbb{N}_I\}$ . Then  $S'^{-1} A[\{\frac{M_i}{a_i}\}_{i \in I}] = S^{-1} A$ .

*Proof.* Using, Fact 2.8, Proposition 2.16 and Fact 2.15, we get

$$S'^{-1}A' = A[\{\frac{M_i}{a_i}\}_{i \in I}][\{\frac{A[\{\frac{M_i}{a_i}\}_{i \in I}]}{a_i}\}_{i \in I}] = A[\{\frac{M_i}{a_i}\}_{i \in I}, \{\frac{A}{a_i}\}_{i \in I}] = A[\{\frac{A}{a_i}\}_{i \in I}] = S^{-1}A.$$

□

PROPOSITION 2.18. Assume that  $a_i = a_j =: b$  for all  $i, j \in I$ , then

$$A[\{\frac{M_i}{a_i}\}_{i \in I}] = A[\frac{\sum_{i \in I} M_i}{b}]$$

*Proof.* This follows from the identity  $(\sum_{\nu \in \mathbb{N}_I, \sum_{i \in I} \nu_i = n} M^\nu) = (\sum_{i \in I} M_i)^n$ . □

PROPOSITION 2.19. Let  $\{d_i\}_{i \in I}$  be positive integers. Let  $I = \coprod_{j \in J} I_j$  be a partition of  $I$ . Assume that, for all  $j \in J$ ,  $M_i = M_{i'} =: M_j$  and  $a_i = a_{i'} =: a_j$  for all  $i, i' \in I_j$ . Assume moreover that, for all  $j \in J$ ,  $\max_{i \in I_j} d_i =: d_j$  exists. Then we have a canonical identification

$$A[\{\frac{M_i}{a_i^{d_i}}\}_{i \in I}] = A[\{\frac{M_j}{a_j^{d_j}}\}_{j \in J}].$$

*Proof.* We define a canonical morphism  $A[\{\frac{M_i}{a_i^{d_i}}\}_{i \in I}] \rightarrow A[\{\frac{M_j}{a_j^{d_j}}\}_{j \in J}]$ . For  $j \in J$ ,  $i \in I_j$ ,  $n \in \mathbb{N}$ ,  $m \in L_i^n$  it is given by  $\frac{m}{(a_i^{d_i})^n} \mapsto \frac{m(a_j^{d_j - d_i})^n}{(a_j^{d_j})^n}$ . This determines it entirely. It is injective and surjective. □

FACT 2.20. Let  $\nu \in \mathbb{N}_I$ . We have  $L^\nu A[\{\frac{M_i}{a_i}\}_{i \in I}] = a^\nu A[\{\frac{M_i}{a_i}\}_{i \in I}]$ .

*Proof.* Obviously  $a^\nu A[\{\frac{M_i}{a_i}\}_{i \in I}] \subset L^\nu A[\{\frac{M_i}{a_i}\}_{i \in I}]$ . Let  $y \in L^\nu$  and  $\frac{x}{a^\alpha} \in A[\{\frac{M_i}{a_i}\}_{i \in I}]$ , the formula

$$y \frac{x}{a^\alpha} = a^\nu \frac{yx}{a^{\alpha+\nu}}$$

now shows that  $L^\nu A[\{\frac{M_i}{a_i}\}_{i \in I}] = a^\nu A[\{\frac{M_i}{a_i}\}_{i \in I}]$ . □

PROPOSITION 2.21. (Universal property) If  $\chi : A \rightarrow B$  is a morphism of semirings such that  $\chi(a_i)$  is a non-zero-divisor and generates  $\chi(L_i)B$  for all  $i \in I$ , then there exists a unique morphism  $\chi'$  of  $A$ -semirings  $A[\{\frac{M_i}{a_i}\}_{i \in I}] \rightarrow B$ . The morphism  $\chi'$  sends  $\frac{l}{a^\nu}$  ( $\nu \in \mathbb{N}_I, l \in L^\nu$ ) to the unique element  $b \in B$  such that  $\chi(a^\nu)b = \chi(l)$ .

*Proof.* The element  $b$  in the statement is unique because  $\chi(a^\nu)$  is a non-zero-divisor for all  $\nu \in \mathbb{N}_I$ . Clearly, the map  $\chi'$  defined in the statement is a morphism of  $A$ -semirings. Now let  $\phi$  be an other morphism of  $A$ -semirings  $A[\{\frac{L_i}{a_i}\}_{i \in I}] \rightarrow B$ . We have

$$\chi(a^\nu)\phi(\frac{l}{a^\nu}) = \phi(\frac{l}{a^\nu})\phi(a^\nu) = \phi(l) = \chi(l).$$

This implies  $\chi'(\frac{l}{a^\nu}) = \phi(\frac{l}{a^\nu})$ . □

DEFINITION 2.22. The blowup algebra, or the Rees algebra, associated to  $A$  and  $\{L_i\}_{i \in I}$  is the  $\mathbb{N}_I$ -graded  $A$ -semiring

$$\text{Bl}_{\{L_i\}_{i \in I}} A = \bigoplus_{\nu \in \mathbb{N}_I} L^\nu$$

where the summand  $L^\nu$  is placed in degree  $\nu \in \mathbb{N}_I$ .

Let  $e_1 = (1, 0, \dots), e_2 = (0, 1, 0, \dots), \dots, e_i = (0, \dots, 0, 1, 0, \dots), \dots$  be the canonical basis of the free  $\mathbb{N}$ -semimodule  $\mathbb{N}_I$ . Recall that  $a_i \in L_i$  for  $i \in I$ . Denote  $a_{i,i}$  the element  $a_i$  seen as an element of degree  $e_i$  in the Rees algebra  $\text{Bl}_{\{L_i\}_{i \in I}} A$ . Let  $S$  be the multiplicative subset of  $\text{Bl}_{\{L_i\}_{i \in I}} A$  generated by  $\{a_{i,i}\}_{i \in I}$ . Let  $(\text{Bl}_{\{L_i\}_{i \in I}} A)[S^{-1}]$  be the localization of the multi-Rees algebra relatively to  $S$ . This  $A$ -semiring inherits a  $\mathbb{Z}_I$ -grading given, for any  $l \in L^\nu$ , by

$$\deg\left(\frac{l}{a_{1,1}^{\alpha_1} \dots a_{i,i}^{\alpha_i}}\right) = \sum_{i \in I} (\nu_i - \alpha_i) e_i \in \mathbb{Z}_I := \bigoplus_{i \in I} \mathbb{Z}.$$

FACT 2.23. *We have a canonical identification of  $A$ -semirings*

$$A\left\{\frac{M_i}{a_i}\right\}_{i \in I} = \left[ (\text{Bl}_{\{L_i\}_{i \in I}} A)[S^{-1}] \right]_{\deg=(0, \dots, 0, \dots)}$$

where the right-hand part is obtained as degree zero elements in  $(\text{Bl}_{\{L_i\}_{i \in I}} A)[S^{-1}]$ .

*Proof.* This is tautological.  $\square$

FACT 2.24. *Assume that  $I = \{1, \dots, k\}$ . Then*

$$A\left\{\frac{M_i}{a_i}\right\}_{i \in I} = \left[ \text{Bl}_{L_1, \dots, L_k} A \left[ ((a_1 \dots a_k)^\mathbb{N})^{-1} \right] \right]_{\deg=(0, \dots, 0)}.$$

*Proof.* Recall that  $\text{Bl}_{L_1, \dots, L_k} A \left[ ((a_1 \dots a_k)^\mathbb{N})^{-1} \right]$  is the localization of  $\text{Bl}_{L_1, \dots, L_k} A$  relatively to  $(a_1 \dots a_k)^\mathbb{N}$ . The element  $a_1 \dots a_k$  has degree  $(1, \dots, 1)$  in  $\text{Bl}_{L_1, \dots, L_k} A$ , so an element in  $\text{Bl}_{L_1, \dots, L_k} A \left[ ((a_1 \dots a_k)^\mathbb{N})^{-1} \right]_{\deg=(0, \dots, 0)}$  is represented by a fraction  $\frac{l}{(a_1 \dots a_k)^d}$  with  $l \in L_1^d \dots L_k^d$ . Now the map  $\frac{l}{a^\nu} \mapsto \frac{a_1^{d-\nu_1} \dots a_k^{d-\nu_k} l}{(a_1 \dots a_k)^d}$ , where  $\nu \in \mathbb{N}^k$  and  $d = \max\{\nu_i | i \in \{1, \dots, k\}\}$ , provides the identification.  $\square$

PROPOSITION 2.25. *Assume that  $I = \{1, \dots, k\}$  is finite. Then we have a canonical identification of  $A$ -semirings*

$$A\left\{\frac{M_i}{a_i}\right\}_{i \in I} = A\left[\frac{\sum_{i \in I} (M_i \cdot \prod_{j \in I \setminus \{i\}} a_j)}{a_1 \dots a_k}\right].$$

*Proof.* Let us provide a map

$$\phi : A\left[\frac{\sum_{i \in I} (M_i \cdot \prod_{j \in I \setminus \{i\}} a_j)}{a_1 \dots a_k}\right] \rightarrow A\left\{\frac{M_i}{a_i}\right\}_{i \in I}.$$

The semiring  $A\left[\frac{\sum_{i \in I} (M_i \cdot \prod_{j \in I \setminus \{i\}} a_j)}{a_1 \dots a_k}\right]$  is generated as  $A$ -semiring by  $\frac{\sum_{i \in I} (M_i \cdot \prod_{j \in I \setminus \{i\}} a_j)}{a_1 \dots a_k}$ , we now define a map  $\phi$  via (for  $m_i \in M_i, i \in I$ ):

$$\phi\left(\frac{\sum_{i \in I} (m_i \cdot \prod_{j \in I \setminus \{i\}} a_j)}{a_1 \dots a_k}\right) = \frac{\sum_{i \in I} (m_i \cdot \prod_{j \in I \setminus \{i\}} a_j)}{a_1 \dots a_k}.$$

This is well-defined and  $\phi$  is a morphism of  $A$ -semirings. It is easy to prove that  $\phi$  is injective and surjective.  $\square$

REMARK 2.26. Assume that  $A = \mathbb{N}[X, Y]$ . The formal symbol  $\frac{X}{3}$  does not define an element in  $A\left[\frac{(2X)+(3Y)}{6}\right]$ . The formal symbol  $\frac{2X}{6}$  defines an element in  $A\left[\frac{(2X)+(3Y)}{6}\right]$ . The formal symbol  $\frac{X}{3}$  defines an element in  $A\left[\frac{(X)}{3}, \frac{(Y)}{2}\right]$ . The identification of Proposition 2.25 identifies  $\frac{X}{3}$  and  $\frac{2X}{6}$ .

LEMMA 2.27. Write  $I = \text{colim}_{J \subset I} J$  as a filtered colimit of sets. We have a canonical identification of  $A$ -semirings

$$A[\{\frac{M_i}{a_i}\}_{i \in I}] = \text{colim}_{J \subset I} A[\{\frac{M_j}{a_i}\}_{i \in J}]$$

where the transition maps are given by Fact 2.14.

*Proof.* For each  $J \subset I$ , Fact 2.14 gives a canonical morphism  $A[\{\frac{M_i}{a_i}\}_{i \in J}] \rightarrow A[\{\frac{M_i}{a_i}\}_{i \in I}]$  of  $A$ -semirings. These morphisms are compatible with transition maps. So we have a canonical  $A$ -morphism

$$\phi : \text{colim}_{J \subset I} A[\{\frac{M_i}{a_i}\}_{i \in J}] \rightarrow A[\{\frac{M_i}{a_i}\}_{i \in I}].$$

The map  $\phi$  is surjective because for any  $\nu \in \mathbb{N}_I$ , there exists a subset  $J \subset I$  such that  $\nu \in \mathbb{N}_J$  (recall that  $\mathbb{N}_J \subset \mathbb{N}_I$ ). It is easy to check injectivity.  $\square$

PROPOSITION 2.28. Let  $T$  be an ideal of  $A$ . Assume that we have a commutative diagram of

$$\begin{array}{ccc} & A/T & \\ \varphi \nearrow & & \nwarrow \phi \\ A[\{\frac{M_i}{a_i}\}_{i \in I}] & \xleftarrow{f} & A \end{array} \quad \text{where } \phi \text{ is the quotient map. Assume that } \phi(a_i)$$

is a non-zero-divisor for all  $i \in I$ . Then

$$\ker(\varphi) = \sum_{\nu \in \mathbb{N}_I} \frac{L^\nu \cap T}{a^\nu} \subset A[\{\frac{M_i}{a_i}\}_{i \in I}].$$

*Proof.* Let  $\nu \in \mathbb{N}_I$  and  $\frac{m}{a^\nu} \in A[\{\frac{M_i}{a_i}\}_{i \in I}]$ , i.e.  $m \in L^\nu$ . We have

$$\phi(a^\nu) \varphi\left(\frac{m}{a^\nu}\right) = \varphi\left(\frac{a^\nu}{1}\right) \varphi\left(\frac{m}{a^\nu}\right) = \varphi\left(\frac{m}{1}\right) = \varphi(f(m)) = \phi(m).$$

Now assume  $\varphi\left(\frac{m}{a^\nu}\right) = 0$ , then  $\phi(m) = 0$  and so  $m \in L^\nu \cap T$ . This shows that  $\ker(\varphi) \subset \sum_{\nu \in \mathbb{N}_I} \frac{L^\nu \cap T}{a^\nu}$ . Reciprocally assume  $m \in L^\nu \cap T$ . Then  $\phi(m) = 0$ . This implies  $\varphi\left(\frac{m}{a^\nu}\right) = 0$  because  $\phi(a^\nu)$  is a non-zero-divisor by assumption.  $\square$

COROLLARY 2.29. We proceed with the notation from Proposition 2.28 and assume that  $I = \{i\}$ ,  $a_i = b_i^k$  for some  $k \in \mathbb{N}$ ,  $b_i \in A$  and that  $T = M_i$ . Then  $\ker(\varphi)$  is the ideal of  $A[\frac{M_i}{a_i}]$  generated by  $\frac{M_i}{a_i}$ , moreover  $A[\frac{M_i}{b_i^k}][\frac{\ker(\varphi)}{b_i^d}] = A[\frac{M_i}{b_i^{k+d}}]$  for any  $d \in \mathbb{N}$ .

*Proof.* Clearly,  $\frac{M_i}{a_i} \subset \ker(\varphi)$ , so it is enough to prove that  $\ker(\varphi)$  is included in the ideal generated by  $\frac{M_i}{a_i}$ . So let  $n \in \mathbb{N}_I = \mathbb{N}$  and let  $\frac{m}{a_i^n} \in \frac{L_i^n \cap M_i}{a_i^n}$ . If  $n = 0$ , then  $\frac{m}{a_i^n} = m \in M_i$  and it clearly belongs to the ideal generated by  $\frac{M_i}{a_i}$ . If  $n \geq 1$ , then  $\frac{m}{a_i^n} \in \frac{L_i^n \cap M_i}{a_i^n}$  and it clearly belongs to the ideal

generated by  $\frac{M_i}{a_i}$ . Now we prove the equality  $A[\frac{M_i}{b_i^k}][\frac{\ker(\varphi)}{b_i^d}] = A[\frac{M_i}{b_i^{k+d}}]$  and finish the proof

$$\begin{aligned} A[\frac{M_i}{b_i^k}][\frac{\ker(\varphi)}{b_i^d}] &= A[\frac{M_i}{b_i^k}][\frac{\frac{M_i}{b_i^k}A[\frac{M_i}{b_i^k}]}{b_i^d}] \\ &= A[\frac{M_i}{b_i^k}][\frac{M_iA[\frac{M_i}{b_i^k}]}{b_i^{k+d}}] \\ \text{by Proposition 2.16} &= A[\frac{M_i}{b_i^k}, \frac{M_i}{b_i^{k+d}}] \\ \text{by Proposition 2.19} &= A[\frac{M_i}{b_i^{k+d}}]. \end{aligned}$$

□

### 2.3 More properties of dilatations of rings

We proceed with the notation from §2.1 and we now assume that the **monoid**  $(A, +)$  is a **group**, i.e. that  $A$  is a unital commutative ring. By Fact 2.2,  $A[\{\frac{M_i}{a_i}\}_{i \in I}]$  is an  $A$ -ring.

**PROPOSITION 2.30.** *Let  $f : A \rightarrow B$  be an  $A$ -ring. Put  $N_i = f(M_i)B$  and  $b_i = f(a_i)$  for  $i \in I$ . Then  $B[\{\frac{N_i}{b_i}\}_{i \in I}]$  is the quotient of  $B \otimes_A A[\{\frac{M_i}{a_i}\}_{i \in I}]$  by the ideal  $T_b$  of elements annihilated by some element in  $b^{\mathbb{N}_I} := \{b^\nu | \nu \in \mathbb{N}_I\}$ .*

*Proof.* We can assume  $a_i \in M_i$  and  $b_i \in N_i$ . Let  $B'$  be the quotient of  $B \otimes_A A[\{\frac{M_i}{a_i}\}_{i \in I}]$  by  $T_b$ . The ring map

$$B \otimes_A A[\{\frac{M_i}{a_i}\}_{i \in I}] \rightarrow B[\{\frac{N_i}{b_i}\}_{i \in I}]$$

is surjective and annihilates  $a^{\mathbb{N}_I}$ -torsion as elements in  $b^{\mathbb{N}_I}$  are non-zero-divisors in  $B[\{\frac{N_i}{b_i}\}_{i \in I}]$ . Hence we obtain a surjective map  $B' \rightarrow B[\{\frac{N_i}{b_i}\}_{i \in I}]$ . To see that the kernel is trivial, we construct an inverse map. Namely, let  $z = \frac{y}{b^\nu}$  be an element of  $B[\{\frac{N_i}{b_i}\}_{i \in I}]$ , i.e.  $y \in N^\nu$  for some  $\nu \in \mathbb{N}_I$ . Write  $y = \sum x_i s_i$  with  $x_i \in M^\nu$  and  $s_i \in B$ . We map  $z$  to the class of  $\sum s_i \otimes \frac{x_i}{a_i^\nu}$  in  $B'$ . This is well defined because we claim that an element of the kernel of the map  $B \otimes_A M^\nu \rightarrow N^\nu$  is annihilated by  $a^\nu$  hence maps to zero in  $B'$ . We now prove the claim of the previous assertion. Let  $\sum_j (s_j \otimes m_j)$  be in the kernel of the map as before ( $s_j \in B, m_j \in M^\nu$  for all  $j$ ), so that  $\sum_j s_j f(m_j) = 0$ ; we have  $a^\nu \sum_j (s_j \otimes m_j) = \sum_j (s_j \otimes m_j a^\nu) = \sum_j (s_j f(m_j) \otimes a^\nu) = (\sum_j s_j f(m_j)) \otimes a^\nu = 0$ . This finishes the proof. □

**COROLLARY 2.31.** *We proceed with the notation from Proposition 2.30 and assume  $f : A \rightarrow B$  is flat. Then  $T_b = 0$ , in other words we have a canonical isomorphism*

$$B[\{\frac{N_i}{b_i}\}_{i \in I}] = B \otimes_A A[\{\frac{M_i}{a_i}\}_{i \in I}].$$

*Proof.* Since  $f$  is flat, the map  $\phi : A[\{\frac{M_i}{a_i}\}_{i \in I}] \rightarrow B \otimes_A A[\{\frac{M_i}{a_i}\}_{i \in I}]$  is flat. Since  $\phi$  is flat, the image of any non-zero-divisor element in  $A[\{\frac{M_i}{a_i}\}_{i \in I}]$  under the map  $\phi$  is a non-zero-divisor. So  $T_b = 0$  and Proposition 2.30 finishes the proof. □

**DEFINITION 2.32.** Assume that  $a_i$  is a non-zero divisor for all  $i \in I$ . Let  $J_i$  be the ideal of  $A$  generated by  $a_i$ , it is invertible. We consider the  $A$ -algebra

$$C_L^J A \stackrel{\text{def}}{=} \bigoplus_{\nu \in \mathbb{N}_I} L^\nu \otimes J^{-\nu},$$

and we call it the associated conic algebra.

**PROPOSITION 2.33.** *We proceed with the notation from Definition 2.32. Let  $\zeta$  be the ideal of  $C_L^J A$  generated by elements  $\rho^\alpha - 1$  for  $\alpha \in \mathbb{N}_I$  where  $\rho_i \in C_L^J A$  is the image of  $1 \in A$  under  $A \cong J_i \otimes J_i^{-1} \subset L_i \otimes J_i^{-1} \subset C_L^J A$  for any  $i \in I$ . We have a canonical isomorphism of  $A$ -algebras*

$$(C_L^J A)/\zeta \longrightarrow A[\{\frac{L_i}{a_i}\}_{i \in I}].$$

*Proof.* Let  $t_i = a_i^\vee$  be the generator for  $J_i^{-1}$ , dual to  $a_i$  for all  $i \in \mathbb{N}_I$ . We have a natural morphism of rings given explicitly by

$$\psi : C_L^J A \longrightarrow A[\{\frac{L_i}{a_i}\}_{i \in I}], \sum_{\nu \in \mathbb{N}_I} l_\nu \otimes t^\nu \mapsto \sum_{\nu \in \mathbb{N}_I} \frac{l_\nu}{a^\nu}.$$

The morphism  $\psi$  is surjective and  $\zeta \subset \ker \psi$ . It is enough to prove that  $\ker \psi \subset \zeta$ . Let  $X = \sum_{\nu \in c} l_\nu \otimes t^\nu \in C_L^J A$  where  $l_\nu \in L^\nu$ , and  $c$  is a finite subset of  $\mathbb{N}_I$ . Let  $\beta \in \mathbb{N}_I$  defined by  $\beta_i = \max_{\nu \in c} \nu_i$  for all  $i \in I$ . Then we have

$$\psi(X) = \sum_{\nu \in c} \frac{l_\nu}{a^\nu} = \frac{\sum_{\nu \in c} a^{\beta-\nu} l_\nu}{a^\beta}.$$

Assume  $X \in \ker \psi$ , then  $\sum_{\nu \in c} a^{\beta-\nu} l_\nu = 0$  because  $a_i$  are non-zero-divisors  $i \in I$ . So we are allowed to write

$$X = \left( \sum_{\nu \in c} l_\nu \otimes t^\nu \right) - \left( \sum_{\nu \in c} a^{\beta-\nu} l_\nu \right) \otimes t^\beta = \sum_{\nu \in c} \left[ (l_\nu \otimes t^\nu) (1 - (a^{\beta-\nu} \otimes t^{\beta-\nu})) \right].$$

This finishes the proof since  $a^{\beta-\nu} \otimes t^{\beta-\nu} = \rho_1^{\beta_1-\nu_1} \dots \rho_i^{\beta_i-\nu_i} \dots$ .  $\square$

**REMARK 2.34.** We note that the ideal  $\zeta$  appearing in Proposition 2.33 is in fact generated by  $\{\rho_i - 1\}_{i \in I}$ . To see this, use for example that in any ring and for any elements  $\rho, \sigma$  in the ring we have  $\rho^n - 1 = (\rho - 1)(\rho^{n-1} + \dots + \rho + 1)$  and  $\rho\sigma - 1 = (\rho - 1)(\sigma + 1) + (\sigma - 1) - (\rho - 1)$ .

**FACT 2.35.** *Let  $R$  be a ring and assume that  $f : R \rightarrow A$  is a morphism of rings. Let  $\{r_i\}_{i \in I}$  be elements in  $R$  and assume that  $a_i = f(r_i)$  for all  $i \in I$ . Let  $R_r = R[\{\frac{R}{r_i}\}_{i \in I}]$  be the localization of  $R$  at  $\{r_i\}_{i \in I}$ . For any  $i \in I$ , let  $M_i \otimes r_i^{-1} \subset A \otimes_R R_r$ . Then  $A[\{\frac{M_i}{a_i}\}_{i \in I}]$  identifies with the  $A$ -subalgebra of  $A \otimes_R R_r$  generated by  $\{M_i \otimes r_i^{-1}\}_{i \in I}$  and  $A$ .*

*Proof.* By [StP, Tag 00DK] and Fact 2.8, we get  $A \otimes_R R_r = (A \otimes_R R)_r = A[\{\frac{A}{a_i}\}_{i \in I}]$ . Moreover  $M_i \otimes r_i^{-1} \subset A \otimes_R R_r$  corresponds to  $\frac{M_i}{a_i} \subset A[\{\frac{A}{a_i}\}_{i \in I}]$ . Now Fact 2.5 finishes the proof.  $\square$

**REMARK 2.36.** We discussed before that dilatations of rings provide a formalism unifying localizations of rings and affine blowups of rings (recall that affine blowups algebras are studied in [StP, Tag 052P]). We now check on the ring  $\mathbb{Z}[X]$  that dilatations strictly generalize localizations and affine blowups. So assume that  $A = \mathbb{Z}[X]$ . Assume that  $I = \mathbb{I} := \{1, 2, 3, 4, 5, \dots\}$ ,  $M_i = (X)$  and  $a_i = i$ . Then  $A[\{\frac{M_i}{a_i}\}_{i \in \mathbb{I}}]$  is equal to the subring  $\mathbb{Z}[\{\frac{X}{i}\}_{i \in \mathbb{I}}]$  of  $\mathbb{Q}[X]$  generated by  $\{\frac{X}{i}\}_{i \in \mathbb{I}}$ . Now let  $S$  be a submonoid of  $(\mathbb{Z}[X], \times)$ , let  $N \subset \mathbb{Z}[X]$  be an ideal and let  $Q$  be an element in  $\mathbb{Z}[X]$ . We first prove that  $S^{-1}A \neq A[\{\frac{M_i}{a_i}\}_{i \in \mathbb{I}}]$ . If  $S$  contains 0, then  $S^{-1}A = 0$  and we are done. So we assume that  $0 \notin S$ . The  $\mathbb{Z}[X]$ -rings  $S^{-1}\mathbb{Z}[X]$  and  $\mathbb{Z}[\{\frac{X}{i}\}_{i \in \mathbb{I}}]$  are  $\mathbb{Z}[X]$ -isomorphic if and only if they coincide in  $\mathbb{Q}(X)$ . Assume first that  $S \subset \mathbb{Z}$ . If  $S \subset \{1, -1\}$ , we are done. If  $S \not\subset \{1, -1\}$ , then  $S^{-1}A$  contains an integer such that  $n \neq 1, -1$  and  $n$  is invertible in  $S^{-1}A$ , this shows that  $S^{-1}A \neq \mathbb{Z}[\{\frac{X}{i}\}_{i \in \mathbb{I}}]$ . Now if  $S \not\subset \mathbb{Z}$ , then  $S$  contains a non-constant polynomial

$P$ . The element  $\frac{1}{p}$  does not belong to  $\mathbb{Z}[\{\frac{X}{i}\}_{i \in \mathbb{I}}]$  because  $\mathbb{Z}$  is a domain. So  $S^{-1}A \neq A[\{\frac{M_i}{a_i}\}_{i \in I}]$ . We now prove that  $A[\frac{N}{Q}] \neq A[\{\frac{M_i}{a_i}\}_{i \in I}]$ . We can assume  $Q \neq 0$ . The  $\mathbb{Z}[X]$ -rings  $\mathbb{Z}[X][\frac{N}{Q}]$  and  $\mathbb{Z}[\{\frac{X}{i}\}_{i \in \mathbb{I}}]$  are  $\mathbb{Z}[X]$ -isomorphic if and only if they coincide in  $\mathbb{Q}(X)$ . There is a prime number  $p$  that does not divide  $Q$  in the factorial ring  $\mathbb{Z}[X]$ . Assume that we have an isomorphism of  $A$ -rings  $A[\frac{N}{Q}] = \mathbb{Z}[\{\frac{X}{i}\}_{i \in \mathbb{I}}]$ , then there is an integer  $k$  and an element  $n \in N^k$  such that  $\frac{X}{p} = \frac{n}{Q^k}$ , this implies that  $pn = XQ^k$ . This implies that  $p$  divides  $X$ . This is absurd. So  $A[\frac{N}{Q}] \neq A[\{\frac{M_i}{a_i}\}_{i \in I}]$ .

### 3. Multi-centered dilatations in the absolute setting

In this section, we define multi-centered dilatations and prove some properties.

#### 3.1 Definitions

Let  $S$  be a scheme. An  $S$ -space is an  $S$ -algebraic space. Let us fix an  $S$ -space  $X$ . For the convenience of the reader, we recall some basic notations and well-known facts.

NOTATION 3.1. Let  $Clo(X)$  be the set of closed  $S$ -subspaces of  $X$ . Recall that  $Clo(X)$  corresponds to quasi-coherent ideals of  $\mathcal{O}_X$  via [StP, Tag 03MB]. Let  $IQCoh(\mathcal{O}_X)$  denote the set of quasi-coherent ideals of  $\mathcal{O}_X$ . It is clear that  $(IQCoh(\mathcal{O}_X), +, \times, 0, \mathcal{O}_X)$  is a semiring. So we obtain a semiring structure on  $Clo(X)$ , usually denoted by  $(Clo(X), \cap, +, X, \emptyset)$ . For clarity, we now recall directly operations on  $Clo(X)$ . Given two closed subspaces  $Y_1, Y_2$  given by ideals  $\mathcal{J}_1, \mathcal{J}_2$ , their sum  $Y_1 + Y_2$  is defined as the closed subspace given by the ideal  $\mathcal{J}_1 \mathcal{J}_2$ . Moreover, if  $n \in \mathbb{N}$ , we denote by  $nY_1$  the  $n$ -th multiple of  $Y_1$ . The set of locally principal closed subspaces of  $X$  (cf. [StP, Tag 083B]), denoted  $Pri(X)$ , forms a submonoid of  $(Clo(X), +)$ . Effective Cartier divisors of  $X$ , denoted  $Car(X)$ , form a submonoid of  $(Pri(X), +)$ . Note that  $Car(X)$  is a face of  $Pri(X)$ . We have an other monoid structure on  $Clo(X)$  given by intersection, this law is denoted  $\cap$ . The operation  $\cap$  corresponds to the sum of quasi-coherent sheaves of ideals. The set  $Clo(X)$  endowed with  $\cap, +$  is a semiring whose neutral element for  $+$  is  $\emptyset$  and whose neutral element for  $\cap$  is  $X$ . Let  $C \in Car(X)$ , a non-zero-divisor (for  $+$ ) in the semiring  $Clo(X)$ . Let  $Y, Y' \in Clo(X)$ . If  $C + Y$  is a closed subspace of  $C + Y'$ , then  $Y$  is a closed subspace of  $Y'$ . Moreover if  $C + Y = C + Y'$ , then  $Y = Y'$ . Let  $f : X' \rightarrow X$  be a morphism of  $S$ -spaces, then  $f$  induces a morphism of semirings  $Clo(f) : Clo(X) \rightarrow Clo(X'), Y \mapsto Y \times_X X'$ , moreover  $Clo(f)$  restricted to  $(Pri(X), +)$  factors through  $(Pri(X'), +)$ , this morphism of monoids is denoted  $Pri(f)$ . Let  $Y_1, Y_2 \in Clo(X)$ , we write  $Y_1 \subset Y_2$  if  $Y_1$  is a closed subspace of  $Y_2$ . We obtain a poset  $(Clo(X), \subset)$ . Let  $Y_1, Y_2, Y_3 \in Clo(X)$ , if  $Y_1 \subset Y_2$  and  $Y_1 \subset Y_3$  then  $Y_1 \subset Y_2 \cap Y_3$ . Let  $Y_1, Y_2 \in Clo(X)$ , then  $(Y_1 \cap Y_2) \subset Y_1$  and  $Y_1 \subset (Y_1 + Y_2)$ . Finally, if  $Y = \{Y_e\}_{e \in E}$  is a subset of  $Clo(X)$  and if  $\nu \in \mathbb{N}^E$ , we put  $Y^\nu = \{\nu_e Y_e\}_{e \in E}$  and if moreover  $\nu \in \mathbb{N}_E$ , we put  $\nu Y = \sum_{e \in E} \nu_e Y_e$ .

REMARK 3.2. Be careful that the operation  $+$  on  $Clo(X)$  is not the operation  $\cup$  of [StP, Tag 0C4H]. Recall that  $+$  corresponds to multiplication of ideals whereas  $\cup$  corresponds to intersection of ideals.

REMARK 3.3. We proceed with the notation from Notation 3.1. In general the image of the map  $Pri(f)|_{Car(X)}$  is not included in  $Car(X')$ .

DEFINITION 3.4. Let  $D = \{D_i\}_{i \in I}$  be a subset of  $Clo(X)$ .

- (i) Let  $\text{Spaces}_X^{D\text{-reg}}$  be the category of  $S$ -algebraic spaces  $f : T \rightarrow X$  over  $X$  such that for any  $i \in I$ ,  $T \times_X D_i$  is a Cartier divisor in  $T$ .



- (ii) If  $X = S$  is a scheme, let  $\text{Sch}_X^{D\text{-reg}}$  be the category of  $X$ -schemes  $f : T \rightarrow X$  such that for any  $i \in I$ ,  $T \times_X D_i$  is a Cartier divisor in  $T$ .

If  $T' \rightarrow T$  is flat and  $T \rightarrow X$  is an object in  $\text{Spaces}_X^{D\text{-reg}}$  or  $\text{Sch}_X^{D\text{-reg}}$ , so is the composition  $T' \rightarrow T \rightarrow X$  by [StP, Tag 083Z] and [StP, Tag 020O]. In particular, the categories  $\text{Spaces}_X^{D\text{-reg}}$  and  $\text{Sch}_X^{D\text{-reg}}$  can be equipped with the fpqc/fppf/étale/Zariski Grothendieck topology so that the notion of sheaves is well-defined.

FACT 3.5. Let  $D = \{D_i\}_{i \in I}$  be a subset of  $\text{Clo}(X)$ .

- (i) Let  $f : T \rightarrow X$  be an object in  $\text{Spaces}_X^{D\text{-reg}}$ . Then for any  $\nu \in \mathbb{N}_I$ , the space  $T \times_X \nu D$  is a Cartier divisor in  $T$ , namely  $\nu(T \times_X D)$ .
- (ii) Assume that  $\#I$  is finite, then  $\text{Spaces}_X^{D\text{-reg}}$  equals  $\text{Spaces}_X^{\sum_{i \in I} D_i}$ .

*Proof.* (i) This follows from the fact that  $(\text{Clo}(f), +)$  is a morphism of monoids and the fact that  $\text{Car}(X)$  is a submonoid of  $\text{Clo}(X)$  (e.g. cf. the discussion in Notation 3.1).

- (ii) This follows from the fact that  $\text{Car}(X)$  is a face of the monoid  $\text{Pri}(X)$  (e.g. cf. the discussion in Notation 3.1). □

DEFINITION 3.6. A multi-center in  $X$  is a set  $\{[Y_i, D_i]\}_{i \in I}$  such that

- (i)  $Y_i$  and  $D_i$  belong to  $\text{Clo}(X)$ ,
- (ii) there exists an affine étale covering  $\{U_\gamma \rightarrow X\}_{\gamma \in \Gamma}$  of  $X$  such that  $D_i|_{U_\gamma}$  is principal for all  $i \in I$  and  $\gamma \in \Gamma$  (in particular  $D_i$  belongs to  $\text{Pri}(X)$  for all  $i$ ).

In other words a multi-center  $\{[Y_i, D_i]\}_{i \in I}$  is a set of pairs of closed  $S$ -spaces such that locally each  $D_i$  is principal.

REMARK 3.7. Let  $\{Y_i, D_i\}_{i \in I}$  such that  $Y_i \in \text{Clo}(X)$  and  $D_i \in \text{Pri}(X)$  for any  $i \in I$ . Assume that  $I$  is finite, then  $\{[Y_i, D_i]\}_{i \in I}$  is a multi-center in  $X$ , i.e. the second condition in Definition 3.6 is satisfied.

We now fix a multi-center  $\{[Y_i, D_i]\}_{i \in I}$  in  $X$ . Denote by  $\mathcal{M}_i, \mathcal{J}_i$  the quasi-coherent sheaves of ideals in  $\mathcal{O}_X$  so that  $Y_i = V(\mathcal{M}_i), V(\mathcal{J}_i) = D_i$ . We put  $Z_i = Y_i \cap D_i$  and  $\mathcal{L}_i = \mathcal{M}_i + \mathcal{J}_i$  so that  $Z_i = V(\mathcal{L}_i)$  for any  $i \in I$ . We put  $Y = \{Y_i\}_{i \in I}$ ,  $D = \{D_i\}_{i \in I}$  and  $Z = \{Z_i\}_{i \in I}$ . We now introduce dilatations  $\mathcal{O}_X$ -algebras by glueing (cf. [StP, Tag 04TP]).

DEFINITION AND PROPOSITION 3.8. The dilatation of  $\mathcal{O}_X$  with multi-center  $\{[\mathcal{M}_i, \mathcal{J}_i]\}_{i \in I}$  is the quasi-coherent  $\mathcal{O}_X$ -algebra  $\mathcal{O}_X \left[ \left\{ \frac{\mathcal{M}_i}{\mathcal{J}_i} \right\}_{i \in I} \right]$  obtained by glueing as follows. The quasi-coherent  $\mathcal{O}_X$ -algebra  $\mathcal{O}_X \left[ \left\{ \frac{\mathcal{M}_i}{\mathcal{J}_i} \right\}_{i \in I} \right]$  is characterized by the fact that its restriction, on any étale  $S$ -morphism  $\varphi : U \rightarrow X$  such that  $U$  is an affine scheme and each  $D_i$  is principal on  $U$  and generated by  $a_{iU}$ , is given by

$$\left( \mathcal{O}_X \left[ \left\{ \frac{\mathcal{M}_i}{\mathcal{J}_i} \right\}_{i \in I} \right] \right) \Big|_U = \Gamma(U, \mathcal{O}_X) \left[ \left\{ \frac{\widetilde{\Gamma(U, \mathcal{M}_i)}}{a_{iU}} \right\}_{i \in I} \right]$$

where  $\widetilde{\phantom{x}}$  is given by [StP, Tag 01I7] and [StP, Tag 03DT] (we work with small étale sites).

*Proof.* By Definition 3.6, the affine schemes  $U \rightarrow X$  satisfying the conditions in the statement form an étale covering of  $X$ . Now Proposition 3.8 follows from [StP, Tag 03M0] and Corollary 2.31. □



Let  $\mathrm{Bl}_{\{\mathcal{L}_i\}_{i \in I}} \mathcal{O}_X = \bigoplus_{\nu \in \mathbb{N}_I} \mathcal{L}^\nu$  denote the multi-Rees algebra, it is a quasi-coherent  $\mathbb{N}_I$ -graded  $\mathcal{O}_X$ -algebra. By localization, we get a quasi-coherent  $\mathcal{O}_X$ -algebra  $(\mathrm{Bl}_{\{\mathcal{L}_i\}_{i \in I}} \mathcal{O}_X)[\{\mathcal{J}_i^{-1}\}_{i \in I}]$  (locally, we invert a generator of  $\mathcal{J}_i$ , for each  $i \in I$ ). This  $\mathcal{O}_X$ -algebra inherits a grading giving local generators of  $\mathcal{J}_i$  degree  $e_i$ .

FACT 3.9. *We have a canonical identification of quasi-coherent  $\mathcal{O}_X$ -algebras*

$$\mathcal{O}_X \left[ \left\{ \frac{\mathcal{M}_i}{\mathcal{J}_i} \right\}_{i \in I} \right] = \left[ (\mathrm{Bl}_{\{\mathcal{L}_i\}_{i \in I}} \mathcal{O}_X)[\{\mathcal{J}_i^{-1}\}_{i \in I}] \right]_{\mathrm{deg}=(0, \dots, 0, \dots)},$$

where the right-hand side is obtained as the subsheaf of degree zero elements in  $(\mathrm{Bl}_{\{\mathcal{L}_i\}_{i \in I}} \mathcal{O}_X)[\{\mathcal{J}_i^{-1}\}_{i \in I}]$ . In particular  $\mathcal{O}_X \left[ \left\{ \frac{\mathcal{M}_i}{\mathcal{J}_i} \right\}_{i \in I} \right] = \mathcal{O}_X \left[ \left\{ \frac{\mathcal{L}_i}{\mathcal{J}_i} \right\}_{i \in I} \right]$ .

*Proof.* This follows from Fact 2.23. □

DEFINITION 3.10. The dilatation of  $X$  with multi-center  $\{[Y_i, D_i]\}_{i \in I}$  is the  $X$ -affine algebraic space over  $S$

$$\mathrm{Bl}_Y^D X \stackrel{\mathrm{def}}{=} \mathrm{Spec}_X(\mathcal{O}_X \left[ \left\{ \frac{\mathcal{M}_i}{\mathcal{J}_i} \right\}_{i \in I} \right]).$$

REMARK 3.11. Fact 3.9 implies that  $\mathrm{Bl}_Y^D X = \mathrm{Bl}_Z^D X$ .

FACT 3.12. *Assume that  $X = S$  is a scheme. Then  $\mathrm{Bl}_Y^D X$  is a scheme.*

*Proof.* We have an affine morphism  $\mathrm{Bl}_Y^D X \rightarrow X$ , now the fact follows from [StP, Tag 03WG]. □

NOTATION 3.13. *We will also use the notation  $\mathrm{Bl}_{\{Y_i\}_{i \in I}}^{\{D_i\}_{i \in I}} X$  and  $\mathrm{Bl}_{\{Y_i\}_{i \in I}}^{\{D_i\}_{i \in I}} X$  to denote  $\mathrm{Bl}_Y^D X$ . If  $I = \{i\}$  is a singleton we also use the notation  $\mathrm{Bl}_{Y_i}^{D_i} X$ . If  $K \subset I$ , we sometimes use the notation  $\mathrm{Bl}_{\{Y_i\}_{i \in K}, \{Y_i\}_{i \in I \setminus K}}^{\{D_i\}_{i \in K}, \{D_i\}_{i \in I \setminus K}} X$ . If  $I = \{1, \dots, k\}$ , we use the notation  $\mathrm{Bl}_{Y_1, \dots, Y_k}^{D_1, \dots, D_k} X$ . Etc.*

DEFINITION 3.14. Let  $X$  be a scheme or an algebraic space over a scheme  $S$ . We say that a morphism  $f : X' \rightarrow X$  is a dilatation morphism if  $f$  is equal to  $\mathrm{Bl}_{\{Y_i\}_{i \in I}}^{\{D_i\}_{i \in I}} X \rightarrow X$  for some multi-center  $\{[Y_i, D_i]\}_{i \in I}$ .

### 3.2 Exceptional divisors

We proceed with the notation from § 3.1.

PROPOSITION 3.15. *As closed subspaces of  $\mathrm{Bl}_Y^D X$ , one has, for all  $\nu \in \mathbb{N}_I$ ,*

$$\mathrm{Bl}_Y^D X \times_X \nu Z = \mathrm{Bl}_Y^D X \times_X \nu D,$$

which is an effective Cartier divisor on  $\mathrm{Bl}_Y^D X$ .

*Proof.* Our claim is étale local on  $X$ . We reduce to the affine case and apply 2.7 and 2.20. □

### 3.3 Universal property

We proceed with the notation from § 3.1. As  $\mathrm{Bl}_Y^D X \rightarrow X$  defines an object in  $\mathrm{Spaces}_X^{D\text{-reg}}$  by Proposition 3.15, the contravariant functor

$$\mathrm{Spaces}_X^{D\text{-reg}} \rightarrow \mathrm{Sets}, \quad (T \rightarrow X) \mapsto \mathrm{Hom}_{X\text{-Spaces}}(T, \mathrm{Bl}_Y^D X) \tag{3.1}$$

together with  $\mathrm{id}_{\mathrm{Bl}_Y^D X}$  determines  $\mathrm{Bl}_Y^D X \rightarrow X$  uniquely up to unique isomorphism. The next proposition gives the universal property of dilatations.

PROPOSITION 3.16. *The dilatation  $\mathrm{Bl}_Y^D X \rightarrow X$  represents the contravariant functor  $\mathrm{Spaces}_X^{D\text{-reg}} \rightarrow \mathrm{Sets}$  given by*

$$(f: T \rightarrow X) \mapsto \begin{cases} \{*\}, & \text{if } f|_{T \times_X D_i} \text{ factors through } Y_i \subset X \text{ for } i \in I; \\ \emptyset, & \text{else.} \end{cases} \quad (3.2)$$

*Proof.* Note that the condition  $f|_{T \times_X D_i}$  factors through  $Y_i \subset X$  is equivalent to the condition  $f|_{T \times_X D_i}$  factors through  $Z_i \subset X$ , because  $Z_i = Y_i \cap D_i$ . Let  $F$  be the functor defined by (3.2). If  $T \rightarrow \mathrm{Bl}_Y^D X$  is a map of  $X$ -spaces, then the structure map  $T \rightarrow X$  restricted to  $T \times_X D_i$  factors through  $Z_i \subset X$  by Proposition 3.15. This defines a map

$$\mathrm{Hom}_{X\text{-Spaces}}(-, \mathrm{Bl}_Y^D X) \longrightarrow F \quad (3.3)$$

of contravariant functors  $\mathrm{Spaces}_X^{D\text{-reg}} \rightarrow \mathrm{Sets}$ . We want to show that (3.3) is bijective when evaluated at an object  $T \rightarrow X$  in  $\mathrm{Spaces}_X^{D\text{-reg}}$ . As (3.3) is a morphism of étale sheaves, we reduce to the case where both  $X$  and  $T$  are affine and  $J_i$  is principal for all  $i \in I$ . Now Proposition 2.21 finishes the proof.  $\square$

PROPOSITION 3.17. *Put  $f: \mathrm{Bl}_Y^D X \rightarrow X$ . Then the morphism of monoids  $\mathrm{Clo}(f)|_{\mathrm{Car}(X)}$  factors through  $\mathrm{Car}(\mathrm{Bl}_Y^D X)$ . In other words, any effective Cartier divisor  $C \subset X$  is defined for  $f$ , i.e. the fiber product  $C \times_X \mathrm{Bl}_Y^D X \subset \mathrm{Bl}_Y^D X$  is an effective Cartier divisor (cf. [StP, Tag 01WV]).*

*Proof.* We reduce to the case where  $X = \mathrm{Spec}(A)$  is affine and apply Fact 3.17.  $\square$

PROPOSITION 3.18. *Let  $J$  be a subset of  $I$ . There exists a unique  $X$ -morphism*

$$\varphi: \mathrm{Bl}_{\{Y_j\}_{j \in I}}^{\{D_j\}_{j \in I}} X \rightarrow \mathrm{Bl}_{\{Y_j\}_{j \in J}}^{\{D_j\}_{j \in J}} X.$$

*Proof.* This follows from Propositions 3.16 and 3.15.  $\square$

REMARK 3.19. Proposition 3.18 is the spaces version of Fact 2.14.

REMARK 3.20. Proposition 3.25 will refine Proposition 3.18 and show that  $\varphi$  is in fact a dilatation map (cf. Definition 3.14).

PROPOSITION 3.21. *Let  $K \subset I$  and assume  $Z_i = D_i$  is a Cartier divisor in  $X$ , for all  $i \in K$ . Then*

$$\mathrm{Bl}_{\{Z_j\}_{j \in I}}^{\{D_j\}_{j \in I}} X = \mathrm{Bl}_{\{Z_j\}_{j \in I \setminus K}}^{\{D_j\}_{j \in I \setminus K}} X.$$

*Proof.* Both sides belong to  $\mathrm{Spaces}_X^{D\text{-reg}}$  by Propositions 3.17 and 3.15, so it is enough to show that they agree when evaluated at any  $f: T \rightarrow X \in \mathrm{Spaces}_X^{D\text{-reg}}$ . We have

$$\begin{aligned} \mathrm{Bl}_{\{Z_j\}_{j \in I}}^{\{D_j\}_{j \in I}} X(T) &= \begin{cases} \{*\}, & \text{if } f|_{T \times_X D_i} \text{ factors through } Z_i \subset X \text{ for } i \in I; \\ \emptyset, & \text{else.} \end{cases} \\ &= \begin{cases} \{*\}, & \text{if } f|_{T \times_X D_i} \text{ factors through } Z_i \subset X \text{ for } i \in I \setminus K; \\ \emptyset, & \text{else.} \end{cases} \\ &= \mathrm{Bl}_{\{Z_j\}_{j \in I \setminus K}}^{\{D_j\}_{j \in I \setminus K}} X(T). \end{aligned}$$

$\square$

REMARK 3.22. Proposition 3.21 is the spaces version of Proposition 2.14.

PROPOSITION 3.23. Assume  $D_i = D_j =: D$  for all  $i, j \in I$ . Then

$$\mathrm{Bl}\{Y_i\}_{i \in I}^D X = \mathrm{Bl}_{\cap_{i \in I} Y_i}^D X.$$

*Proof.* Both sides belong to  $\mathrm{Spaces}_X^{D\text{-reg}}$  by Proposition 3.15, so it is enough to show that they agree when evaluated at any  $f : T \rightarrow X \in \mathrm{Spaces}_X^{D\text{-reg}}$ . We have

$$\begin{aligned} \mathrm{Bl}\{Y_i\}_{i \in I}^D X(T) &= \begin{cases} \{*\}, & \text{if } f|_{T \times_X D} \text{ factors through } Z_i \subset X \text{ for } i \in I; \\ \emptyset, & \text{else.} \end{cases} \\ &= \begin{cases} \{*\}, & \text{if } f|_{T \times_X D} \text{ factors through } \cap_{i \in I} Z_i \subset X; \\ \emptyset, & \text{else.} \end{cases} \\ &= \mathrm{Bl}_{\cap_{i \in I} Y_i}^D X(T). \end{aligned}$$

□

REMARK 3.24. Proposition 3.23 is the spaces version of Proposition 2.18.

PROPOSITION 3.25. Let  $J$  be a subset of  $I$  and put  $K = I \setminus J$ . Then

$$\mathrm{Bl}\{Y_i\}_{i \in I}^{D_i} X = \mathrm{Bl}\left\{_{Y_k \times_X \mathrm{Bl}\{Y_i\}_{i \in J}^{D_i} X}^{D_k \times_X \mathrm{Bl}\{Y_i\}_{i \in J}^{D_i} X}\right\}_{k \in K} \mathrm{Bl}\{Y_i\}_{i \in J}^{D_i} X.$$

This in particular gives the unique  $X$ -morphism

$$\mathrm{Bl}\{Y_i\}_{i \in I}^{D_i} X \rightarrow \mathrm{Bl}\{Y_i\}_{i \in J}^{D_i} X$$

of Proposition 3.18.

*Proof.* The right hand side is well-defined (e.g. cf. [StP, Tag 053P]). Using Proposition 3.15 and Proposition 3.17, one obtains that the right hand side is in  $\mathrm{Spaces}_X^{D\text{-reg}}$ . So it is enough to see that both sides coincide when evaluated at any  $f : T \rightarrow X \in \mathrm{Spaces}_X^{D\text{-reg}}$ . This follows from Proposition 3.16. □

### 3.4 Multi-centered dilatations and mono-centered dilatations

We proceed with the notation from § 3.1.

PROPOSITION 3.26. Write  $I = \mathrm{colim}_{J \subset I} J$  as a filtered colimit of sets where transition maps are given by inclusions of subsets. We have a canonical identification

$$\mathrm{Bl}\{Y_i\}_{i \in I}^{D_i} X = \lim_{J \subset I} \mathrm{Bl}\{Y_i\}_{i \in J}^{D_i} X$$

where transition maps are described in Propositions 3.18 and 3.25. On the right-hand side the direct limit is on the category of  $S$ -spaces over  $X$ .

*Proof.* By [StP, Tag 07SF] the limit exists. For each  $J \subset I$ , Propositions 3.18 and 3.25 give us a  $X$ -morphism  $\mathrm{Bl}\{Y_i\}_{i \in I}^{D_i} X \rightarrow \mathrm{Bl}\{Y_i\}_{i \in J}^{D_i} X$ , so we get an  $X$ -morphism  $\phi : \mathrm{Bl}\{Y_i\}_{i \in I}^{D_i} X \rightarrow \lim_{J \subset I} \mathrm{Bl}\{Y_i\}_{i \in J}^{D_i} X$ . To prove that  $\phi$  is an isomorphism, we reduce to the affine case where the result follows from Proposition 2.27. □

PROPOSITION 3.27. Assume that  $\#I = k$  is finite. We fix an arbitrary bijection  $I = \{1, \dots, k\}$ . We have a canonical isomorphism of  $X$ -spaces

$$\mathrm{Bl}\{Y_i\}_{i \in I}^{D_i} X \cong \mathrm{Bl}_{(\mathrm{Bl}\dots) \times_X Y_k}^{(\mathrm{Bl}\dots) \times_X D_k} \left( \dots \mathrm{Bl}_{(\mathrm{Bl}\dots) \times_X Y_3}^{(\mathrm{Bl}\dots) \times_X D_3} \left( \mathrm{Bl}_{(\mathrm{Bl}_{Y_1}^{D_1} X) \times_X Y_2}^{(\mathrm{Bl}_{Y_1}^{D_1} X) \times_X D_2} (\mathrm{Bl}_{Y_1}^{D_1} X) \right) \right).$$

*Proof.* By induction on  $k$  using Proposition 3.25.  $\square$

PROPOSITION 3.28. Assume that  $\#I = k$  is finite. We fix an arbitrary bijection  $I = \{1, \dots, k\}$ . We have a canonical isomorphism of  $X$ -spaces

$$\mathrm{Bl}_{\{Y_i\}_{i \in I}}^{\{D_i\}_{i \in I}} X \cong \mathrm{Bl}_{\bigcap_{i \in I} (Y_i + D_1 + \dots + D_{i-1} + D_{i+1} + \dots + D_k)}^{D_1 + \dots + D_k} X.$$

*Proof.* Since  $\mathrm{Car}(X)$  is a face of the monoid  $\mathrm{Pri}(X)$ , the right-hand side belongs to  $\mathrm{Spaces}_X^{D\text{-reg}}$ . Let  $f : \mathrm{Bl}_{\bigcap_{i \in I} (Y_i + D_1 + \dots + D_{i-1} + D_{i+1} + \dots + D_k)}^{D_1 + \dots + D_k} X \rightarrow X$  be the dilatation morphism. Let us prove that  $f^{-1}(D_i) \subset f^{-1}(Y_i)$  for all  $i \in \{1, \dots, k\}$ . By Proposition 3.16,

$$f^{-1}\left(\bigcap_{i \in I} Y_i + D_1 + \dots + D_{i-1} + D_{i+1} + \dots + D_k\right) = f^{-1}(D_1 + \dots + D_k)$$

is a Cartier divisor in  $f^{-1}(X)$ . Moreover by the discussion in Notation 3.1

$$\begin{aligned} & f^{-1}\left(\bigcap_{i \in I} Y_i + D_1 + \dots + D_{i-1} + D_{i+1} + \dots + D_k\right) \\ &= \bigcap_{i \in I} f^{-1}(Y_i) + f^{-1}(D_1) + \dots + f^{-1}(D_{i-1}) + f^{-1}(D_{i+1}) + \dots + f^{-1}(D_k), \text{ and} \end{aligned}$$

$$f^{-1}(D_1 + \dots + D_k) = f^{-1}(D_1) + \dots + f^{-1}(D_k).$$

So for any  $i \in \{1, \dots, k\}$ , we have

$$f^{-1}(D_1) + \dots + f^{-1}(D_k) \subset f^{-1}(Y_i) + f^{-1}(D_1) + \dots + f^{-1}(D_{i-1}) + f^{-1}(D_{i+1}) + \dots + f^{-1}(D_k).$$

Since  $f^{-1}(D_l)$  is a Cartier divisor for any  $l \in \{1, \dots, k\}$ , this implies  $f^{-1}(D_i) \subset f^{-1}(Y_i)$ . So we obtain an  $X$ -morphism  $\phi : \mathrm{Bl}_{\bigcap_{i \in I} (Y_i + D_1 + \dots + D_{i-1} + D_{i+1} + \dots + D_k)}^{D_1 + \dots + D_k} X \rightarrow \mathrm{Bl}_{\{Y_i\}_{i \in I}}^{\{D_i\}_{i \in I}} X$ . To check that  $\phi$  is an isomorphism, it is enough to prove that there is an  $X$ -morphism

$$\varphi : \mathrm{Bl}_{\{Y_i\}_{i \in I}}^{\{D_i\}_{i \in I}} X \rightarrow \mathrm{Bl}_{\bigcap_{i \in I} (Y_i + D_1 + \dots + D_{i-1} + D_{i+1} + \dots + D_k)}^{D_1 + \dots + D_k} X.$$

To build  $\varphi$ , we consider the map  $f' : \mathrm{Bl}_{\{Y_i\}_{i \in I}}^{\{D_i\}_{i \in I}} X \rightarrow X$  and check that  $f'^{-1}(D_1 + \dots + D_k) \subset f'^{-1}(\bigcap_{i \in I} (Y_i + D_1 + \dots + D_{i-1} + D_{i+1} + \dots + D_k))$ . This is easy because  $f'^{-1}(D_i) \subset f'^{-1}(Y_i)$  for all  $i$ . An other method to prove Proposition 3.28 is to build  $\phi$  or  $\varphi$  and then reduce to the affine case an apply Proposition 2.25.  $\square$

### 3.5 Functoriality

We proceed with the notation from § 3.1. Let  $X'$  and  $\{[Y'_i, D'_i]\}_{i \in I}$  be another datum as in § 3.1. As usual, put  $Z'_i = Y'_i \cap D'_i$ . A morphism  $f : X' \rightarrow X$  such that, for all  $i \in I$ , its restriction to  $D'_i$  (resp.  $Z'_i$ ) factors through  $D_i$  (resp.  $Z_i$ ), and such that  $f^{-1}(D_i) = D'_i$ , induces a unique morphism  $\mathrm{Bl}_{Y'}^{D'} X' \rightarrow \mathrm{Bl}_Y^D X$  such that the following diagram of  $S$ -spaces

$$\begin{array}{ccc} \mathrm{Bl}_{Y'}^{D'} X' & \longrightarrow & \mathrm{Bl}_Y^D X \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X. \end{array}$$

commutes. This follows directly from Proposition 3.16.

REMARK 3.29. This is the spaces version of Fact 2.11.

### 3.6 Base change

We proceed with the notation from §3.1. Let  $X' \rightarrow X$  be a map of  $S$ -spaces, and denote by  $Y'_i, Z'_i, D'_i \subset X'$  the preimage of  $Y_i, Z_i, D_i \subset X$ . Then  $D'_i \subset X'$  is locally principal for any  $i$  so that the dilatation  $\mathrm{Bl}_{Y'}^D X' \rightarrow X'$  is well-defined. By §3.5 there is a canonical morphism of  $X'$ -spaces

$$\mathrm{Bl}_{Y'}^D X' \longrightarrow \mathrm{Bl}_Y^D X \times_X X'. \quad (3.4)$$

LEMMA 3.30. *If  $\mathrm{Bl}_Y^D X \times_X X' \rightarrow X'$  is an object of  $\mathrm{Spaces}_{X'}^{D\text{-reg}}$ , then (3.4) is an isomorphism.*

*Proof.* Our claim is étale local on  $X$  and  $X'$ . We reduce to the case where both  $X = \mathrm{Spec}(B)$ ,  $X' = \mathrm{Spec}(B')$  are affine, and  $J_i = (b_i)$  is principal for all  $i$ . We denote  $Z'_i = \mathrm{Spec}(B'/L'_i)$  and  $D'_i = \mathrm{Spec}(B'/J'_i)$ . Then  $J'_i = (b'_i)$  is principal as well where  $b'_i$  is the image of  $b_i$  under  $B \rightarrow B'$ . We need to show that the map of  $B'$ -algebras  $B' \otimes_B B[\frac{L}{b}] \rightarrow B'[\frac{L'}{b'}]$  is an isomorphism. However, this map is surjective with kernel the  $b'^{\mathbb{N}_I}$ -torsion elements by Lemma 2.30. As  $b'_1, \dots, b'_i, \dots$  are non-zero-divisors in  $B' \otimes_B B[\frac{L}{b}]$  by assumption, the lemma follows.  $\square$

COROLLARY 3.31. *If the morphism  $X' \rightarrow X$  is flat and satisfies a property  $\mathcal{P}$  which is stable under base change, then  $\mathrm{Bl}_{Y'}^D X' \rightarrow \mathrm{Bl}_Y^D X$  is flat and satisfies  $\mathcal{P}$ .*

*Proof.* Since flatness is stable under base change the projection  $p: \mathrm{Bl}_Y^D X \times_X X' \rightarrow \mathrm{Bl}_Y^D X$  is flat and has property  $\mathcal{P}$ . By Lemma 3.30, it is enough to check that the closed subspace  $\mathrm{Bl}_Y^D X \times_X D'_i$  defines an effective Cartier divisor on  $\mathrm{Bl}_Y^D X \times_X X'$  for all  $i$ . But this closed subscheme is the preimage of the effective Cartier divisor  $\mathrm{Bl}_Y^D X \times_X D_i$  under the flat map  $p$ , and hence is an effective Cartier divisor as well by [StP, Tag 083Z].  $\square$

### 3.7 Relation to multi-centered affine projecting cone

We proceed with the notation from §3.1 and assume that  $\{D_i\}_{i \in I}$  belong to  $\mathrm{Car}(X)$ . In this case, we can also realize  $\mathrm{Bl}_Y^D X$  as a closed subscheme of the multi-centered affine projecting cone associated to  $X, Z$  and  $D$ .

DEFINITION 3.32. The *affine projecting cone  $\mathcal{O}_X$ -algebra* with multi-center  $\{[Z_i = V(\mathcal{L}_i), D_i = V(\mathcal{J}_i)]\}_{i \in I}$  is

$$\mathrm{C}_{\mathcal{L}}^{\mathcal{J}} \mathcal{O}_X \stackrel{\mathrm{def}}{=} \bigoplus_{\nu \in \mathbb{N}_I} \mathcal{L}^{\nu} \otimes \mathcal{J}^{-\nu}.$$

The *affine projecting cone* of  $X$  with multi-center  $\{[Z_i, D_i]\}_{i \in I}$  is

$$\mathrm{C}_Z^D X \stackrel{\mathrm{def}}{=} \mathrm{Spec}(\mathrm{C}_{\mathcal{L}}^{\mathcal{J}} \mathcal{O}_X).$$

PROPOSITION 3.33. *The dilatation  $\mathrm{Bl}_Z^D X$  is the closed subscheme of the affine projecting cone  $\mathrm{C}_Z^D X$  defined by the equations  $\{\varrho_i - 1\}_{i \in I}$ , where for all  $i \in I$ ,  $\varrho_i \in \mathrm{C}_{\mathcal{L}}^{\mathcal{J}} \mathcal{O}_X$  is the image of  $1 \in \mathcal{O}_X$  under the map*

$$\mathcal{O}_X \cong \mathcal{J}_i \otimes \mathcal{J}_i^{-1} \subset \mathcal{L}_i \otimes \mathcal{J}_i^{-1} \subset \mathrm{C}_{\mathcal{L}}^{\mathcal{J}} \mathcal{O}_X.$$

*Proof.* We may work locally and the proposition follows from Proposition 2.33 and Remark 2.34.  $\square$

### 3.8 Relation to Proj of multi-graded algebras and multi-centered blowups

We proceed with the notation from §3.1 and assume that  $X = \mathrm{Spec}(A)$  is an affine scheme but we expect the content of this section can be adapted to algebraic spaces. We assume moreover that

$I = \{1, \dots, k\}$  is finite. We have  $\mathbb{N}_I = \mathbb{N}^k$ . We refer to [BS07, §2] for the construction of the Proj associated to  $\Omega$ -graded  $R$ -algebras where  $\Omega$  is a finitely generated abelian group and  $R$  is a ring. This construction should globalize to arbitrary  $\Omega$ -graded quasi-coherent algebras. It should also work for algebraic spaces, e.g. cf. [StP, Tag 085P] for a hint. We plan to study this later. Recall that  $Z_1, \dots, Z_k$  are closed subspaces of  $X$  with ideals  $L_1, \dots, L_k$ . Recall that  $Z_i \subset D_i$  for all  $i$ . We assume that each  $D_i$  is principal and given by elements  $a_i \in A$ . We introduce the following definition. Recall that  $\text{Bl}_{L_1, \dots, L_k} A$  is canonically  $\mathbb{N}^k$ -graded, and in particular  $\mathbb{Z}^r$ -graded.

DEFINITION 3.34. The blowup of  $X$  with multi-center  $Z = (Z_1, \dots, Z_k)$  is the scheme

$$\text{Bl}_Z X \stackrel{\text{def}}{=} \text{Proj}(\text{Bl}_{L_1, \dots, L_k} A).$$

PROPOSITION 3.35. The map  $\text{Bl}_{Z_1, \dots, Z_k} X \rightarrow X$  is a final object of  $\text{Sch}_X^{Z_1, \dots, Z_k\text{-reg}}$ .

*Proof.* We adapt [GW, Chap 13, pages 414-415]. By definition  $\text{Bl}_Z X = \bigcup_{\{f \text{ relevant}\}} D_+(f)$  (cf. [BS07, Def. 2.2]). We have  $D_+(f) = \text{Spec} A[\frac{L_1}{f}, \dots, \frac{L_k}{f}]$ . Now 2.7, 2.20 and 2.21 finish the proof.  $\square$

PROPOSITION 3.36. Let  $1 \leq r \leq k$  and let  $Z'_i = \text{Bl}_{Z_{r+1}, \dots, Z_k} X \times_X Z_i$  for  $1 \leq i \leq r$ . Then we have a canonical isomorphism of  $X$ -schemes

$$\text{Bl}_{Z'_1, \dots, Z'_r} \text{Bl}_{Z_{r+1}, \dots, Z_k} X = \text{Bl}_{Z_1, \dots, Z_k} X.$$

*Proof.* The right-hand side belongs to  $\text{Sch}_X^{Z_1, \dots, Z_k\text{-reg}}$ . We now observe that the left-hand side also belongs to  $\text{Sch}_X^{Z_1, \dots, Z_k\text{-reg}}$ . Indeed, let  $g$  be the composition

$$\text{Bl}_{Z'_1, \dots, Z'_r} \text{Bl}_{Z_{r+1}, \dots, Z_k} X \rightarrow \text{Bl}_{Z_{r+1}, \dots, Z_k} X \rightarrow X.$$

Obviously  $g^{-1}(Z_i)$  is a Cartier divisor in  $\text{Bl}_{Z'_1, \dots, Z'_r} \text{Bl}_{Z_{r+1}, \dots, Z_k} X$  for  $1 \leq i \leq r$ . We claim that  $g^{-1}(Z_i)$  is also a Cartier divisor in  $\text{Bl}_{Z'_1, \dots, Z'_r} \text{Bl}_{Z_{r+1}, \dots, Z_k} X$  for  $r+1 \leq i \leq k$ . To see this, we observe that the preimage of any Cartier divisor of  $\text{Bl}_{Z_{r+1}, \dots, Z_k} X$  in  $\text{Bl}_{Z'_1, \dots, Z'_r} \text{Bl}_{Z_{r+1}, \dots, Z_k} X$  is a Cartier divisor, this follows from the local definition of Proj and Proposition 3.17. So both sides belong to  $\text{Sch}_X^{Z_1, \dots, Z_k\text{-reg}}$ . It is easy to construct an  $X$ -morphism  $\text{Bl}_{Z'_1, \dots, Z'_r} \text{Bl}_{Z_{r+1}, \dots, Z_k} X \leftarrow \text{Bl}_{Z_1, \dots, Z_k} X$ . This shows that the left-hand side is also (cf. Proposition 3.35) a final object of  $\text{Sch}_X^{Z_1, \dots, Z_k\text{-reg}}$ . This finishes the proof.  $\square$

PROPOSITION 3.37. We have a canonical isomorphism of  $X$ -schemes

$$\text{Bl}_{Z_1 + \dots + Z_k} X = \text{Bl}_{Z_1, \dots, Z_k} X.$$

*Proof.* This follows from [StP, Tag 085Y].  $\square$

FACT 3.38. The dilatation  $\text{Bl}_Z^D X$  is the open subscheme of the blowup  $\text{Bl}_Z X$  defined by  $D_+(a_1 \dots a_k)$  (cf. [BS07, page 6] for the notation  $D_+(-)$ ).

*Proof.* The identity  $A[\frac{L_1}{a_1}, \dots, \frac{L_k}{a_k}] = (\text{Bl}_{L_1, \dots, L_k} A)_{(a_1 \dots a_k)}$ , given by Fact 2.24, is a proof.  $\square$

#### 4. Iterated multi-centered dilatations

We proceed with the notation from § 3.1. Let  $\nu, \theta \in \mathbb{N}^I$  such that  $\theta \leq \nu$ , i.e.  $\theta_i \leq \nu_i$  for all  $i \in I$ .

PROPOSITION 4.1. There is a unique  $X$ -morphism

$$\varphi_{\nu, \theta} : \text{Bl}_Y^{D^\nu} X \rightarrow \text{Bl}_Y^{D^\theta} X.$$

*Proof.* Let  $\varphi_\nu : \mathrm{Bl}_Y^{D^\nu} X \rightarrow X$ ,  $\varphi_\theta : \mathrm{Bl}_Y^{D^\theta} X \rightarrow X$  be the dilatation maps. Let  $i \in I$ , if  $\theta_i = 0$ , then  $\theta_i D_i = X$  and so  $\varphi_\nu^{-1}(\theta_i D_i) = \mathrm{Bl}_Y^{D^\nu} X$  is a Cartier divisor in  $\mathrm{Bl}_Y^{D^\nu} X$ . If  $\theta_i > 0$ , then  $\nu_i > 0$  and  $\varphi_\nu^{-1}(\nu_i D_i) = \nu_i \varphi_\nu^{-1}(D_i)$  is Cartier and so  $\varphi_\nu^{-1}(D_i)$  is Cartier because  $\mathrm{Car}(\mathrm{Bl}_Y^{D^\nu} X)$  is a face of  $\mathrm{Pri}(\mathrm{Bl}_Y^{D^\nu} X)$  (cf. the discussion in Notation 3.1). Consequently  $\varphi_\nu^{-1}(\theta_i D_i) = \theta_i \varphi_\nu^{-1}(D_i)$  is a Cartier divisor. So we proved that  $\mathrm{Bl}_Y^{D^\nu} X$  belongs to  $\mathrm{Spaces}_X^{D^\theta\text{-reg}}$ . We now use Proposition 3.16. Let  $i \in I$ , we have  $\varphi_\nu^{-1}(\theta_i D_i) \subset \varphi_\nu^{-1}(\nu_i D_i) \subset Y \cap \nu_i D_i$ . This finishes the proof.  $\square$

Assume now moreover that  $\nu, \theta \in \mathbb{N}_I \subset \mathbb{N}^I$ . We will prove that, under some assumptions,  $\varphi_{\nu, \theta}$  is a dilatation morphism with explicit descriptions. We need the following observation.

PROPOSITION 4.2. *Assume that we have a commutative diagram of  $S$ -spaces*

$$\begin{array}{ccc} B & \xrightarrow{f'} & C \\ & \searrow f & \swarrow F \\ & & X \end{array} .$$

*Assume that  $F$  is affine and  $f$  is a closed immersion. Then  $f'$  is a closed immersion.*

*Proof.* A closed immersion is affine, so by [StP, Tag 08GB],  $f'$  is affine. Using [StP, Tag 03M4], we reduce to the case where  $B, C$  and  $X$  are affine (taking an étale covering of  $X$  by affine schemes). Now the assertion is clear because closed immersions of affine schemes correspond to surjective morphisms at the level of rings.  $\square$

COROLLARY 4.3. *Assume that we have a commutative diagram of  $S$ -spaces*

$$\begin{array}{ccc} B & \xrightarrow{f'} & \mathrm{Bl}_Y^D X \\ & \searrow f & \swarrow \\ & & X \end{array}$$

*where the right-hand side morphism is the dilatation map. Assume that  $f$  is a closed immersion. Then  $f'$  is a closed immersion.*

*Proof.* Clear by Proposition 4.2.  $\square$

We now assume that  $Z_i \subset D_i$  is a Cartier divisor inclusion for all  $i \in I$ .

Let  $\mathcal{D}_i$  be the canonical diagram of closed immersions

$$\begin{array}{ccc} Y_i & \longrightarrow & \mathrm{Bl}_{Y_i}^{\nu_i D_i} X \\ \uparrow & \square & \uparrow \\ Z_i & \longrightarrow & D_i \end{array}$$

obtained by Propositions 3.16 and 4.3.

LEMMA 4.4. *Assume  $I = \{i\}$  and let  $n_i \in \mathbb{N}$ . We have an identification*

$$\mathrm{Bl}_Y^{(\nu_i + n_i) D_i} X = \mathrm{Bl}_{Y_i}^{n_i D_i} \mathrm{Bl}_{Y_i}^{\nu_i D_i} X.$$

*Proof.* Using Propositions 3.16 and 4.1 and the discussion in Notation 3.1, it is easy to prove that there is a unique  $\mathrm{Bl}_{Y_i}^{\nu_i D_i} X$ -morphism  $\mathrm{Bl}_Y^{(\nu_i + n_i) D_i} X \rightarrow \mathrm{Bl}_{Y_i}^{n_i D_i} \mathrm{Bl}_{Y_i}^{\nu_i D_i} X$ . To prove that it is an isomorphism, we reduce to the affine case and apply Corollary 2.29  $\square$

Let  $f_i$  be the canonical morphism (e.g. cf. 3.18, 3.25 or 4.1)

$$\mathrm{Bl}_Y^{D^\nu} X \rightarrow \mathrm{Bl}_{Y_i}^{\nu_i D_i} X.$$

We denote by  $Y_i \times_{\mathrm{Bl}_{Y_i}^{\nu_i D_i} X} \mathrm{Bl}_Y^{D^\nu} X$  the fiber product obtained via the arrows given by  $f_i$  and  $\mathcal{D}_i$ .

We use similarly the notation  $D_i \times_{\mathrm{Bl}_{Y_i}^{\nu_i D_i} X} \mathrm{Bl}_Y^{D^\nu} X$ .

LEMMA 4.5. *Let  $i \in I$  and let  $n_i \in \mathbb{N}$ . Let  $\gamma_i \in \mathbb{N}_I$  be  $(0, \dots, 0, n_i, 0, \dots)$  where  $n_i$  is in place  $i$ . We have an identification*

$$\mathrm{Bl}_Y^{D^{\nu+\gamma_i}} X = \mathrm{Bl}_{Y_i \times_{\mathrm{Bl}_{Y_i}^{\nu_i D_i} X} \mathrm{Bl}_Y^{D^\nu} X}^{n_i D_i \times_{\mathrm{Bl}_{Y_i}^{\nu_i D_i} X} \mathrm{Bl}_Y^{D^\nu} X} \mathrm{Bl}_Y^{D^\nu} X.$$

In particular we have a canonical dilatation morphism

$$\varphi_{\nu+\gamma_i, \nu} : \mathrm{Bl}_Y^{D^{\nu+\gamma_i}} X \rightarrow \mathrm{Bl}_Y^{D^\nu} X.$$

*Proof.* We have

$$\begin{aligned} & \mathrm{Bl}_{Y \times_{\mathrm{Bl}_{Y_i}^{\nu_i D_i} X} \mathrm{Bl}_Y^{D^\nu} X}^{n_i D_i \times_{\mathrm{Bl}_{Y_i}^{\nu_i D_i} X} \mathrm{Bl}_Y^{D^\nu} X} \mathrm{Bl}_Y^{D^\nu} X \\ \text{by Proposition 3.25} &= \mathrm{Bl}_{Y \times_{\mathrm{Bl}_{Y_i}^{\nu_i D_i} X} \mathrm{Bl}_Y^{D^\nu} X}^{n_i D_i \times_{\mathrm{Bl}_{Y_i}^{\nu_i D_i} X} \mathrm{Bl}_Y^{D^\nu} X} \mathrm{Bl}_{\{Y_j \times_X \mathrm{Bl}_{Y_i}^{\nu_i D_i} X\}_{j \in I \setminus \{i\}}}^{\{\nu_j D_j \times_X \mathrm{Bl}_{Y_i}^{\nu_i D_i} X\}_{j \in I \setminus \{i\}}} \mathrm{Bl}_{Y_i}^{\nu_i D_i} X \\ \text{by Proposition 3.25} &= \mathrm{Bl}_{Y_i, \{Y_j \times_X \mathrm{Bl}_{Y_i}^{\nu_i D_i} X\}_{j \in I \setminus \{i\}}}^{n_i D_i, \{\nu_j D_j \times_X \mathrm{Bl}_{Y_i}^{\nu_i D_i} X\}_{j \in I \setminus \{i\}}} \mathrm{Bl}_{Y_i}^{\nu_i D_i} X \\ \text{by Proposition 3.25 and Lemma 4.4} &= \mathrm{Bl}_{\{Y_j \times_X \mathrm{Bl}_{Y_i}^{(\nu_i+n_i) D_i} X\}_{j \in I \setminus \{i\}}}^{\{\nu_j D_j \times_X \mathrm{Bl}_{Y_i}^{(\nu_i+n_i) D_i} X\}_{j \in I \setminus \{i\}}} \mathrm{Bl}_{Y_i}^{(\nu_i+n_i) D_i} X \\ \text{by Proposition 3.25} &= \mathrm{Bl}_Y^{D^{\nu+\gamma_i}} X. \end{aligned}$$

□

PROPOSITION 4.6. *Recall that  $\theta \leq \nu$ . Put  $\gamma = \nu - \theta$ . Put  $K = \{i \in I \mid \gamma_i > 0\}$ . We have an identification*

$$\mathrm{Bl}_Y^{D^\nu} X = \mathrm{Bl}_{\{Y_i \times_{\mathrm{Bl}_{Y_i}^{\theta_i D_i} X} \mathrm{Bl}_Y^{D^\theta} X\}_{i \in K}}^{\{\gamma_i D_i \times_{\mathrm{Bl}_{Y_i}^{\theta_i D_i} X} \mathrm{Bl}_Y^{D^\theta} X\}_{i \in K}} \mathrm{Bl}_Y^{D^\theta} X.$$

In particular the unique  $X$ -morphism

$$\varphi_{\nu, \theta} : \mathrm{Bl}_Y^{D^\nu} X \rightarrow \mathrm{Bl}_Y^{D^\theta} X$$

of Proposition 4.1 is a dilatation map.

*Proof.* We prove the first assertion by induction on  $\#\{i \in I \mid \nu_i > 0\}$ . If  $k = 1$  the assertion follows from Lemma 4.4. The passage from  $k - 1$  to  $k$  follows from Lemma 4.5 and Proposition 3.25. □

It is now natural to introduce the following terminology.

DEFINITION 4.7. For any  $\nu \in \mathbb{N}^k$ , let us consider

$$\mathrm{Bl}_Y^{D^\nu} X = \mathrm{Bl}_{\{Y_i\}_{i \in I}}^{\{\nu_i D_i\}_{i \in I}} X$$



and call it the  $\nu$ -th iterated dilatation of  $X$  with multi-center  $\{Y_i, D_i\}_{i \in I}$ .

**PROPOSITION 4.8.** *Let  $\nu \in \mathbb{N}_I$ . Assume  $D_i = D_j =: D_0$  and  $Y_i = Y_j =: Y_0$  for all  $i, j \in K := \{i \in I \mid \nu_i > 0\}$ . Put  $m = \max_{i \in I} \nu_i$ . Then*

$$\mathrm{Bl}_Y^{D^\nu} X = \mathrm{Bl}_{Y_0}^{mD_0} X.$$

*Proof.* We prove the assertion by induction on  $k = \#K$ . If  $k = 1$  the assertion is trivial. We assume that the assertion is true for length  $k - 1$ . We fix a bijection  $K = \{1, \dots, k\}$ . We can assume  $\nu_1 \leq \nu_i$  for all  $i$ . Put  $\underline{\nu}_1 = (\nu_1, \dots, \nu_1)$ . Put  $\gamma = \nu - \underline{\nu}_1$ . Put  $K' = \{i \in K \mid \gamma_i \neq 0\}$ . Note that  $\#K' < k$ . We have

$$\begin{aligned} \mathrm{Bl}_Y^{D^\nu} X &= \\ \text{by Proposition 4.6} &= \mathrm{Bl}_{\{Y_1 \times_{\mathrm{Bl}_{Y_1}^{\nu_1 D_1} X} \mathrm{Bl}_Y^{D^{\underline{\nu}_1} X}\}_{i \in K'}} \{\gamma_i D_1 \times_{\mathrm{Bl}_{Y_1}^{\nu_1 D_1} X} \mathrm{Bl}_Y^{D^{\underline{\nu}_1} X}\}_{i \in K'} \mathrm{Bl}_Y^{D^{\underline{\nu}_1} X} \\ \text{by Proposition 3.23} &= \mathrm{Bl}_{\{Y_1\}_{i \in K'}}^{\{\gamma_i D_1\}_{i \in K'}} \mathrm{Bl}_{Y_1}^{\nu_1 D_1} X \\ &\quad \text{by induction} = \mathrm{Bl}_{Y_1}^{(m-\nu_1)D_1} \mathrm{Bl}_{Y_1}^{\nu_1 D_1} X \\ &\quad \text{by Lemma 4.4} = \mathrm{Bl}_{Y_1}^{mD_1} X. \end{aligned}$$

□

## 5. Multi-centered dilatations along multiples of a single divisor

Let  $X$  be an  $S$ -space. We fix a locally principal closed subscheme  $D \subset X$ . Let  $Y_0, Y_1, \dots, Y_i, \dots, Y_k$  be closed  $S$ -subspaces of  $X$  such that  $D \cap Y_i \subset Y_i$  is a Cartier divisor for all  $i$ . We assume moreover that  $Y_0 \subset Y_i$  for  $i \in \{1, \dots, k\}$ . Let  $s_0, s_1, \dots, s_k \in \mathbb{N}$  be integers. We claim that we have a canonical closed immersion  $Y_0 \rightarrow \mathrm{Bl}_{Y_0, \dots, Y_k}^{s_0 D, \dots, s_k D} X$ . This follows from Propositions 3.16 and 4.3 observing that the map  $Y_0 \rightarrow X$  restricted to  $s_i D$  factors through  $Y_i \cap s_i D$  for any  $i$ . We now use the notation  $\mathrm{Bl}_{Y_0, \dots, Y_k}^{s_0, \dots, s_k} X$  to denote  $\mathrm{Bl}_{Y_0, \dots, Y_k}^{s_0 D, \dots, s_k D} X$ , this is harmless because we fixed  $D$  once for all.

**FACT 5.1.** *Assume that  $X = \mathrm{Spec}(A)$  is affine. Assume that  $D = \mathrm{Spec}(A/(a))$  and  $Y_i = \mathrm{Spec}(A/M_i)$  for  $i \in I$ . Then the ideal  $L$  of  $A' := A[\frac{M_0}{a^{s_0}}, \frac{M_1}{a^{s_1}}, \dots, \frac{M_k}{a^{s_k}}]$  corresponding to the canonical closed immersion  $Y_0 \rightarrow \mathrm{Bl}_{Y_0, Y_1, \dots, Y_k}^{s_0, s_1, \dots, s_k} X$  is the ideal  $\langle \frac{M_0}{a^{s_0}}, \frac{M_1}{a^{s_1}}, \dots, \frac{M_k}{a^{s_k}} \rangle$  of  $A'$  generated by  $\frac{M_0}{a^{s_0}}, \frac{M_1}{a^{s_1}}, \dots, \frac{M_k}{a^{s_k}}$ .*

*Proof.* We have a commutative diagram of rings

$$\begin{array}{ccc} A & \xrightarrow{f} & A/M_0 \\ & \searrow & \nearrow f' \\ & & A[\frac{M_0}{a^{s_0}}, \frac{M_1}{a^{s_1}}, \dots, \frac{M_k}{a^{s_k}}] \end{array}$$

where  $f$  and  $f'$  are surjective. It is enough to show that  $\ker(f')$  equals  $\langle \frac{M_0}{a^{s_0}}, \frac{M_1}{a^{s_1}}, \dots, \frac{M_k}{a^{s_k}} \rangle$ . Since  $D \cap Y_0 \subset Y_0$  is a Cartier divisor, the element  $a$  is a non-zero-divisor in  $A/M_0$ . This implies that any power of  $a$  is a non-zero-divisor in  $A/M_0$ . Now let  $m_i \in M_i$ , the identity

$$0 = f(m_i) = f'(m_i) = f'(a^{s_i})f'(\frac{m_i}{a^{s_i}})$$

implies that  $f'(\frac{m_i}{a^{s_i}}) = 0$ . So  $\langle \frac{M_0}{a^{s_0}}, \frac{M_1}{a^{s_1}}, \dots, \frac{M_k}{a^{s_k}} \rangle$  is included in  $\ker(f')$ . Reciprocally let  $x = \frac{m}{(a^{s_0})^{\nu_0} \dots (a^{s_k})^{\nu_k}}$  be in  $A'$ , with  $m \in M^\nu$ . If  $\nu \neq (0, \dots, 0)$ , then  $x$  belongs to  $\ker(f')$ . If  $\nu = (0, \dots, 0)$ , then  $x$  belongs to  $A$  moreover  $x$  belongs to  $\ker(f')$  if and only if  $x$  belongs to  $M_0$ . Now  $M_0$  is included in  $\langle \frac{M_0}{a^{s_0}}, \frac{M_1}{a^{s_1}}, \dots, \frac{M_k}{a^{s_k}} \rangle$ . This shows that  $\ker(f') \subset \langle \frac{M_0}{a^{s_0}}, \frac{M_1}{a^{s_1}}, \dots, \frac{M_k}{a^{s_k}} \rangle$ .  $\square$

PROPOSITION 5.2. *Let  $0 \leq t \leq s_0$  be an integer. We have a canonical identification*

$$\mathrm{Bl}_{Y_0}^t \mathrm{Bl}_{Y_0, Y_1, \dots, Y_k}^{s_0, s_1, \dots, s_k} X = \mathrm{Bl}_{Y_0, Y_1, \dots, Y_k}^{s_0+t, s_1+t, \dots, s_k+t} X.$$

*Proof.* Using Propositions 3.16 and 4.1 and the discussion in Notation 3.1, we get a morphism from left to right. We then reduce to the case where  $X = \mathrm{Spec}(A)$  is affine and use the notation of Fact 5.1. It is enough to show that we have a canonical identification of rings

$$(A[\frac{M_0}{a^{s_0}}, \frac{M_1}{a^{s_1}} \dots \frac{M_k}{a^{s_k}}])[\frac{L}{a^t}] = A[\frac{M_0}{a^{s_0+t}}, \frac{M_1}{a^{s_1+t}} \dots \frac{M_k}{a^{s_k+t}}].$$

Since  $M^\nu \subset L^{\nu_0+\dots+\nu_k}$ , we can define a map  $\Psi$  sending

$$\frac{m}{(a^{s_0+t})^{\nu_0} \dots (a^{s_k+t})^{\nu_k}} \in A[\frac{M_0}{a^{s_0+t}}, \frac{M_1}{a^{s_1+t}} \dots \frac{M_k}{a^{s_k+t}}]$$

to

$$\frac{m}{(a^t)^{\nu_0+\dots+\nu_k}} \in (A[\frac{M_0}{a^{s_0}}, \frac{M_1}{a^{s_1}} \dots \frac{M_k}{a^{s_k}}])[\frac{L}{a^t}].$$

The map  $\Psi$  is a morphism of rings. It is injective and surjective. This finishes the proof.  $\square$

FACT 5.3. *Let  $A$  be a ring. Let  $P = A[T, X_1, \dots, X_n]$  be the polynomial algebra in  $n+1$  variables. Let  $d_1, \dots, d_n \in \mathbb{N}$ . Then there is a canonical identification of  $A$ -algebras*

$$P[x_1, \dots, x_n]/(X_1 - T^{d_1}x_1, \dots, X_n - T^{d_n}x_n) = P[\frac{(X_1)}{T^{d_1}}, \dots, \frac{(X_n)}{T^{d_n}}].$$

*Proof.* The map given by  $x_i \mapsto \frac{X_i}{T^{d_i}}$  is well-defined and surjective. The source and target of our map are  $T$ -torsion free and the map is an isomorphism after inverting  $T$  by Corollary 2.17.  $\square$

PROPOSITION 5.4. *Let  $A$  be a ring. Let  $a \in A$ . Let  $g_1, \dots, g_n$  be elements in  $A$  where  $n \in \mathbb{N}$ . Let  $d_1, \dots, d_n \in \mathbb{N}$ . There is a surjection*

$$A[x_1, \dots, x_n]/(g_1 - a^{d_1}x_1, \dots, g_n - a^{d_n}x_n) \rightarrow A[\frac{(g_1)}{a^{d_1}}, \dots, \frac{(g_n)}{a^{d_n}}]$$

whose kernel is the  $a$ -power torsion in the source.

*Proof.* Consider the map  $P = A[T, X_1, \dots, X_n] \rightarrow A$  sending  $T$  to  $a$  and  $X_i$  to  $g_i$  for  $1 \leq i \leq n$ . By Fact 5.3, we have  $P[x_1, \dots, x_n]/(X_1 - T^{d_1}x_1, \dots, X_n - T^{d_n}x_n) = P[\frac{(X_1)}{T^{d_1}}, \dots, \frac{(X_n)}{T^{d_n}}]$ . Now we use Proposition 2.30 to finish the proof.  $\square$

PROPOSITION 5.5. *Let  $A$  be a ring. Let  $a, g_1, \dots, g_n$  be a  $H_1$ -regular sequence in  $A$  (cf. [StP, Tag 062E] for  $H_1$ -regularity). Let  $d_1, \dots, d_n$  be positive integers. The dilatation algebra identifies with a quotient of a polynomial algebra as follows*

$$A[\frac{(g_1)}{a^{d_1}}, \dots, \frac{(g_n)}{a^{d_n}}] = A[x_1, \dots, x_n]/(g_1 - a^{d_1}x_1, \dots, g_n - a^{d_n}x_n).$$

*Proof.* We can assume that  $d_i > 0$  for all  $i$  by Fact 2.10. By Proposition 5.4, it is enough to show that the right-hand side is  $a$ -torsion free. We adapt the proof of [StP, Tag 0BIQ]. We claim that the sequence  $(a, g_1 - a^{d_1}x_1, \dots, g_n - a^{d_n}x_n)$  is  $H_1$ -regular in  $A[x_1, \dots, x_n]$ . Namely, the map

$$(a, g_1 - a^{d_1}x_1, \dots, g_n - a^{d_n}x_n) : A[x_1, \dots, x_n]^{\oplus(1+n)} \rightarrow A[x_1, \dots, x_n]$$

used to define the Koszul complex on  $a, g_1 - a^{d_1}x_1, \dots, g_n - a^{d_n}x_n$  is isomorphic to the map

$$(a, g_1, \dots, g_n) : A[x_1, \dots, x_n]^{\oplus(1+n)} \rightarrow A[x_1, \dots, x_n]$$

used to define the Koszul complex on  $a, g_1, \dots, g_n$  via the isomorphism  $\Theta$

$$A[x_1, \dots, x_n]^{\oplus(1+n)} \rightarrow A[x_1, \dots, x_n]^{\oplus(1+n)}$$

sending  $(P_0, P_1, \dots, P_n)$  to

$$(P_0 - \sum_{i=1}^n a^{n_i-1}x_i P_i, P_1, P_2, \dots, P_n);$$

this follows from the identity

$$aP_0 + \sum_{i=1}^n (g_i - a^{d_i}x_i)P_i = a(P_0 - \sum_{i=1}^n a^{d_i-1}x_i P_i) + \sum_{i=1}^n g_i P_i.$$

By [StP, Tag 0624] these Koszul complexes are isomorphic. By [StP, Tag 0629] the Koszul complex  $K$  on  $(a, g_1 - a^{d_1}x_1, \dots, g_n - a^{d_n}x_n)$  is the cone on  $a : L \rightarrow L$  where  $L$  is the Koszul complex on  $(g_1 - a^{d_1}x_1, \dots, g_n - a^{d_n}x_n)$ , since  $H_1(K) = 0$ , we conclude that  $a : H_0(L) \rightarrow H_0(L)$  is injective, so the right-hand side is  $a$ -torsion free.  $\square$

## 6. Some flatness and smoothness results

Let  $S$  be a scheme and let  $C \subset S$  be a Cartier divisor in  $S$ . Let  $X$  be a scheme over  $S$ . Let  $D$  be the closed subscheme of  $X$  given by  $X \times_S C$ . Let  $X_j \subset D$  be closed subschemes for  $1 \leq j \leq d$ . We are now making the following assumption. We assume that locally over  $S$ ,  $X$  the following conditions are satisfied

- (i)  $S = \text{Spec}(R)$ ,  $C = \text{Spec}(R/a)$  and  $X = \text{Spec}(A)$ ,
- (ii) there exists a sequence  $g_1, \dots, g_n \in A$  such that  $a, g_1, \dots, g_n$  is a  $H_1$ -regular sequence in  $A$ ,
- (iii) there exists a sequence  $1 \leq i_1 < i_2 < \dots < i_j < \dots < i_d = n$  such that

$$\begin{aligned} X_1 &= \text{Spec}(A/(g_1, \dots, g_{i_1})) \\ X_2 &= \text{Spec}(A/(\overset{\pm}{g}_1, \dots, \overset{\pm}{g}_{i_1}, g_{i_1+1}, \dots, g_{i_2})) \\ &\quad \vdots \\ X_j &= \text{Spec}(A/(\overset{\pm}{g}_1, \dots, \overset{\pm}{g}_{i_{j-1}}, g_{i_{j-1}+1}, \dots, g_{i_j})) \\ &\quad \vdots \\ X_d &= \text{Spec}(A/(\overset{\pm}{g}_1, \dots, \overset{\pm}{g}_{i_{d-1}}, g_{i_{d-1}+1}, \dots, g_n)) \end{aligned}$$

where  $\pm$  over a symbol means that this symbol possibly appears but not necessarily. We put  $U_j = \text{Spec}(A/(g_1, \dots, g_{i_j}))$  for  $1 \leq i \leq j$ .

Let  $m_1 \geq \dots \geq m_j \geq \dots \geq m_d \geq 0 \in \mathbb{N}$  be integers.

PROPOSITION 6.1. (i) *If  $X/S$  is flat and if moreover one of the following holds:*

- (a)  $X_j \rightarrow S$  is flat and  $S, X$  are locally noetherian for  $1 \leq j \leq d$ ,
- (b)  $X_j \rightarrow S$  is flat and  $X_j \rightarrow S$  is locally of finite presentation,
- (c) the local rings of  $S$  are valuation rings,

then  $\mathrm{Bl}_{X_1, \dots, X_d}^{m_1, \dots, m_d} X \rightarrow S$  is flat.

- (ii) If  $X \rightarrow S$  is smooth and (with the local notation of the assumption)  $U_j \times_S \mathrm{Spec}(R/(a^{m_j})) \rightarrow \mathrm{Spec}(R/(a^{m_j}))$  is smooth for all  $1 \leq j \leq d$ , then  $\mathrm{Bl}_{X_1, \dots, X_d}^{m_1, \dots, m_d} X \rightarrow S$  is smooth.

*Proof.* This is local on  $S, X$ . We use notations used to state the assumption before the statement. Proposition 2.19 implies that we can and do assume that  $X_j = U_j$  for all  $1 \leq j \leq d$ .

- (i) We prove the assertions by induction on  $d$ . If  $d = 1$  this follows from [MRR20, Proposition 2.16]. By Proposition 3.25 we have  $\mathrm{Bl}_{X_1, \dots, X_d}^{m_1, \dots, m_d} X = \mathrm{Bl}_{X'_d}^{m_d} \mathrm{Bl}_{X_1, \dots, X_{d-1}}^{m_1, \dots, m_{d-1}} X$  where  $X'_d = \mathrm{Bl}_{X_1, \dots, X_{d-1}}^{m_1, \dots, m_{d-1}} X \times_X X_d$ . Proposition 5.5 implies that  $\mathrm{Bl}_{X_1, \dots, X_{d-1}}^{m_1, \dots, m_{d-1}} X$  identifies with the spectrum of  $P/\mathfrak{J}$  where  $P = A[x_1, \dots, x_{i_{d-1}}]$  and

$$\mathfrak{J} = (g_1 - a^{m_1} x_1, \dots, g_{i_1} - a^{m_1} x_{i_1}, g_{i_1+1} - a^{m_2} x_{i_1+1}, \dots, g_{i_2} - a^{m_2} x_{i_2}, \dots, g_{i_{d-1}} - a^{m_{d-1}} x_{i_{d-1}}).$$

We claim that the sequence given by  $g_{i_{d-1}+1}, \dots, g_{i_d}$  is  $H_1$ -regular in  $P/(\mathfrak{J} + (a^{m_d}))$ . Let us prove the claim. Since  $m_d \leq m_j$  for all  $1 \leq j \leq d-1$ , the ideal  $\mathfrak{J} + (a^{m_d})$  of  $A$  is equal to  $(a^{m_d}, g_1, \dots, g_{i_{d-1}})$ . So  $P/(\mathfrak{J} + (a^{m_d}))$  identifies with  $(A/(a^{m_d}, g_1, \dots, g_{i_{d-1}}))[x_1, \dots, x_{i_{d-1}}]$ . Now since  $(a, g_1, \dots, g_n)$  is  $H_1$ -regular in  $A$ , we know that  $(a^{m_d}, g_1, \dots, g_n)$  is  $H_1$ -regular in  $A$  by [StP, Tag 062G]. So  $g_{i_{d-1}+1}, \dots, g_{i_d}$  is a  $H_1$ -regular sequence in  $A/(a^{m_d}, g_1, \dots, g_{i_{d-1}})$  by [StP, Tag 068L]. This implies that  $g_{i_{d-1}+1}, \dots, g_{i_d}$  is a  $H_1$ -regular sequence in  $(A/(a^{m_d}, g_1, \dots, g_{i_{d-1}}))[x_1, \dots, x_{i_{d-1}}]$ . This finishes to prove our claim. We now apply [MRR20, Proposition 2.16] to deduce the assertions for  $d$ .

- (ii) We prove the assertion by induction on  $d$ . If  $d = 1$ , this is [MRR20, Proposition 2.16]. We now assume that the assertion is true for  $d-1$ . Using [MRR20, Proposition 2.16] and Proposition 3.25, it is enough to show that  $Z'_d := (X_d \times_X \mathrm{Bl}_{X_1, \dots, X_d}^{m_1, \dots, m_d} X) \times_S (\mathrm{Spec}(R/(a^{m_d})))$  is smooth over  $\mathrm{Spec}(R/(a^{m_d}))$ . We have

$$Z'_d = X_d \times_S \mathrm{Spec}(R/(a^{m_d})) \times_{\mathrm{Spec}(R/(a^{m_d}))} \mathrm{Bl}_{X_1, \dots, X_d}^{m_1, \dots, m_d} X \times_S \mathrm{Spec}(R/(a^{m_d})).$$

We computed  $\mathrm{Bl}_{X_1, \dots, X_d}^{m_1, \dots, m_d} X \times_S \mathrm{Spec}(R/(a^{m_d}))$  in (i) and proved that it is the spectrum of  $A/(a^{m_d}, g_1, \dots, g_{i_{d-1}})[x_1, \dots, x_{i_{d-1}}]$ . Now  $X_d \times_S \mathrm{Spec}(R/(a^{m_d}))$  is the spectrum of  $A/(a^{d_m}, g_1, \dots, g_n)$ . Consequently  $Z'_d$  is the spectrum of  $(A/(a^{d_m}, g_1, \dots, g_n))[x_1, \dots, x_{i_{d-1}}]$ . So by assumption  $A/(a^{d_m}, g_1, \dots, g_n)$  is smooth over  $R/(a^{d_m})$ . Moreover any polynomial algebra is smooth over its base ring. A composition of smooth morphisms is smooth. So the composition

$$R/(a^{d_m}) \rightarrow A/(a^{d_m}, g_1, \dots, g_n) \rightarrow (A/(a^{d_m}, g_1, \dots, g_n))[x_1, \dots, x_{i_{d-1}}]$$

is smooth. This finishes the proof.  $\square$

## 7. Néron blowups

Néron blowups are dilatations in the group case. We introduce and study them in this section. Let  $S$  be a scheme, and let  $G \rightarrow S$  be a group scheme. Let  $C_1, \dots, C_k \subset S$  be locally principal closed subschemes. Put  $D_i = G|_{C_i} = G \times_S C_i$  and  $D = (D_1, \dots, D_k)$ . Let  $H_i \subset D_i$  be a closed subgroup scheme over  $C_i$  for all  $i \in \{1, \dots, k\}$  and let  $H = (H_1, \dots, H_k)$ . Let  $\mathcal{G} := \mathrm{Bl}_H^D G \rightarrow G$  be the associated dilatation. We call  $\mathcal{G} \rightarrow S$  the Néron blowup of  $G$  with multi-center  $H, D$ . By Lemma 3.15 the structure morphism  $\mathcal{G} \rightarrow S$  defines an object in  $\mathrm{Sch}_S^{C\text{-reg}}$ .

FACT 7.1. *Let  $\mathcal{G} \rightarrow S$  be the above Néron blowup.*

(1) The scheme  $\mathcal{G} \rightarrow S$  represents the contravariant functor  $\text{Sch}_S^{C\text{-reg}} \rightarrow \text{Sets}$  given for  $T \rightarrow S$  by the set of all  $S$ -morphisms  $T \rightarrow G$  such that the induced morphism  $T|_{C_i} \rightarrow G|_{C_i}$  factors through  $H_i \subset G|_{C_i}$  for all  $i \in \{1, \dots, k\}$ .

(2) Let  $T \rightarrow S$  be an object in  $\text{Sch}_S^{C\text{-reg}}$ , then as subsets of  $G(T)$

$$\mathcal{G}(T) = \bigcap_{i \in \{1, \dots, k\}} (\text{Bl}_{H_i}^{G_i} G)(T).$$

(3) The map  $\mathcal{G} \rightarrow G$  is affine. Its restriction over  $S_i$  factors as  $\mathcal{G}_i \rightarrow H_i \subset D_i$  for all  $i$

*Proof.* Part (1) is a reformulation of Proposition 3.16. Assertion (2) is immediate using (1). Finally (3) is immediate from Proposition 3.15.  $\square$

By virtue of Fact 7.1 the (forgetful) map  $\mathcal{G} \rightarrow G$  defines a subgroup functor when restricted to the category  $\text{Sch}_S^{C\text{-reg}}$ . As  $\mathcal{G} \rightarrow S$  is an object in  $\text{Sch}_S^{C\text{-reg}}$ , it is a group object in this category. Now if  $X \rightarrow S$  and  $Y \rightarrow S$  are two flat morphisms in  $\text{Sch}_S^{C\text{-reg}}$ , then the product of  $X$  and  $Y$  in the category  $\text{Sch}_S^{C\text{-reg}}$  exists and is equal to the product of  $X$  and  $Y$  in the full category of  $S$ -schemes. In particular if the Néron blowup  $\mathcal{G} \rightarrow S$  is flat, then it is equipped with the structure of a group scheme such that  $\mathcal{G} \rightarrow G$  is a morphism of  $S$ -group schemes.

REMARK 7.2. In general the product of two objects in  $\text{Sch}_S^{C\text{-reg}}$  is not equal to the product of the same objects seen in the full category of  $S$ -schemes (in fact the product of these objects in the full category of  $S$ -schemes need not be in  $\text{Sch}_S^{C\text{-reg}}$ ). Proposition 3.35 allows to compute products in  $\text{Sch}_S^{C\text{-reg}}$ .

The following Proposition generalizes the fact that congruence groups are normal subgroups, it is related to Example 1.4 (note that the proof of [Yu01, Lemma 1.4] is not correct).

PROPOSITION 7.3. Assume that  $k = 1$ ,  $C = C_1$  is a Cartier divisor in  $S$  and  $G \rightarrow S$  is flat. Let  $\eta : H \rightarrow G$  be a morphism of group schemes over  $S$  such that  $H \rightarrow S$  is flat. Let  $K \subset G$  be a closed subgroup scheme over  $S$  such that  $K \rightarrow S$  is flat and  $\text{Bl}_K^C G \rightarrow S$  is flat (and in particular a group scheme). Assume that  $H_C$  commutes with  $K_C$  in the sense that the morphism  $H_C \times_C K_C \rightarrow G_C$ ,  $(h, k) \mapsto \eta(h)k\eta(h)^{-1}$  equals the composition morphism  $H_C \times_C K_C \rightarrow K_C \subset G_C$ ,  $(h, k) \mapsto k$ . Then  $H$  normalizes  $\text{Bl}_K^C G$ , more precisely the solid composition map

$$\begin{array}{ccc} H \times_S \text{Bl}_K^C G & \xrightarrow{\text{Id} \times \text{Bl}} & H \times_S G & \xrightarrow{h, g \mapsto \eta(h)g\eta(h)^{-1}} & G \\ & \searrow \text{dashed} & & \searrow \text{Bl} & \\ & & \text{Bl}_K^C G & & \end{array}$$

factors uniquely through  $\text{Bl}_K^C G$ .

*Proof.* Let  $\phi$  be the solid composition map, we claim that it belongs to  $\text{Sch}_G^{G\text{-reg}}$ . Let us prove this claim. The map  $\theta : H \times_S G \xrightarrow{h, g \mapsto \eta(h)g\eta(h)^{-1}} G$  is flat. Indeed it is the composition of an isomorphism, namely  $H \times_S G \xrightarrow{\text{Id} \times \theta} H \times_S G$ , with a flat morphism, namely the projection on the second factor  $H \times_S G \rightarrow G$ . So  $\theta^{-1}(G_C)$  is a Cartier divisor in  $H \times_S G$  (note that  $G_C$  is a Cartier divisor in  $G$  because  $G \rightarrow S$  is flat). Now Lemma 3.30 and 3.17 shows that  $(\text{Id} \times \text{Bl})^{-1}(\theta^{-1}(G_C))$  is a Cartier divisor. This finishes to prove the claim. Now by Proposition 3.16,  $\phi$  factors uniquely through  $\text{Bl}_K^C G$  if and only if  $\phi|_{G_C}$  factors through  $K_C$ . The following diagram, obtained using

Proposition 3.15, finishes the proof

$$\begin{array}{ccccc}
(H \times_S \mathrm{Bl}_K^C G) \times_G G_C & \longrightarrow & (H \times_S G) \times_G G_C & \longrightarrow & G_C \\
\parallel & & \parallel & & \parallel \\
H_C \times_C (\mathrm{Bl}_K^C G \times_G G_C) & \longrightarrow & H_C \times_C G_C & \longrightarrow & G_C \\
& \searrow \text{dashed} & \nearrow & & \nearrow \\
& H_C \times_C K_C & \dashrightarrow & K_C & 
\end{array}$$

□

## 8. Congruent isomorphisms

Let  $(\mathcal{O}, \pi)$  be an henselian pair where  $\pi \subset \mathcal{O}$  is an invertible ideal. Let  $S = \mathrm{Spec}(\mathcal{O})$  and let  $C = \mathrm{Spec}(\mathcal{O}/\pi)$ .

**THEOREM 8.1.** *Let  $G$  be a separated and smooth group scheme over  $S$ . Let  $H_0, H_1, \dots, H_k$  be closed subgroup schemes of  $G$  such that  $H_0 = e_G$  is the trivial subgroup. Let  $s_0, s_1, \dots, s_k$  and  $r_0, r_1, \dots, r_k$  be in  $\mathbb{N}$  such that*

- (i)  $s_i \geq s_0$  and  $r_i \geq r_0$  for all  $i \in \{0, \dots, k\}$
- (ii)  $r_i \geq s_i$  and  $r_i - s_i \leq s_0$  for all  $i \in \{0, \dots, k\}$ .

Assume that  $G$  is affine or  $\mathcal{O}$  is local. Assume that the regularity condition (RC) introduced below is satisfied (cf. Definition 8.2). Then we have a canonical isomorphism

$$\mathrm{Bl}_{H_0, H_1, \dots, H_k}^{s_0, s_1, \dots, s_k} G / \mathrm{Bl}_{H_0, H_1, \dots, H_k}^{r_0, r_1, \dots, r_k} G \cong \mathrm{Lie}(\mathrm{Bl}_{H_0, H_1, \dots, H_k}^{s_0, s_1, \dots, s_k} G) / \mathrm{Lie}(\mathrm{Bl}_{H_0, H_1, \dots, H_k}^{r_0, r_1, \dots, r_k} G).$$

*Proof.* For  $i \in \{0, \dots, k\}$ , put  $t_i = r_i - s_i$ . As a first step in our proof, we assume that  $t_i = t_j =: t$  for all  $i, j \in \{0, \dots, k\}$ . Proposition 5.2 shows that

$$\mathrm{Bl}_{H_0, H_1, \dots, H_k}^{r_0, r_1, \dots, r_k} G = \mathrm{Bl}_{H_0}^{2t} \mathrm{Bl}_{H_0, H_1, \dots, H_k}^{s_0-t, \dots, s_k-t} G \text{ and } \mathrm{Bl}_{H_0, H_1, \dots, H_k}^{s_0, s_1, \dots, s_k} G = \mathrm{Bl}_{H_0}^t \mathrm{Bl}_{H_0, H_1, \dots, H_k}^{s_0-t, \dots, s_k-t} G.$$

Put  $G' = \mathrm{Bl}_{H_0, H_1, \dots, H_k}^{s_0-t, \dots, s_k-t} G$ . The scheme  $G'$  is smooth over  $S$  by (RC). By [MRR20, Theorem 4.3], we have a canonical isomorphism

$$\mathrm{Bl}_{H_0}^t G' / \mathrm{Bl}_{H_0}^{2t} G' \cong \mathrm{Lie}(\mathrm{Bl}_{H_0}^t G') / \mathrm{Lie}(\mathrm{Bl}_{H_0}^{2t} G').$$

This finishes the proof of the case where  $t_i = t_j$  for all  $i, j \in \{1, \dots, k\}$ . Now we prove the general case. Put  $t_m = \max_{i \in \{0, \dots, k\}} t_i$ . The isomorphism [MRR20, Theorem 4.3] is functorial in  $G$ . Applying this functoriality to the morphism  $\mathrm{Bl}_{\{H_i\}_{0 \leq i \leq k}}^{s_i-t_m} G \rightarrow \mathrm{Bl}_{\{H_i\}_{0 \leq i \leq k}}^{r_i-2t_m} G$  and with the

integers given by the inequality  $0 \leq \frac{2t_m}{2} \leq t_m \leq 2t_m$ , we get a canonical commutative diagram

$$\begin{array}{ccc}
& \text{Bl}_{\{H_i\}_{0 \leq i \leq k}}^{\{s_i\}_{0 \leq i \leq k}} G(\mathcal{O}) / \text{Bl}_{\{H_i\}_{0 \leq i \leq k}}^{\{r_i\}_{0 \leq i \leq k}} G(\mathcal{O}) & \\
& \nearrow & \searrow \\
\text{Bl}_{\{H_i\}_{0 \leq i \leq k}}^{\{s_i\}_{0 \leq i \leq k}} G(\mathcal{O}) / \text{Bl}_{\{H_i\}_{0 \leq i \leq k}}^{\{s_i+t_m\}_{0 \leq i \leq k}} G(\mathcal{O}) & \longrightarrow & \text{Bl}_{\{H_i\}_{0 \leq i \leq k}}^{\{r_i-t_m\}_{0 \leq i \leq k}} G(\mathcal{O}) / \text{Bl}_{\{H_i\}_{0 \leq i \leq k}}^{\{r_i\}_{0 \leq i \leq k}} G(\mathcal{O}) \\
\parallel & & \parallel \\
\text{Lie}(\text{Bl}_{\{H_i\}_{0 \leq i \leq k}}^{\{s_i\}_{0 \leq i \leq k}} G)(\mathcal{O}) / \text{Lie}(\text{Bl}_{\{H_i\}_{0 \leq i \leq k}}^{\{s_i+t_m\}_{0 \leq i \leq k}} G)(\mathcal{O}) & \rightarrow & \text{Lie}(\text{Bl}_{\{H_i\}_{0 \leq i \leq k}}^{\{r_i-t_m\}_{0 \leq i \leq k}} G)(\mathcal{O}) / \text{Lie}(\text{Bl}_{\{H_i\}_{0 \leq i \leq k}}^{\{r_i\}_{0 \leq i \leq k}} G)(\mathcal{O}) \\
& \searrow & \nearrow \\
& \text{Lie}(\text{Bl}_{\{H_i\}_{0 \leq i \leq k}}^{\{s_i\}_{0 \leq i \leq k}} G)(\mathcal{O}) / \text{Lie}(\text{Bl}_{\{H_i\}_{0 \leq i \leq k}}^{\{r_i\}_{0 \leq i \leq k}} G)(\mathcal{O}). & 
\end{array}$$

The injectivity of the two hookarrows follows from Remark 1.3. This identifies

$$\text{Bl}_{\{H_i\}_{0 \leq i \leq k}}^{\{s_i\}_{0 \leq i \leq k}} G(\mathcal{O}) / \text{Bl}_{\{H_i\}_{0 \leq i \leq k}}^{\{r_i\}_{0 \leq i \leq k}} G(\mathcal{O})$$

and

$$\text{Lie}(\text{Bl}_{\{H_i\}_{0 \leq i \leq k}}^{\{s_i\}_{0 \leq i \leq k}} G)(\mathcal{O}) / \text{Lie}(\text{Bl}_{\{H_i\}_{0 \leq i \leq k}}^{\{r_i\}_{0 \leq i \leq k}} G)(\mathcal{O})$$

inside the right part of the diagram.  $\square$

DEFINITION 8.2. Let  $G, \{H_i, s_i, r_i\}_{0 \leq i \leq k}$  be as in Theorem 8.1. Put  $t_m = \max_{i \in \{0, \dots, k\}} (r_i - s_i)$ . We introduce the following regularity condition

$$(RC) \text{Bl}_{\{H_i\}_{0 \leq i \leq k}}^{\{s_i-t_m\}_{0 \leq i \leq k}} G \text{ and } \text{Bl}_{\{H_i\}_{0 \leq i \leq k}}^{\{r_i-2t_m\}_{0 \leq i \leq k}} G \text{ are smooth over } S.$$

We recall that Proposition 6.1 offers a way to check (RC) in many cases. We finish with the following result.

COROLLARY 8.3. *Let  $G$  be a separated and smooth group scheme over  $S$ . Let  $H_0 \subset H_1 \subset \dots \subset H_k$  be closed subgroup schemes of  $G$  such that  $H_i$  is smooth over  $S$  for  $0 \leq i \leq d$  and  $H_0 = e_G$ . Let  $s_0, s_1, \dots, s_k$  and  $r_0, r_1, \dots, r_k$  be in  $\mathbb{N}$  such that*

- (i)  $s_i \geq s_0$  and  $r_i \geq r_0$  for all  $i \in \{0, \dots, k\}$
- (ii)  $r_i \geq s_i$  and  $r_i - s_i \leq s_0$  for all  $i \in \{0, \dots, k\}$ .

Assume that  $G$  is affine or  $\mathcal{O}$  is local. Then we have a canonical isomorphism

$$\text{Bl}_{H_0, H_1, \dots, H_k}^{s_0, s_1, \dots, s_k} G / \text{Bl}_{H_0, H_1, \dots, H_k}^{r_0, r_1, \dots, r_k} G \cong \text{Lie}(\text{Bl}_{H_0, H_1, \dots, H_k}^{s_0, s_1, \dots, s_k} G) / \text{Lie}(\text{Bl}_{H_0, H_1, \dots, H_k}^{r_0, r_1, \dots, r_k} G).$$

*Proof.* By Theorem 8.1, it is enough to check the condition (RC), it follows from [SGA3, Exp. III, Proposition 4.15] and Proposition 6.1.  $\square$

REMARK 8.4. Note that the result of the proof of [SGA3, Exp. III, Proposition 4.15] is stronger than its statement. Indeed the statement uses Koszul-regularity and the proof shows regularity (in the terminology of [StP, Tag 063J]).

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