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Julien Trevisan

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Limit laws in the lattice problem. IV. The special case of $\mathbb{Z}^d$

Julien Trevisan

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Résumé

Nous étudions l'erreur du nombre de points du réseau $\mathbb{Z}^d$ qui tombent dans un hypercube centré autour de 0 dilaté et translaté et dont les axes sont parallèles aux axes de coordonnées. Nous montrons que si t, le facteur de dilatation, est distribué selon la mesure de probabilité $\frac{1}{T}\rho(\frac{t}{T})dt$ avec ρ une densité de probabilité sur $[0,1]$, l'erreur, normalisée par t^{d-1}, converge en loi lorsque $T \to \infty$ dans le cas où la translation est de la forme $X = (x, \cdots, x)$ et dans le cas où les coordonnées de X sont indépendantes entre elles, indépendantes de t et distribuées selon la loi uniforme sur $[-\frac{1}{2}, \frac{1}{2}]$. Dans les deux cas, on calcule par ailleurs la fonction caractéristique de la loi limite.

Abstract

We study the error of the number of points of the lattice $\mathbb{Z}^d$ that fall into a dilated and translated hypercube centred around 0 and whose axis are parallel to the axis of coordinates. We show that if t, the factor of dilatation, is distributed according to the probability measure $\frac{1}{T}\rho(\frac{t}{T})dt$ with ρ being a probability density over $[0,1]$ the error, when normalized by t^{d-1}, converges in law when $T \to \infty$ in the case where the translation is of the form $X = (x, \cdots, x)$ and in the case where the coordinates of X are independent between them, independent from t and distributed according to the uniform law over $[-\frac{1}{2}, \frac{1}{2}]$. In both cases, we compute the characteristic function of the limit law.

1 Introduction

Let P be a measurable subset of $\mathbb{R}^d$ of non-zero finite Lebesgue measure. We want to evaluate the following cardinal number when $t \to \infty$:

$$N(tP + X, L) = |(tP + X) \cap L|$$

where $X \in \mathbb{R}^d$, L is a lattice of $\mathbb{R}^d$ and $tP+X$ denotes the set P dilated by a factor t relatively to 0 and then translated by the vector X.
Under mild regularity conditions on the set P, one can show that :

$$N(tP + X, L) = t^d \frac{\mathrm{Vol}(P)}{\mathrm{Covol}(L)} + o(t^d)$$

where $o(f(t))$ denotes a quantity such that, when divided by $f(t)$, it goes to 0 when $t \to \infty$ and where $\mathrm{Covol}(L)$ is the volume of a fundamental set of the lattice L.
We are interested in the error term

$$\mathcal{R}(tP + X, L) = N(tP + X, L) - t^d \frac{\mathrm{Vol}(P)}{\mathrm{Covol}(L)}.$$

In the case where $d = 2$ and where P is the unit disk $\mathbb{D}^2$, Hardy's conjecture in [3] stipulates that we should have for all $\epsilon > 0$,

$$\mathcal{R}(t\mathbb{D}^2, \mathbb{Z}^2) = O(t^{\frac{1}{2}+\epsilon})$$

where $Y = O(X)$ means that there exists $D > 0$ such that $|Y| \leqslant D|X|$.
One of the result in this direction has been established by Iwaniec and Mozzochi in [6]. They have proven that for all $\epsilon > 0$,

$$\mathcal{R}(t\mathbb{D}^2, \mathbb{Z}^2) = O(t^{\frac{7}{11}+\epsilon}).$$

This result has been recently improved by Huxley in [5]. Indeed, he has proven that :

$$\mathcal{R}(t\mathbb{D}^2, \mathbb{Z}^2) = O(t^K \log(t)^\Lambda)$$

where $K = \frac{131}{208}$ and $\Lambda = \frac{18627}{8320}$.
In dimension 3, Heath-Brown has proven in [4] that :

$$\mathcal{R}(t\mathbb{D}^3, \mathbb{Z}^3) = O(t^{\frac{21}{16}+\epsilon}).$$

These last two results are all based on estimating what are called *exponential sums*. Furthermore, they only tackle "deterministic" cases.
Another result about this problem has been established by Bleher, Cheng, Dyson and Lebowitz in [2]. Let ρ be a probability density on $[0, 1]$. They took interest in what is happening when the factor of dilatation t is distributed according to the probability measure $\frac{1}{T}\rho(\frac{t}{T})dt$ and when $T \to \infty$. Their result states as following :

Theorem ([2])**.** *There exists a probability density p on $\mathbb{R}$ such that for every piecewise continuous and bounded function $g : \mathbb{R} \longrightarrow \mathbb{R}$,*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T g(\frac{\mathcal{R}(t\mathbb{D}^2, \mathbb{Z}^2)}{\sqrt{t}})\rho(\frac{t}{T})dt = \int_{\mathbb{R}} g(x)p(x)dx.$$

Furthermore p can be extended as an analytic function over $\mathbb{C}$ and verifies that for every $\epsilon > 0$,

$$p(x) = O(e^{-|x|^{4-\epsilon}})$$

when $x \in \mathbb{R}$ and when $|x| \to \infty$.

We want to follow this approach on another problem. Namely, let us give $a > 0$ and let's define the following set

$$\mathcal{C}(a) = \{x = (x_1, \cdots, x_d) \in \mathbb{R}^d \ |\forall i \in [1, d],\ |x_i| \leqslant a\}. \tag{1}$$

In that case, with ρ being a probability density over $[0, 1]$, we want to study the possible convergence in distribution of the quantity $\frac{\mathcal{R}(t\mathcal{C}(a)+X,\mathbb{Z}^d}{t^{d-1}}$. We already proved such a result when the dimension d was equal to 2. Here, we are going to prove the two following theorems (that constitute a generalization of the previous result) :

Theorem 1. *For all $x \in \mathbb{R}$, when $t \in [0,T]$ is distributed according to the probability density $\frac{1}{T}\rho(\frac{\cdot}{T})$ on $[0,T]$ then, when $T \to \infty$, $\frac{\mathcal{R}(t\mathcal{C}(a)+X,\mathbb{Z}^d)}{t^{d-1}}$ converges in law with $X = (x, \cdots, x)$. Furthermore, the limit law has the following characteristic function*

$$\varphi(u) = \frac{\sin(d2^{d-1}uy) + \sin(d2^{d-1}u(1-y))}{d2^{d-1}u}$$

with $y = |t_{2,0} - t_{1,0}|$ where $t_{2,0}$ is the first $t \geqslant 0$ such that $-t + x \in \mathbb{Z}$ and $t_{1,0}$ is the first $t \geqslant 0$ such that $t + x \in \mathbb{Z}$. In fact, $y = |1 - 2\{x\}|$ où $\{x\}$ désigne la partie fractionnaire de x.

Theorem 2. *Let's assume that $x_1, \cdots, x_d$ are independent random variables distributed according to the uniform distribution over $[-\frac{1}{2}, \frac{1}{2}]$. Let's assume also that $t \in [0,T]$ is distributed according to the probability density $\frac{1}{T}\rho(\frac{\cdot}{T})$ on $[0,T]$. Let's suppose that t and $x_1, \cdots, x_d$ are independent betweem them then, when $T \to \infty$, $\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}$ converges in distribution with $X = (x_1, \cdots, x_d)$.*
Furthermore, the limit law has the following characteristic function :

$$\varphi(u) = (2\frac{1-\cos(2^{d-1}u)}{(2^{d-1}u)^2})^d.$$

In particular, we see that the normalization in these two cases of the error $\mathcal{R}$ is of order t^{d-1}. Furthermore, the two cases studied here are two extreme cases : the case of Theorem 1 is a case where all the x_i are linked (in fact, they are all equal) whereas the case of Theorem 2 is a case where all the x_i are independent between them.
Before beginning, let's observe that it is enough to prove Theorem 1 and Theorem 2 in the case where $\rho = \mathbf{1}_{[0,1]}$ (see, for example, the proof of Theorem 4.2 in [1]). So, in the rest of the article, we are going to suppose that $\rho = \mathbf{1}_{[0,1]}$.
Furthermore, regarding Theorem 1, we can suppose also that $a = 1$. Indeed, instead of considering t, we can consider the new variable $\tilde{t} = at$ that will be distributed according to the probability measure $\frac{1}{aT}\rho(\frac{t}{aT})dt$. In the next section we are going to give a bit of heuristic about Theorem 1 and Theorem 2.

2 A bit of heuristic and plan of the paper

First, let's say that the normalization of $\mathcal{R}(t\mathcal{C}(1) + X, \mathbb{Z}^d)$ by t^{d-1} is quite natural. Indeed, to within a multiplicative factor, it corresponds to the surface measure of $\partial(t\mathcal{C}(1) + X)$.
This normalization appears when looking at the following expression of $\mathcal{R}(t\mathcal{C}(1) + X, \mathbb{Z}^d)$:

$$\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}} = \frac{\sum_{i=1}^d (2t)^{i-1}(\lfloor t+x_i \rfloor - \lceil -t+x_i \rceil + 1 - 2t)\prod_{j=i+1}^d(\lfloor t+x_j \rfloor - \lceil -t+x_j \rceil + 1)}{t^{d-1}}$$

with $X = (x_1, \cdots, x_d)$ (see Proposition 2 and Equation 7).

Plan of the paper. After having proved this expression of $\mathcal{R}(t\mathcal{C}(1) + X, \mathbb{Z}^d)$, we show in section 3 that the study of $\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}$ can be reduced to the study of a simpler quantity which is $\Delta(t, X)$ (see Proposition 1 for the definition of $\Delta(t, X)$).
Then, in section 4, we give the proof of Theorem 1. In fact, the case of Theorem 1 corresponds to a case where the expression $\Delta(t, X)$ (and the expression of $\mathcal{R}(t\mathcal{C}(1) + X, \mathbb{Z}^d)$) is simpler. This simple expression is used to compute the characteristic function of $\Delta(t, X)$ (see Proposition 4). We conclude by using Levy's continuity theorem (see Theorem 3).
In section 5, we also compute the characteristic function of $\Delta(t, X)$ in the case of Theorem

2 (see Proposition 6). In fact, the key of Theorem 2 is the independence of the variables $t, x_1, \cdots, x_d$ which enables us to make this computation. Other computations, with other distributions for the variables $x_1, \cdots, x_d$, but always with the independence theorem, could be made. We, again, conclude by using Levy's continuity theorem.

The next section is dedicated to reduce $\mathcal{R}(t\mathcal{C}(1) + X, \mathbb{Z}^d)$ when we study its asymptotical behaviour.

3 Simplification of the study of $\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}$

The main object of this section is to prove the following proposition :

Proposition 1. *One has that :*

$$\frac{\mathcal{R}(t\mathcal{C}(1) + X, \mathbb{Z}^d)}{t^{d-1}} - \Delta(t, X) \underset{t\to\infty}{\to} 0 \tag{2}$$

where $\Delta(t, X)$ is defined by

$$\Delta(t, X) = 2^{d-1} \sum_{i=1}^{d} (\lfloor t + x_i \rfloor - \lceil -t + x_i \rceil + 1 - 2t) \tag{3}$$

with $X = (x_1, \cdots, x_d)$ and the convergence in Equation (2) is uniform in $X \in \mathbb{R}^d$.

It is a proposition that enables to do some reduction about the asymptotical study of $\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}$. The main idea is that, in this case, everything can be computed quite easily, it is only a matter of definitions.

3.1 An expression of $\mathcal{R}(t\mathcal{C}(1) + X, \mathbb{Z}^d)$

The main object of this subsection is to prove the following proposition :

Proposition 2. *We have for every $X \in \mathbb{R}^d$, for every $t > 0$, that*

$$\mathcal{R}(t\mathcal{C}(1) + X, \mathbb{Z}^d) = \prod_{i=1}^{d} (\lfloor t + x_i \rfloor - \lceil -t + x_i \rceil + 1) - (2t)^d \tag{4}$$

where $X = (x_1, \cdots, x_d)$.

The proof is quite straightforward.

Proof. Let $X = (x_1, \cdots, x_d) \in \mathbb{R}^d$. Let $t > 0$.
One has that :

$$\begin{aligned} N(t\mathcal{C}(1) + X, \mathbb{Z}^d) &= \left(\sum_{\substack{(n_1,\cdots,n_d)\in\mathbb{Z}^d \\ \forall i\in[1,d],\ -t+x_i\leqslant n_i\leqslant t+x_i}} 1 \right) \\ &= \prod_{i=1}^{d} (\lfloor t + x_i \rfloor - \lceil -t + x_i \rceil + 1). \end{aligned} \tag{5}$$

according to Equation (1).
Furthermore, one has that :

$$\mathrm{Vol}(t\mathcal{C}(1)+X)=(2t)^d \tag{6}$$

So, Equation (5) and Equation (6) and the definition of $\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)$ give us Equation (4). □

With Equation (4), one has that :

$$\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}=\frac{\sum_{i=1}^{d}(2t)^{i-1}(\lfloor t+x_i\rfloor-\lceil -t+x_i\rceil+1-2t)\prod_{j=i+1}^{d}(\lfloor t+x_j\rfloor-\lceil -t+x_j\rceil+1)}{t^{d-1}}. \tag{7}$$

Thanks to this last remark, the asymptotical study of $\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}$ with t distributed on $[0,T]$ according to the probability measure $\frac{1}{T}\rho(\frac{t}{T})dt$ is going to be reduced to the study of a simpler quantity. It is the object of the next subsection.

3.2 Reduction of the study of $\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}$

The main object of this subsection is to prove Proposition 1. The proof is quite straightforward and lie on the definitions of $\lfloor\cdot\rfloor$ and of $\lceil\cdot\rceil$ and on Proposition 2.

Proof of Proposition 1. For every $t>0$, for every $x\in\mathbb{R}$,

$$t+x-1<\lfloor t+x\rfloor\leqslant t+x \tag{8}$$

and

$$-t+x\leqslant\lceil -t+x\rceil<-t+x+1. \tag{9}$$

From Equation (8) and Equation (9), one gets that :

$$2t-1<(\lfloor t+x\rfloor-\lceil -t+x\rceil+1)\leqslant 2t+1 \tag{10}$$

So, from this last equation and from Equation (7), one has that, when $t\to\infty$,

$$\left|\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}-\Delta(t,X)\right|\leqslant\sum_{i=1}^{d-1}2^{i-1}\frac{1}{t^{d-i}}=O(\frac{1}{t}). \tag{11}$$

So, one gets the wanted result. □

Thanks to Proposition 1, we see that the asymptotical study of $\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}$ can be reduced to the study of $\Delta(t,X)$. We are going to use this fact in the next two sections.

4 Proof of Theorem 1

The main object of this section is to prove Theorem 1. We are going to use the reduction that was mentioned before (see Proposition 1). In the case of Theorem 1, the expression of $\Delta(t,X)$ is simple and the proof of Theorem 1 is only a matter of computation of a characteristic function.

4.1 Reduction of the study of $\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}$

The main object of this subsection is to prove the following proposition :

Proposition 3. *For every $x \in \mathbb{R}$, for every $g \in C_c(\mathbb{R})$,*

$$\int_{t=0}^{T} (g\left(\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}\right) - g\left(\Delta(t,X)\right))\frac{1}{T}dt \underset{T\to\infty}{\to} 0 \tag{12}$$

where $X = (x, \cdots, x)$ and where $\Delta(t,X)$ was defined in Proposition 1.

It should be noted in this case that

$$\Delta(t,X) = d2^{d-1}(\lfloor t+x \rfloor - \lceil -t+x \rceil + 1 - 2t). \tag{13}$$

The proof of Proposition 3 is quite straightforward and based on Proposition 1.

Proof. One has for every $0 < \kappa < \frac{1}{2}$:

$$\begin{aligned} &|\int_{t=0}^{T} (g\left(\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}\right) - g\left(\Delta(t,X)\right))\frac{1}{T}dt| \leqslant 2\|g\|_\infty \int_0^\kappa dt \\ &+ \int_{\kappa T}^{T} |g\left(\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}\right) - g\left(\Delta(t,X)\right)|\frac{1}{T}dt \end{aligned} \tag{14}$$

and, because $g \in C_c(\mathbb{R})$, it is a uniformly continuous function and so one has, because of Proposition 1, that

$$\limsup_{T\to\infty} \int_{\kappa T}^{T} |g\left(\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}\right) - g\left(\Delta(t,X)\right)|\frac{1}{T}dt = 0 \tag{15}$$

So, Equation (14) and Equation (15) give us that for every $0 < \kappa \leqslant \frac{1}{2}$:

$$\limsup_{T\to\infty} |\int_{t=0}^{T} (g\left(\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}\right) - g\left(\Delta(t,X)\right))\frac{1}{T}dt| \leqslant 2\|g\|_\infty \int_0^\kappa dt. \tag{16}$$

By making κ go to 0, one gets the wanted result. □

In the next subsection, we are going to compute the characteristic function of $\Delta(t,X)$ to within a multiplicative factor.

4.2 Computation of characteristic function

Before stating the main proposition of this section, we need to make some observations and put in place some notations.
Let's call $t_{1,0} < \cdots < t_{1,l}$ the different times $t \in [0,T]$ such that $t + x \in \mathbb{Z}$.
In the same way, let's call $t_{2,0} < \cdots < t_{2,h}$ the different times $t \in [0,T]$ such that $-t + x \in \mathbb{Z}$.
Let's observe that for every $i \in \{0, \cdots, l-1\}$, $t_{1,i+1} - t_{1,i} = 1$ and that for every $j \in \{0, \cdots, h-1\}$, $t_{2,j+1} - t_{2,j} = 1$.
As a consequence, one has necessarily that $t_{2,0} \in [t_{1,0}, t_{1,1}[$ or $t_{1,0} \in [t_{2,0}, t_{2,1}[$ and $h = l$ or

$h = l - 1$ or $h = l + 1$.
Let's set :

$$y = |t_{1,0} - t_{2,0}| \tag{17}$$

and

$$\tilde{\Delta}(t,x) = (\lfloor t + x \rfloor - \lceil -t + x \rceil + 1 - 2t). \tag{18}$$

By the way, let's remark that for all $x \in \mathbb{R}$:

$$y = |1 - 2\{x\}| \tag{19}$$

where $\{x\}$ stands for the fractional part of the real x.
Then, with these notations, one has that :

Proposition 4. *For every $x \in \mathbb{R}$, one has that the characteristic function $\varphi_{\tilde{\Delta}(\cdot,x)}$ of $\tilde{\Delta}(t,x)$, with t being distributed according to $\frac{1}{T}\mathbf{1}_{[0,T]}dt$ verifies that for every $u \in \mathbb{R}$,*

$$\varphi_{\tilde{\Delta}(\cdot,x)}(u) = \frac{h}{uT}(\sin(uy) + \sin(u(1-y))) + O(\frac{1}{T})$$

where the O is uniform in $x \in \mathbb{R}$.
As a consequence, when $T \to \infty$, for every $u \in \mathbb{R}$, one has that :

$$\varphi_{\tilde{\Delta}(\cdot,x)}(u) \underset{T\to\infty}{\to} \frac{\sin(uy) + \sin(u(1-y))}{u}.$$

The proof consists basically in cutting the interval $[0,T]$ into subintervals where all the quantities that intervene in the computation can be expressed simply.

Proof. By symmetry, we can, and we will, suppose that $t_{2,0} \in [t_{1,0}, t_{1,1}[$. Let $u \in \mathbb{R}$.
One has then that :

$$\mathbb{E}(e^{iu\tilde{\Delta}(t,x)}) = \sum_{i=0}^{h-1} \int_{t_{1,i}}^{t_{2,i}} e^{iu\tilde{\Delta}(t,x)} \frac{1}{T} dt + \int_{t_{2,i}}^{t_{1,i+1}} e^{iu\tilde{\Delta}(t,x)} \frac{1}{T} dt + O(\frac{1}{T}) \tag{20}$$

where the O corresponds to the rest of the integral that is calculated on a union of two intervals of respective lengths at most 2.
Let $i \in \{0, \cdots, h-1\}$.
Then one has :

$$\int_{t_{1,i}}^{t_{2,i}} e^{iu\tilde{\Delta}(t,x)} \frac{1}{T} dt = \int_{t_{1,i}}^{t_{2,i}} e^{iu(t_{1,i}+t_{2,i}-2t)} \frac{1}{T} dt \tag{21}$$

according to the Equation (18).
So, one gets that :

$$\int_{t_{1,i}}^{t_{2,i}} e^{iu\tilde{\Delta}(t,x)} \frac{1}{T} dt = \frac{\sin(uy)}{uT} \tag{22}$$

where one conveys that $\frac{sin(0)}{0} = 1$ and one has this last equation because for all $j \in \{0, \cdots, h-1\}$, $y = t_{2,0} - t_{1,0} = t_{2,j} - t_{1,j}$.
In a similar way, one gets that :

$$\int_{t_{2,i}}^{t_{1,i+1}} e^{iu\tilde{\Delta}(t,x)} \frac{1}{T} dt = \frac{\sin(u(1-y))}{uT} \tag{23}$$

because $1 - y = t_{1,i+1} - t_{2,i}$.
So, with Equation (20), Equation (22) and Equation (23), one gets that :

$$\mathbb{E}(e^{iu\tilde{\Delta}(t,x)}) = \sum_{i=0}^{h-1} \frac{\sin(uy)}{uT} + \frac{\sin(u(1-y))}{uT} + O(\frac{1}{T}) = \frac{h}{uT}(\sin(uy) + \sin(u(1-y))) + O(\frac{1}{T}) \tag{24}$$

By using the fact that $\lim_{T\to\infty} \frac{h}{T} = 1$, one gets from equation (24) that :

$$\mathbb{E}(e^{iu\tilde{\Delta}(t,x)}) \underset{T\to\infty}{\to} \frac{\sin(uy) + \sin(u(1-y))}{u}. \tag{25}$$

□

4.3 Conclusion

To conclude the proof of Theorem 2, we need to recall the Lévy's continuity theorem.

Theorem 3. *Let's give us $(X_n)_{n\geqslant 1}$ a sequence of real random variables and let's call $(\phi_n)_{n\geqslant 1}$ the associated sequence of their characteristic functions.*
Let's suppose that the sequence $(\varphi_n)_{n\geqslant 1}$ converges point wisely to some function φ.
Then, it is equivalent to say that there exists X a real random variable such that (X_n) converges in law towards X and to say that the function φ is continuous at the point $t = 0$.
Furthermore, if the last condition is realized, φ is the characteristic function of such a X.

We can now conclude the proof of Theorem 2.

Proof of Theorem 2. Because of Proposition 3, it is enough to study the asymptotic convergence in law, when $T \to \infty$, of the quantity $\Delta(t, X)$ with $X = (x, \cdots, x)$ and t being distributed according to the density $\frac{1}{T}\mathbf{1}_{[0,T]}(t)dt$.
The fact that

$$\Delta(t, X) = d2^{d-1} \tag{26}$$

and Proposition 4 and Theorem 3 give us that $\Delta(t, X)$ converges in law, when $T \to \infty$, and the characteristic function of the limit law is given by

$$\varphi(u) = \frac{\sin(d2^{d-1}uy) + \sin(d2^{d-1}u(1-y))}{d2^{d-1}u} \tag{27}$$

with $y = 1 - 2\{x\}$ according to Equation (19). □

5 Proof of Theorem 2

The main object of this section is to prove Theorem 2. We are going to use the reduction that was mentioned before (see Proposition 1). In the case of Theorem 2, the proof is only a matter of computation of the characteristic function of $\Delta(t, X)$ and here it can be easily dealt with thanks to the independence between the x_i and thanks to the independence between the x_i and t.

5.1 Reduction of the study of $\frac{\mathcal{R}(t\mathcal{C}(1)+X,\mathbb{Z}^d)}{t^{d-1}}$

The main object of this subsection is the following proposition :

Proposition 5. *One has that :*

$$\frac{\mathcal{R}(t\mathcal{C}(1) + X, \mathbb{Z}^d)}{t^{d-1}} - \Delta(t, X) \underset{T\to\infty}{\overset{\mathbb{P}}{\to}} 0 \tag{28}$$

when $T \to \infty$ and when t is distributed according to $\frac{1}{T}\mathbf{1}_{[0,T]}(t)dt$ and $X = (x_1, \cdots, x_d)$ is distributed according to $U([-\frac{1}{2}, \frac{1}{2}])^{\otimes^d}$. $\underset{T\to\infty}{\overset{\mathbb{P}}{\to}}$ signifies that the convergence occurs in probability.

Proof. It is a direct consequence of Proposition 1. □

Because of the independence of the x_i between them and with t, it is convenient for us to calculate the characteristic function of $\tilde{\Delta}(t, x_1)$, $\varphi_{\tilde{\Delta}(t,x_1)}$ (because also the x_i are identically distributed). It is the object of the next subsection.

5.2 Computation of the characteristic function $\varphi_{\tilde{\Delta}(t,x_1)}$

The main object of this subsection is to prove the following proposition :

Proposition 6. *For x a real random variable distributed according to the probability measure $\mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}(x)dx$ and being independent from t, with t being distributed according to the probability measure $\frac{1}{T}\mathbf{1}_{[0,T]}(t)dt$, one has that the characteristic function of $\tilde{\Delta}(t, x)$, $\varphi_{\tilde{\Delta}(t,x)}$ verifies that*

$$\varphi_{\tilde{\Delta}(t,x)}(u) = 2\frac{h}{T}\frac{1-\cos(u)}{u^2} + O(\frac{1}{T}) \tag{29}$$

As a consequence, one has that $\varphi_{\tilde{\Delta}(t,x)}(u) \underset{T\to\infty}{\to} 2\frac{1-\cos(u)}{u^2}$.

The proof is basically a computation that uses Proposition 4 :

Proof. According to Proposition 4 and because x and t are independent from one another, one has that

$$\varphi_{\tilde{\Delta}(t,x)}(u) = \mathbb{E}\left(\frac{h}{uT}(\sin(u|1-2\{x\}|) + \sin(u(1-|1-2\{x\}|)) + O(\frac{1}{T})\right) \tag{30}$$

because of Equation (19) and the O is uniform in x (h can be, and is, chosen so that it does not depend on x).
Two quick computations give us that :

$$\mathbb{E}\left(\sin(u|1-2\{x\}|)\right) = \frac{1-\cos(u)}{u} \tag{31}$$

and

$$\mathbb{E}\left(\sin(u(1-|1-2\{x\}|))\right) = \frac{1-\cos(u)}{u}. \tag{32}$$

So, with these last two equations and Equation (30), one has that :

$$\varphi_{\tilde{\Delta}(t,x)}(u) = 2\frac{h}{T}\frac{1-\cos(u)}{u^2} + O(\frac{1}{T}). \tag{33}$$

By using the fact that $\frac{h}{T} \to 1$ when $T \to \infty$, one gets finally that

$$\varphi_{\tilde{\Delta}(t,x)}(u) \underset{T\to\infty}{\to} 2\frac{1-\cos(u)}{u^2}. \tag{34}$$

□

5.3 Conclusion

We have now all the necessary tools to prove Theorem 2.

Proof of Theorem 2. Proposition 5 gives us that it is enough to prove $\Delta(t, X)$ converges in law, when $T \to \infty$. So, we are going to calculate the characteristic function of $\Delta(t, X)$.
One has, because t and the x_i are independent random variables :

$$\varphi_{\Delta(t,X)}(u) = \prod_{i=1}^{d} \varphi_{\tilde{\Delta}(t,x_i)}(2^{d-1}u) \tag{35}$$

and because $\Delta(t, X) = 2^{d-1} \sum_{i=1}^{d} \tilde{\Delta}(t, x_i)$.
Furthermore, the x_i are identically distributed according to the probability measure $\mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}$.
So Proposition 6 and Equation (35) give us that :

$$\varphi_{\Delta(t,X)}(u) = \prod_{i=1}^{d} 2\frac{h}{T}\frac{1-\cos(2^{d-1}u)}{(2^{d-1}u)^2} + O(\frac{1}{T}). \tag{36}$$

So, by making T goes to ∞, one has that :

$$\varphi_{\Delta(t,X)}(u) \underset{T\to\infty}{\to} (2\frac{1-\cos(2^{d-1}u)}{(2^{d-1}u)^2})^d. \tag{37}$$

Theorem 3 gives us then the wanted result. □

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