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Asymptotic properties of $AD(1, n)$ model and its maximum likelihood estimator

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Abstract

This paper deals with the problem of global parameter estimation of affine diffusions in $\mathbb{R}_+ \times \mathbb{R}^n$ denoted by $AD(1, n)$ where n is a positive integer which is a subclass of affine diffusions introduced by Duffie et al in [14]. The $AD(1, n)$ model can be applied to the pricing of bond and stock options, which is illustrated for the Vasicek, Cox-Ingersoll-Ross and Heston models. Our first result is about the classification of $AD(1, n)$ processes according to the subcritical, critical and supercritical cases. Then, we give the stationarity and the ergodicity theorems of this model and we establish asymptotic properties for the maximum likelihood estimator in both subcritical and a special supercritical cases.

1 Introduction

The set of affine processes contains a large class of important Markov processes such as continuous state branching processes. One of the most important Markov affine models is that of an affine diffusion model which is well characterized on the state space $\mathbb{R}_+^m \times \mathbb{R}^n$ with m and n are non-negative integers (see, e.g., [14]). In this paper, we study the subclass $AD(1, n)$ models where AD stands for affine diffusion, the parameter 1 refers to the dimension of the first component, $m = 1$, which is the Cox-Ingersoll-Ross (CIR) process and n is the dimension of the vector containing the rest of components. Hence, we consider an $AD(1, n)$ process $Z = (Y, X)^\top$ in the d -dimensional canonical state space $\mathcal{D} := [0, \infty) \times (-\infty, \infty)^n$, for $d = n + 1$ and n is a positive integer, strong solution of the following stochastic differential equation (SDE)

$$\begin{cases} dY_t = (a - bY_t) dt + \rho_{11}\sqrt{Y_t} dB_t^1 \\ dX_t = (m - \kappa Y_t - \theta X_t) dt + \sqrt{Y_t}\tilde{\rho} dB_t, \end{cases} \quad t \in [0, \infty), \quad (1)$$

where $B_t = (B_t^1, \dots, B_t^d)^\top$, $t \in [0, \infty)$, is a d -dimensional independent Brownian vector, $(Y_0, X_0)^\top$ is an arbitrary initial value independent of B such that $\mathbb{P}(Y_0 \in (0, \infty)) = 1$, $a \in (0, \infty)$, $m \in (-\infty, \infty)^n$,

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$b \in (-\infty, \infty)$, $\kappa \in (-\infty, \infty)^n$, θ is an $n \times n$ real matrix and $\tilde{\rho} = [\rho_{J1} \ \rho_{JJ}]$, where $J = \{2, \dots, d\}$, $\rho_{J1} = (\rho_{i1})_{i \in J}$ and $\rho_{JJ} = (\rho_{ij})_{i,j \in J}$ which are introduced via the block representation of the $d \times d$ positive definite triangular low matrix $\rho = \begin{pmatrix} \rho_{11} & \rho_{1J} \\ \rho_{J1} & \rho_{JJ} \end{pmatrix}$ satisfying $\sum_{j=1}^d \rho_{ij}^2 = \sigma_i^2$, for all $i \in \{1, \dots, d\}$, where $\sigma = (\sigma_1, \dots, \sigma_d)^\top \in (0, \infty)^d$. Note that, on one hand, according to [14], ρ is admissible in the sense that $AD(1, n)$ is an affine diffusion. On the other hand, the first component of the model is a CIR process which is also a continuous state branching process with branching mechanism $by + \frac{\rho_{11}^2}{2}y^2$ and immigration mechanism ay and conditionally on $\sigma(Y_t, t \in [0, \infty))$ the second component is an n -dimensional Vasicek process.

The affine models, in particular the $AD(1, n)$ one, has been employed in finance since the last decades and they have found growing interest due to their computational tractability as well as their capability to capture empirical evidence from financial time series, see, e.g., Duffie et al [14] and Filipovic and Mayerhofer in [17], where they gave more financial applications of affine models and proved existence and uniqueness through stochastic invariance of the canonical state space. These applications are illustrated for different known diffusion models as Vasicek, Cox-Ingersoll-Ross and Heston models. Further, stationarity and ergodicity studies for affine diffusion processes found a particular interest by many authors, see, e.g., [4, 10, 18, 19, 25, 26, 27] and [43], where they give sufficient conditions for the existence of a unique stationary distribution and for the ergodic property. Moreover, several papers have studied the drift parameter estimation in different models, see, e.g., [2, 3, 5, 6, 7, 9, 15] and the references therein. It should be noted that statistical inferences of $AD(1, n)$ model were not considered and more generally, inferences of affine multidimensional diffusions are rarely treated. We cite, as well, [1, Chapter 4] and [13], where they introduced some examples and simulations of multidimensional affine diffusions in financial and econometric fields.

Related to our work, from a statistical point of view, in [6] and [7], Ben Alaya and Kebaier considered the CIR model, called also the square-root model, which refers to the model (1) with $n = 0$. they showed asymptotic properties of the maximum likelihood estimators (MLE) in the ergodic case ($b > 0$) and non-ergodic cases ($b < 0$ and $b = 0$).

More later, in [3], Barczy et al studied an affine two factor model, which is a submodel of (1) with $n = 1$, $\kappa = \mathbf{0}$ and $\rho = \mathbf{I}_2$. They showed some asymptotic properties of both maximum likelihood and least squares estimators of drift parameters in the ergodic case based on continuous time observations and proved there strong consistency and asymptotic normality. Slightly more generally, in [9], Bolyog and Pap studied asymptotic properties of conditional least squares estimators for the drift parameters of two-factor affine diffusions, which corresponds to the case $n = 1$ in our model (1) by adding a third Wiener process to the SDE related to X . In the subcritical case ($b > 0$ and $\theta > 0$), the special critical case ($b = 0$ and $\theta = 0$) and the special supercritical case ($\theta < b < 0$), they proved different types of consistency and asymptotic normality of the conditional least square estimators under some additional assumptions on the drift and the diffusion parameters.

From ergodicity point of view, recently in 2020 (see [26]), Jin et al gave a general condition for the stationarity of jump affine processes on a general canonical state space $[0, \infty)^m \times (-\infty, \infty)^n$, for non-negative integers m and n . They proved the existence of a limit distribution of the process which can be identified through its characteristic function. Furthermore, for the same model, in [18], Friesen et al. studied the existence and the regularity of transition densities and proved the exponential ergodicity in the total variation distance under a Hörmander type condition and other assumptions on the parameters.

The aim of this paper is twofold. On one hand, we will study the stationarity and the ergodicity of the $AD(1, n)$ model. On the other hand, we will apply the obtained results to establish consistency and asymptotic normality of the drift parameter maximum likelihood estimator. The paper is organized as follows, in the second section, we introduce at first all the principal notations and tools we need, then we give a classification of the model (1) with respect to b and the eigenvalues of θ according to the asymptotic behavior of $\mathbb{E}(Z_t)$ as t tends to infinity. Essentially, we define the subcritical case when $\mathbb{E}(Z_t)$ converges, the critical case when $\mathbb{E}(Z_t)$ has a polynomial growth and the supercritical case when $\mathbb{E}(Z_t)$ has an exponential growth, for more details see Proposition 2.1. In the third section, using affine properties of our process, we give sufficient conditions for the existence of a unique stationary distribution $Z_\infty = (Y_\infty, X_\infty)$ given by the Fourier-Laplace transform

$$\mathbb{E} \left(e^{-\lambda Y_\infty + i\mu^\top X_\infty} \right) = \exp \left(a \int_0^\infty \mathcal{K}_s(-\lambda, \mu) ds + i\mu^\top \theta^{-1} m \right),$$

where λ is a complex number with a positive real part, μ is a n -dimensional real vector and $\mathcal{K} : t \mapsto \mathcal{K}_t(-\lambda, \mu)$ is a complex-valued time function satisfying the Riccati equation (25), for more details see Theorem 3.1. In the fourth section, we establish the exponential ergodicity of $(Y_t, X_t)_{t \in [0, \infty)}$ using the so-called Foster-Lyapunov criteria, see [36, page 535, Section 6], namely, we prove that there exists $\delta \in (0, \infty)$ and $B \in (0, \infty)$ such that

$$\sup_{|g| \leq V+1} |\mathbb{E}(g(Y_t, X_t) | (Y_0, X_0) = (y_0, x_0)) - \mathbb{E}(g(Y_\infty, X_\infty))| \leq B(V(y_0, x_0) + 1)e^{-\delta t},$$

for all $t \in [0, \infty)$ and for all Borel measurable functions $g : \mathcal{D} \rightarrow (-\infty, \infty)$, where $(y_0, x_0) \in \mathcal{D}$ and V is a chosen norm-like function on \mathcal{D} , for more details see Theorem 4.1. Note that our result seems to be new and not covered by [18, Theorem 1.3]. The last section is devoted to the study of statistical inferences of the drift parameters maximum likelihood estimators constructed using Lipster and Shiryaev [33, page 297] associated to the $AD(1, n)$ model. We study its consistency and its asymptotic behavior in both subcritical and a special supercritical cases (see Theorem 5.1 and Theorem 5.2). We close the paper with an appendix, where we recall the strong law of large numbers and the central limit theorem for martingales.

2 Preliminary

At first, we start by fixing the used notations. Let \mathbb{N} , \mathbb{R} , \mathbb{R}_+ , \mathbb{R}_{++} , \mathbb{R}_- , \mathbb{R}_{--} and \mathbb{C} denote the sets of non-negative integers, real numbers, non-negative real numbers, positive real numbers, non-positive real numbers, negative real numbers and complex numbers, respectively and let $\mathcal{D} = \mathbb{R}_+ \times \mathbb{R}^n$, for $n \in \mathbb{N} \setminus \{0\}$. For $x, y \in \mathbb{R}$, we will use the notation $x \wedge y := \min(x, y)$ and $x \vee y := \max(x, y)$. For all $z \in \mathbb{C}$, $\text{Re}(z)$ denotes the real part of z and $\text{Im}(z)$ denotes the imaginary part of z . Let us denote, for $p, q \in \mathbb{N} \setminus \{0\}$, by $\mathcal{M}_{p,q}$ the set of $p \times q$ real matrices, \mathcal{M}_p the set of $p \times p$ real matrices, \mathbf{I}_p the identity matrix in \mathcal{M}_p , $\mathbf{0}_{p,q}$ the null matrix in $\mathcal{M}_{p,q}$, $\mathbf{0}_p := \mathbf{0}_{p,1}$ and $\mathbf{1}_p \in \mathcal{M}_{p,1}$ is the 1-vector of size p . We denote, for all diagonalizable matrix $A \in \mathcal{M}_p$, by $\text{eig}(A)$ the column vector containing all the eigenvalues of A and by $\lambda_i(A)$, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$, the i^{th} , the smallest and the largest eigenvalues of A , respectively. For $m, n \in \mathbb{N} \setminus \{0\}$, we write $I = \{1, \dots, m\}$ and $J = \{m+1, m+n\}$ and for all $x \in \mathbb{R}^{m+n}$, we write $\mathbf{x}_I = (x_i)_{i \in I}$ and $\mathbf{x}_J = (x_j)_{j \in J}$. Throughout this paper, we use the notation

$$A = \begin{bmatrix} A_{II} & A_{IJ} \\ A_{JI} & A_{JJ} \end{bmatrix}$$

for $A \in \mathcal{M}_{m+n}$ where $A_{II} = (a_{i,j})_{i,j \in I}$, $A_{IJ} = (a_{i,j})_{i \in I, j \in J}$, $A_{JI} = (a_{i,j})_{i \in J, j \in I}$ and $A_{JJ} = (a_{i,j})_{i,j \in J}$. For all matrices A_1, \dots, A_k , $\text{diag}(A_1, \dots, A_k)^\top$ denotes the block matrix containing the matrices A_1, \dots, A_k in its diagonal and for all matrix \mathbf{M} , let us denote $\|\mathbf{M}\| := \sqrt{\max \text{eig}(\mathbf{M}\mathbf{M}^\top)}$ with \mathbf{M}^\top denotes the transpose matrix of \mathbf{M} and for all vector $\mathbf{x} = (x_1, \dots, x_p) \in \mathbb{R}^p$, we use the notation $\|\mathbf{x}\| := \sum_{i=1}^p |x_i|$ and

$\|\mathbf{x}\|_2^2 := \sum_{i=1}^p x_i^2$. We denote by \otimes the Kronecker product defined, for all matrices $\mathbf{A} = (a_{ij})_{1 \leq i \leq p, 1 \leq j \leq q}$ and \mathbf{B} ,

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1q}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{p1}\mathbf{B} & \cdots & a_{pq}\mathbf{B} \end{bmatrix}$$

and by \oplus the Kronecker sum defined by $\mathbf{A} \oplus \mathbf{B} = A \otimes \mathbf{I} + \mathbf{I} \otimes B$. We define the vec-operator applied on a matrix \mathbf{A} denoted by $\text{vec}(\mathbf{A})$, which stacks its columns into a column vector. Some of associated matrix properties used in this paper are given, for all matrices with suitable dimensions $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and \mathbf{D} , by

$$(\mathbf{A} \otimes \mathbf{C})(\mathbf{B} \otimes \mathbf{D}) = \mathbf{A}\mathbf{B} \otimes \mathbf{C}\mathbf{D}, \quad \text{vec}(\mathbf{A}\mathbf{B}\mathbf{C}) = (\mathbf{C}^\top \otimes \mathbf{A})\text{vec}(\mathbf{B}), \quad e^{\mathbf{A} \oplus \mathbf{B}} = e^{\mathbf{A}} \otimes e^{\mathbf{B}}, \quad (2)$$

(for more details, see, e.g., [38] and [21]). We use the notations $H_{\mathbf{x}}(f)$ and $\nabla_{\mathbf{x}}f$ for the Hessian matrix and the gradient column vector with respect to the parameter vector \mathbf{x} of a function $f \in \mathcal{C}^2(\mathcal{D}, \mathbb{R})$ which denotes the set of twice continuously differentiable real-valued functions on \mathcal{D} . By $\mathcal{C}_c^2(\mathcal{D}, \mathbb{R})$, we denote the set of functions in $\mathcal{C}^2(\mathcal{D}, \mathbb{R})$ with compact support. Eventually, we will denote the convergence in probability, in distribution and almost surely by $\xrightarrow{\mathbb{P}}$, $\xrightarrow{\mathcal{D}}$, $\xrightarrow{a.s.}$ respectively. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with the augmented filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ corresponding to $(B_t)_{t \in \mathbb{R}_+}$ and a given initial value $Z_0 = (Y_0, X_0)^\top$ being independent of $(B_t)_{t \in \mathbb{R}_+}$ such that $\mathbb{P}(Y_0 \in \mathbb{R}_{++}) = 1$. Note that $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ satisfies the usual conditions, i.e., the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ is right-continuous and \mathcal{F}_0 contains all the \mathbb{P} -null sets in \mathcal{F} . For all $t \in \mathbb{R}_+$, we use the notation $\mathcal{F}_t^Y := \sigma(Y_s; 0 \leq s \leq t)$ for the σ -algebra generated by $(Y_s)_{s \in [0, t]}$.

Secondly, in the statistical part, we write the model (1) using a matrix representation as follows

$$dZ_t = \Lambda(Z_t)\tau dt + \sqrt{Z_t^1} \rho dB_t, \quad t \in \mathbb{R}_+, \quad (3)$$

where $Z_t = (Z_t^1, \dots, Z_t^d)^\top = (Y_t, X_t^1, \dots, X_t^n)^\top$, for all $t \in \mathbb{R}_+$, $\Lambda(Z_t) = \begin{bmatrix} \Lambda_1(Z_t) & \mathbf{0}_{1, n(d+1)} \\ \mathbf{0}_{n, 2} & \mathbf{I}_n \otimes K(Z_t) \end{bmatrix}$, with $\Lambda_1(Z_t) = (1, -Z_t^1)$ and $K(Z_t) = (1, -Z_t^1, \dots, -Z_t^d)$ and the $d^2 + 1$ -dimensional vector τ is stacking in turn the unknown drift parameters of Y, X^1, \dots, X^n into a column vector, namely,

$$\tau = (a, b, m_1, \kappa_1, \theta_{11}, \dots, \theta_{1n}, \dots, m_n, \kappa_n, \theta_{n1}, \dots, \theta_{nn})^\top.$$

For the third part of this section, we come back to the first representation of the model given by relation (1) and we present a classification result of the process $Z = (Y, X)^\top$ related to its first moment.

Proposition 2.1. *Let $a \in \mathbb{R}_{++}$, $b \in \mathbb{R}$, $m, \kappa \in \mathbb{R}^n$ and let θ be a real diagonalizable matrix in \mathcal{M}_n . Suppose that θ is either a positive definite, a negative definite or a zero matrix with eigenvalues different to b or all equal to b . Let $Z = (Y, X)^\top$ be the unique strong solution of the SDE (1) and suppose that Z_0 is integrable. Then, for all $t \in \mathbb{R}_+$, we have*

1. for $b \in \mathbb{R}_{++}$, $\mathbb{E}(Y_t) = \frac{a}{b} + \mathbf{O}(e^{-bt})$ and

$$\mathbb{E}(X_t) = \begin{cases} \theta^{-1}m - \frac{a}{b}\theta^{-1}\kappa + \mathbf{O}(e^{-(\lambda_{\min}(\theta) \wedge b)t})\mathbf{1}_n, & \lambda_{\min}(\theta) \in \mathbb{R}_{++}, \\ t \left(m - \frac{a}{b}\kappa \right) + \mathbf{O}(1)\mathbf{1}_n, & \lambda_{\min}(\theta) = \lambda_{\max}(\theta) = 0, \\ e^{-t\theta} (\mathbb{E}(X_0)(\theta - b\mathbf{I}_n)^{-1}\kappa + \Xi_0) + \mathbf{O}(1)\mathbf{1}_n, & \lambda_{\max}(\theta) \in \mathbb{R}_{--}, \end{cases}$$

2. for $b = 0$, $\mathbb{E}(Y_t) = at + \mathbf{O}(1)$ and

$$\mathbb{E}(X_t) = \begin{cases} -t\theta^{-1}\kappa + \mathbf{O}(1)\mathbf{1}_n, & \lambda_{\min}(\theta) \in \mathbb{R}_{++}, \\ -\frac{t^2}{2}a\kappa + \mathbf{O}(t)\mathbf{1}_n, & \lambda_{\min}(\theta) = \lambda_{\max}(\theta) = 0, \\ e^{-t\theta} (\mathbb{E}(Y_0)\theta^{-1}\kappa + \mathbb{E}(X_0) - \theta^{-1}m - a\theta^{-2}\kappa) + \mathbf{O}(t)\mathbf{1}_n, & \lambda_{\max}(\theta) \in \mathbb{R}_{--}, \end{cases}$$

3. for $b \in \mathbb{R}_{--}$, $\mathbb{E}(Y_t) = \left(\mathbb{E}(Y_0) - \frac{a}{b} \right) e^{-bt} + \mathbf{O}(1)$ and

$$\mathbb{E}(X_t) = \begin{cases} e^{-bt} \left(\left(\frac{a}{b} - \mathbb{E}(Y_0) \right) (\theta - b\mathbf{I}_n)^{-1} \kappa \right) + \mathbf{O}(1)\mathbf{1}_n, & \lambda_{\min}(\theta) \in \mathbb{R}_{++}, \\ e^{-bt} \left(\frac{1}{b} \mathbb{E}(Y_0) \kappa + \mathbb{E}(X_0) - \frac{a}{b^2} \kappa \right) + \mathbf{O}(t)\mathbf{1}_n, & \lambda_{\min}(\theta) = \lambda_{\max}(\theta) = 0, \\ e^{-bt} \left(-\mathbb{E}(Y_0)(\theta - b\mathbf{I}_n)^{-1} \kappa + \frac{a}{b} (\theta - b\mathbf{I}_n)^{-1} \kappa \right) + \mathbf{O}(e^{-\lambda_{\min}(\theta)t})\mathbf{1}_n, & b < \lambda_{\min}(\theta) \leq \lambda_{\max} < 0, \\ te^{-bt} \left(-\mathbb{E}(Y_0) \kappa + \frac{a}{b} \kappa \right) + \mathbf{O}(e^{-bt})\mathbf{1}_n, & \lambda_{\min}(\theta) = \lambda_{\max}(\theta) = b, \\ e^{-t\theta} (\mathbb{E}(Y_0)(\theta - b\mathbf{I}_n)^{-1} \kappa + \Xi_0) + \mathbf{O}(e^{-bt})\mathbf{1}_n, & \lambda_{\max}(\theta) < b, \end{cases}$$

where $\Xi_0 = \mathbb{E}(X_0) - \theta^{-1}m + \frac{a}{b}\theta^{-1}\kappa - \frac{a}{b}(\theta - b\mathbf{I}_n)^{-1}\kappa$.

Proof. First by applying Itô's formula on the processes $(e^{bt}Y_t)_{t \in \mathbb{R}_+}$ and $(e^{t\theta}X_t)_{t \in \mathbb{R}_+}$, we obtain, for $t \in \mathbb{R}_+$,

$$Y_t = e^{-bt}Y_0 + a \int_0^t e^{-b(t-s)} ds + \rho_{11} \int_0^t e^{-b(t-s)} \sqrt{Y_s} dB_s^1. \quad (4)$$

and

$$X_t = e^{-t\theta}X_0 + \int_0^t e^{-(t-s)\theta} m ds - \int_0^t e^{-(t-s)\theta} Y_s \kappa ds + \int_0^t \sqrt{Y_s} e^{-(t-s)\theta} \tilde{\rho} dB_s. \quad (5)$$

By taking the expectation in the two sides of equations (4) and (5) and thanks to Fubini-Tonelli it is easy to check that

$$\mathbb{E}(Y_t) = e^{-bt}\mathbb{E}(Y_0) + a \int_0^t e^{-bu} du = \begin{cases} \frac{a}{b} + (\mathbb{E}(Y_0) - \frac{a}{b}) e^{-bt}, & b \neq 0, \\ \mathbb{E}(Y_0) + at, & b = 0, \end{cases} \quad (6)$$

and

$$\mathbb{E}(X_t) = e^{-t\theta}\mathbb{E}(X_0) + \int_0^t e^{-s\theta} m ds - \int_0^t e^{-(t-s)\theta} \left(e^{-bs}\mathbb{E}(Y_0) + a \int_0^s e^{-bu} du \right) \kappa ds. \quad (7)$$

Thanks to (6), it is easy to get the behavior of $\mathbb{E}(Y_t)$ for the three cases ($b \in \mathbb{R}_{++}$, $b = 0$ and $b \in \mathbb{R}_{--}$). Concerning $\mathbb{E}(X_t)$ we distinguish two cases. For $b = 0$, if $\lambda_{\min}(\theta) = \lambda_{\max}(\theta) = 0$, the relation (7) becomes $\mathbb{E}(X_t) = \mathbb{E}(X_0) + t(m - \mathbb{E}(Y_0)\kappa) - \frac{1}{2}t^2 a\kappa$ and we obtain the second assertion of item 2. Otherwise, we have $\mathbb{E}(X_t) = e^{-t\theta}\mathbb{E}(X_0) + \theta^{-1}(I_n - e^{-t\theta})m - (\mathbb{E}(Y_0)(1 - e^{-t\theta}) + at)\theta^{-1}\kappa + a\theta^{-2}(I_n - e^{-t\theta})\kappa$. Its behavior depends on the sign of the eigenvalues of θ and it is easy to obtain the first and the third assertions of item 2. Now for $b \neq 0$, if $\lambda_{\min} = \lambda_{\max} = 0$, the relation (7) becomes $\mathbb{E}(X_t) = \mathbb{E}(X_0) + t\left(m - \frac{a}{b}\kappa\right) + \left(\frac{1}{b}\mathbb{E}(Y_0) - \frac{a}{b^2}\right)(e^{-bt} - 1)\kappa$ and the behavior depends on the sign of b : for $b \in \mathbb{R}_{++}$, we obtain the second assertion of item 3 and for $b \in \mathbb{R}_{--}$, we obtain the second assertion of item 3, else if $\lambda_{\min} = \lambda_{\max} = b$, the relation (7) becomes $\mathbb{E}(X_t) = e^{-bt}\mathbb{E}(X_0) + \frac{1}{b}(1 - e^{-bt})m - \frac{a}{b^2}(1 - e^{-bt})\kappa + \left(\frac{a}{b} - \mathbb{E}(Y_0)\right)te^{-bt}\kappa$ and we obtain a part of the first assertion of item 1 and the fourth assertion of item 3, otherwise, $\mathbb{E}(X_t) = e^{-t\theta}\mathbb{E}(X_0) + \left(\frac{a}{b} - \mathbb{E}(Y_0)\right)(\theta - bI_n)^{-1}(e^{-bt}I_n - e^{-t\theta})\kappa + \theta^{-1}(I_n - e^{-t\theta})(m - \frac{a}{b}\kappa)$ which covers the rest of cases in item 1 and item 3. This completes the proof of Proposition 2.1. \square

Remark 2.1. If θ is an arbitrary matrix in \mathcal{M}_n admitting joint positive, negative or zero eigenvalues, then

$$\mathbb{E}(X_t) = (QP)^{-1}\mathbb{E}(\Upsilon_t), \quad (8)$$

for all $t \in \mathbb{R}_+$, where $\Upsilon_t = QPX_t$ with $P \in \mathcal{M}_n$ is the change-of-basis matrix transforming θ into a diagonal matrix D , $Q = (Q_1, Q_2, Q_3, Q_4, Q_5)^\top \in \mathcal{M}_n$ is a permutation matrix (a non-singular square binary matrix, called also a doubly stochastic matrix, that allows us to rearrange the eigenvalues of θ on the diagonal of the matrix D) such that, for all $i, j \in \{1, \dots, 5\}$, the rows of Q_i and the rows of Q_j are orthogonal when $i \neq j$. Note that for all $k \in \{1, \dots, 5\}$, $Q_k \in \mathcal{M}_{n_k, n}$ where $n_k \in \mathbb{N}$ satisfying $\sum_{k=1}^5 n_k = n$, and $\Upsilon_t = (\Upsilon_t^1, \Upsilon_t^2, \Upsilon_t^3, \Upsilon_t^4, \Upsilon_t^5)^\top$ with $\Upsilon_t^k \in \mathbb{R}^{n_k}$ is the solution of the following SDE

$$d\Upsilon_t^k = (Q_k P m - Q_k P \kappa \Upsilon_t - \tilde{D}_k \Upsilon_t^k) dt + \sqrt{Y_t} Q_k P \tilde{\rho} dB_t, \quad t \in \mathbb{R}_+,$$

where $\tilde{D}_k \in \mathcal{M}_{n_k}$ satisfies $\lambda_{\min}(\tilde{D}_1) \in \mathbb{R}_{++}$, $\lambda_{\min}(\tilde{D}_2) = \lambda_{\max}(\tilde{D}_2) = 0$, $\lambda_{\max}(\tilde{D}_3) \in \mathbb{R}_{--}$ with $\lambda_{\min}(\tilde{D}_3) > b$ when $b \in \mathbb{R}_{--}$, $\lambda_{\max}(\tilde{D}_4) \in \mathbb{R}_{--}$ with $\lambda_{\min}(\tilde{D}_4) = \lambda_{\max}(\tilde{D}_4) = b$ when $b \in \mathbb{R}_{--}$ and $\lambda_{\max}(\tilde{D}_5) \in \mathbb{R}_{--}$ with $\lambda_{\max}(\tilde{D}_5) < b$ when $b \in \mathbb{R}_{--}$. Consequently, $\mathbb{E}(\Upsilon_t) = (\mathbb{E}(\Upsilon_t^1), \dots, \mathbb{E}(\Upsilon_t^5))^\top$ and for each $k \in \{1, \dots, 5\}$, $\mathbb{E}(\Upsilon_t^k)$ satisfies one of the items 1, 2 or 3 by replacing m, κ and θ by $Q_k P m, Q_k P \kappa$ and \tilde{D}_k , respectively. In fact, in this case, we can choose a permutation matrix $Q \in \mathcal{M}_n$ and a change-of-basis matrix P to diagonalize θ as follows

$$\theta = (QP)^{-1}\tilde{D}(QP) = P^{-1} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{bmatrix}^{-1} \begin{bmatrix} \tilde{D}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{D}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \tilde{D}_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \tilde{D}_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \tilde{D}_5 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{bmatrix} P.$$

Based on the asymptotic behavior of the expectation $\mathbb{E}(Z_t)$ as t tends to ∞ given in Proposition 2.1, we introduce a classification of $AD(1, n)$ processes defined by the SDE (3).

Definition 2.1. Let $(Z_t)_{t \in \mathbb{R}_+}$ be the unique strong solution of (3) satisfying $\mathbb{P}(Z_0^1 \in \mathbb{R}_{++}) = 1$. Suppose that θ is either a positive definite, a negative definite or a zero matrix with eigenvalues different to b or all equal to b . We call $(Z_t)_{t \in \mathbb{R}_+}$ *subcritical* if $b \wedge \lambda_{\min}(\theta) \in \mathbb{R}_{++}$, i.e., when $\mathbb{E}(Z_t)$ converges as $t \rightarrow \infty$, *critical* if either $b \in \mathbb{R}_+$ and $\lambda_{\min}(\theta) = \lambda_{\max}(\theta) = 0$ or $b = 0$ and $\lambda_{\min}(\theta) \in \mathbb{R}_{++}$, i.e., when $\mathbb{E}(Z_t)$ has a polynomial expansion and *supercritical* if $b \wedge \lambda_{\max}(\theta) \in \mathbb{R}_{--}$, i.e., when $\mathbb{E}(Z_t)$ has an exponential expansion.

In the following section we state the stationarity theorem of $(Z_t)_{t \in \mathbb{R}_+}$ defined by the last $AD(1, n)$ model (3) and its proof.

3 Stationarity

In a special case where $n = 1$, the study of existence of stationary distributions for $AD(1, 1)$ process was treated by Barczy M. et al. in [4] with $\kappa = 0$ and it is called affine two-factor model. Analogously, it was proved by Bolyog B. and Pap G. in [10] for affine two-factor diffusions where $\kappa \in \mathbb{R}$. By the following theorem, we extend these studies for the $AD(1, n)$ process $Z = (Y, X)^\top$ defined by the model (1).

Theorem 3.1. Let us consider the $AD(1, n)$ model (1) with $a \in \mathbb{R}_+$, $b \in \mathbb{R}_{++}$, $m \in \mathbb{R}^n$, $\kappa \in \mathbb{R}^n$, $\theta \in \mathcal{M}_n$ a diagonalizable positive definite matrix and $Z_0 = (Y_0, X_0)^\top$ is a random initial value independent of $(B_t)_{t \in \mathbb{R}_+}$ satisfying $\mathbb{P}(Y_0 \in \mathbb{R}_+) = 1$.

1) Then, $Z_t \xrightarrow{\mathcal{D}} Z_\infty$ as $t \rightarrow \infty$, where $Z_\infty = (Y_\infty, X_\infty)^\top$ and the distribution of Z_∞ is given by

$$\mathbb{E}\left(e^{\nu^\top Z_\infty}\right) = \mathbb{E}\left(e^{-\lambda Y_\infty + i\mu^\top X_\infty}\right) = \exp\left(a \int_0^\infty \mathcal{K}_s(-\lambda, \mu) ds + i\mu^\top \theta^{-1} m\right), \quad (9)$$

for $\nu = (-\lambda, i\mu) \in \mathcal{U}_1 \times \mathcal{U}_2$, where $\mathcal{U}_1 := \{u \in \mathbb{C} : \operatorname{Re}(u) \in \mathbb{R}_-\}$ and $\mathcal{U}_2 := \{u \in \mathbb{C}^n : \operatorname{Re}(u) = 0\}$ and for all $(u_1, u_2) \in \mathcal{U}_1 \times \mathbb{R}^n$, the function $t \mapsto \mathcal{K}_t(u_1, u_2)$ is the unique solution of the following (deterministic) non-linear Riccati partial differential equation (PDE)

$$\begin{cases} \frac{\partial \mathcal{K}_t}{\partial t}(u_1, u_2) &= \frac{\rho_{11}^2}{2} \mathcal{K}_t^2(u_1, u_2) - \left(b - i\rho_{11}\rho_{J1}^\top e^{-t\theta^\top} u_2\right) \mathcal{K}_t(u_1, u_2) - i\kappa^\top e^{-t\theta^\top} u_2 \\ &\quad - \frac{1}{2} \left(\operatorname{vec}(\rho_{JJ}\rho_{JJ}^\top)\right)^\top e^{-t(\theta^\top \oplus \theta^\top)}(u_2 \otimes u_2) - \frac{1}{2} \left(\rho_{J1}^\top e^{-t\theta^\top} u_2\right)^2 \\ \mathcal{K}_0(u_1, u_2) &= u_1. \end{cases} \quad (10)$$

2) In addition, if Z_0 has the same distribution as Z_∞ given by (9), then $(Z_t)_{t \in \mathbb{R}_+}$ is strictly stationary.

In order to prove this theorem we consider the following technical lemma:

Lemma 3.1. We can assume, without loss of generality, that $\rho_{J1} = \mathbf{0}_n$ in Theorem 3.1.

The proof of this Lemma will be postponed after the theorem proof.

Proof. (Proof of $(Z_t)_{t \in [0, T]}$, for some $T \in \mathbb{R}_{++}$ $(Z_t)_{t \in [0, T]}$, for some $T \in \mathbb{R}_{++}$ Theorem 3.1) According to Lemma 3.1, we can assume that $\rho_{J1} = \mathbf{0}_n$ and hence consider the model

$$\begin{cases} dY_t = (a - bY_t) dt + \rho_{11} \sqrt{Y_t} dB_t^1, \\ dX_t = (m - \kappa Y_t - \theta X_t) dt + \sqrt{Y_t} \rho_{JJ} dW_t, \quad \text{where } W_t = (B_t^2, \dots, B_t^d). \end{cases} \quad (11)$$

1) The proof of the first assertion follows four main steps:

Step 1: the first step is devoted to convert the asymptotic study for a triplet process $\tilde{Z} = (Y, U, V)$ instead of $Z = (Y, X)$, where, for all $t \in \mathbb{R}_+$,

$$U_t := \int_0^t Y_s e^{-(t-s)\theta} \kappa \, ds \quad \text{and} \quad V_t := \text{vec} \left(\int_0^t Y_s e^{-(t-s)\theta} \rho_{JJ} \rho_{JJ}^\top e^{-(t-s)\theta^\top} \, ds \right),$$

and to write its Fourier-Laplace transform. For $(\lambda, \mu), (y_0, x_0) \in \mathcal{D}$ and $t \in \mathbb{R}_+$, by the explicit expression of X_t given by (5) and the tower property, the conditional Fourier-Laplace transform of (Y_t, X_t) , namely $\mathbb{E} \left(e^{-\lambda Y_t + i\mu^\top X_t} | (Y_0, X_0) = (y_0, x_0) \right)$ is equal to

$$\mathbb{A}_t(\mu, \theta, m, x_0) \mathbb{E} \left(\exp \left(-\lambda Y_t - i\mu^\top U_t \right) \mathbb{E} \left(\exp \left(i\mu^\top \int_0^t \sqrt{Y_s} e^{-(t-s)\theta} \rho_{JJ} \, dW_s \right) | \mathcal{F}_t^Y \right) | Y_0 = y_0 \right), \quad (12)$$

where $\mathbb{A}_t(\mu, \theta, m, x_0) = \exp \left(i\mu^\top e^{-t\theta} x_0 + i\mu^\top \theta^{-1} m - i\mu^\top \theta^{-1} e^{-t\theta} m \right)$. Further, by the independence of Y and W , the second property on the Kronecker operators for matrices in (2) and the definition of V_t , we deduce that the conditional expectation $\mathbb{E} \left(\exp \left(i\mu^\top \int_0^t \sqrt{Y_s} e^{-(t-s)\theta} \rho_{JJ} \, dW_s \right) | \mathcal{F}_t^Y \right)$ is equal to

$$\mathbb{E} \left(\exp \left(-\frac{1}{2} \mu^\top \int_0^t Y_s e^{-(t-s)\theta} \rho_{JJ} \rho_{JJ}^\top e^{-(t-s)\theta^\top} \, ds \mu \right) \right) = \mathbb{E} \left(\exp \left(-\frac{1}{2} (\mu^\top \otimes \mu^\top) V_t \right) \right). \quad (13)$$

By combining (12) and (13) together we obtain

$$\mathbb{E} \left(e^{-\lambda Y_t + i\mu^\top X_t} | (Y_0, X_0) = (y_0, x_0) \right) = \mathbb{A}_t(\mu, \theta, m, x_0) \mathbb{E} \left(e^{-\lambda Y_t - i\mu^\top U_t - \frac{1}{2} (\mu^\top \otimes \mu^\top) V_t} | Y_0 = y_0 \right). \quad (14)$$

Note that the process $\tilde{Z} = (Y, U, V)$ introduced above is solution to

$$\begin{cases} dY_t &= (a - bY_t) \, dt + \rho_{11} \sqrt{Y_t} \, dB_t^1, \\ dU_t &= (Y_t \kappa - \theta U_t) \, dt, \\ dV_t &= (Y_t \text{vec}(\rho_{JJ} \rho_{JJ}^\top) - (\theta \oplus \theta) V_t) \, dt, \end{cases} \quad t \in \mathbb{R}_+,$$

with initial values $Y_0, \mathbf{0}_n$ and $\mathbf{0}_{n^2}$, which is also an affine process, see e.g. Filipovic in [17, Theorem 3.2]. Hence, according to Definition 2.1 in [17], we have for all $T \in \mathbb{R}_+$, $0 \leq t \leq T$ and $u = (u_1, u_2, u_3) \in \mathcal{U} := \mathcal{U}_1 \times \mathcal{U}_2 \times \mathcal{U}_3$, where $\mathcal{U}_3 := \{u \in \mathbb{R}^{n^2} \mid \exists \tilde{u} \in \mathcal{U}_2; u = \tilde{u} \otimes \tilde{u}\}$, the following affine transform formula

$$\mathbb{E} \left(e^{u^\top \tilde{Z}_T} | \mathcal{F}_t \right) = \mathbb{E} \left(e^{u_1 Y_T + u_2^\top U_T + u_3^\top V_T} | \mathcal{F}_t \right) = \exp \left(\phi(T-t, u) + \psi(T-t, u)^\top \tilde{Z}_t \right), \quad (15)$$

where $\psi = (\psi_1, \psi_2, \psi_3)^\top$ is the solution of the following Riccati system

$$\begin{cases} \frac{\partial \psi_1}{\partial t}(t, u) &= \frac{\rho_{11}^2}{2} \psi_1^2(t, u) - b\psi_1(t, u) + \kappa^\top \psi_2(t, u) + (\text{vec}(\rho_{JJ} \rho_{JJ}^\top))^\top \psi_3(t, u), \\ \frac{\partial \psi_2}{\partial t}(t, u) &= -\theta^\top \psi_2(t, u), \\ \frac{\partial \psi_3}{\partial t}(t, u) &= -(\theta^\top \oplus \theta^\top) \psi_3(t, u), \\ \psi_1(0, u) &= u_1, \quad \psi_2(0, u) = u_2, \quad \psi_3(0, u) = u_3, \end{cases} \quad (16)$$

and ϕ is defined, for all $u \in \mathcal{U}$ and $t \in \mathbb{R}_+$, by $\phi(t, u) = a \int_0^t \psi_1(s, u) \, ds$. Using the explicit solutions $\psi_2(t, u) = e^{-t\theta^\top} u_2$ and $\psi_3(t, u) = e^{-t(\theta^\top \oplus \theta^\top)} u_3$, we deduce that ψ_1 solves the following PDE

$$\begin{cases} \frac{\partial \psi_1}{\partial t}(t, u) &= \frac{\rho_{11}^2}{2} \psi_1^2(t, u) - b\psi_1(t, u) + \kappa^\top e^{-t\theta^\top} u_2 + (\text{vec}(\rho_{JJ} \rho_{JJ}^\top))^\top e^{-t(\theta^\top \oplus \theta^\top)} u_3, \\ \psi_1(0, u) &= u_1. \end{cases} \quad (17)$$

for $u \in \mathcal{U}$. Consequently, by taking $t = 0$ in (15), we obtain

$$\mathbb{E} \left(e^{u_1 Y_t + u_2^\top U_t + u_3^\top V_t} | Y_0 = y_0 \right) = \exp \left(a \int_0^t \psi_1(s, u) \, ds + y_0 \psi_1(t, u) \right). \quad (18)$$

Step 2: we prove that if $u_1 \in \mathcal{U}_1$, we have $\psi_1(t, u) \in \mathcal{U}_1$, for all $t \in \mathbb{R}_+$ and $(u_2, u_3) \in \mathcal{U}_2 \times \mathcal{U}_3$. Indeed, using the fact that $u_3^\top V_t \in \mathbb{R}_-$ since

$$u_3^\top V_t = -\frac{1}{2}(\mu^\top \otimes \mu^\top) V_t = -\frac{1}{2} \mu^\top \int_0^t Y_s e^{-(t-s)\theta} \rho_{JJ} \rho_{JJ}^\top e^{-(t-s)\theta^\top} \, ds \, \mu$$

and $\int_0^t Y_s e^{-(t-s)\theta} \rho_{JJ} \rho_{JJ}^\top e^{-(t-s)\theta^\top} \, ds$ is a positive semi-definite matrix, we deduce that

$$\left| \mathbb{E} \left(e^{u_1 Y_t + u_2^\top U_t + u_3^\top V_t} | Y_0 = y_0 \right) \right| \leq \mathbb{E} \left(e^{\operatorname{Re}(u_1) Y_t + u_3^\top V_t} | Y_0 = y_0 \right) \leq 1, \quad (19)$$

for $t \in \mathbb{R}_+$ and $u \in \mathcal{U}$. Consequently, combining the equations (18) and (19), we obtain

$$\left| \exp \left(a \int_0^t \psi_1(s, u) \, ds + y_0 \psi_1(t, u) \right) \right| = \exp \left(a \operatorname{Re} \left(\int_0^t \psi_1(s, u) \, ds \right) + y_0 \operatorname{Re}(\psi_1(t, u)) \right) \leq 1,$$

for all $t \in \mathbb{R}_+$ and $u \in \mathcal{U}$. Now, looking to the ordinary differential equation (17), it's obvious that its solution ψ_1 does not depend on the parameters a and y_0 . Therefore, we are allowed to take $a = 0$ and choose $y_0 \in \mathbb{R}_{++}$ which implies that $\psi_1(t, u) \in \mathcal{U}_1$ when $u_1 \in \mathcal{U}_1$, for all $t \in \mathbb{R}_+$ and $(u_2, u_3) \in \mathcal{U}_2 \times \mathcal{U}_3$.

Step 3: Let $\mathcal{U}' = \mathcal{U}_1 \times \mathbb{R}^n \times \mathcal{U}_3$. Now, we concentrate on the construction of an upper-bound for the modulus of the function \tilde{K} defined through ψ_1 as follows

$$\mathbb{R}_+ \times \mathcal{U}' \ni (t, u_1, u_2, u_3) \mapsto \tilde{K}_t(u_1, u_2, u_3) := \psi_1(t, (u_1, -iu_2, u_3)),$$

which satisfies, for all $u = (u_1, u_2, u_3) \in \mathcal{U}'$,

$$\frac{\partial \tilde{K}_t}{\partial t}(u_1, u_2, u_3) = \frac{\rho_{11}^2}{2} \tilde{K}_t^2(u_1, u_2, u_3) - b \tilde{K}_t(u_1, u_2, u_3) - i \kappa^\top e^{-t\theta^\top} u_2 + (\operatorname{vec}(\rho_{JJ} \rho_{JJ}^\top))^\top e^{-t(\theta^\top \oplus \theta^\top)} u_3, \quad (20)$$

and

$$\mathbb{E} \left(e^{u_1 Y_t - i u_2^\top U_t + u_3^\top V_t} | Y_0 = y_0 \right) = \exp(\tilde{g}_t(u) + y_0 \tilde{K}_t(u)), \quad (21)$$

where $\tilde{g}_t(u) := a \int_0^t \tilde{K}_s(u) \, ds$. Let the real functions $\mathbb{R}_+ \times \mathcal{U}' \ni (t, u) \mapsto v_t(u) := -\operatorname{Re}(\tilde{K}_t(u))$ and $\mathbb{R}_+ \times \mathcal{U}' \ni (t, u) \mapsto w_t(u) := \operatorname{Im}(\tilde{K}_t(u))$ satisfying the following Riccati system

$$\begin{cases} \frac{\partial v_t}{\partial t}(u) &= -\frac{\rho_{11}^2}{2} v_t^2(u) - b v_t(u) + \frac{\rho_{11}^2}{2} w_t^2(u) - (\operatorname{vec}(\rho_{JJ} \rho_{JJ}^\top))^\top e^{-t(\theta^\top \oplus \theta^\top)} u_3, \\ \frac{\partial w_t}{\partial t}(u) &= -(b + \rho_{11}^2 v_t(u)) w_t(u) - \kappa^\top e^{-t\theta^\top} u_2, \\ v_0(u) &= -\operatorname{Re}(u_1), \quad w_0(u) = \operatorname{Im}(u_1), \end{cases} \quad t \in \mathbb{R}_+, \quad (22)$$

and a general solution of w_t takes the form

$$w_t(u) = C e^{-\int_0^t f_z(u) \, dz} - e^{-\int_0^t f_z(u) \, dz} \int_0^t e^{\int_0^s f_z(u) \, dz} \kappa^\top e^{-s\theta^\top} u_2 \, ds,$$

with $f_t(u) := b + \rho_{11}^2 v_t(u)$, for $t \in \mathbb{R}_+$ and C is a real constant. Taking into account the initial value $w_0(u) = \operatorname{Im}(u_1)$, we obtain $C = \operatorname{Im}(u_1)$. Consequently, for all $t \in \mathbb{R}_+$,

$$|w_t(u)| \leq \|u_1\| e^{-\int_0^t f_z(u) \, dz} + \|\kappa\| \|u_2\| \int_0^t e^{-\lambda_{\min}(\theta)s - \int_s^t f_z(u) \, dz} \, ds.$$

Thanks to the fact that $f_t(u) \geq b \in \mathbb{R}_+$, for all $t \in \mathbb{R}_+$ and $u \in \mathcal{U}'$,

$$\begin{aligned} \int_0^t e^{-\lambda_{\min}(\theta)s - \int_s^t f_z(u) dz} ds &\leq \int_0^t e^{-\lambda_{\min}(\theta)s - (t-s)b} ds \\ &= \begin{cases} \frac{e^{-\lambda_{\min}(\theta)t} - e^{-bt}}{b - \lambda_{\min}(\theta)} \leq \frac{e^{-(\lambda_{\min}(\theta) \wedge b)t}}{|b - \lambda_{\min}(\theta)|}, & b \neq \lambda_{\min}(\theta), \\ te^{-bt} \leq e^{-\frac{b}{2}t} \sup_{t \in \mathbb{R}_+} te^{-\frac{b}{2}t} = \frac{2}{eb} e^{-\frac{b}{2}t}, & b = \lambda_{\min}(\theta). \end{cases} \end{aligned}$$

Thus, for all $t \in \mathbb{R}_+$ and $u \in \mathcal{U}'$, we obtain

$$|w_t(u)| \leq C_3(u)e^{-C_2t}, \quad (23)$$

with $C_3(u) := \|u_1\| + \|\kappa\| \|u_2\| \left(\frac{1}{|b - \lambda_{\min}(\theta)|} \mathbb{1}_{\{b \neq \lambda_{\min}(\theta)\}} + \frac{2}{eb} \mathbb{1}_{\{b = \lambda_{\min}(\theta)\}} \right)$ and $C_2 = \lambda_{\min}(\theta) \wedge \frac{b}{2}$.

Combining the first PDE of (22) with the relation (23), we get $\frac{\partial v_t}{\partial t}(u) \leq -bv_t(u) + C_4(u)e^{-C_2t}$, with $C_4(u) = \frac{\rho_{11}^2}{2} C_3(u)^2 + \|\text{vec}(\rho_{JJ} \rho_{JJ}^\top)\| \|u_3\|$, and $v_0(u) = -\text{Re}(u_1)$. By the comparison theorem, we derive the inequality $v_t(u) \leq \tilde{v}_t(u)$, for all $t \in \mathbb{R}_+$ and $u \in \mathcal{U}'$, where \tilde{v}_t is solution of the inhomogeneous linear differential equation $\frac{\partial \tilde{v}_t}{\partial t}(u) = -b\tilde{v}_t(u) + C_4(u)e^{-C_2t}$, and $\tilde{v}_0(u) = -\text{Re}(u_1)$. In a similar

way as in the previous calculation, we obtain $\tilde{v}_t(u) = -\text{Re}(u_1)e^{-bt} + C_4(u)e^{-C_2t} \int_0^t e^{-(b-C_2)s} ds$. Using the fact $b - C_2 \geq \frac{b}{2} \in \mathbb{R}_{++}$, we get

$$0 \leq v_t(u) \leq \tilde{v}_t(u) = -\text{Re}(u_1)e^{-bt} + C_4(u) \frac{e^{-C_2t} - e^{-bt}}{b - C_2} \leq C_5(u)e^{-C_2t},$$

with $C_5(u) = -\text{Re}(u_1) + \frac{2}{b}C_4(u)$. Finally, we obtain

$$|\tilde{K}_t(u)| = \sqrt{v_t(u)^2 + w_t(u)^2} \leq C_1(u)e^{-C_2t}, \quad (24)$$

for all $t \in \mathbb{R}_+$ and $u \in \mathcal{U}'$, with $C_1(u) := \sqrt{C_5^2(u) + C_3^2(u)}$.

Step 4: In order to complete the proof of our first result given by relations (9) and (10), we prove the following assertions:

- i) for all $u \in \mathcal{U}'$ and $y_0 \in \mathbb{R}_+$, $\lim_{t \rightarrow \infty} [y_0 \tilde{K}_t(u) + \tilde{g}_t(u)] = a \int_0^\infty \tilde{K}_s(u) ds =: \tilde{g}_\infty(u)$.
- ii) the function $\mathcal{U}' \ni u \mapsto \tilde{g}_\infty(u)$ is continuous.

At first, using the equation (24), we deduce that $\lim_{t \rightarrow \infty} y_0 \tilde{K}_t(u) = 0$, then thanks to Lebesgue's dominated convergence theorem, we get $\lim_{t \rightarrow \infty} \int_0^t \tilde{K}_s(u) ds = \int_0^\infty \tilde{K}_s(u) ds$. Secondly, in order to prove ii), let $(u^{(n)})_{n \in \mathbb{N}}$ a sequence in \mathcal{U}' such that $\lim_{n \rightarrow \infty} u^{(n)} = u \in \mathcal{U}'$. Similarly, thanks to equation (24), the continuity of $\mathcal{U}' \ni u \mapsto \tilde{K}_t(u)$ with $t \in \mathbb{R}_+$ and the Lebesgue's dominated convergence theorem, we deduce that $\lim_{n \rightarrow \infty} \int_0^\infty \tilde{K}_s(u^{(n)}) ds = \int_0^\infty \tilde{K}_s(u) ds$, which shows the continuity of $\mathcal{U}' \ni u \mapsto \tilde{g}_\infty(u)$. It is worth to note that since the function \tilde{K}_t does not depend on the parameters a and y_0 as unique solution of the differential equation (20), its continuity can be proved by taking $a = 0$ and $y_0 \in \mathbb{R}_{++}$ in relation (21).

Finally, Let us consider $(u_1, u_2, u_3) = (-\lambda, \mu, -\frac{1}{2}\mu \otimes \mu)$. Hence, thanks to the independence of (Y_0, X_0) and $(B_t)_{t \in \mathbb{R}_+}$, we use first the law of total expectation, then the dominated convergence

theorem and we finish with the assertion i) to deduce that the limit as t tends to infinity of the Fourier-Laplace transform $\mathbb{E}\left(e^{-\lambda Y_t + i\mu^\top X_t}\right)$ is equal to

$$\begin{aligned} \lim_{t \rightarrow \infty} \int_0^\infty \int_{\mathbb{R}^n} \mathbb{A}_t(\mu, \theta, m, x_0) \mathbb{E}\left(e^{-\lambda Y_t - i\mu^\top U_t - \frac{1}{2}(\mu^\top \otimes \mu^\top) V_t} | Y_0 = y_0\right) \mathbb{P}_{(Y_0, X_0)}(dy_0, dx_0) \\ = \exp\left(\tilde{g}_\infty\left(-\lambda, \mu, -\frac{1}{2}\mu \otimes \mu\right) + i\mu^\top \theta^{-1} m\right). \end{aligned}$$

The second assertion insures the application of Lévy's continuity theorem to prove the convergence $Z_t \xrightarrow{\mathcal{D}} Z_\infty$ and to characterize the distribution of $Z_\infty = (Y_\infty, X_\infty)$ by its Fourier-Laplace transform as the limit function obtained above. We complete the proof of the first result in the case $\rho_{J1} = \mathbf{0}_n$ by considering the function $\mathbb{R}_+ \times \mathcal{U}_1 \times \mathbb{R}^n \ni (t, u_1, u_2) \mapsto K_t(u_1, u_2) := \tilde{K}_t(u_1, u_2, -\frac{1}{2}u_2 \otimes u_2)$ and writing the associated Riccati equation from relation (20) given by

$$\begin{cases} \frac{\partial K_t}{\partial t}(u_1, u_2) = \frac{\rho_{11}^2}{2} K_t^2(u_1, u_2) - b K_t(u_1, u_2) - i\kappa^\top e^{-t\theta^\top} u_2 - \frac{1}{2}(\text{vec}(\rho_{JJ}\rho_{JJ}^\top))^\top e^{-t(\theta^\top \oplus \theta^\top)}(u_2 \otimes u_2), \\ K_0(u_1, u_2) = u_1. \end{cases} \quad (25)$$

which is nothing but the relation (10) for $\rho_{J1} = \mathbf{0}_n$.

- 2) In order to prove the strict stationarity (translation invariance of the finite dimensional distributions), we show that for all $t \in \mathbb{R}_+$, the distribution of (Y_t, X_t) is translation invariant and has the same distribution of (Y_∞, X_∞) . First, since according to (14) and (21), we have

$$\mathbb{E}\left(e^{-\lambda Y_t + i\mu^\top X_t} | (Y_0, X_0) = (y_0, x_0)\right) = \exp\left(y_0 K_t(-\lambda, \mu) + i\mu^\top e^{-t\theta} x_0 + g_t(-\lambda, \mu)\right),$$

where $g_t(-\lambda, \mu) = a \int_0^t K_s(-\lambda, \mu) ds + i\mu^\top \theta^{-1}(1 - e^{-t\theta})m$, with K solution of the Riccati equation (25), from the first part of the theorem, it is enough to check that for all $t \in \mathbb{R}_+$, $\lambda \in \mathbb{R}_+$ and $\mu \in \mathbb{R}^n$, we have

$$\mathbb{E}\left(\exp(K_t(-\lambda, \mu)Y_\infty + i\mu^\top e^{-t\theta} X_\infty + g_t(-\lambda, \mu))\right) = \exp\left(a \int_0^\infty K_s(-\lambda, \mu) ds + i\mu^\top \theta^{-1} m\right).$$

As $K_t(-\lambda, \mu) \in \mathcal{U}_1$ for all $t \in \mathbb{R}_+$ and $(-\lambda, \mu) \in \mathcal{U}_1 \times \mathbb{R}^n$, again from the first part of the theorem, the expectation above is equal to

$$\begin{aligned} \exp\left(a \int_0^\infty K_s(K_t(-\lambda, \mu), e^{-t\theta^\top} \mu) ds + i\mu^\top e^{-t\theta} \theta^{-1} m + g_t(-\lambda, \mu)\right) \\ = \exp\left(a \left(\int_0^\infty K_s(K_t(-\lambda, \mu), e^{-t\theta^\top} \mu) ds + \int_0^t K_s(-\lambda, \mu) ds\right) + i\mu^\top \theta^{-1} m\right). \end{aligned} \quad (26)$$

By simple comparison of the two terms, it is enough to check that for all $(-\lambda, \mu) \in \mathcal{U}_1 \times \mathbb{R}^n$ $\int_t^\infty K_s(-\lambda, \mu) ds = \int_0^\infty K_s(K_t(-\lambda, \mu), e^{-t\theta^\top} \mu) ds$, which holds if

$$K_s(K_t(-\lambda, \mu), e^{-t\theta^\top} \mu) = K_{t+s}(-\lambda, \mu), \quad (27)$$

for all $s, t \in \mathbb{R}_+$ and $(-\lambda, \mu) \in \mathcal{U}_1 \times \mathbb{R}^n$. Since, by the help of matrix properties given in relation (2), the functions $\mathbb{R}_+ \ni s \mapsto K_s(K_t(-\lambda, \mu), e^{-t\theta^\top} \mu)$ and $\mathbb{R}_+ \ni s \mapsto K_{t+s}(-\lambda, \mu)$, $s \in \mathbb{R}_+$ and $(-\lambda, \mu) \in \mathcal{U}_1 \times \mathbb{R}^n$, satisfy the same differential equation (25) with the same initial value $K_t(-\lambda, \mu)$, we conclude using the uniqueness of the solution. Finally, using the fact that $(Z_t)_{t \in \mathbb{R}_+}$ is a time-homogeneous Markov process and the tower property, we deduce that the process $(Z_t)_{t \in \mathbb{R}_+}$ is strictly stationary which completes the second part of the proof.

□

Proof. (Proof of Lemma 3.1) Let us consider the process

$$(Y_t, \tilde{X}_t)^\top = A(Y_t, X_t)^\top, \quad \text{with } A = \begin{bmatrix} 1 & \mathbf{0}_n^\top \\ -\frac{1}{\rho_{11}}\rho_{J1} & \mathbf{I}_n \end{bmatrix}. \quad (28)$$

It is easy to check that $(Y_t, \tilde{X}_t)^\top$ is an $AD(1, n)$ process with affine drift

$$A \begin{bmatrix} a \\ m \end{bmatrix} - A \begin{bmatrix} b & \mathbf{0}_n^\top \\ \kappa & \theta \end{bmatrix} A^{-1} \begin{bmatrix} Y_t \\ \tilde{X}_t \end{bmatrix} = \begin{bmatrix} a - bY_t \\ m - \frac{a}{\rho_{11}}\rho_{J1} - (\kappa - \frac{1}{\rho_{11}}(b\mathbf{I}_n - \theta)\rho_{J1})Y_t - \theta\tilde{X}_t \end{bmatrix}$$

and a diffusion matrix $\begin{bmatrix} \rho_{11}^2 Y_t & \mathbf{0}_n^\top \\ \mathbf{0}_n & Y_t \rho_{JJ} \rho_{JJ}^\top \end{bmatrix}$. Then we can apply the results of Theorem 3.1 on the process $(Y_t, \tilde{X}_t)^\top$ with the associated drift parameters $(a, b, \tilde{m}, \tilde{\kappa}, \theta)$ and $\tilde{\rho}$, where $\tilde{m} = m - \frac{a}{\rho_{11}}\rho_{J1} \in \mathbb{R}^n$, $\tilde{\kappa} = \kappa - \frac{1}{\rho_{11}}(b\mathbf{I}_n - \theta)\rho_{J1} \in \mathbb{R}^n$ and $\tilde{\rho} = \begin{bmatrix} \rho_{11} & \mathbf{0}_n^\top \\ \mathbf{0}_n & Y_t \rho_{JJ} \end{bmatrix}$. Hence, the relation (10) can be extended to

$$\begin{aligned} \mathbb{E} \left(e^{-\lambda Y_\infty + i\mu^\top X_\infty} \right) &= \mathbb{E} \left(e^{(-\lambda, i\mu) A^{-1} (Y_\infty, \tilde{X}_\infty)^\top} \right) = \mathbb{E} \left(e^{(-\lambda + i\frac{1}{\rho_{11}}\mu^\top \rho_{J1}) Y_\infty + i\mu^\top \tilde{X}_\infty} \right) \\ &= \exp \left(a \int_0^\infty K_s \left(-\lambda + i\frac{1}{\rho_{11}}\mu^\top \rho_{J1}, \mu \right) ds + i\mu^\top \theta^{-1} \left(m - \frac{a}{\rho_{11}}\rho_{J1} \right) \right) \\ &= \exp \left(a \int_0^\infty \mathcal{K}_s(-\lambda, \mu) ds + i\mu^\top \theta^{-1} m \right), \end{aligned}$$

with $\mathcal{K}_t(u_1, u_2) = K_t \left(u_1 + i\frac{1}{\rho_{11}}u_2^\top \rho_{J1}, \mu \right) - i\frac{1}{\rho_{11}}u_2^\top e^{-t\theta} \rho_{J1}$, for all $(u_1, u_2) \in \mathcal{U}_1 \times \mathbb{R}^n$, where K_t is the function associated to the process (Y_t, \tilde{X}_t) solution of the Riccati equation (10) with parameters $(a, b, \tilde{m}, \tilde{\kappa}, \theta)$ and $\tilde{\rho}$. This function satisfies $\frac{\partial \mathcal{K}_t}{\partial t}(u_1, u_2) = \frac{\partial K_t}{\partial t} \left(u_1 + i\frac{1}{\rho_{11}}u_2^\top \rho_{J1}, u_2 \right) + i\frac{1}{\rho_{11}}u_2^\top \theta e^{-t\theta} \rho_{J1}$. Hence, using the dynamic of K_t and its relation with \mathcal{K}_t , it is easy to get $\frac{\partial \mathcal{K}_t}{\partial t}(u_1, u_2)$ is equal to

$$\begin{aligned} \frac{\rho_{11}^2}{2} \left(\mathcal{K}_t(u_1, u_2) + i\frac{1}{\rho_{11}}\rho_{J1}^\top e^{-t\theta^\top} u_2 \right)^2 - b \left(\mathcal{K}_t(u_1, u_2) + i\frac{1}{\rho_{11}}\rho_{J1}^\top e^{-t\theta^\top} u_2 \right) + i\frac{1}{\rho_{11}}\rho_{J1}^\top e^{-t\theta^\top} \theta^\top u_2 \\ - i \left(\kappa^\top - \frac{1}{\rho_{11}}\rho_{J1}^\top (b\mathbf{I}_n - \theta^\top) \right) e^{-t\theta^\top} u_2 - \frac{1}{2} \left(\text{vec}(\rho_{JJ} \rho_{JJ}^\top) \right)^\top e^{-t(\theta^\top \oplus \theta^\top)} (u_2 \otimes u_2). \end{aligned}$$

By a simple calculation, the last equation gives exactly the PDE (10). Moreover, we have also the initial value $\mathcal{K}_0(u_1, u_2) = K_0 \left(u_1 + i\frac{1}{\rho_{11}}u_2^\top \rho_{J1}, u_2 \right) - i\frac{1}{\rho_{11}}u_2^\top \rho_{J1} = u_1$, this completes the extension of the first part of the theorem. For the second part, the strict stationarity of the process $(Y_t, X_t)_{t \in \mathbb{R}_+}$ is deduced by the one of $(Y_t, \tilde{X}_t)_{t \in \mathbb{R}_+}$. □

4 Exponential Ergodicity

In the subcritical case, the following theorem states the exponential ergodicity for the process $(Z_t)_{t \in \mathbb{R}_+}$ defined by the SDE (1) and extends the results found by Barczy et al. in [4] and Bolyog and Pap in [10] for the special cases $n = 1$. As a consequence, a strong law of large numbers given by the relation (30) is obtained by proposition 2.5 of Bhattacharya in [8].

Theorem 4.1. *Let us consider the $AD(1, n)$ model (1) with $a \in \mathbb{R}_{++}$, $b \in \mathbb{R}_{++}$, $m \in \mathbb{R}^n$, $\kappa \in \mathbb{R}^n$ and $\theta \in \mathcal{M}_n$ a diagonalizable positive definite matrix with initial random values $Z_0 = (Y_0, X_0)^\top$ independent of*

$(B_t)_{t \in \mathbb{R}_+}$ satisfying $\mathbb{P}(Y_0 \in \mathbb{R}_{++}) = 1$. Then the process Z is exponentially ergodic, namely, there exists $\delta \in \mathbb{R}_{++}$, $B \in \mathbb{R}_{++}$ and $r \in \mathbb{R}_{++}$ such that

$$\sup_{|g| \leq V+1} |\mathbb{E}(g(Z_t)|Z_0 = z_0) - \mathbb{E}(g(Z_\infty))| \leq B(V(z_0) + 1)e^{-\delta t}, \quad (29)$$

for all $t \in \mathbb{R}_+$ and $z_0 = (y_0, x_0)^\top \in \mathcal{D}$, where the supremum is running for Borel measurable functions $g : \mathcal{D} \rightarrow \mathbb{R}$, $V(y, x) := y^2 + r \|x\|_2^2$, for all $(y, x) \in \mathcal{D}$, and $Z_\infty = (Y_\infty, X_\infty)^\top$ is defined by (9). Moreover, for all Borel measurable functions $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\mathbb{E}(|f(Z_\infty)|) < \infty$, we have

$$\mathbb{P} \left(\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(Z_s) \, ds = \mathbb{E}(f(Z_\infty)) \right) = 1. \quad (30)$$

Proof. At first, we assume that $\rho_{J1} = \mathbf{0}_n$. In order to prove the exponential ergodicity given by (29), we use the Foster-Lyapunov criteria, see [36, Theorem 6.1]. Hence, it is enough to check the three following assertions:

- (i) $(Z_t)_{t \in \mathbb{R}_+}$ is a Borel right process (defined as in Gettoor [20, page 55] and Sharpe [39, page 38]).
- (ii) For the skeleton chain $(Z_k)_{k \in \mathbb{N}}$ all compact sets are petite (skeleton chains and petite sets are defined as in Meyn and Tweedie [35, pages 491 and 500] and [34, pages 550], respectively).
- (iii) There exists $c \in \mathbb{R}_{++}$ and $d \in \mathbb{R}$ such that the inequality

$$(\mathcal{A}_k V)(z) \leq -cV(z) + d, \quad z \in \mathcal{O}_k,$$

holds for all $k \in \mathbb{N}$, where $\mathcal{O}_k := \{z \in \mathcal{D} : \|z\|_2 < k\}$, for each $k \in \mathbb{N}$, \mathcal{A}_k denotes the extended generator of the process $(Z_t^{(k)})_{t \in \mathbb{R}_+}$ given by

$$Z_t^{(k)} := \begin{cases} Z_t, & t < T_k, \\ (0, k, \dots, k)^\top, & t \geq T_k, \end{cases}$$

for $t \in \mathbb{R}_+$, where the stopping time T_k is defined by $T_k := \inf\{t \in \mathbb{R}_+ : Z_t \in \mathcal{D} \setminus \mathcal{O}_k\}$. Note that we can choose instead of $(0, k, \dots, k)^\top$, any other point in $\mathcal{D} \setminus \mathcal{O}_k$.

In order to prove the first assertion, thanks to Meyn and Tweedie [35, page 498], it is enough to check that $(Z_t)_{t \in \mathbb{R}_+}$ is a weak Feller process with continuous sample paths satisfying strong Markov property (the Feller and weak Feller properties are given in Meyn and Tweedie [35, section 3.1]). By Duffie et al. [14, proposition 8.2 or Theorem 2.7], the $AD(1, n)$ model (1), as an affine process, is a Feller Markov process. Besides, since $(Z_t)_{t \in \mathbb{R}_+}$ has continuous sample paths almost surely, it is automatically a strong Markov process (see, e.g., Chung [11, Theorem 1, page 56]). For the second assertion, since $(Z_k)_{k \in \mathbb{N}}$ is Feller, it is sufficient, by Proposition 6.2.8 in Meyn and Tweedie [37], to check that it is irreducible. Note that the irreducibility of $(Z_k)_{k \in \mathbb{N}}$ holds if the conditional distribution of $Z_1 = (Y_1, X_1)^\top$ given $Z_0 = (Y_0, X_0)^\top$ is absolutely continuous with respect to the Lebesgue measure on \mathcal{D} such that the conditional density function $f_{Z_1|Z_0}$ is positive on \mathcal{D} . Indeed, the Lebesgue measure on \mathcal{D} is σ -finite and if B is a Borel set in \mathcal{D} with positive Lebesgue measure, then

$$\mathbb{E} \left(\sum_{k=0}^{\infty} \mathbf{1}_B(Z_k) | Z_0 = z_0 \right) \geq \mathbb{P}(Z_1 \in B | Z_0 = z_0) = \int_B f_{Z_1|Z_0}(z|z_0) \, dz,$$

for all $z_0 \in \mathcal{D}$. The existence of this conditional density with the required property can be checked as follows. By taking $t = 1$ in the expression (5), we consider the vector $(Y_1, \check{X}_1)^\top$, where $\check{X}_1 = (\check{X}_1^1, \dots, \check{X}_1^n)^\top$ is the random part of X_1 given by $\check{X}_1 := -\int_0^1 Y_s e^{s\theta} \kappa \, ds + \int_0^1 \sqrt{Y_s} e^{s\theta} \rho_{JJ} \, dW_s$. Since $Y_1 \geq 0$, it is sufficient to check that, for all $x \in \mathbb{R}^n$, $(Y_1, \check{X}_1)^\top$ has a positive density function on \mathcal{D} . Note that the

conditional distribution of \check{X}_1 given $(Y_t)_{t \in [0,1]}$ is a normal distribution with mean $m_Y = -\int_0^1 Y_s e^{s\theta} \kappa \, ds$ and covariance matrix $C_Y = \int_0^1 Y_s e^{s\theta} \rho_{JJ} \rho_{JJ}^\top e^{s\theta^\top} \, ds$. In order to compute the distribution function of $(Y_1, \check{X}_1)^\top$, if we set $p(u)$ this conditional density function, then for all $(y, x) \in \mathcal{D}$, we have

$$\begin{aligned} \mathbb{P}\left(Y_1 \leq y, \check{X}_1^1 \leq x_1, \dots, \check{X}_1^n \leq x_n\right) &= \mathbb{E}\left(\mathbf{1}_{\{Y_1 \leq y\}} \mathbb{P}\left(\check{X}_1^1 \leq x_1, \dots, \check{X}_1^n \leq x_n \mid (Y_t)_{t \in [0,1]}\right)\right) \\ &= \mathbb{E}\left(\mathbf{1}_{\{Y_1 \leq y\}} \int_{\prod_{i=1}^n (-\infty, x_i)} p(u) \, du\right), \end{aligned}$$

where $p(u) = \frac{(2\pi)^{-n/2}}{\sqrt{\det(C_Y)}} \exp\left(-\frac{1}{2}(u - m_Y)^\top C_Y^{-1}(u - m_Y)\right)$, for all $u \in \mathbb{R}^n$. By conditioning on Y_1 and using Fubini-Tonelli property, we get

$$\mathbb{P}\left(Y_1 \leq y, \check{X}_1^1 \leq x_1, \dots, \check{X}_1^n \leq x_n\right) = \int_0^y \int_{\prod_{i=1}^n (-\infty, x_i)} \mathbb{E}(p(u) \mid Y_1 = z) f_{Y_1}(z) \, dz \, du, \quad (31)$$

where f_{Y_1} denotes the density function of Y_1 given that $Y_0 = y_0 \in \mathbb{R}_{++}$ defined, for all $y \in \mathbb{R}$, by

$$f_{Y_1}(y) = \frac{2be^{b(2a+1)}}{e^b - 1} \left(\frac{y}{y_0}\right)^{a-\frac{1}{2}} \exp\left(\frac{-2b(y_0 + e^b y)}{e^b - 1}\right) I_{2a-1}\left(\frac{2b\sqrt{y_0 y}}{\sinh(\frac{b}{2})}\right) \mathbf{1}_{y \in \mathbb{R}_{++}},$$

where $I_{2a-1}(x) = \sum_{m=0}^{\infty} \frac{1}{m! \Gamma(m+2a)} \left(\frac{x}{2}\right)^{2m+2a-1}$, for all $x \in \mathbb{R}_{++}$. As f_{Y_1} is a positive density (see, e.g., [24]) and $p(u)$ is finite and positive, the random vector $(Y_1, \check{X}_1)^\top$ has a positive density on \mathcal{D} . Concerning the last assertion, note that the extended generator, according to [18, Theorem 1.1], is given by

$$(\mathcal{A}_k V)(y, x) = y + nry + 2(ay - by^2) + 2rx^\top(m - \kappa y - \theta x),$$

for all $(y, x)^\top \in \mathcal{O}_k$, $k \in \mathbb{N}$. Therefore, by simple calculation, we get

$$(\mathcal{A}_k V)(y, x) + cV(y, x) \leq -c_2 \left(y + \frac{r}{c_2} \kappa^\top x\right)^2 - x^\top c_3 x + c_1 \left(y + \frac{r}{c_2} \kappa^\top x\right) + c_4 x,$$

with $c_1 = 1 + nr + 2a$, $c_2 = 2b - c$, $c_3 = r(2\lambda_{\min}(\theta) - c)\mathbf{I}_n - \frac{r^2}{c_2} \kappa \kappa^\top$ and $c_4 = 2rm^\top - r \frac{c_1}{c_2} \kappa^\top$. Next, in order to get $c_2 \in \mathbb{R}_{++}$ and c_3 a positive definite matrix, we choose $0 < c < 2(\lambda_{\min}(\theta) \wedge b)$ and $0 < r < \frac{(2\lambda_{\min}(\theta) - c)(2b - c)}{\lambda_{\max}(\kappa \kappa^\top)}$. Furthermore, since c_3 is symmetric, then $c_3 = \tilde{c}_3 \tilde{c}_3^\top$, with \tilde{c}_3 is positive definite matrix. Hence, we can bounded $(\mathcal{A}_k V)(y, x) + cV(y, x)$ by

$$-c_2 \left(y + \frac{r}{c_2} \kappa^\top x - \frac{c_1}{2c_2}\right)^2 - \left(x^\top \tilde{c}_3 - \frac{1}{2} c_4 (\tilde{c}_3^{-1})^\top\right) \left(x^\top \tilde{c}_3 - \frac{1}{2} c_4 (\tilde{c}_3^{-1})^\top\right)^\top + d,$$

with $d = \frac{c_1^2}{4c_2} + \frac{1}{4} c_4 c_3^{-1} c_4^\top$. This completes the proof of the exponential ergodicity theorem in the special case $\rho_{J_1} = \mathbf{0}_n$. In the general case, we use the transformation (28) introduced in the proof of lemma 3.1. Let g be a Borel measurable function satisfying $|g(y, x)| \leq V(y, x) + 1$, for all $(y, x)^\top \in \mathcal{D}$, then by putting $h := g \circ A^{-1}$, it is easy to check that, for all $(y, \tilde{x})^\top \in \mathcal{D}$, we have

$$|h(y, \tilde{x})| \leq V(A^{-1}(y, x)) = y^2 + r \left\| \frac{y}{\rho_{11}} \rho_{J_1} + \tilde{x} \right\|_2^2 + 1 \leq C(V(y, \tilde{x}) + 1),$$

with $C = \left(1 + 2r \frac{\|\rho_{J1}\|_2^2}{\rho_{11}^2}\right) \vee 2$. Consequently, using the inequality (29) for the process $(Y_t, \tilde{X}_t)_{t \in \mathbb{R}_+}$ with the Borel measurable function $\frac{1}{C}h$, we get

$$\begin{aligned} |\mathbb{E}(g(Y_t, X_t) | (Y_0, X_0) = (y_0, x_0)) - \mathbb{E}(g(Y_\infty, X_\infty))| &\leq BC \left(y_0^2 + r \left\| x_0 - \frac{y_0}{\rho_{11}} \rho_{J1} \right\|_2^2 + 1 \right) e^{-\delta t} \\ &\leq BC^2 (V(y_0, x_0) + 1) e^{-\delta t}. \end{aligned}$$

This completes the proof by changing BC^2 to B . \square

The main goal of the next section is to study the maximum likelihood estimator of the drift parameters in the subcritical and a special supercritical cases using the results obtained in the previous sections.

5 Maximum likelihood estimation

In this section, let us recall that for statistical estimations with continuous observations, we always suppose that the diffusion parameter ρ is known, see e.g. [32, page 50]. In fact, using an arbitrarily short continuous time observation of the process $(Z_t)_{t \in [0, T]}$, for some $T \in \mathbb{R}_{++}$, the diffusion coefficient is $\sigma(Z_t, t \in [0, T])$ -measurable. In fact, we have $\rho\rho^\top = \left(\int_0^T Y_s ds\right)^{-1} \langle Z \rangle_T$ and for all $i, j \in \{1, \dots, d\}$, the bracket $(\langle Z \rangle_T)_{i,j} = \langle Z^i, Z^j \rangle_T$ can be easily approximated by $\sum_{\ell=1}^{p_m} (Z_{t_\ell^m}^i - Z_{t_{\ell-1}^m}^i)(Z_{t_\ell^m}^j - Z_{t_{\ell-1}^m}^j)$, where $t_0^m = 0 \leq t_1^m \leq \dots \leq t_{p_m}^m = T$ is a sequence of subdivisions of $[0, T]$ satisfying $\sup_{1 \leq \ell \leq p_m} |t_\ell^m - t_{\ell-1}^m| \xrightarrow{m \rightarrow \infty} 0$.

Note that this convergence holds in probability and as $\rho\rho^\top$ is symmetric and positive definite, the exact expression of ρ can be obtained using the Cholesky decomposition.

5.1 Existence and uniqueness of MLE

Let $T \in \mathbb{R}_{++}$. In this subsection, we formulate a proposition about existence and uniqueness of the MLE τ_T^* of τ based on the continuous observations $(Z_t)_{t \in [0, T]}$ using Lipster and Shiryaev [33].

Proposition 5.1. *Let $a \in \mathbb{R}_{++}$, $b \in \mathbb{R}$, $m, \kappa \in \mathbb{R}^n$ and $\theta \in \mathcal{M}_n$. Let $(Z_t)_{t \in \mathbb{R}_+}$ be the unique strong solution of the SDE (3) with initial random values $Z_0 = (Y_0, X_0)^\top$ independent of $(B_t)_{t \in \mathbb{R}_+}$ satisfying $\mathbb{P}(Y_0 \in \mathbb{R}_{++}) = 1$. Then, for each $T \in \mathbb{R}_{++}$ there exists a unique MLE of τ almost surely having the form*

$$\tau_T^* = \left(\int_0^T \Lambda(Z_s)^\top S(Z_s)^{-1} \Lambda(Z_s) ds \right)^{-1} \int_0^T \Lambda(Z_s)^\top S(Z_s)^{-1} dZ_s, \quad (32)$$

where, for all $s \in [0, T]$, $S(Z_s) = Z_s^1 \rho \rho^\top$.

Proof. Let $(\tilde{Z}_t)_{t \in \mathbb{R}_+}$ solution to $d\tilde{Z}_t = \sqrt{\tilde{Z}_t^1} \rho dB_t$, for all $t \in \mathbb{R}_+$, with initial random value \tilde{Z}_0 satisfying $\mathbb{P}(\tilde{Z}_0 = Z_0) = 1$. Hence, using the relation (7.138) in Lipster and Shiryaev [33, page 297] applied on the processes $(Z_t)_{t \in [0, T]}$ and $(\tilde{Z}_t)_{t \in [0, T]}$, we deduce the associated likelihood ratio $L(\tau, (Z_t)_{t \in [0, T]}) = \frac{d\mathbb{P}_Z}{d\mathbb{P}_{\tilde{Z}}}((Z_t)_{t \in [0, T]})$. Taking its logarithm, we get the log-likelihood ratio given by

$$\ln L(\tau, (Z_t)_{t \in [0, T]}) = \int_0^T \tau^\top \Lambda(Z_s)^\top S(Z_s)^{-1} dZ_s - \frac{1}{2} \int_0^T \tau^\top \Lambda(Z_s)^\top S(Z_s)^{-1} \Lambda(Z_s) \tau ds. \quad (33)$$

Since the MLE is the vector τ_T^* that maximizes the likelihood L , we compute the gradient of $\ln L$,

$$\nabla_\tau \ln L(\tau, (Z_t)_{t \in [0, T]}) = \int_0^T \Lambda(Z_s)^\top S(Z_s)^{-1} dZ_s - \int_0^T \Lambda(Z_s)^\top S(Z_s)^{-1} \Lambda(Z_s) ds \tau$$

and its Hessian matrix

$$H_\tau \ln L(\tau, (Z_t)_{t \in [0, T]}) = - \int_0^T \Lambda(Z_s)^\top S(Z_s)^{-1} \Lambda(Z_s) ds.$$

As the later matrix is negative definite, we have on one hand, the MLE of τ is the unique zero of $\nabla_\tau \ln L(\tau, (Z_t)_{t \in [0, T]})$ and on the other hand, we deduce that the matrix $\int_0^T \Lambda(Z_s)^\top S(Z_s)^{-1} \Lambda(Z_s) ds$ is invertible which completes the proof. \square

5.2 Consistency of the MLE: subcritical case

Proposition 5.2. *Let $a > \frac{\sigma_1^2}{2}$, $b \in \mathbb{R}_{++}$, $m, \kappa \in \mathbb{R}^n$ and $\theta \in \mathcal{M}_n$ a diagonalizable positive definite matrix. Let $(Z_t)_{t \in \mathbb{R}_+}$ be the unique strong solution of the SDE (3) with initial random values $Z_0 = (Y_0, X_0)^\top$ independent of $(B_t)_{t \in \mathbb{R}_+}$ satisfying $\mathbb{P}(Y_0 \in \mathbb{R}_{++}) = 1$. Then the MLE τ_T^* of τ given by the relation (32) is strongly consistent, i.e. $\mathbb{P}\left(\lim_{T \rightarrow \infty} \tau_T^* = \tau\right) = 1$.*

Proof. By the help of the SDE (3) associated to the process $(Z_t)_{t \in [0, T]}$, we deduce that the drift parameter vector is written as follows

$$\tau = \left(\int_0^T \Lambda(Z_s)^\top S^{-1}(Z_s) \Lambda(Z_s) ds \right)^{-1} \left(\int_0^T \Lambda(Z_s)^\top S^{-1}(Z_s) dZ_s - \int_0^T \sqrt{Z_s^1} \Lambda(Z_s)^\top S^{-1}(Z_s) \rho dB_s \right).$$

Hence, the error term $\tau_T^* - \tau$ has the form $\langle M \rangle_T^{-1} M_T$, where $(M_t)_{t \in [0, T]}$ is a Brownian martingale defined, for all $t \in [0, T]$, by $M_t = \int_0^t \frac{1}{\sqrt{Z_s^1}} (\rho^{-1} \Lambda(Z_s))^\top dB_s$, with the quadratic variation $\langle M \rangle_t = \int_0^t \frac{1}{Z_s^1} (\rho^{-1} \Lambda(Z_s))^\top \rho^{-1} \Lambda(Z_s) ds$. By simple matrix calculation tools, we are able to explicit the expression of each component of $M_T = (M_T^1, \dots, M_T^{d^2+1})^\top$, thus we obtain

$$M_T^1 = \int_0^T \frac{1}{\sqrt{Z_s^1}} \sum_{i=1}^d (\rho^{-1})_{i,1} dB_s^i, \quad M_T^2 = - \int_0^T \sqrt{Z_s^1} \sum_{i=1}^d (\rho^{-1})_{i,1} dB_s^i$$

and for all $k \in \{0, \dots, d-2\}$ and $j \in \{0, \dots, d\}$, we get $M_T^{3+j+k(d+1)} = - \int_0^T \frac{1}{\sqrt{Z_s^1}} Z_s^j \sum_{i=k+2}^d (\rho^{-1})_{i,k+2} dB_s^i$,

with taking $Z_s^0 = -1$. We can write the error

$$\tau_T^* - \tau = (D_T \langle M \rangle_T)^{-1} (D_T M_T),$$

where D_T is a diagonal matrix containing, up to a constant, the inverses of the brackets of M_T^i , $i \in \{1, \dots, d^2+1\}$, namely, $D_T = \text{diag}(V_T)$ where $V_T = (V_T^1, \dots, V_T^{d^2+1})^\top$ is the random vector defined as follows

$$V_T^1 = \left(\int_0^T \frac{1}{Z_s^1} ds \right)^{-1}, \quad V_T^2 = \left(\int_0^T Z_s^1 ds \right)^{-1}$$

and for all $k \in \{0, \dots, d-2\}$ and $i \in \{0, \dots, d\}$, $V_T^{3+i+k(d+1)} = \left(\int_0^T \frac{(Z_s^i)^2}{Z_s^1} ds \right)^{-1}$. Now at first, as $a > \frac{\sigma_1^2}{2}$,

we have $\mathbb{E}\left(\frac{1}{Z_\infty^1}\right) = \frac{2b}{2a - \sigma_1^2} < \infty$, $\mathbb{E}(Z_\infty^1) = \frac{a}{b} < \infty$ and $\mathbb{E}\left(\frac{(Z_\infty^i)^2}{Z_\infty^1}\right) \leq \mathbb{E}((Z_\infty^i)^{2\alpha})^{\frac{1}{\alpha}} \mathbb{E}\left(\frac{1}{(Z_\infty^1)^\beta}\right)^{\frac{1}{\beta}}$ which is finite for all $\alpha \in \mathbb{R}_{++}$ and $\beta \in]0, 2\alpha[$ such that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, thanks to [6, Proposition 1] for the

negative moment of Z_∞^1 and to the inequality of Burkholder Davis Gundy applied on the martingale part of the expression (5) of X_t . Secondly, we deduce using Theorem 4.1 that

$$\begin{aligned} \frac{1}{T} \int_0^T \frac{1}{Z_s^1} ds &\xrightarrow{a.s.} \frac{2b}{2a - \sigma_1^2}, \quad \text{as } T \rightarrow \infty, \\ \frac{1}{T} \int_0^T Z_s^1 ds &\xrightarrow{a.s.} \frac{a}{b}, \quad \text{as } T \rightarrow \infty, \\ \frac{1}{T} \int_0^T \frac{(Z_s^i)^2}{Z_s^1} ds &\xrightarrow{a.s.} \mathbb{E} \left(\frac{(Z_\infty^i)^2}{Z_\infty^1} \right), \quad \text{as } T \rightarrow \infty, \text{ for all } i \in \{2, \dots, d\}. \end{aligned} \quad (34)$$

Consequently, by Theorem 1 in the appendix, we get $D_T M_T \xrightarrow{a.s.} \mathbf{0}_{d^2+1}$ as T tends to infinity. Hence it remains to prove that $D_T \langle M \rangle_T = (T D_T) \left(\frac{1}{T} \langle M \rangle_T \right)$ converges almost surely to an invertible limit matrix, as T tends to infinity. On one hand, we have $T D_T$ converges by the relation (34) and its limit is invertible, on the other hand, up to a constant, the components of $\langle M \rangle_T$ have one of the following terms: $T, \int_0^T \frac{1}{Z_s^1} ds, \int_0^T Z_s^1 ds, \int_0^T \frac{(Z_s^i)^2}{Z_s^1} ds, i \in \{2, \dots, d\}, \int_0^T \frac{Z_s^i}{Z_s^1} ds, i \in \{2, \dots, d\}$ and $\int_0^T \frac{Z_s^i Z_s^j}{Z_s^1} ds, i, j \in \{2, \dots, d\}$ with $i \neq j$. Hence, similarly as above, since $\mathbb{E} \left(\frac{Z_\infty^i}{Z_\infty^1} \right) \leq \mathbb{E} \left((Z_\infty^i)^\alpha \right)^{\frac{1}{\alpha}} \mathbb{E} \left(\frac{1}{(Z_\infty^1)^\beta} \right)^{\frac{1}{\beta}}$ and $\mathbb{E} \left(\frac{Z_\infty^i Z_\infty^j}{Z_\infty^1} \right) \leq \mathbb{E} \left((Z_\infty^i)^\alpha \right)^{\frac{1}{\alpha}} \mathbb{E} \left((Z_\infty^j)^\gamma \right)^{\frac{1}{\gamma}} \mathbb{E} \left(\frac{1}{(Z_\infty^1)^\beta} \right)^{\frac{1}{\beta}}$ which are finite for all $\alpha, \gamma \in \mathbb{R}_{++}$ and $\beta \in]0, 2a[$ such that $\frac{1}{\alpha} + \frac{1}{\gamma} + \frac{1}{\beta} = 1$, we deduce by Theorem 4.1 that

$$\begin{aligned} \frac{1}{T} \int_0^T \frac{Z_s^i}{Z_s^1} ds &\xrightarrow{a.s.} \mathbb{E} \left(\frac{Z_\infty^i}{Z_\infty^1} \right), \quad \text{as } T \rightarrow \infty, \text{ for all } i \in \{2, \dots, d\}, \\ \frac{1}{T} \int_0^T \frac{Z_s^i Z_s^j}{Z_s^1} ds &\xrightarrow{a.s.} \mathbb{E} \left(\frac{Z_\infty^i Z_\infty^j}{Z_\infty^1} \right), \quad \text{as } T \rightarrow \infty, \text{ for all } i \neq j \in \{2, \dots, d\}. \end{aligned} \quad (35)$$

By relations (34) and (35), we conclude that

$$\frac{1}{T} \langle M \rangle_T \xrightarrow{a.s.} \mathbb{E} \left(\frac{1}{Z_\infty^1} \Lambda(Z_\infty)^\top (\rho^{-1})^\top \rho^{-1} \Lambda(Z_\infty) \right), \quad \text{as } T \rightarrow \infty.$$

Next, we need to prove that this limit matrix is invertible. To do that, it is sufficient to prove that it is positive definite, so let $y \in \mathbb{R}^{d^2+1}$ a non-null vector and consider $\mathbb{E} \left(\frac{1}{Z_\infty^1} y^\top \Lambda(Z_\infty)^\top (\rho^{-1})^\top \rho^{-1} \Lambda(Z_\infty) y \right)$. Since, in one hand, the vector $\Lambda(Z_\infty) y$ is a combination of $(1, Z_\infty^1, \dots, Z_\infty^d)$ and on the other hand, $(Z_\infty^1, \dots, Z_\infty^d)^\top$ has a density thanks to the strict stationarity of $(Z_t)_{t \in \mathbb{R}_+}$ and to relation (31), then almost surely it is different to zero. We complete the proof of Theorem 5.2 using the positive definite property of the matrix $(\rho^{-1})^\top \rho^{-1}$. \square

5.3 Asymptotic behavior of the MLE: subcritical case

In the sequel, in order to study the asymptotic behavior of the MLE τ_T^* of τ , we will use the central limit theorem (CLT) for martingales, see Theorem (2) in the appendix.

Theorem 5.1. *Let $a > \frac{\sigma_1^2}{2}$, $b \in \mathbb{R}_{++}$, $m, \kappa \in \mathbb{R}^n$ and $\theta \in \mathcal{M}_n$ a diagonalizable positive definite matrix. Let $(Z_t)_{t \in \mathbb{R}_+}$ be the unique strong solution of the SDE (3) with initial random values $Z_0 = (Y_0, X_0)^\top$ independent of $(B_t)_{t \in \mathbb{R}_+}$ satisfying $\mathbb{P}(Y_0 \in \mathbb{R}_{++}) = 1$. Then the MLE τ_T^* of τ given by (32) is asymptotically normal, namely*

$$\sqrt{T}(\tau_T^* - \tau) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \mathcal{V}), \quad \text{as } T \rightarrow \infty,$$

where \mathcal{V} is the inverse matrix of $\mathbb{E} \left(\frac{1}{Z_\infty^1} \Lambda(Z_\infty)^\top (\rho^{-1})^\top \rho^{-1} \Lambda(Z_\infty) \right)$.

Proof. By the martingale presentation of the error term introduced in the proof of Proposition 5.2 above we have

$$\sqrt{T}(\tau_T^* - \tau) = \left(\frac{1}{T} \langle M \rangle_T \right)^{-1} \frac{1}{\sqrt{T}} M_T.$$

Furthermore, we have established that

$$\left(\frac{1}{T} \langle M \rangle_T \right)^{-1} \xrightarrow{a.s.} \mathcal{V}, \quad \text{as } T \rightarrow \infty. \quad (36)$$

Hence, by the central limit theorem for martingales (see Theorem 2 in the appendix), we deduce that

$$\frac{1}{\sqrt{T}} M_T \xrightarrow{\mathcal{D}} \mathcal{N}(0, \mathcal{V}^{-1}), \quad \text{as } T \rightarrow \infty.$$

This completes the proof. \square

In the following subsection we will treat one subclass of the supercritical case for which the process associated to the $AD(1, n)$ model 3 is non-ergodic and we will call it "a special supercritical case".

5.4 A special supercritical case

In this part, we consider the $d^2 - n$ -dimensional vector $\tilde{\tau} = (b, \kappa_1, \theta_{11}, \dots, \theta_{1n}, \dots, \kappa_n, \theta_{n1}, \dots, \theta_{nn})^\top$ as the unknown drift parameter vector. Next, we formulate a proposition about the unique existence of the MLE $\tilde{\tau}$ of τ based on the continuous observations $(Z_t)_{t \in [0, T]}$, for $T \in \mathbb{R}_{++}$, in a special supercritical case.

Remark 5.1. *In the supercritical case, the MLE of a and m are not even weakly consistent due to the presence of the integral V_T^1 defined in the proof of Proposition 5.2 in its error terms (see [7, Proposition 4]). For that reason, in what follows the parameter $c = (a, m)^\top$ is supposed to be known. However, the MLE of b is strongly consistent and, for all $i, j \in \{1, \dots, n\}$, the MLE of κ_i and θ_{ij} are weakly consistent, see Theorem 5.2. In the remaining cases, in a similar way by assuming that all non-consistent drift parameters to be known, we are able to study statistical estimations related to the $AD(1, n)$ model.*

Next, we present a second version of Proposition 5.1, when $c = (a, m)^\top$ is supposed to be known.

Proposition 5.3. *Let $c = (a, m)^\top \in \mathbb{R}_{++} \times \mathbb{R}^n$ a known parameter vector, $b \in \mathbb{R}$, $\kappa \in \mathbb{R}^n$ and $\theta \in \mathcal{M}_n$. Let $(Z_t)_{t \in \mathbb{R}_+}$ be the unique strong solution of the SDE (3) with initial random values $Z_0 = (Y_0, X_0)^\top$ independent of $(B_t)_{t \in \mathbb{R}_+}$ satisfying $\mathbb{P}(Y_0 \in \mathbb{R}_{++}) = 1$. Then, for each $T \in \mathbb{R}_{++}$ there exists a unique MLE of τ almost surely having the form*

$$\tilde{\tau}_T^* = \left(\int_0^T \tilde{\Lambda}(Z_s)^\top S(Z_s)^{-1} \tilde{\Lambda}(Z_s) \, ds \right)^{-1} \left(- \int_0^T \tilde{\Lambda}(Z_s)^\top S(Z_s)^{-1} dZ_s + \int_0^T \tilde{\Lambda}(Z_s)^\top S(Z_s)^{-1} c \, ds \right),$$

where, for all $s \in [0, T]$, $S(Z_s) = Z_s^1 \rho \rho^\top$ and $\tilde{\Lambda}(Z_s) = \begin{pmatrix} Z_s^1 & \mathbf{0}_{n(n+1)}^\top \\ \mathbf{0}_n & I_n \otimes Z_s^\top \end{pmatrix}$.

Proof. Let $(\tilde{Z}_t)_{t \in \mathbb{R}_+}$ solution to $d\tilde{Z}_t = \sqrt{\tilde{Z}_t^1} \rho \, dB_t$, for all $t \in \mathbb{R}_+$, with initial random value \tilde{Z}_0 satisfying $\mathbb{P}(\tilde{Z}_0 = Z_0) = 1$. Hence, using the relation (7.138) in Lipster and Shiryaev [33, page 297] applied on the processes $(Z_t)_{t \in [0, T]}$ and $(\tilde{Z}_t)_{t \in [0, T]}$, we deduce the associated log-likelihood ratio given by

$$\ln L(\tilde{\tau}, (Z_t)_{t \in [0, T]}) = \int_0^T (c - \tilde{\Lambda}(Z_s) \tilde{\tau})^\top S(Z_s)^{-1} dZ_s - \frac{1}{2} \int_0^T (c - \tilde{\Lambda}(Z_s) \tilde{\tau})^\top S(Z_s)^{-1} (c - \tilde{\Lambda}(Z_s) \tilde{\tau}) \, ds.$$

with gradient vector

$$\nabla_{\tilde{\tau}} \ln L(\tilde{\tau}, (Z_t)_{t \in [0, T]}) = - \int_0^T \tilde{\Lambda}(Z_s)^\top S(Z_s)^{-1} dZ_s + \int_0^T \tilde{\Lambda}(Z_s)^\top S(Z_s)^{-1} (c - \tilde{\Lambda}(Z_s) \tilde{\tau}) \, ds.$$

Since the associated Hessian matrix $H_{\tilde{\tau}}L(\tilde{\tau}) = -\int_0^T \tilde{\Lambda}(Z_s)^\top S(Z_s)^{-1} \tilde{\Lambda}(Z_s) ds$ is negative definite, we deduce that the MLE $\tilde{\tau}_T^*$ is the unique solution of $\nabla_{\tilde{\tau}}L(\tilde{\tau}) = \mathbf{0}_d$. this completes the proof. \square

Remark 5.2. *Similarly in spirit to Proposition (5.2) and to Theorem (5.1), the asymptotic properties of the MLE $\tilde{\tau}_T^*$ related to the drift parameter vector $\tilde{\tau}$ are can be proved in the subcritical case.*

In order to study the asymptotic behavior of the MLE $\tilde{\tau}_T^*$, we need the following integral version of Kronecker Lemma, see [31, Lemma B.3.2]:

Lemma 5.1. *Let $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a measurable function and for all $T \in \mathbb{R}_+$, $G(T) = \int_0^T g(t) dt$. If $\lim_{T \rightarrow \infty} G(T) = \infty$, then for every bounded and measurable function $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ for which the limit $f(\infty) := \lim_{t \rightarrow \infty} f(t)$ exists, we have*

$$\lim_{T \rightarrow \infty} \frac{1}{G(T)} \int_0^T g(t)f(t) dt = f(\infty).$$

The next theorem states asymptotic behavior of the MLE $\tilde{\tau}_T^*$ in a special supercritical case.

Theorem 5.2. *Let $c = (a, m)^\top \in \mathbb{R}_{++} \times \mathbb{R}^n$ a known parameter vector, $\kappa \in \mathbb{R}^n$, $\theta \in \mathcal{M}_n$ a diagonalizable negative definite matrix and $b \in (\lambda_{\max}(\theta), 0)$. Let $(Z_t)_{t \in \mathbb{R}_+}$ be the unique strong solution of the SDE (3) with initial random values $Z_0 = (Y_0, X_0)^\top$ independent of $(B_t)_{t \in \mathbb{R}_+}$ satisfying $\mathbb{P}(Y_0 \in \mathbb{R}_{++}) = 1$. Let P a modal matrix transforming θ to the diagonal matrix D . Suppose that $\text{diag}(P^{-1}m)P^{-1}\kappa \in \mathbb{R}_-^n$. Then*

$$Q_T(\tilde{\tau}_T^* - \tilde{\tau}) \xrightarrow{\mathcal{D}} (\eta\eta^\top)^{-1}\eta\xi, \quad \text{as } T \rightarrow \infty, \quad (37)$$

where $Q_T = \begin{bmatrix} e^{-\frac{b}{2}T} & \mathbf{0}_{n(n+1)}^\top \\ \mathbf{0}_{n(n+1)} & I_n \otimes \tilde{Q}_T \end{bmatrix}$ with $\tilde{Q}_T = \text{diag}\left(e^{-\frac{b}{2}T}, e^{\frac{(b-2\lambda_{\min}(\theta))}{2}T}, \dots, e^{\frac{(b-2\lambda_{\min}(\theta))}{2}T}\right)$, η is defined

through the almost sure limit $Q_T^{-1}\langle \tilde{M} \rangle_T Q_T^{-1} \xrightarrow{\text{a.s.}} \eta\eta^\top$, as $T \rightarrow \infty$, with $\langle \tilde{M} \rangle_T = \int_0^T \tilde{\Lambda}(Z_s)^\top S(Z_s)^{-1} \tilde{\Lambda}(Z_s) ds$ and ξ is a $d^2 - n$ -dimensional standard normally distributed random vector independent of η .

Proof. By writing the SDE associated to the process Z as follows

$$dZ_t = (c - \tilde{\Lambda}(Z_t)\tilde{\tau}) dt + \sqrt{Z_t^1} \rho dB_t,$$

it easy to check that, for $T \in \mathbb{R}_{++}$, we have

$$\tilde{\tau} = \left(\int_0^T \tilde{\Lambda}(Z_s)^\top S(Z_s)^{-1} \tilde{\Lambda}(Z_s) ds \right)^{-1} \left(- \int_0^T \tilde{\Lambda}(Z_s)^\top S(Z_s)^{-1} dZ_s + \int_0^T \tilde{\Lambda}(Z_s)^\top S(Z_s)^{-1} c ds + \int_0^T \tilde{\Lambda}(Z_s)^\top S(Z_s)^{-1} \sqrt{Z_s^1} \rho dB_s \right)$$

and hence the error term is written as $\tilde{\tau}_T^* - \tilde{\tau} = \langle \tilde{M} \rangle_T^{-1} \tilde{M}_T$, with $\tilde{M}_T = \int_0^T \frac{1}{\sqrt{Z_s^1}} \tilde{\Lambda}(Z_s)^\top (\rho^{-1})^\top dB_s$ and

$\langle \tilde{M} \rangle_T = \int_0^T \tilde{\Lambda}(Z_s)^\top S(Z_s)^{-1} \tilde{\Lambda}(Z_s) ds$. Moreover, the components of \tilde{M}_T can be written explicitly, for all $k \in \{1, \dots, d\}$ and $j \in \{1, \dots, n\}$, as follows

$$\tilde{M}_T^1 = \int_0^T \sqrt{Z_s^1} \sum_{i=1}^d (\rho^{-1})_{i,1} dB_s^j \quad \text{and} \quad \tilde{M}_T^{1+k+(j-1)d} = \int_0^T \frac{Z_s^k}{\sqrt{Z_s^1}} \sum_{i=1}^d (\rho^{-1})_{i,j} dB_s^j.$$

Consequently, the components of $\langle \tilde{M} \rangle_T$ are taking, to within a constant, the integral form $\int_0^T \frac{Z_s^i Z_s^j}{Z_s^1} ds$, where $i, j \in \{1, \dots, d\}$. Next, since $b \in (\lambda_{\max}(\theta), 0)$ and $\text{diag}(P^{-1}m)P^{-1}\kappa \in \mathbb{R}^n$, then thanks to Lemma 4.4 and Lemma 7.2 of [9], we obtain respectively the two following results: **first result**: there exists a random variable G_1 such that $e^{bt} Z_t^1 \xrightarrow{a.s.} G_1$, as $t \rightarrow \infty$ and **second result**: for all $i \in \{1, \dots, n\}$, there exists a random variable \tilde{G}_i such that $e^{\lambda_i(\theta)t} \tilde{X}_t^i \xrightarrow{a.s.} \tilde{G}_i$, as $t \rightarrow \infty$, where $(\tilde{X}_t)_{t \in \mathbb{R}_+}$ is the unique strong solution of the following SDE

$$d\tilde{X}_t = (\tilde{m} - \tilde{\kappa}Y_t - D\tilde{X}_t) dt + \sqrt{\tilde{Y}_t} \tilde{\rho} dB_t,$$

with initial value $\tilde{X}_0 = P^{-1}X_0$, where $\tilde{m} = P^{-1}m$, $\tilde{\kappa} = P^{-1}\kappa$ and $\tilde{\rho} = P^{-1}\rho$.

Now, using the fact that $X_t = P\tilde{X}_t$, for all $t \in \mathbb{R}_+$, we get for all $i \in \{1, \dots, n\}$, $X_t^i = \sum_{k=1}^n P_{i,k}(\theta) \tilde{X}_t^k$.

Since $e^{\lambda_k(\theta)t} \tilde{X}_t^k \xrightarrow{a.s.} \tilde{G}_k$, as $t \rightarrow \infty$, for all $i \in \{1, \dots, n\}$, we obtain

$$e^{\lambda_{\min}(\theta)t} X_t^i = \sum_{k=1}^n P_{i,k}(\theta) e^{\lambda_{\min}(\theta)t} \tilde{X}_t^k \xrightarrow{a.s.} P_{i,i_0}(\theta) \tilde{G}_{i_0} =: G_i(\theta),$$

where $i_0 \in \{1, \dots, n\}$ is the index associated to the smallest eigenvalue $\lambda_{\min}(\theta)$. Hence, thanks to Lemma 5.1, for all $i, j \in \{1, \dots, n\}$, we obtain

$$e^{bT} \int_0^T Z_s^1 ds = -\frac{1}{b}(1 - e^{bT}) \frac{\int_0^T e^{-bs} e^{bs} Z_s^1 ds}{\int_0^T e^{-bs} ds} \xrightarrow{a.s.} -\frac{G_1}{b}, \quad \text{as } T \rightarrow \infty,$$

$$e^{\lambda_{\min}(\theta)T} \int_0^T Z_s^i ds = -\frac{1}{\lambda_{\min}(\theta)}(1 - e^{\lambda_{\min}(\theta)T}) \frac{\int_0^T e^{-\lambda_{\min}(\theta)s} e^{\lambda_{\min}(\theta)s} Z_s^i ds}{\int_0^T e^{-\lambda_{\min}(\theta)s} ds} \xrightarrow{a.s.} \frac{-G_i(\theta)}{\lambda_{\min}(\theta)}, \quad \text{as } T \rightarrow \infty,$$

and

$$e^{-(b-2\lambda_{\min}(\theta))T} \int_0^T \frac{Z_s^i Z_s^j}{Z_s^1} ds = \frac{1}{b-2\lambda_{\min}(\theta)} \left(1 - e^{-(b-2\lambda_{\min}(\theta))T}\right) \frac{\int_0^T e^{(b-2\lambda_{\min}(\theta))s} \frac{e^{\lambda_{\min}(\theta)s} Z_s^i e^{\lambda_{\min}(\theta)s} Z_s^j}{e^{bs} Z_s^1} ds}{\int_0^T e^{(b-2\lambda_{\min}(\theta))s} ds} \xrightarrow{a.s.} \frac{G_i(\theta)G_j(\theta)}{(b-2\lambda_{\min}(\theta))G_1}, \quad \text{as } T \rightarrow \infty,$$

Consequently, the expression of Q_T guaranteeing the almost sure convergence of $Q_T^{-1} \langle \tilde{M} \rangle_T Q_T^{-1}$ is deduced and the almost sure limit matrix $\eta\eta^\top$ of $Q_T^{-1} \langle \tilde{M} \rangle_T Q_T^{-1}$ does exist. Moreover, By Theorem 2, we get $Q_T^{-1} \tilde{M}_T \xrightarrow{\mathcal{D}} \eta\xi$, as $T \rightarrow \infty$, where ξ is a $d^2 - n$ -dimensional standard normally distributed random vector independent of η . Then, by Slutsky's Lemma, we have

$$(Q_T^{-1} \tilde{M}_T, Q_T^{-1} \langle \tilde{M} \rangle_T Q_T^{-1}) \xrightarrow{\mathcal{D}} (\eta\xi, \eta\eta^\top), \quad \text{as } T \rightarrow \infty.$$

Finally, we obtain $Q_T(\tilde{\tau}_T^* - \tilde{\tau}) = (Q_T^{-1} \langle \tilde{M} \rangle_T Q_T^{-1})^{-1} (Q_T^{-1} \tilde{M}_T) \xrightarrow{\mathcal{D}} (\eta\eta^\top)^{-1} \eta\xi$, as $T \rightarrow \infty$. \square

Appendix

In what follows we recall some limit theorems for continuous local martingales used in the study of the asymptotic behavior of the MLE of τ . First we recall a strong law of large numbers for continuous local martingales, see Liptser and Shiryaev [33].

Theorem 1. (Liptser and Shiryaev (2001)) Let $(\Omega, \mathcal{F}, (\mathcal{F})_{t \in \mathbb{R}_+}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. Let $(M_t)_{t \in \mathbb{R}_+}$ be a square-integrable continuous local martingale with respect to the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ such that $\mathbb{P}(M_0 = 0) = 1$. Let $(\xi_t)_{t \in \mathbb{R}_+}$ be a progressively measurable process such that

$$\mathbb{P}\left(\int_0^t \xi_u^2 d\langle M \rangle_u < \infty\right) = 1, \quad t \in \mathbb{R}_+,$$

and

$$\int_0^t \xi_u^2 d\langle M \rangle_u \xrightarrow{a.s.} \infty, \quad \text{as } t \rightarrow \infty,$$

where $(\langle M \rangle_t)_{t \in \mathbb{R}_+}$ denotes the quadratic variation process of M . Then

$$\frac{\int_0^t \xi_u dM_u}{\int_0^t \xi_u^2 d\langle M \rangle_u} \xrightarrow{a.s.} 0, \quad \text{as } t \rightarrow \infty.$$

If $(M_t)_{t \in \mathbb{R}_+}$ is a standard Wiener process, the progressive measurability of $(\xi_t)_{t \in \mathbb{R}_+}$ can be relaxed to measurability and adaptedness to the filtration $(\mathcal{F})_{t \in \mathbb{R}_+}$.

The second theorem is about the asymptotic behaviour of continuous multivariate local martingales, see [42, Theorem 4.1].

Theorem 2. For $p \in \mathbb{N} \setminus \{0\}$, let $M = (M_t)_{t \in \mathbb{R}_+}$ be a p -dimensional square-integrable continuous local martingale with respect to the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ such that $\mathbb{P}(M_0 = 0) = 1$. Suppose that there exists a function $Q : [t_0, \infty) \rightarrow \mathcal{M}_p$ with some $t_0 \in \mathbb{R}_+$ such that $Q(t)$ is an invertible (non-random) matrix for all $t \in \mathbb{R}_+$, $\lim_{t \rightarrow \infty} \|Q(t)\| = 0$ and

$$Q(t)\langle M \rangle_t Q(t)^\top \xrightarrow{\mathbb{P}} \eta\eta^\top, \quad \text{as } t \rightarrow \infty,$$

where η is a random matrix in \mathcal{M}_p . Then, for each random matrix $A \in \mathcal{M}_{k,l}$, $k, l \in \mathbb{N} \setminus \{0\}$, defined on $(\Omega, \mathcal{F}, (\mathcal{F})_{t \in \mathbb{R}_+}, \mathbb{P})$, we have

$$(Q(t)M_t, A) \xrightarrow{\mathcal{D}} (\eta Z, A), \quad \text{as } t \rightarrow \infty,$$

where Z is a p -dimensional standard normally distributed random vector independent of (η, A) .

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