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# Weak Uniqueness for a PDE with Boundary Condition

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## Abstract

In a previous work [6], it has been proved that the joint law of a diffusion and the running supremum of its first component is absolutely continuous and that its density satisfies a weak partial differential equation with boundary condition. In this work, we prove uniqueness of the solution of that PDE.

## 1 Introduction and statement of the PDE

In finance, a barrier option is a type of option whose pay-off depends on whether or not the underlying asset exceeds a certain price. An asset is typically modelled using a stochastic differential equation, and in theory, the barrier can be any curve. Yet, theoretical challenges already arise when studying the joint distribution of  $(V_t)_{t \geq 0} = (M_t, X_t)_{t \geq 0}$ , where:

$$\begin{aligned}dX_t &= B(X_t)dt + \sigma(X_t)dW_t, \quad X_0 \sim f_0, \\M_t &= \sup_{s \leq t} X_s^1,\end{aligned}$$

$X_0$  being a random variable, independent of  $W$ , with density  $f_0$ . Many papers, dating as far back as in 1987 [7], have been devoted to understanding the distribution of  $(V_t)_{t \geq 0}$ . However, in early works, the emphasis was on characterizing the joint distribution, in order to deduce theoretical properties (for instance regularity of hitting times or local times). From an applicative point of view, even though these results are very informative, they offer little to no practicality for real life purposes like pricing or hedging.

For instance in [3], authors default to giving (sharp) upper and lower bounds. Therefore, continuous effort has been deployed to better understand the joint distribution. We can notably cite [2] that provides an approximation of the distribution of  $(V_t)_{t \geq 0}$ , in order to improve computation time in Montecarlo schemes. Another result we can quote for context is one obtained by Hayashi and Kohatsu-Higa [10], establishing that the distribution of  $(V_t)_{t \geq 0}$  is absolutely continuous. Such result is expected, since in the simplest case of the Brownian motion and its running supremum  $(W_t^*, W_t)_{t \geq 0}$ , where  $W_t^* = \sup_{s \leq t} W_s$ , an explicit expression

for that density has been given since [12] (see also [11] for a more recent reference). Nevertheless, for the Brownian motion, arguments are quite different, as they rely on the reflection principle which is not true for general diffusions any more.

Still, knowing that the law of  $V_t$  for  $t > 0$ , is absolutely continuous, with density  $p_V(\cdot; t)$   $t > 0$ ; one can naturally ask how far is the density of  $(W_t^*, W_t)_{t \geq 0}$  from the actual density. In other words, one can look for expansion for the density of  $(V_t)_{t \geq 0}$ . In [5], and subsequently in [6], such program has been implemented. In arbitrary dimension and when the diffusion coefficient  $\sigma = I_d$ , the authors use a Girsanov transformation eliminating the drift and relate the law of  $(V_t)_{t \geq 0}$  to the law of  $(W_t^*, W_t)_{t \geq 0}$ , through a Malliavin integration by parts. In dimension 1, a Lamperti transform can be done prior to Girsanov to handle a diffusion coefficient. However, the general case is still for now out of reach with this method. The main result of [5] is the absolute continuity of distribution of  $(V_t)_{t \geq 0}$ , and the arguments have been further expanded upon in [6], where the authors obtain a weak PDE for the law of  $(V_t)_{t \geq 0}$ :

$$(1.1) \quad \int_{\mathcal{T}} \Phi(m, x) p(m, x; t) dm dx = \int_{\mathbb{R}^d} \Phi(m, m, \tilde{x}) f_0(m, \tilde{x}) dm d\tilde{x} + \int_0^t \int_{\mathcal{T}} p(m, x; s) \mathcal{L} \Phi(m, x) dm dx ds + \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} \partial_m \Phi(m, m, \tilde{x}) p(m, m, \tilde{x}; s) dm d\tilde{x} ds,$$

where  $\Phi$  is a test function belonging to a suitable class of functions,  $\mathcal{T}$  is the subset of  $\mathbb{R} \times \mathbb{R}^d$ :

$$\mathcal{T} := \{(m, x) \in \mathbb{R} \times \mathbb{R}^d ; m \geq x^1\},$$

and the operator  $\mathcal{L}$  is the generator of  $(X_t)_{t \geq 0}$  and acts on the variable  $x$  as follows:

$$\mathcal{L}f(x) = B(x) \nabla f(x) + \frac{1}{2} \Delta f(x).$$

In what follows, we extend the operator  $\mathcal{L}$  to functions  $\Phi$  in  $C^2(\mathbb{R}^{d+1}, \mathbb{R})$  as

$$\mathcal{L}(\Phi)(m, x) = B^i(x) \partial_{x_i} \Phi(m, x) + \frac{1}{2} \partial_{x_i x_i}^2 \Phi(m, x)$$

where we use the Einstein's convention.

In this paper, we are interested in proving that PDE (1.1) has a unique solution in a suitable functional space  $\mathcal{X}$  defined below. Our proof relies on a variation of constant method see [1].

**Remark 1.1 (Notations)** *Throughout this paper we will use the notation  $x = (x^1, \tilde{x})$  when no ambiguity is possible, in order to single out the behavior on the first component. Similarly, we write  $dx = dx^1 d\tilde{x}$  to shorten notations.*

*Let us denote  $C_c^2(\mathbb{R}^d)$  the functions  $C^2$  class with compact support and  $C_b^2(\mathbb{R}^d)$  the bounded functions  $C^2$  class with bounded derivatives.*

*Finally,  $p_V(t, \cdot)$  denotes the density of the probability law of process  $V_t = (M_t, X_t)$ .*

**Definition 1.1** Let  $T > 0$ . Define  $\mathcal{X}$  to be the space of the real functions on  $]0, T] \times \mathcal{T}$  satisfying the following items:

- (a)  $\sup_{t \in ]0, T]} \int_{\mathbb{R}^d} \left[ \int_{\mathbb{R}} |p(m, x; t)| dm \right]^2 dx < \infty$ .
- (b) The application  $(m, \tilde{x}; s) \mapsto p(m, m, \tilde{x}; s) \in L^1(]0, T]; L^2(\mathbb{R}^d))$
- (c) For all  $t \in ]0, T]$ ,  $\sup_{u > 0} p(m, m - u, \tilde{x}; t) \in L^1(\mathbb{R}^d)$ ; and for almost surely all  $(m, \tilde{x}) \in \mathbb{R}^d$ ,  $x^1 \mapsto p(m, x^1, \tilde{x}; t)$  is continuous on  $] - \infty, m[$  and  $\lim_{u \rightarrow 0^+} p(m, m - u, \tilde{x}; t)$  exists and is denoted by  $p(m, m, \tilde{x}; t)$ .

**Theorem 1.2** Assume that  $B \in C_b^1(\mathbb{R}^d, \mathbb{R}^d)$ . Let  $T > 0$ , and  $f_0 \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$  with positive values such that  $\int_{\mathbb{R}^d} f_0(x) dx = 1$ . Then,  $p_V$  belongs to  $\mathcal{X}$  and is the unique solution in  $\mathcal{X}$  of (1.1).

It should be emphasized that this PDE is set in dimension  $d + 1$ , but it is degenerate, in the sense that there are no derivative related to the component  $m$ . Moreover, the presence of a boundary term takes us away from the classical equations investigated in textbooks such as Evans [8].

This paper is organized as follow. In Section 2, we prove that the solution of (1.1) with fixed initial condition is unique in a certain class of function  $\mathcal{X}$ . To do so, we rely on a Theorem by Ball [1], whose assumptions are checked in Section 3. Then Section 4 proves that  $p_V$  is an element of set  $\mathcal{X}$ . Finally an appendix with some tools is provided.

## 2 Uniqueness for weak solution of PDE in the set $\mathcal{X}$

The aim of this section is the proof of the following theorem.

**Theorem 2.1** Let  $T > 0$  and  $B \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ . In the set  $\mathcal{X}$  given in Definition 1.1, the PDE (1.1) with initial condition  $f_0 \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$  admits at most one solution.

In this section, we work up the tools allowing us to obtain uniqueness to the solution of the PDE (1.1). Naturally, we consider  $p_1$  and  $p_2$ , two solutions in  $\mathcal{X}$  to (1.1) with the same initial condition, and show that their difference is zero. Having the same initial condition, for  $q = p_1 - p_2$ , we derive the following equation:

$$(2.1) \quad \int_{\mathcal{T}} \Phi(m, x) q(m, x; t) dm dx = \int_0^t \int_{\mathcal{T}} q(m, x; s) \mathcal{L} \Phi(m, x) dm dx ds + \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} \partial_m \Phi(m, m, \tilde{x}) q(m, m, \tilde{x}; s) dm d\tilde{x} ds$$

for all test function  $\Phi \in C_c^2(\mathbb{R}^{d+1})$ . We can chose in particular  $\Phi(m, x) = H(m)F(x)$ , for  $(m, x) \in \mathcal{T}$ , where  $H \in C_c^1(\mathbb{R})$  and  $F \in C_c^2(\mathbb{R}^d)$ . And since there is no second derivative with respect to  $m$ , we introduce

$$(2.2) \quad q_H(x; t) = \int_{m \geq x^1} \mathbf{1}_{\{x^1 \leq m\}} H(m) q(m, x; t) dm, \quad x \in \mathbb{R}^d, \quad t \in [0, T]$$

where  $H \in C_b^1(\mathbb{R})$ .

**Remark 2.2** *Since the solutions to the PDE (1.1) have support over  $\mathcal{T}$ , we can omit the indicator  $\{m \geq x^1\}$  in the previous integral, and simply write:  $q_H(x; t) = \int_{\mathbb{R}} H(m) q(m, x; t) dm$ .*

## 2.1 Reduction to a variation of constant

In this section, we show how to reduce the proof of Theorem 2.1 to the application of a result due to Ball [1], which can be understood as a variation of constant method.

We have the following proposition.

**Proposition 2.3** *Let  $q \in \mathcal{X}$  be a solution of (1.1) with null initial condition and  $H \in C_b^2(\mathbb{R}, \mathbb{R})$  with compact support derivative. The function  $q_H(\cdot; \cdot)$  satisfies the equation in a weak sense:*

$$q_H(x; t) = \int_0^t \mathcal{L}^* q_H(x; s) ds + \frac{1}{2} \int_0^t \partial_m H(x_1) q(x_1, x_1, \tilde{x}; s) ds, \quad t \in [0; T]$$

where  $\mathcal{L}^*$  is

$$\mathcal{L}^*(\Phi)(x) = -B(x) \cdot \nabla \Phi(x)(x) - f(x) \operatorname{div} B(x) + \frac{1}{2} \Delta \Phi(x).$$

Moreover, for all  $t \in [0, T]$ ,  $q_H(\cdot, t) \in L^2(\mathbb{R}^d)$ .

**Proof:** Let  $q \in \mathcal{X}$  such that  $q(\cdot, 0) = 0$ . Using Item (a) of Definition 1.1,

$$\sup_{t \in [0, T]} \int_{\mathbb{R}^d} \left[ \int_{\mathbb{R}} |q(m, x; t)| dm \right]^2 dx < \infty,$$

and the fact that  $H$  is bounded

$$\int_{\mathbb{R}^d} \left[ \int_{\mathbb{R}} |H(m)| |q(m, x; t)| dm \right]^2 dx < \infty, \quad \forall t > 0.$$

Then,  $q_H(\cdot; t) \in L^2(\mathbb{R}^d)$ .

• For a while, assume that  $H$  is  $C^2$  with compact support.

We start from Equation (2.1) satisfied by  $q$ , and plug the choice  $\Phi(m, x) = H(m)F(x)$  into (2.1)

with  $F \in C_c^2(\mathbb{R}^d)$ . Since  $H$  and  $F$  are with compact supports, we can use Fubini's theorem to exchange integrals, the left hand side becomes:

$$\begin{aligned} \int_{\mathcal{T}} \Phi(m, x) q(m, x; t) dm dx &= \int_{\mathbb{R}^d} F(x) \left( \int_{m \geq x^1} H(m) q(m, x; t) dm \right) dx \\ &= \int_{\mathbb{R}^d} F(x) q_H(x; t) dx. \end{aligned}$$

Now considering the right hand side of (2.1), we have for the first contribution:

$$\int_0^t \int_{\mathcal{T}} q(m, x; s) \mathcal{L} \Phi(m, x) dm dx ds = \int_0^t \int_{\mathcal{T}} q(m, x; s) \mathcal{L} (H(m)F)(x) dm dx ds.$$

Note that the operator  $\mathcal{L}$  acts on the variable  $x$ , we thus have:

$$\begin{aligned} \int_0^t \int_{\mathcal{T}} q(m, x; s) \mathcal{L} \Phi(m, x) dm dx ds &= \int_0^t \int_{\mathbb{R}^d} \int_{m \geq x^1} q(m, x; s) H(m) \mathcal{L} F(x) dm dx ds \\ &= \int_0^t \int_{\mathbb{R}^d} q_H(x; s) \mathcal{L} F(x) dx ds \end{aligned}$$

Finally considering the second term in the right hand side of (2.1), since  $\partial_m \Phi(m, m, \tilde{x}) = \partial_m \Phi(m, x^1, \tilde{x})|_{x^1=m}$ , that is, we first compute the derivative of  $\Phi$  with respect to the first component, and evaluate at  $(m, m, \tilde{x}) \in \mathbb{R} \times \mathbb{R}^d$ , the second term becomes up to the factor  $\frac{1}{2}$ :

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^d} \partial_m \Phi(m, m, \tilde{x}) q(m, m, \tilde{x}; s) dm d\tilde{x} ds &= \int_0^t \int_{\mathbb{R}^d} H'(m) F(m, \tilde{x}) q(m, m, \tilde{x}; s) dm d\tilde{x} ds \\ &= \int_{\mathbb{R}^d} \left( \int_0^t H'(m) q(m, m, \tilde{x}; s) ds \right) F(m, \tilde{x}) dm d\tilde{x} \\ &= \int_0^t \int_{\mathbb{R}^d} H'(m) F(m, \tilde{x}) q(m, m, \tilde{x}; s) dm d\tilde{x} ds. \end{aligned}$$

Thus,

$$\int_{\mathbb{R}^d} F(x) q_H(x; t) dx = \int_0^t \int_{\mathbb{R}^d} \mathcal{L}(F)(x) q_H(x; s) dx ds + \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} H'(x^1) F(x) q(x^1, x; s) dx ds.$$

• Now, we assume that  $H$  is  $C_b^2$  with compact support derivative. The idea is to recover the previous case ( $C^2$  with compact support) via a truncation argument. Consider a  $C^2$  function  $f$  defined on  $[0, 1]$ , with  $f(0) = 0$ ,  $f(1) = 1$  and  $f'(0) = f'(1) = f''(0) = f''(1) = 0$ . For instance,  $f$  can be the polynomial function degree 5:  $f(x) = 6x^5 - 15x^4 + 10x^3$ . Let  $(\xi_n)_{n \in \mathbb{N}}$  be a sequence of  $C_c^2$  functions defined as

$$\xi_n(m) = 1 \quad \forall m \in [-n, n], \quad \xi_n(m) = 0 \quad \forall |m| \geq 2n,$$

$$\xi_n(m) = f\left(\frac{m+2n}{n}\right) \quad \forall m \in (-2n, -n), \quad \xi_n(m) = f\left(\frac{2n-m}{n}\right) \quad \forall m \in (n, 2n).$$

In other words,  $(\xi_n)_{n \in \mathbb{N}}$  are a sequence of  $C_c^2$  functions approximating an indicator function. We let  $H_n = H\xi_n$ , which is now  $C^2$  with compact support.

Using Cauchy Schwartz inequality

$$\|q_H(\cdot; t) - q_{H_n}(\cdot; t)\|_{L^2}^2 \leq \|H\|_\infty^2 \int_{\mathbb{R}^d} \left[ \int_{\mathbb{R}} |1 - \xi_n(m)| |q(m, x; t)| dm \right]^2 dx$$

and

$$\int_0^T \|q_H(\cdot; t) - q_{H_n}(\cdot; t)\|_{L^2}^2 dt \leq \|H\|_\infty^2 \int_0^T \int_{\mathbb{R}^d} \left[ \int_{\mathbb{R}} |1 - \xi_n(m)| |q(m, x; t)| dm \right]^2 dx dt.$$

Then, using Lebesgue dominated theorem  $(q_{H_n}(\cdot; t))_{n \in \mathbb{N}}$  converges to  $q_H$  in  $L^2(\mathbb{R}^d)$  for all  $t \in [0, T]$  and  $(q_{H_n})_{n \in \mathbb{N}}$  converges to  $q_H$  in  $L^1([0, T], dt; L^2(\mathbb{R}^d))$ .

Note that

$$H'_n(m) = H(m)\xi'_n(m) + H'(m)\xi_n(m), \quad \forall m \in \mathbb{R}.$$

Let  $n$  be large enough such that the support of the function  $H'$  is in  $(-n, n)$  so  $\forall m \in [-n, +n]$ ,  $H'_n(m) = H'(m)$ . Moreover, for  $|m| \geq 2n$ ,  $H'_n(m) = 0$  and for  $2n \geq |m| \geq n$ ,  $H'_n(m) = \xi'_n(m)H(m)$  and  $H$  is bounded.

Thus we have :

$$\left| (H'_n(x^1) - H'(x^1))q(x^1, x; s) \right| = \mathbf{1}_{n \leq |x^1| \leq 2n} |\xi'_n(x^1)H(x^1)q(x^1, x; s)|$$

We now use that if  $x^1$  satisfies  $n \leq |x^1| \leq 2n$  then  $H'_n(x^1) = H(x^1)\frac{1}{n}f'(\frac{x+2n}{n})$  where  $f'$  and  $H$  are bounded, since the above integrand satisfies  $|(H'_n(x^1) - H'(x^1))q(x^1, x; s)| \leq \frac{c}{n}|q(x^1, x; s)|$ .

Given Item (b) of Definition 1.1  $\int_0^T \sqrt{\int_{\mathbb{R}^d} q^2(x^1, x; s) dx ds} < \infty$ , hence when  $n$  goes to infinity, we get:

$$\int_0^T \int_{\mathbb{R}^d} \left| (\xi'_n(x^1)H(x^1) - H'(x^1))q(x^1, x; s) \right| dx ds \xrightarrow{n \rightarrow +\infty} 0.$$

We point out that in fact,  $\int_0^T \sqrt{\int_{\mathbb{R}^d} |q(x^1, x; s)| dx ds} < \infty$  would be sufficient.

Thus,  $(H'_n(x^1)q(x^1, x; s))_n$  converges in  $L^1([0, T] \times \mathbb{R}^d)$  toward  $H'(x^1)q(x^1, x; s)$ .

Let  $F$  be a  $C_c^2$  function. From the compact support derivative case we have

$$\int_{\mathbb{R}^d} F(x)q_{H_n}(x; t) dx = \int_0^t \int_{\mathbb{R}^d} \mathcal{L}(F)(x)q_{H_n}(x; s) dx ds + \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} H'_n(x)F(x)q(x^1, x; s) dx ds.$$

Using the fact  $F$  is  $C^2$  with compact support,  $F$  and  $\mathcal{L}(F)$  are bounded with compact support thus in  $L^2(\mathbb{R}^d)$ .

$$\int_{\mathbb{R}^d} F(x)q_H(x; t) dx = \lim_n \int_{\mathbb{R}^d} F(x)q_{H_n}(x; t) dx,$$

$$\int_0^t \int_{\mathbb{R}^d} \mathcal{L}(F)(x)q_H(x; s)dx ds = \lim_n \int_0^t \int_{\mathbb{R}^d} \mathcal{L}(F)(x)q_{H_n}(x; s)dx ds.$$

Moreover, since  $(H'_n(x^1)q(x^1, x; s))_n$  converges in  $L^1([0, T] \times \mathbb{R}^d)$  toward  $H'(x^1)q(x^1, x; s)$ ,

$$\lim_n \int_0^t \int_{\mathbb{R}^d} H'_n(x^1)F(x)q(x^1, x; s)dx^1 ds = \int_0^t \int_{\mathbb{R}^d} H'(x^1)F(x)q(x^1, x; s)dx^1 ds$$

and

$$\int_{\mathbb{R}^d} F(x)q_H(x; t)dx = \int_0^t \int_{\mathbb{R}^d} \mathcal{L}(F)(x)q_H(x; s)dx ds + \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} H'(x^1)F(x)q(x^1, x; s)dx ds.$$

The function  $q_H(x; \cdot)$  satisfies the equation  $dx$  almost surely:

$$q_H(x; t) = \int_0^t \mathcal{L}^* q_H(x; s)ds + \frac{1}{2} \int_0^t H'(x^1)q(x^1, x^1, \tilde{x}; s)ds.$$

where  $\mathcal{L}^*$  is

$$\mathcal{L}^*(\Phi)(x) = -B(x) \cdot \nabla \Phi(x) - f(x) \operatorname{div} B(x) + \frac{1}{2} \Delta \Phi(x)$$

so the proof is concluded. •

**Remark 2.4** *The adjoint operator*

$$\mathcal{L}^* f(x) = -B(x) \cdot \nabla f(x) - f(x) \sum_{k=1}^d \partial_k B^k(x) + \frac{1}{2} \Delta f(x),$$

is densely closed and is the generator of the semi-group  $Q$  with kernel  $\Gamma$ :

$$(2.3) \quad Q_t(f)(x) := E \left[ f(X_t^x) \exp \left( - \int_0^t \sum_{k=1}^d \partial_k B^k(X_u^x) du \right) \right] = \int f(y) \Gamma(x, y; t) dy$$

for  $0 \leq t \leq T$ ,  $x \in \mathbb{R}^d$  and  $f$  bounded.

This remark is proved below in Lemma 5.1.

Now, let us quote the following theorem (variation of constants), due to Ball [1].

**Theorem 2.5** *Let  $A$  a densely closed linear operator on a Banach space  $X$  which generates a strongly continuous semi-group  $Q$  on  $X$  of bounded linear operators  $Q_t$  bounded on  $X$ . Let  $f$  belonging to  $L^1([0, T], X)$  and  $u \in C([0, T], X)$  be a weak solution of the PDE:*

$$(2.4) \quad u'(t) = Au(t) + f(t), \quad t \in (0, T], \quad u(0) = x \in D(A) \subset X,$$

where  $D(A)$  is the domain of operator  $A$ . In that case, the unique solution of (2.4) is expressed as

$$(2.5) \quad u(t) = Q(t)x + \int_0^t Q(t-s)f(s)ds, \quad \forall t \in [0, T].$$



We will check that the assumptions of the theorem are satisfied in our case in Section 3 below. We use Ball theorem with:  $X := L^2(\mathbb{R}^d)$ ,  $A := \mathcal{L}^*$ ,  $u(t) := q_H(\cdot; t)$ ,  $f : t \mapsto \frac{1}{2}H'(x^1)q(x^1, x^1, \tilde{x}; t)$  and  $x = 0$ . The conclusion of the theorem reads in our case:

$$(2.6) \quad q_H(x; t) = \int_{\mathbb{R}} H(m)q(m, x; t)dm = \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} H'(y^1)q(y^1, y; s)\Gamma(x, y; t-s)dsdy$$

where  $\Gamma$  is the transition probability density defined in (2.3).

From there, we work to prove that  $q = 0$ . The difficulty of the calculation is to relate  $q(m, x; t)$  on the left to its value on the diagonal  $m = x^1$ , which is the quantity appearing in the right hand side of (2.6) after application of Ball's theorem.

We can prove the following result:

**Proposition 2.6** *Let  $B \in C_b^1(\mathbb{R}^d, \mathbb{R}^d)$  and  $q \in \mathcal{X}$  be a solution of (1.1) with null initial condition. The following identity holds  $\forall t \in [0, T]$  and  $dy^1 d\tilde{x}$  almost surely:*

$$(2.7) \quad q(y^1, y^1, \tilde{x}; t) = -\frac{1}{2} \int_0^t \int_{\mathbb{R}^{d-1}} \partial_{x^1} \Gamma(y^1, \tilde{x}, y^1, \tilde{y}, t-s)q(y^1, y^1, \tilde{y}; s)d\tilde{y}ds.$$

This identity is a first tool from which we can deduce that  $q = 0$ , giving Theorem 2.1. We break down the proof of the above identity in two steps.

**First step:** going back to (2.6), we multiply both sides by  $F(\tilde{x}) \in C_c(\mathbb{R}^{d-1})$  and integrate in  $dx^1 d\tilde{x} = dx$ :

$$(2.8) \quad \int_{\mathbb{R}^{d+1}} F(\tilde{x})H(m)q(m, x; t)dmdx = \frac{1}{2} \int_0^t \int_{\mathbb{R}^{2d}} F(\tilde{x})H'(y^1)q(y^1, y; s)\Gamma(x, y, t-s)dsdydx.$$

We have the following decomposition:

**Lemma 2.7** *It holds that  $\forall t \in [0, T]$ :*

$$(2.9) \quad \int_{\mathbb{R}^d} F(\tilde{x})H(m)q(m, m, \tilde{x}; t)dmd\tilde{x} = \int_0^t \int_{\mathbb{R}^{2d-1}} F(\tilde{x})H'(y_1)\Gamma(y^1, \tilde{x}, y^1, \tilde{y}; t-s)q(y^1, y^1, \tilde{y}; s)dsdy^1 d\tilde{y}d\tilde{x}$$

$$(2.10) \quad - \int_0^t \int_{\mathbb{R}^{2d-1}} F(\tilde{x})H(y_1)\partial_{x^1} \Gamma(y^1, \tilde{x}, y^1, \tilde{y}; t-s)q(y^1, y^1, \tilde{y}; s)dsdy^1 d\tilde{y}d\tilde{x}.$$

Notice that in Lemma 2.7, the functions  $\Gamma$  and  $q$  are considered on the diagonal  $m = y^1$ , as opposed to Equation (2.8). To achieve this, the idea of the proof is to localize the variable  $m$  in (2.8) around  $x^1$  using an approximation of the identity.

**Proof:** Observe, that using Item (c) of Definition 1.1,  $x^1 \mapsto q_H(x^1, \tilde{x}; t)$  is continuous  $d\tilde{x}$  almost surely .

According to Lemma 5.5  $d\tilde{x}$  almost surely  $x^1 \mapsto \int_0^t \int_{\mathbb{R}^d} H'(y^1)q(y^1, y, s)\Gamma(x, y; t - s)dyds$  is continuous. We are in position to localize in  $x^1$  using the following compact support approximation of unity. Let  $\phi$  be the  $C^\infty$  function with compact support in  $[0, 1]$  defined by

$$\phi(u) = : \begin{cases} a \exp -\frac{1}{1-u^2}, & \text{if } |u| < 1, \\ 0 & \text{else} \end{cases}$$

Then,  $\phi$  is non negative, even, and  $a$  is chosen such that  $\int_{\mathbb{R}} \phi(u)du = 1$ .

For fixed  $x^1 \in \mathbb{R}$ , we consider:

$$\phi_\varepsilon(m) = \frac{1}{\varepsilon} \phi\left(\frac{m - x^1}{\varepsilon}\right)$$

From properties of  $\phi$ , we can derive:

$$(2.11) \quad \lim_{\varepsilon \rightarrow 0} \phi_\varepsilon(m)dx^1 = \delta_m(dx^1).$$

**Remark 2.8** *This convergence which allows us to localize  $q$  around the diagonal is made rigorous in our setting using Lebesgue dominated convergence theorem. We refer to Lemmas 5.4 and 5.6 of Section 5 for more details. Specifically, Lemma 5.4 coupled with Lemma 5.5 yields (2.9) and with Lemma 5.6 yields (2.10).*

Going back to (2.8), applied to  $m \mapsto \phi_\varepsilon(m)H(m)$  instead of  $H$ , we get:

$$(2.12) \quad \begin{aligned} & \int_{\mathbb{R}^{d+1}} F(\tilde{x})\mathbf{1}_{\{x^1 < m\}} \left( \phi_\varepsilon(m)H(m) \right) q(m, x; t) dmdx \\ &= \frac{1}{2} \int_0^t \int_{\mathbb{R}^{2d}} F(\tilde{x})\mathbf{1}_{\{x^1 < m\}} \left( \phi_\varepsilon(y^1)H(y^1) \right)' q(y^1, y; s) \Gamma(x, y, t - s) dsdydx. \end{aligned}$$

On the left hand side of (2.12), letting  $\varepsilon \rightarrow 0$ , from (2.11) and using the continuity  $x^1 \mapsto q(m, x^1, \tilde{x}; t)$  for all almost  $(m, \tilde{x})$  and all  $t$  (see third part of Item (c) of Definition 1.1), we get:

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^{d+1}} F(\tilde{x}) \left( \phi_\varepsilon(m)H(m) \right) q(m, x; t) dmdx = \frac{1}{2} \int_{\mathbb{R}^d} F(\tilde{x})H(m)q(m, m, \tilde{x}; t) dmd\tilde{x}.$$

On the right hand side of (2.12), we compute the derivative:

$$(2.13) \quad \begin{aligned} & \int_0^t \int_{\mathbb{R}^{2d}} F(\tilde{x}) \left( \phi_\varepsilon(y^1)H(y^1) \right)' q(y^1, y; s) \Gamma(x, y, t - s) dsdydx \\ &= \int_0^t \int_{\mathbb{R}^{2d}} F(\tilde{x}) \phi_\varepsilon(y^1)H'(y^1)q(y^1, y; s) \Gamma(x, y, t - s) dsdydx \end{aligned}$$

$$+ \int_0^t \int_{\mathbb{R}^{2d}} F(\tilde{x}) \phi'_\varepsilon(y^1) H(y^1) q(y^1, y; s) \Gamma(x, y, t-s) ds dy dx.$$

According to Garroni [9] (3.35) page 187

$$(2.14) \quad \Gamma(x, y; t) = \Gamma_0(x, y; t) + \Gamma_1(x, y; t) \text{ where } \Gamma_0(x, y; t) = \frac{e^{-\frac{\|x-y\|^2}{2t}}}{\sqrt{2\pi t}^d},$$

and for all  $y \in R^d$ ,  $t > 0$   $x \mapsto \Gamma_1(x, y; t) \in C^1(\mathbb{R}^d)$ . Moreover, from Lemma 3.3 page 184, estimation (3.25) of Garroni, for  $\alpha \in ]0, 1[$  there exist some positive constants  $C$  and  $c$  such that

$$(2.15) \quad |\Gamma_1(x, y; t)| \leq C t^{-\frac{d}{2} + \alpha} e^{-\frac{\|x-y\|^2}{ct}}.$$

$$(2.16) \quad |\partial_{x^1} \Gamma_1(x, y; t)| \leq C t^{-\frac{d+1}{2} + \alpha} e^{-\frac{\|x-y\|^2}{ct}}.$$

Now, letting  $\varepsilon \rightarrow 0$  and using Lemma 5.4 with  $k = 2d - 1$ , and the continuity of  $x^1 \mapsto \Gamma(x^1, \tilde{x}, y; t)$ ,  $\forall t \in [0, T]$  the first term on the right hand side converges:

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^{2d}} F(\tilde{x}) \phi_\varepsilon(y^1) H'(y^1) q(y^1, y; s) \Gamma(x, y, t-s) ds dy dx \\ \xrightarrow{\varepsilon \rightarrow 0} & \int_0^t \int_{\mathbb{R}^{2d-1}} F(\tilde{x}) H'(y^1) q(y^1, y^1, \tilde{y}; s) \Gamma(y^1, \tilde{x}, y^1, \tilde{y}, t-s) ds dy d\tilde{x}, \end{aligned}$$

which gives (2.9) in Lemma 2.7 above.

For the second term in (2.13) we look at the integral in  $dy^1$ , we need to integrate:

$$\int_0^t \int_{\mathbb{R}^{2d}} F(\tilde{x}) \phi'_\varepsilon(y^1) H(y^1) q(y^1, y; s) \Gamma(x, y, t-s) ds dy dx.$$

Then, since  $\int_{\mathbb{R}} \phi'_\varepsilon(y^1) \Gamma_0(x, y, t-s) dx^1 = 0$ , we have

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^{2d}} F(\tilde{x}) \phi'_\varepsilon(y^1) H(y^1) q(y^1, y; s) \Gamma(x, y, t-s) ds dy dx \\ = & \int_0^t \int_{\mathbb{R}^{2d}} F(\tilde{x}) \phi'_\varepsilon(y^1) H(y^1) q(y^1, y; s) \Gamma_1(x, y, t-s) ds dy dx \end{aligned}$$

which we can do by performing an integration by parts over  $dx^1$  on  $\mathbb{R}$ :

$$\int_{\mathbb{R}} \phi'_\varepsilon(y^1) \Gamma_1(x^1, \tilde{x}, y^1, \tilde{y}, t-s) dx^1 = - \int_{\mathbb{R}} \phi_\varepsilon(y^1) \partial_{x^1} \Gamma_1(x^1, \tilde{x}, y^1, \tilde{y}, t-s) dx^1$$

since when  $x^1 \rightarrow \infty$ ,  $\phi_\varepsilon(x^1)$  goes to 0.

Letting now  $\varepsilon \rightarrow 0$ , using estimation (2.16), we obtain:

$$\int_{\mathbb{R}} \phi_\varepsilon(y^1) \partial_{x^1} \Gamma_1(x^1, \tilde{x}, y^1, \tilde{y}, t-s) dx^1 \xrightarrow{\varepsilon \rightarrow 0} \partial_{x^1} \Gamma_1(y^1, \tilde{x}, y^1, \tilde{y}, t-s).$$

Note that according to the definition of  $\Gamma$ ,  $\Gamma_1$  and  $\Gamma_0$  (see (2.14))

$$\partial_{x^1} \Gamma_1(m, \tilde{x}, m, \tilde{y}, t-s) = \partial_{x^1} \Gamma(m, \tilde{x}, m, \tilde{y}, t-s).$$

Consequently, using Lemma 5.6, we have  $\forall t \in [0, T]$ :

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^{2d}} F(\tilde{x}) \phi'_\varepsilon(y^1) H(y^1) q(y^1, y; s) \Gamma(x, y, t-s) ds dy dx \\ & \xrightarrow{\varepsilon \rightarrow 0} - \int_0^t \int_{\mathbb{R}^{2d-1}} F(\tilde{x}) H(y^1) q(y^1, y; s) \partial_{x^1} \Gamma(y^1, \tilde{x}, y^1, \tilde{y}, t-s) ds dy dx, \end{aligned}$$

which gives (2.10) in Lemma 2.7 above. •

**Remark 2.9** *We need to use dominated convergence theorem on the functions*

$$x^1 \rightarrow q(m, x^1, \tilde{x}; t) ; \Gamma(x^1, \tilde{x}, m, \tilde{y}; t-s) ; \partial_{x^1} \Gamma(x^1, \tilde{x}, m, \tilde{y}; t-s).$$

*in order to justify the Dirac convergence. These functions have to satisfy the continuity when  $x^1 \rightarrow m$  and the uniform integrability. The continuity is clear for  $\Gamma$  and  $\partial_{x^1} \Gamma$  (see e.g. [10]) and for  $q$ , it is exactly Item (c) in Definition 1.1 above. The integrability of  $\sup_{x^1} |\Gamma(x, y; t-s)|$  and  $\sup_{x^1} |\partial_{x^1} \Gamma(x, y; t-s)|$  is deduced from [9] page 171 and (3.25) page 184. Moreover, we also rely on the fact that  $F$  has compact support in  $\mathbb{R}^{d-1}$ .*

**Second step:** we prove that the term (2.9) in the right hand side of Lemma 2.7 actually vanish. Indeed, this term is exactly the integral with respect to  $dy^1$  of the product of  $H'(y^1)$  and expression below (2.17):

**Lemma 2.10** *We have  $\forall t \in [0, T]$ ,  $dy^1$  almost surely*

$$(2.17) \quad \int_0^t \int_{\mathbb{R}^{2d-2}} F(\tilde{x}) q(y^1, y^1, \tilde{y}; s) \Gamma(y^1, \tilde{x}, \tilde{y}; t-s) ds d\tilde{x} d\tilde{y} = 0.$$

**Proof:** First of all, observe that from (2.6), taking  $H(m) = 1$  for all  $m \geq 0$  yields:

$$(2.18) \quad \int_{\mathbb{R}} q(m, x; t) dm = \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} H'(y^1) q(y^1, y; s) \Gamma(x, y; t-s) ds dy = 0.$$

Consider now for  $\Psi$  a test function, following choice for  $H$ :

$$H(m) = \int_{-\infty}^m \Psi(y_1) dy^1,$$

In that case, Equation (2.6) becomes :

$$\int_{\mathbb{R}^2} \mathbf{1}_{y_1 < m} \Psi(y_1) q(m, x, t) dm dy^1 = \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} \Psi(y^1) q(y^1, y, s) \Gamma(x, y, t - s) ds dy,$$

for any  $\Psi$  so by identification  $dy_1$  almost surely:

$$\int_{\mathbb{R}} \mathbf{1}_{y_1 < m} q(m, x, t) dm = \frac{1}{2} \int_0^t \int_{\mathbb{R}^{d-1}} q(y^1, y^1, \tilde{y}, s) \Gamma(x, y^1, \tilde{y}; t - s) ds d\tilde{y}.$$

But recall from (2.18)  $\int_{\mathbb{R}} q(m, x, t) dm = 0$ , hence:

$$0 = \int_{\mathbb{R}} q(m, x, t) dm = \int_{\mathbb{R}} \mathbf{1}_{y_1 < m} q(m, x, t) dm + \int_{\mathbb{R}} \mathbf{1}_{y_1 > m} q(m, x, t) dm.$$

Therefore, we get:

$$- \int_{\mathbb{R}} \mathbf{1}_{y_1 > m} q(m, x, t) dm = \frac{1}{2} \int_0^t \int_{\mathbb{R}^{d-1}} q(y^1, y^1, \tilde{y}; s) \Gamma(x, y^1, \tilde{y}; t - s) ds d\tilde{y}.$$

We multiply that last identity by  $F \in C_k^1(\mathbb{R}^{d-1})$  and integrate with respect to  $d\tilde{x}$ . Recall that  $q$  has support in  $\mathcal{T}$ , i.e.  $q(m, x^1, \tilde{x}; t) = 0$  when  $x^1 > m$ , we get:

(2.19)

$$- \int_{\mathbb{R}^d} F(\tilde{x}) \mathbf{1}_{y_1 > m > x^1} q(m, x; t) dm d\tilde{x} = \frac{1}{2} \int_{\mathbb{R}^{2d-2}} F(\tilde{x}) \int_0^t q(y^1, y^1, \tilde{y}, s) \Gamma(x, y^1, \tilde{y}; t - s) ds d\tilde{y} d\tilde{x}$$

and letting  $x^1$  increase to  $y^1$ , using item (a) of the Definition 1.1 the limit of left hand is null  $dy^1$  almost surely. According to the definition of  $\Gamma$  (see (2.14)) and estimation (2.15), the limit of the right hand side is

$$\frac{1}{2} \int_{\mathbb{R}^{2d-2}} F(\tilde{x}) \int_0^t q(y^1, y^1, \tilde{y}, s) \Gamma(y^1, \tilde{x}, y^1, \tilde{y}; t - s) ds d\tilde{y} d\tilde{x}$$

$dy^1$  almost surely. See also Lemma 5.5 for a proof. •

Going back to the conclusion of Lemma 2.7 and cancelling the first term, we get  $\forall t \in [0, T]$ :

$$\begin{aligned} & \int_{\mathbb{R}^d} F(\tilde{x}) H(y^1) q(y^1, y^1, \tilde{x}; t) dy^1 d\tilde{x} \\ &= -\frac{1}{2} \int_0^t \int_{\mathbb{R}^{2d-1}} F(\tilde{x}) H(y_1) \partial_{x^1} \Gamma(y^1, \tilde{x}, y^1, \tilde{y}, t - s) q(y^1, y^1, \tilde{y}; s) ds dy^1 d\tilde{y} d\tilde{x}, \end{aligned}$$

which gives by identification over  $H(y^1) F(\tilde{x}) dy^1 d\tilde{x}$  almost surely:

$$q(y^1, y^1, \tilde{x}; t) = -\frac{1}{2} \int_0^t \partial_{x^1} \Gamma(y^1, \tilde{x}, y^1, \tilde{y}, t - s) q(y^1, y^1, \tilde{y}; s) d\tilde{y} ds.$$

which proves the identity (2.7) in Proposition 2.6. •

## 2.2 Proof of Theorem 2.1

Let us show how to conclude to  $q = 0$ . Taking the absolute values in Proposition 2.6 gives:

$$(2.20) \quad |q(y^1, y^1, \tilde{x}; t)| \leq \frac{1}{2} \int_0^t \int_{\mathbb{R}^{d-1}} |q(y^1, y^1, \tilde{y}, s) \partial_{x^1} \Gamma_1(y^1, \tilde{x}, y^1, \tilde{y}, t-s)| ds d\tilde{y}.$$

Equation (3.25) page 184 [9] with  $l = 1, 0 < \alpha < 1$  gives the following Gaussian estimate for  $\Gamma$  and its derivatives:

$$|\partial_{x^1} \Gamma_1(y^1, \tilde{x}, y^1, \tilde{y}, t-s)| \leq C_T \frac{1}{\sqrt{(t-s)^{d+\alpha-1}}} \exp\left(-\frac{\|\tilde{x} - \tilde{y}\|^2}{c(t-s)}\right).$$

We use this bound then we integrate (2.20) with respect to  $d\tilde{x}$ :

$$(2.21) \quad \int_{\mathbb{R}^{d-1}} |q(y^1, y^1, \tilde{x}; t)| d\tilde{x} \leq \frac{1}{2} \int_0^t \int_{\mathbb{R}^{2d-2}} |q(y^1, y^1, \tilde{y}, s)| C_T \frac{1}{\sqrt{(t-s)^{d+\alpha-1}}} \exp\left(-\frac{\|\tilde{x} - \tilde{y}\|^2}{c(t-s)}\right) ds d\tilde{y} d\tilde{x}.$$

Solving on the right hand the integral with respect to  $d\tilde{x}$ , there exists a constant  $K(c, d)$  such that:

$$(2.22) \quad \int_{\mathbb{R}^{d-1}} |q(y^1, y^1, \tilde{x}; t)| d\tilde{x} \leq \frac{1}{2} C_T K(c, d) \int_0^t \int_{\mathbb{R}^{d-1}} |q(y^1, y^1, \tilde{y}, s)| \frac{1}{\sqrt{(t-s)^\alpha}} ds d\tilde{y}.$$

Define for convenience  $f(y^1, t) := \int |q(y^1, y^1, \tilde{x}, t)| d\tilde{x}$ , the last identity becomes:

$$f(y^1, t) \leq d_T \int_0^t f(y^1, s) \frac{ds}{\sqrt{(t-s)^\alpha}} = d_T f(y^1, \cdot) * g_\alpha(t)$$

for some constant  $d_T > 0$  and where  $g_\alpha : s \mapsto \frac{1}{\sqrt{s^\alpha}}$ . Remark that the ratio  $\frac{g_\alpha(s)}{g_1(s)} \leq T^{(1-\alpha)/2}$  since  $0 < \alpha < 1$  and  $s \in [0, T]$ . So the inequality can be written as  $f(y^1, t) \leq d_T T^{(1-\alpha)/2} f(y^1, \cdot) * g_1(t)$ . We can now iterate this inequality to get for all  $n \geq 1$ :

$$f(y^1, \cdot) \leq d_T T^{(1-\alpha)/2} f(y^1, \cdot) * g_1 \leq \dots \leq (d_T T^{(1-\alpha)/2})^n f(y^1, \cdot) * g_1^{n*}.$$

That iterated convolution can now be dealt with using Gamma functions (see details in Lemma 5.3 below), denoting  $D_T = d_T T^{(1-\alpha)/2}$  and recalling that  $t-s \leq t \leq T$ :

$$f(y^1, t) \leq \frac{(D_T \Gamma(\frac{1}{2}))^n}{\Gamma(\frac{n}{2})} \int_0^t f(y^1, s) (t-s)^{\frac{n-2}{2}} ds \leq \frac{(D_T \Gamma(\frac{1}{2}) \sqrt{T})^n}{\Gamma(\frac{n}{2}) T} \int_0^t f(y^1, s) ds.$$

Finally, Lemma 5.2 below shows that the integral of  $f(y^1, s)$  over  $s$  is finite  $dy^1$  almost surely which yields  $f(y^1, t) = 0$ : Indeed,  $\forall a > 0$ , the ratio  $\frac{a^n}{\Gamma(\frac{n}{2})}$  goes to 0 when  $n \rightarrow \infty$  using Stirling formula.

As a conclusion, we have shown that  $\int |q(y^1, y^1, \tilde{y}, t)| d\tilde{y} = 0$   $dy^1$  almost surely, yielding  $q(y^1, y^1, \tilde{y}, s) = 0$   $d\tilde{y}$  almost surely, and applying Ball (2.6):

$$\int_{\mathbb{R}} H(m)q(m, \cdot, t)dm = \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} H'(y^1)q(y^1, y, s)\Gamma(\cdot, y, t-s)dsdy = 0$$

thus  $\forall t \in [0, T] \leq T$  and for all test function  $H$ , we proved that  $\int H(m)q(m, \cdot; t)dm = 0$  meaning the function  $q(\cdot, \cdot; t) = 0$ .

This concludes the proof of Theorem 2.1 up to the verification of assumptions in Theorem 2.5, what is done is the next section. •

### 3 Checking assumptions of Theorem 2.5

In this section, we check that the assumptions of Ball's Theorem 2.5 actually holds in our setting. The Banach space  $X$  is  $L^2(\mathbb{R}^d)$ . We have to prove that

- $\mathcal{L}^*$  is a densely closed linear operator on  $X$ , and  $\mathcal{L}^*$  generates a strongly continuous on  $X$  semi-group  $(Q_t)_{t \leq T}$  such that the operators  $Q_t$  are bounded on  $L^2(\mathbb{R}^d)$ , this is done in Lemma 5.1.
- $f$  belongs to  $L^1([0, T], X)$  where  $f(t, x) = \frac{1}{2}H'(x^1)q(x^1, x^1, \tilde{x}; t)$ .
- $u \in C([0, T], X)$ ; where  $u = q_H$ .

#### 3.1 The function $f$ belongs to $L^1([0, T], X)$

Recall that this function is defined as

$$f : t \mapsto \left( x \mapsto \frac{1}{2} \partial_m H(x^1) q(x^1, x^1, \tilde{x}; t) \right).$$

Since  $H'$  is bounded, this is a consequence of Item (b) satisfied by  $q \in \mathcal{X}$ .

#### 3.2 The function $u = q_H$ belongs to $\mathcal{C}([0, T], X)$ , where $X = L^2(\mathbb{R}^d)$ .

**Lemma 3.1** *Let  $q$  satisfies Items (a) and (b), meaning*

$$(3.1) \quad \sup_{t \in [0, T]} \int_{\mathbb{R}^d} \left[ \int_{\mathbb{R}} |q(m, x, t)| dm \right]^2 dx < \infty$$

$$(3.2) \quad \int_0^t ds \sqrt{\int_{\mathbb{R}^d} |q(x_1, x; s)|^2 dx} < \infty.$$

then  $q_H(x; t) := \int_{\mathbb{R}} H(m)q(m, x; t)dm$  satisfies  $q_H \in C([0, T], L^2(\mathbb{R}^d))$ .

**Proof:**

**Step 1:**  $\sup_{t \in [0, T]} \|q_H(\cdot, t)\|_{L^2(\mathbb{R}^d)} < +\infty$ .

Remark that for any  $t \in [0, T]$

$$\int_{\mathbb{R}^d} q_H^2(x, t) dx = \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}} H(m) q(m, x; t) dm \right)^2 dx \leq \|H\|_\infty^2 \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}} |q(m, x; t)| dm \right)^2 dx$$

and using  $\sup_{t \in [0, T]}$  on both hands, Step 1 is proved according to Assumption (3.1)=Item (a).

**Step 2:** for all  $\varphi \in L^2(\mathbb{R}^d)$ ,  $t \mapsto \int_{\mathbb{R}^d} \varphi(x) q_H(x, t) dx$  is continuous.

• Firstly we consider  $\varphi \in C_c^2(\mathbb{R}^d)$  and  $\psi(m, x) = H(m)\varphi(x)$ . According to the PDE (1.1) (recall that  $q = p^1 - p^2$  so the first term is null)

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x) q_H(x, t) dx &= \int_{\mathbb{R}^{d+1}} \varphi(x) H(m) q(m, x; t) dm dx = \\ &= \int_0^t \int_{\mathbb{T}} q(m, x; s) H(m) \mathcal{L}\varphi(x) dm dx ds + \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} H'(m) \varphi(m, \tilde{x}) q(m, m, \tilde{x}; s) dm d\tilde{x} ds \end{aligned}$$

where we identify (using Lebesgue Theorem)

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x) q_H(x, t) dx &= \\ &= \int_0^t \int_{\mathbb{R}^d} q_H(x, u) \mathcal{L}\varphi(x) dx du + \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} H'(m) \varphi(m, \tilde{x}) q(m, m, \tilde{x}; u) dm d\tilde{x} du. \end{aligned}$$

Using Cauchy-Schwartz, boundness of  $H'$  and denoting  $\tilde{q}(x, u) = q(x^1, x^1, \tilde{x}; u)$ :

$$(3.3) \quad \left| \int_{\mathbb{R}^d} \varphi(x) [q_H(x, t) - q_H(x, s)] dx \right| \leq \int_s^t \|q_H(\cdot, u)\|_{L^2(\mathbb{R}^d)} \|\mathcal{L}\varphi\|_{L^2(\mathbb{R}^d)} du + \frac{1}{2} \int_s^t \|H'\|_\infty \|\varphi\|_{L^2(\mathbb{R}^d)} \|\tilde{q}(\cdot, u)\|_{L^2(\mathbb{R}^d)} du.$$

Since when  $f \in L^1[0, T]$ ,  $t \mapsto \int_0^t f(s) ds$  is continuous, and according to Step 1 and Assumption (3.2) on  $\tilde{q}$  then:

$$t \mapsto \int_0^t \|q_H(\cdot, u)\|_{L^2(\mathbb{R}^d)} \|\mathcal{L}\varphi\|_{L^2(\mathbb{R}^d)} du + \frac{1}{2} \int_0^t \|H'\|_\infty \|\varphi\|_{L^2(\mathbb{R}^d)} \|\tilde{q}(\cdot, u)\|_{L^2(\mathbb{R}^d)} du.$$

is continuous. Then, for all  $\varepsilon > 0$ , there exists  $\eta > 0$ , such that for  $|t - s| < \eta$

$$\int_s^t \|q_H(\cdot, u)\|_{L^2(\mathbb{R}^d)} \|\mathcal{L}\varphi\|_{L^2(\mathbb{R}^d)} du + \frac{1}{2} \int_0^t \|H'\|_\infty \|\varphi\|_{L^2(\mathbb{R}^d)} \|\tilde{q}(\cdot, u)\|_{L^2(\mathbb{R}^d)} du \leq \varepsilon,$$

and according to estimation (3.3)

$$\left| \int_{\mathbb{R}^d} \varphi(x) [q_H(x, t) - q_H(x, s)] dx \right| < \varepsilon$$



so yields the continuity of the map  $t \mapsto \int_{\mathbb{R}^d} \varphi(x) q_H(x, t) dx$  for  $\varphi \in C_c^2(\mathbb{R}^d)$ .

• Secondly, let  $\varphi \in L^2(\mathbb{R}^d)$ . There exists for all  $\varepsilon > 0$  a function  $\varphi_\varepsilon \in C_c^2(\mathbb{R}^d)$  such that  $\|\varphi - \varphi_\varepsilon\|_{L^2(\mathbb{R}^d)} \leq \varepsilon$ .  
Let  $0 < s < t$ :

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x)(q_H(x, t) - q_H(x, s)) dx &= \int_{\mathbb{R}^d} (\varphi - \varphi_\varepsilon)(x) q_H(x, t) \\ &- \int_{\mathbb{R}^d} (\varphi - \varphi_\varepsilon)(x) q_H(x, s) dx + \int_{\mathbb{R}^d} \varphi_\varepsilon(x) [q_H(x, t) - q_H(x, s)] dx. \end{aligned}$$

Using the continuity of  $t \mapsto \int_{\mathbb{R}^d} \varphi_\varepsilon(x) q_H(x, t) dx$  due to  $\varphi_\varepsilon \in C_c^2$ , there exists  $\eta$  such that  $t - s \leq \eta$  yields

$$\left| \int_{\mathbb{R}^d} \varphi_\varepsilon(x) [q_H(x, t) - q_H(x, s)] dx \right| \leq \varepsilon.$$

We now use for any  $u \in [0, T]$

$$\left| \int_{\mathbb{R}^d} (\varphi - \varphi_\varepsilon)(x) q_H(x, u) dx \right| \leq \|\varphi - \varphi_\varepsilon\|_2 \sup_{u \in [0, T]} \left[ \int_{\mathbb{R}^d} (q_H(x, u))^2 dx \right]^{\frac{1}{2}}.$$

Thus for  $0 \leq s \leq t \leq T$ ,  $t - s < \eta$  gathering these three bounds

$$\left| \int_{\mathbb{R}^d} \varphi(x) (q_H(x, t) - q_H(x, s)) dx \right| \leq \varepsilon (1 + 2 \sup_{u \in [0, T]} \left[ \int_{\mathbb{R}^d} (q_H(x, u))^2 dx \right]^{\frac{1}{2}})$$

which concludes the uniform continuity of the map  $t \mapsto \int_{\mathbb{R}^d} \varphi(x) q_H(x, t) dx$  for  $\varphi \in L^2(\mathbb{R}^d)$ .

**Step 3:**  $q_H \in C([0, T], L^2(\mathbb{R}^d))$ .

The previous step gives the weak continuity, the first step the uniform bound in  $L^2$ , so the continuity in  $C([0, T], L^2(\mathbb{R}^d))$  is proved. •

## 4 The density $p_V$ is an element of space $\mathcal{X}$

Let us denote  $p_V(\cdot; t, x_0)$  the density of law of  $(M_t, X_t)$  when the initial condition is  $X_0 = x_0$ . We recall that  $f_0$  is the square integrable density of  $X_0$ . Thus

$$p_V(m, x; t) = \int_{\mathbb{R}^d} p_V(m, x; t, x_0) f_0(x_0) dx_0.$$

The aim in this section is to check that the function  $p_V$  defined above satisfies Items (a) (b) and (c) in Definition 1.1.

**Proposition 4.1** *Item (a) is satisfied by  $p_V$ :*

$$\sup_{t \in [0, T]} \int_{\mathbb{R}^d} \left[ \int_{\mathbb{R}} |p_V(m, x, t)| dm \right]^2 dx < \infty$$

**Proof:** According to [6] (page 26 (ii)) there exists  $C_T$  such that

$$(4.1) \quad p_V(m, x; t, x_0) \leq C_T \phi_{d+1}(m - x_0^1, m - x^1, \tilde{x} - \tilde{x}_0; 2t)$$

where  $\phi_d(u, t) = \left(\frac{1}{\sqrt{2\pi t}}\right)^d e^{-\frac{\|u\|^2}{2t}}$ ,  $u \in \mathbb{R}^d$ .

**Remark 4.1** Considering two  $\mathbb{R}^d$  value Gaussian independent variables  $X \sim \mathcal{N}(a, s)$  and  $Y \sim \mathcal{N}(b, r)$ , the sum  $-X + Y \sim \mathcal{N}(b - a, s + r)$  and by convolution:

$$\int_{\mathbb{R}^d} \phi_d(x - a, s) \phi_d(x - b, r) dx = \phi_d(b - a, r + s).$$

We have to bound  $\forall t \in ]0, T]$ :

$$(4.2) \quad \begin{aligned} A_t &:= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}} p_V(m, x; t) dm \right|^2 dx \\ &\leq C_T^2 \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}} \int_{\mathbb{R}^d} \phi_{d+1}(m - x_0^1, m - x^1, \tilde{x} - \tilde{x}_0; 4t) f_0(x_0) dx_0 dm \right|^2 dx. \end{aligned}$$

Integrating with respect to  $m$  and using Remark 4.1, we get:

$$\int_{\mathbb{R}} \phi_{d+1}(m - x_0^1, m - x^1, \tilde{x} - \tilde{x}_0; 4t) dm = \phi_1(x^1 - x_0^1, 8t) \phi_{d-1}(\tilde{x} - \tilde{x}_0, 4t).$$

Thus, we have :

$$\begin{aligned} &\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}} p_V(m, x; t) dm \right|^2 dx \leq \\ &C_T^2 \int_{\mathbb{R}^{3d}} \phi_1(x^1 - x_0^1, 8t) \phi_1(x^1 - y_0^1, 8t) \phi_{d-1}(\tilde{x} - \tilde{x}_0, 4t) \phi_{d-1}(\tilde{x} - \tilde{y}_0, 4t) f_0(x_0) f_0(y_0) dx dy_0. \end{aligned}$$

Then integrating with respect to  $x^1, \tilde{x}$  and using once again Remark 4.1:

$$\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}} p_V(m, x; t) dm \right|^2 dx \leq C_T^2 \int_{\mathbb{R}^{2d}} \phi_1(x_0^1 - y_0^1, 16t) \phi_{d-1}(\tilde{x}_0 - \tilde{y}_0, 8t) f_0(x_0) f_0(y_0) dx_0 dy_0.$$

Let  $z = x_0 - y_0$  so  $y_0 = x_0 - z$ , the upper bound, using  $\sup(\phi_k(y, t), \phi_k(y, 2t)) \leq (\sqrt{2})^k \phi_k(y, 2t)$ , with respect to a multiplicative constant is  $\int_{\mathbb{R}^{2d}} \phi_d(z, 16t) f_0(x_0) f_0(x_0 - z) dz dx_0$ .

We first integrate with respect to  $x_0$  so Cauchy-Schwartz inequality yields the upper bound:

$$\int_{\mathbb{R}^d} \phi_d(z, 16t) \left( \sqrt{\int_{\mathbb{R}} f_0^2(x_0) dx_0 \int_{\mathbb{R}} f_0^2(x_0 - z) dx_0} \right) dz$$

then using the change of variable  $u = x_0 - z$  in the last factor under the square root, since  $\int_{\mathbb{R}^d} \phi_d(z, 16t) dz = 1$  the upper bound is

$$A_t \leq \sqrt{\int_{\mathbb{R}} f_0^2(x_0) dx_0 \int_{\mathbb{R}} f_0^2(u) du},$$

meaning  $\|f_0\|_{L^2}^2 < \infty$  according to assumption on  $f_0$  in Theorem 1.2. •

**Proposition 4.2** *Item (b) is satisfied by  $p_V$ :*

$$\int_0^t ds \sqrt{\int_{\mathbb{R}^d} |p_V(x^1, x; s)|^2 dx} < \infty.$$

Proof : Using (4.1) with  $m = x^1$

$$\begin{aligned} p_V(x^1, x; t) &\leq C_T \int_{\mathbb{R}^d} \phi_{d+1}(x^1 - x_0^1, 0, \tilde{x} - \tilde{x}_0; 4t) f_0(x_0) dx_0 \\ &= \frac{C_T}{\sqrt{8\pi t}} \int_{\mathbb{R}^d} \phi_d(x^1 - x_0^1, \tilde{x} - \tilde{x}_0; 4t) f_0(x_0) dx_0. \end{aligned}$$

We operate the  $L^2$  norm

$$\begin{aligned} &\int_{\mathbb{R}^d} (p_V(x^1, x; t))^2 dx \\ &\leq \frac{C_T^2}{8\pi t} \int_{\mathbb{R}^{3d}} \phi_d(x^1 - x_0^1, \tilde{x} - \tilde{x}_0; 4t) \phi_d(x^1 - y_0^1, \tilde{x} - \tilde{y}_0; 4t) f_0(x_0) f_0(y_0) dx_0 dy_0 dx. \end{aligned}$$

We use Remark 4.1 and we integrate with respect to  $dx$

$$\int_{\mathbb{R}^d} (p_V(x^1, x; t))^2 dx \leq \frac{C_T^2}{8\pi t} \int_{\mathbb{R}^{2d}} \phi_d(x_0^1 - y_0^1, \tilde{x}_0 - \tilde{y}_0; 8t) f_0(x_0) f_0(y_0) dx_0 dy_0.$$

We operate the change of variable  $z = x_0 - y_0$

$$\int_{\mathbb{R}^d} (p_V(x^1, x; t))^2 dx \leq \frac{C_T^2}{8\pi t} \int_{\mathbb{R}^{2d}} \phi_d(z; 8t) f_0(x_0) f(x_0 - z) dx_0 dz.$$

Using Cauchy-Schwartz inequality for the measure  $dx_0$ :

$$\int_{\mathbb{R}^d} (p_V(x^1, x; t))^2 dx \leq \frac{C_T^2}{8\pi t} \int_{\mathbb{R}^d} \phi_d(z; 8t) \|f_0\|_{L^2(\mathbb{R}^d)}^2 dz = \frac{C_T^2}{8\pi t} \|f_0\|_{L^2(\mathbb{R}^d)}^2.$$

The integrability in time of this  $L^2(\mathbb{R}^d)$ -norm yields Item (b):

$$\int_0^T dt \sqrt{\int_{\mathbb{R}^d} (p_V(x^1, x; t))^2 dx} \leq \int_0^T dt \frac{C_T}{\sqrt{2\pi t}} \|f_0\|_{L^2(\mathbb{R}^d)} < \infty.$$

•

## 4.1 $p_V$ satisfies Item (c) in Definition 1.1

Actually Item (c) is stronger than Hypothesis 2.1 in [6], satisfied in case  $A = I_d$  (our present case) and  $d = 1$ . For the sake of readability, we recall this assumption here: for all  $t > 0$ , the density with respect to the Lebesgue measure of the distribution of the random vector  $(M_t, X_t)$ , denoted by  $p_V$ , satisfies:

(i)  $(t, m, \tilde{x}) \rightarrow \sup_{u>0} p_V(m, m - u, \tilde{x}; t)$  belongs to  $L^1([0, T] \times \mathbb{R}^d, dt dm d\tilde{x})$ .

(ii) for all  $t$ , almost surely in  $(m, \tilde{x}) \in \mathbb{R}^d$ ,  $\lim_{u \rightarrow 0^+} p_V(m, m - u, \tilde{x}; t)$  exists and is denoted by  $p_V(m, m, \tilde{x}; t)$ .

Property (ii) is not efficient to recover Item(c) since we have to prove the continuity of  $x^1 \mapsto p_V(m, x^1, \tilde{x}; t)$  in all  $\mathcal{T}$ , not only on the boundary.

**Proposition 4.3** *Item (c) is satisfied by  $p_V$ : For all  $t \in ]0, T]$ ,  $\sup_{u>0} p(m, m - u, \tilde{x}; t) \in L^1(\mathbb{R}^d)$ ; and for almost surely in  $(m, \tilde{x}) \in \mathbb{R}^d$ ,  $x^1 \mapsto p(m, x^1, \tilde{x}; t)$  is continuous on  $] - \infty, m[$  and  $\lim_{u \rightarrow 0^+} p(m, m - u, \tilde{x}; t)$  exists and is denoted by  $p(m, m, \tilde{x}; t)$ .*

**Proof:** The proof is a consequence of the following lemmas and propositions. First, we prove the integrability assumption.

**Lemma 4.4** *For all  $t \in ]0, T]$ ,  $\sup_{u>0} p_V(m, m - u, \tilde{x}; t) \in L^1(\mathbb{R}^d)$ .*

**Proof:** Recall that  $p_V(m, x; t) = \int_{\mathbb{R}^d} p_V(m, x, t; x_0) f_0(x_0) dx_0$  where  $p_V(m, x, t; x_0)$  is the density of the law of  $(M_t, X_t)$  when the initial condition is  $X_0 = x_0$ . Moreover, recall (4.1)

$$|p_V(m, x; t; x_0)| \leq C_T \phi_{d+1}(m - x_0^1, m - x^1, \tilde{x} - \tilde{x}_0; 2t) \mathbf{1}_{m > \max(x^1, x_0^1)}.$$

Then, using the fact that  $e^{-\frac{u^2}{4t}} \leq 1$  we obtain  $\forall t \in ]0, T]$

$$\sup_{u>0} p_V(m, m - u, \tilde{x}; t) \leq \frac{C_T}{\sqrt{4\pi t}} \int_{\mathbb{R}^d} \phi_d(m - x_0^1, \tilde{x} - \tilde{x}_0; 2t) f_0(x_0) dx_0.$$

Integrating with respect to  $m$  and  $\tilde{x}$  we obtain  $\forall t \in ]0, T]$ :

$$\int_{\mathbb{R}^d} \sup_{u>0} p_V(m, m - u, \tilde{x}; t) dm d\tilde{x} \leq \frac{C_T}{\sqrt{4\pi t}} \int_{\mathbb{R}^d} f_0(x_0) dx_0 = \frac{C_T}{\sqrt{4\pi t}} < \infty.$$

This achieves the proof of Lemma 4.4. •

The following Lemma is Proposition 4.5 of [6].

**Lemma 4.5** *For all  $(t, m, \tilde{x}) \in ]0, T] \times \mathbb{R}^d$ ,  $\lim_{u \rightarrow 0^+} p_V(m, m - u, \tilde{x}; t)$  exists and is denoted by  $p_V(m, m, \tilde{x}; t)$ .*

Now, we turn to the continuity of  $p_V$  for  $x^1 \in ] - \infty, m[$ .

**Proposition 4.6** *For all  $(t, m, \tilde{x}) \in ]0, T] \times \mathbb{R}^d$ ,  $x^1 \mapsto p_V(m, x^1, \tilde{x}; t)$  is continuous on  $] - \infty, m[$ .*

**Proof:** This proposition extends the results of Proposition 4.5 of [6]. Its proof follows almost the same lines and is splitted in three lemmas. We firstly recall Proposition 4.2 of [6]:

**Lemma 4.7** For all  $t > 0$ ,

$$(4.3) \quad p_V = p_0 - \sum_{k=1}^{d+1} (p^{k,\alpha} + p^{k,\beta})$$

where the various  $p$  are defined as

$$\begin{aligned} p_0(m, x; t) &:= \int_{\mathbb{R}^d} p_{W^{*1}, W}(m - x_0^1, x - x_0; t) f_0(x_0) dx_0, \\ p^{k,\alpha}(m, x; t) &:= \int_0^t \int_{\mathbb{R}^{d+1}} \mathbf{1}_{m>b} B^k(a) \partial_k p_{W^{*1}, W}(m - a^1, x - a; t - s) p_V(b, a; s) db da ds, \\ p^{k,\beta}(m, x; t) &:= \int_0^t \int_{\mathbb{R}^{d+1}} \mathbf{1}_{m>b} B^k(a) \partial_k p_{W^{*1}, W}(b - a^1, x - a; t - s) p_V(m, a; s) db da ds \end{aligned}$$

where  $\partial_k$  is the derivative with respect to  $k = m, x^1, \dots, x^d$ , and  $B^m = B^1$ .

Using the same lines as the proof of Lemma 4.5 of [6], one can prove the following Lemma:

**Lemma 4.8** For all  $(t, m, \tilde{x})$ ,  $x_1 \mapsto p_0(m, x^1, \tilde{x}; t)$  is continuous on  $] - \infty, m[$ .

Using the same lines as the proof of Lemma 4.6 of [6], one can prove the following Lemma:

**Lemma 4.9** For all  $(t, m, \tilde{x})$ ,  $x_1 \mapsto p^{k,\alpha}(m, x^1, \tilde{x}; t)$  is continuous on  $] - \infty, m[$  for  $k = m, 1, \dots, d$ .

Unfortunately, for the continuity of the maps  $p^{k,\beta}$  we can not follow the same lines as the proof of Lemma 4.7 of [6], but we can prove the following.

**Lemma 4.10** For all  $(t, m, \tilde{x})$ ,  $x_1 \mapsto p^{k,\beta}(m, x^1, \tilde{x}; t)$  is continuous on  $] - \infty, m[$  for  $k = m, 1, \dots, d$ .

**Proof:** Instead of using explicit computations as the proof of Lemma 4.7 of [6], we use uniform integration. Let us introduce the measure  $\nu$  on  $[0, t] \times \mathbb{R}^{2d+1}$  defined by

$$\nu(ds db da dx_0) = \mathbf{1}_{]0, t]}(s) \mathbf{1}_{a^1 < b < m} p_V(m, a, s; x_0) f_0(x_0) ds da db dx_0.$$

According to estimation (4.1)

$$\begin{aligned} \nu([0, t] \times \mathbb{R}^{2d+1}) &\leq C_T \int_0^t \int_{\mathbb{R}^{2d+1}} \mathbf{1}_{a^1 < b < m} \phi_{d+1}(m - x_0^1, m - a^1, \tilde{a} - \tilde{x}_0; 2s) f_0(x_0) ds da db dx_0 \\ &\leq C_T \int_0^t \int_{\mathbb{R}^{2d}} (m - a^1)_+ \phi_{d+1}(m - x_0^1, m - a^1, \tilde{a} - \tilde{x}_0; 2s) f_0(x_0) ds da dx_0 \end{aligned}$$

after integrating with respect to  $b$ . Integrating with respect to  $\tilde{a}$ :

$$\nu([0, t] \times \mathbb{R}^{2d+1}) \leq C_T \int_0^t \int_{\mathbb{R}^2} (m - a^1)_+ \phi_2(m - x_0^1, m - a^1; 2s) f_0(x_0) ds da^1 dx_0.$$

Then integrating with respect to  $a^1$

$$(4.4) \quad \begin{aligned} \nu([0, t] \times \mathbb{R}^{2d+1}) &\leq C_T \int_0^t \int_{\mathbb{R}} \sqrt{\frac{s}{\pi}} \phi_1(m - x_0^1; 2s) f_0(x_0) ds dx_0 \\ &\leq \frac{C_T}{2\pi} \int_0^t \int_{\mathbb{R}} f_0(x_0) dx_0 ds \leq \frac{TC_T}{2\pi} < \infty. \end{aligned}$$

That means that  $\nu$  is a finite measure on  $[0, t] \times \mathbb{R}^{2d+1}$ . Note that

$$p^{k,\beta}(m, x^1, \tilde{x}; t) = \int_{[0, T] \times \mathbb{R}^{2d+1}} \mathbf{1}_{b < m} B^k(a) \partial_k p_{W^*1, W}(b - a^1, x - a; t - s) \nu(ds db da dx_0).$$

According to identity (57) of [6]

$$p_{W^*1, W}(b - a^1, x^1 - a^1; t) = -2\partial_{x^1} \phi_d(2b - a^1 - x^1, \tilde{x} - \tilde{a}; t) \mathbf{1}_{b \geq \max(a^1, x^1)}.$$

Then,  $x^1 \mapsto \partial_k p_{W^*1, W}(b - a^1, x^1 - a^1, \tilde{x} - \tilde{a}; t)$  is continuous for  $x^1 < b$ .

Assume for a while that for  $1 < \varepsilon < \frac{d+3}{d+2}$ , that

$$(4.5) \quad \sup_{x^1 < m} I(x_1) < \infty,$$

$$\text{where } I(x_1) = \int_{[0, t] \times \mathbb{R}^{2d+1}} |B^k(a) \partial_k p_{W^*1, W}(b - a^1, x^1 - a^1, \tilde{x} - \tilde{a}; t)|^\varepsilon \nu(ds db da dx_0)$$

then using uniform integrability, namely Vitali convergence Theorem, for all  $(t, m, \tilde{x})$ ,  $x_1 \mapsto p^{k,\beta}(m, x^1, \tilde{x}; t)$  is continuous on  $] - \infty, m[$  for  $k = m, 1, \dots, d$ . We now turn to the proof of (4.5). Recall estimation (55) of Lemma A.2 of [6]: there exists a constant  $D$  such that for  $x = b, a^1, \dots, a^d$

$$|\partial_x p_{W^*1, W}(b, a; t)| \leq \frac{D}{\sqrt{t}} \phi_{d+1}(b, b - a^1, \tilde{a} : 2t) \mathbf{1}_{b > \max(a^1, 0)}.$$

Since  $\varepsilon > 1$ ,  $\phi_{d+1}^\varepsilon(\cdot; t) \leq [\sqrt{2\pi t}]^{(1-\varepsilon)(d+1)} \phi_{d+1}(\cdot; t)$  and  $B$  is bounded

$$|B^k(a) \partial_x p_{W^*1, W}(b - a^1, x^1 - a^1, \tilde{x} - \tilde{a}; t - s)|^\varepsilon \leq \frac{D^\varepsilon \|B\|_\infty^\varepsilon}{\sqrt{(t-s)^\gamma}} \phi_{d+1}(b - a^1, b - x^1, \tilde{a} : 2(t-s)),$$

where  $\gamma = \varepsilon + (\varepsilon - 1)(d+1) < 2$ .

Then, up to a multiplicative constant, using once again (4.1) to bound the density of the measure  $\nu$ ,

$$\begin{aligned} I(x_1) &\leq \int_{[0, t] \times \mathbb{R}^{2d+1}} \frac{D^\varepsilon \|B\|_\infty^\varepsilon}{\sqrt{(t-s)^\gamma}} \phi_{d+1}(b - a^1, b - x^1, \tilde{a} - \tilde{x} : 2(t-s)) \\ &\quad C_T \phi_{d+1}(m - x_0^1, m - a^1, \tilde{a} - \tilde{x}_0; 2s) f_0(x_0) ds da db dx_0. \end{aligned}$$

Using Remark 4.1  $\int_{\mathbb{R}^d} \phi_d(x-a; s) \phi_d(x-b; r) dx = \phi_d(b-a; s+r)$ .  
 We use this trick several times: we integrate with respect to  $a$  so

$$I(x_1) \leq \int_{[0,t] \times \mathbb{R}^{d+2}} \frac{D^\varepsilon \|B\|_\infty^\varepsilon}{\sqrt{(t-s)^\gamma}} \phi_1(b-x^1; 2(t-s)) \phi_1(m-x_0^1; 2s) \phi_{d-1}(\tilde{x}-\tilde{x}_0; 4t) \phi_1(b-m; 4t) f_0(x_0) ds db dx_0,$$

then, we integrate with respect to  $b$  :

$$I(x_1) \leq \int_{[0,t] \times \mathbb{R}^{d+2}} \frac{D^\varepsilon \|B\|_\infty^\varepsilon}{\sqrt{(t-s)^\gamma}} \phi_1(m-x^1; 2(3t-s)) \phi_1(m-x_0^1; 2s) \phi_{d-1}(\tilde{x}-\tilde{x}_0; 2t) f_0(x_0) ds dx_0.$$

Since  $s < t$  then  $3t-s > t$  then

$$\phi_1(m-x_1; 2(3t-s)) \phi_{d-1}(\tilde{x}-\tilde{x}_0; 2t) \phi_1(m-x_0^1; 2s) \leq [\sqrt{4\pi t}]^{-d} [\sqrt{4\pi s}]^{-1}$$

and

$$I(x_1) \leq \int_0^t \int_{\mathbb{R}^d} \frac{D^\varepsilon \|B\|_\infty^\varepsilon}{\sqrt{(4\pi t)^d} \sqrt{(t-s)^\gamma} \sqrt{4\pi s}} f_0(x_0) dx_0 ds.$$

Since  $\int_{\mathbb{R}^d} f_0(x_0) dx_0 = 1$ ,  $\forall x^1 < m$ :

$$I(x_1) \leq \int_0^t \frac{D^\varepsilon \|B\|_\infty^\varepsilon}{\sqrt{(4\pi t)^d} \sqrt{(t-s)^\gamma} \sqrt{4\pi s}} ds$$

is finite since  $\gamma < 2$  and so  $\sup_{x^1 < m} I(x_1) < \infty$ , and the proof of (4.5) is achieved, so the one of Lemma 4.10. •

The proof of Proposition 4.6 is achieved using Lemmas 4.7 to 4.10. •

The proof of Proposition 4.3 is then a consequence of Lemmas 4.4 and 4.5 and Proposition 4.6.

The following concludes this section:

**Proposition 4.11** *The density of probability  $p_V$ , density of the law of  $(M_t, X_t)$  where*

$$\begin{aligned} X_t &= X_0 + \int_0^t B(X_s) ds + W_t, \\ M_t &= \sup_{s \leq t} X_t^1, \end{aligned}$$

*with  $X_0$  independent of  $W$  with density  $f_0 \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ , is an element of the set  $\mathcal{X}$ .*

This is a consequence of Propositions 4.1, 4.2, 4.3. Finally Theorem 1.2 is proved. •

## 5 Tools

In this section, we recall and prove some technical results needed throughout this paper.

**Lemma 5.1** *The adjoint operator*

$$\mathcal{L}^* f(x) = -B(x) \cdot \nabla f(x) - f(x) \sum_{k=1}^d \partial_k B^k(x) + \frac{1}{2} \Delta f(x),$$

is closed and generates a semi-group  $(Q_t)_{t \geq 0}$  that is strongly continuous and bounded with kernel  $\Gamma$ . Besides, for  $0 \leq t \leq T$ ,  $x \in \mathbb{R}^d$  and  $f$  measurable and bounded,

$$(5.1) \quad Q_t(f)(x) := E \left[ f(Y_t^x) \exp \left( - \int_0^t \sum_{k=1}^d \partial_k B^k(Y_u^x) du \right) \right] = \int_{\mathbb{R}^d} f(y) \Gamma(x, y; t) dy.$$

where  $dY_t^x = -B(Y_t^x)dt + dW_t$ ,  $Y_0^x = x$ .

**Proof:** (i) The fact that  $\mathcal{L}^*$  generates  $(Q_t)_{t \geq 0}$  is a consequence of Itô's formula, more specifically, the Feynman-Kac representation.

The operator  $\mathcal{L}^*$  is densely closed, as a consequence of  $B \in C_b^1$ .

(ii) The semi-group property can be deduced from the definition of  $(Q_t)_{t \geq 0}$  above and the Markov property of the process  $Y$ :

$$Y_t^{Y_s^x} \stackrel{(d)}{=} Y_{t+s}^x.$$

Indeed, we write:

$$\begin{aligned} Q_t \circ Q_s(f)(x) &= E \left[ Q_s(f)(Y_t^x) \exp \left( - \int_0^t \sum_{k=1}^d \partial_k B^k(Y_u^x) du \right) \right] \\ &= E \left[ E \left[ f(Y_s^{Y_t^x}) \exp \left( - \int_0^s \sum_{k=1}^d \partial_k B^k(Y_u^{Y_t^x}) du \right) \right] \exp \left( - \int_0^t \sum_{k=1}^d \partial_k B^k(Y_u^x) du \right) \right] \\ &= E \left[ f(Y_{t+s}^x) \exp \left( - \underbrace{\int_0^s \sum_{k=1}^d \partial_k B^k(Y_{t+u}^x) du}_{= \int_t^{t+s} \sum_{k=1}^d \partial_k B^k(Y_u^x) du} \right) \exp \left( - \int_0^t \sum_{k=1}^d \partial_k B^k(Y_u^x) du \right) \right] \\ &= E \left[ f(Y_{t+s}^x) \exp \left( - \int_0^{t+s} \sum_{k=1}^d \partial_k B^k(Y_u^x) du \right) \right] = Q_{t+s}(f)(x). \end{aligned}$$

(iii) We now deal with the boundness of  $Q_t$ : Because  $\text{div} B$  is bounded, we clearly have:

$$|Q_t(f)(x)| \leq CE [ \|f(Y_t^x)\| ].$$



Let

$$Z_t := \exp \left[ \int_0^t \sum_{k=1}^d B^k(Y_s^x) dW_s^k - \frac{1}{2} \int_0^t \|B(Y_s^x)\|^2 ds \right]$$

and  $\mathbb{Q}$  be the probability measure such that

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = Z_t$$

then

$$|Q_t(f)(x)| \leq C E_{\mathbb{Q}} [ |f(Y_t^x)| Z_t^{-1} ].$$

Note that

$$Z_t^{-1} = \exp \left[ - \sum_{k=1}^d \int_0^t B^k(Y_s^x) dY_s^x - \frac{1}{2} \int_0^t \|B(Y_s^x)\|^2 ds \right].$$

According to the Girsanov Theorem,  $Y^x - x$  is a  $\mathbb{Q}$  Brownian motion issued from 0 and

$$\begin{aligned} |Q_t(f)(x)| &\leq C E_{\mathbb{P}} \left[ |f(x + W_t)| \exp \left[ - \int_0^t \sum_{k=1}^d B^k(x + W_s) dW_s^k - \frac{1}{2} \int_0^t \|B(x + W_s)\|^2 ds \right] \right] \\ &\leq C \sqrt{E_{\mathbb{P}} [ |f(x + W_t)|^2 ] E_{\mathbb{P}} \left[ \exp \left[ -2 \int_0^t \sum_{k=1}^d B^k(x + W_s) dW_s^k - \int_0^t \|B(x + W_s)\|^2 ds \right] \right]} \\ &\leq C e^{\|B\|^2 T/2} \sqrt{E_{\mathbb{P}} [ |f(x + W_t)|^2 ]} \end{aligned}$$

since

$$\exp \left[ -2 \int_0^t \sum_{k=1}^d B^k(x + W_s) dW_s^k - 2 \int_0^t \|B(x + W_s)\|^2 ds \right]$$

is a martingale and  $B$  is bounded. Thus, using Tonelli Theorem we have that  $(Q_t)_{t \geq 0}$  is a bounded operator in  $L^2$ :

$$\begin{aligned} \|Q_t(f)\|_{L^2}^2 &\leq C^2 e^{\|B\|^2 T} \int_{\mathbb{R}^d} E [ |f(x + W_t)|^2 ] dx \\ (5.2) \quad &= C^2 e^{\|B\|^2 T} E \left[ \int_{\mathbb{R}^d} |f(x + W_t)|^2 dx \right] = C^2 e^{\|B\|^2 T} \|f\|_{L^2}^2. \end{aligned}$$

(iv) Finally, let us say a few words on the strong continuity. First, we expand:

$$|Q_t(f)(x) - f(x)| = \left| \mathbb{E} \left( f(Y_t^x) \exp \left( - \int_0^t \sum_{k=1}^d \partial_k B^k(Y_u^x) du \right) - f(x) \right) \right|$$

$$\begin{aligned}
&\leq \left| \mathbb{E} \left[ \left( f(Y_t^x) - f(x) \right) \exp \left( - \int_0^t \sum_{k=1}^d \partial_k B^k(Y_u^x) du \right) \right] \right| \\
&\quad + \left| \mathbb{E} \left[ f(x) \left( \exp \left( - \int_0^t \sum_{k=1}^d \partial_k B^k(Y_u^x) du \right) - 1 \right) \right] \right| \\
&= I(t, x) + II(t, x)
\end{aligned}$$

Next, since  $\text{div}B$  is bounded, we can use the dominated convergence theorem:

- Point-wise convergence:

$$f(x) \left( \exp \left( - \int_0^t \sum_{k=1}^d \partial_k B^k(Y_u^x) du \right) - 1 \right) \xrightarrow{t \rightarrow 0} 0$$

- Domination:

$$\left| f(x) \left( \exp \left( - \int_0^t \sum_{k=1}^d \partial_k B^k(Y_u^x) du \right) - 1 \right) \right| \leq |f(x)|, \quad dx \otimes d\mathbb{P} \text{ integrable}$$

Thus, we have

$$\int |II(t, x)|^2 dx \xrightarrow{t \rightarrow 0} 0.$$

Next, and again, since  $\text{div}B$  is bounded, we can proceed as in (iii) to get:

$$I(t, x) = \left| \mathbb{E} \left[ \left( f(Y_t^x) - f(x) \right) \exp \left( - \int_0^t \sum_{k=1}^d \partial_k B^k(Y_u^x) du \right) \right] \right| \leq C_T \mathbb{E}_{\mathbb{P}} \left( f(x + W_t) - f(x) \right),$$

and thus conclude using the strong continuity of the Brownian semigroup (note that is is enough to obtain continuity in a dense subspace, due to the domination condition (5.2)).

Let  $f \in L^2(\mathbb{R})$  and  $\varepsilon > 0$ , there exists  $g \in L^2 \cap C(\mathbb{R}^d, \mathbb{R})$  such that  $\|g - f\|_{L^2} < \varepsilon$  and  $t_0$  such that for  $0 \leq t < t_0$ ,  $\|Q_t(g) - g\|_{L^2} \leq \varepsilon$ . Then for  $0 \leq t < t_0$ , and using (5.2)

$$\begin{aligned}
\|Q_t(f) - f\|_{L^2} &\leq \|Q_t(f - g)\|_{L^2} + \|f - g\|_{L^2} + \|Q_t(g) - g\|_{L^2} \\
&\leq 2\varepsilon + C e^{\|B\|^2 T/2} \|f - g\|_{L^2}.
\end{aligned}$$

Thus,  $(Q_t)_{t \geq 0}$  that is strongly continuous. •

The following lemmas are a collection of tools needed in Section 2 to prove uniqueness with null initial condition.

**Lemma 5.2** *For all  $t \leq T$ , almost surely with respect to  $y^1 \in \mathbb{R}$ ,  $\int_0^t f(y^1, s) ds < \infty$  where  $f(y^1, s) := \int_{\mathbb{R}^{d-1}} |q(y^1, y^1, \tilde{x}; s)| d\tilde{x}$ .*

**Proof:**

$$\int_0^T f(y^1, s) ds = \int_0^T \int_{\mathbb{R}^{d-1}} |q(y^1, y^1, \tilde{x}; s)| d\tilde{x} ds.$$

Using Item (b) in Definition 1.1 yields  $\int_0^T \int_{\mathbb{R}^d} |q(y^1, y^1, \tilde{x}; s)| dy^1 d\tilde{x} ds < \infty$ . Tonelli Theorem is used since  $f \geq 0$ :

$$\int_{\mathbb{R}} \int_0^T f(y^1, s) ds dy^1 = \int_0^T \int_{\mathbb{R}^d} |q(y^1, y^1, \tilde{x}; s)| dy^1 d\tilde{x} ds < \infty.$$

So the integrand  $\int_0^T f(y^1, s) ds$  is finite  $dy^1$  almost surely. •

**Lemma 5.3** Let  $g(t) := \frac{1}{\sqrt{t}}$ . Then for all  $n \geq 0$ ,  $g^{*n}(t) = t^{\frac{n-2}{2}} \frac{(\Gamma(\frac{1}{2}))^n}{\Gamma(\frac{n}{2})}$ .

**Proof:** Actually, this is satisfied for  $n = 1$  since the formula provides  $g(t) = \frac{1}{\sqrt{t}}$ .

We now assume the property is satisfied for any  $k \leq n$ :  $g^{*k}(t) = t^{\frac{k-2}{2}} \frac{(\Gamma(\frac{1}{2}))^k}{\Gamma(\frac{k}{2})}$ , and we compute

$$g^{*(n+1)}(t) = \int_0^t s^{\frac{n-2}{2}} \frac{(\Gamma(\frac{1}{2}))^n}{\Gamma(\frac{n}{2})} \frac{1}{\sqrt{t-s}} ds = \frac{(\Gamma(\frac{1}{2}))^n}{\Gamma(\frac{n}{2})} \int_0^t \frac{s^{\frac{n-2}{2}}}{\sqrt{t-s}} ds.$$

In the integral we recognize a Beta function after the change of variable  $s = tu$ :

$$\int_0^t \frac{s^{\frac{n-2}{2}}}{\sqrt{t-s}} ds = t^{\frac{n-1}{2}} \int_0^1 \frac{u^{(n-2)/2}}{\sqrt{1-u}} du = B\left(\frac{n}{2}, \frac{1}{2}\right) t^{\frac{n-1}{2}} = \frac{\Gamma(n/2)\Gamma(\frac{1}{2})}{\Gamma(\frac{n+1}{2})} t^{\frac{n-1}{2}}.$$

Thus the lemma is proved:

$$g^{*(n+1)}(t) = \frac{(\Gamma(\frac{1}{2}))^n}{\Gamma(\frac{n}{2})} \frac{\Gamma(n/2)\Gamma(\frac{1}{2})}{\Gamma(\frac{n+1}{2})} t^{\frac{n-1}{2}}.$$
•

**Lemma 5.4** Let  $\phi(u) := \begin{cases} a \exp -\frac{1}{1-u^2}, & \text{if } u \in ]-1, 1[ \\ 0 & \text{else} \end{cases}$ , and  $\phi_\varepsilon(x) = \frac{1}{\varepsilon} \phi(\frac{x}{\varepsilon})$ ,  $a$  such that

$\int \phi(u) du = 1$ . Then when  $\varepsilon \rightarrow 0$  there is a convergence to Dirac measure:  $\lim_{\varepsilon \rightarrow 0} \phi_\varepsilon(m - x^1) dx^1 = \delta_m(dx^1)$  meaning for all  $f$  continuous with respect to  $x^1$  and satisfying  $\sup_{x^1} |f(x^1, x; t)| \in L^1([0, T] \times \mathbb{R}^{k+1})$ ,  $\forall m$  :

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\mathbb{R}^{k+1}} f(x^1, x; t) \phi_\varepsilon(m - x^1) dx^1 dx dt \rightarrow \int_0^T \int_{\mathbb{R}^k} f(m, x; t) dx dt.$$

**Proof:** We operate the change of variable  $u = \frac{m-x^1}{\sqrt{\varepsilon}}$

$$\int_0^T \int_{\mathbb{R}^{k+1}} [f(x^1, \tilde{x}; t) - f(m, \tilde{x}; t)] \phi_\varepsilon(x^1 - m) dx^1 d\tilde{x} dt =$$

$$\int_0^T \int_{\mathbb{R}^{k+1}} [f(m - u\sqrt{\varepsilon}, \tilde{x}; t) - f(m, \tilde{x}; t)] \phi(u) du d\tilde{x} dt.$$

Since  $f$  is continuous with respect to  $x^1$  and  $\sup_{x^1} |f(x^1, \tilde{x}; t)| \in L^1([0, T] \times \mathbb{R}^k)$  the dominated Lebesgue Theorem is applied so yields the expected limit.  $\bullet$

**Lemma 5.5** *Let  $G$  be a bounded function with compact support, then  $dtd\tilde{x}$  almost surely  $x^1 \mapsto \int_0^t \int_{\mathbb{R}^d} G(y_1) q(y^1, y; s) \Gamma(x^1, \tilde{x}, y; t - s) d\tilde{y} ds$  is continuous.*

**Proof :** For an easier reading, we here recall the previous (2.14), (2.15), (2.16):

$$\Gamma(x, y; t) = \Gamma_0(x, y; t) + \Gamma_1(x, y; t) \text{ where } \Gamma_0(x, y; t) = \frac{e^{-\frac{\|x-y\|^2}{2t}}}{\sqrt{2\pi t}^d},$$

and for all  $y \in \mathbb{R}^d$ ,  $t > 0$   $x \mapsto \Gamma_1(x, y; t) \in C^1(\mathbb{R}^d)$ . Moreover, for  $\alpha \in ]0, 1[$  there exists some positive constants  $C$  and  $c$  such that

$$|\Gamma_1(x, y; t)| \leq Ct^{-\frac{d}{2}+\alpha} e^{-\frac{\|x-y\|^2}{ct}} ; \quad |\partial_{x^i} \Gamma_1(x, y; t)| \leq Ct^{-\frac{d+1}{2}+\alpha} e^{-\frac{\|x-y\|^2}{ct}}.$$

First, note that for all  $(\tilde{x}, y, t - s)$ ,  $x^1 \mapsto \Gamma(x^1, \tilde{x}, y; t - s)$  is continuous. Second we will prove that

$$(5.3) \quad \int_0^T \int_0^t \int_{\mathbb{R}^{2d-1}} |G(y_1) q(y^1, y; s)| \sup_{x^1} |\Gamma(x^1, \tilde{x}, y; t - s)| d\tilde{x} dy ds dt < \infty.$$

According to estimation (2.15),

$$|\Gamma(x^1, \tilde{x}, y; t - s)| \leq C(t - s)^{-\frac{d}{2}} e^{-\frac{\|\tilde{x}-\tilde{y}\|^2}{ct}} \quad \forall (\tilde{x}, y, t) \in \mathbb{R}^{2d-1} \times [0, T].$$

Integrating with respect to  $\tilde{x}$  we obtain

$$\begin{aligned} & \int_0^T \int_0^t \int_{\mathbb{R}^{2d-1}} |G(y_1) q(y^1, y; s)| \sup_{x^1} |\Gamma(x^1, \tilde{x}, y; t - s)| d\tilde{x} dy ds dt \\ & \leq Cc^{d-1} \int_0^T \int_0^t \int_{\mathbb{R}^d} |G(y_1) q(y^1, y; s)| (t - s)^{-1/2} dy ds dt \end{aligned}$$

Using the fact that  $G$  is continuous and has compact support and Cauchy-Schwartz inequality,

$$\begin{aligned} & \int_0^T \int_0^t \int_{\mathbb{R}^{2d-1}} |G(y_1) q(y^1, y; s)| \sup_{x^1} |\Gamma(x^1, \tilde{x}, y; t - s)| d\tilde{x} dy ds dt \\ & \leq Cc^{d-1} \|G\|_{L^2} \int_0^T \int_0^t \sqrt{\int_{\mathbb{R}^d} q^2(y^1, y, s) dy} (t - s)^{-1/2} ds dt \end{aligned}$$

Using Fubini theorem, we obtain

$$\int_0^T \int_0^t \int_{\mathbb{R}^{2d-1}} |G(y_1) q(y^1, y; s)| \sup_{x^1} |\Gamma(x^1, \tilde{x}, y; t - s)| d\tilde{x} dy ds dt$$

$$\leq Cc^{d-1}\|G\|_{L^2} \int_0^T \sqrt{\int_{\mathbb{R}^d} q^2(y^1, y, s) dy} \int_s^T (t-s)^{-1/2} dt ds$$

Then,

$$(5.4) \quad \int_0^T \int_0^t \int_{\mathbb{R}^{2d-1}} |G(y_1)q(y^1, y; s)| \sup_{x^1} |\Gamma(x^1, \tilde{x}, y; t-s)| d\tilde{x} dy ds dt \\ \leq 2T^{1/2} Cc^{d-1} \|G\|_{L^2} \int_0^T \sqrt{\int_{\mathbb{R}^d} q^2(y^1, y, s) dy} ds < \infty$$

according to Item (b) of Definition 1.1. That means that  $d\tilde{x}dt$  almost surely

$$(5.5) \quad \int_0^t \int_{\mathbb{R}^d} |G(y_1)q(y^1, y; s)| \sup_{x^1} |\Gamma(x^1, \tilde{x}, y; t-s)| dy ds < \infty.$$

Third, for all  $(\tilde{x}, t)$  such that (5.5) holds, according to the dominated convergence theorem  $x^1 \mapsto \int_0^t \int_{\mathbb{R}^d} G(y_1)q(y^1, y; s)\Gamma(x^1, \tilde{x}, y; t-s)d\tilde{y}ds$  is continuous. •

**Lemma 5.6** *Let  $F$  be a continuous function with compact support,  $dt dy^1$  almost surely  $x^1 \mapsto \int_0^t \int_{\mathbb{R}^{2d-2}} F(\tilde{x})q(y^1, y^1, \tilde{y}, s)\Gamma(x^1, \tilde{x}, y^1, \tilde{y}; t-s)d\tilde{y}dsd\tilde{x}$  is continuous.*

**Proof:** Let  $G$  be the indicator of the interval  $[n, n]$ . According to estimation (5.4) and since  $F$  is bounded, almost surely  $dt dy^1$  on  $[0, T] \times [-n, n]$

$$(5.6) \quad \int_0^t \int_{\mathbb{R}^{2d-2}} |F(\tilde{x})| |q(y^1, y^1, \tilde{y}, s)| \sup_{x^1} |\Gamma(x^1, \tilde{x}, y^1, \tilde{y}; t-s)| d\tilde{y}dsd\tilde{x} < \infty.$$

Since  $(\tilde{x}, y, t-s) \mapsto \Gamma(x^1, \tilde{x}, y; t-s)$  is continuous, using Lebesgue dominated theorem for  $(t, y_1)$  such that (5.6) holds,  $x^1 \mapsto \int_0^t \int_{\mathbb{R}^{2d-2}} F(\tilde{x})q(y^1, y^1, \tilde{y}, s)\Gamma(x^1, \tilde{x}, y^1, \tilde{y}; t-s)d\tilde{y}dsd\tilde{x}$  is continuous.

Since  $\cup_n [-n, n] = \mathbb{R}$  this achieves the proof of Lemma 5.6. •

## 6 Conclusion and perspectives

In this work we have shown uniqueness to equation (1.1). Along with previous work on this matter, namely Proposition 4.3 in [6], where a series expansion is obtained, a numerical approximation of this density could be proposed, based on the joint density of the Brownian motion and its running supremum.

Moreover, in [4] is obtained an analogous PIDE in the case of Lévy processes. The uniqueness of a solution to such a PIDE should be accessible with the same kind of method.

Finally, we would like to point out the similarities between in the methods we used in this work together with [5], and the parametrix technique. In its simplest form, the parametrix method is a continuity method that approximates the solution of an equation with the solution of the same equation, but with constant coefficients (frozen equation). The quality of the approximation is then controlled using the difference of infinitesimal generator. This method can be used to derive a series expansion for the solution of the PDE, and it has been pointed out that the estimates giving the convergence can also be used to derive uniqueness to the martingale problem (see e.g. Menozzi [13]). However, in our case, it can be shown that the process  $(W_t, W_t^*)_{t \geq 0}$  does not have a generator. Upcoming work should investigate the similarities between the parametrix derivation and the method used here, that are based on a combination of Girsanov transform and Malliavin calculus.

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