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► **To cite this version:**

Emmylou Haffner. Going to the source(s) of sources in mathematicians' drafts. M. Zack et D. Waszek. Research in History and Philosophy of Mathematics. Annals of the Canadian Society for History and Philosophy of Mathematics, Birkhäuser, pp.83-110, 2024, 978-3-031-46192-7. hal-04028943

HAL Id: hal-04028943

<https://hal.science/hal-04028943>

Submitted on 20 Feb 2024

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Going to the source(s) of sources in mathematicians' drafts

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Postprint, please do not cite this version

Abstract

The mathematical text in its published form, as we are most used to reading it, is a carefully structured and polished means of communicating results to the scientific community. It is, as Reuben Hersh put it, the ‘front’ of mathematics. In this paper, I propose to look at the ‘back’ of mathematics, at what happens in the privacy of drafts, which can certainly be seen as the mathematician’s laboratory. Considering that these preliminary texts are a part of the mathematical practice — and indeed a crucial one — I will show that they allow us to understand the shaping of mathematics in deep and significant ways. Using a selection of examples, I will focus on questions related to the materiality of mathematical texts, how textual elements and mathematical practices work with each other, the processes of writing in mathematics, and the choices made in writing a text deemed suitable for communication to the scientific community.

1 Introduction

In modern and contemporary mathematics, the first means of diffusion of (new) mathematical knowledge is very often the written text. Such texts, which are the core medium for mathematicians, but also for historians and philosophers of mathematics are, of course, the result of a fairly long process. This process is often made invisible to readers, who are presented with a polished, reconstructed and restructured version of the author’s initial work.

In [Hersh(1991)], Reuben Hersh suggested, using an idea from sociologist Erving Goffman, that there is a “front” and a “back” of mathematics: areas in which the public is admitted, areas in which it is not. Similarly, albeit more provocatively, in 1963, Nobel Prize winner Peter Medawar gave a talk entitled “Is the scientific paper a fraud?”¹ in which he suggested that the norms of writing for scientific publication did not reflect how research truly happened, in particular because of the very rigid structure of the papers, the setup of a narrative falsely reconstructing the research process, and the dehumanization of the discovery process. In this paper, I would like to make the case for looking at the “back” of mathematics, the parts of mathematics which are not written for the public, and argue that, as Medawar

¹[Medawar(1991)]

stated, our conception and understanding of mathematical research are biased by such publication norms, and that epistemological and methodological reflections tend to miss what really happens in the research process. To do so, I will be using a specific kind of archival material, namely mathematicians' drafts.²

A core methodological point of my analysis will be the rejection of a supposed divide between the *contents* of scientific texts and their *materiality*. Such an approach was in parts initiated by Karine Chemla's suggestion to jointly use history of mathematics and history of text. In the introduction of her 2004 book, *History of science, history of text*, she explains that the starting point of this approach is "the hypothesis that [scientific] texts, as such, are to be presumed to be historical objects in every respect":

In a first, weaker sense, [texts] are historical objects simply because they were produced at different times and within given working communities. However, in a second, stronger sense [...] they are historical objects because they were produced, as texts, at the same time as the concepts, results and theories which they contain were, and essentially contribute to the shaping and sharing of these ideas.

In other words, we reject the presupposition that, once concepts, results or theories have been obtained by other ways, in an immaterial space, they are merely transcribed in a textual form that remains indifferent to them. (...) [We assume that] the texts elaborated in the course of the practice of science belong, as such, to the outcome of the scientific work, along with the concepts, results, or theories, in intimate interaction with which they were shaped. [Chemla(2004), viii]

In this light, it should be clear that mathematicians' drafts are tools for their research. They are their laboratory, the place where we can witness the steps of their research, some of which were erased from the publication. As such, drafts give us an internal view of mathematical practice, which can be different from the idealised image of a very normed, somewhat rigid, text, which we have become used to – especially in contemporary mathematics – and they allow us to better understand the shaping of mathematical ideas, results or theories. Not only do these sources provide us with additional elements on their author's mathematics, of crucial importance for the history of mathematics,³ but they also force us to reconsider some of what we thought we knew by studying only published works. For example, in [Descotes(2010)], Descotes presents a mathematical manuscript of Pascal's, discovered while working on the *Recueil original* of Pascal's *Pensées*, which reveals – unlike what his published works suggest – that Pascal used symbolism and computations to support his geometrical reasoning. Historians of mathematics are also familiar with Carl Siegel's and Harold Edwards' arguments against an image of Riemann as a purely conceptual mathematician:⁴ relying on his *Nachlass*, they argued that Riemann's so-called conceptual mathematics, in fact, relied deeply on computations – or, as Edwards puts it, Riemann "did not venture into these higher realms without doing a lot of serious computation to lay the groundwork

²I will consider as *drafts* all *private* research manuscripts, not written to be communicated to another reader. It will not matter, here, whether they are 'just' working manuscripts without specific publication intents or part of a process that led to a published work.

³See, for example, [Lützen(1990), Knobloch(2004), Sauer and Schütz(2020), Bustamante(2022)]

⁴[Siegel(1932), Edwards(2010)]

for his flights” [Edwards(2010), 64]. In my own work,⁵ I showed that a similar observation can, in fact, be made for Dedekind’s researches on modules and *Dualgruppen* (equivalent to our concept of lattice). Rather than flying into the abstract set-theoretic realms of *Dualgruppen* right away, or even as soon as he could, Dedekind laid a rather thick ground of computations, concrete examples, tables, diagrams, etc., seemingly going against his statements that mathematics should be essentially “conceptual”.⁶

At the same time, it is important to acknowledge that drafts are a very specific type of manuscripts, insofar as they show intermediate states of writing, and, most often, were not written to be read by anybody but their author. Hence, they require caution in interpreting them. In addition to careful critical historical analysis, it is useful to consider methods from literary studies, which have been studying writers’s drafts for a long time, and in particular the approach called “*critique génétique*”, genetic criticism:

The term *critique génétique* was coined by Louis Hay in 1979 in the title of a collection of essays, *Essais de critique génétique*. Genetic critics focus on the temporal dimension of writing and regard a work of literature as a process rather than a product. The end result remains inextricably bound up with its textual memory, that is, the numerous textual transformations that preceded its publication. [...] [Their] main objective [...] is to examine the writing mechanisms and to reconstruct the genesis of the literary work by studying notebooks, manuscripts, typescripts, and proofs, focusing on the “potentiality” of writing. The confrontation of a published text with all its previous versions gives the reader an idea of what it might have become. Genetic critics study the contingencies of the writing process as research objects in their own right, independent from the publication of a finished product (a “corrected” text). [Van Hulle(2009), 3]

In this paper, I will raise several issues in these directions. I will propose to discuss textual practices developed by mathematicians in their research. I will do so using (mostly) a selection of drafts from Leibniz (Gottfried Wilhelm Leibniz – Bibliothek Niedersächsische Landesbibliothek Hannover, abbreviated in LH), Richard Dedekind (Cod. Ms. Dedekind, Niedersächsische Staats-und Universitätsbibliothek Göttingen), and Élie Cartan (fonds 38J, Académie des Sciences de Paris). The selection of sources, here, is admittedly subjective. It should be clear that it is not at all meant as an exhaustive look into textual and mathematical practices in drafts, but as a way to open the door and encourage us to look behind the scene whenever possible. Drafts have been overlooked as historical sources because they show temporary states of mathematics. I believe that this is what makes them most interesting.⁷

⁵[Haffner(2018a), Haffner(2023)]

⁶Another example of a new light shed by studying drafts is Clare Moriarty’s recent paper on the draft of Maclaurin’s *Treatise on Fluxions* and what it reveals of the initial ideological concerns of its author [Moriarty(2022)]. There is a number of recent or ongoing works using archival material akin to drafts, many of which I will cite in this paper – there is also a considerably larger number of available (yet often unexploited) notebooks and drafts that are awaiting exploration. My own project funded by the Émergence(s) program of the Ville de Paris, “Brouillons mathématiques” gathers a small team of historians and philosophers focused on such issues (see <http://www.item.ens.fr/brouillons-mathematiques-projet-emergences-2022-2026/>).

⁷Of course, archives contain many unpublished works, in more or less finished states. As we know,

2 Written tools and artefacts for research

In mathematical *drafts*, even more so than in mathematical texts in general, the variety of writing practices is striking. Indeed, drafts being a private space of writing, the writing in drafts is freed from practical constraints related to printing, and freed as well from a number of disciplinary, sociological norms for publication. In their drafts, the mathematician is free to write however and wherever they like. Compared to publications (and in large part to letters as well) which are written to be *read*, drafts present striking material characteristics. Each page has its own semiotics. This material singularity is important, as it testifies to the multiplicity of writing practices used by mathematicians throughout their work. It is also striking to observe that this diversity of practices can be observed in the work of one and the same mathematician as well, suggesting that different textual practices are used for different reasons and produce different sorts of knowledge. And drafts are, of course, a privileged place to observe mathematical invention, in particular the production of notations, signs, written tools and artefacts for research, which will be the focus of this section.

As should be clear, writing, in its different forms and shapes, is the mathematician's main tool to convey and materialize thought. How a mathematician writes gives us concrete information about the conceptual and textual paths followed, and provides precious clues to understand their processes of working and the genesis of mathematical concepts and texts. Hence, it is essential to pay attention to the uses, and indeed the changes in uses of notations, tables, diagrams and other written tools, as well as computations, hypotheses, proofs, and so on, as they testify of the mathematical work *as it is being done*.

I have written elsewhere about Dedekind's drafts,⁸ and the shaping of written tools and artefacts to accompany and support his mathematical practice. I have highlighted the genesis – or more exactly geneses – of his notations for computing with modules in a way that is not only efficient but also allows him to put forward structural properties of *Dualgruppen*. Directly related to these notations and their role in computations, is Dedekind's use of spatialization of writing: how he arranged his writing on the paper sheet to guide computations, how he used spatial arrangements of writing to clarify and arrange the results of said computations, and how he devised spatial tools, in particular columns and tables, but also diagrams, to both support his research and conceptualize his results. This is striking not only in that his careful crafting of notations and written visualization tools is, indeed, efficient both computationally and visually to explore the theory of modules and generalize it to the more abstract structure of *Dualgruppen*, but also because Dedekind famously advocated for mathematics relying on “concepts” and internal characteristics, rather than on “representations”. Drafts thus reveal large parts of Dedekind's mathematical practice that were effectively hidden when writing the version of his work he considered suited for communication to the scientific community.

Rather than repeat myself with Dedekind's drafts, I would like to point out how some of these aspects of draft writing can be found in other authors. As there are obvious limits of space in this paper, I will concentrate on two authors, namely

many of them have been published and served greatly in advancing our understanding of the history of mathematics. This is not my focus in this paper. I would rather like to focus, here, on aspects of mathematics *in the making*, on temporary steps before the completion or final stages of a readable or publishable text.

⁸[Haffner(2018a), Haffner(2023)]

G. W. Leibniz⁹ and Élie Cartan.

2.1 Cartan’s notational changes

Élie Cartan’s notebooks show how notational choices are ingrained in personal practice. Cartan, a prominent French mathematician of the late 19th to mid-20th century, left a rather large archive, which is kept at the Académie des Sciences in Paris (fonds 38J). In this archive, there are over fifty notebooks, covering most periods of his career from 1893 to 1947.¹⁰ Those are beautiful sources, but Cartan’s mathematics is notoriously difficult, and so are his notebooks. Nevertheless, they provide a compelling inside view of his works, with mathematical research, reading notes, lecture plans, bibliographies, administrative notes, and, more rarely, some personal notes. Cartan’s notebooks do not follow any sort of thematic order, and all the above types of work – as well as the various mathematical subjects he investigated – are blended together in the notebooks. As such, his notebooks are testimonies of a moment of his work. One of the most remarkable aspects of these notebooks is the amount of extremely dense computations, and Cartan’s use of it to understand and explore various mathematical subjects. Of course, any sort of computing is deeply reliant on notations, but it is interesting to note that, in Cartan’s notebooks, changing notations played an important role in his mathematical practice. As it would be difficult to do justice to Cartan’s drafts in such a limited space, let me concentrate on a typical (and understandable) case of switching notations to further mathematical understanding: when reading and taking notes about other people’s works.¹¹

For a friendly example, let us look at Cartan’s notes on Dirk Struik’s 1922 doctoral dissertation *Grundzüge der Mehrdimensionalen Differentialgeometrie* (Foundations of multidimensional differential geometry).¹² Cartan’s notes start on p. 51, in section “6. Parallel V_{n-1} ” (where V_{n-1} are $n - 1$ -dimensional manifolds with arbitrary quadratic measure). Below is a correspondence between the beginning of Struik’s and Cartan’s texts, with similar colors indicating corresponding contents:¹³

⁹While I do not pretend to possess any sort of expertise on Leibniz’s manuscripts as I might have been able to build for Dedekind’s, the material available is too amazing to be overlooked. My remarks on Leibniz have greatly benefited from my discussions with members of the ERC project PHILUUMM, to whom I am very grateful.

¹⁰See <http://eliecartanpapers.ahp-numerique.fr/> and [Haffner(2017a)].

¹¹On note taking as a private scientific practice, see [Bustamante(2020), Bustamante(2022)].

¹²[Struik(1922)]

¹³Unless stated otherwise, translations are mine

Struik

Let the parameter lines x^{a_1} be geodesic lines and x^{a_1} , their arc length, calculated from the corresponding intersection point with a determined V_{n-1} parameter e.g. with $x^{a_1} = 0$. Then, there is (71) $g_{a_1 a_1} = e'_{a_1} \cdot e'_{a_1} = 1$; and the $\infty^1 V_{n-1} : x^{a_1} = \text{const.}$ are the locus of the end-points of the same arcs, which are traced on the geodesic x^{a_1} -lines from $x^{a_1} = 0$.

Then, e'_{a_1} is the unit vector, and it satisfies the equation:

$$(72) \quad e'_{a_1} \cdot \nabla e'_{a_1} = 0$$

or

$$(73) \quad a_{a_1 a_1} a^\lambda = a_{a_1 a_1} a_\lambda = 0$$

From (71) follows, according to (32):

$$(74) \quad a_{a_1 \lambda} a^{a_1} = a_{\lambda a_1} a_{a_1} = 0.$$

Thus:

$$(75) \quad \frac{\partial g_{a_1 \lambda}}{\partial x^{a_1}} = a_{a_1 a_1} a_\lambda + a_{\lambda a_1} a_{a_1} = 0$$

and the $g_{a_1 \lambda}$ are also independent from x^{a_1} .

Now, we assume that the parameter lines x^{a_1} are orthogonal to $x^{a_1} = 0$. Then, there is for $x^{a_1} = 0$:

$$(76) \quad e'_{a_1} \cdot e'_\lambda = g_{a_1 \lambda} = 0. \text{ Since according to}$$

(75) the $g_{a_1 \lambda}$ are not dependent of x^{a_1} , they are all zero:

$$(77) \quad g_{a_1 \lambda} = 0,$$

and the parameter lines are also orthogonal to all $\infty^1 V_{n-1} : x^{a_1} = \text{const.}$

[Struik(1922), 51]

Struik then gives the following theorem:

If through each point of a V_{n-1} the geodesic lines normal to it are drawn and on them from the V_{n-1} equal arcs are traced, then the locus of the end points is again a V_{n-1} normal to the geodesic lines. [Struik(1922), 51]

Cartan's version unfolds and reformulates the contents in the following way:

If the V_{n-1} intersect equal arcs of the geodesics, we have (3) and the equations of (4) become (according to (2))

$$d(e_i e_n) = \sum (\gamma_{i n}^k - \gamma_{n i}^k) e_k e_n$$

linear and homogeneous eq[ua]tions such that if [for] $(V_{n-1})_0$ the $e_i e_n$ are null, they are null for all the V_{n-1} . Hence, by placing constant arcs on the geodesics normal to $(V_{n-1})_0$, the locus of the endpoints is again a V_{n-1} normal to the geodesics.

Conversely, if the V_{n-1} are orthogonal to the (\mathcal{C}) , then we have, according to (2) $\gamma_{n i}^i = 0$ according to (4) $\gamma_{i n}^i = 0$ hence, the [equations] (3) are verified, and the V_{n-1} decompose equal arcs of the geodesics. (Fonds 38J, 1-54, 11)

Cartan

Consider a family of V_{n-1} and a congruence of geodesics forming filling the entire space with all the V_n .

Consider in one point e_n the unit vector tangent to the geodesic (\mathcal{C}) , e_1, e_2, \dots, e_n the vectors tangent to the V_{n-1} .

We express the hypotheses in the following way

1. For[?] $\omega^i = 0$ ($i < n$), we have $de_n = 0$, which gives us

$$(1) \quad \gamma_{n n}^i = 0 \quad \gamma_{n n}^n = 0 \quad (i = 1, \dots, n-1)$$

2. $d(e_n^2) = 0 \quad \omega_n^n + \sum_k \omega_n^k e_k e_n = 0$, which gives us in

$$(2) \quad \gamma_{n n}^i + \sum_k \gamma_{n i}^k + e_k e_n = 0$$

This being set, let us express the fact that the arcs of geodesics comprised between two given V_{n-1} are all equal, that is, $\delta(ds) = 0$ (d on V_{n-1} , δ on the \mathcal{C}) or $\delta(\omega_n^b(d)) = 0$.

We have $\delta\omega_n^n(d) = \delta\omega_n^n(d) - d\omega_n^n(\delta) = \sum_{i=1}^{i=n-1} \omega^i(\delta) - \omega^n(d)\omega_n^n(\delta)$ which gives us the $n-1$ equations

$$(3) \quad \gamma_{i n}^i - \gamma_{n i}^i = 0.$$

Let us express now that one of the V_{n-1} being orthogonal trajectory [?] of \mathcal{C} , the other ones are as well. Now, we have $d(e_i e_n) = \omega_i^n + \sum_k \omega_i^k e_k e_n + \omega_n^n e_i e_n + \sum_k \omega_n^k e_i e_k$ or, according to

$$(1) \quad d(e_i e_n) = \gamma_{i n}^i + \sum_k \gamma_{i n}^k e_k e_n$$

(Fonds 38J, 1-54, 10)

Cartan’s notes continue for several pages, but it would takes us too far to follow his reading (and rewriting) of Struik.¹⁴ What matters, here, is the observation that to better understand, and to a certain extent appropriate, what he is reading, Cartan reformulates it in his own notation. In many cases, Cartan merely translates results and proofs in his own language, so his notational changes might as well be a question of taste.¹⁵ Still, it suggests further that this notational adjustments help reframing and clarifying notions and results. This is further supported by the fact that it happens fairly often that, in his own computations, Cartan interrupts himself and “start[s] again differently” or “reformulate[s]” his computations.

2.2 A very cursory glimpse in Leibniz’s drafts

Leibniz’s drafts provide a wealth of material to study written tools and artefacts. Let me start with notations.¹⁶ A well-known example of witnessing the birth of notation is the introduction of the dx notation, in *Methodi tangentium inversae exempla* (Examples of the inverse method of tangents (1675), A VII, 5, 324),¹⁷ in which Leibniz introduces both the \int sign and the dx notation. The signs are introduced (and used, for the first page and half) in the following way:

$$\int \overline{wz} = \frac{y^2}{2} \text{ or } wz = \frac{y^2}{2d}$$

[In the margin:] \int sum, d difference. [Leibniz and Child(2008), 93 (the marginal note is not included in Child’s translation)]

On the next page, in the midst of his reasoning, a marginal note states a notational change:

Remark dx . is the same as $\frac{x}{d}$., that is, the difference between two closest x . (A VII, 5, 324¹⁸)

And from there, Leibniz uses the dx notation. This notational change is certainly neither trivial nor inconsequential – despite being made in a marginal note – for his calculus, considering the emphasis he put on his symbolism.

That being said, Leibniz’s drafts also provide us with an important warning *not* to put too much weight on notations alone in general. An example of this (but certainly not the only one) is the manuscript *Elemens du calcul* (LH 35, 4, 12, Bl. 1-2),¹⁹ a text on the foundations of algebra with a list of axioms for elementary algebra (on numbers and magnitudes). It opens with:

¹⁴The interested reader can look directly at Cartan’s notes here: <http://eliecartanpapers.ahp-numerique.fr/items/show/37>.

¹⁵This is not to say that his reading of his contemporaries’ works never led to important breakthroughs in his own research, as his reading of Weyl’s or of Einstein’s works testify.

¹⁶Notations in Leibniz’s mathematics have been fairly well studied, first in [Cajori(1929)], but also in [Serfati(2005), Knobloch(2010), Trunk(2016), Waszek(2018), Gentil(2021)]. Yet the sheer amount of manuscripts available makes this question a still largely to be explored one

¹⁷Citations for Leibniz manuscripts are from the so-called *Akademie* edition [Leibniz(1923 -)]. VII designates the seventh series on mathematical writings. 5 means the fifth volume. This precise text is translated and commented in [Leibniz and Child(2008)] (see also [Hofmann(1974)], chap. 13).

¹⁸Child’s translation does not reproduce this note – although both Gerhardt’s original edition and the 2008 edition by the Leibniz-Akademie do.

¹⁹See [Rabouin(2021)].

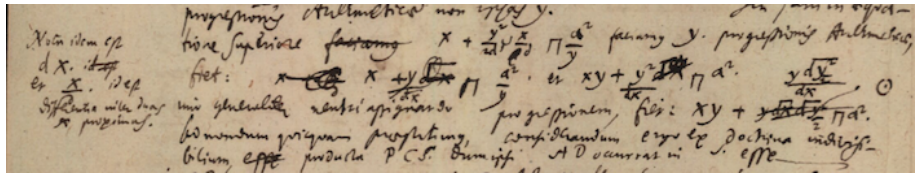


Figure 1: LH 35, 5, 9 Bl. 2 .

Any number can be marked by a letter, an imaginary number, ou by any other character one wishes, of those there are many; as long as this character is not already assigned to some meaning in the calculus in which this number must enter. Hence, a number, known or unknown, determined or undetermined, can be marked by a , b , x , y , A , B , C , \odot , \mathcal{D} , φ , even 10, 11, 12, etc. 100, 101, etc.²⁰ but these numbers then only signify letters and are used to mark some relation that exist between the numbers we mark. And we must, then, distinguish them by some mark from the numbers they usually signify. (LH 35, 4, 12, Bl. 1)

More interesting is the definition of several signs for some of the operations, as well as for parentheses (for which he uses a comma, a vinculum or proper parentheses):

- (5) The Number $a \cap b$ or $a.b$, or ab is the product of a multiplication of the number a by the number b . (...)
- (6) The number $a \cup b$ or $a : b$, or $\frac{a}{b}$, is a number which multiplied by b gives a . (LH 35, 4, 12, Bl. 2)

In his list of axioms, Leibniz uses notations indifferently, writing, for example:

- (7) $0.a = 0$
 $0 \cap 1 = 0$ $0 \cap 2 = 0$ $0 \cap 3 = 0$ etc.
- (8) $1a = a$
 $1 \cap 1 = 1$ $1 \cap 2 = 2$ $1 \cap 3 = 3$ etc. (*Ibid.*)

This is not uncommon in Leibniz’s mathematical practice, as in a number of his contemporaries, such as Descartes or Van Roomen, who switch notations without a qualm, both in private papers and in (some) letters. This suggests that even if one has the impression that a notation is more efficient than another, actors worked, apparently without trouble, with several notations, prompting us to be careful about the so-called ‘power’ of notations.

Leibniz’s manuscripts, of course, harvest a wealth of unpublished works in which he develops such written tools.²¹ Knobloch, in [Knobloch(2004)] in particular, showed very well how, for Leibniz, the use of “text as process” (as stated in the title of his paper) supports a use of writing in which text “serv[es] the art of invention, [...] the visualization of his thoughts, theorems, and proofs”, how it is “used to fix insights” to “think by writing” [Knobloch(2004), 51]. Textual artefacts are, for Leibniz – and certainly for many mathematicians – effectively *tools*:

²⁰This is a reference to Leibniz’s attempts to denote the coefficients of equations by sequences of numbers in order to make writing determinants easier.

²¹See, among others, [Knobloch(2004)], [Trunk(2016), Gentil(2021)] on ambiguous signs, or Arilès Remaki’s rich PhD dissertation [Remaki(2021)].

Tables, illustrations, and figures play a crucial role in Leibniz's mathematical thinking. They enable him to find rules, laws, and regularities; in other words, they serve the art of invention, sometimes successfully, and sometimes not. (*Ibid.*, 52)

This is very explicitly stated by Leibniz himself:

[T]he art of inventing famous theorems, regarding the intelligible, consists in painting or hearing their representations, because they themselves cannot be painted or heard. (...) And in observing some sensible beauties in them. They will enable us to understand a theorem or the property of the intelligible thing, or at least that which is of such a nature that it produces, so to say, these apparitions if it is expressed by these characters. (A VII 1, 598. Quoted and translated in [Knobloch(2004), 63])

Leibniz's seemingly deep trust in the efficiency of such written tools should, of course, be taken with a grain of salt – especially since it concerns, here, a diagrammatic representation of the distribution of prime numbers, which Leibniz hoped to use to find a law of distribution of prime numbers (see [Knobloch(2004), 57-63]). Yet, it is undeniable in Leibniz's drafts that writing, its spatialization and its dynamics, is a multifold tool indispensable for mathematical research.

Among the most powerful of such tools are tables, which, according to Knobloch, have two main roles, a dynamic role and a static one:

While the dynamic role of tables [...] consists in serving the art of invention, the static role of tables serves to avoid repeated calculations at future times. According to Leibniz, such a table is the best way of fixing the results which have been calculated once and for all. The tables can be used to find a needed value. While the purpose of the first use of tables, as we were told, is to reduce tables to calculations, the purpose of the second use of tables is to replace calculations by tables. (*Ibid.*, 64)

Knobloch cites a number of examples on number partitions, permutations when computing the powers of polynomials, symmetric functions, and more. More recently, Arilès Remaki showed, in [Remaki(2021)], the complexity and richness of Leibniz's tabular practice(s) in questions related to combinatorics, to quadratures and to exponentiation. Tables, he argues, are “both a general mechanism for analysis and for computation, a mode of presentation, or a classification tool” (*ibid.*, 592). Tables support Leibniz's mathematical research, largely as a powerful heuristic tool – to answer given questions but also to ask new questions – but also, in some instances, as a tool for proofs. Remaki shows how tables can function as simple lists, but also as “exposing the procedure to generate” elements (*ibid.*, 313); how using and inventing tables relies on a “deep reflection on the notion of disposition”, that is, how to arrange its elements; how tables are used to “study the relations between terms and places in order to extract a rule, a universal law, which characterizes all the terms” (*ibid.*, 314); and how “it is thus possible that the inductive practice of tables constituted, in a first time, for the young Leibniz, an apodictic mode both satisfying and suitable to determine a result without doubt” (*ibid.*, 590).

In many aspects, these observations about Leibniz's tabular practices, and more largely about his use of written artefacts and spatialization of writing, are in line

with those I made on Dedekind's use of columns, tables and diagrammatic representations, in his researches on modules and *Dualgruppen* [Haffner(2023)]. For Dedekind, tables, lists and diagrams are parts of a larger toolbox developed to support his computations. They help him organize his computations and their results, and visualize properties, relations and patterns. Tables, in particular, are both a result – since they contain the ordered results of computations – and a device for further computations – since they help shorten and verify the computations. Such textual elements of practice give us clues to better understand research practices, discovery practices, which are largely invisible in publications even though they are an essential part of the mathematician's activity. And while they are, most of the time, erased from published works (especially since the 19th century), such practices are, in fact, a fundamental and even founding part of the theoretical activity. This also supports the idea that mathematics is not solely an activity of *proving*, but indeed can contain elements of experimental practice.

2.3 Computations and experimentations

Experimental aspects of mathematical research appear salient and fairly striking when working with drafts.²² Experimentations in mathematics is not a new subject, nor is it a subject specific to drafts.²³ However, drafts do show a certain prevalence of experimentations and inductive approaches among mathematicians who are not, in fact, classified as experimental mathematicians (neither by themselves, nor by commentators). Leibniz advocated for calculations as a sort of experimentation (*La vraie methode* (1677), A VI 4, 4-5), and although he recognized that induction is sometimes misleading, his manuscripts show a rather abundant use of this approach.²⁴ Even Dedekind, who not only advocated against computations as grounding mathematics but openly disliked them, can be observed as developing research processes that are primarily experimental computations, which he uses to explore properties of his objects. Indeed, as I have mentioned, Dedekind's drafts on module and *Dualgruppe* theories are filled with computations, and they tell us that *through computations*, Dedekind was observing the laws, trying to identify which ones are general, which ones are fundamental, by trial and error. It is strikingly different from his publications.²⁵

To give more substance to this picture, let me illustrate it with some excerpts of Cartan's notebooks. As I mentioned earlier, Cartan's notebooks reveal the extent to which his mathematical practice relied on computations. It is fairly clear

²²While I will concentrate on computations for mainly incidental reasons – they are the approach favored by the authors I am studying –, it should be clear that similar observations can certainly be made concerning diagrammatic experimentations. There are instances in the history of mathematics, of course: Leibniz used diagrams, as [Knobloch(2004)] shows well, and we also find diagrams in Riemann's, Hurwitz's, or even Dedekind's drafts. Mathematician and theoretical computer scientist Viviane Pons, who works on algebraic combinatorics at the Laboratoire de Recherche en Informatique at the Université Paris-Saclay, argues for an experimental approach in mathematics (see <https://www.youtube.com/watch?v=3LZiZKgVjaU> or <https://www.lri.fr/~pons/docs/EAUMP/introduction.pdf>), and her own notebooks – which are visible in these links and which she showed me personally – are covered in diagrams, sometimes for dozens of pages.

²³Of course, experimental mathematics is an approach argued for by a number of mathematicians (see, for example, the journal titled and dedicated to *Experimental mathematics*). For historical sources, see [Echeverria(1992), Echeverria(1996), Goldstein(2008), Goldstein(2011)].

²⁴See [Remaki(2021), 245], for example.

²⁵Riemann's computations can certainly also be interpreted as experimentations, to some extent.

that Cartan used computations as a way to explore and experiment. Computations appear to be his main, most important tool to think about and explore a given mathematical subject. Unlike Leibniz, Cartan did not make any statement about using something akin to the inductive method. As observers, though, there is a number of clues that point towards this interpretation of his drafts. His notebooks showcase a recurring method: in a given mathematical framework but not necessarily with a precise starting question, Cartan studies the behavior of the relevant mathematical objects through computations, observes the properties that are brought out, makes an hypothesis, and attempts to prove it. To observe this, we are lucky to have not only Cartan's many pages of computations, but also his very explicit annotations: what he wants to study, when he makes a mistake, when he notices something interesting, when he changes approaches, when he tries and succeeds or fails to prove something... Cartan is engaged in a dialogue with himself,²⁶ in his drafts, and through this inner dialogue, we can follow the steps of his reasoning.

For example, on pages 149 sqq of 38J 1-37, in researches on what he called “*Espaces de plans*” (spaces of planes, a type of manifold), Cartan considers the links between the coefficients of the structure equation and those giving the decomposition into linear (and later irreducible) groups of the group of the surface. He starts with the structure equation

$$\omega_{h00}\omega_{abc} = \sum A_{\alpha\beta\gamma}\omega_{\alpha\beta\gamma}$$

and sets up to prove the following theorem:

Theorem: if $\omega_{h00}\omega_{abc} = \sum A_{\alpha\beta\gamma}\omega_{\alpha\beta\gamma}$, the coefficients $A_{\alpha\beta\gamma}$ are the same as those giving the decomposition of the product of 2 linear groups $(a_1x_1 + a_2x_2 + \dots)^h$ and $(a_1x_1 + \dots)^{a-b}(a_2x_2 + \dots)^{b-c}$. (*Ibid.*, 150)

From there, he looks briefly at the “general case” and states a result which he comments as being “very important”:

In the formula $\omega_{hk0}\omega_{abc} = \sum A_{\alpha\beta\gamma}\omega_{\alpha\beta\gamma}$, the coefficients $A_{\alpha\beta\gamma}$ are the same as those giving the decomposition of the product of 2 irreducible groups of dominant weights $hx + ky$, $ax + by + cz$. (*Ibid.*, 151)

The next step is to consider a higher dimension, that is $p = 4$ (since the previous pages considered the case $p = 3$). After some computations, and a proof modelled explicitly on the previous one, Cartan is able to show the theorem “if one of the factors is ω_{h000} ”. He then proves that it is also true “if one of the factors is ω_{hk00} ” and as well for ω_{hkl0} . From there, he concludes:

This is probably general. (*Ibid.*, 153)

a statement which he briefly justifies.²⁷

Computations can also raise new questions in Cartan's notebooks. In 38J 1-38, for example, after some very short computations under the title *Réseaux à invariants ponctuels et tangentiels égaux* (Lattices with punctual and tangential invariants), Cartan arrives at a conclusion he calls a “paradox”:

²⁶This is something that Knobloch also notes in Leibniz's drafts [Knobloch(2004), 77]. It is certainly not the case of every mathematician.

²⁷I could not locate a related result in Cartan's publications (but I could have missed it). The notebook contain a number of researches on manifolds and generalised spaces, some related to these pages, many not. See <http://eliecartanpapers.ahp-numerique.fr/items/show/21>.

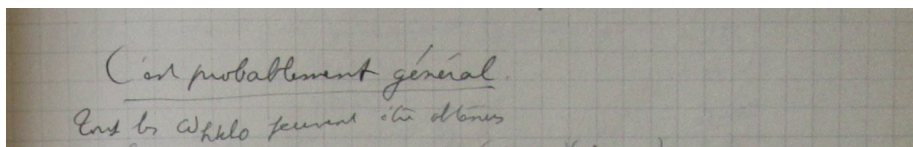


Figure 2: Fonds 38J, 1-37, 153.

If these are surfaces R and if the lattice is that of projective def[ormations], the asymptotes of (A_1) do not correspond to that of (A_2) . (38J, 1-38, 267)

where a surface R is a projectively deformable surface, and A_1, A_2 are the Laplace transforms of a point A . After a few lines of computations justifying this “paradox”, Cartan notes that it is “Something to study” and:

It seems that the conjugate lattice of the first also has punctual tangential invariants. Curious! (*Ibid.*, 268)

Cartan continues these researches further in the notebook,²⁸ but following them would lead us too far. Let me just mention that it is related to surfaces R and Jonas surfaces and the researches that eventually led to what he called “surfaces E ” in [Cartan(1944)].

3 Processes of redaction

In the previous section, I focused on written practices related to what one would call the *research* process – the back of mathematics. In the following, I would like to consider the process of writing a readable, fully redacted text – the elaboration of what would be the front of mathematics. These two aspects arguably represent different processes, different practices of writing. These modes of writing are complementary in the creative mathematical process, and likely both indispensable.

In writing a text that would be not only readable but hopefully publishable, at least that could be communicated to the scientific community, a lot of ‘external’ criteria come into play, such as disciplinary and sociological norms of what text is acceptable, and practical aspects related to printing. These are not questions I will try to address, here. Rather, I will continue to focus on internal textual aspects, and analyze the process of writing such a text. Of course, it could be tempting to distinguish between the type of documents used in the first part of this paper as being part of the *discovery* process, and the redaction of readable texts as a *justification* process. One of the core goal of the following paragraphs is to show that it is fairly more complicated than that. First, I will use an example from Dedekind’s archive to illustrate that conceptual clarification and thinking by writing are still at play in these steps of writing. Then, I will propose a selection of examples from recent secondary literature to further illustrate the possibilities offered by a genetic analysis of mathematical texts.

²⁸<http://eliecartanpapers.ahp-numerique.fr/items/show/22>

3.1 Dedekind’s terminological fine-tuning in *Was sind und was sollen die Zahlen?*

Dedekind’s famous essay on the natural numbers *Was sind und was sollen die Zahlen?*²⁹ was published in 1888. In this book, Dedekind wants to provide a general, rigorous definition for the sequence of natural numbers. He defines an abstract concept named “simply infinite system”, of which the sequence of natural numbers is an instance. To do so, he develops a set-theoretical arsenal, based on two concepts: mappings, which he calls *Abbildungen*, and sets, which he calls *Systeme*.

We know from Dedekind’s correspondence with Heinrich Weber, and from his archive, that an early version of *Zahlen* was started in the 1870s, which he left aside for about ten years. In Cod. Ms. Dedekind III-1, we find three sets of texts that can be identified as drafts of *Zahlen*:³⁰ pp. 1-13, second draft (1887); pp. 15-27, first draft (1872-78);³¹ pp. 30-45, third draft (1887).³² The first set of manuscripts dated from 1872-78 is a bit complex: there are three texts following, completing and correcting each other. The three together do not make a coherent whole, there are repetitions and gaps. The drafts are organized in a two-column layout, with the left column containing the main text, and the right column containing corrections and additions – a fairly standard layout for Dedekind’s redacted texts. In the right column, some additions and corrections seem to be more or less simultaneous with the writing, while some can be traced to the second or third draft (by the use of notations only introduced in those later versions). Material aspects and contentual aspects allow us to understand the chronology of the writing and better understand its stratigraphy. The sets of 1887 drafts are more standard. Dedekind uses the two column presentation in all of these drafts.³³

My aim, in the following, is not to offer strikingly new insights on Dedekind’s concept of number, but to look at the genesis of the *text* itself, at the process of writing. As the text itself is fairly long, I will focus on the concept of mapping, which appears early in the text and does not require much context.³⁴

In 1872-77 In the first layer, Dedekind does not introduce the notion of mapping but of “distinctly (*deutlich*) mappable”, which is a one-to-one correspondance:

²⁹[Dedekind(1888)]. I will refer to it as *Zahlen*.

³⁰See [Sieg and Schlimm(2005)] for an analysis of the changes in Dedekind’s concept of number reflected in these manuscripts.

³¹These documents were published in [Dugac(1976), 293-308]

³²A lot of the pages of this third draft are notes. In those that are redactions, many are crossed out, and some are written with a pencil. This is not the last version in the sense of the version sent to the printer (which, as far as I know, we do not have).

³³In fact, this type of layout is fairly standard, we find it in some of Leibniz’s texts, in some of Cartan’s, in Borel’s notebooks. . .

³⁴The concept of mapping is an important concept in Dedekind’s mathematics, used in many areas of his works, and which is subjected to a number of transformations through its uses in number theory, function theory, set theory... It is not my purpose, here, to analyse the changes of the concept of *Abbildung*, I only wish to study its definition in the versions of *Zahlen*.

A system S is called distinctly mappable in a system T if for each thing contained in S (*origin*) a (associated, corresponding) thing in T (image) can be specified such that to different origins correspond different images.

(Cod. Ms. Dedekind III 1, 16v)

[Late addition] Mapping φ of a system SA into a system T . To any thing a of A corresponds (is determined by a) a thing $a|\varphi = b$ of the system B . Distinctness of application $\varphi : a'|\varphi = b$ and $a''|\varphi = b$ different if a' et a'' different.

The definition of *Abbildung* added in the right column uses a notation introduced *only* in the third layer of the draft, so it was likely added later.

In the second layer, Dedekind does not define either mappable systems nor mappings. He was likely satisfied with the definition of mappable system given previously. He does define “image” for sets and studies some properties of mappings. A notion of “distinct” (*deutlich*) mapping is introduced later, somewhat in the middle of the text.

In the third layer, Dedekind gives an actual definition of *Abbildung* in the main text, with, again, the notion of image and its properties as stated in the second draft. “Distinct mappings” are moved up after mappings, making their introduction less *ad hoc* and more part of a structured paragraph on mappings.

1. [Definition]. By a mapping φ of a system S , we understand a law according to which for any thing a contained in S is generated a determined thing, which must be called the image of a and denoted by the sign $a|\varphi$. (Cod. Ms. Dedekind III 1, 23r)

This is followed by the definition of images (as sets), and by:

4. Definition. A mapping φ of a system S is called distinct if the images of $a|\varphi, b|\varphi$ of different things a, b contained in S are themselves different. (*Ibid.*)

Note, here, a shift in focus: sets were the central concept in the previous definition, but in this new one, Dedekind is giving more importance to mappings.

In 1887 For the 1887 version, I will concentrate on the first draft (and second overall). This draft is the most complete of the last two, and contains most of the conceptual and textual modifications. Mappings are defined in §2, with a number of additions (Cod. Ms. Dedekind III 1, 2v). The page is again divided in two columns, the left-hand side one for the main text, the right-hand side one for the corrections. The right-hand side column is, in this part of the text, almost completely filled (see Figure 3), showing corrections and additions, with the substantial addition of a new section – I will come back to that. The material aspects of the manuscript immediately suggest that there was a number of back and forth to write these paragraphs. As a matter of fact, these back and forth give us a beautiful view of the conceptual clarification of (some aspects of) the *Abbildung* concept.

First, Dedekind gave the definition of an *Abbildung* (Figure 4, in the frame number 1 in red):

Definition and notation. By a *mapping* φ of a system S , we mean a law according to which to each determined element s of a system S belongs a determined thing denoted by $\varphi(s)$ (or $\varphi(s)$ corresponds to the element s , $\varphi(s)$ is generated by from s), $\varphi(s)$ is called the *image* of s (and s is the original image [*Urbild*] (?), the model [*Vorbild*] (?); the origin [*Original*] (?), the source [*Ursprung*] (?) of s).

Since for the time being (in this §.) we deal only with S and with a certain mapping φ of it of a certain system S , simple large Latin letters shall denote exclusively parts of S , and for convenience the image $\varphi(s)$ shall be denoted more briefly by s' .

Then, he gave propositions 14 to 17 (Figure 4, in the frame number 2 in blue)³⁵. In this paragraph, Dedekind added two “Nota Bene” (highlighted in purple in Figure 4). Their place in the text shows clearly that they were not added later, but their being in square brackets suggests a specific status, which remains relatively unclear. Indeed, these NB add important information on the reciprocity of the propositions in relation with a yet-to-be-introduced notion of *distinctness* (*Deutlichkeit*). As such, they give clues about what is expected to follow in the text.

14. Definition. A' represents the system (image of A), whose elements are the images a' of the elements a of A .

15. [Proposition]. If $A3B$, then $A'3B'$ (from 1. 14. 1.). – [NB: the converse is allowed only if it is a mapping such that we will call a distinct [*deutlich*] later.]

16. [Proposition]. The image of $\mathfrak{M}(A, B, C \dots)$ whose system is, according to 7., a part of S , is $\mathfrak{M}(A', B', C')$. Proof. (...)

17. Proposition. The image of $\mathfrak{G}(A, B, C)$ is a part of $\mathfrak{G}(A', B', C')$. Proof (...) [NB: with the hypothesis of the distinctness [*Deutlichkeit*] of the mapping, the proposition will later be made stronger, see 15.]

In the margin, Dedekind wrote the definition of a “distinct mapping” (*deutliche Abbildung*), and three propositions numbered 15*, 15** and 17* (green frame number 3 in Figure 4). We will see, a little further down the line, why it is clear that this part was written before the paragraph just above it. The definition is the following:

The mapping of the system S is called distinct [*deutlich*] when two different elements a, b from the system S also have different images a', b' . Under this condition, the following is true:

15*. Proposition. If $A'3B'$, then $A3B$.

15**. Proposition. If $A' = B'$, then $A = B$.

17* Proposition. $\mathfrak{G}(A', B', C' \dots)$ is the image of $\mathfrak{G}(A, B, C)$.

One may notice that the definition, unlike the definition of a mapping, and most of the propositions in the text, is not numbered. This, along with the specific numbering of the following propositions, suggests that this was written as a complement to the definition 13 of an *Abbildung* and, of course, *after* propositions 14-17 written in the left column. The propositions 15*, 15**, 17* are the ones alluded to in the NB. Hence, with these propositions, the NB are obsolete but Dedekind did not delete them.

Dedekind then develops some properties of distinct mappings (yellow frame number 4 in Figure 4):

In each determined mapping of a system S is also contained a determined mapping of each part T of S ; this latter is determined by the fact that the image of each element t contained in T is the same thing t' which t , as an element of S , has as its image. If the mapping of S

³⁵There is little difference between these propositions and the published ones.

is distinct [*deutlich*], the mapping of T contained in it is also distinct [*deutlich*].

Each distinct [*deutlich*] mapping of a part T of S is contained in a distinct [*deutlich*] mapping of S (or can be extended into one), by taking as the image of each element of S not contained in T this element itself.

Let me now turn to the paragraph written above the definition of a *deutliche Abbildung* (orange frame number 5 in Figure 4). This was likely added later than the paragraphs just described (frames 1 to 4 on Figure 4). The fact that the writing is pressed down is a clue that it was added later, but more importantly the contents of this paragraph significantly change the text. Indeed, it contains not only a correction of the structure of both paragraphs, but also a change of terminology:

§2.a. Similarity of a mapping, similar systems

For this, best a special § !!!

Instead of distinct [*deutliche*] better similar [*ähnliche*] mapping! Converse or reverse thereof. The definition of similar systems [10 July 1887] A, B (there is a similar mapping of A such that B is the image of A). Then it will state later: S is called infinite if it is similar to a proper part of itself.

The change of terminology is significant, as it shows a shift in Dedekind's conception. "Distinct" suggests that the property that interests him is the fact that the images of two distinct elements are two distinct elements. By switching to "similar", it suggests that what is of interest is the fact that the mapping (here, a bijection since it is from S into itself) preserves such properties.

Finally, Dedekind added three propositions at the bottom (brown frame number 6 in Figure 4), which all contain the word *ähnlich*, and were thus written after this addition. In the left column:

If A, B are similar systems, then each proper part of A is also similar to proper part of B .

If A and B are similar systems, then each system C similar to B is also similar to BA .

In the right column:

Moreover, there is something to say about the composition of mappings: $B = \varphi(A), C = \psi(B) = \psi\varphi(A)$. If φ and ψ are similar mappings, then so is $\psi\varphi$.

More interesting, even, than this change of terminology mid-definition is the fact that these modifications were not made at the same moment(s) of the writing process. Indeed, the next few pages of the text still contain *deutlich* and *Deutlichkeit*. Following the terminological change provides a lot of information about the profoundly non-linear nature of this writing process.

A first (but neither the strongest nor the deepest) clue of this non-linearity, is the fact that in the paragraph §3 on "mappings of a system onto itself", the numbering of propositions and definitions does not take into account the propositions 15* to 17*, it starts at 18 rather than 21. (It does not take into account the additional paragraph either.) The paragraph itself does not consider distinct / similar

S. 2. Abbildung eines Systems \mathcal{F} .

13. Erklärung mit Begründung. Jedes eines Ab-
 bildung φ eines Systems \mathcal{F} ist ein Gesetz
 von Elementen s des Systems \mathcal{F} zu einem bestimmten
 Element s' des Systems \mathcal{F} zu bestimmen,
 mit $\varphi(s)$ zu bezeichnen und dies $\varphi(s)$ (oder
 $\varphi(s)$ auspricht das Element s , $\varphi(s)$ ist
~~das~~ s zugehört); $\varphi(s)$ heißt das Bild
 von s (mit s Abbildung φ), Urbild (s), Original,
 Urbild (s') von $\varphi(s)$. In φ ist das Gesetz
 (von diesem φ) eines bestimmten Elementes s
 mit einem bestimmten Element s' zu
 verbinden, so sollen mit φ die
 Abbildung φ und die Abbildung φ von
 \mathcal{F} mit dem bestimmten Element s verbunden
 werden.

14. Erklärung. φ bedeutet die Abbildung, die
 Element s des Systems \mathcal{F} zu einem
 Element s' des Systems \mathcal{F} zuordnet. Die
 Abbildung φ ist ein Gesetz, das ein
 Element s des Systems \mathcal{F} zu einem
 Element s' des Systems \mathcal{F} zuordnet.
 15. Satz: Ist φ die Abbildung, die
 Element s des Systems \mathcal{F} zu einem
 Element s' des Systems \mathcal{F} zuordnet,
 so ist $\varphi(s)$ das Bild von s und
 s das Urbild von $\varphi(s)$.

I. $M' = M$ (mit s, s', \dots).
 Da M ein Element des Systems \mathcal{F} ist,
 so ist $\varphi(M)$ ein Element des Systems \mathcal{F} .
 II. $M' = M$ (mit s, s', \dots).
 Da M ein Element des Systems \mathcal{F} ist,
 so ist $\varphi(M)$ ein Element des Systems \mathcal{F} .
 17. Satz: Das Bild von φ (mit s, s', \dots)
 ist $\varphi(s)$.

18. Satz: Ist φ die Abbildung, die
 Element s des Systems \mathcal{F} zu einem
 Element s' des Systems \mathcal{F} zuordnet,
 so ist $\varphi(s)$ das Bild von s und
 s das Urbild von $\varphi(s)$.

S. 2. a. Äquivalenz einer Abbildung,
 Äquivalenzsysteme.

19. Satz: Ist φ die Abbildung, die
 Element s des Systems \mathcal{F} zu einem
 Element s' des Systems \mathcal{F} zuordnet,
 so ist $\varphi(s)$ das Bild von s und
 s das Urbild von $\varphi(s)$.

20. Satz: Ist φ die Abbildung, die
 Element s des Systems \mathcal{F} zu einem
 Element s' des Systems \mathcal{F} zuordnet,
 so ist $\varphi(s)$ das Bild von s und
 s das Urbild von $\varphi(s)$.

21. Satz: Ist φ die Abbildung, die
 Element s des Systems \mathcal{F} zu einem
 Element s' des Systems \mathcal{F} zuordnet,
 so ist $\varphi(s)$ das Bild von s und
 s das Urbild von $\varphi(s)$.

22. Satz: Ist φ die Abbildung, die
 Element s des Systems \mathcal{F} zu einem
 Element s' des Systems \mathcal{F} zuordnet,
 so ist $\varphi(s)$ das Bild von s und
 s das Urbild von $\varphi(s)$.

Figure 4: Cod. Ms. Dedekind III 1, 2v., modified by E. H.

mappings.³⁶ On the other hand, at the end of §3, Dedekind wrote proposition 36 and three additional propositions, making it 39, and the paragraph §4, on the definition of the finite and the infinite, does start at definition number 40. Paragraph §4 then seems to take into account the additions that were made in the previous paragraph (but only in §3), and adjusted his numbering. This further suggests that the additions in §2 – at least starting from the propositions 15* to 17* – were written relatively late in the process, maybe even after writing §4.

In this paragraph, Dedekind defines an infinite system using *deutlich* and referring to §2 (not §2a):

40. Definition. S is called an infinite system, if there is a distinct mapping from S , such that the image of S is a proper part of S ; in the opposite case S is called a finite system. (Cod. Ms. Dedekind III 1, 3v, transl. altered in [Sieg and Schlimm(2005), 149])

Likewise, in the entire §4, Dedekind uses the word *deutlich*. We have here a clue that Dedekind, at that point was working with the definition of *Deutlichkeit*, but only with that definition – hence, before the correction of terminology and of the structure of the text. This is a further clue about the chronology of the corrections in §2.

In the interest of honesty, and to underline how cautious it is necessary to be when interpreting written traces in this context, let me mention that the numbering of propositions in paragraph §5 starts at 50, while §4, which starts at 40, only contains five numbered propositions and definitions. It would be tempting to assume that Dedekind decided to count all additional propositions, including in notes and in §2a., but this is unlikely (and does not round up to 50!). Indeed, the following paragraphs all start at a multiple of ten (the next highest after the last number in the previous paragraph, so if a paragraph ends at proposition 56, the first proposition of the next paragraph will be 60). It is, in fact, more likely that Dedekind gave up on the idea of following the numbers from paragraph to paragraph in this version of the text. Still, since it was not the case up to §4, we can nevertheless hypothesize that the additions in §2 were done after §4 – the additional notes to the 4th paragraph use the word *deutlich* – or maybe during.

Paragraph §5 allows us to close this investigation of Dedekind turning *distinct* into *similar*. The page, again, bears traces of significant changes. In particular, the right column contains corrections of the contents and title initially written in the left column (see Figure 5).³⁷

The left column states the following:

§5 The sequence of natural numbers

If S is an infinite system, there is thus a distinct mapping φ of S such that the image S' is a proper part of S . Then, I choose an element of S which is not in S' , and I examine the property of the image-chain N of this element. I disregard the accidental special nature of the elements of N completely here, retain only their distinguishability and only consider the mutual relations in which they stand to one another according to the former mapping, I name the elements n of N natural

³⁶However, the published version gives a definition of the mapping of a system onto itself that mentions the *Abbildung* being either similar or not.

³⁷The blue and purple circles were added by me to locate the use of *deutlich* and *ähnlich*, as explained below.

numbers or numbers for short, and N the sequence of natural [ordinal]
numbers. (Cod. Ms. Dedekind III 1, 4r)

This first definition, rather informal, thus uses the word *deutlich*. It is also used at the end of the paragraph. Note that this definition does not contain the abstract concept of simply infinite system, the core of Dedekind's theory of natural numbers. The correction in the right column reads:

§5 The simply infinite systems.
(Sequence of natural numbers)

Definition. A system N is called simply infinite when there exists a similar mapping φ of N ~~which fulfils the following conditions~~ in itself, namely such that N is the image-chain of an element not contained in $\varphi(N)$. Any such mapping φ of N can be called an order [*Anordnung*] [numerical] of N . The elements of N (with respect to this order) can be called natural numbers.

51. Propositions. In any infinite system S , a simply infinite part N is contained.

N a simple sequence or order. (*Ibid.*)

This is indeed a deep and crucial correction, as it drastically changes how the sequence of natural numbers is defined. However, this is not my focus, here. Rather let me draw the reader's attention to the fact that Dedekind uses the term *ähnlich*, and not *deutlich*. The *deutlich* used in the left paragraph is the last we meet, after that, Dedekind uses *ähnlich* / *Ähnlichkeit*. In fact, we can observe in that paragraph a correction that was likely added *after* the new definition, as it uses *Ähnlichkeit*. See Figure 5, where the occurrences of *deutlich* are circled in blue and those *ähnlich* and *Ähnlichkeit* are circled in purple.

It is relatively safe to state that this terminology change was made at this moment of the writing process. Both the additional properties and the change of terminology thus happened fairly late after to the initial introduction of the notion. Finally, there is a reference to *Ähnlichkeit* and to §2 in §5 (Cod. Ms. Dedekind III 1, 5r, Definition 60), which tells us that making the paragraph on similar mappings a paragraph of its own was likely not a decision made exactly at the same moment as changing the definition. But it was, indeed, made while writing the text – not once it was finished –, as we also find references to §2a in §6 (Cod. Ms. Dedekind III 1, 9r, Proposition 100). We would need more clues to assess exactly the timing of that change, which we do not have. Still, we are able to point quite a narrow time frame, and to underline how Dedekind was thinking by writing. The non-linear chronology of his writing is striking, and shows that a core basic concept (the distinct / similar mapping) was not fully clarified when Dedekind wrote this version of the text, and only came close to its full final form when Dedekind arrived at the definition of a simply infinite system – the most central concept of the essay.

3.2 Some additional examples

A genetic analysis of the writing process can also provide other types of valuable insights. Let me give, here, a selection of examples where a genetic approach has given or is giving such insights.

In my own (ongoing) work on the edition of Riemann's *Gesammelte mathematische Werke und wissenschaftlicher Nachlass* by Dedekind and Heinrich Weber, I show how a joint use of the Dedekind-Weber correspondence and Dedekind's and Riemann's manuscripts (Cod. Ms. Riemann, Niedersächsische Staats- und Universitätsbibliothek Göttingen) allows to unfold a multi-layered construction of the texts, in which the editors themselves played a significant role by correcting and completing the texts.³⁸

In [Joffredo(2019)], Thierry Joffredo gives a beautiful example of using the methods of genetic criticism, along with a careful codicological analysis of the manuscripts and a thorough investigation of Cramer's correspondence, thus reconstituting a textual and conceptual biography of the *Introduction à l'analyse des lignes courbes algébriques*. Doing so, he “shed[s] light on the steps of writing the book, the variations of the perimeter of its subject and the strengthening of the author's demands, the consideration of his contemporaries' works and of correspondences, its initial goals and the various functions (scientific, social, scholarly) [the author] gave his work in construction, the structural changes of the book through the versions of its manuscript” [Joffredo(2019), 237].

In [Krömer(2023)], Ralf Krömer uses Bourbaki's archive to investigate the history of duality in the works of members of the Bourbaki group, both individually and as a collective. To better understand the shaping of their collective reflection on duality, Krömer proposes to reconstruct the writing process of the Chapter II of the book *Algèbre* of Bourbaki's *Éléments de mathématique*. To do so, he uses the successive drafts of the *Éléments*, called *Rédactions*, together with the minutes of the meetings of the group and the vast correspondence network of the members.³⁹ Each *Rédaction* constitutes a different state of the text, and each is written by a different author. In some *Rédactions*, the author is identified – for this case, identified authors are Jean Dieudonné, Charles Ehresmann, and Claude Chevalley – but some *Rédactions* do not provide this information. The drafts circulate among Bourbaki members, are widely discussed (in meetings and in letters) and rewritten accordingly, making it a truly collective text, with a sometimes complex genesis. Krömer tracks the changes made on two propositions on duality in vector spaces (and the solution of linear equations). He notes a number of controversial points related to duality which, throughout the collective writing process, shed light on how the individual conceptions of duality of the members collide, and he highlights how the resolution (or lack thereof) of their disagreements (in particular, a debate between Dieudonné and Chevalley, who give very different definitions of duality [Krömer(2023), 280 sqq]) eventually leads to the final published version. Krömer's reconstitution of the writing process allows him to raise important points on the conception of duality (in particular “whether duality is necessarily symmetric [...] [a] problem [...] closely tied to [...] the relation of the algebraic and the topological dual” (*ibid.*, 293)) in that context, and on the “rather difficult functioning of the Bourbaki group at that time” (*ibid.*, 282). It also sheds (additional) light on important epistemological and historical issues, such as the debate around choosing the ‘right’ generality, and the acceptance (or avoidance) of the axiom of choice.

³⁸See [Haffner(2017b), Haffner(2018b), Haffner(2021)] and hopefully, more to come.

³⁹See <http://archives-bourbaki.ahp-numerique.fr/>.

The reader will not be surprised to read that many examples of complex redaction processes can be found in Leibniz’s archive. In [Remaki(2021), 179 sqq], Remaki reconstitutes Leibniz’s first attempts to answer Huyghens’ challenge to compute the sum of the inverse of triangular numbers, using the only two manuscripts addressing that question: *De Summa Numerorum Triangularium Reciprocorum* (A VII, 3, N.1) and *Differentiae Numerorum Harmonicum et Reciprocorum Triangularium* (A VII, 3, N.2), in which Leibniz tries two different (but ultimately inefficient) methods: decomposition of fractions and the method of differences.⁴⁰ By analyzing the material and textual clues in both of Leibniz’s manuscripts, Remaki unfolds a possible chronology of writing that shows how Leibniz’s work on this question is spread over two manuscripts (LH 35 3B 10 Bl. 5 and LH 35 12 2 Bl. 197), with parts of his computations “inserted in another text on algebra” [Remaki(2021), 180]. The manuscript (LH 35 3B 10 Bl. 5) contains the first method and elements of the second one on differences. The manuscript (LH 35 12 2 Bl. 197) contains the second method, in between unrelated researches on algebra. The reconstitution of the writing process, albeit hypothetical, suggests that these two manuscripts should be read together, as the elements of the method of differences in the first manuscript could be, in fact, the end of the researches in the second one (see [Remaki(2021), 181]). This is all the more insightful as it shows, as Remaki underlines, how underestimated “the schemas of differences and diagrams in general” had been (*ibid.*, 181).

Studying Leibniz’s manuscripts with a genetic approach also allows unveiling past editorial choices that deeply modified the text, which can have crucial repercussions on our reading of it. In a joint work with David Rabouin, we are considering such cases, in relation to reflections on how to best edit such texts so as to faithfully showcase their genesis, insofar as it gives us critical material to understand Leibniz’s thought. A typical example is the edition of the *De Arte characteristica inventoriaque analytica combinatoriave in mathesis universalis* (A VI, 4, 315-331)⁴¹ which, in its edited version, looks like a text that went through three preliminary versions (*Ansatz* 1 to 3) and a definitive one (*Endgültige Fassung*). The manuscript (LH 35, 1, 27, Bl. 3 sqq) tells a completely different story: there is only one text, with several layers of corrections on the first folio (3r and 3v), whose entire contents was subsequently crossed out, making the page 4r the *de facto* beginning of the text – but this page was *already* written in the first version (i.e., before the corrections in the first folio). Hence, rather than several versions of the text, we have a unique text with a relatively complex stratigraphy of corrections. Of course, the edition of such a text is not simple, if one wishes to offer a critical, exhaustive edition of the drafts. One reason is that such layers of text cannot be accurately represented statically. Another reason is the status of a completely deleted part of text, such as the first folio: how can we, in fact, edit it as a *variant*?⁴²

⁴⁰These are Leibniz’s first mathematical works, in which we witness the beginning of his method of differences. In these manuscripts, Leibniz only attempt to compute the said sum, but later he tries to generalize his approach to pyramidal numbers. These works are also identified as the origin for Leibniz’s harmonic triangle.

⁴¹The title is not Leibniz’s but the editors’.

⁴²On (the difficulties of) editing Leibniz, see also [Costa and Pasini(2019)].

4 Concluding remarks

Drafts open doors to the mathematician’s workshop, they reveal practices and methods which are often erased from the published text and, as such, offer access to the working processes of mathematicians. By studying the materiality of writings, invention of written artefacts, tools and notations, inductive and experimental aspects of research, etc., we gain invaluable information on the shaping of mathematical knowledge and its multiple forms. This rich material gives us keys to better understand the genesis of knowledge and texts, to better understand the links between the steps of conceptual development and the different modes of writing, to better understand the variety of means involved in the production of new mathematical knowledge.

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