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An epistemological gap between Analysis and Calculus: the case of Nathan

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The terms Analysis and Calculus are widely used in mathematics. It seems that the professional community dealing with research on the teaching and learning of analysis and calculus is gradually realizing that Mathematical Analysis and Calculus are not one and are not the same subject, no matter how closely related they are. We agree that there is a substantial difference between them, leading to genuine didactical challenges. The study reported below provides empirical evidence supporting this claim.

Keywords: Teaching and learning of analysis and calculus, Integral, Fundamental Theorem of Calculus, Meanings

INTRODUCTION

Being mathematics educators, we teach mathematics. Some of our students are future mathematicians. Some of our students are future engineers. Some of our students don't know yet what they want to do after graduation. Considering this, we should ask ourselves: what are we teaching to whom?

Let us consider the known controversies between Newton and Leibniz (Hall, 1920; Meli, 1993; Garber, 2008). Newton's approach was to use mathematics as a tool to explain and further understand natural phenomena. Leibniz's approach to mathematics was intra-mathematical, math for math's sake. He sought to better understand and to investigate mathematical objects and their abstract structures.

In this paper, we will distinguish between Mathematical Analysis and Calculus, building on the work of Topic Study Group 12 "Research and development in the teaching and learning of Calculus" (ICME, 2004) and of the Working Group "Didactic contrasts between Calculus and Analysis" (PME-NA, 2021). The difference and possible tension between Mathematical Analysis and Calculus is under discussion in the professional community (additionally to the groups mentioned above see, for example, Katz & Tall, 2012; Moreno, 2014). As there is currently no solid theoretical framework for this issue, we will describe our own developed approach.

By Mathematical Analysis, we invoke the formulations of Cauchy, Weierstrass, etc. Mathematical Analysis therefore deals with functions, limits, variables. This is done in a logical-symbolic and formal way. On the other hand, Calculus deals with quantities that vary in magnitude, rate of change and accumulation. The quantities covary with each other and have dimensions and units. Calculus requires a variational way of thinking within a natural extra-mathematical context. In this paper we will focus on integration and accumulation, it is therefore important to note that "The central idea of

Calculus to quantify accumulation is not that of antiderivative; it is ‘isolate in small intervals-multiply-add’” (PME-NA, 2021). The multiplication is a multiplication of the relevant average rate of change by the length of the small interval.

It seems that the professional community dealing with research on the teaching and learning of analysis and calculus, is gradually realizing that Mathematical Analysis and Calculus are not one and are not the same subject, no matter how closely related they are (see for example: Katz & Tall, 2012; Moreno, 2014; PME-NA, 2021; Rogers, 2005). Analysis is more pure mathematics. Calculus is more applied mathematics. We agree that there is a substantial difference between them, leading to genuine didactical challenges. The study reported below provides empirical evidence supporting this claim.

THE STUDY

Nathan is a valued and experienced teacher at Middle and High School (Advanced Level). Nathan has completed several Analysis courses at university and is very skilled in the subject.

Nathan volunteered to be interviewed in the framework of a research project aiming at identifying students’ meanings for rate of change (RoC) and accumulation.

The semi-structured interview with Nathan included the following tasks:

1. Evaluating the accumulated amount of cash, given a graph of the cash flow function.
2. Calculating the length of a curve representing a smooth function.
3. Finding the mass of a thin wire, given its mass density function.

The interview was recorded and transcribed.

FINDINGS

First task – accumulated cash given the RoC graph

In the first task, Nathan was given a graph of cash flow at a certain bank as a function of time from 8:30 to 11:30. Nathan was able to give reasoned and appropriate estimates regarding the behavior of the accumulated cash in this scenario, by effectively linking the accumulated cash with the area between the graph and the x-axis. This included, for example, an estimation of when the bank had less cash than the initial sum (**Error! Reference source not found.**; inscriptions in blue and orange are Nathan’s), which was given to be 5 million Shekels.

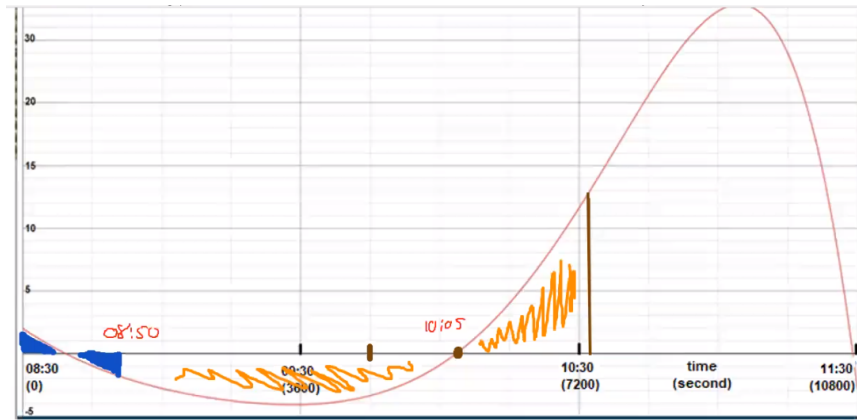


Figure 1: The area marked orange shows when the bank has less cash than initially

Even though Nathan identified the area as representing the amount of cash, and used this to correctly answer the questions he was asked, the collapse metaphor (Oehrtman, 2009) was prevalent throughout his explanations (translated from Hebrew) – “How much money exited the bank? The sum of the length of this segment and the length of this segment and the length of this segment, etc. etc., an integral on the graph of the function between 8:40 and 10:00” (**Error! Reference source not found.**).

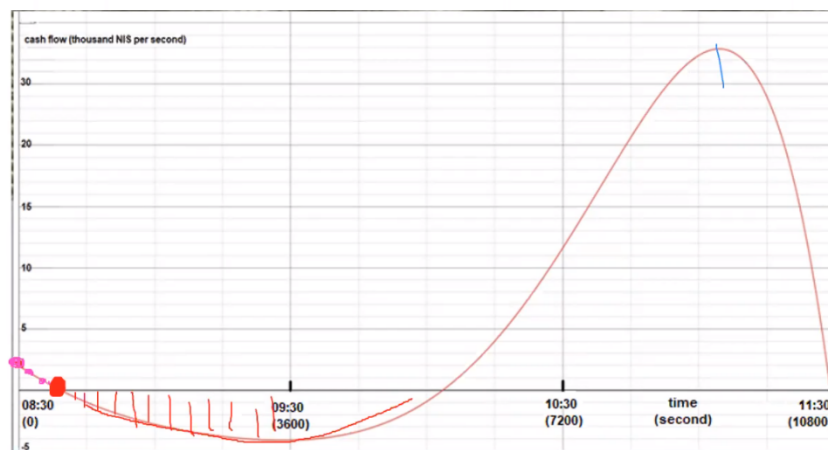


Figure 2: Nathan explains that the amount of money exiting the bank is the sum of the lengths of the red lines

Nathan’s conception of integral is based on notions of limit. This is evident from both his use of the collapse metaphor, which is a way of reasoning about limits, and from the mathematical terms he invokes (for more details regarding the collapse metaphor manifestation in Nathan’s case, see Noah-Sella et.al., 2022). Example of this can be seen in the following two exchanges:

Nathan: My logic is that when Δx approaches zero, or is even equal to zero, the size of the – I don’t want to say rectangle, it’s a line. It has no width. It’s just a line, and since the width of this line is zero, when we add up all of these lines, we will get the area trapped under the curve.

Interviewer: Can you explain to me what adding up lines is?

Nathan: Adding up lines? Or attaching lines?

Interviewer: Whichever is more convenient for you.

Nathan: Adding up, actually adding up the y-coordinate of every, I mean adding up the lengths of all the lines, meaning adding up the y-coordinate of all the points, the infinite number of points.

Interviewer: You're adding up an infinite number of values, how are you doing that in practice?

Nathan: Using limits, and using an integral on a graph of a function.

...

Nathan: [The integral] sums the y-values when Δx approaches zero. I mean, that's the visualization I have in my head for an integral, that's how I perceive it, like rectangles with zero width – straight perpendiculars.

Interviewer: Is there such a thing as a rectangle whose width is zero?

Nathan: There is a straight line. A rectangle whose width approaches zero. It comes from Riemann and Darboux sums.

Two examples how to interpret points on the graph had been presented by the interviewer in the introduction of the item: “at 8:30 am (time $x = 0$ seconds), $y = 2$, that is 2000 Shekel per second entered the bank's accounts; at 9:30 (time $x = 3600$ seconds), $y = -4.1$, that is 4100 Shekel per second left the bank's accounts”. Although Nathan read these statements aloud accurately, when he recalled the first statement, he omitted the “per second” from the units of measurement.

Although Nathan answered all of the questions correctly, and provided detailed justifications involving the areas, when the interview concluded he expressed a lack of confidence that stems from not having an algebraic representation of the cash flow function – “I felt like I was missing something. The lack of ability to see the function, even if I don't know how to integrate it, even if I can't actually feel it, something here felt very amorphic, and made me somewhat insecure”.

In response, Nathan was presented with the algebraic representation of the cash flow function. He exclaimed – “Great! This I can work with!” and proceeded to say “Now I feel like I have something to fall back on, if I'm completely at a loss. [...] If I want to substantiate the answer I gave, and to make sure I answer correctly, I know I have the analytical tools, the actual analytical tools to deal with this thing”.

Second task – evaluating the length of a curve

In this task Nathan was asked to evaluate the length of a curve given both the formula and graph of a function, and a closed line segment. This question has no extra-mathematical context and is well suited for a Mathematical Analysis approach.

Nathan described how he intends to find the length: “I want to take two points, calculate the distance between them, and make Δx approach zero. Then I'll get the length of a

single segment, and that's going to be for any pair of points on the graph of the function. If I integrate this, I will get the length of this segment along the graph of the function”.

At first, Nathan struggled with the development of a formula for the length of the curve. This prompted the interviewer to ask for an estimated length. While one might expect Nathan to divide the curve into segments of width h , and repeat the above calculation for some finite h , he instead gave an upper and lower bound using secants and tangents. In spite of his initial struggles, Nathan eventually managed to derive the formula for the general case (Figure 3). While it is possible that he had learned the formula at university, his behavior and responses during the interview suggest that he is at the very least reformulating it, and not reciting it from memory.

$$h = \Delta x$$

$$\lim_{h \rightarrow 0} \sqrt{(f(x+h) - f(x))^2 + h^2} = \lim_{h \rightarrow 0} \sqrt{\left(\frac{f(x+h) - f(x)}{h}\right)^2 + 1} \cdot h$$

$$= \sqrt{f'(x)^2 + 1} \Delta x$$

אם נסכם את כל הסיגמנטים נקבל את האורך של הקטע

$$\int_a^b \sqrt{f'(x)^2 + 1} dx$$

Figure 3: Nathan’s solution to find the length of a curve (Hebrew text translation: When we add up all the lengths of the segments we will get the length of the segment)

It seems that Nathan dealt with the second task using a well-established approach from Analysis. He approximated the required length by summing the lengths of small chords. He calculated the length of each small chord using the Pythagorean theorem. Then he used a limit for the purpose of formalizing the idea that the length of the chord is getting progressively smaller (Figure 3). As a result, Nathan gets a formula

$(\lim_{h \rightarrow 0} \sqrt{(f(x+h) - f(x))^2 + h^2})$ that structurally (!) doesn’t fit with Nathan’s knowledge regarding Riemann sums – an expression multiplied by Δx . This is where Nathan started to struggle with the task. Eventually, Nathan succeeded to manipulate the formula algebraically to get a known (to him) representation of the limit of Riemann sum $(\lim_{h \rightarrow 0} \sqrt{(f'(x))^2 + 1} \Delta x)$. Only after this process, he was ready “to replace” this representation by definite integral.

Third task – finding the mass given its density

In this question, Nathan was asked to calculate the mass of a straight wire, given its density (**Error! Reference source not found.**; for the original task in Hebrew, see Figure 5). At first, Nathan immediately replies with the correct integral, reasoning that “at point X, if the density is [the length of the segment] AX then the mass is x, and therefore $\int_0^8 x \, dx$ will give the mass of the wire”. Though his result is correct, he does not distinguish between density and mass in his explanation. When the interviewer reminds him of the units of measurement, a conflict arises for Nathan which causes him to lose confidence: “I’m in a spiral with myself”.

Below is an 8 meters long wire AB.

A _____ B

This wire is made from cutting edge material. The density of the material is not constant.

The density can be calculated in the following way: the density (measured in grams per meter) at any point X on the wire is numerically equal to the length of the segment AX.

Find the mass of the wire (in grams).

Figure 4: Mass from density task

In response to the conflict, Nathan retracts his answer and decides to change course, attempting to find a function representing the mass at each point. This causes him within two minutes to exclaim “all of my confidence is lost”, citing that “the density versus mass thing is really confusing me” and “I can’t decide with myself what will be the mass at a specific point, I want to take a certain point, express the mass there, and this function will be the integral”. He further explains this by saying that if the mass at every point is given by $g(x)$ then the total mass will be $\int_0^8 g(x) \, dx$ (**Error! Reference source not found.**).

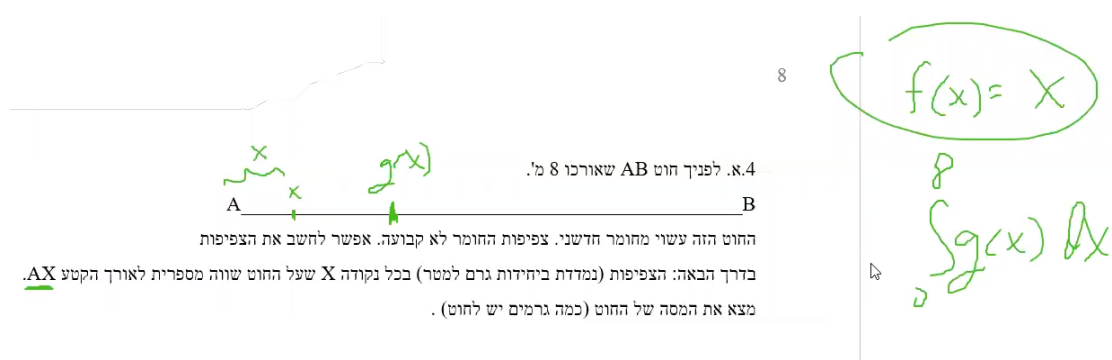


Figure 5: $f(x)$ denotes the density at each point. $g(x)$ denotes the mass at each point, and the integral is the mass of the entire wire

DISCUSSION

Nathan's conception of integral is greatly influenced by the collapse metaphor. In the second task, he invokes the collapse metaphor in the beginning – "I want to take all the points and add them up". When asked what adding up points means, he describes a procedure of calculating the distance between two points on the graph and taking the limit as Δx approaches zero. This illustrates his use of the collapse metaphor to make sense of limits. Thus, Nathan's thought process is rooted in limits, which is a characteristic of Analysis.

Nathan's use of the collapse metaphor seems to lead him to develop the misconception that integrating is summing the y -values of the integrand. Thus, to him the integrand does not represent the rate of change of the quantity accumulated, and dx is a notation and not a quantity, and therefore unitless. This might explain why Nathan changed the given RoC function's units of measurement to Shekel in the first task, and to grams in the third task. It appears that he unwittingly omitted "per second" and "per meter" respectively to amend these quantities to better fit his own meaning of integration and accumulation. Unsurprisingly, when this omission was pointed out to him in the last task, a cognitive conflict followed. This is consistent with Nathan having a background in Analysis, since variable quantities in Calculus have units, while in Mathematical Analysis variables and functions are dimensionless.

It is also important to contemplate why Nathan performed better in the first and second task than in the third. In the second task, one might suggest that the absence of extra-mathematical context means that units of measurement will not be a cause of conflict. However, the first task is set within an extra-mathematical context. We suggest that the graph given in the first task allowed Nathan to convert the problem to a geometric problem involving areas, thus circumventing the need to identify the given function as a rate of change function. This is important to note, since rate of change is a concept central to Calculus. In the third task, since there was no graph given, relating to quantities, rate of change and accumulation was necessary. Consequently, Nathan was unable to complete the task.

The connection between the collapse metaphor and Mathematical Analysis can also be seen in Nathan's use of language. When explaining the dimensional collapse, he isn't referring to the extra-mathematical context of the task, to rate of change nor to accumulation, but to function, limit, and Riemann and Darboux sums. This suggests that Nathan's meaning for integral was formed within Mathematical Analysis.

In addition, Mathematical Analysis deals mainly with functions, whose properties are analyzed mostly using algebraic and symbolic tools. Thus, Nathan's wish for the formula of a function in the first task, and his reaction upon receiving it support the notion that he is more at ease in Mathematical Analysis than in Calculus.

This claim is also supported by Nathan's performance in the second task. When asked to find the length of a curve, he uses formulas and limits. While the development of these formulas could be interpreted as based on the practice of "isolate in small

intervals-multiply-add”, which is a Calculus practice, when explaining his reasoning, it appears that Nathan operated with the intent of “isolate in small intervals-add”, with the multiplication arising as a byproduct, due to his familiarity with the structure of Riemann sums. The multiplication is essential to ‘Calculus thinking’ since it embodies accumulating from a rate of change. Furthermore, when prompted for an estimate of the length he did not use the same principle. One might expect that Nathan would divide the curve into small segments, approximate their length using a chord, and sum the lengths of all the chords, thus using accumulation reasoning. Instead of dividing the curve into smaller segments, he examined the curve as a whole, looking for an upper and a lower bound for the entire length, using a tangent and a secant. This suggests that he views the length of the curve as a static value (Analysis), and not as an accumulated quantity (Calculus). Consequently, it seems that the Riemann sum technique is only available to Nathan in an algebraic or symbolic context, central to Mathematical Analysis, and not in a numerical context, central to Calculus. Finding the formula for the general case, rather than finding a solution for the given curve is in accordance with the preference for formal proof in Mathematical Analysis.

Considering all of the above, we argue that Nathan shows a high proficiency in Mathematical Analysis problems, and difficulties in Calculus problems, strengthening the assertion that these are epistemologically distinct.

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