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# Personal Meanings versus Basic Mental Models for the Integral

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*While there are parallels between what has been called, in the literature, Basic Mental Models (BMMs, Grundvorstellungen) and what has been called Personal Meanings, there are fundamental differences between them. In this paper, we work out some of these differences, using the notion of integral as example. Roughly summarized, our findings are that BMMs, including individual ones, are epistemological whereas Personal Meanings are cognitive. Here epistemological refers to a content analysis, often from a didactic point of view, and hence is informative, for example, for curriculum developers; cognitive refers to individual students' personal conceptions, and hence is of interest, among others, to teachers.*

*Keywords: Teaching and learning of analysis and calculus, Grundvorstellungen (Basic Mental Models), Personal Meanings, Integral.*

## INTRODUCTION

Greefrath et al. (2021) have introduced Basic Mental Models (in German: Grundvorstellungen; henceforth BMMs) for the definite integral. They claim that students use their individual BMMs when they solve a problem. On the other hand, Thompson (2013) has introduced the notion of students' personal meanings and has shown many students and teachers lack such meanings for some central basic notions of mathematics. The question arises, what do students use when they solve a problem or answer a question, in our case about integrals; specifically, how is it possible that students who lack meanings use individual BMMs? We conclude that there are fundamental differences between individual BMMs and personal meanings. The aim of this paper (and its research question) is to clarify the differences and similarities between individual BMMs and personal meanings.

## THEORETICAL BACKGROUND

### Basic Mental Models (BMMs, Grundvorstellungen)

A fundamental idea produced and used by the German tradition of didactics of mathematics is the notion of Grundvorstellung, often translated into English as Basic Mental Model or BMM. According to Greefrath et al. (2021),

Normative BMMs are interpretations of a mathematical concept that learners should generally and ideally develop. These BMMs are identified by didactic analyses of the mathematical concept... They can be used as educational guidelines and to specify learning

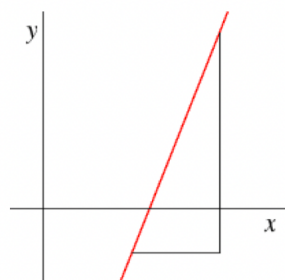
objectives for mathematics lessons. This can provide orientation to teachers when designing and organizing their lessons. (p. 650)

That is, BMMs are theoretical constructs based on a content analysis, and as such they can be useful in the determination of learning objectives and the design of instructional materials. For the notion of definite integral, Greefrath et al. (2021) have identified four BMMs: area, reconstruction, average and accumulation. For example, the accumulation BMM “considers the definite integral of a function as the limit of a sum with a large number of small terms” (p. 650), which are products, and hence this BMM emphasizes the process of integration. In this paper, we relate to only three of these four BMMs because the average BMM did not play a role in any of our data.

In addition to normative BMMs, Greefrath et al. (2021) also consider individual BMMs, which “are the specific manifestations of normative BMMs in a person” (p. 654). They developed an instrument for assessing students’ individual BMMs for integral; when using this instrument, students are presented with an argument that uses one of the four BMMs and are asked whether the argument reflects the student’s own line of thought. The instrument has a high validity for the students’ choice of one of the four BMMs, as shown by expert evaluation.

### Students’ personal meanings

Taking a Piagetian approach, Thompson (2016) focuses on what comes to a person’s mind upon encountering a situation; meaning is what the person imbues to the situation. A person’s meaning in a situation is what comes to the person’s mind immediately, together with what is ready to come to mind next. Thus, meaning also has an implicative nature. The meaning of an understanding is the space of implications that the current understanding mobilizes—actions, operations, or schemes that the person’s current understanding suggests.



**Figure 1: The slope item**

To make this more concrete, we present the following item about slope developed by Thompson (2016). Subjects are shown the graph in Figure 1 and told that it represents the relationship between two co-varying quantities P and Q, whose measures  $x$  and  $y$  are related by  $y = mx + b$ . Students are first asked to give an approximate numerical value of  $m$  given that the  $x$  – axis and the  $y$  – axis use the same scale. In a second question, they are asked for the numerical value of  $m$  if the scale of the  $y$ -axis is changed so that the distance between 0 and 1 is two times as large as the original one.

Thompson (2016) presents results from high school teachers to the slope item. We have informally presented the same item to mathematics education MA students. The results were similar. Most respondents give values between 2 and 3 to the first question. Most of these respondents doubled or halved their value in response to the second question.

The value of  $m$  in the relationship between the co-varying quantities P and Q does not depend on the scales of the axes in the graph used to represent this relationship. Hence, respondents whose meaning of slope in the above situation is the ratio between the relative changes in P and in Q, will not change the value of  $m$  from the first to the second question. Respondents who change the value of  $m$  from the first to the second question, hold a meaning for slope as a property of the triangle representing the relationship rather than a meaning of slope as a property of the relationship itself.

### **INDIVIDUAL BMMs ARE NOT PERSONAL MEANINGS**

Our motivation for writing this paper is a question addressed to us repeatedly when presenting initial results from a project investigating Israeli high school students' personal meanings for derivative and integral (e.g., Elias et al., 2022; Noah-Sella et al., 2022). We were asked why we did not use individual BMMs as theoretical framework for our research. This question appears to be well taken since individual BMMs are, at least in the case of the definite integral, well elaborated, an instrument for assessing them exists, and this instrument has been used with a large sample of first year mathematics students in German universities: Why should we not use the existing instrument, whose reliability and validity have been shown, and administer it to a suitable sample of Israeli high school students?

The brief version of our answer to this question is that BMMs do well what they are designed to do namely to reveal students' knowledge about mathematical concepts, and about integrals specifically. Such knowledge may be assumed to have been constructed (or memorized) during an epistemic journey led by a teacher. However, BMMs have not been designed to assess students' personal meanings. In the remainder of this paper, we will present three examples illustrating differences between BMMs and personal meanings. We will conclude the paper by giving a more elaborate answer to the question why we do not use BMMs as theoretical framework.

### **EXAMPLES**

The examples for students' personal meanings presented below have been selected from among more than 50 task-based interviews held for the purpose of designing a set of questionnaires to assess high school students' meanings for derivative and integral. Using Thompson's definition of personal meaning, we decided on the following five criteria as being potentially indicative of the interviewee's personal meaning.

*Distinctive language.* Utterances may reveal personal meaning if they refer to mathematical notions in terms distinctive to the interviewee. By this we mean that the terms have not been used by the interviewer or in the task, and were not prompted in any way, but are the interviewee's own terms.

*Repetition.* Utterances may allude to personal meaning if they contain terms that repeat themselves, whether within a certain task, or across different tasks and contexts. Repetition signifies that the concepts associated with these terms are readily available and relevant to the interviewee, and therefore are a salient part of their thinking.

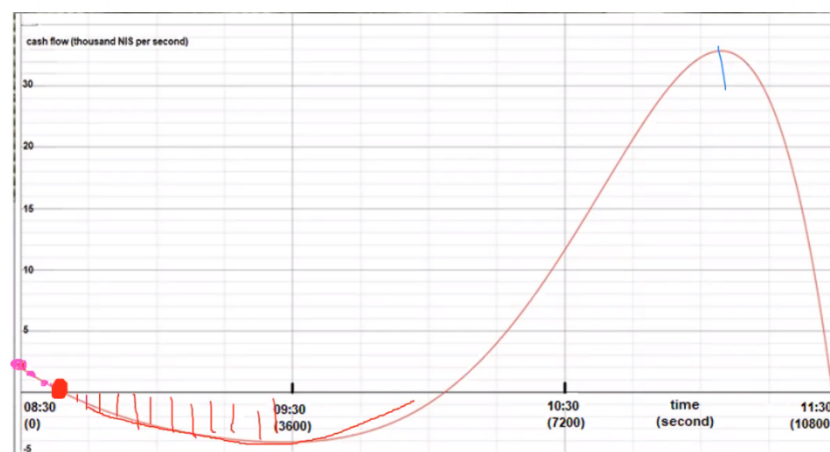
*Reasoning.* Utterances that are intended to explain or justify mathematical notions or claims are indicative of the interviewee's thought process, and therefore may point to personal meaning.

*Unexpectedness.* If an utterance by the interviewee is unexpected for the interviewer, then it was not solicited by the interviewer or the task and may be an expression of the interviewee's personal meaning.

*Statement of opinion.* Utterances explicitly qualified by the interviewee as their own belief, opinion or interpretation will be regarded as indicative of personal meaning. This includes, for example, utterances containing phrases such as "to me", "in my opinion", "the way I understand it".

## Nathan

Nathan is an experienced high school teacher. He had completed several analysis courses at university and is very skilled in the subject. He was interviewed with the aim of clarifying his personal meaning for integral (see Noah-Sella et al., 2022, for details). The three tasks used in the interview with Nathan dealt with the length of a segment of the graph of a function, with the mass of a wire, given its mass density, and with the amount of money in a bank, given the cash flow.



**Figure 2: Nathan's markings on the cash flow graph**

The cash flow task was presented in a graphical setting, leading Nathan to draw vertical (red) lines from the horizontal time-axis down to the graph representing the cash flow (Figure 2, between about time 8:50 and 9:30), and to explain

Nathan: My logic is that when  $\Delta x$  approaches zero, or is even equal to zero, the size of the – I don't want to say rectangle, it's a line - it has no width. It's just a

line, and since its width is zero, when we add up all these lines, we will get the area under the curve.

Nathan sums little bits, thus using the accumulation BMM. He “adds up” all the lines to obtain the area and interprets the area as amount of cash, thus using the area BMM. And when asked how he would add up the lengths in practice, he said “adding up the y-coordinates of all the points, the infinite number of points”. To Nathan, that was what the integral was doing, and for the computation he would find an antiderivative and use the second part of the fundamental theorem, that is the reconstruction BMM. Thus, Nathan’s personal meaning for integral encompassed the three BMMs.

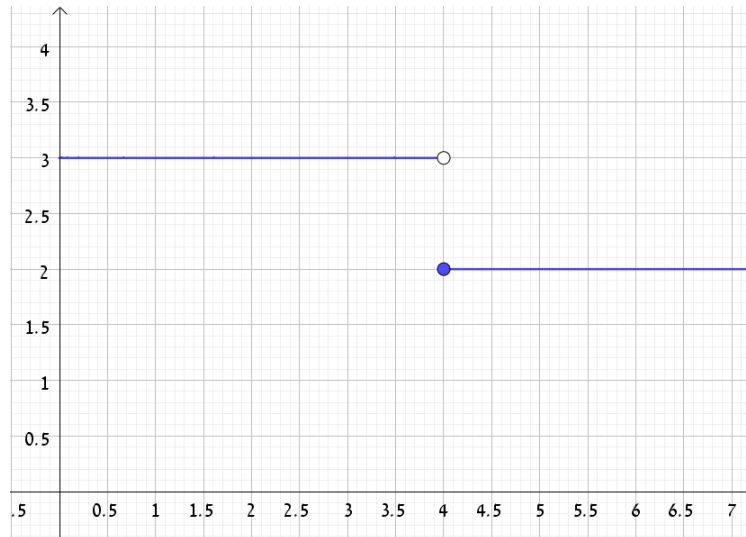
Nathan’s personal meaning for integral was consistent across tasks. When asked about the curve length, Nathan wanted to “take all the points and add them up”, which he elaborated as “take two points, calculate the distance between them, and make  $\Delta x$  approach zero. Then I’ll get the length of a single segment... If I integrate this, I will get the length of this segment”. Nathan first lets  $\Delta x$  approach zero, and then “integrates” or sums the values. We mention in passing that Nathan’s explanations are completely in line with what Oehrtman (2019) has called the collapse metaphor.

What exactly Nathan imagines adding became most clear in the mass from density task; in this task, Nathan claimed that the given mass density function did not serve him to find the mass of the wire; rather that he needed a function giving the mass at each point, and then he could integrate these point masses to find the mass of the wire.

We conclude that Nathan’s personal meaning for integration is that the integral is a technique that sums the values of the integrand at each point of the interval of integration. From Nathan’s explanation in the curve length task, it appears that he may have developed this meaning in an effort to tackle the role of limit in the definition of the integral. He uses this meaning explicitly in combination with the accumulation BMM in all three tasks; he combines it with the area BMM in the cash flow task; and he combines it with the reconstruction BMM in both, the cash flow and the curve length task. He is familiar with the three BMMs and uses them freely in his explanations. But what he interprets the integral to be, to do and to mean is to add values of the integrand. This interpretation is independent of the BMMs and reaches across all three of them.

## **Oren**

Oren is a 12<sup>th</sup> grade student who is taking advanced track mathematics. The task on which Oren’s interview was based relates to  $\int_0^x f(t)dt$ , where  $f$  is a step function with two positive values (Figure 3).



**Figure 3: the graph presented to Oren**

Oren related to  $\int_0^x f(t)dt$  as an “accumulating integral” and distinguished it from a “regular integral”. When asked what accumulates, Oren explained that “what accumulates are values” and “I like to look at it as with some image, for example accumulation of money in the bank. If we accumulate positive values, the amount of money grows.” The discontinuity of  $f$  did not disturb Oren – he explained that the integral simply did not accumulate a value there, whereas at all other  $x$ ’s it does. So, one might conclude that Oren’s individual BMM for integral was accumulation. However, this was not at all the case. When presented with a hypothetical student K who used antiderivatives in the same situation, Oren explained that “the integral is the opposite of derivative, and that’s what K did when he took the antiderivative. If it were a regular integral, I would agree with him, but since it is an accumulating integral, he makes a mistake”. Similarly, when presented with a hypothetical student A who used area in the given situation, Oren explained that “the integral is the area caught between the graph – the ceiling – and the x-axis - the floor. What’s between them, that’s really our integral”. But for Oren, an accumulating integral is not represented by area: “When I think about an accumulating integral, I don’t think about area; I think about values that I accumulate, y-values.”

Finally, when presented with a student C who added a constant of integration, Oren was “in a dilemma how to relate to the constant of integration. In a regular integral, we add a constant of integration. I am not sure whether we do that in an accumulating integral; my intuition says no. In an accumulating integral, you don’t speak about a primitive function, to which you could add an arbitrary constant that would then disappear when you differentiate. I don’t agree with C. He would be correct if we spoke about a regular integral, but we don’t.”

Like Nathan, Oren flexibly relates to the three BMMs, rather than preferring one of them. However, there is a disconnect for him. He links the expression  $\int_0^x f(t)dt$  to the “accumulating integral” or the accumulation BMM, which is separate from and

behaves differently from a "regular integral" that can be thought about using the area BMM or the reconstruction BMM.

In contrast to Nathan who linked the three individual BMMs by a common way of thinking about them, Oren separates the accumulation BMM as being a completely different entity with different properties from the "regular" area and reconstruction BMMs. Oren's personal meaning consists of two distinct notions of integral, "regular" and "accumulating". His meaning for integral does not reside in an individual BMM but rather in how he views, connects, or separates the three BMMs. The "regular" integral is evaluated by means of an antiderivative (reconstruction) and returns the value of the area of a fixed static region enclosed by the graph of the function and the x-axis. The "accumulating" integral, on the other hand, has nothing to do with area or reconstruction; it sums values.

### **Nadia**

Nadia was interviewed on an item parallel to Oren's, about  $g(x) = \int_0^x f(t)dt$ , where  $f$  is a step function with two positive values (Figure 3). Her item was slightly different, however: The integral was described as representing the area under the graph of  $f$ . Nadia was presented with the thinking of 5 hypothetical student. With the thinking of two of them she did not at all identify; we discuss the other three.

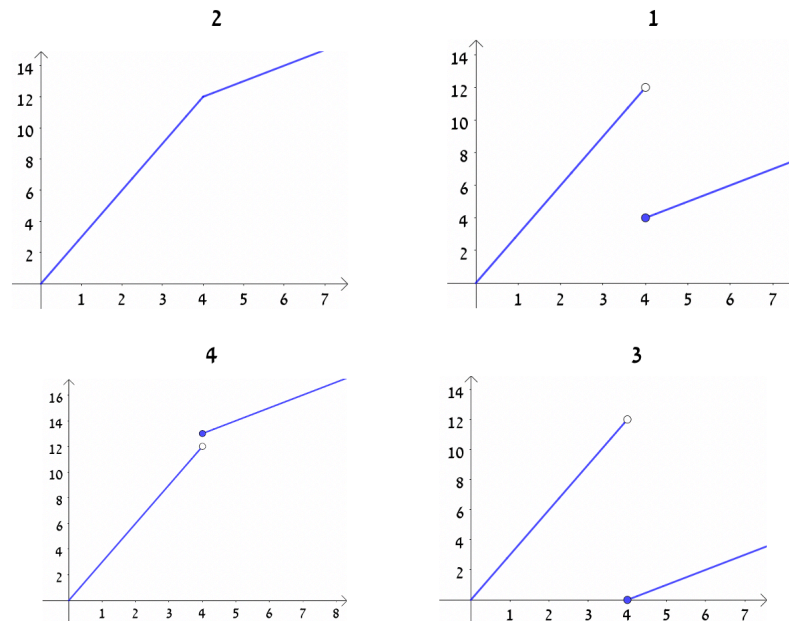
Hypothetical student V said that to find the integral, one needs to find the antiderivative in each of the two subdomains. Nadia explained V's reasoning as follows: V splits the domain into two subdomains, notices that  $f$  is the derivative of  $g$ , and hence does, in each subdomain, the operation inverse to differentiation, which is finding the antiderivative. Hypothetical student T said that area accumulates at a certain rate up the point of discontinuity and then continues to accumulate at a different rate. Nadia's reaction was that she was familiar with both ways, the one of V and the one of T, but that she slightly prefers V's way. Hypothetical student P looked at the integral as describing area accumulation like T but claims that the accumulation starts again from 0 at the point of discontinuity. Nadia points out the similarity between P and T, but not the difference. She identifies with both, the thinking of T and of P, but not as closely as with the thinking of V.

Our interpretation, so far, is that Nadia appears to be at ease with and able to explain the area BMM, the reconstruction BMM and the accumulation BMM, with a slightly higher affinity for the reconstruction BMM than for the other two.

Next, Nadia was presented with four potential graphs for  $g$  (Figure 4). All graphs are positive and have the correct slopes; the only difference between them is at the point of discontinuity of  $f$ . Nadia was asked, which of the graphs represent the function  $g(x)$ . Nadia kept vacillating between the three discontinuous graphs for about 10 minutes, giving no clear reasons for her choices, and struggling: "it's a bit difficult to explain" and "it feels strange". At one stage, she explained that "it can't start from a lower point as in 1 and 3" because "if you do an integral, if you separate the two parts..., you like,



have to add one to the other”, and later “because we do the addition... there is a jump between them because it can’t be 100% continuous”. Indeed, she preferred, at different times, Graphs 1, 3 and 4, but never Graph 2.



**Figure 4: The graphs proposed to Nadia for  $g(x)$**

As the conversation went on, she ventured “it depends how I look at this... you can take it as a question about definite integral, and you can take it as question about area; and each one has a very different meaning”; “I say they [the graphs] can all be correct; it’s just a question of how I try to look at them”; “Graph 3, for example, can be if I look at the first segment, and then I like restart the area”; “I choose Graph 1 because I don’t completely restart from zero”; and she concluded with “[if the horizontal axis represents  $x$  and the vertical one  $g(x)$ ] then I would choose Graph 4”.

Nadia’s only definite argument was against Graph 2; her exclusion of Graph 2, as well as her acceptance of Graphs 1 and 3, contradict accumulative thinking. We conclude that her earlier support for the way of thinking of students T and P may have been only declarative. The BMMs of area, and reconstruction play an explicit and recurring role in Nadia’s thinking about the graphs; however, she expresses a clear disconnection between definite integral and area, claiming that they lead to different graphs (209); these different graphs are incompatible with each other, as well as with a meaning of area for the definite integral – the growth of area can’t be discontinuous.

Nadia’s meaning for integral contains elements of all three BMMs, but the elements from each BMM lead her to ways of thinking that are incompatible with the other BMMs. For example, Nadia’s seemingly strong grounding in the reconstruction BMM was too weak to play a role for her when choosing a graph. Nadia’s meaning for integral at the time of the interview cannot be associated with a specific BMM, nor with a coherent fusion of two or all three BMMs.

## CONCLUSION

BMMs and meanings are not orthogonal, as the examples show. In Oren's meaning for integral, all three BMMs play a role, but this meaning decomposes into two notions for integral, something that might not have become evident, had we only looked for BMMs. BMMs may also blend rather incoherently in a student's mind as for Nadia. On the other hand, a student's meaning for integral may be consistent across all three BMMs, as for Nathan; Nathan has a strong but mathematically erroneous meaning for integral that is quite independent of the BMMs. The meaning of none of the three students can be clearly associated with one BMM; quite the contrary: for each of the students, all three BMMs play some role, and these roles differ greatly from student to student.

The test developed by Greefrath et al. (2021) lets students choose between 4 options. This is useful for the purpose for which the test has been designed. But as pointed out in the previous paragraph, the test is not likely to reveal students' meanings. Students' meanings are not neat enough to be categorized into 4 slots that have been defined by theoretical content analysis. In addition, the BMM a student uses may strongly depend on the context of the situation presented to the student. For Oren, the area BMM played a minor role. For Nadia, who was presented with what may appear to an expert to be the same problem (except that the area interpretation was mentioned in the description of the integral), the area BMM played a central role.

As epistemological constructs, BMMs are useful for comparing students' meanings to desired meanings for integral, for assessing how close students' meanings are (or are not) to the desired meanings offered by the BMMs. We note that Greefrath et al. (2021) themselves explicitly distinguish between BMMs and concept images; this seems a reasonable conclusion. We conclude that the role of BMMs in assessing students' meanings that differ from the desired meanings is limited.

The researcher who intends to investigate meanings should certainly be aware of BMMs and ask whether and how these meanings are related to BMMs. But when we ask about a student's meaning, we do not, or not mainly ask in terms of which models students think with. We ask how the student thinks about and with these models. In the BMM instrument, Nathan might have expressed the same high appreciation for the area and for the accumulation answers; and we would not have learned how he thinks about the integral, namely as a sum of values (or lengths) rather than a sum of products or areas (or small rectangles).

The BMM instrument is doing well what it was designed to do, but it does not assess students' meanings. BMMs have an epistemological role in the design of instruction. Designers will want to introduce several BMMs sequentially and coherently. Going beyond epistemology, Thompson (2013) has pointed out the importance of meanings for mathematics teaching and learning to become productive. The investigation of meanings requires suitable tools, and the development of these tools requires intensive interviewing (Thompson, 2016). It is a complex long-term effort.

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