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# Student Teachers' Development of Introductory ODE Learning Units - Subject-specific and further Challenges

Reinhard Hochmuth<sup>1</sup>, and Jana Peters<sup>1</sup>

<sup>1</sup>Leibniz University Hannover, Faculty of Mathematics and Physics, Germany,  
[hochmuth@idmp.uni-hannover.de](mailto:hochmuth@idmp.uni-hannover.de)

*What mathematical knowledge do student teachers for grammar schools have at the end of their studies and how do students succeed in linking their knowledge with subject-didactic considerations in the development of online learning units? This paper reports observations on both questions from the context of subject-specific and subject-didactic courses in German Master programs. The observations show that students generally have considerable difficulties in using standard knowledge from Analysis and Linear Algebra lectures. The linking of mathematical knowledge with subject-didactic considerations also poses considerable challenges. In view of the findings, we will argue that it might be helpful to broaden the view in research regarding transition issues and, in particular, to consider institutional-societal conditions.*

*Keywords: teacher education, ordinary differential equations, online learning units, subject-specific knowledge, institutional-societal conditions.*

## INTRODUCTION

After more than 100 years, Felix Klein's dictum of the double discontinuity has not lost its relevance. Both, Klein's critical diagnosis of the actual teaching of mathematics in schools and that, in order to improve the situation, it is the task of the university to train future teachers as well as possible, still seem to be true. Klein's goals have been taken up in various ways in the last decades, for example, concerning capstone courses near the end of university studies (Winsløw & Grønbaek, 2014): Students should be shown connections between university mathematics they have already learned in order to use them meaningfully as a resource for their professional lives. Klein's suggestion of specific bridging courses at the beginning of university studies is also being taken up in many places in Germany by a variety of measures (Hochmuth et al., 2022).

A premise of Klein's (1908) Elementary Mathematics from a Higher Standpoint course is that the students are well versed regarding university mathematics and were, in principle, able to represent mathematics as a science in its own right to grammar school students. Now, in this respect, it can be asked whether the actually learned knowledge from the basic first year courses of Analysis and Linear Algebra is in fact available to students in such a way that they can use it flexibly and adequately in order to acquire new mathematics that is closely related to school mathematical knowledge. This requirement addresses both techniques and rationales of content and concepts, such as representations of functions or of solutions of linear equations, derivatives as rates of change or tangent slopes. In addition to the availability of subject-specific knowledge, it is also a question of whether and how students, who have already attended substantial parts of their compulsory courses in subject didactics and pedagogy, can use this

mathematical knowledge in preparing and reflecting online learning units. Surprisingly, the state of knowledge and the related subject-didactic ability to act with regard to academic mathematics strongly linked to school mathematics among student teachers, who have more or less completed their studies, has not been investigated in detail.

This paper reports and discusses observations regarding those questions. We consider Master courses, in which students are asked to prepare online learning units on selected topics about Ordinary Differential Equations (ODEs) and accompanying essays reflecting the units from a subject didactical point of view. Elementary aspects of ODEs are a suitable choice because they use concepts from both Analysis and Linear Algebra in a way that has strong links to subject-didactic reflections of school mathematics. In addition, there are many contexts of use, for example in physics or biology, with models that can be assessed as school-related (e.g. harmonic oscillator, pendulum, and predator-prey models). Last but not least, there is a wide variety of literature that presents ODEs at different levels and didactically diverse ways: deductive and concept orientated (e.g. Hirsch et al., 2012), application-oriented (e.g. Bryan, 2021) or even inquiry oriented (Rasmussen et al., 2018). The courses considered here represent an opportunity to exploit the potential of advanced mathematics in mathematics teacher education addressed by Hochmuth (2022), opening up a view of mathematics that most student teachers do not encounter in current courses. One important didactic goal of the Master courses was the exploration and learning of the subject-specific preparation of mathematical knowledge for online learning units that are oriented towards concepts of inquiry-orientated learning (Artigue & Blomhøj, 2013; Jaworski, Gómez-Chacón & Hochmuth, 2021). Of course, it is also an empirical question which potentials students actually realise under the current restrictive institutional-societal conditions of study programs and the contradictions induced by this in the relationship between learning opportunities and learning resistances. Our qualitative analysis of the students' developments provides some evidence with regard to those issues shedding light on the state of knowledge and subject-didactic abilities.

The contribution is structured as follows: The next section provides information on the teaching-learning context of the courses. In particular, mathematical and subject-didactic pre-knowledge, the literature provided and knowledge taught about ODEs, and the objectives of the learning units to be developed by the students are addressed. In the following section, in view of the above reflections, two research questions are formulated and briefly outlined regarding theoretical backgrounds and methodological issues. Observations regarding the research questions are then presented in the successive sections. In the concluding section, results are discussed and possible conclusions as well as further research issues are addressed.

## **TEACHING-LEARNING CONTEXT AND DATA**

The observations reported are about Master courses for student teachers for grammar school at two German Universities. All students had successfully attended not only the basic courses on Analysis and Linear Algebra, but also courses about Numeric,

Stochastic, Geometry and Algebra. Specific pre-knowledge about ODEs was only available for a few. Such knowledge then came, for example, from the teaching of the exponential function in Analysis or of normal forms for matrices  $A$  and observations regarding  $e^A$  in Linear Algebra, or from their eventual second study subject Physics or another science. Therefore, introductory teaching units about ODEs were provided, which were mainly based on application-oriented literature (Bryan, 2021) and on inquiry-oriented presentations (Rasmussen et al., 2018; Gómez-Chacón et al. 2021). Fundamental theorems about unique solvability, the continuous dependence of initial conditions and parameters, and the stability of hyperbolic equilibrium points were mentioned and illustrated, but not proved. Instead, emphasis was placed on phase diagrams (in 1D and 2D), detailed phenomena-oriented treatments of linear systems (esp. equilibrium solutions, asymptotic behaviour) as well as applications such as the harmonic oscillator, the string pendulum, and predator-prey systems. Following the IODE course (Rasmussen et al., 2018), the notion of rate-of-change equation and directional- and vector fields were dominant. Elementary tasks mobilising changes of representation (terms, solution curves, phase diagrams) and their respective use and interpretation in application contexts were focussed.

In the first half of the semester, the described contents were taught in lectures with integrated exercise units. From week to week, the tasks of the exercise units were to be worked on and, in addition, a few tasks introducing new contents should be explored. In the second half of the semester, eight groups of three to four students each were accompanied in the development of online learning units. The learning units to be developed should cover introductory and slightly advanced topics including linear systems (with emphasis on 2D), harmonic oscillator (modelling various phenomena including resonance, possibly double oscillator), predator-prey models and bifurcations (in 1D and in 2D exemplified by pitchfork and Hopf bifurcation). In addition to the literature already mentioned, specific parts from (Chow et al., 2012; Hirsch, Smale & Devaney, 2012) and diverse internet resources, such as a school project work on the justification of periodic solutions of the Lotka-Volterra model, were provided or referred to. In addition, possibilities for the use of digital tools like GeoGebra and Applets in Wikis were introduced in the first half of the course. Moreover, subject-didactic concepts for the preparation of mathematics for teaching units in general (Barzel et al., 2012; Hußmann & Prediger., 2016) and inquiry-orientated units in particular were recalled or introduced (Winter, 1989; Jessen, 2017) with specific foci on representations and types of linking mathematical knowledge.

The exam consisted of the preparation of an online learning unit and an accompanying essay. Based on subject-didactic reflections, the essay should describe, explain and justify the respective preparation of the mathematical content and the methodical-didactic design of the learning unit. The following observations mainly refer to the integrated exercise units, the development of the online learning units and the units themselves. The accompanying essays are considered insofar as design elements to be

recognised in learning units are reflected with respect to their overall inquiry orientation.

## RESEARCH QUESTIONS

Based on the data described in the preceding section, two research questions are considered in the successive sections. In addition to formulating the research questions, we also sketch the respective theoretical backgrounds. The data analyses have been guided by qualitative content analysis (Gläser & Laudel, 2009).

**RQ 1:** What students' deficits from Analysis and Linear Algebra can be observed in the exploration of knowledge about ODEs?

The deficits concern not only factual knowledge about mathematical objects, symbols and their definitions, but especially techniques and rationales and their interrelationship. In particular, it is about concepts, such as the concept of derivative or the flexible use of different representations of functions. Our view is oriented towards the 4T-model of the Anthropological Theory of the Didactics, although analyses in this respect are not be made explicit in this contribution, if only for reasons of space.

**RQ 2:** What kind of linking of mathematics to application contexts using ODEs can be observed in the student's elaborations?

One focus of the courses was on the mathematical description of basic phenomena that could be assigned e.g. to Physics or Biology. Dynamic and structural properties of the phenomena should have motivated the use of concepts from Analysis and Linear Algebra. Otherwise, phenomena also served to interpret and validate achieved mathematical results. The students were familiar with the basic structure of the modelling cycle and respective modelling tasks presented in school as well as from their subject-didactic courses. Our theoretical background is complemented by insights from studies on the use of mathematics in engineering and beyond (Hochmuth & Peters, 2021; 2022).

### ON RQ 1: SUBJECT-SPECIFIC DEFICITS REGARDING ANALYSIS AND LINEAR ALGEBRA

Students were given the task of relating rate-of-change equations to representations of slope fields and to justify their assignments. Considerable hurdles were observed both in the interpretation of the equations and in their representation by slope fields. Regarding for example equations like  $\frac{dy}{dt} = t - 1$  (Rasmussen et al., 2018, p. 1.5) students could hardly detach themselves from drawing the right-hand side function itself into a coordinate system and then integrating it directly. Students failed in sketching the slopes, that are, in this case, independent of  $y$  (!), or, finally, in sketching the solutions that depend on a constant. Also expressions like  $\frac{dy}{dt} = y^2 - t^2$  (Rasmussen et al., 2018, p. 1.5) turned out to be rather difficult. Students got confused by the simultaneous occurrence of the variables  $t$  and  $y$ . Basically, in each point

$(t, y)$  of the coordinate system a straight line with unit length and the right slope must be drawn. Against the background of the drawn slope field, it is then a matter of detaching oneself from point-by-point interpretation of both the symbolic and the iconic representation and of passing over to local or global conceptions of functions, i.e. to consider  $y$  as a function dependent on  $t$ , or to think about graphs of functions whose tangents correspond to the drawn straight line segments of the slope field.

The tasks require a flexible handling of punctual, local and global perspectives on functions and their respective iconic and symbolic representations. In this respect the teacher students seem to be on a similar level of knowledge as college students, targeted by the IODE material, although they already completed their mathematical study. Against the background of Klein's second discontinuity, subject-specific knowledge underlying didactic considerations in the sense of basic ideas of derivatives and functions hardly seemed to be available. From this observation, it is comprehensible why in the development of the online learning units but also in lesson plans for schools about e.g. derivatives, didactic-methodical considerations are seldom substantiated by subject matter, but instead pedagogical considerations dominate (Hochmuth & Peters, submitted).

While students were essentially able to deal with vector fields in the generic case of planar linear ODEs with constant coefficients, i.e. cases where the matrices possess linear independent rows and the origin is the unique equilibrium solution, and to use in such cases publicly available GeoGebra applets, the planar system  $\frac{dx}{dt} = -3x - \frac{1}{2}y$ ,  $\frac{dy}{dt} = 6x + y$  leads to severe hurdles in getting an overview of the solutions and in particular about the equilibrium solutions. No group was able to come up with a complete answer. This task also originated from the IODE course (Rasmussen et al., 2018, p. 10.13): The respective goal is to explore what happens if the matrix belonging to the right side has linear dependent rows, i.e., the kernel of the linear mapping is a one-dimensional subspace (the trivial case of a zero matrix is excluded from considerations), which geometrically represents a straight line through the origin consisting of equilibrium points. The other non-equilibrium solutions can then be represented by straight lines parallel to each other intersecting the straight line of equilibrium points and, since the corresponding eigenvalue is -2, converge to the intersection points. Of course, the task could systematically be solved by a more or less canonical approach. But such an approach was only sketched in the lecture and not trained. The intention of giving this task was to enable students to combine knowledge about 2D-matrices (or linear mappings) with geometric and analytical considerations. On the one hand, such ideas already play a role in school mathematics and, on the other hand, they form the subject-specific basis of related didactic considerations. Unlike college students, the student teachers have successfully passed courses that provide such knowledge in several contexts (multivariable Analysis, Linear Algebra, Analytical Geometry). However, it was hardly possible for the students to use such knowledge in this new context. Of course, one does not really need such taught

knowledge to explore the given situation, instead might reflect about it in an elementary and direct way, but this was not possible either. The latter is the concern of the task in the IODE context: The insight into such phenomena should motivate to take a closer structural look and to discover structural reasons for such solution patterns. Conversely, the goal of basic university lectures like Analysis and Linear Algebra was to provide structural knowledge for ordering such phenomena systematically. But neither such a transfer, nor successful ad hoc explorations appear in the students' works.

A particularly rough case of misunderstanding related to Linear Algebra showed up in the following assertion presented in a submitted learning unit: The linear system  $x' = Ax$  is solvable if  $\text{rg}(A) = \text{rg}(A|x')$  holds. Obviously the later could formally be noticed, but actually makes no sense.

In the situation of considering elementary situations of bifurcations the following nonlinear system was considered (Hirsch et al., 2012, pp. 162):

$$x' = \frac{1}{2}x - y - \frac{1}{2}(x^3 + y^2x), \quad y' = x + \frac{1}{2}y - \frac{1}{2}(y^3 + x^2y).$$

With respect to this system, the linearisation at the origin (0,0) should first be determined and the corresponding local phase diagram be sketched. No group of students could find the linearisation directly from the given equations, which hints on a missing conceptual understanding of, e.g., the Taylor expansion in higher dimension. Instead, they tried (many calculation errors occurred) to calculate the Jacobi matrix formally. The eigenvalues are  $1/2 \pm i$ , which means that the Hartman-Grobman-Theorem is applicable and locally the phase portrait could be sketched, i.e., locally the solutions of this system spiral away from the origin. Then the students should transform the system to polar coordinates  $(r, \theta)$ , which leads to

$$r' = \frac{r(1 - r^2)}{2}, \quad \theta' = 1.$$

Neither during group work through the course nor as homework a correct solution was worked out by any student. For some of the students, hurdles start with the fact that they did not realise that the chain rule had to be applied. And if this was recognised, it could not correctly be executed. The chain rule for functions of several variables is a standard content of Analysis 2 and polar coordinates are typically treated at the latest in the context of the substitution rule for multidimensional integration. Interestingly, students with Physics as second subject also failed. Because of all these hurdles, the interesting global structure of the solutions of the nonlinear system, which would remain undiscovered by focusing only on the local considerations around the origin, could not be appreciated and not adequately presented in developed learning units. Likewise, it could not be appreciated that such insights were possible by (rather simple) qualitative considerations in the context of phase diagrams, in particular without explicitly calculating solutions of the (in the beginning) complicated-looking system of differential equations.

Summarising, the presented examples show considerable technical deficits, which made it impossible to acquire and represent rationales in the somewhat more advanced field of ODEs. Generally, the students could not rely on conceptual knowledge with respect to either multidimensional Analysis or Linear Algebra. Both in the course and its exercise units and in the elaboration of the learning units, these deficits obstructed an appreciation and appropriate use of illustrative representations such as vector fields and phase diagrams. This suggests that students moved away from inquiry oriented and rationales-focused presentations of ODEs in their own elaborations of learning units and returned to small-stepped and calculation-focused elaborations.

## **ON RQ 2: USING MATHEMATICS IN APPLICATION CONTEXTS AND MODELING**

ODEs allow to place mathematics in the context of everyday but also in physical, biological, chemical or technical contexts. The literature used in the course presented extensive chapters on topics from these areas and demonstrated how the qualitative approach focused on in the course lead to interesting insights, often without complex calculations and usually without solving the equations explicitly, which in fact is often not possible in the case of ODEs. Several application contexts are also used in the IODE material to promote and motivate a basic understanding of concepts and mathematical relationships. Thereby, concepts such as rate-of-change equations and tools like phase diagrams, allow students to understand the mathematical concepts even without explicit knowledge from application fields. They potentially enable students to get insights in dynamic interrelationships and, above all, phenomena to be modelled on the basis of everyday ideas. However, it is also obvious that basic mathematical deficits as addressed in the previous section make such epistemic processes significantly more difficult, which can be seen in several rather problematic derivations of models in the developed learning units.

An interesting observation in this context is that insight into mathematical deficits was averted by the teacher students by locating the hurdles instead in insufficient knowledge of application fields, such as Physics. This might be related to characteristics of known modelling cycles (Blum & Leiss, 2005) and dominant ideas of “applicationism” (Barquero et al., 2011), which suggest that modelling essentially takes place in an extra-mathematical world: If modelling does not work out, then, of course, it is due to missing knowledge in the extra-mathematical world. In other words: Underlying ideas about the role and the use of mathematics in application contexts turn out to be a kind of ideological obstacle, here possibly for the purpose of psychical relief in view of the experience of failure. Of course, this is a hypothesis which has to be reviewed in further research.

Conversely, short-circuited argumentations transferring directly from modelling contexts to mathematics can be found: in learning units and accompanying essays we find a lack of noted needs for justification and proof of such argumentations. Thus, besides the phenomenon of a strong separation between context and mathematics, we also found phenomena of instant identifications, hence an implicitly assumed identity

of reality and mathematical model. Related to this, there is also the phenomenon that mathematical results are sometimes directly applied to reality. The possibility that, in addition to explicit assumptions made, further and uncontrolled assumptions are included in models, and that every calculated and proven property thus also represents a validation possibility of the respective model as such, is only acknowledged in principle, but rarely considered in practice.

Summarising, any finely woven interweaving of various mathematical discourses related to slightly different ways of talking and doing mathematics as well as justifying validity (Hochmuth & Peters, 2021), which could and should also have subject-didactic relevance in school contexts, cannot be observed.

## **DISCUSSION AND OUTLOOK**

There is only few research so far about the academic mathematics knowledge of teacher students at the end of their studies and its availability for developing subject didactically reflected learning units. This paper focuses Master's programmes in which teacher students have to mobilise knowledge from basic lectures in a way that is professionally relevant both to the subject and the subject-didactic. Thus, these courses are in the context of the second discontinuity addressed by Klein. The focus of the reported observations was on (non-) available knowledge from introductory courses about Analysis and Linear Algebra as well as their use in inner and extra-mathematical contexts. Regarding subject-specific knowledge, there are considerable deficits with respect to both techniques and rationales. It seems that a central premise of Klein's concerning his Elementary Mathematics from the Higher Standpoint, namely reliably available university knowledge, is hardly given. Problematic claims of a life-world orientation expressed by the students contribute to questionable results with regard to the use of mathematics even in the context of simple application situations. Moreover, the subject-specific deficits at least add to the fact that teaching materials that have been clearly prepared in the sense of an inquiry orientation and in which ideas are to be introduced and used in a concept-oriented way are transformed by the students into small-stepped, calculation-oriented learning units.

In recent years, the focus in university mathematics education research has been on the first discontinuity, i.e., the transition from school to university. Against a broad background of theoretical and empirical analyses, a wide variety of measures has been developed and established (Hochmuth et al., 2021; Hochmuth et al., 2022). However, if one looks at our results also from the point of view of observations in (Hochmuth & Peters, submitted), where we reflected on problematic aspects of the prevailed societal determined formation of learning processes, one is led to the following questions: Are the approaches and orientation of those measures adequately specified? The observed deficits do indeed show that students can mobilise very little university mathematics knowledge after three years of university mathematics studies that are successful in the sense of the examination requirements. How do the measures work in this respect, and, in particular, could they potentially contribute to teacher students never gaining access to university mathematics?

Similar study conditions at other universities at least suggest that the reported phenomena are not exceptional, although our insights do not allow statements regarding their frequency or representativeness. In order to systematically deepen our observations, more research with a substantial subject-specific reference and with a critical view on institutionalised teaching-learning relationships seems necessary. In view of the officially successful study efforts a crucial question concerns the following: How must university teaching be constituted in which “learning processes are possible in which, beyond [...] mechanisms of influence and control, real [...] experiences and insights can be gained” (Holzkamp, 1991)?<sup>1</sup>

## REFERENCES

- Artigue, M., & Blomhøj, M. (2013). Conceptualizing inquiry-based education in mathematics. *ZDM* 45(6), 797-810.
- Barquero, B., Bosch, M. & Gascón, J. (2011). ‘Applicationism’ as the dominant epistemology at university level. In M. Pytlak, T. Rowland, E. Swoboda (Eds.), *Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education CERME7* (pp. 1938–1948). University of Rzeszów.
- Barzel, B., Holzäpfel, L., Leuders, T., & Streit, C. (2012). *Mathematik unterrichten: Planen, durchführen, reflektieren*. Cornelsen.
- Blum, W. & Leiss, D. (2005). How do students and teachers deal with mathematical modelling problems? The example “Sugarloaf”. *ICTMA 12 Proceedings* (S. 222-231).
- Bryan, K. (2021). *Differential Equations: A Toolbox for Modeling the World*. Cornwall, SIMIODE.
- Chow, S. N., & Hale, J. K. (2012). *Methods of bifurcation theory* (Vol. 251). Springer Science & Business Media.
- Gläser, J., & Laudel, G. (2009). *Experteninterviews und qualitative Inhaltsanalyse als Instrumente rekonstruierender Untersuchungen*. VS Verlag.
- Gómez-Chacón, I. M., Hochmuth, R., Jaworski, B., Rebenda, J. & Ruge, J. (Eds.) (2021). *Inquiry in University Mathematics Teaching and Learning: The PLATINUM Project*. Masaryk University Press.
- Hirsch, M. W., Smale, S., & Devaney, R. L. (2012). *Differential equations, dynamical systems, and an introduction to chaos*. Academic press.
- Hochmuth, R. (2022). Analysis tasks based on a theorem in Nonlinear Approximation theory. *International Journal of Mathematical Education in Science and Technology*, in press.

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<sup>1</sup> Wie muss eine Hochschullehre beschaffen sein, in der „Lernprozesse möglich sind, in denen jenseits [...] Beeinflussungs- und Kontrollmechanismen wirkliche [...] folgenreiche Erfahrungen und Einsichten gewinnbar sind“ (Holzkamp, 1991).

- Hochmuth, R., Biehler, R., Schaper, N., & Liebendörfer, M. (2022). *Unterstützungsmaßnahmen in mathematikbezogenen Studiengängen – Eine anwendungsorientierte Darstellung verschiedener Konzepte, Praxisbeispiele und Untersuchungsergebnisse*. Springer Fachmedien.
- Hochmuth, R., Broley, L., & Nardi, E. (2021). Transitions to, across and beyond university. In V. Durand-Guerrier, R. Hochmuth, E. Nardi, C. Winsløw (Eds.), *Research and Development in University Mathematics Education* (pp. 191-215). Routledge.
- Hochmuth, R., & Peters, J. (2021). On the analysis of mathematical practices in signal theory courses. *International Journal of Research in Undergraduate Mathematics Education*, 7(2), 235-260.
- Hochmuth, R., & Peters, J. (2022). About two epistemological related aspects in mathematical practices of empirical sciences. In Y. Chevallard, B. B. Farràs, M. Bosch, I. Florensa, J. Gascón, P. Nicolás, & N. Ruiz-Munzón (Eds.), *Advances in the Anthropological Theory of the Didactic* (pp. 327-342). Birkhäuser.
- Hochmuth, R., & Peters, J. (submitted). About the Use of QA-Maps in the Development of Lesson Plans by Student Teachers.
- Holzkamp, K. (1991). Lehren als Lernbehinderung? *Forum Kritische Psychologie* 27, 5-22.
- Holzkamp, K. (1993). *Lernen: Subjektwissenschaftliche Grundlegung*. Campus.
- Hußmann, S., & Prediger, S. (2016). Specifying and structuring mathematical topics. *Journal für Mathematik-Didaktik*, 37(1), 33-67.
- Jaworski, B., Gómez-Chacón, I. M., & Hochmuth, R. (2021). Conceptual Foundations of the PLATINUM Project. In I. M. Gómez-Chacón, R. Hochmuth, B. Jaworski, J. Rebenda, & J. Ruge (Eds.), *Inquiry in University Mathematics Teaching and Learning: The PLATINUM Project* (pp. 7 – 27). Masaryk University Press.
- Jessen, B. E. (2017). How to generate autonomous questioning in secondary mathematics teaching. *Recherches en didactique des mathématiques*, 37(2/3), 217-245.
- Klein, F. (1908). *Elementarmathematik vom höheren Standpunkte aus*. I. B.G. Teubner.
- Rasmussen, C., Keene, K., Dunmyre, J., & Fortune, N. (2018). *Inquiry oriented differential equations (course materials)*. <https://iode.wordpress.ncsu.edu>.
- Winsløw, C., & Grønbaek, N. (2014). Klein's double discontinuity revisited. *Recherches en didactique des mathématiques*, 34(1), 59-86.
- Winter, H. (1989). *Entdeckendes Lernen im Mathematikunterricht*. Vieweg.