Local behaviour of the solutions of the Chipot-Weissler equation

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Abstract We study the local properties of positive solutions of the equation $-\Delta u = u^p - m\,|\nabla u|^q$ in a punctured domain $\Omega \setminus \{0\}$ of $\mathbb{R}^N$ or in a exterior domain $\mathbb{R}^N \setminus B_{r_0}$ in the range $\min\{p, q\} > 1$ and $m > 0$. We prove a series of a priori estimates depending p and q, and of the sign of $q - \frac{2p}{p+1}$ and $q - p$. Using various techniques we obtain removability results for singular sets and we give a precise description of behaviour of solutions near an isolated singularity or at infinity in $\mathbb{R}^N$.

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Contents

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1 Introduction

The aim of this paper is to study the local properties of solutions of

$$\mathcal{L}_{m,p,q}u := -\Delta u + m|\nabla u|^q - |u|^{p-1}u = 0 \qquad \text{in } \Omega, \tag{1.1}$$

where m is a nonnegative real number, $p, q \geq 1$ and Ω is either a punctured domain if we are interested in isolated singularities, or an exterior domain if we study the asymptotic behaviour of solutions. This equation has been introduced by Chipot and Weissler [19] in connection with the associated evolution problem

$$\partial_t u + \mathcal{L}_{m,p,q}u = 0 \qquad \text{in } \Omega \times (0,T). \tag{1.2}$$

Its study has been developed in the radial case in [20] and completed in [35]. A very deep research of radial ground states has been carried on by Serrin and Zou in [30] and [31]. Several non-existence results of positive, not necessarily radial, supersolutions in an exterior domain have been obtained in [1] and [2].

The interest of the operator $\mathcal{L}_{m,p,q}$ lies in the presence of two reaction terms which are acting in opposite directions and are of a different nature. The following exponents play a key role in the study of asymptotics of solutions of (1.1),

$$\alpha = \frac{2}{p-1}\,,\ \beta = \frac{2-q}{q-1}\,,\ \gamma = \frac{q}{p-q} \text{ if } q \neq p \ \text{ and } \ \sigma = (p+1)q - 2p. \tag{1.3}$$

When $q = \frac{2p}{p+1}$ the equation (1.1) is invariant under the transformation T_ℓ defined by

$$T_\ell[u](x) = \ell^\alpha u(\ell x). \tag{1.4}$$

This critical value of q plays a fundamental role in the analysis of the solutions. If $1 < q < \frac{2p}{p+1}$, the source term is dominant for large values of u e.g. near a singular point, and the behaviour of singular solutions is modelled by the *Lane-Emden equation*

$$-\Delta u - u^p = 0. \tag{1.5}$$

If $\frac{2p}{p+1} < q < p$, the diffusion is negligeable and the behaviour of singular solutions is modelled by an *eikonal equation*

$$u^p - m|\nabla u|^q = 0. \tag{1.6}$$

Notice that in this equation the sign of $p-q$ is fundamental and makes the distinction between the existence or the non-existence of singular solutions. Another equation which plays a crucial role is the *Riccatti equation*

$$-\Delta u + m|\nabla u|^q = 0. \tag{1.7}$$

For this equation the value of q with respect to 2 is the key element. Finally, if $q = \frac{2p}{p+1}$ no reaction term is dominant and the value of m becomes fundamental as the following result proved in [8] shows it:

Theorem A *Let $N \geq 2$, $1 < p < \frac{N+2}{N-2}$ and $q = \frac{2p}{p+1}$. Then there exist two positive constants $c = c_(N,p)$ and m_0 such that for any real number m verifying $|m| \leq m_0$, any positive solution u of (1.1) in Ω satisfies*

$$u(x) + |\nabla u(x)|^{\frac{2}{p+1}} \leq c\left(\mathrm{dist}\,(x,\partial\Omega)\right)^{-\alpha} \qquad \text{for all } x \in \Omega. \tag{1.8}$$

As a consequence there exists no positive solution (called ground state) in $\mathbb{R}^N$.

An a priori estimate holds by a perturbation method for positive solutions, for all values of m whenever $1 < p < \frac{N+2}{N-2}$, and the following result is obtained in [29].

Theorem B *Let $N \geq 2$, $1 < p < \frac{N+2}{N-2}$ and $1 < q < \frac{2p}{p+1}$. For any $m \in \mathbb{R}$ there exists a positive constant $c = c(N,p,q,m)$ such that any positive solution u of (1.1) in Ω satisfies*

$$u(x) + |\nabla u(x)|^{\frac{2}{p+1}} \leq c\left(1 + \left(\mathrm{dist}\,(x,\partial\Omega)\right)^{-\alpha}\right) \qquad \text{for all } x \in \Omega. \tag{1.9}$$

Up to now, these two results were the only ones known concerning a priori estimates for general nonnegative solutions when $m > 0$. In the present article we prove new upper estimates for positive solutions u of (1.1) either in a punctured domain $B_{r_0} \setminus \{0\}$ or in an exterior domain $\Omega = B^c_{r_0}$.

The next statements extend previous results concerning positive supersolutions proved in [1]. If u is a positive continuous function defined either in $B_{r_0} \setminus \{0\}$ or in $B^c_{r_0}$, we set

$$\mu(r) = \inf_{|x|=r} u(x), \tag{1.10}$$

and we prove the following estimates valid in the case $1 < q < p$.

Theorem 1.1 *Let $N \geq 1$, $p, q > 1$ and $m > 0$.*
1- Let u be a C^2 positive supersolution of (1.1) in $B^c_{r_0}$, then
1-(i) If $\frac{2p}{p+1} < q < p$ there exists $C = C(N,p,q,u) > 0$ such that

$$\mu(r) \leq Cr^{-\alpha} \quad \text{for all } r \geq 2r_0. \tag{1.11}$$

1-(ii) If $1 < q \leq \frac{2p}{p+1}$ there exists $C = C(N,p,q,u) > 0$ such that

$$\mu(r) \leq Cr^{-\gamma} \quad \text{for all } \ r \geq 2r_0. \tag{1.12}$$

1-(iii) If $1 < p \le q$ *and* $\mu(|x|)$ *is bounded, then* (1.12) *is still satisfied.*
2- Let u *be a positive supersolution of* (1.1) *in* $B_{r_0} \setminus \{0\}$*, then*
2-(i) If $\frac{2p}{p+1} \le q < p$ *there exists* $C = C(N,p,q,u) > 0$ *such that*

$$\mu(r) \le Cr^{-\gamma} \quad \text{for all } 0 < r \le \tfrac{r_0}{2}. \tag{1.13}$$

2-(ii) If $1 < q < \frac{2p}{p+1}$ *there exists* $C = C(N,p,q,u) > 0$ *such that*

$$\mu(r) \le Cr^{-\alpha} \quad \text{for all } 0 < r \le \tfrac{r_0}{2}. \tag{1.14}$$

All the estimates on $\mu(r)$ will play a crucial role for the study of radial solutions of (1.1) see [13].

In the case $q \ge p$, the upper estimates are no more satisfied. The next result points out a dichotomy for estimates of positive supersolutions in an exterior domain when $q \ge p$.

Theorem 1.2 *Let* $N \ge 2$ *and* $1 < p \le q$*. If* u *is any positive supersolution of* (1.1) *in* $B^c_{r_0}$*, then for any* $\rho > r_0$ *there exists* c_ρ*,* C_ρ*,* C'_ρ*,* $C''_\rho > 0$ *such that, for* $|x| \ge \rho$*,*
(i) either

$$u(x) \ge \begin{cases} X_m |x|^{\frac{q}{q-p}} \left(1 - \frac{C_\rho}{|x|}\right)_+ & \text{if } q > p \\ c_\rho e^{m^{-\frac{1}{m}}|x|} & \text{if } q = p, \end{cases} \tag{1.15}$$

where $X_m = (m|\gamma|^q)^{\frac{1}{p-q}}$*,*
(ii) or $p > \frac{N}{N-2}$ *and*

$$\begin{aligned} &(a) && \mu(|x|) \le C'_\rho [x]^{-\alpha} \\ &(b) && u(x) \ge C''_\rho [x]^{2-N}. \end{aligned} \tag{1.16}$$

When $q > p$, the function $U(x) = X_m |x|^{|\gamma|}$ is a C^1 subsolution of (1.1) in $\mathbb{R}^N$, a fact which shows the optimality of the lower estimate.

In the case $q > p$ we prove a series of new estimates of *solutions*, by a delicate combination of Bernstein, Keller-Osserman methods and Moser iterative scheme. The general Bernstein estimates will play a fundamental role in the description of the behaviour of positive solutions near an isolated singularity or at infinity in $\mathbb{R}^N$.

Theorem 1.3 *Let* $q > p > 1$*,* $m > 0$ *and* u *be a nonnegative solution of* (1.1) *in a domain* $G \subset \mathbb{R}^N$*. Then*
1- If $G = B_{r_0} \setminus \{0\}$*, there exists* $c > 0$ *depending on* N, p, q *and* $\|u\|_{L^\infty(B_{r_0} \setminus B_{\frac{3r_0}{4}})}$ *such that*

$$|\nabla u(x)| \le c|x|^{-\frac{1}{q-1}} \quad \text{for all } 0 < |x| \le \frac{r_0}{2}. \tag{1.17}$$

2- If $G = B^c_{r_0}$*, there exists* $c > 0$ *depending on* N*,* p*,* q *and* $\|u\|_{L^\infty(B_{2r_0} \setminus B_{r_0})}$ *such that*

$$|\nabla u(x)| \le c|x|^{\frac{p}{q-p}} \quad \text{for all } |x| \ge 2r_0. \tag{1.18}$$

Note that in $B_{r_0} \setminus \{0\}$ the dominant effect comes from the Riccatti equation, while it comes from the eikonal equation in $B^c_{r_0}$. However it concerns solutions which may blow-up at infinity. When $q < p$, the *eikonal equation* plays a fundamental role in the proof of the next result which uses all the previous techniques involved in the proof of Theorem 1.3 above combined with the doubling Lemma method of [24].

Theorem 1.4 *Let $p > 1$, $m > 0$ and $r_0 > 0$.*
1- Let $1 < q < \frac{2p}{p+1}$. If u is a positive solution of (1.1) in $B^c_{r_0}$ satisfying

$$\lim_{|x|\to\infty} u(x) = 0, \tag{1.19}$$

then there exists a positive constant $C = C(N, p, q, u, r_0, m)$ such that

$$u(x) \le C|x|^{-\frac{q}{p-q}} \text{ and } |\nabla u(x)| \le C|x|^{-\frac{p}{p-q}} \tag{1.20}$$

for all $x \in B^c_{2r_0}$.
2- Let $\frac{2p}{p+1} < q < p$. Any u positive solution u of (1.1) in $B_{r_0} \setminus \{0\}$ satisfies (1.20) for all $x \in B_{\frac{r_0}{2}} \setminus \{0\}$ for some constant $C = C(N, p, q, u, r_0, m) > 0$.

In a forthcoming article [13] we prove the existence of infinitely many different radial solutions satisfying the decay estimate (1.20) by a combination of ODE and dynamical systems approach.

The following result is the counterpart at infinity Theorems A and B.

Theorem 1.5 *Let $1 < p < \frac{N+2}{N-2}$, $m > 0$ and u be a positive solution of (1.1) in $B^c_{r_0}$ ($r_0 > 0$) satisfying*

$$\lim_{|x|\to\infty} u(x) = 0. \tag{1.21}$$

Assume
(i) either $\frac{2p}{p+1} < q \le 2$ and m is arbitrary,
(ii) or $q = \frac{2p}{p+1}$ and $m \le \epsilon_0$ for some $\epsilon_0 > 0$ depending on N and p.
Then there exists a positive constant $C = C(N, p, q, u, r_0, m)$ such that

$$u(x) \le C|x|^{-\frac{2}{p-1}} \text{ and } |\nabla u(x)| \le C|x|^{-\frac{p+1}{p-1}} \quad \text{for all } x \in B^c_{2r_0}. \tag{1.22}$$

Thanks to the estimates of Theorem 1.3 we can prove removability results for singularities of positive solutions of (1.1).

Theorem 1.6 *Let $N \ge 2$, $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain containing 0. If $1 \le p < q$ and $q \ge \frac{N}{N-1}$, any nonnegative solution $u \in C^2(\Omega \setminus \{0\})$ of (1.1) in $\Omega \setminus \{0\}$ can be extended as a weak solution of the same equation in Ω and it belongs to $L^\infty_{loc}(\Omega) \cap W^{1,q}_{loc}(\Omega) \cap H^1_{loc}(\Omega)$.*

This result admits extensions for removability of more general sets included in a domain $\Omega \subset \mathbb{R}^N$ in two completely different directions. Using a geometric construction as in [32] we prove:

Theorem 1.7 *Let $N \geq 3$, $\Omega \subset \mathbb{R}^N$ be a bounded domain, $\Sigma \subset \Omega$ a k-dimensional compact complete submanifold ($0 \leq k \leq N-2$), $m > 0$ and $1 \leq p < q$ such that $q \geq \frac{codim(\Sigma)}{codim(\Sigma)-1}$. Then any positive solution of (1.1) in $\Omega \setminus \Sigma$ is locally bounded and can be extended as a weak solution in Ω.*

Using capacitary estimates we extend to the case $q > 2$ a previous removability result due to Brezis and Nirenberg [17] obtained in the case $q = 2$.

Theorem 1.8 *Assume $p > 0$, $q \geq \max\{2, p\}$ and $m > 0$. If K is a compact subset of Ω such that $cap_{1,q'}(K) = 0$, then any positive solution of (1.1) in $\Omega \setminus K$ is locally bounded and can be extended as a weak solution in Ω.*

The last Section is devoted to the study of asymptotics of positive solutions, either near a singularity or at infinity. In the case $q < \frac{2p}{p+1}$ the dominant equation for the study of isolated singularity is the Lane-Emden one, and the techniques involved combine energy methods and Fourier analysis. The description of the singular behaviour depends upon the value of p with respect to $\frac{N}{N-2}$ and $\frac{N+2}{N-2}$, and we obtain the complete classification of the possible behaviours of a positive solution near an isolated singularity:

Theorem 1.9 *Let $N \geq 2$, $m > 0$, $1 < p < \frac{N+2}{N-2}$ and $1 < q < \frac{2p}{p+1}$. If u is a nonnegative solution of (1.1) in $B_{r_0} \setminus \{0\}$, then either u is a classical solution of (1.1) in B_{r_0}, or*
1- when $N \geq 3$ and $1 < p < \frac{N}{N-2}$ (resp. $N = 2$ and $p > 1$) there exists $k > 0$ such that $|x|^{N-2}u(x)$ (resp. $-u(x)/\ln|x|$) converges to k when $x \to 0$. Furthermore u satisfies

$$-\Delta u + m|\nabla u|^q - u^p = c_N k \delta_0 \quad \text{in } \mathcal{D}'(B_{r_0}); \tag{1.23}$$

2- when $N \geq 3$ and $p = \frac{N}{N-2}$, $|x|^{N-2}(-\ln|x|)^{\frac{N-2}{2}}u(x)$ converges to $\left(\frac{N-2}{\sqrt{2}}\right)^{N-2}$ when $x \to 0$;

3- when $N \geq 3$ and $\frac{N}{N-2} < p < \frac{N+2}{N-2}$, $|x|^{\alpha}u(x)$ converges to $\omega_0 := \left(\alpha\frac{(N-2)p-N}{p-1}\right)^{\frac{1}{p-1}}$ when $x \to 0$.

In the case $q > p$ the dominant equation near an isolated singularity is the Riccatti equation; the removability result of Theorem 1.6 is no more valid if $1 < q < \frac{N}{N-1}$, and we mainly use a scaling method.

Theorem 1.10 *Let $N \geq 3$, $1 < p < q < \frac{N}{N-1}$, $m > 0$ and u be a nonnegative solution of (1.1) in $B_{r_0} \setminus \{0\}$. Then either u is a classical solution,*
(i) or $|x|^{\beta}u(x)$ converges to $\xi_m := \frac{1}{\beta}\left(\frac{(N-1)q-N}{m(q-1)}\right)^{\frac{1}{q-1}}$ when $x \to 0$,
(ii) or there exists $k > 0$ such that $|x|^{N-2}u(|x|,.) \to c_N k$ in $L^1(S^{N-1})$ when $x \to 0$ and u satisfies

$$-\Delta u + m|\nabla u|^q - u^p = k\delta_0 \qquad \text{in } \mathcal{D}'(B_{r_0}).$$

The asymptotic behaviour of solutions in an exterior domain exhibits also the two types of underlying dominant equations: either the Lane-Emden equation, or the eikonal equation. This depends on the value of q with respect to $\frac{2p}{p+1}$, see Theorem 5.5, Theorem 5.6. The techniques are similar to the ones used in the analysis of isolated singularities but the range of values of q are reversed; a phenomenon which is easily understandable when considering the scaling transformations leaving the underlying equations invariant.

2 Estimates on supersolutions

2.1 Some preliminary results

In the sequel we denote by c or C a generic positive constant the value of which may vary from one occurence to another. When needed we introduce the constants c_i, C_i with $i = 1, 2, ...$, in particular within the development of the proof of a statement. If it is important we precise the parameters (N, p, q, m etc.) on which the various constants depend. In the next result we precise a bootstrap argument some variants of which have already been used in [11], [10] and [6].

Lemma 2.1 *Let d, $h \in \mathbb{R}$ with $0 < d < 1$ and y, Φ be two positive continuous functions defined on $(0, r_0]$ (resp. $[r_0, \infty)$). We assume that there exist $C^*, M > 0$ and $\epsilon_0 \in (0, \frac{1}{8}]$ such that for any $\epsilon \in (0, \epsilon_0]$ and $0 < r \leq \frac{r_0}{2}$ (resp. any $r \geq 2r_0$),*

$$y(r) \leq C^* \epsilon^{-h} \Phi(r) y^d(r(1-\epsilon)) \quad \text{and} \quad \max_{\frac{r}{2} \leq \tau \leq r} \Phi(\tau) \leq M \Phi(r), \tag{2.1}$$

respectively

$$y(r) \leq C^* \epsilon^{-h} \Phi(r) y^d(r(1+\epsilon)) \quad \text{and} \quad \max_{r \leq \tau \leq \frac{3r}{2}} \Phi(\tau) \leq M \Phi(r). \tag{2.2}$$

Then there exists $c_1 = c_1(C^*, M, d, h, \epsilon_0) > 0$ such that

$$y(r) \leq c_1 \left(\Phi(r)\right)^{\frac{1}{1-d}}, \tag{2.3}$$

in $(0, \frac{r_0}{2}]$ (resp. in $[2r_0, \infty)$).

Proof. The result is obvious when $h \leq 0$, so we can suppose $h > 0$. Consider the sequence $\epsilon_n = 2^{-n}\epsilon_0$, $n \geq 0$. Then the series $\sum \epsilon_n$ is convergent and

$$S = \sum_{j=1}^{\infty} \epsilon_j \leq \frac{1}{4}.$$

For $n \geq 1$ we denote $P_n = (1-\epsilon_1)...(1-\epsilon_j)...(1-\epsilon_n)$ and $Q_n = (1+\epsilon_1)...(1+\epsilon_j)...(1+\epsilon_n)$. Clearly the sequence $\{P_n\}$ is decreasing while the sequence $\{Q_n\}$ is increasing. Furthermore

$$Q_n \leq \prod_{j=1}^{\infty} (1+\epsilon_j) := Q \leq e^S \leq e^{\frac{1}{4}} < \frac{3}{2}.$$

Concerning P_n, we have $1-\epsilon_n > \frac{1}{1+2\epsilon_n}$. Therefore

$$P_n \geq \prod_{j=1}^{n}(1+2\epsilon_j)^{-1} \geq e^{-2S} \geq e^{-\frac{1}{2}},$$

which implies $\frac{1}{2} < P_n < 1$. Then , for any $r \in (0, \frac{r_0}{2}]$ (resp. $r \geq 2r_0$) we have that $rP_n \in [\frac{r}{2}, r]$ (resp. $rQ_n \in [r, \frac{3r}{2}]$). First we assume (2.1) and use P_n. Then

$$y(rP_{n-1}) \leq c_2\epsilon_n^{-h}\Phi(rP_{n-1})y^d(rP_n).$$

In particular

$$\begin{cases} y(r) \leq c_2\epsilon_1^{-h}\Phi(r)y^d(rP_1)) \\ y^d(rP_1) \leq c_2^d\epsilon_2^{-hd}\Phi^d(rP_1)y^{d^2}(rP_2)) \\ \vdots \\ y^{d^{n-1}}(rP_{n-1}) \leq c_2^{d^{n-1}}\epsilon_n^{-hd^{n-1}}\Phi^{d^{n-1}}(rP_{n-1})y^{d^n}(rP_n)). \end{cases}$$

By the assumption on Φ, this implies

$$y(r) \leq c_2^{1+d+d^2+\dot{+}d^{n-1}}\epsilon_1^{-h}\epsilon_2^{-hd}\dots\epsilon_n^{-hd^{n-1}}\Phi(r)\phi^d(rP_1)\dots\Phi^{d^{n-1}}(rP_{n-1})y^{d^n}(rP_n),$$

for any $n \geq 2$. Hence for any $n \geq 2$,

$$\begin{aligned} y(r) &\leq (c_2\epsilon_0^{-h})^{1+d+\dots+d^{n-1}}2^{h(1+2d+\dots+nd^{n-1})}\Phi(r)\Phi^d(rP_1)\dots\Phi^{d^{n-1}}(rP_{n-1})y^{d^n}(rP_n) \\ &\leq (c_2\epsilon_0^{-h})^{1+d+\dots+d^{n-1}}2^{h(1+2d+\dots+nd^{n-1})}M^{d+d^2+\dots d^{n-1}}\Phi^{1+d+d^2+\dots d^{n-1}}(r). \end{aligned} \tag{2.4}$$

Letting $n \to \infty$ and using the fact that $P_n \to P > 0$ and $y^{d^n}(rP_n) \to 1$ as $n \to \infty$, since $0 < d < 1$, we obtain

$$y(r) \leq (c_2\epsilon_0^{-h})^{\frac{1}{1-d}}2^{\frac{h}{(1-d)^2}}M^{\frac{d}{1-d}}(\Phi(r))^{\frac{1}{1-d}}. \tag{2.5}$$

If we assume (2.2), the proof of (2.3) in $[2r_0, \infty)$ is similar. □

Next we recall and extend the monotony property dealing with supersolutions of Riccatti equation proved in [1].

Lemma 2.2 *Let $N \geq 2$, $q > 1$ and $u \in C^2(B_{r_0} \setminus \{0\})$ (resp. $u \in C^2(B^c_{r_0})$) be a positive function such that*

$$-\Delta u + |\nabla u|^q \geq 0 \quad \text{in } B_{r_0} \setminus \{0\} \quad (\text{resp. in } B^c_{r_0}).$$

Then the function μ defined by (1.10) *is nonincreasing on $(0, r_0]$ (resp. there exists $r_1 \geq r_0$ such that μ is monotone on $[r_1, \infty)$).*

Proof. The case of an exterior domain is treated in [1, Lemma 5]. In the first case, then for any $r_1 \in (0, r_0)$ and $\delta > 0$ there exists $r_d \in (0, r_1]$ such that for any $0 < r \leq r_\delta$ such that $\mu(r_1) \leq \delta r^{2-N}$ if $N \geq 3$ or $\mu(r_1) \leq \delta|\ln r|$ if $N = 2$. Let $h(x) = \mu(r_1) - \delta|x|^{2-N}$ if $N \geq 3$ (resp. $h(x) = \mu(r_1) - \delta||\ln|x||$ if $N = 2$). Then $u \geq h$ on $\partial B_{r_1} \cup \partial B_r$. By the standard comparison principle [1], [27], $u \geq h$ in $\overline{B}_{r_1} \setminus B_r$. If we let $r \to 0$ we derive $u \geq h$ in $\overline{B}_{r_1} \setminus \{0\}$, and by letting $\delta \to 0$ we finally obtain $u \geq \mu(r_1)$ in $\overline{B}_{r_1} \setminus \{0\}$. In particular this inequality implies $\mu(r) \geq \mu(r_1)$ if $0 < r \leq r_1$. □

2.2 Estimates of the spherical minimum. Proof of Theorem 1.1

In this Section we consider non-necessarily radial supersolutions u of (1.1), either in a punctured or in an exterior domain. We give estimates of the minimum of u on spheres with center 0 $\mu(r) = \min_{|y|=r} u(y)$.

We first consider supersolutions of the exterior problem

$$-\Delta u + m|\nabla u|^q - f(u) = 0 \quad \text{in } B^c_{r_0}, \tag{2.6}$$

where $m > 0$ and f satisfies

(F) *f is a continuous nondecreasing function on $\mathbb{R}_+$ verifying $f(0) = 0$ and $f > 0$ on $(0, \infty)$.*

We recall the following result of [1, Theorems 1, 3, 4].

Theorem C *(1) If $\liminf_{r\to 0} r^{-p} f(r) > 0$ and $1 < p \leq \frac{N}{N-2}$, $q > \frac{2p}{p+1}$, there exists no positive supersolution $u \in C^2(B^c_{r_0})$ of (2.6) such that $\liminf_{|x|\to\infty} u(x) < \infty$.*

(2) If $\liminf_{r\to\infty} r^{-p} f(r) > 0$ and $1 < q < p$, there exists no positive supersolution $u \in C^2(B^c_{r_0})$ of (2.6) such that $\lim_{|x|\to\infty} u(x) = \infty$.

Here we combine a technique developed in [1, Lemma 6] in order to prove Theorem 1.1 with the bootstrap argument of Lemma 2.1.

Lemma 2.3 *Let $m > 0$, $N \geq 1$, $q > 1$ and f satisfying (F). Let $u \in C^2(B^c_{r_0})$ (resp $u \in C^2(B_{r_0} \setminus \{0\})$) be any positive function satisfying*

$$-\Delta u + m|\nabla u|^q \geq f(u) \quad \text{in } B^c_{r_0} \quad (\text{resp. in } B_{r_0} \setminus \{0\}). \tag{2.7}$$

1- Then for any $R \geq 2r_0$ (resp. for any $0 < R \leq \frac{r_0}{2}$) and for any $0 < \epsilon \leq \frac{1}{2}$,

$$\min_{(1-\epsilon)R \leq r \leq (1+\epsilon)R} f(u(r)) \leq c_1 \left(\frac{\mu(R)}{\epsilon^2 R^2} + \frac{\mu^q(R)}{\epsilon^q R^q} \right), \tag{2.8}$$

where $c_1 = c_1(N, q, m) > 0$.

2- As a consequence, any positive C^2 supersolution u of (2.6) in $B^c_{r_0}$ satisfies
(i) either $\lim_{|x|\to\infty} u(x) = \infty$,
(ii) or $\liminf_{|x|\to\infty} u(x) = 0$.

Proof. 1- Let $R \geq 2r_0$ (resp. $0 < R \leq \frac{r_0}{2}$) and $\epsilon \in (0, \frac{1}{2}]$. Let ϕ_ϵ be a smooth nonnegative radial cut-off function defined on $\mathbb{R}_+$, vanishing on $[0, 1-\epsilon] \cup [1+\epsilon, \infty)$ with value 1 on $[1-\frac{\epsilon}{2}, 1+\frac{\epsilon}{2}]$, such that $|\phi'_\epsilon| \leq \frac{C}{\epsilon}\chi_{I_\epsilon}$ and $|\phi''_\epsilon| \leq \frac{C}{\epsilon^2}\chi_{I_\epsilon}$ where $\chi_{I_\epsilon} = [1-\epsilon, 1-\frac{\epsilon}{2}] \cup [1+\frac{\epsilon}{2}, 1+\epsilon]$. We set

$$v(x) = u(x) - \mu(R)\phi_\epsilon(\tfrac{|x|}{R}).$$

There exists $x_{R,\epsilon}$ such that $|x_{R,\epsilon}| = R$ and $u(x_{R,\epsilon}) = \mu(R)$, thus $v(x_{R,\epsilon}) = 0$. If u is defined in $B^c_{r_0}$, we have that $v = u > 0$ in $(B_{R(1-\epsilon)} \cap B^c_{r_0}) \cup B^c_{R(1+\epsilon)}$. If u is defined in $B_{r_0} \setminus \{0\}$, then $v = u > 0$ in $(B_{R(1-\epsilon)} \setminus \{0\}) \cup \left(B_{r_0} \cap B^c_{R(1+\epsilon)}\right)$. Then v achieves its nonpositive minimum at some $\widetilde{x}_{R,\epsilon} \in B_{R(1+\epsilon)} \cap \overline{B}^c_{R(1-\epsilon)}$, where $\nabla v(\widetilde{x}_{R,\epsilon}) = 0$ and $\Delta v(\widetilde{x}_{R,\epsilon}) \geq 0$. Since $v(\widetilde{x}_{R,\epsilon}) \leq 0$ there holds $\mu(|\widetilde{x}_{R,\epsilon}|) \leq \mu(R)$ and

$$\begin{aligned} f(u(\widetilde{x}_{R,\epsilon})) &= -\Delta v(\widetilde{x}_{R,\epsilon}) + m|\nabla v(\widetilde{x}_{R,\epsilon})|^q \\ &= -\mu(R)\Delta\left(\phi_\epsilon(\tfrac{|x|}{R})\right) + m\mu^q(R)\left|\nabla\left(\phi_\epsilon(\tfrac{|x|}{R})\right)\right|^q \\ &\leq c_1\left(\frac{\mu(R)}{\epsilon^2 R^2} + \frac{\mu^q(R)}{\epsilon^q R^q}\right), \end{aligned}$$

where $c_1 = c_1(N,p,q,m) > 0$. Because $u(\widetilde{x}_{R,\epsilon}) \geq \min\limits_{(1-\epsilon)R \leq r \leq (1+\epsilon)R} \mu(r)$, (2.8) follows from the monotonicity of f.

2- From Lemma 2.2, $\mu(r)$ is monotone for large r.
If μ is bounded, then

$$\min_{\frac{R}{2} \leq r \leq 2R} f(\mu(r)) \leq c_3\left(\frac{1}{R^2} + \frac{1}{R^q}\right).$$

Hence $\lim\limits_{R\to\infty} \min\left\{f(\mu(\frac{R}{2})), f(\mu(2R))\right\} = 0$ which implies that $\mu(R) \to 0$ when $R \to \infty$, since f is continuous and vanishes only at 0.
If μ is unbounded, then $\lim\limits_{r\to\infty} \mu(r) = \infty$ which implies $\lim\limits_{|x|\to\infty} u(x) = \infty$. □

Now we assume that $f(u) = u^p$, $p > 1$, and prove Theorem 1.1. We recall that the exponents α, β and γ have been defined at (1.3).

Proof of Theorem 1.1. Let $p, q > 1$ and u be a positive supersolution of (1.1) in $B^c_{r_0}$ (resp. $B_{r_0} \setminus \{0\}$). Let $R \geq 2r_0$ (resp. $0 < R \leq \frac{R}{2}$). From Lemma 2.3, we have that:
if μ is nonincreasing on $[R-\epsilon, R+\epsilon]$, then $\mu(R) \geq u(\widetilde{x}_{R,\epsilon}) \geq \mu(|\widetilde{x}_{R,\epsilon}|) \geq \mu(R(1+\epsilon))$, then

$$\mu^p(R(1+\epsilon)) \leq c_4\left(\frac{\mu(R)}{\epsilon^2 R^2} + \frac{\mu^q(R)}{\epsilon^q R^q}\right) \leq c_4\epsilon^{-h}\left(\frac{\mu(R)}{R^2} + \frac{\mu^q(R)}{R^q}\right) \quad \text{with } h = \max\{2,q\}, \tag{2.9}$$

if μ is nondecreasing on $[R-\epsilon, R+\epsilon]$, then $\mu(R) \geq u(\widetilde{x}_{R,\epsilon}) \geq \mu(|\widetilde{x}_{R,\epsilon}|) \geq \mu(R(1-\epsilon))$, then

$$\mu^p(R(1-\epsilon)) \leq c_4\epsilon^{-h}\left(\frac{\mu(R)}{R^2} + \frac{\mu^q(R)}{R^q}\right). \tag{2.10}$$

Note that for any $c, R > 0$ there holds

$$\frac{\mu^q(R)}{R^q} \leq c\frac{\mu(R)}{R^2} \Longleftrightarrow \mu(R) \leq c^{-\frac{1}{q-1}} R^{-\beta}, \tag{2.11}$$

since $\beta = \frac{2-q}{q-1}$.

1- The exterior problem. From Lemma 2.2, $\mu(r)$ is monotone for $R \geq r_1 \geq r_0$ large enough, so we assume $R > r_1$, and either μ is decreasing or it increases to ∞. In our cases, we claim that μ is decreasing. It holds by assumption if $q \geq p$. When $q < p$ and if μ were increasing, then

$$\mu((1-\epsilon)R) \leq c_5 \epsilon^{-\frac{h}{p}} R^{-\frac{h}{p}} \mu^{\frac{q}{p}}(R),$$

and by Lemma 2.1,

$$\mu(R) \leq c_6 r^{-\frac{h}{p-q}} \quad \text{for } R \geq r_2,$$

contradiction.
Hence μ is decreasing and tends to 0 at infinity by (2.10). Furthermore (2.10) implies

$$\mu^p((1+\epsilon)R) \leq C\epsilon^{-h} R^{-\tilde{h}} \mu(R) \text{ and thus } \mu((1+\epsilon)R) \leq C\epsilon^{-\frac{h}{p}} R^{-\frac{\tilde{h}}{p}} \mu^{\frac{1}{p}}(R) \tag{2.12}$$

with $\tilde{h} = \min\{2, q\}$. Applying again Lemma 2.1 we deduce

$$\mu(R) \leq c_7 R^{-\frac{\tilde{h}}{p-1}}. \tag{2.13}$$

Note that if $q \geq 2$, $\frac{\tilde{h}}{p-1} = \alpha$ and we obtain (1.11). If $1 < q < 2$, then $\tilde{h} = q$ and $\frac{\tilde{h}}{p-1} = \frac{q}{p-1}$ and we encounter two possibilities:
(a) if $\frac{q}{p-1} \geq \beta$, then (1.13) implies

$$\mu(R) \leq c_8 R^{-\beta},$$

and by the equivalence in (2.11)

$$\frac{\mu^q(R)}{R^q} \leq c_8^{1-q} \frac{\mu(R)}{R^2},$$

which in turn implies

$$\mu^p(R(1+\epsilon)) \leq 2c_8 \epsilon^{-2} \frac{\mu(R)}{R^2}.$$

By Lemma 2.1 we obtain (1.11).

This holds in particular when $1 < p \leq q < 2$ which completes the proof of 1-(iii).
(b) Let $A_0 = \frac{q}{p-1} < \beta$. For any $0 < A \leq \beta$ and $\mu(R) \leq c_9 A^{-A}$ we have that

$$\mu^p(2R) \leq c_{10} \left(R^{-(A+2)} + R^{-(A+1)q} \right) = c_{10} R^{-(A+1)q} \left(1 + R^{A(q-1)-(2-q)} \right) \leq 2c_{10} R^{-(A+1)q},$$

so $\mu(2R) \leq c_{11} R^{-\frac{(A+1)q}{p}}$. We define a sequence $\{A_n\}$ by $A_0 = \frac{q}{p-1}$ and

$$A_n = \frac{(A_{n-1}+1)q}{p} \quad \text{for } n \geq 1. \tag{2.14}$$

Then, as long as $A_{n-1} \leq \beta$, we have

$$\mu(2^n R) \leq C_n R^{-A_n}.$$

Furthermore $A_1 - A_0 = \frac{q(q-1)}{p(p-1)}$ and $A_n - A_{n-1} = \frac{q(A_{n-1}-A_{n-2})}{p}$. Therefore the sequence $\{A_n\}$ is increasing.

Proof of 1-(i). For $q > \frac{2p}{p+1}$ we have $\beta < \alpha < \gamma$. If $A_{n-1} < \beta$ for any $n \geq 1$ the sequence $\{A_n\}$ converges to γ, contradiction. Therefore there exists $n_0 \geq 1$ such that $A_{n_0+1} \geq \beta$, so we conclude as in case (a).

Proof of 1-(ii). If $1 < q \leq \frac{2p}{p+1}$, then $\gamma < \alpha < \beta$, and $A_0 < \gamma \leq \beta$ since $q > 1$. So the sequence $\{A_n\}$ is still increasing and it converges to γ. This implies that for any $\theta > 0$, there exists C_θ such that

$$\mu(R) \leq C_\theta R^{-\gamma+\theta} \quad \text{for } R \geq 2r_0.$$

Set $g(r) = r^{-\gamma}$, then

$$g^p(R(1+\epsilon)) \leq R^{-p\gamma} \leq \epsilon^{-q} \frac{g^q(R)}{R^q},$$

since $\gamma = \frac{q}{p-q}$. Recalling that

$$\mu^p(R(1+\epsilon)) \leq c_4 \epsilon^{-q} \left(\frac{\mu(R)}{R^2} + \frac{\mu^q(R)}{R^q} \right),$$

and putting $\phi(R) = \max\{g(R), \mu(R)\}$ we obtain

$$\phi(R(1+\epsilon)) \leq c_{12} \epsilon^{-q} \left(\frac{\mu(R)}{R^2} + \frac{\mu^q(R)}{R^q} + \frac{g^q(R)}{r^q} \right) \leq c_{13} \epsilon^{-q} \left(\frac{\phi(R)}{R^2} + \frac{\phi^q(R)}{R^q} \right).$$

Because $\phi(R) \geq g(R) \geq R^{-\beta}$ as $\gamma \leq \beta$, we have $\frac{\phi(R)}{R^2} \leq \frac{\phi^q(R)}{R^q}$, hence

$$\phi(R(1+\epsilon)) \leq c_{14} \epsilon^{-\frac{q}{p}} R^{-\frac{q}{p}} \phi^{\frac{q}{p}}(R).$$

It follows from Lemma 2.1-(2.3)-(2.16) that $\phi(R) \leq c_{15} R^{-\gamma}$. This is (1.12).

2- The problem in $B_{r_0} \setminus \{0\}$. By Lemma 2.2, μ is nonincreasing and (2.9) holds. If μ is bounded, then it admits a positive limit at 0 and the two estimates in *2* hold. Hence we assume that $\mu(R) \to \infty$ as $R \to 0$. From (2.10)

$$\mu^p(R(1-\epsilon)) \leq c_4 \epsilon^{-h} \left(\frac{\mu(R)}{R^2} + \frac{\mu^q(R)}{R^q} \right),$$

where, we recall it, $h = \max\{2, q\}$. We notice that if (2.11) holds, then

$$\mu((1+\epsilon)R) \leq c_{16} R^{-\frac{2}{p}} \mu^{\frac{1}{p}}(R) \Longrightarrow \mu(R) \leq C' R^{-\alpha},$$

which is the desired estimate in the case $1 < q \leq \frac{2p}{p+1}$. We notice also that the fact that $\mu(R) \to \infty$ as $R \to 0$ implies

$$\mu^p(R(1+\epsilon)) \leq c_4 \epsilon^{-h} \left(\frac{1}{R^2} + \frac{1}{R^q} \right) \mu^q(R) \leq 2 c_4 \epsilon^{-h} R^{-h} \mu^q(R),$$

which in turn yields

$$\mu(R) \leq c_{17} R^{-\frac{h}{p-q}} \quad \text{for } 0 < R \leq r_1 < r_0. \tag{2.15}$$

Hence, if $h = q$, we obtain (1.13).

Proof of 2-(i). Let $2 > q \geq \frac{2p}{p+1}$. Then $\beta \leq \alpha \leq \gamma$, then we start with $\mu(R) \leq R^{-A_0}$ with $A_0 = \frac{2}{p-q} > \gamma$. For any $A > 0$ larger than γ and such that $\mu(R) \leq c_{18}R^{-A}$, there holds

$$\mu^p(\tfrac{R}{2}) \leq c_{19}R^{-(1+A)q},$$

as above since $A > \beta$. The sequence $\{A_n\}$ still defined by (2.14) satisfies

$$\mu\left(\tfrac{R}{2^n}\right) \leq c_n R^{-A_n}$$

as long as $A_{n-1} > \beta$. We have $A_1 - A_0 = \frac{q-(p-q)A_0}{p} < 0$. Since $A_{n+1} - A_n = \frac{q}{p}(A_n - A_{n-1})$, the sequence $\{A_n\}$ is decreasing and it converges to γ. We adapt the technique developed in *1-(ii)*: for any $\theta > 0$ there exists $C_\theta > 0$ such that

$$\mu(R) \leq C_\theta R^{-\gamma-\theta} \quad \text{for } 0 < R \leq \frac{r_0}{2}.$$

Defining $g(R) = R^{-\gamma}$ and $\phi(R) = \max\{g(R), \mu(R)\}$, then we obtain

$$\phi^p(R(1-\epsilon)) \leq c_{20}\epsilon^{-h}\left(\frac{\mu(R)}{R^2} + \frac{\mu^q(R)}{R^q} + \frac{g^q(R)}{R^q}\right) \leq c_{21}\epsilon^{-h}\left(\frac{\phi(R)}{R^2} + \frac{\phi^q(R)}{R^q}\right)$$

Because $\gamma > \beta$ we have $R^{-\beta} \leq R^{-\gamma} \leq \phi(R)$ for $0 < R \leq 1$ which implies that $\frac{\phi(R)}{R^2} \leq \frac{\phi^q(R)}{R^q}$ and

$$\phi^p(R(1-\epsilon)) \leq 2c_{21}\epsilon^{-h}\frac{\phi^q(R)}{R^q}.$$

It follows by Lemma 2.1 that $\phi(R) \leq c_{22}R^{-\gamma}$ and (1.13).

Proof of 2-(ii). If $1 < q < \frac{2p}{p+1}$. Then $\gamma < \beta < \alpha$. We proceed as in case *2-(i)* with the same sequence $\{A_n\}$. We notice that $A_0 = \frac{2}{p-q} > \alpha > \gamma$ since $q > 1$. Then $A_1 < A_0$ and as above $\{A_n\}$ is nonincreasing and converges to γ. As in the proof of *1-(i)* there exists an integer n_0 such that $A_{n_0} \leq \beta$ which in turn implies (2.11), and finally (1.14) holds. □

Remark. From Theorem 1.1 we recover easily the result of Theorem C-(*2*). Indeed, if $f(r) > cr^p$ for $c > 0$ and $r \geq r_1$ and $1 < q < p$, any positive supersolution u of (2.6) in $B^c_{r_1}$ such that $\lim_{|x|\to\infty} u(x) = \infty$ is a supersolution of

$$-\Delta u + m|\nabla u|^q = cu^p$$

in this domain. Then $\lim_{r\to\infty} \mu(r) = 0$ from the upper estimates of Theorem 1.1, contradiction.

2.3 Construction of radial minorant solutions in the exterior problems

The next result extends the construction of [5, Theorem 1.3] and brings precisions to [2, Lemma 4] that we recall below.

Assume $N \geq 2$, $q > 1$ and let $f : (0, \infty) \mapsto \mathbb{R}$ be positive, nondecreasing and continuous. Suppose there exists a positive supersolution u of problem (2.16) *below. Then there exists a positive radial supersolution v of* (2.16)*. In addition, if u does not blow up at infinity, then v is bounded, while if u blows up at infinity, v is bounded from below.*

Our result is the following.

Theorem 2.4 *Let $q > 1$, $m > 0$ and $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a Lipschitz continuous function satisfying assumption (F). Suppose that there exists a positive $C^2(\overline{B}^c_{r_0})$ function u satisfying*

$$-\Delta u + m|\nabla u|^q - f(u) \geq 0 \quad \text{in } B^c_{r_0}, \tag{2.16}$$

then there exists a positive radial and monotone function $v \in C^2(\overline{B}^c_{r_0})$ smaller than u satisfying

$$-\Delta v + m|\nabla v|^q - f(v) = 0 \quad \text{in } B^c_{r_0}, \tag{2.17}$$

such that:
1- $v(r_0) = \min\limits_{|x|=r_0} u(x)$ *and* $\lim\limits_{r\to\infty} v(r) = \infty$, *when* $\lim\limits_{|x|\to\infty} u(x) = \infty$.
2- $0 < v(r_0) = a \leq \min\limits_{|x|=r_0} u(x)$ *and* $\lim\limits_{r\to\infty} v(r) = 0$, *when* $\liminf\limits_{|x|\to\infty} u(x) = 0$, *under the additional condition when $q > 2$,*

$$a < \Theta := \left(\frac{q(N-1)-N}{m(q-1)}\right)^{\frac{1}{q-1}} r_0^{2-N} \int_1^{\frac{\tau}{r_0}} t^{1-N}\left(1 - t^{N-q(N-1)}\right)^{-\frac{1}{q-1}} dt. \tag{2.18}$$

Proof. The proof is based upon an iterative process reminiscent of a method used in [5]. However the technicalities are much more involved and developed in the Appendix. By Lemma 2.3 a positive supersolution u in an exterior domain either tends to ∞ at ∞ or satisfies $\liminf\limits_{|x|\to\infty} u(x) = 0$.

For $\tau > r_0$ we set $\tilde{b}_0 = \inf\limits_{|x|=r_0} u(x)$ and $b_\tau = \inf\limits_{|x|\geq\tau} u(x)$. If $0 < a \leq \tilde{b}_0$ and $0 \leq b \leq b_\tau$ we consider the sequence of radially symmetric functions defined in $B_\tau \cap B^c_{r_0}$ functions $\{v_{k,\tau}\}_{k\in\mathbb{N}}$ such that $v_{0,\tau} \equiv 0$, and for $k \geq 1$

$$\begin{aligned} -\Delta v_{k,\tau} + m|\nabla v_{k,\tau}|^q &= f(v_{k-1,\tau}) && \text{in } B_\tau \cap B^c_{r_0} \\ v_{k,\tau} &= b && \text{in } \partial B_\tau \\ v_{k,\tau} &= a && \text{in } \partial B_{r_0}. \end{aligned} \tag{2.19}$$

If $1 < q \leq 2$, the function $v_{1,\tau}$ exists without any restriction on a and b.
If $q > 2$ we have existence if $a \leq b$ provided $\tau \geq \tau^*$ where τ^* is defined in Lemma 6.1 (2), and if $a > b$ the condition for existence is

$$b < a < b + \Theta.$$

In both case, the function $v_{1,\tau}$ is positive, monotone. and dominated by u.
Next for $k = 2$ we apply the extension [34, Corollary 1.4.5] of the classical result [14, Théorème 2.1]. The function $v_{1,\tau}$ satisfies equation (2.19) with right-hand side 0 instead of $f(v_{k-1,\tau})$. By the maximum principle it is dominated by the supersolution u, thus $f(u) \geq f(v_{1,\tau})$. Then there exists a function $v_{2,\tau}$ which satisfies (2.19) with $k = 2$ and

$$v_{1,\tau} \leq v_{2,\tau} \leq u.$$

Note that this function is unique by the maximum principle. We introduce there the spherical coordinates $(r, \theta) \in \mathbb{R}_+ \times S^{N-1}$ in $\mathbb{R}^N$. Let $\bar{v}_{2,\tau}(r)$ be the spherical average of $v_{2,\tau}(r, .)$ on S^{N-1}.

Since $f(v_{1,\tau})$ is radial, by convexity, $\bar{v}_{2,\tau}$ satisfies

$$\begin{aligned} -\Delta \bar{v}_{2,\tau} + m|\nabla \bar{v}_{2,\tau}|^q \le f(v_{1,\tau}) \quad & \text{in } B_\tau \cap B^c_{r_0} \\ \bar{v}_{2,\tau} = b \quad & \text{in } \partial B_\tau \\ \bar{v}_{2,\tau} = a \quad & \text{in } \partial B_{r_0}. \end{aligned}$$

By the maximum principle we have $\bar{v}_{2,\tau}(r) \le v_{2,\tau}(r,\theta)$ for any r and any θ, which implies that $\bar{v}_{2,\tau} = v_{2,\tau}$, hence $v_{2,\tau}$ is spherically symmetric. Iterating this process, we construct the increasing the sequence $\{v_{k,\tau}\}_{k\in\mathbb{N}}$ of positive spherically symmetric solutions of (2.19) dominated by u in $B_\tau \cap B^c_{r_0}$. For $k \ge 2$ the function $v_{k,\tau}$ cannot have a local minimum, hence if $a \le b$ it is monotone increasing (as a function of $|x|$) and if $a > b$, it is decreasing for $|x|$ close to τ.
Since the sequence $\{v_{k,\tau}\}_{k\in\mathbb{N}}$ is increasing and $v_{k,\tau} \le u$, it converges to some radial positive function $v_{\infty,\tau} := v_\tau$ by Ascoli theorem and v_τ is a positive C^2 solution of

$$\begin{aligned} -\Delta v_\tau + m|\nabla v_\tau|^q = f(v_\tau) \quad & \text{in } B_\tau \cap B^c_{r_0} \\ v_\tau = b \quad & \text{in } \partial B_\tau \\ v_\tau = a \quad & \text{in } \partial B_{r_0}. \end{aligned} \tag{2.20}$$

If $a \ge b$ then necessarily $v_{k,\tau} \le v_{k,\tau'}$ in $B_\tau \cap B^c_{r_0}$ otherwise $v_{k,\tau'}$ would have a local minimum in $B_{\tau'} \cap B_{r_0}$.

Assertion 1. Here $\mu(r) \to \infty$ when $r \to \infty$. Let $r_1 > r_0$ such that $b_\tau > \min\limits_{|x|=r_0} u(x)$ for all $\tau \ge r_1$. Let $v_{\infty,\tau} := v_\tau$ be the solution of (2.20) with $a = \min_{|x|=r_0} u(x)$ and $b = \beta_{r_1}$ and $\tau > \tau^*$ if $q > 2$, which is not a restriction since we aim to let $\tau \to \infty$. Since v_τ cannot have any local minimum in $B_\tau \cap B^c_{r_0}$, we have

$$a \le v_\tau(|x|) \le u(x) \quad \text{for all } x \in B_\tau \cap B^c_{r_0}.$$

By standard ODE techniques, for any $T > r_1$, v_τ is bounded in $C^3(\overline{B}_T \cap B^c_{r_0})$ uniformly with respect to $\tau \ge T+1$. Hence there exists a sequence $\{\tau_n\}$ tending to infinity and a radially symmetric positive function $v \in C^2 B^c_{r_0}$ such that

$$\begin{aligned} -\Delta v + m|\nabla v|^q = f(v) \quad & \text{in } B^c_{r_0} \\ v = a \quad & \text{in } \partial B_{r_0}. \end{aligned} \tag{2.21}$$

Furthermore $a \le v \le u$. By Lemma 2.3 $v(r) \to \infty$ when $r \to \infty$ which proves 1.

Assertion 2. We solve (2.20) with $b = 0$ and $a \le \min_{|x|=r_0} u(x)$ with the additional condition $a < \Theta$ if $q > 2$ and we set $v_{\infty,\tau} := v_\tau$. Then $0 \le v_\tau \le a$ and since the function v_τ cannot have a local minimum in (r_0, τ), we have also that

$$v_\tau(|x|) \le v_{\tau'}(|x|) \le u(x) \quad \text{for all } \tau' > \tau \text{ and } x \in B_\tau \cap B^c_{r_0}.$$

Letting $\tau \to \infty$ we obtain that v_τ converges in the local $C^2(B^c_{r_0})$-topology to some $v \in C^2(B^c_{r_0})$, which satisfies (2.21) and $v(|x|) \le u(x)$ for $x \in B^c_{r_0}$. Therefore $v(r) \to 0$ as $r \to \infty$ and we complete the proof of *2.* □

Corollary 2.5 *Let $N \ge 2$, $m > 0$, $q > \frac{N}{N-1}$ and f be as in Theorem 2.4. Then any positive $C^2(\overline{B}^c_{r_0})$ function u verifying* (2.16) *satisfies*

$$u(x) \ge c|x|^{2-N} \quad \textit{for all } \ x \in B^c_{r_0} \tag{2.22}$$

for some $c > 0$.

Proof. For $r_0 < \tau$, we introduced the function $v_{1,\tau}$ which satisfies

$$\begin{aligned} -v''_{1,\tau} - \frac{N-1}{r} v'_{1,\tau} + m|v_{1,\tau}|^q = 0 \qquad & \text{in } (r_0, \tau) \\ v_{1,\tau}(r_0) = a & \\ v_{1,\tau}(\tau) = 0 & \end{aligned}$$

with $0 < a \leq \min_{|x|=r_0} u(x)$. We have seen therein that $v_{1,\tau}(|x|) \leq u(x)$ for $x \in B_\tau \setminus B_\rho$. If $q > 2$ we choose $a \leq \Theta$. When $\tau \to \infty$, $v_{1,\tau} \uparrow v_{1,\infty}$ and $v := v_{1,\infty}(|x|) \leq u(x)$ in $B^c_{r_0}$. Since $v' \leq 0$, we have

$$v'' + v^p = m|v'|^q - \frac{N-1}{r} v' \geq 0.$$

then

$$E(r) := \left(\frac{v'(r)^2}{2} + \frac{v(r)^{p+1}}{p+1} \right)' \leq 0.$$

Therefore $E(r)$ admits a limit when $r \to \infty$. Because $v(r) \to 0 \geq 0$, this implies that $v'(r)$ admits also a limit $\ell \leq 0$ when $r \to \infty$ and this, limit is necessarily 0 since v is bounded.
Set $w(r) = -r^{N-1} v'$, then $w \geq 0$ and

$$w' + m r^{(1-q)(n-1)} w^q \geq 0.$$

Integrating this equation as it is done in Appendix, we obtain

$$(w^{1-q})'(r) + \frac{m(q-1)}{q(N-1) - N} m (r^{(N-q(N-1)})' \leq 0,$$

which implies by integration

$$w^{1-q}(r) - w^{1-q}(r_1) \leq \frac{m(q-1)}{q(N-1) - N} \left(r_1^{N-q(N-1)} - r^{N-q(N-1)} \right).$$

Therefore $w(r) \geq c_1 > 0$ and $v'(r) \geq -c_1 r^{1-N}$ and thus $v(r) \geq \frac{c_1}{N-2} r^{2-N}$. Because $u(x) \geq v(r)$ for $|x| = r \geq r_0$ this yields (2.22). □

Remark. As a consequence we recover Theorem C-(1) in the case $q > \frac{N}{N-1}$. Indeed, suppose that $f(s) \geq C s^p$ near $s = 0$ and $1 < p \leq \frac{N}{N-2}$. Then if there exists a positive supersolution of (2.6) which is bounded at infinity, then $\liminf_{|x| \to \infty} u(x) = 0$ by Lemma 2.3. Since u is a supersolution of

$$-\Delta u + m|\nabla u|^q = C u^q \quad \text{in } B^c_{r_1}$$

for some $r_1 > r_0$, by Theorem 1.1 and Corollary 2.5 there exists a positive radially symmetric solution v of the above equation such that

$$u(x) \geq v(|x|) \geq c|x|^{2-N} \quad \text{for all } x \in B^c_{r_1}.$$

By Theorem 1.1 we have also $\mu(|x|) \leq C|x|^{-\alpha}$ in $B^c_{r_1}$. This is a contradiction when $p > \frac{N}{N-2}$. When $p = \frac{N}{N-2}$ we set $v(r) = r^{2-N} X(t)$ with $t = \ln r$. Then $c_1 \leq X(t) \leq c_2$ for $t \geq t_1 = \ln r_1$. Hence X is a bounded solution of

$$X'' - (N-2)X' + CX^p - m e^{(N-q(N-1))t} \left(|(N-2)X - X'| \right)^q = 0,$$

and it is straightforward to verify that the ω-limit set of the trajectory $\mathcal{T}_+[v] = \bigcup_{t\geq t_1}\{X(t)\}$ is reduced to $\{0\}$, which is still a contradiction.

2.4 Dichotomy result when $q \geq p$. Proof of Theorem 1.2

In this Section we suppose $q \geq p > 1$. Then there exist supersolutions of (1.1) such that $\lim_{|x|\to\infty} u(x) = \infty$, e.g. $u(x) = e^{\lambda|x|}$ for any $\lambda > 0$ if $q > p$ or λ large enough if $q = p$.

Proof of Theorem 1.2. Our proof is based upon Theorem 2.4 with $f(u) = u^p$. Let u be a positive supersolution of (1.1). From Lemma 2.3, either $u(x) \to \infty$ or $\mu(|x|) \to 0$ when $|x| \to \infty$.

(i) Suppose that $\lim_{|x|\to\infty} u(x) = \infty$. By Theorem 2.4 there exists a radial and increasing function v below u in $B^c_{r_1}$ satisfying

$$\begin{aligned} -v'' - \frac{N-1}{r}v' + mv'^q = v^p \quad \text{in } (r_1,\infty) \\ v(r_1) = \min_{|x|=r_1} u(x) \\ \lim_{r\to\infty} v(r) = \infty. \end{aligned} \tag{2.23}$$

For $\epsilon > 0$ we set $F_\epsilon(r) = v^p(r) - (1+\epsilon)m(v'(r))^q$. This type of function introduced by [30] is fundamental in the study of radial soutions. Then

$$F'_\epsilon(r) = pv'v^{p-1} - q(1+\epsilon)mv''v'^{q-1} = pv'v^{p-1} + q(1+\epsilon)mv'^{q-1}\left(\frac{N-1}{r}v' + v^p - mv'^q\right).$$

If there exists some $r_2 > r_1$ such that $F_\epsilon(r_2) = 0$, then

$$F'_\epsilon(r_2) = pv'v^{p-1} + q(1+\epsilon)mv'^{q-1}\left(\frac{N-1}{r_2}v' + \epsilon mv'^q\right) > 0.$$

This implies that $F_\epsilon(r) > 0$ for all $r > r_2$. As a consequence, $F_\epsilon(r)$ has a constant sign for r large enough.

When $N \geq 3$ we can take $\epsilon = 0$. If $F_0 \leq 0$ for $r > r_2 > r_0$, then $v^p(r) \leq m(v'(r))^q$ which implies

$$v(r) \geq (m|\gamma|^q)^{\frac{1}{p-q}}(r - r_2)^{|\gamma|} \quad \text{for all } r > r_2, \tag{2.24}$$

in the case $q > p$ and

$$v(r) \geq v(r_2)e^{m^{-\frac{1}{m}}(r-r_2)} \quad \text{for all } r > r_2, \tag{2.25}$$

when $q = p$. This yields (1.15).

If $F_0 \geq 0$ for $r > r_2 > r_0$, then $\Delta v \leq 0$ if $|x| > r_2$, and the function $r^{N-1}v'r)$ is nonincreasing on $[r_2,\infty)$, thus $v'(r) \leq cr^{1-N}$. If $N \geq 3$, it implies that $v(r)$ remains bounded, which is a contradiction.

When $N = 2$ we take $\epsilon = 1$. If $F_1(r_3) = 0$ for some r_3, then either F_1 is positive for $r \geq r_3$, which implies

$$-2v'' = \frac{1}{r}v' + v^p + F_2(r) \geq v^p \quad \text{for } r \geq r_2.$$

In such a case, we deduce by multiplying by $v' \geq 0$ that the function $r \mapsto \left(v'^2 + \frac{v^{p+1}}{p+1}\right)(r)$ is nonincreasing, hence bounded, contradiction. If this does not hold, then F_1 is nonpositive for $r \geq r_3$, which yields

$$v(r) \geq \begin{cases} (2m|\gamma|^q)^{\frac{1}{p-q}} (r-r_2)^{-\gamma} & \text{if } r \geq r_2 \text{ when } N \geq 3 \\ v(r_2) e^{(2m)^{-\frac{1}{2m}}(r-r_2)} & \text{if } r \geq r_2 \text{ when } N = 2. \end{cases} \tag{2.26}$$

If we have now $F_0(r) > 0$, then $v'(r) \leq cr^{-1}$ which implies $v(r) \leq c \ln r + d$, which is not compatible with (2.26). Therefore $F_0(r) \leq 0$ which again implies that (1.15) holds.

(ii) Assume now that $\lim_{r\to\infty} \mu(r) = 0$. Inequality (1.16)-(a) follows from Theorem 1.1 (1-iii). Since $q > p > \frac{N}{N-2}$ we have $q > \frac{N}{N-1}$. Thus (1.16)-(b) is a consequence of Corollary 2.5. □

3 Estimates on solutions

3.1 General estimates

A major tool for proving a priori estimates either near an isolated singularity or at infinity is the Keller-Osserman combined with Bernstein method applied to the function $z = |\nabla u|^2$. We recall the variant of Keller-Osserman a priori estimate that we proved in [8].

Lemma 3.1 *Let $q > 1$ $d \geq 0$ and P and Q two continuous functions defined in $B_\rho(a)$ such that $\inf\{P(y) : y \in B_\rho(a)\} > 0$ and $\sup\{Q(y) : y \in B_\rho(a)\} < \infty$. If z is a positive C^1 function defined in $B_\rho(a)$ and such that*

$$-\Delta z + P(y) z^q \leq Q(y) + d\frac{|\nabla z|^2}{z} \quad \text{in } B_\rho(a), \tag{3.1}$$

then there exists a positive constant $C = C(N, q, d) > 0$ such that

$$z(x) \leq C \left(\left(\frac{1}{\rho^2} \frac{1}{\inf\limits_{B_\rho(a)} P} \right)^{\frac{1}{q-1}} + \left(\sup_{B_\rho(a)} \frac{Q}{P} \right)^{\frac{1}{q}} \right) \quad \text{for all } x \in B_{\frac{\rho}{2}}(a). \tag{3.2}$$

In the next statement we show how an upper estimate on $u(x)$ by a power of $|x|$ implies a precise estimate on $|\nabla u(x)|$.

Theorem 3.2 *Let $p, q > 1$, $m > 0$ and $r_0 > 0$.*
1- If u is a positive solution of (1.1) in $B_{r_0} \setminus \{0\}$ where it satisfies

$$|x|^\lambda u(x) \leq c \tag{3.3}$$

for some constant $c > 0$ and some exponent $\lambda > 0$, then there exists $c_1 = c_1(N, p, q, \lambda, c) > 0$ such that

$$|\nabla u(x)| \leq c_1 \left(|x|^{-\frac{1}{q-1}} + |x|^{-\frac{\lambda p}{q}} + |x|^{-\frac{\lambda(p-1)}{2(q-1)}} \right) \quad \text{for all } x \in B_{\frac{r_0}{2}} \setminus \{0\}. \tag{3.4}$$

Furthermore, when $1 < q \leq 2$*, one has an improvement of* (3.4) *under the form*

$$|\nabla u(x)| \leq c_1' |x|^{-(\lambda+1)} \quad \text{for all } x \in B_{\frac{r_0}{2}} \setminus \{0\}, \tag{3.5}$$

for any $\lambda > 0$ *such that* $\lambda \leq \min\{\alpha, \beta\}$.
2- If u *is a positive solution of* (1.1) *in* $B^c_{r_0}$*, then*

$$\limsup_{|x|\to\infty} u(x) < \infty \Longrightarrow \limsup_{|x|\to\infty} |\nabla u(x)| < \infty, \tag{3.6}$$

$$\lim_{|x|\to\infty} u(x) = 0 \Longrightarrow \lim_{|x|\to\infty} |\nabla u(x)| = 0. \tag{3.7}$$

If u *satisfies* (3.3) *in* $B^c_{r_0}$ *for some* $c > 0$ *and* $\lambda > 0$*, then there exists* $c_1 := c_1(N, p, q, \lambda, c) > 0$ *such that*

$$|\nabla u(x)| \leq c_1 \left(|x|^{-\frac{1}{q-1}} + |x|^{-\frac{\lambda p}{q}} + |x|^{-\frac{\lambda(p-1)}{2(q-1)}} \right) \quad \text{for all } x \in B^c_{2r_0}. \tag{3.8}$$

Furthermore, if $1 < q \leq 2$*, one has an improvement of* (3.8) *under the form*

$$|\nabla u(x)| \leq c_2 |x|^{-(\lambda+1)} \quad \text{for all } x \in B^c_{2r_0}, \tag{3.9}$$

for $c_2 := c_2(N, p, q, \lambda, c) > 0$ *for any* $\lambda \geq \max\{\alpha, \beta\}$.

Proof. We use Bernstein method, setting $z(x) = |\nabla u(x)|^2$ and Weitzenböck's formula

$$-\frac{1}{2}\Delta z = |D^2 u|^2 + \langle \nabla(\Delta u), \nabla u \rangle.$$

Using the inequality $|D^2 u|^2 \geq \frac{1}{N}(\Delta u)^2$ and the equation satisfied by u we obtain

$$-\frac{1}{2}\Delta z + \frac{1}{N}(m z^{\frac{q}{2}} - u^p)^2 + \langle \nabla(m z^{\frac{q}{2}} - u^p), \nabla u \rangle \leq 0.$$

Developing this inequality yields

$$-\frac{1}{2}\Delta z + \frac{m^2}{N} z^q + \frac{1}{N} u^{2p} \leq \frac{2m}{N} u^p z^{\frac{q}{2}} + p u^{p-1} z + \frac{mq}{2} z^{\frac{q}{2}-1} \langle \nabla z, \nabla u \rangle.$$

Now for $\epsilon > 0$

$$z^{\frac{q}{2}-1} \langle \nabla z, \nabla u \rangle = z^{\frac{q}{2}-\frac{1}{2}} \langle \frac{\nabla z}{\sqrt{z}}, \nabla u \rangle \leq z^{\frac{q}{2}} \frac{|\nabla z|}{\sqrt{z}} \leq \epsilon z^q + \frac{1}{\epsilon} \frac{|\nabla z|^2}{z},$$

$$u^{p-1} z \leq \epsilon z^q + \epsilon^{-\frac{1}{q-1}} u^{\frac{q(p-1)}{q-1}},$$

and

$$u^p z^{\frac{q}{2}} \leq \epsilon z^q + \frac{1}{\epsilon} u^{2p}.$$

We choose ϵ small enough and get

$$-\Delta z + \frac{m^2}{N} z^q \leq c_3 \frac{|\nabla z|^2}{z} + c_4 u^{2p} + c_5 u^{\frac{q(p-1)}{q-1}} \tag{3.10}$$

where $c_i = c_i(N,p,q,m) > 0$, $i = 3,4,5$. We Apply Lemma 3.1 in $\overline{B}_{2\rho}(a)$, with $\overline{B}_{2\rho}(a) \subset B_{r_0}\backslash\{0\}$ in case 1, or $\overline{B}_{2\rho}(a) \subset \overline{B}^c_{r_0}$ in case 2, we obtain for some positive constant $c_6 := c_6(N,q,m) > 0$,

$$\sup_{B_\rho(a)} z(y) \le c_6 \left(\rho^{-\frac{2}{q-1}} + \sup_{B_{2\rho}(a)} \left(u^{2p} + u^{\frac{q(p-1)}{q-1}} \right)^{\frac{1}{q}} \right), \tag{3.11}$$

which is equivalent to

$$\sup_{B_\rho(a)} |\nabla u(z)| \le c_7 \left(\rho^{-\frac{1}{q-1}} + \sup_{B_{2\rho}(a)} \left(u^{\frac{p}{q}} + u^{\frac{p-1}{2(q-1)}} \right) \right), \tag{3.12}$$

where $c_7 = c_7(N,q,m,c_6) > 0$.

1- Next we assume that $u(x) \le c_8|x|^{-\lambda}$ in $B_{r_0} \setminus \{0\}$. Then (3.12) yields exactely (3.4) with $c_9 = c_9(N,m,p,q,\lambda,c_8) > 0$.

In some cases we can obtain a different estimate which requires $1 < q \le 2$. For $k > 0$ we set

$$u_k(x) = k^\lambda u(kx).$$

Then u_k satisfies

$$-\Delta u_k + mk^{\lambda+2-q(\lambda+1)}|\nabla u_k|^q - k^{\lambda+2-\lambda p}u_k^p = 0 \quad \text{in } B_{k^{-1}r_0}. \tag{3.13}$$

The function u_k is uniformly bounded in the spherical shell $\Gamma_{\frac{r_0}{8},\frac{2r_0}{3}} := \left\{x : \frac{r_0}{8} \le |x| \le \frac{r_0}{2}\right\}$. If we assume that

$$\lambda + 2 - q(\lambda+1) \ge 0 \Longleftrightarrow \lambda \le \tfrac{2-q}{q-1} = \beta \quad \text{and } \lambda + 2 - \lambda p \ge 0 \Longleftrightarrow \lambda \le \tfrac{2}{p-1} = \alpha, \tag{3.14}$$

then we deduce from standard regularity estimates [23] (this is why we need $1 < q \le 2$) that

$$|\nabla u_k(x)| \le c_9 \Longleftrightarrow |\nabla u(kx)| \le c_9 k^{-\lambda-1} \quad \text{for all } x \in \Gamma_{\frac{r_0}{4},\frac{r_0}{2}}. \tag{3.15}$$

This implies in particular

$$|\nabla u(x)| \le c_9|x|^{-\lambda-1} \quad \text{for all } x \in B_{\frac{r_0}{4}} \setminus \{0\}. \tag{3.16}$$

Now, this estimate is better than the one in (3.4) if and only if $\lambda \le \min\{\alpha,\beta\}$ and

$$\lambda + 1 \le \max\left\{ \frac{1}{q-1}, \frac{\lambda p}{q}, \frac{\lambda(p-1)}{2(q-1)} \right\}, \tag{3.17}$$

that means

$$\lambda \le \beta, \text{ or } (q < p \text{ and } \lambda > \gamma), \text{ or } \left(q < \tfrac{p+1}{2} \text{ and } \lambda > \tfrac{2(q-1)}{p+1-2q}\right). \tag{3.18}$$

Hence it is an improvement for any $\lambda \le \min\{\alpha,\beta\}$.

2- We apply (3.12) for $|a| > \rho/2$ with $\rho = \frac{|a|}{4}$, then we get

$$|\nabla u(a)| \le c_{10} \left(|a|^{-\frac{1}{q-1}} + \max_{|x| \ge \frac{|a|}{2}} \left(u^{\frac{p}{q}} + u^{\frac{p-1}{2(q-1)}} \right) \right).$$

Clearly (3.6) and (3.7) follow.
Next we assume $1 < q \leq 2$ and $u(x) \leq c_{10}|x|^{-\lambda}$ in $B^c_{r_0}$, then (3.12) yields precisely (3.8).
Again the function u_k defined previously is uniformly bounded in the spherical shell $\Gamma_{\frac{3r_0}{2},4r_0}$. In order to apply the standard elliptic equations regularity results to (3.13), we need again $1 < q \leq 2$ and

$$\lambda + 2 - q(\lambda + 1) \leq 0 \Longleftrightarrow \lambda \geq \beta \quad \text{and} \quad \lambda + 2 - \lambda p \leq 0 \Longleftrightarrow \lambda \geq \alpha, \tag{3.19}$$

This yields

$$|\nabla u(x)| \leq c_{11}|x|^{-\lambda-1} \quad \text{for all } x \in B^c_{2r_0}. \tag{3.20}$$

This estimate is an improvement of (3.8) if $\lambda \geq \max\{\alpha, \beta\}$ and

$$\lambda + 1 \geq \min\left\{\frac{1}{q-1}, \frac{\lambda p}{q}, \frac{\lambda(p-1)}{2(q-1)}\right\}. \tag{3.21}$$

That means

$$\lambda \leq \beta, \text{ or } (q \geq p \text{ and } \lambda \leq \gamma), \text{ or } \left(q < \tfrac{p+1}{2} \text{ and } \tfrac{\lambda(p+1-2q)}{2(q-1)} < 1\right). \tag{3.22}$$

Hence it is an improvement for any $\lambda \geq \max\{\alpha, \beta\}$. □

3.2 Upper estimates on solutions when $q > p$. Proof of Theorem 1.3

Proof of Theorem 1.3. We apply Lemma 3.1.
1- *Proof of 1-* By change of scale we can assume that $r_0 = 1$. For $0 < \theta < \frac{1}{4}$ we set $\Omega_\theta = B_{1-\theta}\backslash B_\theta$. For $0 < \epsilon < \frac{1}{2}$, we have by (3.12)

$$\max_{\overline{\Omega}_\theta} |\nabla u| \leq C\left(\left(\frac{1}{\theta\epsilon}\right)^{\frac{1}{q-1}} + \max_{\overline{\Omega}_{\frac{\theta}{1+\epsilon}}}\left(u^{\frac{p}{q}} + u^{\frac{p-1}{2(q-1)}}\right)\right), \tag{3.23}$$

and $u^{\frac{p-1}{2(q-1)}} \leq u^{\frac{p}{q}} + 1$ since $q > \frac{2p}{p+1}$. Hence

$$\max_{\overline{\Omega}_\theta} |\nabla u| \leq c_1\left(\left(\frac{1}{\theta\epsilon}\right)^{\frac{1}{q-1}} + 1 + \max_{\overline{\Omega}_{\frac{\theta}{1+\epsilon}}} u^{\frac{p}{q}}\right).$$

Next we estimate u in function of its gradient: for any $x \in \overline{\Omega}_{\frac{\theta}{1+\epsilon}}$,

$$u(x) \leq u\left((1-\theta)\frac{x}{|x|}\right) + \left|x - (1-\theta)\frac{x}{|x|}\right| \max_{y \in [x,(1-\theta)\frac{x}{|x|}]} |\nabla u(y)|.$$

Therefore

$$\max_{\overline{\Omega}_{\frac{\theta}{1+\epsilon}}} u \leq \max_{\overline{B}_1\backslash B_{\frac{1}{2}}} u + \max_{\overline{\Omega}_{\frac{\theta}{1+\epsilon}}} |\nabla u| \leq c_1' + \max_{\overline{\Omega}_{\frac{\theta}{1+\epsilon}}} |\nabla u|.$$

Since $1 \leq \frac{1}{\theta\epsilon}$, we deduce

$$\max_{\overline{\Omega}_\theta} |\nabla u| \leq c_2 \left((\theta\epsilon)^{-\frac{1}{q-1}} + \left(\max_{\overline{\Omega}_{\frac{\theta}{1+\epsilon}}} |\nabla u| \right)^{\frac{p}{q}} \right).$$

We set

$$A(\theta) = \theta^{\frac{1}{q-1}} \max_{\Omega_\theta} |\nabla u|,$$

then $A(\frac{\theta}{1+\epsilon}) \leq A((1-\frac{\epsilon}{2})\theta)$ since $\epsilon, \theta \leq \frac{1}{2}$, hence

$$A(\theta) \leq c_4 \left(\epsilon^{-\frac{1}{q-1}} + \theta^{\frac{q-p}{q(q-1)}} (1+\epsilon)^{\frac{p}{q(q-1)}} \left(A((1-\tfrac{\epsilon}{2})\theta) \right)^{\frac{p}{q}} \right).$$

If we set $F(\theta) = 1 + A(\theta)$ there holds

$$F(\theta) \leq c_5 \epsilon^{-\frac{1}{q-1}} F^{\frac{p}{q}}(A(1-\tfrac{\epsilon}{2})\theta), \tag{3.24}$$

and we can apply the bootstrap result of Lemma 2.1 with $\Phi = 1$, $h = \frac{1}{q-1}$ and $d = \frac{p}{q}$. We deduce that F is bounded, hence

$$\max_{\overline{\Omega}_\theta} |\nabla u| \leq c_6 \theta^{-\frac{1}{q-1}}. \tag{3.25}$$

Thus (1.17) holds.

2- *Proof of 2-* By change of scale we assume again that $r_0 = 1$. For $T > 3$ and $0 < \epsilon < 1/2$ we set

$$\Omega_T = B_T \setminus \overline{B}_1 \ \text{ and } \ \Omega_{T,\epsilon} = B_{T-\epsilon} \setminus \overline{B}_{1+\epsilon}.$$

By (3.12), for any $\rho > 0$ and $x \in B^c_{1+2\rho}$ we have

$$|\nabla u(x)| \leq c_7 \left(\rho^{-\frac{1}{q-1}} + 1 + \max_{\overline{B}_{2\rho}(x)} u^{\frac{p}{q}} \right).$$

Taking $\rho = \frac{\epsilon}{2}$ we get

$$\max_{\overline{\Omega}_{T,\epsilon}} |\nabla u| \leq c_8 \left(\epsilon^{-\frac{1}{q-1}} + 1 + \max_{\overline{\Omega}_T} u^{\frac{p}{q}} \right). \tag{3.26}$$

It is clear that

$$\max_{\overline{\Omega}_T} u \leq \max_{|x|=1} u(x) + T \max_{\overline{\Omega}_T} |\nabla u|.$$

reporting this inequality in (3.26) we obtain that for any $T \geq 1$,

$$1 + \max_{\overline{\Omega}_{T,\epsilon}} |\nabla u| \leq c_9 \epsilon^{-\frac{1}{q-1}} T^{\frac{p}{q}} \left(1 + \max_{\overline{\Omega}_T} |\nabla u| \right)^{\frac{p}{q}}. \tag{3.27}$$

We set $F(T) = 1 + \max_{\overline{\Omega}_T} |\nabla u|$, then

$$\begin{aligned} F(T(1-\epsilon)) &\leq 1 + \max_{1\leq|x|\leq 1+\epsilon} |\nabla u(x)| + \max_{\overline{\Omega}_{T,\epsilon}} |\nabla u| \\ &\leq 1 + \max_{1\leq|x|\leq 2} |\nabla u(x)| + + \max_{\overline{\Omega}_{T,\epsilon}} |\nabla u| \\ &\leq c_{10}\left(\epsilon^{-\frac{1}{q-1}} + 1 + \left(\max_{|x|=1} u(x) + T\max_{\overline{\Omega}_T} |\nabla u|\right)^{\frac{p}{q}}\right) \\ &\leq c_{11}\epsilon^{-\frac{1}{q-1}} T^{\frac{p}{q}} F^{\frac{p}{q}}(T). \end{aligned} \tag{3.28}$$

Using again the bootstrap result of Lemma 2.1 with $d = \frac{p}{q}$ we obtain in particular for $T \geq 2$,

$$F(T) \leq c_{12} T^{\frac{p}{q}\frac{1}{1-\frac{p}{q}}} = c_{12} T^{\frac{p}{q-p}}. \tag{3.29}$$

This implies

$$|\nabla u(x)| \leq c_{13}|x|^{\frac{p}{q-p}}. \tag{3.30}$$

Using (3.30) we get

$$\max_{\overline{\Omega}_T} u \leq \max_{|x|=1} u(x) + T\max_{\overline{\Omega}_T} |\nabla u| \leq c_{14} T^{1+\frac{p}{q-p}} = c_{14} T^{\frac{q}{q-p}},$$

which leads to

$$u(x) \leq c_{14}|x|^{\frac{q}{q-p}} \quad \text{for all } x \in B_3^c. \tag{3.31}$$

□

By integrating the inequalities (1.17) and (1.18), we obtain:

Corollary 3.3 *Under the assumption of Theorem 1.3, any nonnegative solution u of* (1.1) *in G satisfies:*
1- If $G = B_{r_0} \setminus \{0\}$.
1-(i) If $q > \max\{2, p\}$, then u can be extended as a continuous function in B_{r_0}.
1-(ii) If $q = 2 > p$, then there exists a constant $C_1 > 0$ such that

$$u(x) \leq C_1(|\ln|x|| + 1) \quad \textit{for all } x \in B_{\frac{r_0}{2}} \setminus \{0\}. \tag{3.32}$$

1-(iii) If $2 > q > p$, then there exists a constant $C_2 > 0$ such that

$$u(x) \leq C_2|x|^{-\frac{2-q}{q-1}} \quad \textit{for all } x \in B_{\frac{r_0}{2}} \setminus \{0\}. \tag{3.33}$$

2- If $G = B_{r_0}^c$, then there exists a constant $C_3 > 0$ such that

$$u(x) \leq C_3|x|^{\frac{q}{q-p}} \quad \textit{for all } x \in B_{2r_0}^c \setminus \{0\}. \tag{3.34}$$

Remark. The constants C_i in (3.32)-(3.33) (resp. (3.34)) depend on $\sup_{B_{r_0}\setminus B_{\frac{3r_0}{4}}} u(y)$ (resp. $\sup_{B_{2r_0}\setminus B_{r_0}} u(y)$). Up to modifying θ it is possible to reduce that domain of dependance of the constant with respect to u to $\sup_{B_{r_0}\setminus B_{(1-\tau)r_0}} u(y)$ (resp. $\sup_{B_{(1+\tau)r_0}\setminus B_{r_0}} u(y)$ for any $\tau \in (0,1)$.

3.3 Upper estimates on solutions when $q < p$. Proof of Theorem 1.4

We recall the doubling Lemma [24], [29].

Theorem 3.4 *Let (X, d) be a complete metric space, D a non-empty subset of X, Σ a closed subset of X containing D and $\Gamma = \Sigma \setminus D$. Let $M : D \mapsto (0, \infty)$ be a map which is bounded on compact subsets of D and let $k > 0$ be a real number. If $y \in D$ is such that*

$$M(y)\mathrm{dist}\,(y, \Gamma) > 2k,$$

there exists $x \in D$ such that

$$\begin{aligned} &M(x)\mathrm{dist}\,(x, \Gamma) > 2k \\ &M(x) \geq M(y) \\ &M(z) \leq 2M(x) \quad \textit{for all } z \in D \textit{ s.t. } d(z, x) \leq \frac{k}{M(x)}. \end{aligned}$$

Proof of Theorem 1.4-(1). We can assume that $r_0 = 1$. By (3.7), (1.21) implies that $|\nabla u(x)| \to 0$ when $|x| \to \infty$. The estimate (1.20) is equivalent to

$$u(x) \leq C|x|^{-\frac{q}{p-q}} = C|x|^{-\gamma} \tag{3.35}$$

for all $x \in B_2^c$ by (3.4), hence also to

$$u^{\frac{1}{\gamma}}x) + |\nabla u(x)|^{\frac{1}{\gamma+1}} \leq \frac{C}{|x|} \tag{3.36}$$

for all $x \in B_2^c$. We set

$$M(x) := u^{\frac{1}{\gamma}}(x). \tag{3.37}$$

Then $M(x) \to 0$ when $|x| \to \infty$. Let us assume that $|x|^\gamma u(x)$ is unbounded in $B_{2r_0}^c$. Then by Theorem 3.4 applied with $\Sigma = B_2^c$, $D = \overline{B}_2^c$, thus $\Gamma = B_2^c \setminus \overline{B}_2^c = \partial B_2$, and $k = n$, there exists a sequence $\{y_n\} \subset \overline{B}_2^c$ such that $(|y_n| - 2)M(y_n) \to \infty$ when $n \to \infty$. There exists a sequence $\{x_n\} \subset \overline{B}_2^c$ such that

$$\begin{aligned} &|x_n|M(x_n) > (|x_n| - 2)M(x_n) > 2n \\ &M(x_n) \geq M(y_n) \\ &M(z) \leq 2M(x_n) \quad \text{for all } z \in \overline{B}_2^c \text{ s.t. } |z - x_n| \leq \frac{n}{M(x_n)}. \end{aligned} \tag{3.38}$$

Clearly $\{x_n\}$ is unbounded since M is bounded on bounded subsets of B_2^c and, up to extracting a sequence, we can assume that $|x_n| \to \infty$ as $n \to \infty$. We now define

$$u_n(x) = \frac{u(z(x, n))}{M^\gamma(x_n)} \quad \text{with } z(x, n) = x_n + \frac{x}{M(x_n)}. \tag{3.39}$$

Then

$$u_n(0) = 1 \quad \text{and } u_n(x) \leq 2^\gamma \quad \text{for } x \in B_n. \tag{3.40}$$

The main point is to use estimate (3.12) in order to obtain a uniform estimate on ∇u_n. We apply this inequality in $B_{\frac{n}{M(x_n)}}(x_n)$ which yields

$$\max_{z\in B_{\frac{n}{2M(x_n)}}(x_n)} |\nabla u(z)| \le c_7 \left(\left(\frac{n}{2M(x_n)} \right)^{-\frac{1}{q-1}} + \max_{z\in B_{\frac{n}{M(x_n)}}(x_n)} \left(u^{\frac{p}{q}}(z) + u^{\frac{p-1}{2(q-1)}}(z) \right) \right) \tag{3.41}$$

Furthermore $z \in B_{\frac{n}{M(x_n)}}(x_n)$ is equivalent to $|x| \le n$. Similarly, $z \in B_{\frac{n}{2M(x_n)}}(x_n)$ is equivalent to $|x| \le \frac{n}{2}$. If u_n is defined by (3.39), then

$$\nabla u_n(x) = \frac{\nabla u(z(x,n))}{M^{\gamma+1}(x_n)}.$$

We have that $\frac{p}{q} < \frac{p-1}{2(q-1)}$ since $q < \frac{2p}{p+1}$. Combined with the decay estimate (1.19) we infer that

$$\max_{z\in B_{\frac{n}{M(x_n)}}(x_n)} \left(u^{\frac{p}{q}}(z) + u^{\frac{p-1}{2(q-1)}}(z) \right) \le c_8 \max_{z\in B_{\frac{n}{M(x_n)}}(x_n)} u^{\frac{p}{q}}(z). \tag{3.42}$$

We now replace $u(z)$ and $\nabla u(z)$ by their respective value with respect to $u_n(x)$ and $\nabla u_n(x)$ and we get

$$\max_{|x|\le\frac{n}{2}} |\nabla u_n(x)| \le c_9 \left(n^{-\frac{1}{q-1}} \left(M(x_n) \right)^{\frac{1}{q-1}-\gamma-1} + \max_{|x|\le n} u_n^{\frac{p}{q}}(x) \right). \tag{3.43}$$

Because $1 < q < \frac{2p}{p+1}$, $\frac{1}{q-1} - \gamma - 1 > 0$. Since $M(x_n) \to 0$ when $n \to \infty$ it follows that

$$|\nabla u_n(x)| \le c_{10} \quad \text{for all } x \in B_{\frac{n}{2}}. \tag{3.44}$$

Therefore the new constraints are

$$u_n^{\frac{1}{\gamma}}0) = 1 \quad \text{and} \quad u_n(x) + |\nabla u_n(x)| \le 2^\gamma + c_{10} \quad \text{for } x \in B_{\frac{n}{2}}. \tag{3.45}$$

We have also

$$-\Delta u_n(x) = -\frac{\Delta u(z(x,n))}{M^{\gamma+2}(x_n)},$$

hence

$$\begin{aligned} -\Delta u_n(x) &= \frac{u^p(z(x,n)) - m|\nabla u(z(x,n))|}{M^{\gamma+2}(x_n)} \\ &= \frac{M^{\gamma p}(x_n) u_n^p(x) - m M^{(\gamma+1)q}(x_n)|\nabla u_n(x)|}{M^{\gamma+2}(x_n)} \\ &= M^{\gamma(p-1)-2}(x_n) u_n^p - m M^{(\gamma(q-1)-2+q)q}(x_n)|\nabla u_n(x)|^q. \end{aligned}$$

There holds

$$\gamma(p-1) - 2 = \gamma(q-1) - 2 + q = \frac{\sigma}{p-q},$$

and by assumption, $\sigma < 0$. Therefore u_n satisfies

$$-\epsilon_n \Delta u_n(x) = u_n^p - m|\nabla u_n|^q \quad \text{with } \epsilon_n = M^{-\frac{\sigma}{p-q}}(x_n) \to 0 \text{ as } n \to \infty. \tag{3.46}$$

Jointly with the conditions (3.45) there exists a subsequence of $\{u_n\}$ still denoted by $\{u_n\}$ and a function $v \in W^{1,\infty}(\mathbb{R}^N)$ such that u_n converges to v locally uniformly in $\mathbb{R}^N$ and $\nabla u_n \rightharpoonup \nabla v$ for the weak topology of $L^\infty_{loc}(\mathbb{R}^N)$. By a classical viscosity result [21, Proposition IV.1], v is a bounded viscosity solution of

$$m|\nabla v]^q = v^p \quad \text{in } \mathbb{R}^N. \tag{3.47}$$

By [21, Proposition 4.3] (3.43) has a unique viscosity solution which is zero which is not compatible with $v(0) = 1$ by (3.45), which ends the proof.

Proof of Theorem 1.4-(2). We can take that $r_0 = 1$. The proof is still based upon Theorem 3.4 with $\Sigma = \overline{B}_{\frac{1}{2}}$, $D = \overline{B}_{\frac{1}{2}} \setminus \{0\}$ and $\Gamma = \{0\}$. Thus we assume that there exists a solution $u \in C(\overline{B}_1 \setminus \{0\})$, solution of (1.1) in $B_1 \setminus \{0\}$ and a sequence of points $\{y_n\} \subset \overline{B}_1 \setminus \{0\}$ such that

$$|y_n| M(y_n) \geq 2n \tag{3.48}$$

where we have set

$$M(x) = u^{\frac{1}{\gamma}}(x).$$

There exists a sequence $\{x_n\} \subset B_1 \setminus \{0\}$ such that

$$\begin{aligned} &|x_n| M(x_n) > 2n \\ &M(x_n) \geq M(y_n) \\ &M(z) \leq 2M_n(x_n) \quad \text{for all } z \in B_{\frac{n}{M(x_n)}}(x_n). \end{aligned} \tag{3.49}$$

Clearly $x_n \to 0$ as $n \to \infty$. We define u_n by (3.39) and (3.40) holds. The gradient estimate (3.41) is verified and if $z \in B_{\frac{n}{M(x_n)}}(x_n)$, we have $|z| \leq |x_n| + |z - x_n| \leq |x_n| + \frac{n}{M(x_n)}$ which tends to 0 as $n \to \infty$. If we replace $u(z)$ by $u_n(x) = \frac{u(z(x,n))}{M^\gamma(x_n)}$, (3.41) becomes

$$\max_{|x| \leq \frac{n}{2}} |\nabla u_n(x)| \leq c_{11} \left(n^{-\frac{1}{q-1}} \left(M(x_n)\right)^{\frac{1}{q-1} - \gamma - 1} + \max_{|x| \leq n} \left(u_n^{\frac{p}{q}}(x) + \left(M(x_n)\right)^{-\frac{\sigma}{2(q-1)(p-q)}} u_n^{\frac{p-1}{2(q-1)}}(x) \right) \right). \tag{3.50}$$

Notice that $M(x_n) \to \infty$ and $\frac{1}{q-1} - \gamma - 1 = \frac{-\sigma}{(q-1)(p-q)} < 0$. Using (3.40) we obtain

$$\max_{|x| \leq \frac{n}{2}} |\nabla u_n(x)| \leq c_{11} \left(o(1) + 2^{\frac{p}{p-q}} + o(1) \right) \leq c_{12}. \tag{3.51}$$

Hence (3.45) holds with a new constant c_{13}. Equation (3.46) is verified, but now $\sigma > 0$. Hence $\epsilon_n \to 0$ as $n \to \infty$. We conclude by the same argument as the one used in (1). □

Remark. In Theorem 1.4-(2) It is possible to obtain a constant C in estimate (1.20) independent u provided the functions under consideration are uniformly locally bounded from above in $\overline{B}_{r_0} \setminus \{0\}$ in the sense that for any $\epsilon > 0$ there exists $C_\epsilon > 0$ independent of u such that

$$u(x) \leq C_\epsilon \quad \text{for all } x \in B_{r_0} \setminus B_\epsilon. \tag{3.52}$$

This assumption implies that in the proof of Theorem 1.4-2), $M(x_n) \to \infty$ independently of u.

3.4 Asymptotic estimates on decaying solutions in the case $q > \frac{2p}{p+1}$

Using Theorem 3.4, we prove Theorem 1.5.

Proof of Theorem 1.5. We can assume that $r_0 = 1$. By (3.7), $\nabla u(x)$ tends to 0 as $|x| \to \infty$. Estimate (1.22) is equivalent to

$$M(x) := u^{\frac{p-1}{2}}(x) + |\nabla u(x)|^{\frac{p-1}{p+1}} \leq C|x|^{-1} \quad \text{for all } x \in B_2^c. \tag{3.53}$$

Using (1.21) jointly with (3.7) we have that $M(x) \to 0$ as $|x| \to \infty$. Let us assume that for any $C > 0$ inequality (3.53) does not hold; then there exists a sequence $\{y_n\} \subset B_2^c$ such that $\lim_{n\to\infty}(|y_n| - 2)M(y_n) = \infty$. There exists a sequence $\{x_n\} \subset \overline{B}_2^c$ such that 3.38 holds. Clearly $\{x_n\}$ is unbounded since M is bounded on bounded subset of B_2^c and, up to extracting a sequence, we can assume that $|x_n| \to \infty$ as $n \to \infty$. We set

$$u_n(x) = \frac{u(z(x,n)}{M^\alpha(x_n)} \quad \text{with } \; z(x,n) = x_n + \frac{x}{M(x_n)}. \tag{3.54}$$

Then we have $M(x_n)|x_n| > 2n$ and for any $x \in B_n$,

$$M(z(n,x)) = u^{\frac{p-1}{2}}(z(n,x) + |\nabla u|^{\frac{p-1}{p+1}}(z(n,x) \leq 2M(x_n). \tag{3.55}$$

Then

$$\nabla u_n(x) = \frac{\nabla u(z(x,n))}{M^{\alpha+1}(x_n)}\,, \; \Delta u_n(x) = \frac{\Delta u(z(x,n))}{M^{\alpha+2}(x_n)},$$

which implies

$$\begin{aligned}\Delta u_n(x) &= \frac{u^p(z(x,n)) - m|\nabla u|^q(z(x,n))}{M^{\alpha+2}(x_n)} \\ &= \frac{M^{\alpha+2}(x_n)u_n(x) - mM^{(\alpha+1)q}(x_n)|\nabla u(z(x,n))|^q}{M^{\alpha+2}(x_n)}.\end{aligned}$$

Hence u_n satisfies

$$-\Delta u_n = u_n^p - m(M(x_n))^{(\alpha+1)q-\alpha p}|\nabla u_n|^q \quad \text{in } B_n,$$

with the additional condition

$$u_n^{\frac{p-1}{2}}(0) + |\nabla u_n(0)|^{\frac{p-1}{p+1}} = 1.$$

Observe that

$$(\alpha+1)q - \alpha p = \frac{(p+1)q - 2p}{p-1} \geq 0,$$

with equality if $q = \frac{2p}{p+1}$ and strict inequality otherwise. Furthermore

$$u_n^{\frac{p-1}{2}}(x) + |\nabla u_n(x)|^{\frac{p-1}{p+1}} \leq 2 \quad \text{for all } x \in B_n.$$

By standard elliptic equations regularity results [23], the sequence $\{u_n\}$ is eventually locally compact in the $C^1_{loc}(\mathbb{R}^N)$-topology, thus, up to extracting a subsequence, $\{u_n\}$ converges in this topology to some nonnegative $C^1(\mathbb{R}^N)$ function v which satisfies

$$-\Delta v = v^p \quad \text{in } \mathbb{R}^N \tag{3.56}$$

if $q > \frac{2p}{p+1}$ since $M(x_n) \to 0$ as $n \to \infty$, and

$$-\Delta v + m|\nabla v|^q = v^p \quad \text{in } \mathbb{R}^N \tag{3.57}$$

if $q = \frac{2p}{p+1}$. Furthermore $v^{\frac{p-1}{2}}(0) + |\nabla v(0)|^{\frac{p-1}{p+1}} = 1$. Since $1 < p < \frac{N+2}{N-2}$, by Gidas and Spruck result [22] equation (3.56) admits no global positive solution. Concerning (3.57), if $m \leq \epsilon_0$ satisfies no global positive solution can exist by Theorem B. This ends the proof. □

Remark. In the case $q = \frac{2p}{p+1}$, the assumption (1.21) can be relaxed and replaced by

$$\limsup_{|x|\to\infty} u(x) < \infty. \tag{3.58}$$

Actually, if this holds we have by (3.6)

$$\limsup_{|x|\to\infty} |\nabla u(x)| < \infty. \tag{3.59}$$

The function u_n defined by (3.54) satisfies the same equation (1.1) as u and the limit v also. We end the proof as in Theorem 1.5.

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4 Removable singularities

In this Section we give partial extensions to (1.1) of previous results dealing with removability of singularities for equations

$$-\Delta u + m|\nabla u|^q = 0$$

and

$$-\Delta u + m|\nabla u|^2 - u^p \leq 0,$$

obtained respectively in [28] and [17].

4.1 Removable isolated singularities. Proof of Theorem 1.6

Proof of Theorem 1.6. We can assume that $\overline{B}_{r_0} \subset \Omega$ with $r_0 \geq 1$ and $a = 0$. Since (1.17) holds we have

$$|\nabla u(x)| \leq c|x|^{-\frac{1}{q-1}} \quad \text{and} \quad u(x) \leq c_1 + c_2 \begin{cases} |x|^{\frac{q-2}{q-1}} & \text{if } q > 2 \\ |\ln|x|| & \text{if } q = 2 \end{cases} \quad \text{for } 0 < |x| \leq r_0. \tag{4.1}$$

Since $q > p$ and $q \geq \frac{N}{N-1}$, we have that $\nabla u \in L^p(B_{r_0})$, which implies $u^p \in L^1(B_{r_0})$.
Step 1: We claim that $\nabla u \in L^q(B_{r_0})$ and the equation holds in $\mathcal{D}'(B_{r_0})$. Let $\eta_n \in C_0^\infty(B_{r_0} \setminus \{0\})$ such that $\eta_n = 1$ on $B_{r_0/2} \setminus B_{1/n}$, $\eta_n = 0$ if $|x| \leq 1/2n$ and if $|x| \geq 2r_0/3$ and $0 \leq \eta_n \leq 1$. We construct η_n such that $|\nabla \eta_n| \leq cn\mathbf{1}_{B_{1/n}\setminus B_{1/2n}}$. Then

$$\int_{B_{r_0}} \nabla u.\nabla \eta_n dx + m\int_{B_{r_0}} |\nabla u|^q \eta_n dx = \int_{B_{r_0}} u^p \eta_n dx.$$

By Holder's inequality and using (1.17) there holds with $q' = \frac{q}{q-1}$,

$$\left| \int_{B_{r_0}} \nabla u.\nabla \eta_n dx \right| = \left| \int_{B_{1/n}\setminus B_{1/2n}} \nabla u.\nabla \eta_n dx \right| \leq c_2 n^{q'-N}.$$

Since $q \geq \frac{N}{N-1}$, then $q' - N \leq 0$, and the right-hand side is bounded, hence $|\nabla u|^q \in L^1(B_{\frac{r_0}{2}})$ by Fatou's theorem and the first statement follows.
Next consider $\zeta \in C_0^\infty(B_{r_0/2})$ and take $\zeta\eta_n$ as a test function, then

$$\int_{B_{r_0}} (\zeta \nabla u.\nabla \eta_n + \eta_n \nabla u.\nabla \zeta)\, dx + m \int_{B_{r_0}} |\nabla u|^q \zeta \eta_n dx = \int_{B_{r_0}} u^p \zeta \eta_n dx.$$

Since

$$\left| \int_{B_{r_0}} \zeta \nabla u.\nabla \eta_n dx \right| \leq c_3 n^{1-\frac{N}{q'}} \|\zeta\|_{L^\infty} \left(\int_{B_{1/n}\setminus B_{1/2n}} |\nabla u|^q \right)^{\frac{1}{q}}, \tag{4.2}$$

and the left-hand side tends to 0 as $n \to \infty$, we conclude by the dominated convergence theorem that

$$\int_{B_{r_0}} \nabla u.\nabla \zeta dx + m \int_{B_{r_0}} |\nabla u|^q \zeta dx = \int_{B_R} u^p \zeta dx,$$

which proves the second statement.

Step 2: u is bounded. For proving the boundedness assertion we can assume that $\frac{N}{N-1} \leq q < 2$. As a test function we take $\zeta = \eta_n^q$, then

$$q \int_{B_{r_0}} \eta_n^{q-1} \nabla u.\nabla \eta_n dx + m \int_{B_{r_0}} \eta_n^q |\nabla u|^q dx = \int_{B_{r_0}} \eta_n^q u^p dx.$$

We have

$$\begin{aligned} \int_{B_{r_0}} \eta_n^q |\nabla u|^q dx = \int_{B_{r_0}} |\eta_n \nabla u|^q dx &= \int_{B_{r_0}} |\nabla(\eta_n u) - u\nabla \eta_n|^q dx \\ &\geq 2^{1-q} \int_{B_{r_0}} |\nabla(\eta_n u)|^q dx - \int_{B_{r_0}} u^q |\nabla \eta_n|^q dx. \end{aligned}$$

By (4.1)

$$\int_{B_{r_0}} u^q |\nabla \eta_n|^q dx \leq c_4 n^{q'-N} \leq c'$$

as we have already seen it and, from (4.2) there holds

$$\left| \int_{B_{r_0}} \eta_n^{q-1} \nabla u.\nabla \eta_n dx \right| \to 0 \text{ as } n \to \infty.$$

It follows that $\nabla(\eta_n u)$ is bounded in $L^q(B_{r_0})$ independently of n, and by Sobolev inequality,

$$\|\eta_n u\|_{L^{q^*}(B_{r_0})} \leq c'' \quad \text{with } q^* = \frac{Nq}{N-q},$$

which in turn implies that $\|u\|_{L^{q^*}(B_{r_0})} \leq c_1$. Set

$$r_1 = \frac{Nq}{N-q} - p. \tag{4.3}$$

Taking $\eta_n^{q+r_1}(T_k(u))^{r_1}$ as a test function, where $T_k(r) = \min\{r, k\}$ for $r, k > 0$, we obtain

$$r_1 \int_{B_{r_0}\cap\{u<k\}} (T_k(u))^{r_1-1}\eta_n^{q+r_1}|\nabla u|^2 dx + (q+r_1)\int_{B_{r_0}} (T_k(u))^{r_1}\eta_n^{q+r_1-1}\nabla\eta_n.\nabla u dx$$
$$+ m\int_{B_{r_0}} T_k(u^{r_1})|\nabla u|^q\eta_n^{q+r_1}dx = \int_{B_{r_0}} T_k(u^{r_1})u^p\eta_n^{q+r_1}dx.$$

From *Step 1* $|\nabla u| \in L^q(B_{r_0})$, thus

$$\int_{B_{r_0}} (T_k(u))^{r_1}\eta_n^{q+r_1-1}\nabla\eta_n.\nabla u dx \to 0 \quad \text{as } n \to \infty,$$

hence

$$o(1) + m\int_{B_{r_0}} T_k(u^{r_1})|\nabla u|^q\eta_n^{q+r_1}dx \leq \int_{B_{r_0}} T_k(u^{r_1})u^p\eta_n^{q+r_1}dx.$$

Letting successively $n \to \infty$ and $k \to \infty$, we deduce by Fatou's lemma and the monotone convergence theorem that

$$m\int_{B_{r_0}} u^{r_1}|\nabla u|^q\tilde{\eta}^{q+r_1}dx \leq \int_{B_{r_0}} u^{\frac{Nq}{N-q}}\tilde{\eta}^{q+r_1}dx, \tag{4.4}$$

where $\tilde{\eta}^{q+r_1} = \lim_{n\to\infty}\eta_n^{q+r_1}$ belongs to $C_0^\infty(B_{r_0})$ and takes value 1 in $B_{\frac{r_0}{2}}$ and $0 \leq \tilde{\eta} \leq 1$. Since

$$\int_{B_{r_0}} u^{r_1}|\nabla u|^q\tilde{\eta}^{q+r_1}dx = \left(\frac{q}{q+r_1}\right)^q \int_{B_{r_0}} |\tilde{\eta}^{1+\frac{r_1}{q}}\nabla(u^{1+\frac{r_1}{q}})|^q dx$$
$$\geq \left(\frac{q}{r_1+q}\right)^q 2^{1-q}\int_{B_{r_0}} |\nabla(\tilde{\eta}u)^{1+\frac{r_1}{q}}|^q dx - \left(\frac{q}{r_1+q}\right)^q \int_{B_{r_0}} u^{q+r_1}|\nabla\tilde{\eta}|^q dx$$
$$\geq c_{N,q}\left(\frac{q}{r_1+q}\right)^q \left(\int_{B_{r_0}} (\tilde{\eta}u)^{\frac{N(q+r_1)}{N-q}}dx\right)^{\frac{N-q}{N}} - K_1\left(\frac{q}{r_1+q}\right)^q,$$

where

$$K_1 = {r_0}^N \|u\|^{q+r_1}_{L^\infty(B_{r_0}\setminus B_{\frac{r_0}{2}})} \|\nabla\tilde{\eta}\|^q_{L^\infty(B_{r_0})}.$$

This leads to the following inequality

$$mc_{N,q}\left(\frac{q}{r_1+q}\right)^q \|\tilde{\eta}u\|^{q+r_1}_{L^{\frac{N(q+r_1)}{N-q}}(B_{r_0})} - mK_1\left(\frac{q}{r_1+q}\right)^q \leq \left\|\tilde{\eta}^{\frac{(N-q)(q+r_1)}{Nq}}u\right\|^{\frac{Nq}{N-q}}_{L^{\frac{Nq}{N-q}}(B_{r_0})}$$
$$\leq \|\tilde{\eta}u\|^{\frac{Nq}{N-q}}_{L^{\frac{Nq}{N-q}}(B_{r_0})}, \tag{4.5}$$

since $\frac{(N-q)(q+r_1)}{Nq} > 1$ from (4.3) and $q > p$ combined with the fact that $\tilde{\eta} \le 1$.

Next we proceed by induction, setting

$$r_{j+1} = \frac{N(q+r_j)}{N-q} - p \quad \text{for } j \ge 1, \tag{4.6}$$

with explicit value

$$r_{j+1} = \left(\left(\frac{N}{N-q} \right)^{j+1} - 1 \right) \frac{(N-q)r_1}{q}. \tag{4.7}$$

Taking $\eta_n^{q+r_{j+1}} T_k(u^{r_{j+1}})$ for test function and letting successively $n \to \infty$ and $k \to \infty$ we obtain

$$m \int_{B_{r_0}} u^{r_{j+1}} |\nabla u|^q \tilde{\eta}^{q+r_{j+1}} dx \le \int_{B_{r_0}} u^{\frac{N(q+r_j)}{N-q}} \tilde{\eta}^{q+r_{j+1}} dx \le \int_{B_{r_0}} (\tilde{\eta} u)^{\frac{N(q+r_j)}{N-q}} dx. \tag{4.8}$$

Note that for the right-hand side we have used $q + r_{j+1} \ge \frac{N(q+r_j)}{N-q}$ and $\tilde{\eta} \le 1$. Moreover

$$\int_{B_{r_0}} u^{r_{j+1}} |\nabla u|^q \tilde{\eta}^{q+r_{j+1}} dx \ge \left(\frac{q}{r_{j+1}+q} \right)^q \int_{B_{r_0}} |\tilde{\eta}^{1+\frac{r_{j+1}}{q}} \nabla (u^{1+\frac{r_{j+1}}{q}})|^q dx. \tag{4.9}$$

Writing

$$\tilde{\eta}^{1+\frac{r_{j+1}}{q}} \nabla (u^{1+\frac{r_{j+1}}{q}}) = \nabla (\tilde{\eta} u)^{1+\frac{r_{j+1}}{q}} - \frac{q+r_{j+1}}{q} u^{1+\frac{r_{j+1}}{q}} \tilde{\eta}^{\frac{r_{j+1}}{q}} \nabla \tilde{\eta},$$

we have, since $\tilde{\eta} = 1$ in $B_{\frac{r_0}{2}}$ and $0 \le \tilde{\eta} \le 1$, and using Sobolev inequality,

$$\begin{aligned} \left\| \tilde{\eta}^{1+\frac{r_{j+1}}{q}} \nabla (u^{1+\frac{r_{j+1}}{q}}) \right\|_{L^q(B_{r_0})} &\ge \left\| \nabla (\tilde{\eta} u)^{1+\frac{r_{j+1}}{q}} \right\|_{L^q(B_{r_0})} \\ &\quad - \frac{q+r_{j+1}}{q} \|\nabla \tilde{\eta}\|_{L^\infty} \left\| u^{1+\frac{r_{j+1}}{q}} \right\|_{L^q(B_{r_0} \setminus B_{\frac{r_0}{2}})} \\ &\ge c_{N,q} \|\tilde{\eta} u\|^{\frac{q+r_{j+1}}{q}}_{L^{\frac{N(q+r_{j+1})}{N-q}}(B_{r_0})} - \frac{q+r_{j+1}}{q} \|\nabla \tilde{\eta}\|_{L^\infty} \|u\|^{\frac{q+r_{j+1}}{q}}_{L^{q+r_{j+1}}(B_{r_0} \setminus B_{\frac{r_0}{2}})}. \end{aligned} \tag{4.10}$$

Let us assume now that $u \notin L^\infty(B_{r_0})$, otherwise the result follows, then

$$\lim_{j \to \infty} \|\tilde{\eta} u\|_{L^{\frac{N(q+r_{j+1})}{N-q}}(B_{r_0})} = \infty, \tag{4.11}$$

and there exists $j_0 \ge 1$ such that for any $j \ge j_0$,

$$\|\tilde{\eta} u\|_{L^{\frac{N(q+r_{j+1})}{N-q}}(B_{r_0})} \ge 2 \|\nabla \tilde{\eta}\|^{\frac{q}{q+r_{j+1}}}_{L^\infty} \|u\|_{L^{q+r_{j+1}}(B_{r_0} \setminus B_{\frac{r_0}{2}})}; \tag{4.12}$$

as a consequence the right-hand side of (4.10) is bounded from below by

$$\left(c_q - 2^{-\frac{q+r_{j+1}}{q}} \frac{q+r_{j+1}}{q} \right) \|\tilde{\eta} u\|^{\frac{q+r_{j+1}}{q}}_{L^{\frac{N(q+r_{j+1})}{N-q}}(B_{r_0})} \ge \frac{c_{N,q}}{2} \|\tilde{\eta} u\|^{\frac{q+r_{j+1}}{q}}_{L^{\frac{N(q+r_{j+1})}{N-q}}(B_{r_0})} \tag{4.13}$$

for $j \geq j_1 \geq j_0$. Combining (4.8), (4.9) and (4.13) we derive

$$\frac{1}{m}\int_{B_{r_0}} (\tilde{\eta}u)^{\frac{N(q+r_j)}{N-q}} dx \geq \left(\frac{qc_{N,q}}{2(r_{j+1}+q)}\right)^q \|\tilde{\eta}u\|^{q+r_{j+1}}_{L^{\frac{N(q+r_{j+1})}{N-q}}(B_{r_0})}. \tag{4.14}$$

We obtain finally

$$\|\tilde{\eta}u\|_{L^{\frac{N(q+r_{j+1})}{N-q}}(B_{r_0})} \leq \left(\frac{2(r_{j+1}+q)}{qc_{N,q}m^{\frac{1}{q}}}\right)^{\frac{q}{q+r_{j+1}}} \|\tilde{\eta}u\|^{\frac{N(q+r_j)}{(N-q)(q+r_{j+1})}}_{L^{\frac{N(q+r_j)}{N-q}}(B_{r_0})}. \tag{4.15}$$

Put

$$X_j = \ln\left(\|\tilde{\eta}u\|_{L^{\frac{N(q+r_j)}{N-q}}(B_{r_0})}\right).$$

Since

$$\frac{N(q+r_j)}{(N-q)(q+r_{j+1})} = \frac{p+r_{j+1}}{q+r_{j+1}} < 1, \tag{4.16}$$

we deduce

$$X_{j+1} \leq \frac{q}{q+r_{j+1}} \ln\left(\frac{2(r_{j+1}+q)}{qc_q m^{\frac{1}{q}}}\right) + X_j, \tag{4.17}$$

which implies that

$$\ln\left(\|u\|_{L^\infty(B_{\frac{r_0}{2}})}\right) \leq \limsup_{j\to\infty} X_{j+1} \leq X_1 + q\sum_{j=1}^{\infty} \frac{1}{q+r_{j+1}} \ln\left(\frac{2(r_{j+1}+q)}{qc_q m^{\frac{1}{q}}}\right) < \infty, \tag{4.18}$$

by (4.7). This is a contradiction with (4.11), which ends the proof. □

4.2 Removable singular sets

In the following theorem we combine the technique of Theorem 1.6 with the geometric approach based upon the construction of tubular neighbourhoods used in [32] to prove the removability of singular sets contained into a smooth submanifold. The next result proves and completes Theorem 1.7.

Theorem 4.1 *Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain with $N \geq 3$ and $\Sigma \subset \Omega$ be a k-dimensional compact complete smooth submanifold with $1 \leq k \leq N-2$. If $1 \leq p < q$ and $q \geq \frac{N-k}{N-1-k}$, any nonnegative solution $u \in C^2(\Omega \setminus \Sigma)$ of (1.1) in $\Omega \setminus \Sigma$ can be extended as a weak solution of the same equation in Ω which belongs to $L^\infty_{loc}(\Omega) \cap W^{1,q}_{loc}(\Omega) \cap H^1_{loc}(\Omega)$.*

Proof. Step 1: We claim that there exists $r_0 > 0$ and $C = C(N,p,q,m,r_0,\Sigma) > 0$ such that

$$|\nabla u(x)| \leq C(\mathrm{dist}\,(x,\Sigma))^{-\frac{1}{q-1}} \quad \text{for all } x \text{ s.t. } \mathrm{dist}\,(x,\Sigma) \leq r_0. \tag{4.19}$$

For $\delta > 0$ we set

$$TUB_\delta(\Sigma) = \{x \in \mathbb{R}^N : \mathrm{dist}\,(x,\Sigma) < \delta\}.$$

If $\delta \leq \inf\{dist(x,\Sigma) : x \in \Omega^c\}$, we have that $TUB_\delta(\Sigma) \subset \Omega$. Since Σ is smooth with no boundary, there exixts $\delta_0 > 0$ such that the sets $\partial TUB_\delta(\Sigma) = \{x \in \Omega : \text{dist}\,(x,\Sigma) = \delta\}$ are k-dimensional compact complete smooth submanifolds of Ω. We use the ideas of the proof of Theorem 1.3 adapting it to the peculiar geometric configuration. By rescaling we can assume that $\delta_0 = 1$ and for $0 < \theta < \frac{1}{4}$, we set $\Theta_\theta = TUB_{1-\theta}(\Sigma) \setminus TUB_\theta(\Sigma)$. For any $0 < \epsilon < \frac{1}{2}$ we have by (3.23),

$$\max_{\overline{\Theta}_\theta} |\nabla u| \leq c_1 \left((\epsilon\theta)^{-\frac{1}{q-1}} + \max_{\overline{\Theta}_{\frac{\theta}{1+\epsilon}}} \left(u^p + u^{\frac{p-1}{2(q-1)}} \right)^{\frac{1}{q}} \right) \leq c_2 \left((\epsilon\theta)^{-\frac{1}{q-1}} + 1 + \max_{\overline{\Theta}_{\frac{\theta}{1+\epsilon}}} u^{\frac{p}{q}} \right). \quad (4.20)$$

In order to obtain an upper bound on $u(x)$ for $x \in \overline{\Theta}_{\frac{\theta}{1+\epsilon}}$, we join it to some $x_\epsilon \in \partial TUB_1(\Sigma)$ by a smooth curve ω such that $\omega(0) = x$, $\omega(1) = x_\epsilon$. We can choose ω such that $|\omega'(t)| \leq 2$ for all $t \in [0,1]$ and

$$2^{-1}\text{dist}\,(tx + (1-t)x_\epsilon, \Sigma) \leq \text{dist}\,(\omega(t), \Sigma) \leq 2\text{dist}\,(tx + (1-t)x_\epsilon, \Sigma).$$

Then

$$\begin{aligned} u(x) &\leq u(x_\epsilon) + \left| \int_0^1 \nabla u(\omega(t)).\omega'(t)dt \right| \leq u(x_\epsilon) + 2\int_0^1 |\nabla u(\omega(t))|dt \\ &\leq \|u\|_{L^\infty(TUB_1(\Sigma)\setminus TUB_{\frac{1}{2}}(\Sigma))} + 2\max_{\Omega_{\frac{\theta}{1+\epsilon}}} |\nabla u|. \end{aligned} \quad (4.21)$$

Therefore

$$\begin{aligned} \max_{\overline{\Theta}_{\frac{\theta}{1+\epsilon}}} u^{\frac{p}{q}} &\leq c_3 \left(\|u\|^{\frac{p}{q}}_{L^\infty(TUB_1(\Sigma)\setminus TUB_{\frac{1}{2}}(\Sigma))} + \max_{\overline{\Theta}_{\frac{\theta}{1+\epsilon}}} |\nabla u|^{\frac{p}{q}} \right) \\ &\leq c_3 \left(\|u\|^{\frac{p}{q}}_{L^\infty(TUB_1(\Sigma)\setminus TUB_{\frac{1}{2}}(\Sigma))} + \max_{\overline{\Theta}_{(1-\epsilon)\theta}} |\nabla u|^{\frac{p}{q}} \right). \end{aligned} \quad (4.22)$$

We put

$$B(\theta) = \max_{\overline{\Theta}_\theta} \theta^{\frac{1}{q-1}} |\nabla u(z)| \text{ and } F(\theta) = 1 + B(\theta),$$

and we obtain from (4.20) and (4.22)

$$F(\theta) \leq c_4 \epsilon^{-\frac{1}{q-1}} F^{\frac{p}{q}}((1-\epsilon)\theta), \quad (4.23)$$

where c_4 depends on the structural constants and of $\|u\|_{L^\infty(TUB_1(\Sigma)\setminus TUB_{\frac{1}{2}}(\Sigma))}$. It follows from Lemma 2.1 that $B(\theta)$ is bounded independently of θ, which implies (4.19).

In order to derive the upper estimate on u we set $\mu = \sup\{u(y) : y \in \partial TUB_1(\Sigma)\}$. If $0 < \text{dist}\,(x,\Sigma) = t \leq 1$ there exists $z_x \in \Sigma$ and $\xi \in \partial TUB_1(\Sigma)$ such that

$$2^{-1}|tx + (1-t)\xi - z_x| \leq \text{dist}\,(tx + (1-t)\xi, \Sigma) \leq 2|tx + (1-t)\xi - z_x|.$$

Since dist $(\xi, \Sigma) = 1$,

$$\begin{aligned} u(x) &\le \mu + c_5 \int_0^1 |tx + (1-t)\xi - z_x|^{-\frac{1}{q-1}} dt \\ &\le \mu + c_5 \int_0^1 (t\text{dist}\,(x,\Sigma) + (1-t)\text{dist}\,(\xi,\Sigma))^{-\frac{1}{q-1}} = \mu + c_5 \int_0^1 (t\text{dist}\,(x,\Sigma) + 1 - t)^{-\frac{1}{q-1}} \\ &\le \mu + c_5 \frac{q-1}{2-q} \left(1 - \text{dist}\,(x,\Sigma)\right) \left((\text{dist}\,(x,\Sigma))^{\frac{2-q}{q-1}} - 1 \right), \end{aligned}$$

if $q \neq 2$, with an obvious modification if $q = 2$. At end we deduce

$$u(x) \le c_6 \begin{cases} (\text{dist}\,(x,\Sigma))^{\frac{2-q}{q-1}} + C' & \text{for all } x \in TUB_1(\Sigma) \quad \text{if } q \neq 2 \\ |\ln(\text{dist}\,(x,\Sigma))| + C' & \text{for all } x \in TUB_1(\Sigma) \quad \text{if } q = 2. \end{cases} \tag{4.24}$$

Step 2: We claim that $u \in L^p(TUB_1(\Sigma))$ *and* $|\nabla u| \in L^q(TUB_1(\Sigma))$. For such a task we consider test functions $\eta_n \in C_0^\infty(TUB_1(\Sigma))$ with value in $[0,1]$ vanishing in $TUB_{1/(2n)}(\Sigma) \cup TUB^c_{2/3}(\Sigma)$, with value 1 in $TUB_{1/2}(\Sigma) \setminus TUB_{1/n}(\Sigma)$ and such that

$$|\nabla \eta_n(x)| \le c_7 n \mathbf{1}_{TUB_{1/n}(\Sigma) \setminus TUB_{1/2n}(\Sigma)},$$

where the constant $c_7 > 0$ depends on the geometry of Σ. If $q > 2$, u is bounded thus $u^p \in L^1(TUB_1(\Sigma))$. If $\frac{N-k}{N-k-1} \le q \le 2$ we have for $1 > \epsilon > \frac{1}{n}$

$$\begin{aligned} \int_{TUB_\epsilon(\Sigma)} \eta_n u^p dx &\le \int_{TUB_\epsilon(\Sigma) \setminus TUB_{1/2n}(\Sigma)} u^p dx \\ &\le c_8 \int_{1/2n}^{\epsilon} \tau^{-\frac{(2-q)p}{q-1}} \frac{d}{d\tau} Vol(TUB_\tau(\Sigma)) d\tau \\ &\le c_8 \epsilon^{-\frac{(2-q)p}{q-1}} Vol(TUB_\epsilon(\Sigma)) + c_8 \frac{(2-q)p}{q-1} \int_{1/2n}^{\epsilon} \tau^{-\frac{(2-q)p}{q-1}-1} Vol(TUB_\tau(\Sigma)) d\tau. \end{aligned}$$

By Weyl's formula [36]

$$Vol(TUB_\tau(\Sigma)) = \sum_{i=0}^{[k/2]} a_i \tau^{N-k+2i} \tag{4.25}$$

where the a_i are smooth bounded functions near Σ and $[k/2]$ is the integer part of $k/2$. Therefore

$$\int_{1/(2n)}^{\epsilon} \tau^{-\frac{(2-q)p}{q-1}} \frac{d}{d\tau} Vol(TUB_\tau(\Sigma)) d\tau \le C(\epsilon) + c_9 n^{\frac{(2-q)p}{q-1} - N + k}.$$

Since $\frac{(2-q)p}{q-1} < \frac{q}{q-1} \le N - k$, we have that $\frac{(2-q)p}{q-1} - N + k < 0$. Letting $n \to \infty$ we obtain that $u^p \in L^1(TUB_1(\Sigma))$.

For the second assertion we have with the same test function η_n,

$$\int_{TUB_1(\Sigma)} \nabla u . \nabla \eta_n dx + m \int_{TUB_1(\Sigma)} |\nabla u|^q \eta_n dx = \int_{TUB_1(\Sigma)} u^p \eta_n dx.$$

Using (4.19) and (4.25),

$$\left|\int_{TUB_1(\Sigma)} \nabla u.\nabla\eta_n dx\right| \leq Cn^{\frac{q}{q-1}} Vol(TUB_\tau(1/n)) = C'n^{\frac{q}{q-1}+k-N}.$$

By assumption $\frac{q}{q-1} \leq N-k$. Since $u \in L^p(TUB_1(\Sigma))$ we conclude that $|\nabla u| \in L^q(TUB_1(\Sigma))$ by Fatou's lemma.

Step 3: We claim that $u \in L^\infty(TUB_1(\Sigma))$. The proof that u is a weak solution of (1.1) is similar to the one in Theorem 1.6. For obtaining that $u \in L^\infty(TUB_1(\Sigma))$ we use the same test functions η_n as in Step 2, the same sequence $\{r_j\}$ defined by (4.6) and derive (4.13) where B_R is replaced by $TUB_1(\Sigma)$ under the assumption (4.11). And similarly (4.18), again replacing B_R by $TUB_1(\Sigma)$ holds in the same way, we obtain a contradiction.

□

The next theorem extends a previous result of Brezis and Nirenberg [17] that they proved in the case $q = 2$. The technique is completely different from the one used in Theorem 4.1 and based upon capacity theory.

Theorem 4.2 *Let $\Omega \subset \mathbb{R}^N$ $N \geq 2$, be a bounded smooth domain. Assume p and q are real numbers such that $0 < p \leq \max\{2,p\} \leq q$ and $m > 0$. Let $K \subset \Omega$ be a compact set and $u \in C^1(\overline{\Omega} \setminus K)$ be a positive function satisfying*

$$-\Delta u + m|\nabla u|^q - u^p \leq 0 \tag{4.26}$$

in $\Omega \setminus K$ and such that $u \geq \delta > 0$. If $cap_{1,q'}(K) = 0$, then $u \in L^\infty(\Omega)$.

Proof. If $cap_{1,q'}(K) = 0$, then $|K| = 0$ and there exists a sequence $\{\zeta_k\} \subset C_c^\infty(\Omega)$ such that $0 \leq \zeta_k \leq 1$, $\zeta_k = 1$ in a neighborhoood of K such that

$$\lim_{k\to\infty} \||\nabla\zeta_k|\|_{L^{q'}(\Omega)} = 0. \tag{4.27}$$

Furthermore $\zeta_k \to 0$ a.e. in Ω, and we set $\eta_k = 1-\zeta_k$. For $\theta > 0$ let j_θ be a $C^\infty(\mathbb{R})$ nondecreasing function with value 0 on $(-\infty, 0]$ and 1 on $[\theta, \infty)$. We set

$$\lambda(t) = meas\{x \in \Omega : u(x) \geq t\}$$

for $t \geq t_0$ where $t_0 = \sup_{\partial\Omega} u \geq \delta$. Taking $\eta_k^{q'} j_\theta(u-t)u^{-p}$ as a test function, we have

$$\begin{aligned} q' \int_\Omega \eta_k^{q'-1} j_\theta(u-t)u^{-p}\nabla u.\nabla\eta_k dx + \int_\Omega j'_\theta(u-t)u^{-p}|\nabla u|^2\eta_k^{q'}dx \\ - p\int_\Omega \eta_k^{q'} j_\theta(u-t)u^{-p-1}|\nabla u|^2 dx + m\int_\Omega \eta_k^{q'} j_\theta(u-t)u^{-p}|\nabla u|^q dx \leq \int_\Omega \eta_k^{q'} j_\theta(u-t)dx. \end{aligned}$$

Since $j'_\theta \geq 0$, it follows

$$\begin{aligned} q' \int_\Omega \eta_k^{q'-1} j_\theta(u-t)u^{-p}\nabla u.\nabla\eta_k dx - p\int_\Omega \eta_k^{q'} j_\theta(u-t)u^{-p-1}|\nabla u|^2 dx \\ + m\int_\Omega \eta_k^{q'} j_\theta(u-t)u^{-p}|\nabla u|^q dx \leq \int_\Omega \eta_k^{q'} j_\theta(u-t)dx \leq \lambda(t). \end{aligned} \tag{4.28}$$

Step 1: the basic inequality. We set

$$S(t) = \begin{cases} \frac{q}{q-p} t^{\frac{q-p}{q}} & \text{if } p < q \\ \ln t & \text{if } p = q. \end{cases} \tag{4.29}$$

Then $u^{-p}|\nabla u|^q = |\nabla S(u)|^q$ and

$$\begin{aligned} m \int_\Omega \eta_k^{q'} j_\theta(u-t) |\nabla(S(u))|^q dx \leq \lambda(t) + q' \int_\Omega \eta_k^{q'-1} j_\theta(u-t) u^{-p} |\nabla u| |\nabla \eta_k| dx \\ + p \int_\Omega \eta_k^{q'} j_\theta(u-t) u^{-p-1} |\nabla u|^2 dx. \end{aligned} \tag{4.30}$$

We take $t \geq t_1 \geq t_0$ for some t_1 to be fixed, then

$$\begin{aligned} q' \int_\Omega \eta_k^{q'-1} j_\theta(u-t) u^{-p} |\nabla u| |\nabla \eta_k| dx &= q' \int_\Omega \eta_k^{q'-1} j_\theta(u-t) u^{-\frac{p(q-1)}{q}} u^{-\frac{p}{q}} |\nabla u| |\nabla \eta_k| dx \\ &\leq q' t_1^{-\frac{p(q-1)}{q}} \int_\Omega \eta_k^{q'} j_\theta(u-t) |\nabla S(u)| \frac{|\nabla \eta_k|}{\eta_k} dx \\ &\leq q' t_1^{-\frac{p(q-1)}{q}} \left(\frac{\epsilon^q}{q} \int_\Omega \eta_k^{q'} j_\theta(u-t) |\nabla S(u)|^q dx + \frac{1}{q' \epsilon^{q'}} \int_\Omega j_\theta(u-t) |\nabla \eta_k|^{q'} dx \right). \end{aligned} \tag{4.31}$$

We recall that $\sigma = (p+1)q - 2p$. Since $q \geq 2$ we have that $\sigma \geq 2$, with strict inequality if $q > 2$. Therefore

$$\begin{aligned} p \int_\Omega \eta_k^{q'} j_\theta(u-t) u^{-p-1} |\nabla u|^2 dx &= p \int_\Omega \eta_k^{q'} j_\theta(u-t) u^{-\frac{\sigma}{q}} u^{-\frac{2p}{q}} |\nabla u|^2 dx \\ &\leq p t_1^{-\frac{\sigma}{q}} \int_\Omega \eta_k^{q'} j_\theta(u-t) |\nabla S(u)|^2 dx. \end{aligned} \tag{4.32}$$

We first consider the case $q > 2$. We have by Hölder's inequality,

$$\begin{aligned} p \int_\Omega j_\theta(u-t) u^{-p-1} |\nabla u|^2 \eta_k^{q'} dx \leq p t_1^{-\frac{\sigma}{q}} \left(\frac{2\epsilon^q}{q} \int_\Omega j_\theta(u-t) |\nabla S(u)|^q \eta_k^{q'} dx \right. \\ \left. + \frac{q}{(q-2)\epsilon^{\frac{q}{q-2}}} \int_\Omega j_\theta(u-t) \eta_k^{q'} dx \right). \end{aligned} \tag{4.33}$$

We then deduce that

$$\begin{aligned} &\left(m - \epsilon^q \left(\frac{2p}{q} t_1^{-\frac{\sigma}{q}} + \frac{1}{q-1} t_1^{-\frac{p(q-1)}{q}} \right) \right) \int_\Omega \eta_k^{q'} j_\theta(u-t) |\nabla(S(u))|^q dx \\ &\qquad \leq \left(1 + \frac{pq}{(q-2)\epsilon^{\frac{q}{q-2}}} \right) \lambda(t) + \frac{t_1^{-\frac{p(q-1)}{q}}}{\epsilon^{q'}} \int_\Omega j_\theta(u-t) |\nabla \eta_k|^{q'} dx \\ &\qquad \leq \left(1 + \frac{pq}{(q-2)\epsilon^{\frac{q}{q-2}}} \right) \lambda(t) + \frac{t_1^{-\frac{p(q-1)}{q}}}{\epsilon^{q'}} \int_\Omega |\nabla \eta_k|^{q'} dx. \end{aligned} \tag{4.34}$$

Since $cap_{1,q'}(K) = 0$ and $\eta_k \to 1$, we let $k \to \infty$ and obtain

$$\left(m - \epsilon^q \left(\frac{2p}{q} t_1^{-\frac{\sigma}{q}} + \frac{1}{q-1} t_1^{-\frac{p(q-1)}{q}}\right)\right) \int_\Omega j_\theta(u-t)|\nabla(S(u))|^q dx \le \left(1 + \frac{pq}{(q-2)\epsilon^{\frac{q}{q-2}}}\right) \lambda(t), \tag{4.35}$$

having fixed $t_1 \ge t_0$ and $\epsilon > 0$ small enough such that

$$m - \epsilon^q \left(\frac{2p}{q} t_1^{-\frac{\sigma}{q}} + \frac{1}{q-1} t_1^{-\frac{p(q-1)}{q}}\right) \ge \frac{m}{2}.$$

We set

$$\nu(s) = meas\{x \in \Omega : \, S(u(x)) \ge s\}.$$

By letting $\theta \to 0$ we infer that there exists a constant $C_1 > 0$ such that, for $s \ge s_1 = S(t_1)$,

$$\int_\Omega |\nabla(S(u) - s)_+|^q dx \le C_1 \nu(s). \tag{4.36}$$

Before continuing on this inequality, we can look at the case $q = 2$ (which is actually the case considered by Brezis and Nirenberg [17]). Then $\sigma = 2$ and (4.34) is replaced by

$$\left(m - \left(2pt_1^{-1} - \epsilon^2 t_1^{-\frac{p}{2}}\right)\right) \int_\Omega \eta_k^2 j_\theta(u-t)|\nabla(S(u))|^2 dx \le \lambda(t) + \frac{t_1^{-\frac{p}{2}}}{\epsilon^2} \int_\Omega |\nabla \eta_k|^2 dx. \tag{4.37}$$

By choosing ϵ and t_1 we obtain (4.36) with $q = 2$ and a specific constant C_1.

Step 2: end of the proof. We set $w = S(u)$ and by Hölder's inequality since $q > 2$,

$$\begin{aligned} \int_\Omega |\nabla(w-s)_+|^{q'} dx &\le \left(\int_\Omega |\nabla(w-s)_+|^q dx\right)^{\frac{q'}{q}} \left(meas\left\{|\nabla(w-s)_+ > 0|\right\}\right)^{1-\frac{q'}{q}} \\ &\le c_1^{\frac{q'}{q}} \left(\nu(s)\right)^{\frac{q'}{q}} \left(meas\left\{|\nabla(w-s)_+ > 0|\right\}\right)^{1-\frac{q'}{q}} \\ &\le c_1^{\frac{q'}{q}} \nu(s), \end{aligned} \tag{4.38}$$

since $\nabla(w-s)_+ = 0$ a.e. on the set where $(w-s)_+ = 0$. This implies that, up to a set of zero measure, we have $\{|\nabla(w-s)_+ > 0|\} \subset \{(w-s)_+ > 0\}$, thus $meas\left\{|\nabla(w-s)_+ > 0|\right\} \le \nu(s)$. Note that this also holds if $q = 2$. By Sobolev inequality,

$$\left(\int_\Omega (w-s)_+^{q'^*} dx\right)^{\frac{q'}{q'^*}} \le c(N,q) \int_\Omega |\nabla(w-s)_+|^{q'} dx \quad \text{with} \;\; q'^* = \frac{Nq'}{N-q'}, \tag{4.39}$$

if $q' < N$ which is always satisfied except in the case $q = 2 = N$ in which case the modifications are straightforward and left to the reader. Furthermore

$$\int_\Omega (w-s)_+ dx \le \left(\int_\Omega (w-s)_+^{q'^*} dx\right)^{\frac{1}{q'^*}} \left(\nu(s)\right)^{1-\frac{1}{q'^*}}.$$

This yields

$$\int_\Omega (w-s)_+ dx \le c_2 \nu(s))^{1+\frac{1}{N}} \quad \text{for any } s \ge s_1, \tag{4.40}$$

since $1+\frac{1}{q'}-\frac{1}{q'^*}=1+\frac{1}{N}$. Set

$$\phi(s)=\int_\Omega (w-s)_+ dx = \int_s^\infty \nu(\tau)d\tau, \ \text{ hence } \ -\phi'(s)=\nu(s),$$

and (4.40) leads to $\phi(s) \le c_2(-\phi'(s))^{\frac{N+1}{N}}$ and we finally obtain the following differential inequality

$$\phi' + c_2^{\frac{N}{N+1}} \phi^{\frac{N}{N+1}} \le 0 \quad \text{on } [s_1,\infty). \tag{4.41}$$

The solution is explicit:

$$\phi(s) \le \begin{cases} \left((\phi(s_1))^{\frac{1}{N+1}} - \frac{c_2^{\frac{N}{N+1}}}{N}(s-s_1) \right)^{N+1} & \text{if } s_1 \le s \le s_2, \\ 0 & \text{if } s > s_2 \end{cases} \tag{4.42}$$

where

$$s_2 = s_1 + N c_2^{-\frac{N}{N+1}} (\phi(s_1))^{\frac{1}{N+1}}.$$

Hence $(w-s)_+ = 0$ if $s \ge s_2$ which implies the claim. □

Proof of Theorem 1.8. If u is a solution the assumption that $u \ge \delta > 0$ can be replaced by $u \ge 0$ since $u+\delta$ is a subsolution. It is standard that if u is bounded and $cap_{1,q'}(K)$ is zero then it is a weak solution. □

Motivated by the result of Theorem 1.6 when K is a single point, we have the following conjecture.

Conjecture. *Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain. Assume p, q are such that $1 \le p \le q < 2$ and $m > 0$. Let $K \subset \Omega$ be a compact set and $u \in C^1(\overline{\Omega} \setminus K)$ be a nonnegative solution of*

$$-\Delta u + m|\nabla u|^q - u^p = 0 \tag{4.43}$$

in $\Omega \setminus K$. If $cap_{1,q'}(K) = 0$, then u is a weak solution of (4.43) *in Ω and it belongs to $L^\infty(\Omega)$.*

5 Asymptotics of solutions

The natural way for studying the singular or asymptotic behaviour of solutions of (1.1) is to use the spherical coordinates $(r,\theta) \in [0,\infty) \times S^{N-1}$. Denoting $u(x) = u(r,\theta)$, equation (1.1) endows the form

$$-u_{rr} - \frac{N-1}{r}u_r - \frac{1}{r^2}\Delta' u + m\left(u_r^2 + \frac{1}{r^2}|\nabla' u|^2\right)^{\frac{q}{2}} - u^p = 0, \tag{5.1}$$

where Δ' and ∇' represent respectively the Laplace Beltrami operator and the covariant gradient identified with the tangential derivative on the unit sphere. This equation admits separable solutions i.e. solutions under the form $u(r,\theta) = r^{-a}\omega(\theta)$ if and only if $q = \frac{2p}{p+1}$, in which case

$$a = \alpha = \beta = \gamma.$$

Then ω is a nonnegative solution of

$$-\Delta'\omega - \alpha\left(\alpha + 2 - N\right)\omega + m\left(\alpha^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{p}{p+1}} - \omega^p = 0 \quad \text{in } S^{N-1}. \tag{5.2}$$

When $q \neq \frac{2p}{p+1}$, one nonlinear term could dominate the other thus the asymptotics can be described either by the separable solutions of the Lane-Emden equation (1.5) or the Riccatti equation (1.7). For the Lane-Emden equation the separable solutions have the form $u(r,\theta) = r^{-\alpha}\omega(\theta)$ where ω is a positive solution of

$$-\Delta'\omega - \alpha\left(\alpha + 2 - N\right)\omega - \omega^p = 0 \quad \text{in } S^{N-1}, \tag{5.3}$$

while for the Riccatti equation the separable solutions are under the form $u(r,\theta) = r^{-\beta}\omega(\theta)$ where ω is a positive solution of

$$-\Delta'\omega - \beta\left(\beta + 2 - N\right)\omega + m\left(\beta^2\phi^2 + |\nabla'\omega|^2\right)^{\frac{q}{2}} = 0 \quad \text{in } S^{N-1}. \tag{5.4}$$

Separable nonnegative solutions of the eikonal equation (1.8) have the form $u(r,\theta) = r^{-\gamma}\omega(\theta)$ and ω satisfies

$$m\left(\gamma^2\omega^2 + |\nabla'\omega|^2\right)^{\frac{q}{2}} - \omega^p = 0 \quad \text{in } S^{N-1}. \tag{5.5}$$

We recall below some results concerning these equations.

Theorem 5.1 *Let $N \geq 2$, $p, q > 1$ and $m \geq 0$.*
1- Suppose $q = \frac{2p}{p+1}$.
1-a If $N \geq 3$, $p \geq \frac{N}{N-2}$ and $m > 0$ there exists a unique positive constant solution x_m to (5.2).
1-b If $N = 2$ and $p > 1$, or $N \geq 3$ and $1 < p < \frac{N}{N-2}$ there exists no positive constant solution to (5.2) *if $0 \leq m < \mu^*$, a unique positive constant solution x_{μ^*} if $m = \mu^*$ and two positive constant solutions $x_{1,m} < x_{2,m}$ if $m > \mu^*$, where*

$$\mu^* := (p+1)\left(\frac{N - (N-2)p}{2p}\right)^{\frac{p}{p+1}}. \tag{5.6}$$

2- There exist positive solutions to (5.3) *if and only if $p > \frac{N}{N-2}$. Furthermore, if $\frac{N}{N-2} < p < \frac{N+1}{N-3}$, the positive solutions are constant and therefore unique with value*

$$\omega_0 = (\alpha(N-2-\alpha))^{\frac{1}{p-1}} = \left(\alpha\frac{(N-2)p - N}{p-1}\right)^{\frac{1}{p-1}}. \tag{5.7}$$

3- If $m > 0$ and $1 < q < \frac{N}{N-1}$ there exists a unique positive solution to (5.4)*. This solution is constant with value*

$$\xi_m = \frac{1}{\beta}\left(\frac{(N-1)q-N}{m(q-1)}\right)^{\frac{1}{q-1}}. \tag{5.8}$$

If $q \geq \frac{N}{N-1}$ there exists no positive solution to (5.4)*.*
4- If $m > 0$ and $p, q > 1$, $p \neq q$, any positive solution to (5.5) *is constant with value*

$$X_m = (m|\gamma|^q)^{\frac{1}{p-q}}. \tag{5.9}$$

Remark. Assertion 1 is proved in [8, Proposition 6.1], assertion 2 in [22], assertions 3 and 4 are easy consequences of the study of the extrema of a positive smooth solution.

5.1 Isolated singularities

In this Section we obtain the precise behaviour of positive singular solutions of (1.1) in $B_{r_0}\setminus\{0\}$.

5.1.1 Proof of Theorem 1.9

The proof is a delicate combination of various techniques, some new and some other already which have already been used by the authors in several different contexts.
Up to change of scale we assume that $r_0 = 1$. Set

$$u(r,\theta) = r^{-\alpha}v(t,\theta) \quad \text{with } t = \ln r,\ t \leq 0. \tag{5.10}$$

The function v satisfies

$$\begin{aligned} v_{tt} + (N-2-2\alpha)v_t + \alpha\left(\alpha+2-N\right)v + \Delta' v \\ - me^{-\frac{\sigma t}{p-1}}\left((v_t-\alpha v)^2 + |\nabla' v|^2\right)^{\frac{q}{2}} + v^p = 0, \end{aligned} \tag{5.11}$$

in $(-\infty,0]\times S^{N-1}$, recalling that $\sigma = (p+1)q - 2p$. By Theorem B the functions v, v_t and $|\nabla' v|$ is bounded in $(-\infty,0]\times S^{N-1}$. By standard regularity estimates and Ascoli-Arzela theorem the limit set at $-\infty$ of the trajectory of v in $C^2(S^{N-1})$,

$$\mathcal{T}_-[v] = \bigcup_{t\leq 0}\{v(t,.)\},$$

is a non-empty compact connected subset Γ_- of $C^2(S^{N-1})$. Set

$$\mathcal{E}[v](t) = \frac{1}{2}\int_{S^{N-1}}\left(v_t^2 - |\nabla' v|^2 + \alpha\left(\alpha+2-N\right)v^2 + \frac{2}{p+1}|v|^{p+1}\right)dS,$$

then

$$\frac{d}{dt}\mathcal{E}[v](t) = -(N-2-2\alpha)\int_{S^{N-1}} v_t^2 dS - me^{-\frac{\sigma t}{p-1}}\int_{S^{N-1}}\left((v_t-\alpha v)^2 + |\nabla' v|^2\right)^{\frac{q}{2}} v_t dS.$$

Therefore, for any $t < 0$,

$$\begin{aligned}\mathcal{E}[v](t) - \mathcal{E}[v](0) = (N-2-2\alpha)\int_t^0 \int_{S^{N-1}} v_t^2 dS d\tau \\ + m\int_t^0 e^{-\frac{\sigma\tau}{p-1}} \int_{S^{N-1}} \left((v_t - \alpha v)^2 + |\nabla' v|^2\right)^{\frac{q}{2}} v_t dS d\tau.\end{aligned} \tag{5.12}$$

Since $\mathcal{E}[v](t)$ and $\left((v_t - \alpha v)^2 + |\nabla' v|^2\right)^{\frac{q}{2}}$ are uniformly bounded, $N-2-2\alpha \neq 0$ because $p \neq \frac{N+2}{N-2}$ and $\sigma < 0$, this implies that

$$\int_{-\infty}^0 \int_{S^{N-1}} v_t^2 dS d\tau < \infty. \tag{5.13}$$

Since v_t is uniformly continuous on $(-\infty, 0] \times S^{N-1}$, it implies in turn that

$$\lim_{t\to-\infty} \int_{S^{N-1}} v_t^2(t) dS = 0.$$

Multiplying the equation (5.11) by v_{tt}, using the C^2 estimate on v and (5.13) we obtain that

$$\int_{-\infty}^0 \int_{S^{N-1}} v_{tt}^2 dS d\tau < \infty, \tag{5.14}$$

which implies in turn

$$\lim_{t\to-\infty} \int_{S^{N-1}} v_{tt}^2(t) dS = 0.$$

Letting $t \to -\infty$ in (5.11) we conclude that Γ_- is a a non-empty compact connected subset of the set on nonnegative solutions of (5.3).
If $1 < p \leq \frac{N}{N-2}$ we have

$$\lim_{t\to-\infty} v(t,.) = 0 \quad \text{uniformly on } S^{N-1}. \tag{5.15}$$

If $\frac{N}{N-2} < p < \frac{N+2}{N-2}$,

$$\text{either} \quad \lim_{t\to-\infty} v(t,.) = 0 \quad \text{or} \quad \lim_{t\to-\infty} v(t,.) = \omega_0 \quad \text{uniformly on } S^{N-1}. \tag{5.16}$$

where ω_0 is defined by (5.7).
The remaining problem is to analyse the case where $\lim_{t\to-\infty} v(t,.) = 0$. This is delicate and presented in the following lemmas.

Lemma 5.2 *Let $N \geq 3$, $p \in (1,\infty) \setminus \left\{\frac{N}{N-2}, \frac{N+2}{N-2}\right\}$ and $1 < q < \frac{2p}{p+1}$. If u is a nonnegative solution of* (1.1) *in $B_2 \setminus \{0\}$, such that*

$$\lim_{x\to 0} |x|^\alpha u(x) = 0, \tag{5.17}$$

then there exists $\epsilon > 0$ such that

$$u(x) \leq C|x|^{-\alpha+\epsilon} \qquad \text{for all } x \in B_1 \setminus \{0\}. \tag{5.18}$$

Furthermore

$$|\nabla u(x)| \leq C'|x|^{-\alpha-1+\epsilon} \qquad \text{for all } x \in B_1 \setminus \{0\}. \tag{5.19}$$

Proof. The key point is the proof is that under the assumptions on p the coefficients $\alpha(\alpha+2-N)$ and $N-2-2\alpha$ in the equation (5.11) satisfied by the function v defined before are not zero. We note that (5.18) is equivalent to

$$v(t,\theta) \le Ce^{\epsilon t} \qquad \text{for all } (t,\theta) \in (-\infty,0] \times S^{N-1}. \tag{5.20}$$

If (5.20) does not hold we have that

$$\limsup_{t\to-\infty} e^{-\epsilon t}\rho(t) = +\infty \quad \text{for all } \epsilon > 0,$$

where $\rho(t) = \sup\{v(t,\theta) : \theta \in S^{N-1}\}$. We use now a technique introduced in [18, Lemma 2.1]: it is proved that there exists a function $\eta \in C^\infty\big((-\infty,0]\big)$ such that

$$\begin{aligned}
&(i) && \eta > 0,\ \eta' > 0,\ \lim_{t\to-\infty} \eta(t) = 0;\\
&(ii) && 0 < \limsup_{t\to-\infty} \frac{\rho(t)}{\eta(t)} < +\infty;\\
&(iii) && \lim_{t\to-\infty} e^{-\varepsilon t}\eta(t) = +\infty \quad \text{for all } \varepsilon > 0;\\
&(iv) && \left(\frac{\eta'}{\eta}\right)', \left(\frac{\eta''}{\eta}\right)' \in L^1((-\infty,0));\\
&(v) && \lim_{t\to-\infty} \frac{\eta'(t)}{\eta(t)} = \lim_{t\to-\infty} \frac{\eta''(t)}{\eta(t)} = 0.
\end{aligned} \tag{5.21}$$

We define ψ by $v(t,\cdot) = \eta(t)\psi(t,.)$, then

$$\begin{aligned}
\psi_{tt} + K_1\psi_t + K_2\psi + \Delta'\psi - me^{-\frac{\sigma t}{p-1}}\eta^{q-1}\left(\left(\psi_t - \alpha\frac{\eta_t}{\eta}\psi\right)^2 + |\nabla'\psi|^2\right)^{\frac{q}{2}}\\
+ \eta^{p-1}\psi^p = 0 \quad \text{in } (-\infty,0]\times S^{N-1},
\end{aligned} \tag{5.22}$$

where

$$K_1(t) = N-2-2\alpha+2\frac{\eta'}{\eta} \quad \text{and} \quad K_2(t) = \alpha(\alpha+2-N) + (N-2-2\alpha)\frac{\eta'}{\eta} + \frac{\eta''}{\eta}.$$

The function ψ is bounded and by standard regularity estimates it is uniformly bounded in the C^2-topology of $(-\infty,0]\times S^{N-1}$. We set

$$\tilde{\mathcal{E}}[\psi](t) = \frac{1}{2}\int_{S^{N-1}} \left(\psi_t^2 - |\nabla'\psi|^2 - \alpha\left(\alpha+2-N\right)\psi^2\right) dS,$$

then

$$\begin{aligned}
\frac{d}{dt}\tilde{\mathcal{E}}[\psi](t) = &-\left(N-2-2\alpha+2\frac{\eta'}{\eta}\right)\int_{S^{N-1}}\psi_t^2 dS + \left((N-2-2\alpha)\frac{\eta'}{\eta} + \frac{\eta''}{\eta}\right)\int_{S^{N-1}}\psi\psi_t dS\\
&-\eta^{p-1}\int_{S^{N-1}}\psi^p\psi_t dS + me^{-\frac{\sigma t}{p-1}}\eta^{q-1}\int_{S^{N-1}}\left(\left(\psi_t - \alpha\frac{\eta_t}{\eta}\psi\right)^2 + |\nabla'\psi|^2\right)^{\frac{q}{2}}\psi_t dS.
\end{aligned} \tag{5.23}$$

We analyse the different terms in the right-hand side of (5.23):

$$\int_{S^{N-1}} \psi^p \psi_t dS = \frac{1}{p+1}\frac{d}{dt}\int_{S^{N-1}} \psi^{p+1}\eta^{p-1} - \frac{p-1}{p+1}\eta'\eta^{p-2}\int_{S^{N-1}} \psi^{p+1} dS.$$

By the mean value theorem, for any $t<0$ there exists $t^* \in (t,0)$ such that

$$\int_t^0 \int_{S^{N-1}} \eta^{p-1}\int_{S^{N-1}} \psi^p\psi_t dS d\tau = \frac{1}{p+1}\left[\int_{S^{N-1}} \psi^{p+1}\eta^{p-1}\right]_t^0 - \frac{1}{p+1}\left(\eta^{p-1}(0)-\eta^{p-1}(t)\right)\int_{S^{N-1}} \psi^{p+1}(t^*,.)dS,$$

and this expression is bounded independently of $t<0$. Also

$$\left((N-2-2\alpha)\frac{\eta'}{\eta}+\frac{\eta''}{\eta}\right)\int_{S^{N-1}}\psi\psi_t dS = \frac{1}{2}\frac{d}{dt}\left(\left((N-2-2\alpha)\frac{\eta'}{\eta}+\frac{\eta''}{\eta}\right)\int_{S^{N-1}}\psi^2 dS\right) - \frac{1}{2}\left((N-2-2\alpha)\left(\frac{\eta'}{\eta}\right)'+\left(\frac{\eta''}{\eta}\right)'\right)\int_{S^{N-1}}\psi^2 dS.$$

The term involving the gradient is clearly integrable on $(-\infty,0)$. Hence we obtain for any $t<0$,

$$\tilde{\mathcal{E}}[\psi](0)-\tilde{\mathcal{E}}[\psi](t) = -\int_t^0\left(N-2-2\alpha+2\frac{\eta'}{\eta}\right)\int_{S^{N-1}}\psi_t^2 dS d\tau + A(t) \tag{5.24}$$

where $A(t)$ is bounded independently of $t<0$. Because the left-hand side of (5.24) is bounded independently of $t<0$, $\frac{\eta'}{\eta}(\tau)\to 0$ when $\tau\to-\infty$ and $N-2-2\alpha\neq 0$ as $p\neq\frac{N+2}{N-2}$, we infer that

$$\int_{-\infty}^0\int_{S^{N-1}}\psi_t^2 dS d\tau < \infty. \tag{5.25}$$

By uniform continuity, this implies that $\psi_t(t)\to 0$ in $L^2(S^{N-1})$ when $t\to-\infty$. Multiplying the equation satisfied by ψ_{tt} we obtain similarly, using the previous estimate and (5.21)-(iv)-(v) that

$$\int_{-\infty}^0\int_{S^{N-1}}\psi_{tt}^2 dS d\tau < \infty; \tag{5.26}$$

in turn this implies that $\psi_{tt}(t)\to 0$ in $L^2(S^{N-1})$ when $t\to-\infty$. The limit set at $-\infty$ of the trajectory $\mathcal{T}_-[\psi]$ is a connected and compact subset of the set of nonnegative solutions of

$$\alpha(\alpha+2-N)\omega+\Delta'\omega = 0 \quad \text{in } S^{N-1}. \tag{5.27}$$

Since $\alpha(\alpha+2-N)$ is not an eigenvalue of $-\Delta'$ in $W^{1,2}(S^{N-1})$, it follows that $\omega=0$, which contradicts the fact that by (5.21)-(ii) the limit set contains at least one non-zero positive element. Hence (5.18) holds, as for (5.19) it is a consequence of Theorem 3.2. This ends the proof. □

Lemma 5.3 *Let the assumptions of Theorem 1.9 hold, then*
1- If $N \geq 3$ and $1 < p < \frac{N}{N-2}$ (resp. $N = 2$ and $p > 1$) there exists $k \geq 0$ such that $|x|^{N-2}u(x)$ (resp. $-u(x)/\ln|x|$) converges to k when $x \to 0$. Furthermore u satisfies (1.23).
2- If $N \geq 3$ and $\frac{N}{N-2} < p < \frac{N+2}{N-2}$,
2-(i) either $|x|^{\alpha}u(x)$ converges to ω_0 when $x \to 0$,
2-(ii) or u is a classical solution of (1.1) *in B_{r_0}.*

Proof. Since $|x|^{\alpha}u(x) + |x|^{\alpha+1}|\nabla u(x)|$ remains bounded and $q \leq \frac{2p}{p+1}$, we have

$$|x|^2 u^{p-1}(x) + |x||\nabla u(x)|^{q-1} \leq c_1 \quad \text{for all } x \in B_{r_0}. \tag{5.28}$$

Hence Harnack inequality is valid uniformly on any sphere with center 0 (see e.g. [23]) in the sense that

$$\max_{|y|=r} u(y) \leq c_2 \min_{|y|=r} u(y) \quad \text{for all } 0 < r \leq \tfrac{r_0}{2}. \tag{5.29}$$

Step 1: first estimate on the average of v. The second order linear equation

$$X'' + (N - 2 - 2\alpha)X' + \alpha(\alpha + 2 - N)X = 0 \tag{5.30}$$

admits the two linearly independent solutions

$$X_1(t) = e^{\lambda_1 t} \quad \text{and } X_2(t) = e^{\lambda_2 t},$$

where the λ_j are the roots of $P(\lambda) = \lambda^2 + (N - 2 - 2\alpha)\lambda + \alpha(\alpha + 2 - N)$. Note that these roots are explicit:

$$\lambda_1 = \alpha > \lambda_2 = \alpha + 2 - N, \tag{5.31}$$

and $\lambda_2 > 0$ (resp. $\lambda_2 < 0$) if $1 < p < \frac{N}{N-2}$ (resp. $p > \frac{N}{N-2}$). We set

$$H(t,.) = me^{-\frac{\sigma t}{p-1}} \left((v_t - \alpha v)^2 + |\nabla' v|^2\right)^{\frac{q}{2}} - v^p. \tag{5.32}$$

Since $\|v(t,.)\|_{L^\infty(SN-1)} + \|\nabla' v(t,.)\|_{L^\infty(SN-1)} \leq Ce^{\epsilon t}$ by (5.18)-(5.19), there holds

$$\|H(t,.)\|_{L^\infty(SN-1)} \leq c_3 e^{\delta_1 t} \tag{5.33}$$

where

$$\delta_1 = \min\left\{\epsilon p, \epsilon q - \tfrac{\sigma}{p-1}\right\}, \tag{5.34}$$

and $\sigma = (p+1)q - 2p < 0$. Let $\bar{v}(t)$ and $\overline{H}(t)$ be the average respectively of $v(t,.)$ and $H(t,.)$ on S^{N-1}. Then $|\overline{H}(t)| \leq Ce^{\delta_1 t}$. Since

$$\bar{v}'' + (N - 2 - 2\alpha)\bar{v}' + \alpha(\alpha + 2 - N)\bar{v} = \overline{H}(t). \tag{5.35}$$

Assuming that $\delta_1 \neq \lambda_1, \lambda_2$ (which can always be assume up to changing ϵ) the function $\bar{v}$ endows the general form

$$\bar{v}(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t} + C(t)e^{\delta_1 t}, \tag{5.36}$$

for some constants A and B and for some particular solution $C(t)e^{\delta_1 t}$ where C is bounded on $(-\infty, 0]$. This can be checked by the so-called method of "the variation of constants". Therefore, since $v(t,.) \to 0$ when $t \to -\infty$,

$$\bar{v}(t) = \begin{cases} Ae^{\lambda_1 t} + Be^{\lambda_2 t} + C(t)e^{\delta_1 t} & \text{if } 1 < p < \frac{N}{N-2} \\ Ae^{\lambda_1 t} + C(t)e^{\delta_1 t} & \text{if } p > \frac{N}{N-2}. \end{cases} \tag{5.37}$$

This leads us to the second decay estimate (besides the one given by Lemma 5.2)

$$\bar{v}(t) \leq c_4 e^{\theta_1 t} \tag{5.38}$$

where $\theta_1 = \min\{\lambda_2, \delta_1\}$ if $1 < p < \frac{N}{N-2}$ and $\theta_1 = \min\{\lambda_1, \delta_1\}$ if $p > \frac{N}{N-2}$.

Step 2: first a priori estimate on v. The global estimate on v is obtained by using an iterative method based upon the integral representation of the solutions introduced in [15]. We set

$$\mathbb{L} = -\left(-\Delta' + \tfrac{(N-2)^2}{4} I\right)^{\frac{1}{2}}, \tag{5.39}$$

and let $S(t) = e^{t\mathbb{L}}$ be the semigroup of contraction generated by $\mathbb{L}$ in $L^2(S^{N-1})$. Introducing the standard Hilbertian decomposition of $H^1(S^{N-1})$ associated to the operator $-\Delta'$, it is classical that the space $\mathbb{H} = \{\phi \in L^2(S^{N-1}) : \bar{\phi} = 0\}$ is invariant by $\mathbb{L}$, since $\bar{\phi}$ is the orthogonal projection in $H^1(S^{N-1})$ onto $(\ker(-\Delta'))^{\perp} = \mathbb{H}$. Because

$$\inf \sigma(\mathbb{L}\lfloor_{\mathbb{H}}) = \frac{N^2}{4},$$

we have

$$\|S(t)\phi\|_{L^2(S^{N-1})} \leq e^{-\frac{Nt}{2}} \|\phi\|_{L^2(S^{N-1})} \quad \text{for all } t > 0 \text{ and } \phi \in \mathbb{H}, \tag{5.40}$$

and

$$\|S(t)\phi\|_{L^\infty(S^{N-1})} \leq Ce^{-\frac{Nt}{2}} \|\phi\|_{L^\infty(S^{N-1})} \quad \text{for all } t > 0 \text{ and } \phi \in \mathbb{H} \cap L^\infty(S^{N-1}). \tag{5.41}$$

for some $C > 0$. Note that this last inequality is easily obtained by using the Hilbertian decomposition with spherical harmonics. The following representation formula for $v^* = v - \bar{v}$ is proved in [15]:

$$v^*(t,.) = e^{\frac{2\alpha+2-N}{2}t} S(-t)v^*(0,.) - \int_t^0 e^{\frac{2\alpha+2-N}{2}s} S(-s) \int_\infty^0 e^{\frac{N-2\alpha-2}{2}\tau} S(-\tau) H^*(-t-\tau+s, \sigma) d\tau ds \tag{5.42}$$

where $H^*(t,.) = H(t,.) - \overline{H}(t)$. Since

$$\|H^*(t,.)\|_{L^\infty(S^{N-1})} \leq c_3 e^{\delta_1 t} \tag{5.43}$$

by (5.33) where δ_1 is defined in (5.34), we get

$$\|v^*(t,.)\|_{L^\infty(S^{N-1})} \leq c_5 e^{(\alpha+1)t} + c_6 e^{\delta_1 t} \quad \text{for all } t \leq 0. \tag{5.44}$$

Writing $v(t,.) = \bar{v}(t) + v^*(t,.)$ we deduce

$$\|v(t,.)\|_{L^\infty(S^{N-1})} \le c_7 e^{(\alpha+1)t} + c_8 e^{\delta_1 t} + c_9 e^{\theta_1 t} \le c_{10} e^{\theta_1 t} \quad \text{for all } t \le 0, \tag{5.45}$$

where we use the value of θ_1 defined in (5.38) and λ_1, λ_2 given in (5.31). This leads us to an improvement of the decay estimate given by (5.20). Notice also that if $\theta_1 = \lambda_2 = \alpha + 2 - N$ (resp. $\theta_1 = \lambda_1 = \alpha$) when $1 < p < \frac{N}{N-2}$ (resp. $\frac{N}{N-2} < p < \frac{N+2}{N-2}$) we deduce from the definition of v that the function u is smaller that $c_{10}|x|^{2-N}$ (resp. is bounded by c_{10}).

Step 3: a priori estimate on v by iterations. For the sake of understanding we will distinguish two cases according to the sign of $p - \frac{N}{N-2}$.

(i) Let $1 < p < \frac{N}{N-2}$. Since $v(t,.) \le c_{10} e^{\theta_1 t}$, then by Theorem 3.2 that $v(t,.) + |\nabla v(t,.)| \le c_{11} e^{\theta_1 t}$. Therefore

$$\|H(t,.)\|_{L^\infty(S^{N-1})} \le c_{12} e^{\delta_2 t}$$

with

$$\delta_2 = \min\left\{\theta_1 p, \theta_1 q - \tfrac{\sigma}{p-1}\right\}.$$

Since (5.35) holds with H satisfying (5.33) with δ_1 replaced by δ_2, we deduce that

$$\bar{v}(t) = A e^{\lambda_1 t} + B e^{\lambda_2 t} + C(t) e^{\delta_2 t}$$

where A, B are constants and C is bounded which implies $\theta_2 = \min\{\lambda_2, \delta_2\}$. Since (5.35) holds with H satisfying (5.33) with δ_1 replaced by δ_2

$$\bar{v}(t) \le c_{13} e^{\theta_2 t}, \tag{5.46}$$

with $\theta_2 = \min\{\lambda_1, \lambda_2, \delta_2\} = \min\{\lambda_2, \delta_2\}$. The integral representation (5.42) is satisfied by $v^* = v - \bar{v}$ and we obtain as in the previous step that (5.44) holds with δ_1 replaced by δ_2 and finally

$$\|v(t,.)\|_{L^\infty(S^{N-1})} \le c_{14} e^{(\alpha+1)t} + c_{15} e^{\delta_2 t} + c_{16} e^{\theta_2 t} \le c_{17} e^{\theta_2 t} \quad \text{for all } t \le 0. \tag{5.47}$$

If $\theta_2 = \alpha + 2 - N$ we have the desired estimate, otherwise we iterate. We define the sequences

$$\begin{aligned} &(i) && \delta_1 = \min\left\{p\epsilon, q\epsilon - \tfrac{\sigma}{p-1}\right\} \text{ and } \theta_1 = \min\{\lambda_2, \delta_1\} \\ &(ii) && \delta_n = \min\left\{p\theta_{n-1}, q\theta_{n-1} - \tfrac{\sigma}{p-1}\right\} \text{ and } \theta_n = \min\{\lambda_2, \delta_n\}, \end{aligned} \tag{5.48}$$

for all the integers n such that $\delta_n < \lambda_2$. Then $\delta_n, \theta_n > 0$ and the function v satisfies

$$\|v(t,.)\|_{L^\infty(S^{N-1})} \le c_{1,n} e^{(\alpha+1)t} + c_{2,n} e^{\delta_n t} + c_{3,n} e^{\theta_n t} \le c_{4,n} e^{\theta_n t} \quad \text{for all } t \le 0. \tag{5.49}$$

Furthermore

$$\theta_n - \theta_{n-1} = \min\left\{\lambda_2 - \theta_{n-1}, \min\left\{(p-1)\theta_{n-1}, (q-1)\theta_{n-1} - \frac{\sigma}{p-1}\right\}\right\}. \tag{5.50}$$

We assume first that there exists a largest integer n_0 such that $\theta_n < \lambda_2$. Then $\theta_1 < \theta_2 < ... < \theta_n < ...\theta_{n_0}$ and $\theta_{n_0+1} = \lambda_2$.

If such a largest integer does not exist, then $\{\theta_n\}$ is increasing with limit $\theta_\infty \le \lambda_2$. By (5.50), θ_∞ and λ_2 coincide. By (5.48)-(ii), $\{\delta_n\}$ is increasing. For any $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that $\lambda_2 - \epsilon\theta_n < \lambda_2$ for $n \ge n_\epsilon$, hence

$$\delta_{n_\epsilon} > \min\left\{p(\lambda_2 - \epsilon), q\lambda_2 - \epsilon) - \frac{\sigma}{p-1}\right\} > \lambda_2$$

if ϵ is small enough. This implies that $\theta_{n_\epsilon} = \lambda_2$, contradiction. Therefore inequality (5.49) with $n = n_\epsilon$ becomes

$$\|v(t,.)\|_{L^\infty(S^{N-1})} \le c_{18} e^{(\alpha+2-N)t} \quad \text{for all } \ t \le 0. \tag{5.51}$$

(ii) Let $\frac{N}{N-2} < p < \frac{N+2}{N-2}$. The proof differs from the previous one only with very little modifications. Since $\lambda_2 < 0$, (5.48) is replaced by

$$\begin{aligned} &(i) && \delta_1 = \min\left\{p\epsilon, q\epsilon - \tfrac{\sigma}{p-1}\right\} \text{ and } \theta_1 = \min\{\lambda_1, \delta_1\} \\ &(ii) && \delta_n = \min\left\{p\theta_{n-1}, q\theta_{n-1} - \tfrac{\sigma}{p-1}\right\} \text{ and } \theta_n = \min\{\lambda_1, \delta_n\}. \end{aligned} \tag{5.52}$$

Inequality (5.49) holds with the θ_n defined above, and there exists an integer n_ϵ such that $\theta_n = \lambda_1 = \alpha$. Hence

$$\|v(t,.)\|_{L^\infty(S^{N-1})} \le c_{19} e^{\alpha t} \quad \text{for all } \ t \le 0. \tag{5.53}$$

Step 4: convergence. (i) When $1 < p < \frac{N}{N-2}$, the function H defined (5.32) satisfies

$$\|H(t,.)\|_{L^\infty(S^{N-1})} \le c_{20} e^{\tilde{\delta} t} \quad \text{for all } \ t \le 0. \tag{5.54}$$

with $\tilde{\delta} = \min\{\lambda_2 p, \lambda_2 q - \frac{\sigma}{p-1}\}$. Hence $|\overline{H}(t)|$ satisfies the same estimate and $\bar{v}$ can be written as in (5.36) with new coefficients A, B and $C(.)$ under the form

$$\bar{v}(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t} + C(t)e^{\tilde{\delta} t} = Be^{\lambda_2 t} + o(e^{\lambda_2 t}) \quad \text{as } t \to -\infty. \tag{5.55}$$

Since formulas (5.42), (5.43) and (5.44) holds with δ_1 replaced by δ we conclude that

$$\|v^*(t,.)\|_{L^\infty(S^{N-1})} = o(e^{\lambda_2 t}) \quad \text{as } t \to -\infty, \tag{5.56}$$

and finally

$$\lim_{t\to-\infty} e^{(N-2-\alpha)t} v(t,.) = B \quad \text{uniformly on } S^{N-1}. \tag{5.57}$$

Equivalently

$$\lim_{x\to 0} |x|^{N-2} u(x) = B. \tag{5.58}$$

Therefore $u \in L^p(B_{r_0})$. We use the same type of cut-off function η_n used in the proof of Theorem 1.6, except that we assume also that $|\Delta\eta_n| \le cn^2 \mathbf{1}_{B_{1/n}\setminus B_{1/(2n)}}$, and we obtain

$$-\int_{B_{r_0}} u\Delta\eta_n dx + m\int_{B_{r_0}} |\nabla u|^q \eta_n dx = \int_{B_{r_0}} u^p \eta_n dx. \tag{5.59}$$

The right-hand side of (5.59) is bounded from above by $\|u\|^p_{L^p(B_{\frac{2r_0}{3}})}$. We have also

$$\left|\int_{B_{r_0}} u\Delta\eta_n dx\right| \leq c_{21}n^{2-N-2+N} \leq c_{22}.$$

By Fatou's lemma we deduce that $\nabla u \in L^q(B_{\frac{2r_0}{3}})$. Therefore, by the Brezis-Lions Lemma [16] we conclude that there exists k such that (1.23) holds.

If $k = 0$, then $B = 0$ and (5.55) yields

$$\bar{v}(t) \leq c_{23}e^{\tilde{\theta}_1 t}, \tag{5.60}$$

with $\tilde{\theta}_1 = \min\left\{\lambda_1, \tilde{\delta}\right\}$. Using again the representation (5.42) combined with (5.54) we obtain

$$\|v(t,.)\|_{L^\infty(S^{N-1})} \leq c_{24}e^{(\alpha+1)t} + c_{25}e^{\tilde{\delta}t} + c_{26}e^{\tilde{\theta}_1 t} \leq c_{27}e^{\tilde{\theta}_1 t} \quad \text{for all } t \leq 0, \tag{5.61}$$

We define now the sequence

$$\begin{aligned} &(i) && \tilde{\delta}_1 := \tilde{\delta} \text{ and } \tilde{\theta}_1 = \min\{\lambda_1, \tilde{\delta}_1\} \\ &(ii) && \tilde{\delta}_n = \min\left\{p\tilde{\theta}_{n-1}, q\tilde{\theta}_{n-1} - \tfrac{\sigma}{p-1}\right\} \text{ and } \tilde{\theta}_n = \min\{\lambda_1, \tilde{\delta}_n\}, \end{aligned} \tag{5.62}$$

and we have

$$\|v(t,.)\|_{L^\infty(S^{N-1})} \leq Ce^{\tilde{\theta}_n t} \quad \text{for all } t \leq 0. \tag{5.63}$$

By the construction of Step 3-(ii) there exists n^* such that $\tilde{\theta}_n = \lambda_1$ which means that inequality (5.53) holds and $\bar{v}$ satisfies

$$\bar{v}(t) = Be^{\lambda_1 t} + C(t)e^{\tilde{\delta}_{n^*}t}) = Be^{\lambda_1 t} + o(e^{\lambda_1 t}) \quad \text{as } t \to -\infty, \tag{5.64}$$

and

$$\|v^*(.,t)\|_{L^\infty(S^{N-1})} = o(e^{\lambda_1 t}) \quad \text{as } t \to -\infty, \tag{5.65}$$

Hence

$$\lim_{t\to-\infty} e^{-\alpha t}v(t,.) = A \quad \text{uniformly on } S^{N-1}, \text{ equivalently } \lim_{x\to 0} u(x) = A. \tag{5.66}$$

Using again the same type of cut-off function η_n as in the proof of Theorem 1.6 we obtain successively that $|\nabla u| \in L^q(B_{r_0})$ and that u is a classical solution.

(ii) When $\frac{N}{N-2} < p < \frac{N+2}{N-2}$, (5.54) is valid with $\delta = \tilde{\delta} = \min\{\lambda_1 p, \lambda_1 q - \frac{\sigma}{p-1}\}$. Hence the proof of (i) when $A = 0$ applies and we obtain that u is a bounded classical solution.

□

Lemma 5.4 *Let the assumptions of Theorem 1.9 holds with $N \geq 3$ and $p = \frac{N}{N-2}$, then*
(i) either $|x|^{N-2}(-\ln|x|)^{\frac{N-2}{2}}u(x)$ converges to $\left(\frac{N-2}{\sqrt{2}}\right)^{N-2}$ when $x \to 0$,
(ii) or u is a classical solution of (1.1) in B_{r_0}.

Proof. The proof is based upon a combination of several techniques introduced in [33] for analysing the exterior problem

$$-\Delta u + |u|^{\frac{2}{N-2}}u = 0 \qquad \text{in } B^c_{r_0}, \tag{5.67}$$

and adapted in [4] to characterise the isolated singularities of

$$-\Delta u = u^{\frac{N}{N-2}}. \tag{5.68}$$

1- We claim that u satisfies

$$u(x) \le C|x|^{2-N}(-\ln|x|)^{\frac{2-N}{2}} \tag{5.69}$$

for $0 < |x| \le r_1$ *where* $r_1 < \min\left\{1, \frac{r_0}{2}\right\}$.
The function v which is defined by (5.10) with $\alpha = N-2$ here is bounded and satisfies

$$v_{tt} + (2-N)v_t + \Delta' v - me^{-\frac{\sigma t}{p-1}}\left((v_t + (2-N)v)^2 + |\nabla' v|^2\right)^{\frac{q}{2}} + v^{\frac{N}{N-2}} = 0 \tag{5.70}$$

in $(-\infty, 0] \times S^{N-1}$. By (5.15), $v(t,.) \to 0$ uniformly when $t \to -\infty$. The average $\bar v$ satisfies

$$\bar v_{tt} + (2-N)\bar v_t - \mathcal{H}(t) = 0,$$

where

$$\mathcal{H}(t) = \frac{1}{|S^{N-1}|}\int_{S^{N-1}} \left(me^{-\frac{\sigma t}{p-1}}\left((v_t + (2-N)v)^2 + |\nabla' v|^2\right)^{\frac{q}{2}} - v^{\frac{N}{N-2}}\right) dS.$$

Set $s = e^{(N-2)t}$, $z(s,.) = v(t,.)$ and $\bar z(s) = \bar v(t)$, then there holds

$$s^2 \bar z_{ss} - Z_1(s) + Z_2(s) = 0 \quad \text{in } (0, e^{2-N}) \tag{5.71}$$

where

$$Z_1(s) = \frac{ms^{-\frac{\sigma}{(p-1)(N-2)}}}{(N-2)^2|S^{N-1}|}\int_{S^{N-1}} \left[(N-2)^2(sz_s - z)^2 + |\nabla' z|^2\right]^{\frac{q}{2}} dS$$

and

$$Z_2(s) = \frac{1}{(N-2)^2|S^{N-1}|}\int_{S^{N-1}} z^{\frac{N}{N-2}} dS.$$

Using the energy method as in Lemma 5.2 and (5.15) we obtain that

$$\|z(s,.)\|_{L^\infty(S^{N-1})} + \|sz_s(s,.)\|_{L^\infty(S^{N-1})} \to 0 \quad \text{as } s \to 0. \tag{5.72}$$

If $0 < \delta < 1$ the function $s \mapsto w(s) := \bar z(s) + s^\delta$ satisfies

$$s^2 w_{ss} = s^2 \bar z_{ss} + \delta(\delta-1)s^\delta = Z_1(s) - Z_2(s) + \delta(\delta-1)s^\delta. \tag{5.73}$$

We set

$$\delta_0 = \frac{-\sigma}{(N-2)(p-1)} = \frac{2p - q(p+1)}{(N-2)(p-1)} = \frac{N - q(N-1)}{N-2},$$

then $0 < \delta_0 < 1$ since $1 < q < \frac{N}{N-1}$. We take $0 < \delta < \min\left\{\delta_0, \frac{N}{N-2}\right\}$. Then there exists $s_0 > 0$ such that for $0 < s \le s_0$ there holds $Z_1(s) < \frac{\delta(1-\delta)}{2} s^\delta$ which implies

$$s^2 w_{ss} + \frac{\delta(1-\delta)}{2} s^\delta + Z_2(s) \le 0 \quad \text{in } (0, s_0]. \tag{5.74}$$

The function w is therefore concave. Since it vanishes for $s = 0$, it is increasing. We now adapt the proof of [3, Lemma 1] and integrate (5.74) on (s, s_0). Using the fact that $Z_2(s) \ge \frac{1}{(N-2)^2} \bar{z}^{\frac{N}{N-2}}(s)$, we obtain

$$\begin{aligned} w_s(s_0) = w_s(s) + \int_s^{s_0} w_{ss} d\tau &\le w_s(s) - \int_s^{s_0} \left(\frac{\delta(1-\delta)}{2} \tau^{\delta-2} + \frac{Z_2(\tau)}{\tau^2} \right) d\tau \\ &\le w_s(s) - \int_s^{s_0} \left(\frac{\delta(1-\delta)}{2} \tau^{\delta-2} + \frac{\bar{z}^{\frac{N}{N-2}}(\tau)}{(N-2)^2 \tau^2} \right) d\tau. \end{aligned} \tag{5.75}$$

Since

$$w^{\frac{N}{N-2}} \le 2^{\frac{2}{N-2}} \left(\bar{z}^{\frac{N}{N-2}} + s^{\frac{N\delta}{N-2}} \right),$$

we infer that

$$\begin{aligned} w_s(s_0) &\le w_s(s) + \frac{1}{(N-2)(N-2-N\delta)} \left(s^{\frac{N\delta}{N-2}-1} - s_0^{\frac{N\delta}{N-2}-1} \right) \\ &\qquad - \frac{1}{2^{\frac{2}{N-2}} (N-2)^2} \int_s^{s_0} \frac{w^{\frac{N}{N-2}}(\tau)}{\tau^2} d\tau \\ &\le w_s(s) - C_1 \frac{w^{\frac{N}{N-2}}(s)}{s} + C_2 s^{\frac{N\delta}{N-2}-1} + C_1 \frac{w^{\frac{N}{N-2}}(s)}{s_0} - C_2 s_0^{\frac{N\delta}{N-2}-1} \end{aligned} \tag{5.76}$$

for some $C_1, C_2 > 0$.
We claim that

$$w_s(s) - C_1 \frac{w^{\frac{N}{N-2}}(s)}{s} + C_2 s^{\frac{N\delta}{N-2}-1} \ge 0. \tag{5.77}$$

Actually, if it were not true there would exist a sequence $\{s_n\} \subset (0, s_0]$ decreasing to 0 such that

$$w_s(s_n) - C_1 \frac{w^{\frac{N}{N-2}}(s_n)}{s_n} + C_2 s_n^{\frac{N\delta}{N-2}-1} < 0,$$

which would imply

$$w_s(s_0) < C_1 \frac{w^{\frac{N}{N-2}}(s_n)}{s_0} - C_2 s_0^{\frac{N\delta}{N-2}-1}. \tag{5.78}$$

Since $w(s_n) \to 0$, it would follow that $w_s(s_0) < 0$, contradiction.
Next we set

$$\rho(s) = w(s) + c s^{\frac{N\delta}{N-2}},$$

for some $c > 0$ which will be fixed later on. Then, from (5.77)

$$\rho_s(s) \geq C_1 \frac{w^{\frac{N}{N-2}}(s)}{s} + \left(c\frac{N\delta}{N-2} - C_2 \right) s^{\frac{N\delta}{N-2}-1}.$$

Now

$$\rho^{\frac{N}{N-2}}(s) \leq 2^{\frac{2}{N-2}} \left(w^{\frac{N}{N-2}}(s) + c^{\frac{N}{N-2}} s^{\left(\frac{N}{N-2}\right)^2 \delta} \right).$$

Therefore

$$\rho_s(s) \geq C_1 2^{-\frac{2}{N-2}} \frac{\rho^{\frac{N}{N-2}}(s)}{s} + \left(c\frac{N\delta}{N-2} - C_2 \right) s^{\frac{N\delta}{N-2}-1} - C_1 2^{-\frac{2}{N-2}} C^{\frac{N}{N-2}} s^{\left(\frac{N}{N-2}\right)^2 \delta - 1}.$$

Fixing $c = 2C_2 \frac{N-2}{N\delta}$, we deduce that for s small enough,

$$\rho_s(s) \geq C_1 2^{-\frac{2}{N-2}} \frac{\rho^{\frac{N}{N-2}}(s)}{s}, \tag{5.79}$$

which implies by integration,

$$\rho(s) \leq C_3 \left(-\ln s \right)^{\frac{2-N}{2}} \quad \text{on } (0, s_1]. \tag{5.80}$$

2- End of the proof. Set $h(t,.) = (-t)^{\frac{N-2}{2}} v(t,.)$, then h is bounded and it satisfies

$$\begin{aligned} h_{tt} + (N-2)(1+t))\, h_t - \frac{1}{t}\left(h^{\frac{2}{N-2}} - \frac{(N-2)^2}{2} \right) h + \frac{N(N-2)}{4t^2} h \\ - m e^{\frac{\sigma t}{p-1}} (-t)^{\frac{(2-N)q}{2}} \left(\left(h_t - (N-2)\left(1 + \frac{1}{t}\right) h \right)^2 + |\nabla' h|^2 \right)^{\frac{q}{2}} = 0. \end{aligned} \tag{5.81}$$

Using methods introduced in [33], it is proved in [12, Corollary 4.2] that $\left\| h(t,.) - \bar{h}(t) \right\|_{L^\infty(S^{N-1})}$ tends to 0 as $t \to \infty$ and consequently that $h(t,.)$ converges in $C^2(S^{N-1})$ to some limit ℓ and necessarily

$$\ell \in \left\{ 0, \left(\frac{N-2}{\sqrt{2}} \right)^{N-2} \right\}. \tag{5.82}$$

This ends the proof of Lemma 5.4 and consequently of Theorem 1.9. □

Remark 1. The convergence result *3* of Theorem 1.6 can be extended to the case $p \in \left(\frac{N}{N-2}, \frac{N+1}{N-3} \right) \setminus \left\{ \frac{N+2}{N-2} \right\}$ for every positive solution u such that $|x|^\alpha u(x)$ is bounded.

Remark 2. When $p = \frac{N}{N-2}$, the proof of the existence of solutions of (1.1) satisfying

$$\lim_{x \to 0} |x|^{N-2} \left(-\ln |x| \right)^{\frac{N-2}{2}} = \left(\frac{N-2}{\sqrt{2}} \right)^{N-2}$$

is obtained in the radial case in [13] using techniques from dynamical systems theory such as the central manifold.

Remark 3. The description of the behaviour in the case $q = \frac{2p}{p+1}$ exhibits a remarkable complexity which appears out of reach in the general case. The treatment of radial solutions is performed in [9] and shows this complexity.

5.1.2 Proof of Theorem 1.10

Before proving the result we recall that if $q \geq \frac{N}{N-1}$ and $1 < p < q$ any nonnegative solution u of (1.1) in $B_{r_0} \setminus \{0\}$ is a bounded weak solution of (1.1) in B_{r_0} by Theorem 1.6.

Proof. Next we assume $p < q < \frac{N}{N-1}$. By Theorem 1.3 u satisfies

$$|x|u(x) + |\nabla u(x)| \leq c_1 |x|^{-\frac{1}{q-1}}, \tag{5.83}$$

for $0 < |x| \leq r_0$. Since $q > \frac{2p}{p+1}$, this implies that (5.28) holds and therefore u satisfies a uniform Harnack inequality in $B_{\frac{r_0}{2}}$ in the sense that

$$u(x) \leq c_2 u(y) \qquad \text{for all } x, y \in B_{\frac{r_0}{2}} \setminus \{0\} \text{ s.t. } |x| = |y|. \tag{5.84}$$

Case 1. Assume that $|x|^{N-2}u(x)$ is bounded. We cannot apply directly the result of Theorem 3.2 since $q > \frac{2p}{p+1}$ and we define u_ℓ by

$$u_\ell(x) = \ell^{N-2} u(\ell x) \quad \text{for } \ell > 0.$$

Then u_k satisfies

$$-\Delta u_\ell + m\ell^{N-q(N-1)} |\nabla u_\ell|^q - \ell^{N-p(N-2)} u_\ell^p = 0 \qquad \text{in } B_{\frac{r_0}{\ell}}.$$

Since $q < \frac{N}{N-1}$, $N - q(N-1) > 0$, therefore we deduce as in the proof of Theorem 3.2 that ∇u_ℓ satisfies estimate (3.15) with k replaced by ℓ, which implies

$$|\nabla u(x)| \leq c_3 |x|^{1-N} \qquad \text{for all } x \in B_{\frac{r_0}{2}} \setminus \{0\}. \tag{5.85}$$

then

$$|\nabla u|^q \in L^{\frac{N}{N-1}-\epsilon}(B_{r_0}) \quad \text{and} \quad u^p \in L^1(B_{r_0}),$$

for any $\epsilon > 0$. By the Brezis-Lions Lemma [16] there exists $k \geq 0$ such that u satisfies

$$-\Delta u + m|\nabla u|^q = u^p + k\delta_0 \quad \text{in } \mathcal{D}'(B_{r_0}). \tag{5.86}$$

Furthermore, u verifies

$$\lim_{r \to 0} r^{N-2} u(r, .) = c_N k \tag{5.87}$$

in $L^1(S^{N-1})$ and actually uniformly. By comparing u with the radial solution $\tilde{u}_k$ of the Riccatti equation (1.7)

$$-\Delta u + m|\nabla u|^q = k\delta_0 \qquad \text{in } \mathcal{D}'(B_{r_0}) \tag{5.88}$$

vanishing on ∂B_{r_0} (see [7]), we obtain by the maximum principle that $u \geq \tilde{u}_k$. The solution u_k^* of (5.88) with $r_0 = \infty$ and vanishing at infinity is explicit and given in [7, Theorem 3.13] by

$$u_k^*(x) = \int_{|x|}^{\infty} s^{1-N} \left(\frac{q-1}{N-q(N-1)} s^{N-q(N-1)} + c_N k^{1-q} \right)^{-\frac{1}{q-1}} ds. \tag{5.89}$$

Therefore we easily obtain that the solution u verifies

$$u_k^*(x) - C(r_0) \le \tilde{u}_k \le u(x) \quad \text{for all } x \in B_{r_0} \setminus \{0\}, \tag{5.90}$$

for some constant $C(r_0) > 0$.
If $k = 0$, we proceed as in the proof of Lemma 5.3-*Step 4* with the same sequences $\{\tilde{\delta}_n\}$ and $\{\tilde{\theta}_n\}$. With the notations therein, we obtain (5.65) and (5.66) and derive that u is a bounded regular solution.

Case 2. Assume that $|x|^{N-2}u(x)$ is unbounded near $x = 0$. Then there exists a sequence $\{r_n\}$ decreasing to 0 such that

$$\lim_{r_n \to 0} \sup_{|x|=r_n} r_n^{N-2} u(x) = \infty.$$

By (5.84) there holds

$$\lim_{r_n \to 0} \inf_{|x|=r_n} r_n^{N-2} u(x) = \infty.$$

Let $k > 0$, since $|x|^{N-2}\tilde{u}_k(x) = c_N k$, where $\tilde{u}_k$ has been defined in (5.88), for $r_n \le r_{n_k}$, one has $\tilde{u}_k \le u$ in $B_{r_0} \setminus B_{r_n}$ by the maximum principle, which implies that the same inequality holds in $B_{r_0} \setminus \{0\}$. Let $k \to \infty$ implies that

$$\lim_{k \to \infty} \tilde{u}_k := \tilde{u}_\infty \le u \quad \text{in } B_{r_0} \setminus \{0\}.$$

Since (5.90) still holds with $k = \infty$ and combining with [7, Theorem 3.13] we obtain that

$$\xi_m |x|^{-\beta} - C(r_0) \le \tilde{u}_\infty \le u(x) \quad \text{for all } x \in B_{r_0} \setminus \{0\}, \tag{5.91}$$

where ξ_m is expressed by (5.8); indeed it is proved in the above mentioned article that $\lim_{k \to \infty} u_k^* := u_\infty^*(x) = \xi_m |x|^{-\beta}$. This yields

$$\liminf_{x \to 0} |x|^\beta u(x) \ge \xi_m. \tag{5.92}$$

In order to obtain the sharp estimate from above, we define, for $\ell > 0$, $S_\ell[u](x) = \ell^\beta u(\ell x) = u_\ell(x)$ in $B_{\frac{r_0}{\ell}} \setminus \{0\}$, where u_ℓ satisfies

$$-\Delta u_\ell + m|\nabla u_\ell|^q = \ell^{\beta(p-1)-2} u_\ell^p. \tag{5.93}$$

Let

$$\phi^* = \limsup_{|x| \to 0} |x|^\beta u(x) = \lim_{r_n \to 0} r_n^\beta u(r_n, \theta_n),$$

for some sequence $\{(r_n, \theta_n)\} \to (0, \theta_*)$ and set $u_n(x) := u_{r_n}(x)$. Then $\phi^* \ge \xi_m$ by (5.92). The function u_n satisfies

$$-\Delta u_n + m|\nabla u_n|^q = r_n^{2-\beta(p-1)} u_n^p \tag{5.94}$$

in $B_{\frac{r_0}{r_n}} \setminus \{0\}$ and

$$|x| u_n(x) + |\nabla u_n(x)| \le c_4 |x|^{-\frac{1}{q-1}} \quad \text{if } 0 < |x| \le \tfrac{r_0}{2r_n}. \tag{5.95}$$

Since $q > p > \frac{2p}{p+1}$, we have $2 - \beta(p-1) > 0$ and by standard regularity result (see e.g. [23]), there exists a subsequence, still denoted by $\{u_{r_n}\}$, and a C^2 function u^* such that $u_{r_n} \to u^*$ in

the C^2_{loc} topology of $\mathbb{R}^N \setminus \{0\}$. The function u^* is a nonnegative solution of the Riccatti equation (1.7) in $\mathbb{R}^N \setminus \{0\}$ and it tends to 0 at ∞. By [7, Theorem 3.13], either $u^* \equiv 0$, either there exists $k > 0$ such that u^* verifies (5.87), or

$$u^*(x) = \xi_m |x|^{-\beta}, \tag{5.96}$$

where ξ_m is expressed by (5.8). Note that $\xi_m |x|^{-\beta}$ is the maximal positive solution of (1.7) in $\mathbb{R}^N \setminus \{0\}$ which tends to 0 at infinity. Since $u^*(1, \sigma_*) = \phi^* \geq \xi_m$, we obtain that $\phi^* = \xi_m$ which implies

$$\lim_{x \to 0} |x|^\beta u(x) = \xi_m. \tag{5.97}$$

□

Remark. The existence of solutions of (5.86) for any $k > 0$ is proved in the radial case in [13]. We can observe that if $k > 0$ is small enough the existence is straightforward since there exists a solution $\hat{u}_k$ of

$$\begin{array}{rl} -\Delta u - u^p = k\delta_0 & \text{in } \mathcal{D}'(B_{r_0}) \\ u = 0 & \text{in } \partial B_{r_0}, \end{array} \tag{5.98}$$

see [25]. The function $\hat{u}_k$ is a supersolution of (1.1). Since the solution $\tilde{u}_k$ of (5.88) is a subsolution, and both $\hat{u}_k$ and $\tilde{u}_k$ are ordered and have the same behaviour at 0 given by (5.87) it follows that there exists a solution u_k of (1.1) which vanishes on ∂B_{r_0} and satisfies $\tilde{u}_k \leq u_k \leq \hat{u}_k$. Hence it satisfies (5.87) and it is easy to check that it is a solution of (5.86).

5.2 Behaviour at infinity

The asymptotic behaviour of positive solutions of (1.1) in an exterior domain is obtained in some particular cases by using the energy method. Here we make more precise the results contained in Theorem 1.5.

Theorem 5.5 *Let $N \geq 3$, $\frac{N}{N-2} < p < \frac{N-1}{N+3}$, $p \neq \frac{N+2}{N-2}$, $q > \frac{2p}{p+1}$ and $m > 0$. If u is a positive solution of* (1.1) *in $B^c_{r_0}$ satisfying* (1.22) *the following alternative holds.*
(i) Either

$$\lim_{|x| \to \infty} |x|^\alpha u(x) = \omega_0 \tag{5.99}$$

where ω_0 is given by (5.7)*.*
(ii) Or there exists $k > 0$ such that

$$\lim_{|x| \to \infty} |x|^{N-2} u(x) = k. \tag{5.100}$$

Proof. We recall that estimate (1.22) holds when $\frac{N}{N-2} < p < \frac{N+2}{N-2}$ by the doubling method. As in the proof of Theorem 1.9 we set $u(r, \theta) = r^\alpha w(t, \theta)$ with $t = \ln r > 0$ (we can assume that $r_0 < 1$) and w is a bounded solution of (5.11) in $(0, \infty) \times S^{N-1}$. Notice that $\sigma > 0$. The omega-limit set of the trajectory

$$\mathcal{T}_+[v] = \bigcup_{t \geq 0} v(t, .)$$

is a non-empty compact connected subset Γ_+ of $C^2(S^{N-1})$. The energy method used in the proof of Theorem 1.9 applies because $p \neq \frac{N+2}{N-2}$, hence

$$\lim_{t\to\infty} \|v_t(t,.)\|_{L^2(S^{N-1})} = \lim_{t\to\infty} \|v_{tt}(t,.)\|_{L^2(S^{N-1})} = 0.$$

This implies that Γ_+ is a compact and connected subset of the set of nonnegative solutions of (5.3). Since $\frac{N}{N-2} < p < \frac{N+1}{N-3}$, $\Gamma_+ = \{0, X_0\}$ by [22], hence if $X_0 \in \Gamma_+$, then (5.99) holds, otherwise

$$\lim_{|x|\to\infty} |x|^\alpha u(x) = 0. \tag{5.101}$$

In such a case, we obtain by changing t into $-t$ as in the proof of Lemma 5.2, that there exists $\epsilon > 0$ such that

$$v(t,\theta) \leq c_1 e^{-\epsilon t} \quad \text{in } (0,\infty)\times S^{N-1} \Longrightarrow u(x) \leq c_1|x|^{-\alpha-\epsilon} \quad \text{in } B_{r_0}\setminus\{0\}. \tag{5.102}$$

The computations of Lemma 5.3 are still valid, but since $t \to \infty$ the results therein have to be re-interpreted. Since the spherical average $\bar{v}(t)$ of $v(t,.)$ satisfies (5.35), in this equation the right-hand side $\overline{H}(t)$ which satisfies $\overline{H}(t) \leq c_2 e^{-\delta_1 t}$ and δ_1 expressed by (5.34). By the same standard method of "the variation of constants" the expression (5.36) which expressed all the solutions of under the form

$$\bar{v}(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t} + C(t)e^{-\delta_1 t}, \tag{5.103}$$

where A and B are constant and $C(t)$ is a bounded function. The exponents λ_1 and λ_2 are given by (5.31). It is important to notice that $\lambda_2 < 0 < \lambda_1$. Thus, $\bar{v}(t) \to 0$ when $t \to \infty$ implies $A = 0$ and

$$\bar{v}(t) \leq c_3 e^{-\delta_1 t} \quad \text{for } t > 0 \tag{5.104}$$

with δ_1 given by (5.48)-(i). The representation formula (5.42) valid for $v^* = v - \bar{v}$ is replaced by

$$v^*(t,.) = e^{\frac{2\alpha+2-N}{2}t} S(t)v^*(0,.) - \int_0^t e^{\frac{2\alpha+2-N}{2}s} S(s) \int_0^\infty e^{\frac{N-2\alpha-2}{2}\tau} S(\tau) H^*(t+\tau-s,\sigma)d\tau ds \tag{5.105}$$

see [15, (1.14)], where

$$H^*(t,.) = me^{-\frac{\sigma t}{p-1}} \left((v_t - \alpha v)^2 + |\nabla' v|^2\right)^{\frac{q}{2}} - v^p$$
$$- \frac{1}{|S^{N-1}|} \int_{S^{N-1}} \left(me^{-\frac{\sigma t}{p-1}} \left((v_t - \alpha v)^2 + |\nabla' v|^2\right)^{\frac{q}{2}} - v^p\right) dS.$$

Since

$$\|H(t,.)\|_{L^\infty(S^{N-1}} \leq c_4 e^{-\delta_1 t},$$

and (5.41) holds, we deduce that

$$\|v^*(t,.)\|_{L^\infty(S^{N-1})} \leq C_1 e^{-(N-\alpha-1)t} + C_2 e^{-\delta_1 t} \quad \text{for all } t \leq 0. \tag{5.106}$$

Since $v(t,.) = \bar{v}(t) + v^*(t,.)$ we deduce

$$\|v(t,.)\|_{L^\infty(S^{N-1})} \le C_1 e^{-(N-\alpha-1)t} + C_2 e^{-\delta_1 t} + C_3 e^{-\theta_1 t} \le C_4 e^{-\theta_1 t} \quad \text{for all } t \le 0, \tag{5.107}$$

with θ_1 from (5.48)-(i). We iterate the process and, defining δ_n and θ_n by (5.48), we obtain, as long as $\theta_n < \lambda_2$,

$$\|v(t,.)\|_{L^\infty(S^{N-1})} \le C_1 e^{-(N-\alpha-1)t} + C_2 e^{-\delta_n t} + C_3 e^{-\theta_n t} \le C_4 e^{-\theta_n t} \quad \text{for all } t \ge 0, \tag{5.108}$$

Then there exists n^* such that $\theta_{n^*} = \lambda_2 = \alpha + 2 - N$ and this implies that

$$v(t,.) \le C_5 e^{(\alpha+2-N)t}. \tag{5.109}$$

This implies

$$\bar{v}(t) = B e^{\lambda_2 t}(1 + o(1)) \quad \text{as } t \to \infty.$$

Since

$$\|v^*(t,.)\|_{L^\infty(S^{N-1})} := \|v(t,.) - \bar{v}(t)\|_{L^\infty(S^{N-1})} \le C_1 e^{-(N-\alpha-1)t} + C_2 e^{-\delta_{n^*} t}$$

and $\delta_{n^*} = \min\left\{p\theta_{n^*}, q\theta_{n^*} + \dfrac{\sigma}{p-1}\right\} > \theta_{n^*}$, we conclude that

$$\lim_{t\to\infty} e^{(N-2-\alpha)t} v(t,.) = B \quad \text{uniformly on } S^{N-1}, \tag{5.110}$$

which is (5.100) with $k = B$. By Corollary 2.5 we have necessarily $k > 0$. □

Remark. The existence of radial solutions in $B^c_{r_0}$ satisfying (5.100) with $k > 0$ is proved in [2].

The next result completes Theorem 1.4.

Theorem 5.6 *Let $N \ge 3$, $1 < q < \min\{\frac{2p}{p+1}, \frac{N}{N-1}\}$ and $m > 0$. Let u be a positive solution of (1.1) in $B^c_{r_0}$.*

1- Then

$$\liminf_{|x|\to\infty} |x|^\beta u(x) \ge \xi_m. \tag{5.111}$$

2- If $|x|^\beta u(x)$ is bounded, then

$$\lim_{|x|\to\infty} |x|^\beta u(x) = \xi_m. \tag{5.112}$$

Proof. For $\ell \ge 1$ the function $u_\ell(x) = \ell^\beta u(\ell x)$ satisfies (5.93) in $B^c_{\frac{r_0}{\ell}}$ and is bounded therein. Since $q < \frac{2p}{p+1}$, $\beta(p-1) - 2 < 0$, thus we deduce by regularity techniques that

$$|x|u(x) + |\nabla u(x)| \le C|x|^{-\frac{1}{q-1}}. \tag{5.113}$$

This implies that $|x|^2 u^{p-1}(x) + |x||\nabla u(x)|^{q-1} \le C$ in $B^c_{r_0}$, and therefore Harnack inequality holds uniformly in $B^c_{r_0}$ in the sense that

$$\max_{|x|=r} u(x) \le C \min_{|x|=r} u(x) \qquad \text{for all } r \ge r_0. \tag{5.114}$$

Set $\mu = \min_{|z|=1} u(z)$ and define k_μ by

$$\mu = u^*_{k_\mu}(1) = \int_1^\infty \left(\frac{q-1}{N-q(N-1)} s^{N-q(N-1)} + k_\mu^{1-q} \right)^{-\frac{1}{q-1}} s^{N-1} ds. \tag{5.115}$$

Then for any $\epsilon > 0$, $u \geq (u^*_k - \epsilon)_+$ which is a subsolution of the Riccatti equation in B_1^c. This implies that $u \geq u^*_{k_\mu}$ in B_1^c. Since

$$\lim_{|x|\to\infty} |x|^\beta u^*_{k_\mu}(x) = \lim_{|x|\to\infty} \int_{|x|}^\infty \left(\frac{q-1}{N-q(N-1)} s^{N-q(N-1)} + k_\mu^{1-q} \right)^{-\frac{1}{q-1}} s^{N-1} ds = \xi_m, \tag{5.116}$$

actually this limit is independent of k_μ, it follows that

$$\liminf_{|x|\to\infty} |x|^\beta u(x) \geq \xi_m.$$

This implies (5.112).
Set

$$\psi^* = \limsup_{|x|\to\infty} |x|^\beta u(x) = \lim_{r_n\to\infty} r_n^\beta u(r_n, \theta_n)$$

where $\theta_n \in S^{N-1}$ and we can assume that $\theta_n \to \theta^* \in S^{N-1}$. Then $\psi^* \geq \xi_m$. The function $u_{r_n} : x \mapsto r_n^\beta u(r_n x)$ satisfies

$$-\Delta u_{r_n} + m|\nabla u_{r_n}|^q = r_n^{2-\beta(p-1)} u_{r_n}^p = r_n^{\frac{\sigma}{q-1}} u_{r_n}^p \tag{5.117}$$

in $B^c_{\frac{r_0}{r_n}}$. Since $\sigma < 0$, we have that $r_n^{\frac{\sigma}{q-1}} \to 0$. By the local regularity a priori estimates inherited from (5.113) implies that, up to a subsequence still denoted by $\{r_n\}$, u_{r_n} converge in the C^2-local topology of $\mathbb{R}^N \setminus \{0\}$ to a positive solution w of

$$-\Delta w + m|\nabla w|^q = 0 \qquad \text{in } \mathbb{R}^N \setminus \{0\}. \tag{5.118}$$

Because of (5.113) and similarly to the proof of Theorem 1.10 we can use Arzela-Ascoli theorem to infer that up to a subsequence still denoted by $\{r_n\}$, u_{r_n} converges in the C^2_{loc} topology of $\mathbb{R}^N \setminus \{0\}$ to a positive solution of the Riccatti equation (1.7) in $\mathbb{R}^N \setminus \{0\}$ which is a function u^*_k ($0 < k \leq \infty$) given by the expression given by (5.89). Because $\psi^* = w(1) \geq \xi_m = \lim_{k\to\infty} u^*_k(1)$. Hence $\psi^* = \xi_m$ which conclude the proof. □

6 Appendix

In this Section we prove a technical result concerning the existence of positive radial solutions of

$$-v'' - \frac{N-1}{r} v' + m|v'|^q = 0 \tag{6.1}$$

on (r_0, ∞) satisfying non-homogeneous Dirichlet conditions at $r = r_0$ and at infinity.

Lemma 6.1 *Let $q > 1$, $0 < r_0 < \tau$ and $a, b > 0$. Then there exists a solution v of (6.1) on (r_0, τ) satisfying $v(r_0) = a$ and $v(\tau) = b$ if and only if $a = b$, or, if $a \neq b$:*

1- When $a < b$, for any $1 < q \leq 2$ and $\tau > r_0$.

2- When $a < b$, for any $q > 2$ and $\tau \geq \tau^* > r_0$ where τ^* depends on $b - a$.

3- When $a > b$, for any $1 < q \leq 2$ and $\tau > r_0$

4- When $a > b$, for any $q > 2$ and $\tau > r_0$ if and only if

$$a - b < \left(\frac{q(N-1)-N}{m(q-1)}\right)^{\frac{1}{q-1}} r_0^{2-N} \int_1^{\frac{\tau}{r_0}} t^{1-N} \left(1 - t^{N-q(N-1)}\right)^{-\frac{1}{q-1}} dt. \tag{6.2}$$

Proof. If $a = b$ the constant function $v \equiv a$ is a solution. If v_1 and v_2 are solutions of (2.17) and if there exists $\theta > r_0$ such that $v_1'(\theta) = v_2'(\theta)$, then $v_1 = v_2 + v_1(\theta) - v_2(\theta)$ by the Cauchy-Lipschitz theorem. This implies in particular that if v_1 and v_2 are solution either on (r_0, τ) with $v_1(r_0) = v_2(r_0)$ and $v_1(\tau) = v_2(\tau)$, or on (r_0, ∞) with $v_1(r_0) = v_2(r_0)$ and $\lim_{r\to\infty}(v_1(r) - v_2(r)) = 0$, then $v_1 = v_2$. We first consider the problem on (r_0, τ) for some $\tau > r_0$ and if $a, b > 0$ we denote by $v := v_{a,b}$ the solution of (6.1) on (r_0, τ) such that $v(r_0) = a$ and $v(\tau) = b$. Solutions are explicit by setting $w(r) = r^{N-1}v'(r)$, then

$$w' - mr^{(1-q)((N-1))}|w|^q = 0. \tag{6.3}$$

Case 1: $a < b$. If a solution exists it is increasing and we can replace v by $\tilde{v} = v - a$, thus $\tilde{v}(r_0) = 0$ and $\tilde{v}'(r) \geq 0$

$$r^{N-1}\tilde{v}'(r) = \begin{cases} \left[(r_0^{N-1}\tilde{v}'(r_0))^{1-q} - \frac{m(q-1)}{N-q(N-1)}\left(r^{N-q(N-1)} - r_0^{N-q(N-1)}\right)\right]^{-\frac{1}{q-1}} & \text{if } q \neq \frac{N}{N-1} \\ \left[(r_0^{N-1}\tilde{v}'(r_0))^{1-q} - m(q-1)\ln\frac{r}{r_0}\right]^{-\frac{1}{q-1}} & \text{if } q = \frac{N}{N-1}. \end{cases}$$

We set $X := \tilde{v}'(r_0)$ and we study the mapping $r \mapsto \mathcal{T}_X(r)$ defined by

$$\mathcal{T}_X(r) = \int_{r_0}^r s^{1-N}\left[(r_0^{N-1}X)^{1-q} - \frac{m(q-1)}{N-q(N-1)}\left(s^{N-q(N-1)} - r_0^{N-q(N-1)}\right)\right]^{-\frac{1}{q-1}} ds \tag{6.4}$$

if $q \neq \frac{N}{N-1}$, and

$$\mathcal{T}_X^*(r) = \int_{r_0}^r s^{1-N}\left[(r_0^{N-1}X)^{1-q} - m(q-1)\ln\frac{r}{r_0}\right]^{-\frac{1}{q-1}} ds \tag{6.5}$$

$q = \frac{N}{N-1}$.

(i) If $N - q(N-1) > 0$, $\mathcal{T}_X$ is defined for $r_0 \leq r < r_X := \left[\frac{N-q(N-1)}{m(q-1)}(r_0^{N-1}X)^{1-q} + r_0^{N-q(N-1)}\right]^{\frac{1}{N-q(N-1)}}$.

(ii) If $q = \frac{N}{N-1}$, $\mathcal{T}^*(X)$ is defined for $r_0 \leq r < r_X^* := r_0 e^{\frac{1}{m(q-1)}(r_0^{N-1}X)^{1-q}}$.

(iii) If $N - q(N-1) < 0$, $\mathcal{T}_X$ is defined for any $r \geq r_0$ if $X \leq X_0 := \left[\frac{N(q-1)-N}{m(q-1)r_0}\right]^{\frac{1}{q-1}}$, and for

$$r < \tilde{r}_X := \left[1 - \frac{q(N-1)-N}{m(q-1)r_0 X^{q-1}}\right]^{-\frac{1}{q(N-1)-N}} r_0$$

if $X > X_0$.
In case (i) (resp. (ii)), we fix $\tau > r_0$ then the mapping $X \mapsto \mathcal{T}_X(\tau)$ (resp. $X \mapsto \mathcal{T}^*_X(\tau)$) is continuous, increasing and defined provided $\tau < r_X$ (resp. $\tau < r^*_X$), that is

$$X < X_\tau := r_0^{1-N} \left[\frac{m(q-1)}{N-q(N-1)} \left(\tau^{N-q(N-1)} - r_0^{N-q(N-1)} \right) \right]^{-\frac{1}{q-1}}, \tag{6.6}$$

in case (i) and

$$X < X^*_\tau := r_0^{1-N} \left[m(q-1) \ln \frac{\tau}{r_0} \right]^{-\frac{1}{q-1}} \tag{6.7}$$

in case (ii). Furthermore $\mathcal{T}_0(\tau) = \mathcal{T}^*_0(\tau) = 0$ and $\lim_{X \uparrow X_\tau} \mathcal{T}_X(\tau) = \lim_{X \uparrow X^*_\tau} \mathcal{T}^*_X(\tau) = \infty$ since $q \le 2$. As a consequence there exists a unique $\tilde{X} \in (0, X_\tau)$ (resp. $\tilde{X} \in (0, X^*_\tau)$) such that $\mathcal{T}_{\tilde{X}}(\tau)) = b - a$ (resp. $\mathcal{T}^*_{\tilde{X}}(\tau)) = b - a$).
In case (iii) we have in the case $X \le X_0$,

$$\lim_{r \to \infty} \mathcal{T}_X(r) = \begin{cases} \infty & \text{if } N = 2 \\ C_1(X) := \frac{r_0 X}{N-2} \left[1 - \left(\frac{X}{X_0} \right)^{q-1} \right]^{-\frac{1}{q-1}} & \text{if } N \ge 3. \end{cases} \tag{6.8}$$

Since $C_1(0) = 0$ and $C_1(X) \to \infty$ when $X \uparrow X_0$, C_1 is a continuous increasing function from $[0, X_0]$ onto $[0, \infty]$. If $X > X_0$,

$$\lim_{r \to \tilde{r}_X} \mathcal{T}_X(r) = \begin{cases} \infty & \text{if } \frac{N}{N-1} < q \le 2 \\ C_2(X) & \text{if } q > 2, \end{cases} \tag{6.9}$$

where

$$C_2(X) = \left(\frac{q(N-1) - N}{m(q-1)} \right)^{\frac{1}{q-1}} \tilde{r}_X^{\frac{q-2}{q-1}} \int_{\frac{r_0}{\tilde{r}_X}}^1 \left(t^{N-q(N-1)} - 1 \right)^{-\frac{1}{q-1}} t^{1-N} dt. \tag{6.10}$$

For $\tau > r_0$, we introduce again the mapping $X \mapsto \mathcal{T}_X(\tau)$. In view of the last relation in the case $\frac{N}{N-1} < q \le 2$ then for any $b > a$ and $\tau > r_0$ there exists a unique $\tilde{X} > X_0$ such that $\tau < r_{\tilde{X}}$ and $\mathcal{T}_{\tilde{X}}(\tau) = b - a$.
If $q > 2$ and $N \ge 3$, for any $b > a$ there exists $\tau^* > r_0$, depending on $b - a$, such that for any $\tau \ge \tau^*$ there exists $X \le X_0$ such that $\mathcal{T}_X(\tau) = b - a$. We can explicit τ^* by $\tau^* = \tilde{r}_{X^*}$ where X^* is characterized by $C_2(X^*) = b - a$.

Case 2: $a > b$. Then v is decreasing and the method has to be slightly modified in order to obtain a positive solution of $-v'' - \frac{N-1}{r} v' + m|v'|^q = 0$ on (r_0, τ) such that $v(r_0) = a$ and $v(\tau) = b$. By replacing v by $\tilde{v} := v - b$ we look for a solution $\tilde{v}$ vanishing at τ and positive on (r_0, τ). Let $X = \tilde{v}'(r_0)$ then

$$-r^{N-1} \tilde{v}'(r) = \begin{cases} \left[(-r_0^{N-1} X)^{1-q} + \frac{m(q-1)}{N-q(N-1)} \left(r^{N-q(N-1)} - r_0^{N-q(N-1)} \right) \right]^{-\frac{1}{q-1}} & \text{if } q \ne \frac{N}{N-1} \\ \left[(-r_0^{N-1} X)^{1-q} + m(q-1) \ln \frac{r}{r_0} \right]^{-\frac{1}{q-1}} & \text{if } q = \frac{N}{N-1}. \end{cases}$$

We study the mapping $r \mapsto \mathcal{S}_X(r)$ defined by

$$\mathcal{S}_X(r) = a - b - \int_{r_0}^{r} s^{1-N} \left[(-r_0^{N-1}X)^{1-q} + \frac{m(q-1)}{N-q(N-1)} \left(s^{N-q(N-1)} - r_0^{N-q(N-1)} \right) \right]^{-\frac{1}{q-1}} ds \tag{6.11}$$

if $q \neq \frac{N}{N-1}$ and

$$\mathcal{S}^*_X(r) = a - b - \int_{r_0}^{r} s^{1-N} \left[(-r_0^{N-1}X)^{1-q} + m(q-1) \ln \frac{s}{r_0} \right]^{-\frac{1}{q-1}} ds \tag{6.12}$$

if $q = \frac{N}{N-1}$. If $q \leq \frac{N}{N-1}$, these two functions are defined on (r_0, τ). A solution $\tilde{v}$ satisfying the boundary conditions at $r = r_0$ and $r = \tau$ corresponds to the fact that $\mathcal{S}_X(\tau) = 0$ if $q \neq \frac{N}{N-1}$ or $\mathcal{S}^*_X(\tau) = 0$ if $q = \frac{N}{N-1}$.
(i) If $q < \frac{N}{N-1}$ we have

$$\lim_{X \uparrow 0} \mathcal{S}_X(\tau) = a - b \text{ and } \lim_{X \to -\infty} \mathcal{S}_X(\tau) = -\infty, \tag{6.13}$$

because $q < 2$ implies that $\int_{r_0}^{\tau} s^{1-N} \left[\frac{m(q-1)}{N-q(N-1)} \left(s^{N-q(N-1)} - r_0^{N-q(N-1)} \right) \right]^{-\frac{1}{q-1}} ds = \infty$.
(ii) If $q = \frac{N}{N-1}$ we have also

$$\lim_{X \uparrow 0} \mathcal{S}^*_X(\tau) = a - b \text{ and } \lim_{X \to -\infty} \mathcal{S}^*_X(\tau) = -\infty. \tag{6.14}$$

This implies that in these two cases for any $\tau > 0$ there exists a unique $X < 0$ such that $\mathcal{S}_X(\tau) = 0$ or $\mathcal{S}^*_X(\tau) = 0$.
(iii) If $q > \frac{N}{N-1}$, $\mathcal{S}_X(r)$ is defined for any $X \leq 0$ and any $r \in (r_0, \tau)$. We write it under the form

$$\mathcal{S}_X(\tau) = a - b - \int_{r_0}^{\tau} s^{1-N} \left[(-r_0^{N-1}X)^{1-q} + \frac{m(q-1)}{q(N-1)-N} \left(r_0^{N-q(N-1)} - s^{N-q(N-1)} \right) \right]^{-\frac{1}{q-1}} ds \tag{6.15}$$

We have that $\lim_{X \uparrow 0} \mathcal{S}_X(\tau) = a - b$ and $\lim_{X \to -\infty} \mathcal{S}_X(\tau) = -\infty$ if $\frac{N}{N-1} < q \leq 2$; in such case there exists $X_\tau < 0$ such that $\mathcal{S}_{X_\tau}(\tau) = 0$. On the contrary, if $q > 2$, we have

$$\lim_{X \to -\infty} \mathcal{S}_X(\tau) = a - b - \left(\frac{q(N-1)-N}{m(q-1)} \right)^{\frac{1}{q-1}} r_0^{2-N} \int_1^{\frac{\tau}{r_0}} t^{1-N} \left(1 - t^{N-q(N-1)} \right)^{-\frac{1}{q-1}} dt. \tag{6.16}$$

In that case we can find some $X = X(\tau) < 0$ (actually always unique) such that $\mathcal{S}_{X_\tau}(\tau) = 0$ if and only if

$$a - b < \left(\frac{q(N-1)-N}{m(q-1)} \right)^{\frac{1}{q-1}} r_0^{2-N} \int_1^{\frac{\tau}{r_0}} t^{1-N} \left(1 - t^{N-q(N-1)} \right)^{-\frac{1}{q-1}} dt. \tag{6.17}$$

Letting $\tau \to \infty$ we can find $\tau > r_0$ such that (6.17) holds if and only if (6.2) holds. □

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